Breakthrough in understanding of $M^8 - H$ duality

M. Pitkänen,

February 2, 2024
Email: matpitka6@gmail.com.
http://tgdtheory.com/public_html/.
Postal address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland. ORCID: 0000-0002-8051-4364.

Contents

Contents

Abstract

A critical re-examination of M^8-H duality is discussed. M^8-H duality is one of the cornerstones of Topological Geometrodynamics (TGD). The original version of M^8-H duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent or normal space and contains a complex or co-complex surface, it can be mapped to a 4-surface in $H = M^4 \times CP_2$.

Later emerged the idea that octonionic analyticity realized in terms of real polynomials P algebraically continued to polynomials of complexified octonion could fulfill the dream. The vanishing of the real part $Re_Q(P)$ (imaginary part $Im_Q(P)$) in the quaternionic sense would give rise to an associative (co-associative) space-time surface.

The realization of the general coordinate invariance motivated the notion of strong form of holography (SH) in H allowing realization of a weaker form of M^8-H duality by assuming that associativity/co-associativity conditions are needed only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits.

The outcome of the re-examination yielded both positive and negative surprises.

1. Although no interesting associative space-time surfaces are possible, every distribution of normal associative planes (co-associativity) is integrable.
2. Another positive surprise was that Minkowski signature is the only possible option. Equivalently, the image of M^4 as real co-associative subspace of O_c (complex valued octonion norm squared is real valued for them) by an element of local G_2 or rather, its subgroup $SU(3)$, gives a real co-associative space-time surface.
3. The conjecture based on naive dimensional counting, which was not correct, was that the polynomials P determine these 4-D surfaces as roots of $Re_Q(P)$. The normal spaces of these surfaces possess a fixed 2-D commuting sub-manifold or possibly their distribution allowing the mapping to H by M^8-H duality as a whole.

   If this conjecture were correct, strong form of holography (SH) would not be needed and would be replaced with extremely powerful number theoretic holography determining space-time surface from its roots and selection of real subspace of O_c characterizing the state of motion of a particle. erate
4. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [L4, L5, L6] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P = 0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X = 0, Y = 0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [L10, L13].
5. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X = 0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.

   The expectation was wrong! The equations $X = 0$ and $Y = 0$ involve the same(!) complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X = 0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.

   There are solutions $X = 0$ and $Y = 0$ only if the two polynomials considered have a common a_c^2 as a root! Also now the solutions are complex mass shells H^7_c.

How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .

1. The key observation is that G_2 acts as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space M^4 gives a co-associative four-surface as a real projection. Octonion analyticity would correspond to G_2 gauge transformation: this would realize the original idea about octonion analyticity.

2. A co-associative X^4_c satisfying also the conditions posed by the existence of $M^8 - H$ duality is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines $M^8 - H$ duality even if the co-associativity in geometric sense were not satisfied.

   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

   The group $SU(3)$ has interpretation as a Kac-Moody type analog of color group and the map defining space-time surface. This picture conforms with the H-picture in which gluon gauge potentials are identified as color gauge potentials. Note that at QFT limit the gauge potentials are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

3. Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \rightarrow M^4 \subset H$ from an identification to an almost inversion. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.

This picture combined with zero energy ontology leads also to a view about quantum TGD at the level of M^8. A local $SU(3)$ element defining 4-surface in M^8, which suggests a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

1 Introduction

$M^8 - H$ duality [L12, L10, L11, L18] has become a cornerstone of quantum TGD but several aspects of this duality are still poorly understood.

1.1 Development of the idea about $M^8 - H$ duality

A brief summary about the development of the idea is in order.

1. The original version of $M^8 - H$ duality assumed that space-time surfaces in M^8 can be identified as associative or co-associative surfaces. If the surface has associative tangent/normal space and contains a complex co-complex surface, it can be mapped to a 4-surface in $M^4 \times CP_2$.

2. Later emerged the idea that octonionic analyticity realized in terms of a real polynomials P algebraically continued to polynomials of complexified octonion might realize the dream [L4, L5, L6]. The original idea was that the vanishing condition for the real/imaginary part of P in quaternion sense could give rise to co-assocative/associative sense.

   $M^8 - H$ duality concretizes number theoretic vision [L7, L8] summarized as adelic physics fusing ordinary real number based physics for the correlates of sensory experience and various p-adic physics ($p = 2, 3, ...$) as physics for the correlates of cognition. The polynomials of real variable restricted to be rational valued defines an extension or rationals via the roots of the polynomials and one obtains an evolutionary hierachy associated with these extensions increasing in algebraic complexity. These extensions induce extensions of p-adic numbers and the points of space-time surface in M^8 with coordinates in the extension of rationals define cognitive representations as unique discretizations of the space-time surface.

3. The realization of the general coordinate invariance in TGD framework [K6, K3, K10, L23] [L20] motivated the idea that strong form of holography (SH) in H could allow realizing M^8-H duality by assuming associativity/co-associativity conditions only at 2-D string world sheet and partonic 2-surfaces and possibly also at their light-like 3-orbits at which the signature of the induced metric changes from Minkowskian to Euclidian.

1.2 Critical re-examination of the notion

In this article M^8-H duality is reconsidered critically.

1. The healthy cold shower was the learning that quaternion (associative) sub-spaces of quaternionic spaces are geodesic manifolds [A6]. The distributions of quaternionic normal spaces are however always integrable. Hence, co-associativity remains the only interesting option. Also the existence of co-commutative sub-manifolds of space-time surface demanding the existence of a 2-D integrable sub-distribution of subspaces is possible. This learning experience motivated a critical examination of the M^8-H duality hypothesis.

2. The basic objection is that for the conjectured associative option, one must assign to each state of motion of a particle its own octonionic structure since the time axis would correspond to the octonionic real axis. It was however clear from the beginning that there is an infinite number of different 4-D planes O_c in which the number theoretical complex valued octonion inner product reduces to real - the number theoretic counterpart for Riemann metric. In the co-associative case this is the only option. Also the Minkowski signature for the real projection turns out to be the only physically acceptable option. The mistake was to assume that Euclidian regions are co-associative and Minkowskian regions associative: both must be co-associative.

3. The concrete calculation of the octonion polynomial was the most recent step - carried already earlier [L4, L5, L6] but without realizing the implications of the extremely simple outcome. The imaginary part of the polynomial is proportional to the imaginary part of octonion itself. It turned out that the roots $P=0$ of the octonion polynomial P are 12-D complex surfaces in O_c rather than being discrete set of points defined as zeros $X=0, Y=0$ of two complex functions of 2 complex arguments. The analogs of branes are in question. Already earlier 6-D real branes assignable to the roots of the real polynomial P at the light-like boundary of 8-D light-cone were discovered: also their complex continuations are 12-D [L10, L13].

4. P has quaternionic de-composition $P = Re_Q(P) + I_4 Im_Q(P)$ to real and imaginary parts in a quaternionic sense. The naive expectation was that the condition $X=0$ implies that the resulting surface is a 4-D complex surface X^4_c with a 4-D real projection X^4_r, which could be co-associative.

   The expectation was wrong! The equations $X=0$ and $Y=0$ involve the same(!) complex argument o_c^2 as a complex analog for the Lorentz invariant distance squared from the tip of the light-cone. This implies a cold shower. Without any additional conditions, $X=0$ conditions have as solutions 7-D complex mass shells H^7_c determined by the roots of P. The explanation comes from the symmetries of the octonionic polynomial.

   There are solutions $X=0$ and $Y=0$ only if the two polynomials considered have a common a_c^2 as a root! Also now the solutions are complex mass shells H^7_c.

5. How could one obtain 4-D surfaces X^4_c as sub-manifolds of H^7_c? One should pose a condition eliminating 4 complex coordinates: after that a projection to M^4 would produce a real 4-surface X^4 .

   A co-associative X^4_c is obtained by acting with a local SU_3 transformation g to a co-associative plane $M^4 \subset M^8_c$. If the image point $g(p)$ is invariant under $U(2)$, the transformation corresponds to a local CP_2 element and the map defines M^8-H duality even if the co-associativity in geometric sense were not satisfied.

   The co-associativity of the plane M^4 is preserved in the map because G_2 acts as an automorphism group of the octonions. If this map also preserves the value of 4-D complex mass squared, one can require that the intersections of X^4_c with H^7_c correspond to 3-D complex mass shells. One obtains holography with mass shells defined by the roots of P giving boundary data. The condition H images are analogous to Bohr orbits, corresponds to number theoretic holography.

It this, still speculative, picture is correct, it would fulfil the original dream about solving classical TGD exactly in terms of roots for real/imaginary parts of octonionic polynomials in M^8 and by mapping the resulting space-time surfaces to H by M^8-H duality. In particular, strong form of holography (SH) would not be needed at the level of H, and would be replaced with a dramatically stronger number theoretic holography.

Octonionic Dirac equation, which is purely algebraic equation and the counterpart for ordinary Dirac equation in momentum space, serves as a second source of information.

1. The first implication is that O_c has interpretation as an analog of momentum space for quarks: this has profound implications concerning the interpretation. The space-time surface in M^8 would be analog of Fermi ball. The octonionic Dirac equation reduces to the mass shell condition $m^2 = r_n$, where r_n is a root of the polynomial P defining the 4-surface but only in the co-associative case.

2. Cognitive representations are defined by points of M^8 with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta of quarks. In the generic case the cognitive representations are finite. If the points of M^8 correspond to quark momenta, momentum conservation is therefore expected to make the scattering trivial.

   However, a dramatic implication of the reduction of the co-associativity conditions to the vanishing of ordinary polynomials Y is that by the Lorentz invariance of roots of P, the 3-D mass shells ofd M^4 have an infinite number of points in a cognitive representation defined by points with coordinates having values in the extensions of rational defined by P and allowing an interpretation as 4-momenta. This is what makes interesting scattering amplitudes for massive quarks possible.

3. What is the situation for the images of M^4 points under the effective local CP_2 element defined by local $SU(3)$ element g preserving the mass squared and mapping H^3 to $g(H^3)$? If g is expressible in terms of rational functions with rational coefficients, algebraic points are mapped to algebraic points. This is true also in the interior of M^4.

   This would mean a kind of cognitive explosion for massive quark momenta. Without the symmetry one might have only forward scattering in the interior of X^4_r. Note that massless quarks can however arrive at the boundary of CD which also allows cognitive representation with an infinite number of points.

4. In the number theoretic approach, kinematics becomes a highly non-trivial part of the scattering. The physically allowed momenta would naturally correspond to algebraic integers in the extension E of rationals defined by P. Momentum conservation and on-mass-shell conditions together with the condition that momenta are algebraic integers in E are rather strong. The construction of Pythagorean squared generalize to the case of quaternions provides a general solutions to the conditions: the solutions to the conditions are combinations of momenta which correspond to squares of quaternions having algebraic integers as components.

5. The original proposal was that local $G_{2,c}$ element $g(x)$ defines a vanishing holomorphic gauge field and its restriction to string world sheet or partonic 2-surface defines conserved current. M^8-H duality however requires that local $SU(3)$ element with the property that image point is invariant under $U(2)$ is required by M^8-H duality defines $X^4 \subset M^8$.

   In any case, these properties suggest a Yangian symmetry assignable to string world sheets and partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The generators of the Yangian algebra have a representation as Hamiltonians which are in involution. They define conserved charges at the orbits for a Hamiltonian evolution defined by any combination of these the Hamiltonians. ZEO suggests a concrete representation of this algebra in terms of quark and antiquark oscillator operators. This algebra extends also to super-algebra. The co-product of the associated Yangian would give rise to zero energy states defining as such the scattering amplitudes.

1.3 Octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of the ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = qq = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of p p are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

Before continuing, I must apologize for the still fuzzy organization of the material related to $M^8 - H$ duality. The understanding of its details has been a long and tedious process, which still continues, and there are unavoidably inaccuracies and even logical inconsistencies caused by the presence of archeological layers present.

2 The situation before the cold shower

The view about $M^8 - H$ duality before the cold shower - leading to what I dare to call a breakthrough - helps to gain idea about the phenomenological side of $M^8 - H$ duality. Most of the phenomenology survives the transition to a more precise picture. This section is however not absolutely necessary for what follows it.

2.1 Can one deduce the partonic picture from $M^8 - H$ duality?

The M^8 counterparts for partons and their light like orbits in H can be understood in terms of octonionic Dirac equation in M^8 as an analog for the algebraic variant of ordinary Dirac equation at the level of momentum space [L18, L17] but what about the identification of partonic 2-surfaces as interaction vertices at which several partonic orbits meet? Can one deduce the phenomenological view about elementary particles as pairs of wormhole contacts connected by magnetic flux tubes from $M^8 - H$ duality? There is also the question whether partonic orbits correspond to their own sub-CDs as the fact that their rest systems correspond to different octonionic real axes suggests.

There are also some questions which have become obsolote. For instance: qhy should the partonic vertices reside at $t = r_n$ branes? This became obsolste with the realization that M^8 is analogous to momentum space so that the identification as real octonionic coordinate corresponds now to a component of 8-momentum identifiable as energy. Furthermore, the assumption the associativity of the 4-surface in M^8 had to be replaced with-co-associtivity and octonionic real coordinate does not have interpretation as time coordinate is associative surface

$M^8 - H$ duality indeed conforms with the phenomenological picture about scattering diagrams in terms of partonic orbits [L23, L22] [L22, L23] [L23], and leads to the view about elementary particles as pairs of Euclidian wormhole contacts associated with flux tubes carrying monopole flux.

