The work with TGD inspired model for quantum computation led to the realization that von Neumann algebras, in particular hyper-finite factors, could provide the mathematics needed to develop a more explicit view about the construction of M-matrix generalizing the notion of S-matrix in zero energy ontology (ZEO). In this chapter I will discuss various aspects of hyper-finite factors and their possible physical interpretation in TGD framework.

\vm{\it 1. Hyper-finite factors in quantum TGD}\vm

The following argument suggests that von Neumann algebras known as hyper-finite factors (HFFs) of type III$_1$ appearing in relativistic quantum field theories provide also the proper mathematical framework for quantum TGD.

\begin{enumerate}

\item The Clifford algebra of the infinite-dimensional Hilbert space is a von Neumann algebra known as HFF of type II$_1$. Therefore also the Clifford algebra at a given point (light-like 3-surface) of world of classical worlds (WCW) is HFF of type II$_1$. If the fermionic Fock algebra defined by the fermionic oscillator operators assignable to the induced spinor fields (this is actually not obvious!) is infinite-dimensional it defines a representation for HFF of type II$_1$. Super-conformal symmetry suggests that the extension of the Clifford algebra defining the fermionic part of a super-conformal algebra by adding bosonic super-generators representing symmetries of WCW respects the HFF property. It could however occur that HFF of type II_{∞} results.

\item WCW is a union of sub-WCWs associated with causal diamonds (CD) defined as intersections of future and past directed light-cones. One can allow also unions of CDs and the proposal is that CDs within CDs are possible. Whether CDs can intersect is not clear.

\item The assumption that the M^4 proper distance a between the
tips of
CD is quantized in powers of 2 reproduces p-adic length scale
hypothesis but one must also consider the possibility that a can
have all
possible values. Since $SO(3)$ is the isotropy group of CD, the
CDs
associated with a given value of a and with fixed lower tip are
parameterized by the Lobatchevski space $L(a)=SO(3,1)/SO(3)$.
Therefore
the CDs with a free position of lower tip are parameterized by
$M^4\times L(a)$. A possible interpretation is in terms of quantum
cosmology with a identified as cosmic time. Since Lorentz
boosts define a non-compact group, the generalization of so called
crossed
product construction strongly suggests that the local Clifford
algebra of
WCW is HFF of type III$_1$. If one allows all values of a, one
ends up
with $M^4\times M^4_+$ as the space of moduli for WCW.

\item An interesting special aspect of 8-dimensional Clifford
algebra
with Minkowski signature is that it allows an octonionic
representation
of gamma matrices obtained as tensor products of unit matrix 1 and
7-D
gamma matrices γ_k and Pauli sigma matrices by replacing 1
and
γ_k by octonions. This inspires the idea that it might be
possible to end up with quantum TGD from purely number theoretical
arguments. One can start from a local octonionic Clifford algebra in
M^8. Associativity (co-associativity) condition is satisfied if
one
restricts the octonionic algebra to a subalgebra associated with any
hyper-quaternionic and thus 4-D sub-manifold of M^8. This means
that
the induced gamma matrices associated with the K\"ahler action span
a
complex quaternionic (complex co-quaternionic) sub-space at each
point
of the sub-manifold. This associative (co-associative) sub-algebra
can
be mapped a matrix algebra. Together with M^8-H duality this
leads
automatically to quantum TGD and therefore also to the notion of
WCW
and its Clifford algebra which is however only mappable to an
associative (co-associative(algebra and thus to HFF of type II$_1$.

\end{enumerate}

\vm {\it 2. Hyper-finite factors and M-matrix} \vm

HFFs of type III$_1$ provide a general vision about M-matrix.

\begin{enumerate}

\item The factors of type III allow unique modular automorphism Δ^{it} (fixed apart from unitary inner automorphism). This raises the question whether the modular automorphism could be used to define the M-matrix of quantum TGD. This is not the case as is obvious already from the fact that unitary time evolution is not a sensible concept in zero energy ontology.

\item Concerning the identification of M-matrix the notion of state as it is used in theory of factors is a more appropriate starting point than the notion modular automorphism but as a generalization of thermodynamical state is certainly not enough for the purposes of quantum TGD and quantum field theories (algebraic quantum field theorists might disagree!). Zero energy ontology requires that the notion of thermodynamical state should be replaced with its \blockquote{complex square root} abstracting the idea about M-matrix as a product of positive square root of a diagonal density matrix and a unitary S-matrix. This generalization of thermodynamical state -if it exists- would provide a firm mathematical basis for the notion of M-matrix and for the fuzzy notion of path integral.

\item The existence of the modular automorphisms relies on Tomita-Takesaki theorem, which assumes that the Hilbert space in which HFF acts allows cyclic and separable vector serving as ground state for both HFF and its commutant. The translation to the language of physicists states that the vacuum is a tensor product of two vacua annihilated by annihilation oscillator type algebra elements of HFF and creation operator type algebra elements of its commutant isomorphic to it. Note however that these algebras commute so that the two algebras are not hermitian conjugates of each other. This kind of situation is exactly what emerges in zero energy

ontology (ZEO): the two vacua can be assigned with the positive and negative
energy parts of the zero energy states entangled by M-matrix.

