Infinite primes are besides p-adicization and the representation of space-time region as a associative (co-associative) sub-manifold of hyper-octonionic space the basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

\vm{\it 1. Why infinite primes are unavoidable} \vm

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible.
Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at

a lower
level of infinity: at the basic level are finite primes for which one
cannot find any formula.

\vm{\it 2. Two views about the role of infinite primes and physics in TGD
Universe}\vm

Two different views about how infinite primes, integers, and rationals
might be relevant in TGD Universe have emerged. \begin{enumerate}

\item The first speculative view is based on the idea that infinite primes
characterize quantum states of the entire Universe. 8-D complexified
octonions containing hyper-octonions as sub-space would make this
correspondence very concrete since 8-D hyper-octonions have interpretation
as 8-momenta. By quantum-classical correspondence also the decomposition of
space-time surfaces to p-adic space-time sheets should be coded by infinite
hyper-octonionic primes. Infinite primes could even have a representation
as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space.
This view is addmittedly speculative since the notion of octonionic prime makes
sense only for complexified octonions.

\item The second view is based on the idea that infinitely structured
space-time points define space-time correlates of mathematical cognition.
The mathematical analog of Brahman=Atman identity would however suggest
that both views deserve to be taken seriously.

\end{enumerate}

\vm{\it 3. Infinite primes and infinite hierarchy of second quantizations}\vm

The discovery of infinite primes suggested strongly the possibility to
reduce physics to number theory. The construction of infinite

primes can
be regarded as a repeated second quantization of a super-symmetric
arithmetic quantum field theory. Later it became clear that the
process
generalizes so that it applies in the case of quaternionic and
octonionic
primes and their hyper counterparts. This hierarchy of second
quantizations
means an enormous generalization of physics to what might be
regarded a
physical counterpart for a hierarchy of abstractions about
abstractions
about.... The ordinary second quantized quantum physics corresponds
only to
the lowest level infinite primes. This hierarchy can be identified
with the
corresponding hierarchy of space-time sheets of the many-sheeted
space-time.

One can even try to understand the quantum numbers of physical
particles in
terms of infinite primes. In particular, the hyper-quaternionic
primes
correspond four-momenta and mass squared is prime valued for them.
The
properties of 8-D hyper-octonionic primes motivate the attempt to
identify
the quantum numbers associated with CP_2 degrees of freedom in
terms of
these primes. The representations of color group $SU(3)$ are indeed
labelled by two integers and the states inside given representation
by
color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational
infinite
primes. One can however replace classical associativity with quantum
associativity for quantum states assigned with infinite prime. One
can also
decompose rational infinite primes to hyper-octonionic infinite
primes at
lower level of the hierarchy. Physically this would mean that the
number
theoretic 8-momenta have only time-component. This decomposition is
completely analogous to the decomposition of hadrons to its colored
constituents and might be even interpreted in terms of color
confinement.
The interpretation of the decomposition of rational primes to primes
in the
algebraic extensions of rationals, hyper-quaternions, and hyper-
octonions
would have an interpretation as an increase of number theoretical
resolution and the principle of number theoretic confinement could

be seen
as a fundamental physical principle implied by associativity
condition.

\vm{\it 4. Space-time correlates of infinite primes}\vm

Infinite primes code naturally for Fock states in a hierarchy of
super-symmetric arithmetic quantum field theories. Quantum classical
correspondence leads to ask whether infinite primes could also code
for the
space-time surfaces serving as symbolic representations of quantum
states.
This would a generalization of algebraic geometry would emerge and
could
reduce the dynamics of K\"ahler action to algebraic geometry and
organize
4-surfaces to a physical hierarchy according to their algebraic
complexity.
Note that this conjecture should be consistent with two other
conjectures
about the dynamics of space-time surfaces (space-time surfaces as
preferred
extrema of K\"ahler action and space-time surfaces as quaternionic
or
co-quaternionic (as associative or co-associative) 4-surfaces of
hyper-octonion space M^8).