2.2 What happens to the "very special moments in the life of self"?

The original title was "What happens at the "very special moments in the life of self?" but it turned out that "at" must be replaced with "to". The answer to the new question would be "They disappear from the glossary".

The notion of "very special moments in the life of self" (VPM) [L10, L13] makes sense if M^8 has interpretation as an 8-D space-time. M^4 projections of VPMs were originally identified as hyperplanes $t = r_n$, where t is time coordinate and r_n is a root of the real polynomial defining octonionic polynomial as its algebraic continuation.

The interpretation of M^8 as cotangent space of H was considered from the beginning but would suggest the interpretation of M^8 as the analog of momentum space. It is now clear that this interpretation is probably correct and that $M^8 - H$ duality generalizes the momentum-position duality of wave mechanics. Therefore one should speak of $E = r_n$ plane and simply forget the misleading term VMP. VPMs would correspond to constant values of the M^8 energy assignable to M^4 time coordinate.

The identification of space-time surface as co-associative surface with quaternionic normal space containing integrable distribution of 2-D commutative planes essential for $M^8 - H$ duality is also in conflict with the original interpretation. Also the modification of $M^8 - H$ duality in M^4 degrees of freedom forced by Uncertainty Principle [L28] has led to the conclusion that VMPs need not have a well-defined images in H.

2.3 What does SH mean and its it really needed?

SH has been assumed hitherto but what is its precise meaning?

1. Hitherto, SH at the level of H is believed to be needed: it assumes that partonic 2-surfaces and/or string world sheets serve as causal determinants determining X^4 via boundary conditions.

   (a) The normal or tangent space of X^4 at partonic 2-surfaces and possibly also at string world sheets has been assumed to be associative that is quaternionic. This condition should be true at the entire X^4.

   (b) Tangent or normal space has been assumed to contain preferred M^2 which could be replaced by an integrable distribution of $M^2(x) \subset M^4$. At string world sheets only the tangent space can be associative. At partonic 2-surfaces also normal space could be associative. This condition would be true only at string world sheets and partonic 2-surfaces so that only these can be mapped to H by $M^8 - H$ duality and continued to space-time surfaces as preferred extremals satisfying SH.

   The current work demonstrates that although SH could be used at the level of SH, this is not necessary. Co-associativity together with co-commutativity for string world sheets allows the mapping of the real space-time surfaces in M^8 to H implying exact solvability of the classical TGD.

2.4 Questions related to partonic 2-surfaces

There are several questions related to partonic 2-surfaces.

Q1: What are the M^8 pre-images of partons and their light-like partonic orbits in H?

It will be found that the octonionic Dirac equation in M^8 implies that octo-spinors are located to 3-D light-like surfaces Y^3_r - actually light-cone boundary and its 3-D analogs at which number theoretic norm squared is real and vanishes - or to the intersections of X^3_r with the 6-D roots of P in which case Dirac equation trivializes and massive states are allowed. They are mapped to H by $M^8 - H$ duality.

Remark: One can ask whether the same is true in H in the sense that modified Dirac action would be localized to 3-D light-like orbits and 3-D ends of the space-time surfaces at the light-like boundaries of CD having space-like induced metric. Modified Dirac action would be defined by Cherm-Simons term and would force the classical field equations for the bosonic Chern-Simons term. If the interior part of the modified Dirac action is absent, the bosonic action is needed to define the space-time surfaces as extremals. They would be minimal surfaces and universal by their holomorphy and would not depend on coupling parameters so that very general actions can have them as preferred extremals. This issue remains still open.

The naïve - and as it turned out, wrong - guess was that the images of the light-like surfaces should be light-like surfaces in H at the boundaries of Minkowskian and Euclidian regions (wormhole contacts). In the light-like case Y^3_r corresponds to the light-cone boundary so that this would be the case. X^3_r however turns out to correspond to a hyperboloid in M^4 as an analog of a mass shell and is not identifiable as a partonic orbit.

It turned out that the complex surface X^4_c allows real sections in the sense that the number theoretic complex valued metric defined as a complex continuation of Minkowski norm is real at 4-D surfaces: call them Z^4_r. They are bounded by a 3-D region at Z^3_r at which the value of norm squared vanishes. This surface is an excellent candidate for the pre-image of the light-like orbit of partonic 2-surface serving as a topological vertex. One has therefore strings worlds sheets, partonic 2-surfaces and their light-like orbits and they would connect the "mass shells" at X^4_r. All ingredients for SH would be present.

The intersections of Z^3_r with X^3_r identifiable as the section of X^4_r $a = constant$ hyperboloid would give rise to partonic 2-surfaces appearing as topological reaction vertices.

The assumption that the 4-D tangent space at these light-like 3-surfaces is co-associative, would give an additional condition determining the image of this surface in H, so that the boundary conditions for SH would become stronger. One would have boundary conditions at light-like partonic orbits. Note that string world sheets are assumed to have light-like boundaries at partonic orbits.

Q2: Why should partonic 2-surfaces appear as throats of wormhole contact in H? Wormhole contacts do not appear in M^8.

1. In M^8 light-like orbits are places where the Minkowskian signature changes to Euclidian. Does $M^8 - H$ duality map the images of these coinciding roots for Euclidian and Minkowskian branches to different throats of the wormhole contact in H so that the intersection would disappear?

2. This is indeed the case. The intersection of Euclidian and Minkowskian branches defines a single 3-surface but the tangent and normal spaces of branches are different. Therefore their H images under $M^8 - H$ duality for the partonic 2-surface are different since normal spaces correspond to different CP_2 coordinates. These images would correspond to the two throats of wormhole contact so that the H-image by SH is 2-sheeted. One would have wormhole contacts in H whereas in M^8 the wormhole contact would reduce to a single partonic 2-surface.

3. The wormhole contact in H can have only Euclidian signature of the induced metric. The reason is that the M^4 projections of the partonic surfaces in H are identical so that the points with same M^4 coordinates have different CP_2 coordinates and their distance is space-like.

Q3: In H picture the interpretation of space-time surfaces as analogs of Feynman graphs assumes that several partonic orbits intersect at partonic 2-surfaces. This assumption could be of course wrong.This raises questions.

What the pre-images of partonic 2-surfaces are in M^8? Why should several partonic orbits meet at a given partonic 2-surface? Is this needed at all?

The space-time surface X^4_r associated intersects the surface X^6_r associated with different particle - say with different value of mass along 2-D surface. Could this surface be identified as partonic 2-surface X^2_r? This occurs symmetrically so that one has a pair of 2-surfaces X^2_r. What does this mean? Could these surface map to the throats of wormhole contact in H?

Why several partonic surfaces would co-incide in topological reaction vertex at the level of H? At this moment is is not clear whether this is forced by M^8 picture.

Octonionic Dirac equation implies that M^8 has interpretation as analog of momentum space so that interaction vertices are replaced by multilocal vertices representing momenta and propagators become local being in this sense analogous to vertices of QFT. One could of course argue that without the gluing along ends there would be no interactions since the interactions in X^6_r for two

3-surfaces consist in the generic case of a discrete set of points. One could also ask whether the surfaces Y^3_r associated with the space-time surfaces X^4_r associated with incoming particles must intersect along partonic 2-surface rather than at discrete set of points.

The meeting along ends need not be true at the level of M^8 since the momentum space interpretation would imply that momenta do not differ much so that particles should have identical masses: for this to make sense one should assume that the exchanged virtual particles are massless. One other hand, if momenta are light-like for Y^3_r, this might be the case.

Q4: Why two wormhole contacts and monopole flux tubes connecting them at the level of H? Why monopole flux?

1. The tangent spaces of the light-like orbits have different light-like direction. Intuitively, this corresponds to different directions of light-like momenta. Momentum conservation requires more than one partonic orbit changing its direction meeting at partonic 2-surface. By light-likeness, the minimum is 2 incoming and two outgoing lines giving a 4-vertex. This allows the basic vertices involving Ψ and $\overline{Psi}$ at opposite throats of wormhole contacts. Also a higher number of partonic orbits is possible.

2. A two-sheeted closed monopole flux tube having wormhole contacts as its "ends" is suggested by elementary particle phenomenology. Since M^8 homology is trivial, there is no monopole field in M^8. If $M^8 - H$ duality is continuous it maps homologically trivial partonic 2-surfaces to homologically trivial 2-surfaces in H. This allows the wormhole throats in H to have opposite homology charges. Since the throats cannot correspond to boundaries there must be second wormhole contact and closed flux tube.

3. What does the monopole flux for a partonic 2-surface mean at the level of M^8? The distribution of quaternionic 4-D tangent/normal planes containing preferred M^2 and associated with partonic 2-surface in M^8 would define a homologically on-trivial 2-surface in CP_2. The situation is analogous to a distribution of tangent planes or equivalently normal vectors in S^2.

Q4: What is the precise form of $M^8 - H$ duality: does it apply only to partonic 2-surfaces and string world sheets or to the entire space-time surfaces?

$M^8 - H$ duality is possible if the X^4 in M^8 contains also integrable distribution of complex tangent or normal 2-planes at which 4-D tangent space is quaternionic/associative. String world sheets and partonic 2-surfaces define these distributions.

The minimum condition allowed by SH in H is that string world sheets and there is a finite number of partonic 2-surfaces and string world sheets. In this case only these 2-surfaces can be mapped to H and SH assigns to them a 4-D space-time surface. The original hypothesis was that these surfaces define global orthogonal slicings of the X^4 so that $M^8 - H$ duality could be applied to the entire X^4. This condition is probably too strong.

3 Challenging $M^8 - H$ duality

$M^8 - H$ duality involves several alternative options and in the following arguments possibly leading to a unique choice are discuses.

1. Are both associativity and co-associativity possible or is only either of these options allowed? Is it also possible to pose the condition guaranteeing the existence of 2-D complex submanifolds identifiable as string world sheets necessary to map the entire space-time surface from M^8 to H? In other words, is the strong form of holography (SH) needed in M^8 and/or H or is it needed at all?

2. The assignment of the space-time surface at the level of M^8 to the roots of real or imaginary part (in quaternionic sense) of octonionic polynomial P defined as an algebraic continuation of real polynomial is an extremely powerful hypothesis in adelic physics [L8, L7] and would mean a revolution in biology and consciousness theory.

   Does P fix the space-time surface with the properties needed to realize $M^8 - H$ duality or is something more needed? Does the polynomial fix the space-time surface uniquely - one would have extremely strong number theoretic holography - so that one would have number theoretic holography with coefficients of a real polynomial determining the space-time surface?

3. $M^8 - H$ duality involves mapping of $M^4 \subset M^8$ to $M^4 \subset H$. Hitherto it has been assumed that this map is direct identification. The form of map should however depend on the interpretation of M^8. In octonionic Dirac equation M^8 coordinates are in the role of momenta [L18]. This suggests the interpretation of M^8 as an analog of 8-D momentum space. If this interpretation is correct, Uncertainty Principles demands that the map $M^4 \subset M^8 \rightarrow M^4 \subset H$ is analogous to inversion mapping large momenta to small distances.

4. Twistor lift of TGD [K13] is an essential part of the TGD picture. Ttwistors and momentum twistors provide dual approaches to twistor Grassmann amplitudes. Octonionic Dirac equation suggests that M^8 and H are in a similar dual relation. Could $M^8 - H$ duality allow a generalization of twistorial duality to TGD framework?

3.1 Explicit form of the octonionic polynomial

What does the identification of the octonionic polynomial P as an octonionic continuation of a polynomial with real or complexified coefficients imply? In the following I regard M^8_c as O^8_c and consider products for complexified octonions.

Remark: In adelic vision the coefficients of P must be rationals (or at most algebraic numbers in some extension of rationals).

One interesting situation corresponds to the real subspace of O_c spanned by $\{I_0, iI_k\}, = 1, ..7$, with a number theoretic metric signature $(1, -1, -1..., -1)$ of M^8 which is complex valued except at in various reals subspaces.This subspace is associative. The original proposal was that Minkowskian space-time regions as projections to this signature are associative whereas Euclidian regions are co-associative. It however turned out that associative space-time surfaces are physically uninteresting.

The canonical choice $(iI_0, I_1, I_2, iI_3, I_4, iI_5, I_6, iI_7)$ defining the complexification of the tangent space represents a co-associative sub-space realizing Minkowski signature. It turns out that both Minkowskian and Euclidian space-time regions must be co-associative .

3.1.1 Surprises

The explicit calculation of the octonionic polynomial yielded a chilling result. If one poses (co-)associativity conditions as vanishing of the imaginary or real part in quaterionic sense: $Im_Q(P) = 0$ or $Re_Q(P) = 0$, the outcome is that the space-time surface is just M^4 or E^4. Second chilling result is that quaternionic sub-manifolds are geodesic sub-manifolds. This led to the question how to modify the (co-)associativity hypothesis.

