This chapter considers possible answers to the   basic questions of the p-adicization program, which are following.

Some of the basic questions of the p-adicization program are following.

\begin{enumerate}

\item Is there some kind of duality between real and p-adic physics? What is its precise mathematic formulation? In particular, what is the concrete map of p-adic physics in long scales (in real sense) to real physics in short scales? Can one find a rigorous mathematical formulation of the canonical identification induced by the map $p\rightarrow 1/p$ in pinary expansion of p-adic number such that it is both continuous and respects symmetries or one must accept the finite measurement resolution.

Few years after writing this the answer to this question is in terms of the notion of p-adic manifold. Canonical identification serving as its building brick however allows many variants and it seems that quantum arithmetics provides one further variant

\item What is the origin of the p-adic length scale hypothesis suggesting that primes near power of two are physically preferred? Why Mersenne primes seem to be especially important (p-adic mass calculations suggest this)?

This chapter studies some ideas but does not provide a clearcut answer to these questions. The notion of quantum arithmetics obtained is central in this approach.

\end{enumerate}

The starting point of quantum arithmetics is the map $n\rightarrow n_q$ taking integers to quantum integers: $n_q=(q^n-q^{-n})/(q-q^{-1})$. Here $q =exp(i\pi/n)$ is quantum phase defined as a root of unity. From TGD point of view prime roots $q =exp(i\pi/p)$ are of special interest. Also prime prime power roots $q =exp(i\pi/p^n)$ of unity are of interest. Quantum phase can be also generalized to complex number with modulus different from unity.

One can consider several variants of quantum arithmetics. One can regard finite integers as either real or p-adic. In the intersection of \blockquote{real and p-adic worlds} finite integers can be regarded both p-adic and real.

\begin{enumerate}

\item If one regards the integer n real one can keep some information about the prime decomposition of n by dividing n to its prime factors and performing the mapping $p\rightarrow p_q$. The map takes prime first to finite field $G(p,1)$ and then maps it to quantum integer. Powers of p are mapped to zero unless one modifies the quantum map so that p is mapped to p or $1/p$ depending on whether one interprets the outcome as analog of p-adic number or real number. This map can be seen as a modification of p-adic norm to a map, which keeps some information about the prime factorization of the integer. Information about both real and p-adic structure of integer is kept.

\item For p-adic integers the decomposition into prime factors does not make sense. In this case it is natural to use pinary expansion of integer in powers of p and perform the quantum map for the coefficients without decomposition to products of primes $p_1<p$. This map can be seen as a modification of canonical identification.

\item If one wants to interpret finite integers as both real and p-adic then one can imagine the definition of quantum integer obtained by de-compositing n to a product of primes, using pinary expansion and mapping coefficients to quantum integers looks natural. This map would keep information about both prime factorization and also a bout pinary series of factors. One can also decompose the coefficients to prime factors but it is not clear whether this really makes sense since in finite field $G(p,1)$ there are no primes.

\end{enumerate}

One can distinguish between two basic options concerning the definition of quantum integers.

\begin{enumerate}

\item For option I the prime number decomposition of integer is mapped to its quantum counterpart by mapping the primes l to quantum primes $l_q=(q^l-q^{-l})/(q-q^{-1})$, $q= exp(i\pi/p)$ so that image of product is product of images. Sums are {\it not} mapped to sums as is easy to verify. p is mapped to zero for the standard definition of quantum integer. Now p is however mapped to itself or $1/p$ depending on whether one wants to interpret quantum integer as p-adic or real number. Quantum integers generate an algebra with respect to sum and product.

\item Option II one uses pinary expansion and maps the prime factors of coefficients to quantum primes. There seems to be no point in decomposing the pinary coefficients to their prime factors so that they are mapped to standard quantum integers smaller than p.

The quantum primes l_q act as generators of Kac-Moody type algebra defined by powers p^n such that sum is completely analogous to that for Kac-Moody algebra: $a+b= \sum_n a_np^n+\sum b_np^n=\sum_n (a_n+b_n)p^n$. For p-adic numbers this is not the case.

\item For both options it is natural to consider the variant for which one has expansion $n= \sum_k n_kp^{kr}$, $n_k<p^r$, $r=1,2...$. p^k would serve as cutoff.

\end{enumerate}

The notion of quantum matrix group differing from ordinary quantum groups in that matrix elements are commuting numbers makes sense. This group forms a discrete counterpart of ordinary quantum group and its existence suggested by quantum classical correspondence. The existence of this group for matrices with unit determinant is guaranteed by mere ring property since the inverse matrix involves only arithmetic product and sum.

\begin{enumerate} \item The quantum counterparts of special linear groups $SL(n,F)$ exists always. For the covering group $SL(2,C)$ of $SO(3,1)$ this is the case so that 4-dimensional Minkowski space is in a very special position. For orthogonal, unitary, and orthogonal groups the quantum counterpart exists only if the number of powers of p for the generating elements of the quantum matrix group satisfies an upper bound characterizing the matrix group.

\item For the quantum counterparts of $SO(3)$ ($SU(2)$/ $SU(3)$) the orthogonality conditions state that at least some multiples of the prime characterizing quantum arithmetics is sum of three (four/six) squares. For $SO(3)$ this condition is strongest and satisfied for all integers, which are not of form $n= 2^{2r}(8k+7))$. The number $r_3(n)$ of representations as sum of squares is known and $r_3(n)$ is invariant under the scalings $n\rightarrow 2^{2r}n$. This means scaling by 2 for the integers appearing in the square sum representation.

\end{enumerate}

The findings about quantum $SO(3)$ suggest a possible explanation for p-adic length scale hypothesis and preferred p-adic primes.

\begin{enumerate}

\item The idea to be studied is that the quantum matrix group which is discrete is in some sense very large for preferred p-adic primes. If cognitive representations correspond to the representations of quantum matrix group, the representational capacity of cognitive representations is high and this kind of primes are survivors in the algebraic evolution leading to algebraic extensions with increasing dimension. The simple estimates of this chapter restricting the consideration to finite fields ($O(p)=0$ approximation) do not support this idea in the case of Mersenne primes.

\item An alternative idea is that number theoretic evolution leading to algebraic extensions of rationals with increasing dimension favors p-adic primes which do not split in the extensions to primes of the extension. There is also a nice argument that infinite primes which are in one-one correspondence with prime polynomials code for algebraic extensions. These primes code also for bound states of elementary particles. Therefore the stable bound states would define preferred p-adic primes as primes which do not split in the algebraic extension defined by infinite prime. This should select Mersenne primes as preferred ones.

\end{enumerate}