The representation of space-time surfaces as algebraic surfaces in
M^8 is
however too naive idea and the attempt to map hyper-octonionic
infinite
primes to algebraic surfaces seems has not led to any concrete
progress.

The endless updating of quantum TGD might be blamed to be a waste of
time.
The interaction of new ideas with old ones has however again and
again
turned out to be an extremely fruitful process leading to rather
precise
view about how infinite hyper-octonionic rationals can be mapped to
space-time surfaces without ad hoc assumptions. The progress in
quantum
TGD during the second half of the first decade of the new millenium

led to
several new and quite convincing ideas. Mention only zero energy
ontology, the generalization of the imbedding space concept
realizing the
hierarchy of Planck constants, hyper-finite factors and their
inclusions,
and in particular, the realization of quantum classical
correspondence in
terms of measurement interaction term associated with the K\"ahler-
Dirac
action.

The crucial observation is that quantum classical correspondence
allows to
map quantum numbers of WCW spinor fields to space-time
geometry. Therefore, if one wants to map infinite rationals to
space-time
geometry it is enough to map infinite primes to quantum numbers.
This map
can be indeed achieved thanks to the detailed picture about the
interpretation of the symmetries of infinite primes in terms of
standard
model symmetries.

\vm{\it 5. Generalization of ordinary number fields: infinite primes
and
cognition} \vm

Both fermions and p-adic space-time sheets are identified as
correlates of
cognition in TGD Universe. The attempt to relate these two
identifications
leads to a rather concrete model for how bosonic generators of
super-algebras correspond to either real or p-adic space-time sheets
(actions and intentions) and fermionic generators to pairs of real
space-time sheets and their p-adic variants obtained by algebraic
continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads
also to
a generalization of real numbers since an infinite algebra of real
units
defined by finite ratios of infinite rationals multiplied by
ordinary
rationals which are their inverses becomes possible. These units are
not
units in the p-adic sense and have a finite p-adic norm which can be
differ
from one. This construction generalizes also to the case of hyper-
quaternions and -octonions although non-commutativity and in case of
octonions also non-associativity pose technical problems. Obviously
this
approach differs from the standard introduction of infinitesimals in

the
sense that sum is replaced by multiplication meaning that the set of real
and also more general units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time
and imbedding space can be seen as infinitely structured and able to
represent all imaginable algebraic structures. Certainly
counter-intuitively, single space-time point is even capable of
representing the quantum state of the entire physical Universe in its
structure. For instance, in the real sense surfaces in the space of units
correspond to the same real number 1, and single point, which is
structure-less in the real sense could represent arbitrarily
high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely
invisible and is relevant only for the physics of cognition. On the other
hand, one can consider the possibility of mapping the WCW
and WCW spinor fields to the number theoretical anatomies
of a single point of imbedding space so that the structure of this point
would code for the world of classical worlds and for the quantum states of
the Universe. Quantum jumps would induce changes of WCW
spinor fields interpreted as wave functions in the set of number
theoretical anatomies of single point of imbedding space in the ordinary
sense of the word, and evolution would reduce to the evolution of the
structure of a typical space-time point in the system. Physics would reduce
to space-time level but in a generalized sense. Universe would be an
algebraic hologram, and there is an obvious connection both with
Brahman=Atman identity of Eastern philosophies and Leibniz's notion of
monad.

Infinite rationals are in one-one correspondence with quantum states and in
zero energy ontology hyper-octonionic units identified as ratios of the
infinite integers associated with the positive and negative energy parts of

the zero energy state define a representation of WCW spinor fields. The
action of subgroups of SU(3) and rotation group SU(2) preserving
hyper-octonionic and hyper-quaternionic primeness and identification of
momentum and electro-weak charges in terms of components of
hyper-octonionic primes makes this representation unique. Hence
Brahman-Atman identity has a completely concrete realization and fixes
completely the quantum number spectrum including particle masses and
correlations between various quantum numbers.

%\end{abstract}