The vision has been that space-time surfaces can be identified as roots for the imaginary (co-associative) part $Im_Q(O)$ or real part $Im_Q(O)$ of octonionic polynomial using the standard decomposition $(1, e_1, e_2, e_3)$.

1. The naïve counting of dimensions suggests that one obtains 4-D surfaces. The surprise was that also 6-D brane like entities located at the boundary of M^8 light-cone and with topology of 6-sphere S^6 are possible. They correspond to the roots of a real polynomial $P(o)$ for the choice $(1, iI_1, ..., iI_7)$. The roots correspond to the values of the real octonion coordinate interpreted as values of linear M^4 time in the proposal considered. Also for the canonical proposal one obtains a similar result. In O_c they correspond to 12-D complex surfaces X^6_c satisfying the same condition conditions $x_0^2 + r^2 = 0$ and $P(x_0) = 0$.

2. There was also another surprise. As already described, the general form for the octonionic polynomial $P(o)$ induced from a real polynomial is extremely simple and reduces to $X(t^2, r^2)I_0 + iY(t^2, r^2)Im(o)$. There are only two complex variables t and r^2 involved and the solutions of $P = 0$ are 12-D complex surfaces X^6_c in O_c. Also the special solutions have the same dimension.

3. In the case of co-associativity 8 conditions are needed for $Re_Q(P) = 0$: note that $X = 0$ is required. The naive expectation is that this gives a complex manifold X^4_c with 4-D real projection X^4_r as an excellent candidate for a co-associative surface.

The expectation turned out to be wrong: in absence of any additional conditions the solutions are complex 7-dimensional mass shells! This is due to the symmetries of the octonionic polynomials as algebraic continuation of a real polynomial.

4. The solution of the problem is to change the interpretation completely. One must assign to the 7-D complex mass shell H^7_c a 3-D complex mass shell H^3_c.

   One can do this by assuming space-time surface is surface intersecting the7-D mass shell obtained as a deformation of $M^4_c \subset M^8_c$ by acting with local $SU(3)$ gauge transformation and requiring that the image point is invariant under $U(2)$. If the 4-D complex mass squared remains invariant in this transformation, X^4_c intersects H^7_c.

   With these assumptions, a local CP_2 element defines X^4_c and X^4_r is obtained as its real projection in M^4. This definition assigns to each point of M^4 a point of CP_2 so that M^8-H duality is well-defined.

   One obtains holography in which the fixing of 3-D mass shells fixes the 4-surface and also assigns causal diamond with the pair of mass shells with opposite energies. If the space-time surface is analog of Bohr orbit, also its preimage under M^8-H duality should be such and P would determine 4-surface highly uniquely [L30] and one would have number theoretic holography.

3.1.2 General form of P and of the solutions to $P=0$, $Re_Q(P)=0$, and $Im_Q(P)=0$

It is convenient to introduce complex coordinates for O_c since the formulas obtained allow projections to various real sections of O_c.

1. To see what happens, one can calculate o_c^2. Denote o_c by $o_c = tI_0 + \overline{o_c}$ and the norm squared of $\overline{o}$ by r^2, where $r^2 = \sum o_k^2$ where o_k are the complex coordinates of octonion. Number theoretic norm squared for o_c is t^2+r^2 and reduces to a real number in the real sections of O_c. For instance, in the section $(I_1, iI3, iI_5, iI_7)$ the norm squared is $-x_1^2+x_3^2+x_5^2+x_7^2$ and defines Minkowskian norm squared.

   For o^2 one has:

$$o^2 = t^2 - r^2 + 2t\overline{o} \equiv X_2 + \overline{Y}_2 \ \ .$$

   For o^3 one obtains

$$o^3 = tX_2 - \overline{o}\cdot\overline{Y}_2 + t\overline{Y}_2 + X_2\overline{o} \ \ .$$

   Clearly, $Im_Q(o^n)$ has always the same direction as $Im_Q(o)$. Hence one can write in the general case

$$o^n = X + Y\overline{o} \ \ . \tag{3.1}$$

   This trivial result was obtained years ago but its full implications became evident only while preparing the current article. The point is that the solutions to associativity/co-associativity conditions by putting $Re(Q(P)=0$ or $Im_Q(P)=0$ are trivial: just M^4 or E^4. What goes wrong with basic assumptions, will be discussed later.

   Remark: In M^8 sub-space one has imaginary $\overline{o}$ is proportional to the commuting imaginary unit.

2. It is easy to deduce a recursion formula for the coefficients for X and Y for n:th power of o_c. Denote by t the coordinate associated with the real octonion unit (not time coordinate). One obtains

$$\begin{aligned} o_c^n &= X_nI_0 + Y_n\overline{o} \ \ , \\ X_n &= tX_{n-1} - rY_{n-1} \ \ , \\ Y_n &= tY_{n-1} + rX_{n-1} \ \ . \end{aligned} \tag{3.2}$$

In the co-associative case one has $t = 0$ or possibly constant $t = T$ (note that in the recent interpretation t does not have interpretation as time coordinate). The reason is that the choice of octonionic coordinates is unique apart from translation along the real axis from the condition that the coefficients of P remain complex numbers in powers of the new variable.

3. The simplest option correspond to $t = 0$. One can criticize this option since the quaternion-icity of normal space should not be affected if t is constant different from zero. In any case, for $t = 0$ the recursion formula gives for the polynomial $P(o_c)$ the expression

$$P(o_c) = \sum(-1)^n r^{2n}(p_{2n-1}I_0 + p_{2n}\overline{o}) \ . \tag{3.3}$$

Denoting the even and of odd parts of P by P_{even} and P_{odd}, the roots $r_{k,odd}$ of $X = Re(P(o_c))$ are roots P_{odd} and roots $r_{k,even}$ of $Y = Im(P(o_c))$ are roots of P_{even}. Co-associativity gives roots of X and the roots of P as simultaneous roots of P_{odd} and P_{even}. The interpretation of roots is as in general complex mass squared values.

In the general case, the recursion relation would give the solution

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = A^n \begin{pmatrix} t \\ r \end{pmatrix}$$
$$A = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \tag{3.4}$$

One can diagonalize the matrix appearing in the iteration by solving the eigenvalues $\lambda_\pm = t \pm ir$ and eigenvectors $X_\pm = (\pm i, 1)$ and by expressing $(X_1, Y_1) = (t, r)$ in terms of the eigenvectors as $(t, r) = ((it + r)X_+ + (r - it)X_-)/2$. This gives

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (t+ir)^n i - (t-ir)^n i \\ (t+ir)^n + (t-ir)^n \end{pmatrix} \tag{3.5}$$

This gives

$$\begin{array}{l} P(o_c) = P_1 I_0 + P_2\overline{o} \ , \\ P_1(r) = \sum X_n p_n r^{2n} \ , \\ P_2(r) = \sum Y_n p_n r^{2n} \ . \end{array} \tag{3.6}$$

For the restriction to M^4_c, r^2 reduces to complex 4-D mass squared given by the root r_n. In general case r^2 corresponds to complex 8-D mass squared. All possible signatures are obtained by assuming M^8_c coordinates to be either real or imaginary (the number theoretical norm squared is real with this restriction).

3.1.3 How does one obtain 4-D space-time surfaces?

Contrary to the naive expections, the solutions of the vanishing conditions for the $Re_Q(P)$ ($Im_Q(P)$) (real (imaginary) part in quaternionic sense) are 7-D complex mass shells $r^2 = r_{n,1}$ as roots of $P_1(r) = 0$ or $r^2 = r_{n,2}$ of $P_2(r) = 0$ rather than 4-D complex surfaces (for a detailed discussion see [K2]) A solution of both conditions requires that P_1 and P_2 have a common root but the solution remains a 7-D complex mass shell! This was one of the many cold showers during the development of the ideas about $M^8 - H$ duality! It seems that the adopted interpretation is somehow badly wrong. Here zero energy ontology (ZEO) and holography come to the rescue.

1. Could the roots of P_1 or P_2 define only complex mass shells of the 4-D complex momentum space identifiable as M^4_c? ZEO inspires the question whether a proper interpretation of mass shells could be as pre-images of boundaries of cd:s (intersections of future and past directed light-cones) as pairs of mass shells with opposite energies. If this is the case, the challenge would be to understand how X^4_c is determined if P does not determine it.

   Here holography, considered already earlier, suggests itself: the complex 3-D mass shells belonging to X^4_c would only define the 3-D boundary conditions for holography and the real mass shells would be mapped to the boundaries of cds. This holography can be restricted to X^4_R. Bohr orbit property at the level of H suggests that the polynomial P defines the 4-surface more or less uniquely.

2. Let us take the holographic interpretation as a starting point. In order to obtain an X^4_c mass shell from a complex 7-D light-cone, 4 complex degrees of freedom must be eliminated. M^8-H duality requires that X^4_c allows M^4_c coordinates.

   Note that if one has $X^4_c = M^4_c$, the solution is trivial since the normal space is the same for all points and the H image under M^8-H duality has constant $CP_2 = SU(3)/U(2)$ coordinates. X^4_c should have interpretation as a non-trivial deformation of M^4_c in M^8.

3. By M^8-H duality, the normal spaces should be labelled by $CP_2 = SU(3)/U(2)$ coordinates. M^8-H duality suggests that the image $g(p)$ of a momentum $p \in M^4_c$ is determined essentially by a point $s(p)$ of the coset space $SU(3)/U(2)$. This is achieved if M^4_c is deformed by a local $SU(3)$ transformation $p \to g(p)$ in such a way that each image point is invariant under $U(2)$ and the mass value remains the same: $g(p)^2 = p^2$ so that the point represents a root of P_1 or P_2.

   Remark: I have earlier considered the possibility of G_2 and even $G_{2,c}$ local gauge transformation. It however seems that that local $SU(3)$ transformation is the only possibility since G_2 and $G_{2,c}$ would not respect M^8-H duality. One can also argue that only real $SU(3)$ maps the real and imaginary parts of the normal space in the same manner: this is indeed an essential element of M^8-H duality.

4. This option defines automatically M^8-H duality and also defines causal diamonds as images of mass shells $m^2 = r_n$. The real mass shells in H correspond to the real parts of r_n. The local $SU(3)$ transformation g would have interpretation as an analog of a color gauge field. Since the H image depends on g, it does not correspond physically to a local gauge transformation but is more akin to an element of Kac-Moody algebra or Yangian algebra which is in well-defined half-algebra of Kac-Moody with non-negative conformal weights.

The following summarizes the still somewhat puzzling situation as it is now.

1. The most elegant interpretation achieved hitherto is that the polynomial P defines only the mass shells so that mass quantization would reduce to number theory. Amusingly, I started to think about particle physics with a short lived idea that the d'Alembert equation for a scalar field could somehow give the mass spectrum of elementary particles so that the issue comes full circle!

2. Holography assigns to the complex mass shells complex 4-surfaces for which M^8-H duality is well-defined even if these surfaces would fail to be 4-D co-associative. These surfaces are expected to be highly non-unique unless holography makes them unique. The Bohr orbit property of their images in H indeed suggests this apart from a finite non-determinism [L30]. Bohr orbit property could therefore mean extremely powerful number theoretical duality for which the roots of the polynomial determine the space-time surface almost uniquely. $SU(3)$ as color symmetry emerges at the level of M^8. By M^8-H duality, the mass shells are mapped to the boundaries of CDs in H.

3. Do we really know that X^4_r co-associative and has distribution of 2-D commuting subspaces of normal space making possible M^8-H duality? The intuitive expectation is that the

answer is affirmative [A6]. In any case, $M^8 - H$ duality is well-defined even without this condition.

4. The special solutions to $P = 0$, discovered already earlier, are restricted to the boundary of CD_8 and correspond to the values of energy (rather than mass or mass squared) coming as roots of the real polynomial P. These mass values are mapped by inversion to "very special moments in the life of self" (a misleading term) at the level of H as special values of light-cone proper time rather than linear Minkowski time as in the earlier interpretation [L10]. The new picture is Lorenz invariant.

Octonionic Dirac equation as analog of momentum space variant of ordinary Dirac equation forces the interpretation of M^8 as an analog of momentum space and Uncertainty Principle forces to modify the map $M^4 \subset M^8 \to M^4 \subset H$ from identification to inversion. The equations for $Re_Q(P) = 0$ reduce to simultaneous roots of the real polynomials defined by the odd and even parts of P having interpretation as complex values of mass squared mapped to light-cone proper time constant surfaces in H. This leads to the idea that the formulation of scattering amplitudes at M^8 levels provides the counterpart of momentum space description of scattering whereas the formulation at the level of H provides the counterpart of space-time description.

This picture combined with zero energy ontology (ZEO) leads also to a view about quantum TGD at the level of M^8. Local $SU(3)$ element has properties suggesting a Yangian symmetry assignable to string world sheets and possibly also partonic 2-surfaces. The representation of Yangian algebra using quark oscillator operators would allow to construct zero energy states at representing the scattering amplitudes. The physically allowed momenta would naturally correspond to algebraic integers in the extension of rationals defined by P. The co-associative space-time surfaces (unlike generic ones) allow infinite-cognitive representations making possible the realization of momentum conservation and on-mass-shell conditions.

