Infinite primes are besides p-adicization and the representation of space-time surface as an associative (co-associative) sub-manifold of hyper-octonionic space, basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible. Infinite primes generate wild philosophical speculations involved and the fate of speculations is usually sad. There are also amazing analogies with basic quantum physics, which make me to take infinite primes seriously.

\vm{\it 1. Why infinite primes are unavoidable?} \vm

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a

super symmetric quantum field theory. At given level of hierarchy one can
identify the decomposition of space-time surface to p-adic regions with
the corresponding decomposition of the infinite prime to primes at a lower
level of infinity: at the basic level are finite primes for which one
cannot find any formula.

\vm{\it 2. Two views about the role of infinite primes and physics in TGD
Universe}\vm

Two different views about how infinite primes, integers, and rationals
might be relevant in TGD Universe have emerged.

\begin{enumerate}

\item The first view is based on the idea that infinite primes characterize
quantum states of the entire Universe. 8-D hyper-octonions make this
correspondence very concrete since 8-D hyper-octonions have interpretation
as 8-momenta. By quantum-classical correspondence also the decomposition of
space-time surfaces to p-adic space-time sheets should be coded by infinite
hyper-octonionic primes. Infinite primes could even have a representation
as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space.

\item The second view is based on the idea that infinitely structured
space-time points define space-time correlates of mathematical cognition.
The mathematical analog of Brahman=Atman identity would however suggest
that both views deserve to be taken seriously.

\end{enumerate}

\vm{\it 3. Infinite primes and infinite hierarchy of second
quantizations}\vm

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

\vm{\it 4. Infinite primes as a bridge between quantum and classical?}\vm

An important stimulus came from the observation stimulated by algebraic number theory. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes whereas hitherto it was possible to construct explicitly only what might be called generating infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers).

Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

\vm{\it 5. Various equivalent characterizations of space-times as surfaces}

One can imagine several number-theoretic characterizations of the space-time surface.

\begin{enumerate}

\item The approach based on octonions and quaternions suggests that space-time surfaces might correspond to associative or hyper-quaternionic surfaces of hyper-octonionic imbedding space.

\item Space-time surfaces could be seen as absolute minima of the K\"ahler action. The challenge is to prove that this characterization is equivalent with the number theoretical dynamics,

\end{enumerate}

\vm{\it 6. The representation of infinite complex-octonionic primes as 4-surfaces}\vm

The difficulties caused by the Euclidian metric signature of the number theoretical norm forced to give up the idea that space-time surfaces could be regarded as quaternionic sub-manifolds of octonionic space, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions {\it resp.} -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart for the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The notions of hyper-quaternionic and octonionic manifold make sense but it is implausible that $H=M^4\times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces can be assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a locally fixed complex structure (preferred imaginary unit contained by tangent space at each point of HO) are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4\times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

Any hyper-octonion analytic function $OH\rightarrow OH$ defines a function $g: OH\rightarrow SU(3)$ acting as the group of octonion automorphisms leaving a preferred imaginary unit invariant, and g in turn defines a foliation of OH and $H=M^4\times CP_2$ by space-time surfaces. The selection can be local which means that G_2 appears as a local gauge group.

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P: OH\rightarrow OH$ and hence also a foliation of OH and $H=M^4\times CP_2$ by 4-surfaces. Therefore space-time surface can be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO\rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of K\"ahler action.

\vm{\it 7. Generalization of ordinary number fields: infinite primes and cognition} \vm

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by

ordinary
rationals which are their inverses becomes possible. These units are not
units in the p-adic sense and have a finite p-adic norm which can be differ
from one. This construction generalizes also to the case of hyper-quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this
approach differs from the standard introduction of infinitesimals in the
sense that sum is replaced by multiplication meaning that the set of real
units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time
and imbedding space can be seen as infinitely structured and able to
represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its
structure. For instance, in the real sense surfaces in the space of units
correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this
structure is completely invisible and is relevant only for the physics of
cognition. One can say that Universe is an algebraic hologram, and there is
an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

%\end{abstract}