\item There exists infinite number of thermodynamical states related by
modular automorphisms. This must be true also for their possibly existing
\blockquote{complex square roots}. Physically they would correspond to different
measurement interactions meaning the analog of state function collapse
in zero modes fixing the classical conserved charges equal to the quantal
counterparts. Classical charges would be parameters characterizing zero modes.

\end{enumerate}

A concrete construction of M-matrix motivated the recent rather precise
view about basic variational principles is proposed. Fundamental fermions
localized to string world sheets can be said to propagate as massless
particles along their boundaries. The fundamental interaction vertices
correspond to two fermion scattering for fermions at opposite throats of
wormhole contact and the inverse of the conformal scaling generator L_0
would define the stringy propagator characterizing this interaction.
Fundamental bosons correspond to pairs of fermion and antifermion at
opposite throats of wormhole contact. Physical particles correspond to
pairs of wormhole contacts with monopole K\"ahler magnetic flux flowing
around a loop going through wormhole contacts.

\vm{\it 3. Connes tensor product as a realization of finite measurement
resolution} \vm

The inclusions ${\cal N}\subset {\cal M}$ of factors allow an attractive
mathematical description of finite measurement resolution in terms of
Connes tensor product but do not fix M-matrix as was the original
optimistic belief.

\begin{enumerate}

\item In ZEO ${\cal N}$ would create
states experimentally indistinguishable from the original one.
Therefore
${\cal N}$ takes the role of complex numbers in non-commutative
quantum
theory. The space ${\cal M}/{\cal N}$ would correspond to the
operators
creating physical states modulo measurement resolution and has
typically
fractal dimension given as the index of the inclusion. The
corresponding
spinor spaces have an identification as quantum spaces with
non-commutative ${\cal N}$-valued coordinates.

\item This leads to an elegant description of finite measurement
resolution. Suppose that a universal M-matrix describing the
situation for
an ideal measurement resolution exists as the idea about square root
of
state encourages to think. Finite measurement resolution forces to
replace
the probabilities defined by the M-matrix with their ${\cal N}$
\blockquote{averaged}
counterparts. The \blockquote{averaging} would be in terms of the
complex square root
of ${\cal N}$-state and   a direct analog of functionally or path
integral
over the degrees of freedom below measurement resolution defined by
(say)
length scale cutoff.

\item One can construct also directly M-matrices satisfying the
measurement resolution constraint. The condition that ${\cal N}$
acts like
complex numbers on M-matrix elements as far as ${\cal N}$-
\blockquote{averaged}
probabilities are considered is satisfied if M-matrix is a tensor
product
of M-matrix in ${\cal M}({\cal N}$ interpreted as finite-
dimensional space
with a projection operator to ${\cal N}$. The condition that $
{\cal N}$
averaging in terms of a complex square root of ${\cal N}$ state
produces
this kind of M-matrix poses a very strong constraint on M-matrix if
it is
assumed to be universal (apart from variants corresponding to
different
measurement interactions).

\end{enumerate}

\vm{\it 4. Analogs of quantum matrix groups from finite measurement resolution?}\vm

The notion of quantum group replaces ordinary matrices with matrices with non-commutative elements. In TGD framework I have proposed that the notion should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution.

In this article I will consider the notion of quantum matrix inspired by recent view about quantum TGD and it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution. The basic idea is to replace complex matrix elements with operators expressible as products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers.

The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. The weak definition of determinant meaning its development with respect to a fixed row or column does not pose additional conditions. Strong definition of determinant requires its invariance under permutations of rows and columns. The permutation of rows/columns turns out to have interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

The resulting quantum matrices define a more general structure than quantum group but provide a concrete representation and interpretation for quantum group in terms of finite measurement resolution if q is a root of unity. For $q=\pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by sign factor invariant under the permutations of both rows and columns. One could also understand the fractal structure of

inclusion sequences of
hyper-finite factors resulting by recursively replacing operators appearing as matrix elements with
quantum matrices.

\vm {\it 5. Quantum spinors and fuzzy quantum mechanics} \vm

The notion of quantum spinor leads to a quantum mechanical description of
fuzzy probabilities. For quantum spinors state function reduction cannot be
performed unless quantum deformation parameter equals to $q=1$. The reason
is that the components of quantum spinor do not commute: it is however
possible to measure the commuting operators representing moduli squared of
the components giving the probabilities associated with
\blockquote{true} and \blockquote{false}.
The universal eigenvalue spectrum for probabilities does not in general
contain (1,0) so that quantum qbits are inherently fuzzy. State
function reduction would occur only after a transition to q=1 phase and
decoherence is not a problem as long as it does not induce this transition.

%\end{abstract}