3.2 The input from octonionic Dirac equation

The octonionic Dirac equation allows a second perspective on associativity. Everything is algebraic at the level of M^8 and therefore also the octonionic Dirac equation should be algebraic. The octonionic Dirac equation is an analog of the momentum space variant of ordinary Dirac equation and also this forces the interpretation of M^8 as momentum space.

Fermions are massless in the 8-D sense and massive in 4-D sense. This suggests that octonionic Dirac equation reduces to a mass shell condition for massive particle with $q \cdot q = m^2 = r_n$, where $q \cdot q$ is octonionic norm squared for quaternion q defined by the expression of momentum p as $p = I_4 q$, where I_4 is octonion unit orthogonal to q. r_n represents mass shell as a root of P.

For the co-associative option the co-associative octonion p representing the momentum is given in terms of quaternion q as $p = I_4 q$. One obtains $p \cdot p = q\overline{q} = m^2 = r_n$ at the mass shell defined as a root of P. Note that for M^4 subspace the space-like components of p p are proportional to i and the time-like component is real. All signatures of the number theoretic metric are possible.

For associative option one would obtain $qq = m^2$, which cannot be satisfied: q reduces to a complex number $zx + Iy$ and one has analog of equation $z^2 = z^2 - y^2 + 2Ixy = m_n^2$, which cannot be true. Hence co-associativity is forced by the octonionic Dirac equation.

One of the big surprises was that the cognitive representations for both light-like boundary and X_r^4 are not generic meaning that they would consist of a finite set of points but infinite due to the Lorentz symmetry: a kind of cognitive explosion would happen by the Lorentz symmetry. The natural assumption is that for a suitable momentum unit, physical momenta satisfying mass shell conditions are algebraic integers in the extension of rationals defined by P. Periodic boundary conditions in turn suggest that for the physical states the total momenta are ordinary integers and this leads to Galois confinement as a universal mechanism for the formation of bound states.

3.2.1 Hamilton-Jacobi structure and Kähler structure of $M^4 \subset H$ and their counterparts in $M^4 \subset M^8$

The Kähler structure of $M^4 \subset H$, forced by the twistor lift of TGD, has deep physical implications and seems to be necessary. It implies that for Dirac equation in H, modes are eigenstates of only the longitudinal momentum and in the 2 transversal degrees of freedom one has essentially harmonic oscillator states [L28, L27], that is Gaussians determined by the 2 longitudinal momentum components. For real longitudinal momentum the exponents of Gaussians are purely imaginary or purely real.

The longitudinal momentum space $M^2 \subset M^4$ and its orthogonal complement E^2 is in a preferred role in gauge theories, string models, and TGD. The localization of this decomposition leads to the notion of Hamilton-Jacobi (HJ) structure of M^4 and the natural question is how this relates to Kähler structures of M^4. At the level of H spinors fields only the Kähler structure corresponding to constant decomposition $M^2 \oplus E^2$ seems to make sense and this raises the question how the H-J structure and Kähler structure relate. TGD suggests the existence of two geometric structure in M^4: HJ structure and Kähler structure. It has remained unclear whether HJ structure and Kähler structure with covariantly constant self-dual Kähler form are equivalent notions or whether there several H-J structures accompaning the Kähler structure.

In the following I argue that H-J structures correspond to different choices of symplectic coordinates for M^4 and that the properties of $X^4 \subset H$ determined bt M^-H duality make it natural to to choose particular symplectic coordinates for M^4.

Consider first what H-J structure and Kähler structure could mean in H.

1. The H-J structure of $M^4 \subset H$ would correspond to an integrable distribution of 2-D Minkowskian sub-spaces of M^4 defining a distribution of string world sheets $X^2(x)$ and orthogonal distribution of partonic 2-surfaces $Y^2(x)$. Could this decomposition correspond to self-dual covariantly Kähler form in M^4?

   What do we mean with covariant constancy now? Does it mean a separate covariant constancy for the choices of $M^2(x)$ and $Y^2(x)$ or only of their sum, which in Minkowski coordinates could correspond to a constant electric and magnetic fields orthogonal to each other?

2. The non-constant choice of $M^2(x)$ ($E^2(x)$) cannot be covariantly constant. One can write $J(M^4) = J(M^2(x)) \oplus J(E^2(x)$ corresponding to decomposition to electric and magnetic parts. Constancy of $J(M^2(x)$ would require that the gradient of $J(M^2(x)$ is compensated by the gradient of an antisymmetric tensor with square equal to the projector to $M^2(x)$. Same condition holds true for $J(E^2(x))$. The gradient of the antisymmetric tensor would be parallel to itself implying that the tensor is constant.

3. H-J structure can only correspond to a transformation acting on J but leaving $J_{kl}dm^kdm^l$ invariant. One should find analogs of local gauge transformations leaving J invariant. In the case of CP_2, these correspond to symplectic transformations and now one has a generalization of the notion. The M^4 analog of the symplectic group would parameterize various decompositions of $J(M^4)$.

   Physically the symplectic transformations define local choices of 2-D space $E^2(x)$ of transversal polarization directions and longitudinal momentum space M^2 emerging in the construction of extremals of Kähler action.

4. For the simplest Kähler form for $M^4 \subset H$, this decomposition in Minkowski coordinates would be constant: orthogonal constant electric and magnetic fields. This Kähler form extends to its number theoretical analog in M^8. The local $SU(3)$ element g would deform M^4 to $g(M^4)$ and define an element of local CP_2 defining M^8-H duality. g should correspond to a symplectic transformation of M^4.

Consider next the number theoretic counterparts of H-J- and Kähler structures of $M^4 \subset H$ in $M^4 \subset M^8$.

1. In M^4 coordinates H-J structure would correspond to a constant $M^2 \times E^2$ decomposition. In M^4 coordinates Kähler structure would correspond to constant E and B orthogonal to each other. Symplectic transformations give various representations of this structure as H-J structures.

2. The number theoretic analog of H-J structure makes sense also for $X^4 \subset M^8$ as obtained from the distribution of quaternionic normal spaces containing 2-D commutative sub-space

at each point by multiplying then by local unit $I_4(x)$ orthogonal to the quaternionic units $\{1, I_1 = I_2 = I_3\}$ with respect to octonionic inner product. There is a hierarchy of CDs and the choices of these structures would be naturally parameterized by G_2.

This would give rise to a number theoretically defined slicing of $X^4_c \subset M^8_c$ by complexified string world sheets X^2_c and partonic 2-surfaces Y^2_c orthogonal with respect to the octonionic inner product for complexified octonions.

3. In M^8-H duality defined by $g(p) \subset SU(3)$ assigns a point of CP_2 to a given point of M^4. $g(p)$ maps the number theoretic H-J to H-J in $M^4 \subset M^8$. The space-time surface itself - that is $g(p)$ - defines these symplectic coordinates and the local $SU(3)$ element g would naturally define this symplectic transformation.

4. For $X^4 \subset M^8$ g reduces to a constant color rotation satisfying the condition that the image point is $U(2)$ invariant. Unit element is the most natural option. This would mean that g is constant at the mass and energy shells corresponding to the roots of P and the mass shell is a mass shell of M^4 rather than some deformed mass shell associated with images under $g(p)$.

   This alone does not yet guarantee that the 4-D tangent space corresponds to M^4. The additional physically very natural condition on g is that the 4-D momentum space at these mass shells is the same. M^8-H duality maps these mass shells to the boundaries of these cd:s in M^4 (CD= $cd \times CP_2$). This conforms with the identification of zero energy states as pairs of 3-D states at the boundaries of CD.

This generalizes the original intuitive but wrong interpretation of the roots r_n of P as "very special moments in the life of self" [L10].

1. Since the roots correspond to mass squared values, they are mapped to the boundaries of cd with size $L = \hbar_{eff}/m$ by M^8-H duality in M^4 degrees of freedom. During the sequence of SSFRs the passive boundary of CD remains does not shift only changes in size, and states at it remain unaffected. Active boundary is shifted due to scaling of cd.

   The hyperplane at which upper and lower half-cones of CD meet, is shifted to the direction of geometric future. This defines a geometric correlate for the flow of experienced time.

2. A natural proposal is that the moments for SSFRs have as geometric correlates the roots of P defined as intersections of geodesic lines with the direction of 4-momentum p from the tip of CD to its opposite boundary (here one can also consider the possibility that the geodesic lines start from the center of cd). Also energy shells as roots $E = r_n$ of P are predicted. They decompose to a set of mass shells $m_{n.,k}$ with the same $E = r_n$: similar interpretation applies to them.

3. What makes these moments very special is that the mass and energy shells correspond to surfaces in M^4 defining the Lorentz quantum numbers. SSFRs correspond to quantum measurements in this basis and are not possible without this condition. At $X^4 \subset M^8$ the mass squared would remain constant but the local momentum frame would vary. This is analogous to the conservation of momentum squared in general relativistic kinematics of point particle involving however the loss of momentum conservation.

4. These conditions, together with the assumption that g is a rational function with real coefficients, strongly suggest what I have referred to as preferred extremal property, Bohr orbitology, strong form of holography, and number theoretical holography.

In principle, by a suitable choice of M^4 one can make the momentum of the system light-like: the light-like 8-momentum would be parallel to M^4. I have asked whether this could be behind the fact that elementary particles are in a good approximation massless and whether the small mass of elementary particles is due to the presence of states with different mass squares in the zero state allowed by Lorentz invariance.

The recent understanding of the nature of right-handed neutrinos based on M^4 Kähler structure [L27] makes this mechanism un-necessary but poses the question about the mechanism choosing some particular M^4. The conditions that $g(p)$ leaves mass shells and their 4-D tangent spaces invariant provides this kind of mechanism. Holography would be forced by the condition that the 4-D tangent space is same for all mass shels representing inverse images for very special moments of time.

3.2.2 What about string world sheets and partonic 2-surfaces?

One can apply the above arguments also to the identification of 2-D string world sheets and partonic 2-surfaces.

1. One has two kinds of solutions: M^2 and 3-D surfaces of X^4 as analogs of 6-brane. The interpretation for 3-D *resp.* 2-D branes as a light-like 3-surface associated with the octonionic Dirac equation representing mass shell condition *resp.* string world sheet is attractive.

2. M^2 would be replaced with an integrable distribution of $M^2(x)$ in local tangent space $M^4(x)$. The space for the choices of $M^2(x)$ would be S^3 corresponding to the selection of a preferred quaternion imaginary unit equal to the choices of preferred octonion imaginary unit.

   The choices of the preferred complex subspace $M^2(x)$ at a given point would be characterized by its normal vector and parameterized by sphere S^2: the interpretation as a quantization axis of angular momentum is suggestive. One would have space $S^3 \times S^2$. Also now the integrability conditions $de_A = 0$ would hold true.

3. String world sheets could be regarded as analogs of superstrings connecting 3-D brane like entities defined by the light-like partonic orbits. The partonic 2-surfaces at the ends of light-like orbits defining also vertices could correspond to the 3-surfaces at which quaternionic 4-surfaces intersect 6-branes.

3.3 Is (co-)associativity possible?

The number theoretic vision relying on the assumption that space-time surfaces are 8-D complex 4-surfaces in o_c^8 determined as algebraic surfaces for octonionic continuations of real polynomials, which for adelic physics would have coefficients which are rational or belong to an extension of rationals. The projections to subspaces Re^8 of o_c^8 defined as space for which given coordinate is purely real or imaginary so that complexified octonionic norm is real would give rise to real 4-D space-time surfaces. $M^8 - H$ duality would map these surfaces to geometric objects in $M^4 \times CP_2$. This vision involves several poorly understood aspects and it is good to start by analyzing them.

3.3.1 Challenging the notions of associativity and co-associativity

Consider first the notions of associativity *resp.* co-associativity equivalent with quaternionicity *resp.* co-quaternionicity. The original hope was that both options are possible for surfaces of real sub-spaces of O_c ("real" means here that complexified octonionic metric is real).

1. The original idea was that the associativity of the tangent space or normal space of a real space-time surface X^4 reduces the classical physics at the level of M^8 to associativity. Associativity/co-associativity of the space-time surface states that at each point of the tangent-/normal space of the real space-time surface in O is quaternionic. The notion generalizes also to $X_c^4 \subset O_c^8$. (Co-)associativity makes sense also for the real subspaces space of O with Minkowskian signature.

2. It has been however unclear whether (co-)associativity is possible. The cold shower came as I learned that associativity allows only for geodesic sub-manifolds of quaternionic spaces about which octonions provide an example [A6]. The good news was that the distribution of co-associative tangent spaces always defines an integrable distribution in the sense that one can find sub-manifold for which the associative normal space at a given point has tangent space as an orthogonal complement. Should the number theoretic dynamics rely on co-associativity rather than associativity?

3. Minkowskian space-time regions have been assumed to be associative and to correspond to the projection to the standard choice for basis as $\{1, iI_1, iI_2, iI_3\}$. The octonionic units $\{1, I_1, I_2, I_3\}$ define quaternionic units and associative subspace and their products with unit I_4 define the orthogonal co-associative subspace as $\{I_4, I_5 = I4I_1, I_6 = I_4I_2, I_7 = I_4I_3\}$. This result forces either to weaken the notion of associativity or to consider alternative identifications of Minkowskian regions, which can be co-associative: fortunately, there exists a large number of candidates.

The article [A6] indeed kills the idea about the associativity of the space-time surface. The article starts from a rather disappointing observation that associative sub-manifolds are geodesic sub-manifolds and therefore trivial. Co-associative quaternion sub-manifolds are however possible. With a motivation coming from this observation, the article discusses what the author calls RC quaternionic sub-manifolds of quaternion manifolds. For a quaternion manifold the tangent space allows a realization of quaternionic units as antisymmetric tensors. These manifolds are constant curvature spaces and typically homogeneous spaces.

1. Quaternion sub-manifold allows a 4-D integrable distribution of quaternion units. The normal complement of this distribution is expressible in terms of the second fundamental form and the condition that it is trivial implies that the second fundamental form is vanishing so that one has a geodesic submanifold. Quaternionic sub-manifolds are thus too trivial to be interesting. As a diametric opposite, one can also define totally real submanifolds for which the normal space contains a distribution of quaternion units. In this case the distribution is always integrable. This case is much more interesting from the TGD point of view.

2. Author introduces the notion of CR quaternion sub-manifold $N \subset M$, where M is quaternion manifold with constant sectional curvatures. N has quaternion distribution D in its tangent spaces if the action of quaternion units takes D to itself. $D^\perp$ is the co-quaternionic orthogonal complement D in the normal space N. D would take also $D^\perp$ to itself. $D^\perp$ can be expressed in terms of the components of the second fundamental form and vanishes for quaternion sub-manifolds.

3. Author deduces results about CR quaternion sub-manifolds, which are very interesting from the TGD point of view.

   (a) Sub-manifold is CR quaternion sub-manifold only if the curvature tensor of R_M of the embedding space satisfies $R_M(D, D, D^\perp, D\ \ ) = 0$. The condition is trivial if the quaternion manifold is flat. In the case of octonions this would be the case.

   (b) D is integrable only if the second fundamental form restricted to it vanishes meaning that one has a geodesic manifold. Totally real distribution $D^\perp$ is always integrable to a co-associative surface.

   (c) If $D^\perp$ integrates to a minimal surface then N itself is a minimal surface.

Could one consider RC quaternion sub-manifolds in TGD framework? Both octonions and their complexifixation can be regarded as quaternionic spaces. Consider the real case.

1. If the entire D is quaternionic then N is a geodesic sub-manifold. This would leave only E^4 and its Minkowskian variants with various signatures. One could have however 4-D totally real (co-associative) space-time surfaces. Simple arguments will show that the intersections of the conjectured quaternionic and co-quaternionic 4-surfaces have 2- and 3-D intersections with 6-branes.

   Should one replace associative space-time surfaces with CR sub-manifolds with $d \leq 3$ integrable distribution D whereas the co-quaternionic surfaces would be completely real having 4-D integrable $D^\perp$? Could one have 4-D co-associative surfaces for which $D^\perp$ integrates to $n \geq 1$-dimensional minimal surface (geodesic line) and the X^4 itself is a minimal surface?

   Partially associative CR manifold do not allow M^8H duality. Only co-associative surfaces allow it and also their signature must be Minkowskian: the original idea [L12, L4, L5, L6] about Euclidian (Minkowskian) signature for co-associative (associative) surfaces was wrong.

2. The integrable 2-D sub-distributions D defining a distribution of normal planes could define foliations of the X^4 by 2-D surfaces. What springs to mind is foliations by string world sheets and partonic 2 surfaces orthogonal to them and light-like 3-surfaces and strings transversal to them. This expectation is realized.

3.3.2 How to identify the Minkowskian sub-space of O_c?

There are several identifications of subspaces of O_c with Minkowskian signature. What is the correct choice has been far from obvious. Here symmetries come in rescue.

1. Any subspace of O^c with 3 (1) imaginary coordinates and 1 (3) real coordinates has Minkowskian signature in octonionic norm algebraically continued to O_c (complex valued continuation of real octonion norm instead of real valued Hilbert space norm for O_c). Minkowskian regions should have local tangent space basis consisting of octonion units which in the canonical case would be $\{I_1, iI_3, iI_5, iI_7\}$, where i is commutative imaginary unit. This particular basis is co-associative having whereas its complement $\{iI_0, I_2, I_4, I_6\}$ is associative and has also Minkowskian signature.

2. The size of the isometry group of the subspace of M^8_c depends on whether the tangent basis contains real octonion unit 1 or not. The isometry group for the basis containing I_0 is $SO(3)$ acting as automorphisms of quaternions and $SO(k, 3-k)$ when $3-k$ units are proportional to i. The reason is that G_2 (and its complexification $G_{2,c}$) and its subgroups do not affect I_0. For the tangent spaces built from 4 imaginary units I_k and iI_l the isometry group is $SO(k, 4-k) \subset G_{2,c}$.

   The choice therefore allows larger isometry groups and also co-associativity is possible for a suitable choice of the basis. The choice $\{I_1, iI_3, iI_5, iI_7\}$ is a representative example, which will be called canonical basis. For these options the isometry group is the desired $SO(1,3)$ as an algebraic continuation of $SO(4) \subset G_2$ acting in $\{I_1, I_3, I_5, I_7\}$, to $SO(1,3) \subset G_{2,c}$.

   Also Minkowskian signature - for instance for the original canonical choice $\{I_0, iI_1, iI_2, iI_3\}$ - can have only $SO(k, 3-k)$ as isometries. This is the basic objection against the original choice $\{I_0, iI_1, iI_2, iI_3\}$. This identification would force the realization of $SO(1,3)$ as a subgroup of $SO(1,7)$. Different states of motion for a particle require different octonion structure with different direction of the octonion real axis in M^8. The introduction of the notion of moduli space for octonion structures does not look elegant. For the option $\{I_1, iI_3, iI_5, iI_7\}$ only a single octonion structure is needed and $G_{2,c}$ contains $SO(1,3)$.

   Note that also the signatures $(4,0)$, $(0,4)$ and $(2,2)$ are possible and the challenge is to understand why only the signature (1,3) is realized physically.

Co-associative option is definitely the only physical alternative. The original proposal for the interpretation of the Minkowski space in terms of an associative real sub-space of M^4 had a serious problem. Since time axis was identified as octonionic real axis, one had to assign different octonion structure to particles with non-parallel moment: $SO(1,7)$ would relate these structures: how to glue the space-time surfaces with different octonion structures together was the problem. This problem disappears now. One can simply assign to particles with different state of motion real space-time surface defined related to each other by a transformation in $SO(1,3) \subset G_{2,c}$.

3.3.3 The condition that $M^8 - H$ duality makes sense

The condition that $M^8 - H$ duality makes sense poses strong conditions on the choice of the real sub-space of M^8_c.

1. The condition that tangent space of O_c has a complexified basis allowing a decomposition to representations of $SU(3) \subset G_2$ is essential for the map to $M^8 \to H$ although it is not enough. The standard representation of this kind has basis $\{\pm iI_0 + I_1\}$ behaving like $SU(3)$ singlets $\{I_2 + \epsilon iI_3, I_4 + \epsilon iI_5, \epsilon I_6 \pm iI_7\}$ behaves like SU(3) triplet 3 for $\epsilon = 1$ and its conjugate $\overline{3}$ for $\epsilon = -1$. $G_{2,c}$ provides new choices of the tangent space basis consistent with this choice. $SU(3)$ leaves the direction I_1 unaffected but more general transformations act as Lorentz

transformation changing its direction but not leaving the M^4 plane. Even more general $G_{2,c}$ transformations changing M^4 itself are in principle possible.

Interestingly, for the canonical choice the co-associative choice has $SO(1,3)$ as isometry group whereas the complementary choice failing to be associative correspond to a smaller isometry group $SO(3)$. The choice with M^4 signature and co-associativity would provide the highest symmetries. For the real projections with signature $(2,2)$ neither consistent with color structure, neither full associativity nor co-associativity is possible.

2. The second essential prerequisite of M^8-H duality is that the tangent space is not only (co-)associative but contains also (co-)complex - and thus (co-)commutative - plane. A more general assumption would be that a co-associative space-time surface contains an integrable distribution of planes $M^2(x)$, which could as a special case reduce to M^2.

   The proposal has been that this integrable distribution of $M^2(x)$ could correspond to string sheets and possibly also integrable orthogonal distribution of their co-complex orthogonal complements as tangent spaces of partonic 2-surfaces defining a slicings of the space-time surface. It is now clear that this dream cannot be realized since the space-time surface cannot be even associative unless it is just E^4 or its Minkowskian variants.

3. As already noticed, any distribution of the associative normal spaces integrates to a co-associative space-time surface. Could the normal spaces also contain an integrable distribution of co-complex planes defined by octonionic real unit 1 and real unit $I_k(x)$, most naturally I_1 in the canonical example? This would give co-commutative string world sheet. Commutativity would be realized at the 2-D level and associativity at space-time level. The signature of this plane could be Minkowskian or Euclidian. For the canonical example $\{I_1, iI_3, iI_5, iI_7\}$ the 2-D complex plane in quaternionic sense would correspond to $(a\times 1, +n_2I_2+n_4I_6+n_6I_6,$ where the unit vector n_i has real components and one has $a=1$ or $a=i$ is forced by the complexification as in the canonical example.

   Since the distribution of normal planes integrates to a 4-surface, one expects that its sub-distribution consting of commutative planes integrates to 2-D surface inside space-time surface and defines the counterpart of string worlds sheet. Also its normal complement could integrate to a counterpart of partonic 2-surface and a slicing of space-time surface by these surfaces would be obtained.

4. The simplest option is that the commutative space does not depend on position at X^4. This means a choice of a fixed octonionic imaginary unit, most naturally I_1 for the canonical option. This would make $SU(3)$ and its sub-group $U(2)$ independent of position. In this case the identification of the point of $CP_2 = SU(3)/U(2)$ labelling the normal space at a given point is unique.

   For a position dependent choice $SU(3)(x)$ it is not clear how to make the specification of $U(2)(x)$ unique: it would seem that one must specify a unique element of $G_2(x)$ relating $SU(3)(x)$ to a choice at special point x_0 and defining the conjugation of both $SU(3)(x)$ and $U(2)(x)$. Otherwise one can have problems. This would also mean a unique choice for the direction of time axis in O and fixing of $SO(1,3)$ as a subgroup of $G_{2,c}$. Also this distribution of associative normal spaces is integrable. Physically this option is attractive but an open question is whether it is consistent with the identification of space-time surfaces as roots $Re_Q(P)=0$ of P.

3.3.4 Co-associativity from octonion analyticity or/and from G_2 holography?

Candidates for co-associative space-time surfaces X^4_r are defined as restrictions X^4_r for the roots X^4_c of the octonionic polynomials such that the O_c coordinates in the complement of a real co-associative sub-space of O_c vanish or are constant. Could the surfaces X^4_r or even X^4_c be co-associative?

1. X^4_r is analogous to the image of real or imaginary axis under a holomorphic map and defines a curve in complex plane preserving angles. The tangent vectors of X^4_r and X^4_c involve gradients of all coordinates of O_c and are expressible in terms of all octonionic unit vectors. It is not

obvious that their products would belong to the normal space of X_r^4 a strong condition would be that this is the case for X_c^4.

2. Could octonion analyticity in the proposed sense guarantee this? The products of octonion units also in the tangent space of the image would be orthogonal to the tangent space. Ordinary complex functions preserve angles, in particular, the angle between x- and y-axis is preserved since the images of coordinate curves are orthogonal. Octonion analyticity would preserve the orthogonality between tangent space vectors and their products.

3. This idea could be killed if one could apply the same approach to associative case but this is not possible! The point is that when the real tangent space of O_c contains the real octonion unit, the candidate for the 4-D space-time surface is a complex surface X_c^2. The number theoretic metric is real only for 2-D X_r^2 so that one obtains string theory with co-associativity replaced with co-commutativity and $M^4 \times CP_2$ with $M^2 \times S^2$. One could of course ask whether this option could be regarded as a "sub-theory" of the full theory.

My luck was that I did not realize the meaning of the difference between the two cases first and realized that one can imagine an alternative approach.

1. G_2 as an automorphism group of octonions preserves co-associativity. Could the image of a co-associative sub-space of O_c defined by an octonion analytic map be regarded as an image under a local G_2 gauge transformation. $SU(3) \subset G_2$ is an especially interesting subgroup since it could have a physical interpretation as a color gauge group. This would also give a direct connection with $M^8 - H$ duality since $SU(3)$ corresponds to the gauge group of the color gauge field in H.

2. One can counter-argue that an analog of pure gauge field configuration is in question at the level of M^8. But is a pure gauge configuration for $G_{2,c}$ a pure gauge configuration for G_2? The point is that the $G_{2,c}$ connection $g^{-1}\partial_\mu g$ trivial for $G_{2,c}$ contains by non-linearity cross terms from $g_2g, c = g_{2,1} + ig_{2,2}$, which are of type $Re = X[g_{2,1}, g_{2,1}] - X[g_{2,2}, g_{2,2}] = 0$ and $Im = iZ[g_{2,1}, g_{2,2}] = 0$. If one puts $g_{2,2}$ contributions to zero, one obtains $Re = X[g_{2,1}, g_{2,1}]$, which does not vanish so that $SU(3)$ gauge field is non-trivial.

3. X_r^4 could be also obtained as a map of the co-associative M^4 plane by a local $G_{2,c}$ element. It will turn out that $G_{2,c}$ could give rise to the speculated Yangian symmetry [L3] at string world sheets analogous to Kac-Moody symmetry and gauge symmetry and crucial for the construction of scattering amplitudes in $M8$.

4. The decomposition of the co-associative real plane of O_c should contain a preferred complex plane for $M^8 - H$ duality to make sense. $G_{2,c}$ transformation should trivially preserve this property so that SH would not be necessary at H side anymore.

There is a strong motivation to guess that the two options are equivalent so that $G_{2,c}$ holography would be equivalent with octonion analyticity. The original dream was that octonion analyticity would realize both associative and co-associative dynamics but was exaggeration!

3.3.5 Does one obtain partonic 2-surfaces and strings at boundaries of ΔCD_8?

It is interesting to look for the dimensions of the intersections of the light-like branes at the boundary of CD_8 giving rise to the boundary of CD_4 in M^4 to see whether it gives justification for the existing phenomenological picture involving light-like orbits of partonic 2-surfaces connected by string world sheets.

1. Complex light-cone boundary has dimension $D = 14$. $P = 0$ as an additional condition at δCD_8 gives 2 complex conditions and defines a 10-D surface having 5-D real projections.

2. The condition $Im_Q(P) = 0$ gives 8 conditions and gives a 2-D complex surface with 1-D real projection. The condition $Re_Q(P) = 0$ gives 3 complex conditions since $X = 0$ is already satisfied and the solution is a 4-D surface having 2-D real projection. Could the interpretation be in terms of the intersection of the orbit of a light-like partonic surface with the boundary of CD_8?

3. Associativity is however not a working option. If only co-associative Minkowskian surfaces allowing mapping to H without SH are present then only 4-D space-time surfaces with Minkowskian signature, only partonic 2-surfaces and their light-like orbits would emerge from co-associativity.

   This option would not allow string world sheets for which there is a strong intuitive support. What could a co-complex 2-surface of a co-associative manifold mean? In the co-associative case the products of octonion imaginary units are in the normal space of space-time surface. Could co-complex surface $X^2_c \subset X^4_c$ be defined by an integrable co-complex sub-distribution of co-associative distribution. The 4-D distribution of normal planes is always integrable.

   Could the 2-D sub-distributions of co-associative distribution integrate trivially and define slicings by string world sheets or partonic 2-surfaces. Could the distribution of string distributions and its orthogonal complement be both integrable and provide orthogonal slicings by string world sheets and partonic 2-surfaces? String world sheets with Minkowskian signature should intersect the partonic orbits with Euclidian signature along light-like lines. This brings in mind the orthogonal grid of flow lines defined by the $Re(f) = 0$ and $Im(f) = 0$ lines of an analytic function in plane.

4. In this picture the partonic 2-surfaces associated with light-like 3-surface would be physically unique and could serve as boundary values for the distributions of partonic 2-surfaces. But what about string world sheets connecting them? Why would some string world sheets be exceptional? String world sheets would have a light-like curve as an intersection with the partonic orbit but this is not enough.

   Could the physically special string world sheets connect two partonic surfaces? Could the string associated with a generic string world sheet be like a flow line in a hydrodynamic flow past an obstacle - the partonic 2-surface? The string as a flowline would go around the obstacle along either side but there would be one line which ends up to the object.

Interactions would correspond geometrically to the intersections of co-associative space-time surfaces X^4_r associated with particles and corresponding to different real sub-spaces of O_c related by Lorentz boost in $SO(1,3) \subset G_{2,c}$. In the generic case the intersection would be discrete. In the case that X and Y have a common root the real surfaces $X^4_r \subset X^6_r$ associated with quarks and depending on their state of motion would reside inside the same 6-D surface X^6_r and have a 2-D surface X^2_r as intersection. Could this surface be interpreted as a partonic 2-surface? One must however bear in mind that partonic 2-surfaces as topological vertices are assumed to be non-generic in the sense that the light-like partonic orbits meet at them. At the level of H, the intersections would be partonic 2-surfaces X^2 at which the four 3-D partonic orbits would meet along their ends. Does this hold true at the level of M^8? Or can it hold true even at the level H?

The simplest situation corresponds to 4 external quarks. There are 6 different intersections. Not all of them are realized since a given quark can belong only to a single intersection. One must have two disjoint pairs -say 12 and 34. Most naturally positive *resp.* negative energy quarks form a pair. These pairs are located in different half-cones. The intersections would give two partonic 2-surfaces and this situation would be generic. This suggests a modification of the description of particle reaction in M^8 . $M^8 - H$ duality suggests a similar description in H.

3.3.6 What could be the counterparts of wormhole contacts at the level of M^8?

The experience with H, in particular the presence of extremals with Euclidian signature of the induced metric and identified as building bricks of elementary particles, suggest that also the light-like 3-surfaces in M^8_c could have a continuation with an Euclidian signature of the number theoretic metric with norm having real values only for the projections to planes allowing real coordinates.

The earlier picture has been that the wormhole contacts as CP_2 type extremals correspond to co-associative regions and their exteriors to associative regions. If one wants $M^8 - H$ duality in strong form and thus without need for SH, one should assume that both these regions are co-associative.

1. The simplest option is that the real Minkowskian time coordinate becomes imaginary. Instead of the canonical (I_1, iI_3, iI_5, iI_7) the basis would be (iI_1, iI_3, iI_5, iI_7) having Euclidian

signature and SO(4) as isometry group. The signature would naturally change at light-like 3-surface the time coordinate along light-like curves becomes zero - proper time for photon vanishes - and can ransforms continuously from real to imaginary.

2. Wormhole contacts in H behave like pairs of magnetic monopoles with monopole charges at throats. If one does not allow point-like singularity, the monopole flux must go to a parallel Minkowskian space-time sheet through the opposite wormhole throat. Wormhole contact with effective magnetic charge would correspond in M^8_c to a distribution of normal 4-planes at the partonic 2-surfaces analogous to the radial magnetic field of monopole at a sphere surrounding it. To avoid singularity of the distribution, there must be another light-like 3-surface M^8 such that its partonic throat has a topologically similar distribution of normal planes.

In the case of X^3_c dimension does not allow co-quaternion structure: could they allow 4-D co-associative sub-manifolds? It will be found that this option is not included since co-associative tangent space distributions in a quaternion manifold (now O) are always integrable.

3.4 Octonionic Dirac equation and co-associativity

Also the role of associativity concerning octonionic Dirac equation in M^8 must be understood. It is found that co-associativity allows very elegant formulation and suggests the identification of the points appearing as the ends of quark propagator lines in H as points of boundary of CD representing light-like momenta of quarks. Partonic vertices would involve sub-CDs and momentum conservation would have purely geometric meaning bringing strongly in mind twistor Grassmannian approach [**?**, **?**, **?**]. I have discussed the twistor lift of TGD replacing twistors as fields with surfaces in twistor space having induced twistor structure in [K13, K11, K14] [L15, L16].

3.4.1 Octonionic Dirac equation

The following arguments lead to the understanding of co-associativity in the case of octonion spinors. The constant spinor basis includes all spinors but the gamma matrices appearing in the octonionic Dirac equation correspond to co-associative octonion units.

1. At the level of O_c the idea about massless Dirac equation as partial differential equation does not make sense. Dirac equation must be algebraic and the obvious idea is that it corresponds to the on mass shell condition for a mode of ordinary Dirac equation with well-define momentum: $p^k\gamma_k\Psi = 0$ satisfying $p^kp_k = 0$. This suggests that octonionic polynomial P defines the counterpart of $p^k\gamma_k$ so that gamma matrices γ_k would be represented as octonion components. Does this make sense?

2. Can one construct octonionic counterparts of gamma matrices? The imaginary octonion units I_k indeed define the analogs of gamma matrices as $\gamma_k \equiv iI_k$ satisfying the conditions $\{\gamma_k, \gamma_l\} = 2\delta_{kl}$ defining Euclidian gamma matrices. The problem is that one has $I_0I_lk + I_kI_0 = 2I_k$. One manner to solve the problem would be to consider tensor products $I_0\sigma_3$ and $I_k\sigma_2$ where σ_3 and $sigma_2$ are Pauli's sigma matrices with anti-commutation relations $\{\sigma_i, \sigma_j\} = \delta_{i,j}$. Note that I_k do not allow a matrix representation.

   Co-associativity condition suggests an alternative solution. The restriction of momenta to be co-associative and therefore vanishing component p^0 as octonion, would selects a sub-space spanned by say the canonical choice $\{I_2, iI3, iI_5, iI_7\}$ satisfying the anticommutation relations of Minkowskian gamma matrices. Octonion units do not allow a matrix representation because they are not associative. The products for a co-associative subset of octonion units are however associative $(a(bc) = (ab)c$ so that they can be mapped to standard gamma matrices in Minkowski space. Co-associativity would allow the representation of 4-D gamma matrices as a maximal associative subset of octonion units.

3. What about octonionic spinors. The modes of the ordinary Dirac equation with a well-defined momentum are obtained by applying the Dirac operator to an orthogonal basis of constant spinors u_i to give $\Psi = p^k\gamma_ku_i$. Now the counterparts of constant spinors u_i would naturally

be octonion units $\{I_0, I_k\}$: this would give the needed number 8 of real spinor components as one has for quark spinors.

Dirac equation reduces to light-likeness conditions $p^k p_k = 0$ and p_k must be chosen to be real - if pk are complex, the real and imaginary parts of momentum are parallel. One would obtain an entire 3-D mass shell of solution and a single mode of Dirac equation would correspond to a point of this mass shell.

Remark: Octonionic Dirac equation is associative since one has a product of form $(p_k\gamma_k)^2 u_i$ and octonion products of type $x^2 y$ are associative.

4. p^k would correspond to the restriction of $P(o_c)$ to M^4 as sub-space of octonions. Since co-associativity implies $P(o_c) = Y(o_c)o_c$ restricted to counterpart of M^4 (say subspace spanned by $\{I_2, iI3, iI_5, iI_7\}$), Dirac equation reduces to the condition $o^k o_k = 0$ in M^4 defining a light-cone of M^4. This light-cone is mapped to a curved light-like 3-surface X^3 in o_c as $o_c \rightarrow P(o_c) = Yo_c$. $M^8 - H$ duality maps points of space-time surface on M^8 H and therefore the light-cone of M^4 corresponds to either light-like boundary of CD. It seems that the image of X^3 in H has M^4 projection to the light-like boundary of CD.

   Co-associative space-time surfaces have 3-D intersections X^3 with the surface $P = 0$: the conjecture is that X^3 corresponds to a light-like orbit of partonic 2-surfaces in H at which the induced metric signature changes. At X^3 one has besides $X = 0$ also $Y = 0$ so that octonionic Dirac equation $P(o_c)\Psi = P^k I_k \Psi = Yp^k I_k \Psi = 0$ is trivially satisfied for all momenta $p^k = o^k$ defined by the M^4 projections of points of X^3 and one would have $P^k = Yp^k = 0$ so that the identification of P^k as 4-momentum would not allow to assign non-vanishing momenta to X^3. The direction of p^k is constrained only by the condition of belonging to X^3 and the momentum would be in general time-like since X^3 is inside future light-cone.

   $Y = 0$ condition conforms with the proposal that X^3 defines a boundary of Minkowskian and Euclidian region: Euclidian mass shell condition for real P^k requires $P^k = 0$. The general complex solution to $P^2 = 0$ condition is $P = P_1 + iP_2$ with $P_1^2 = P_2^2$.

A single mode of Dirac equation with a well-defined value of p^k as the analog of 4-momentum would correspond to a selection of single time-like point at X^3 or light-like point at the light-like boundary of CD. X^3 intersects light-cone boundary as part of boundary of 7-D light-cone. The picture about scattering amplitudes - consistent with the view about cognitive representations as a unique discretization of space-time surface - is that quarks are located at discrete points of partonic 2-surfaces representing the ends of fermionic propagator lines in H and that one can assign to them light-like momenta.

3.4.2 Challenging the form of $M^8 - H$ duality for the map $M^4 \subset M^8$ to $M^4 \subset H$

The assumption that the map $M^4 \subset M^8$ to $M^4 \subset H$ in $M^8 - H$ duality is a simple identification map has not been challenged hitherto.

1. Octonionic Dirac equation forces the identification of M^8 as analog of 8-D momentum space and the earlier simple identification is in conflict with Uncertainty Principle. Inversion allowed by conformal invariance is highly suggestive: what comes first in mind is a map $m^k \rightarrow \hbar_{eff} m^k / m^k m_k$.

   At the light-cone boundary the map is ill-defined. Here on must take as coordinate the linear time coordinate m^0 or equivalently radial coordinate $r_M = m^0$. In this case the map would be of fporm $t \rightarrow \hbar_{eff}/m^0$: m^0 has interpretation as energy of massless particle.

   The map would give a surprisingly precise mathematical realization for the intuitive arguments assigning to mass a length scale by Uncertainty Principle.

2. Additional constraints on $M^8 - H$ duality in M^4 degrees of freedom comes from the following argument. The two half-cones of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T - o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T - o)$: the space-time surfaces at half-cones would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - \overline{o}^2$ only, the condition is identically satisfied for both options.

There are two options for the identification of the coordinate t.

Option a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time.

At the level of M^8, The M^4 projection of CD_8 is a union of future and past directed light-cones with a common tip rather than CD_4. Both incoming and outgoing momenta have the same origin automatically. This identification is the natural one at the level of M^8.

Option b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The half-cone at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.

If **Option a)** is realized at the level of M^8 and **Option b)** at the level of H, as seems natural, a time translation $m^0 \rightarrow m^0 + 2T$ of the past directed light-cone in $M^4 \subset H$ is required in order to to give upper half-cone of CD_4.

3. The map of the momenta to embedding space points does not prevent the interpretation of the points of M^8 as momenta also at the level of H since this information is not lost. One cannot identify p^k as such as four-momentum neither at the level of M^8 nor H as suggested by the naïve identification of the Cartesian factors M^4 for M^8 and H. This problem is circumvented by a conjugation in M^8_c changing the sign of 3-momentum. The light-like momenta along the light-cone boundary are non-physical but transform to light-like momenta arriving into light-cone as the physical intuition requires.

   Therefore the map would have in the interior of light-cone roughly the above form but there is still a question about the precise form of the map. Does one perform inversion for the M^4 projection or does one take M^4 projection for the inversion of complex octonion. The inversion of M^4 projection seems to be the more plausible option. Denoting by $P(o_c)$ the real M^4 projection of X^4 point one therefore has:

$$P(o_c) \rightarrow \hbar_{eff} \frac{\overline{P(o_c)}}{P(o_c) \cdot P(o_c)} \quad . \tag{3.7}$$

   Note that the conjugation changes the direction of 3-momentum.

   At the light-cone boundary the inversion is ill-defined but Uncertainty Principle comes in rescue, and one can invert the M^4 time coordinate:

$$Re(m^0) = t \rightarrow \hbar_{eff} \frac{1}{t} \quad . \tag{3.8}$$

A couple of remarks are in order.

1. The presence of $\hbar_{eff}$ instead of $\hbar$ is required by the vision about dark matter. The value of $\hbar_{eff}/h_0$ is given by the dimension of extension of rationals identifiable as the degree of P.
2. The image points $\overline{p}^k$ in H would naturally correspond to the ends of the propagator lines in the space-time representation of scattering amplitudes.

The information about momenta is not lost in the map. What could be the interpretation of the momenta $\overline{p}^k$ at the level of H?

1. Super-symplectic generators at the partonic vertices in H do not involve momenta as labels. The modes of the embedding space spinor field assignable to the ground states of super-symplectic representations at the boundaries of CD have 4-momentum and color as labels.

The identification of $\overline{p}^k$ as this momentum label would provide a connection with the classical picture about scattering events.

At the partonic 2-surfaces appearing as vertices, one would have a sum over the ground states (spinor harmonics). This would give integral over momenta but M^8-H duality and number theoretic discretization would select a finite subset and the momentum integral would reduce to a discrete sum. The number of M^8 points with coordinates in a given extension of rationals is indeed finite.

2. $M^4 \subset M^8$ could be interpreted as the space of 4-momenta labeling the spinor harmonics of M^8. Same would apply at the level of H: spinor harmonics would correspond to the ground states of super-symplectic representations.

3. The interpretation of the points of M^4_c as complex 4-momenta inspires the question whether the interpretation of the imaginary part of the momentum squared in terms of decay decay width so that M^8 picture would code even information about the dynamics of the particles.

4 How to achieve periodic dynamics at the level of $M^4 \times CP_2$?

Assuming M^8-H duality, how could one achieve typical periodic dynamics at the level of H - at least effectively?

It seems that one cannot have an "easy" solution to the problem?

1. Irreducible polynomials which are products of monomials corresponding to roots r_n which are in good approximation evenly spaced $r_n = r_0 + nr_1\Delta r_n$ would give "very special moments in the life of self" as values of M^4 time which are evenly spaced [L12, L10]. This could give rise to an effective periodicity but it would be at the level of M^8, not H, where it is required.

2. Is it enough that the periodic functions are *only* associated with the spinor harmonics of H involved with the construction of scattering amplitudes in H [L22]? For the modified Dirac equation [K16] the periodic behavior is possible. Note also that the induced spinors defining ground states of super-symplectic representations are restrictions of second quantized spinors of H proportional to plane waves in M^4. These solutions do not guarantee quantum classical correspondence.

4.1 The unique aspects of Neper number and number theoretical universality of Fourier analysis

Could one assume more general functions than polynomials at the level of H? Discrete Fourier basis is certainly an excellent candidate in this respect but does it allow number theoretical universality?

1. Discrete Fourier analysis involves in the Euclidian geometry periodic functions $exp(2\pi x)$, n integer and in hyperbolic geometry exponential functions $exp(kx)$.

Roots of unity $exp(i2\pi/n)$ allow to generalize Fourier analysis. The p-adic variants of $exp(ix)$ exist for rational values of $x = k2\pi/n$ for $n = K$ if $exp(i2\pi/K)$ belongs to the extension of rationals. $x = k = 2pi/n$ does not exist as a p-adic number but $exp(x) = exp(i2\pi/n)$ can exist as phase replacing x as coordinate in extension of p-adics. One can therefore define Fourier basis $\{exp(inx)|n \in Z\}$ which exist at discrete set of rational points $x = k/n$

Neper number e is also p-adically exceptional in that e^p exists as a p-adic number for all primes p. One has a hierarchy of finite-D extensions of p-adic numbers spanned by the roots $e^{1/n}$. Finiteness of cognition might allow them. Hyperbolic functions $exp(nx)$, $n = 1, 2...$ would have values in extension of p-adic number field containing $exp(1/N)$ in a discrete set of points $\{x = k/N|k \in Z\}$.

2. (Complex) rationality guarantees number theoretical universality and is natural since CP_2 geometry is complex. This would correspond to the replacement $x \rightarrow exp(ix)$ or $x \rightarrow exp(x)$ for powers x^n. The change of the signature by replacing real coordinate x with ix would automatically induce this change.

3. Exponential functions are in a preferred position also group theoretically. Exponential map maps $g \rightarrow exp(itg)$ the points of Lie algebra to the points of the Lie group so that the tangent space of the Lie algebra defines local coordinates for the Lie group. One can say that tangent space is mapped to space itself. M^4 defines an Abelian group and the exponential map would mean replacing of the M^4 coordinates with their exponential, which are p-adically more natural. Ordinary Minkowski coordinates have both signs so that they would correspond to the Lie algebra level.

4. CP_2 is a coset space and its points are obtained as selected points of $SU(3)$ using exponentiation of a commutative subalgebra t in the decomposition $g = h + t + \bar{t}$ in the Lie-algebra of $SU(3)$. One could interpret the CP_2 points as exponentials and the emergence of exponential basis as a basis satisfying number theoretical universality.

4.2 Are CP_2 coordinates as functions of M^4 coordinates expressible as Fourier expansion

Exponential basis is not natural at the level of M^8. Exponential functions belong to dynamics, not algebraic geometry, and the level H represents dynamics.

It is the dependence of CP_2 coordinates on M^4 coordinates, where the periodicity is needed. The map of the tangent spaces of $X^4 \subset M^8$ to points of CP_2 is slightly local since it depends on the first derivatives crucial for dynamics. Could this bring in dynamics and exponential functions at the level of H?

These observations inspire the working hypothesis that CP_2 points as functions of M^4 coordinates are expressible as polynomials of hyperbolic and trigonometric exponentials of M^4 coordinates.

Consider now the situation in more detail.

1. The basis for roots of e would be characterized by integer K in $e^{1/K}$. This brings in a new parameter characterizing the extension of rationals inducing finite extensions of p-adic numbers. K is analogous to the dimension of extension of rationals: the p-adic extension has dimension $d = Kp$ depending on the p-adic prime explicitly.

2. If CD size T is given, $e^{-T/K}$ defines temporal and spatial resolution in H. K or possibly Kp could naturally correspond to the gravitational Planck constant [L9] [K1] [?] $K = n_{gr} = \hbar_{gr}/h_0$.

3. In [L24] many-sheetedness with respect to CP_2 was proposed to correspond to flux tubebundles in M^4 forming quantum coherent structures. A given CP_2 point corresponds to several M^4 points with the same tangent space and their number would correspond to the number of the flux tubes in the bundle.

   Does the number of these points relate to K or Kp? p-Adic extension would have finite dimension $d = Kp$. Could $d = Kp$ be analogous to a degree of polynomial defining the dimension of extension of rationals? Could this be true in p-adic length scale resolution $O(p^2) = 0$ The number of points would be Kp and very large. For electron one has $p = M_{127} = 2^{127} - 1$.

4. The dimension n_A Abelian extension associated with EQ would naturally satisfy $n_A = K$ since the trigonometric and hyperbolic exponentials are obtained from each other by replacing a real coordinate with an imaginary one.

5. There would be two effective Planck constants. $h_{eff} = nh_0$ would be defined by the degree n of the polynomial P defining $X^4 \subset M^8$. $\hbar_{gr} = n_{gr}h_0$ would define infra-red cutoff in M^4 as the size scale of CD in $H = M^4 \times CP_2$. n *resp.* $n_{gr} = Kp$ would characterize many-sheetedness in M^4 *resp.* CP_2 degrees of freedom.

4.3 Connection with cognitive measurements as analogs of particle reactions

There is an interesting connection to the notion of cognitive measurement [L24, L25, L26].

1. The dimension n of the extension of rationals as the degree of the polynomial $P = P_{n_1} \circ P_{n_2} \circ ...$ is the product of degrees of degrees n_i: $n = \prod_i n_i$ and one has a hierarchy of Galois groups G_i associated with $P_{n_i} \circ ....$ G_{i+1} is a normal subgroup of G_i so that the coset space $H_i = G_i/G_{i+1}$ is a group of order n_i. The groups H_i are simple and do not have this kind of decomposition: simple finite groups appearing as building bricks of finite groups are classified. Simple groups are primes for finite groups.

2. The wave function in group algebra $L(G)$ of Galois group G of P has a representation as an entangled state in the product of simple group algebras $L(H_i)$. Since the Galois groups act on the space-time surfaces in M^8 they do so also in H. One obtains wave functions in the space of space-time surfaces. G has decomposition to a product (not Cartesian in general) of simple groups. In the same manner, $L(G)$ has a representation of entangled states assignable to $L(H_i)$ [L24, L26].

This picture leads to a model of analysis as a cognitive process identified as a cascade of "small state function reductions" (SSFRs) analogous to "weak" measurements.

1. Cognitive measurement would reduce the entanglement between $L(H_1)$ and $L(H_2)$, the between $L(H_2)$ and $L(H_3)$ and so on. The outcome would be an unentangled product of wave functions in $L(H_i)$ in the product $L(H_1) \times L(H_2) \times ....$ This cascade of cognitive measurements has an interpretation as a quantum correlate for analysis as factorization of a Galois group to its prime factors. Similar interpretation applies in M^4 degrees of freedom.

2. This decomposition could correspond to a replacement of P with a product $\prod_i P_i$ of polynomials with degrees $n = n_1 n_2 ...$, which is irreducible and defines a union of separate surfaces without any correlations. This process is indeed analogous to analysis.

3. The analysis cannot occur for simple Galois groups associated with extensions having no decomposition to simpler extensions. They could be regarded as correlates for irreducible primal ideas. In Eastern philosophies the notion of state empty of thoughts could correspondto these cognitive states in which SSFRs cannot occur.

4. An analogous process should make sense also in the gravitational sector and would mean the splitting of $K = n_A$ appearing as a factor $n_{gr} = Kp$ to prime factors so that the sizes of CDs involved with the resulting structure would be reduced. This process would reduce to a simultaneous measurement cascade in hyperbolic and trigonometric Abelian extensions. The IR cutoffs having interpretation as coherence lengths would decrease in the process as expected. Nature would be performing ordinary prime factorization in the gravitational degrees of freedom.

Cognitive process would also have a geometric description.

1. For the algebraic EQs, the geometric description would be as a decay of n-sheeted 4-surface with respect to M^4 to a union of n_i-sheeted 4-surfaces by SSFRs. This would take place for flux tubes mediating all kinds of interactions.

   In gravitational degrees of freedom, that is for trascendental EQs, the states with $n_{gr} = Kp$ having bundles of Kp flux tubes would deca to flux tubes bundles of $n_{gr,i} = K_i p$, where K_i is a prime dividing K. The quantity $log(K)$ would be conserved in the process and is analogous to the corresponding conserved quantity in arithmetic quantum field theories (QFTs) and relates to the notion of infinite prime inspired by TGD [K12].

2. This picture leads to ask whether one could speak of cognitive analogs of particle reactions representing interactions of "thought bubbles" i.e. space-time surfaces as correlates of cognition. The incoming and outgoing states would correspond to a Cartesian product of simple subgroups: $G = \prod_i^{\times} H_i$. In this composition the order of factors does not matter and the situation is analogous to a many particle system without interactions. The non-commutativity in general case leads to ask whether quantum groups might provide a natural description of the situation.

3. Interestingly, Equivalence Principle is consistent with the splitting of gravitational flux tube structures to smaller ones since gravitational binding energies given by Bohr model in $1/r$ gravitational potential do not depend on the value of $\hbar_{gr}$ if given by Nottale formula $\hbar_{gr} = GMm/v_0$ [L29]. The interpretation would be in terms of spontaneous quantum decoherence taking place as a decay of gravitational flux tube bundles as the distance from the source increases.

4.4 Still some questions about M^8-H duality

There are still on questions to be answered.

1. The map $p^k \rightarrow m^k = \hbar_{eff}p^k/p\cdot p$ defining M^8-H duality is consistent with Uncertainty Principle but this is not quite enough. Momenta in M^8 should correspond to plane waves in H.

   Should one demand that the momentum eigenstate as a point of cognitive representation associated with $X^4\subset M^8$ carrying quark number should correspond to a plane wave with momentum at the level of $H = M^4\times CP_2$? This does not make sense since $X^4\subset CD$ contains a large number of momenta assignable to fundamental fermions and one does not know which of them to select.

2. One can however weaken the condition by assigning to CD a 4-momentum, call it P. Could one identify P as

   (a) the total momentum assignable to either half-cone of CD

   (b) or the sum of the total momenta assignable to the half-cones?

The first option does not seem to be realistic. The problem with the latter option is that the sum of total momenta is assumed to vanish in ZEO. One would have automatically zero momentum planewave. What goes wrong?

1. Momentum conservation for a single CD is an ad hoc assumption in conflict with Uncertainty Principle, and does not follow from Poincare invariance. However, the sum of momenta vanishes for non-vanishing planewave when defined in the entire M^4 as in QFT, not for planewaves inside finite CDs. Number theoretic discretization allows vanishing in finite volumes but this involves finite measurement resolution.

2. Zero energy states represent scattering amplitudes and at the limit of infinite size for the large CD zero energy state is proportional to momentum conserving delta function just as S-matrix elements are in QFT. If the planewave is restricted within a large CD defining the measurement volume of observer, four-momentum is conserved in resolution defined by the large CD in accordance with Uncertainty Principle.

3. Note that the momenta of fundamental fermions inside half-cones of CD in H should be determined at the level of H by the state of a super-symplectic representation as a sum of the momenta of fundamental fermions assignable to discrete images of momenta in $X^4\subset H$.

4.4.1 M^8-H-duality as a generalized Fourier transform

This picture provides an interpretation for M^8-H duality as a generalization of Fourier transform.

1. The map would be essentially Fourier transform mapping momenta of zero energy as points of $X^4\subset CD\subset M^8$ to plane waves in H with position interpreted as position of CD in H. CD and the superposition of space-time surfaces inside it would generalize the ordinary Fourier transform . A wave function localized to a point would be replaced with a superposition of space-time surfaces inside the CD having interpretation as a perceptive field of a conscious entity.

2. $M^8 - H$ duality would realize momentum-position duality of wave mechanics. In QFT this duality is lost since space-time coordinates become parameters and quantum fields replace position and momentum as fundamental observables. Momentum-position duality would have much deeper content than believed since its realization in TGD would bring number theory to physics.

4.4.2 How to describe interactions of CDs?

Any quantum coherent system corresponds to a CD. How can one describe the interactions of CDs? The overlap of CDs is a natural candidate for the interaction region.

1. CD represents the perceptive field of a conscious entity and CDs form a kind of conscious atlas for M^8 and H. CDs can have CDs within CDs and CDs can also intersect. CDs can have shared sub-CDs identifiable as shared mental images.

2. The intuitive guess is that the interactions occur only when the CDs intersect. A milder assumption is that interactions are observed only when CDs intersect.

3. How to describe the interactions between overlapping CDs? The fact the quark fields are induced from second quantized spinor fields in in H *resp.* M^8 solves this problem. At the level of H, the propagators between the points of space-time surfaces belonging to different CDs are well defined and the systems associated with overlapping CDs have well-defined quark interactions in the intersection region. At the level of M^8 the momenta as discrete quark carrying points in the intersection of CDs can interact.

4.4.3 Zero energy states as scattering amplitudes and subjective time evolution as sequence of SSFRs

This is not yet the whole story. Zero energy states code for the ordinary time evolution in the QFT sense described by the S-matrix. What about subjective time evolution defined by a sequence of "small" state function reductions (SSFRs) as analogs of "weak" measurements followed now and then by BSFRs? How does the subjective time evolution fit with the QFT picture in which single particle zero energy states are planewaves associated with a fixed CD.

1. The size of CD increases at least in statistical sense during the sequence of SSFRs. This increase cannot correspond to M^4 time translation in the sense of QFTs. Single unitary step followed by SSFR can be identified as a scaling of CD leaving the passive boundary of the CD invariant. One can assume a formation of an intermediate state which is quantum superposition over different size scales of CD: SSFR means localization selecting single size for CD. The subjective time evolution would correspond to a sequence of scalings of CD.

2. The view about subjective time evolution conforms with the picture of string models in which the Lorentz invariant scaling generator L_0 takes the role of Hamiltonian identifiable in terms of mass squared operator allowing to overcome the problems with Poincare invariance. This view about subjective time evolution also conforms with super-symplectic and Kac-Moody symmetries of TGD.

   One could perhaps say that the Minkowski time T as distance between the tips of CDs corresponds to exponentiated scaling: $T = exp(L_0 t)$. If t has constant ticks, the ticks of T increase exponentially.

The precise dynamics of the unitary time evolutions preceding SSFRs has remained open.

1. The intuitive picture that the scalings of CDs gradually reveal the entire 4-surface determined by polynomial P in M^8: the roots of P as "very special moments in the life of self" would correspond to the values of time coordinate for which SSFRs occur as one new root emerges. These moments as roots of the polynomial defining the space-time surface would correspond to scalings of the size of both half-cones for which the space-time surfaces are mirror images. Only the upper half-cone would be dynamical in the sense that mental images as sub-CDs appear at "geometric now" and drift to the geometric future.

2. The scaling for the size of CD does *not* affect the momenta associated with fermions at the points of cognitive representation in $X^4 \subset M^8$ so that the scaling is not a genuine scaling of M^4 coordinates which does not commute with momenta. Also the fact that L_0 for super symplectic representations corresponds to mass squared operator means that it commutes with Poincare algebra so that M^4 scaling cannot be in question.

3. The Hamiltonian defining the time evolution preceding SSFR could correspond to an exponentiation of the sum of the generators L_0 for super-symplectic and super-Kac Moody representations and the parameter t in exponential corresponds to the scaling of CD assignable to the replaced of root r_n with root r_{n+1} as value of M^4 linear time (or energy in M^8). L_0 has a natural representation at light cone boundaries of CD as scalings of light-like radial coordinate.

4. Does the unitary evolution create a superposition over all over all scalings of CD and does SSFR measure the scale parameter and select just a single CD?

   Or does the time evolution correspond to scaling? Is it perhaps determined by the increase of CD from the size determined by the root r_n as "geometric now" to the root r_{n+1} so that one would have a complete analogy with Hamiltonian evolution? The scaling would be the ratio r_{n+1}/r_n which is an algebraic number.

   Hamiltonian time evolution is certainly the simplest option and predicts a fixed arrow of time during SSFR sequence. L_0 identifiable essentially as a mass squared operator acts like conjugate for the logarithm of the logarithm of light-cone proper time for a given half-cone.

   One can assume that L_0 as the sum of generators associated with upper and lower half-cones if the fixed state at the lower half-cone is eigenstate of L_0.

How does this picture relate to p-adic thermodynamics in which thermodynamics isdetermined by partition function which would in real sector be regarded as a vacuum expectation value of an exponential $exp(iL_0t)$ of a Hamiltonian for imaginary time $t = i\beta$ $\beta = 1/T$ defined by temperature. L_0 is proportional to mass squared operator.

1. In p-adic thermodynamics temperature T is dimensionless parameter and $\beta = 1/T$ is integer valued. The partition function as exponential $exp(-H/T)$ is replaced with $p^{\beta L_0)}$, $\beta = n$, which has the desired behavior if L_0 has integer spectrum. The exponential form $e^{L_0/T_R)}$, $\beta_R = nlog(p)$ equivalent in the real sector does not make sense p-adically since the p-adic exponential function has p-adic norm 1 if it exists p-adically.

2. The time evolution operator $exp(-iL_0t)$ for SSFRs (t would be the scaling parameter) makes sense for the extensions of p-adic numbers if the phase factors for eigenstates are roots of unity belonging to the extension. $t = 2\pi k/n$ since L_0 has integer spectrum. SSFRs would define a clock. The scaling $exp(t) = exp(2\pi k/n)$ is however not consistent with the scaling by r_{n-1}/r_n.

   Both the temperature and scaling parameter for time evolution by SSFRs would be quantized by number theoretical universality. p-Adic thermodynamics could have its origins in the subjective time evolution by SSFRs.

3. In the standard thermodynamics it is possible to unify temperature and time by introducing a complex time variable $\tau = t + i\beta$, where $\beta = 1/T$ is inverse temperature. For the space-time surface in complexified M^8, M^4 time is complex and the real projection defines the 4-surface mapped to H. Could thermodynamics correspond to the imaginary part of the time coordinate?

   Could one unify thermodynamics and quantum theory as I have indeed proposed: this proposal states that quantum TGD can be seen as a "complex square root" of thermodynamics. The exponentials $U = exp(\tau L_0/2)$ would define this complex square root and thermo-dynamical partition function would be given by $UU^\dagger = exp(-\beta L_0)$.

5 Can one construct scattering amplitudes also at the level of M^8?

$M^8 - H$ duality suggests that the construction is possible both at the level of H and M^8. These pictures would be based on differential geometry on one hand and algebraic geometry and number theory on the other hand. The challenge is to understand their relationship.

5.1 Intuitive picture

H picture is phenomenological but rather detailed and M^8 picture should be its pre-image under $M^8 - H$ duality. The following general questions can be raised.

1. Can one construct the counterparts of the scattering amplitudes also at the level of M^8?
2. Can one use $M^8 - H$ duality to map scattering diagrams in M^8 to the level of H?

Consider first the notions of CD and sub-CD.

1. The intuitive picture is that at the level of H that one must surround partonic vertices with sub-CDs, and assign the external light-like momenta with the ends of propagator lines from the boundaries of CD and other sub-CDs. The incoming momenta $\overline{p}^k$ would be assigned to the boundary of sub-CD.
2. What about the situation in M^8? Sub-CDs must have different origin in the general case since the momentum spectrum would be shifted. Therefore the sub-CDs have the same tip - either upper or lower tip, and have as their boundary part of either boundary of CD. A hierarchy of CDs associated with the same upper or lower tip is suggestive and the finite maximal size of CD in H gives IR cutoff and the finite maximal size of CD in M^8 gives UV cutoff.
3. Momentum conservation at the vertices in M^8 could decompose the diagram to sub-diagrams for which the momentum conservation is satisfied. On the basis of QFT experience, one expects that there are some minimal diagrams from which one can construct the diagram: in the TGD framework this diagram would describe 4-quark scattering. The condition that the momenta belong to the extension of rationals gives extremely strong constraints and it is not clear that one obtains any solutions to the conditions unless one poses some conditions on the polynomials assigned with the two boundaries of CD.

   The two half-cones (HCs) of CD contain space-time surfaces in M^8 as roots of polynomials $P_1(o)$ and $P_2(2T - o)$ which need not be identical. The simplest solution is $P_2(o) = P_1(2T - o)$: the space-time surfaces at HCs would be mirror images of each other. This gives $P_1(T, Im_R(o)) = P_1(T - Im_R(o))$ Since P_1 depends on $t^2 - r^2$ only, the condition is identically satisfied for both options.

   There are two options for the identification of the coordinate t.

   Option (a): t is identified as octonionic real coordinate o_R identified and also time coordinate as in the original option. In the recent option octonion o_R would correspond to the Euclidian analog of time coordinate. The breaking of symmetry from $SO(4)$ to $SO(3)$ would distinguish t as a Newtonian time. The M^4 projection of CD_8 gives a union of future and past directed light-cones with a common tip rather than CD_4 in M^4 at the level of M^8 . Both incoming and outgoing momenta have the same origin automatically. This identification seems to be the natural one at the level of M^8.

   Option (b): t is identified as a Minkowski time coordinate associated with the imaginary unit I_1 in the canonical decomposition $\{I_1, iI_3, iI_5, iI_7\}$. The HC at $o = 0$ would be shifted to $O = (0, 2T, 0...0)$ and reverted. M^4 projection would give CD_4 so that this option is consistent with ZEO. This option is natural at the level of Hbut not at the level of M^8.

   If Option (a) is realized at the level of M^8 and Option b) at the level of H, as seems natural, a time translation of the past directed light-cone by T in $M^4 \subset H$ is required to give CD_4. The momentum spectra of the two HCs differ only by sign and at least a scattering diagram in which all points are involved is possible. In fact all the pairs of subsets with opposite momenta are allowed. These however correspond to a trivial scattering. The decomposition to say 4-vertices with common points involving momentum space propagator suggests a decomposition into sub-CDs. The smaller the sub-CDs at the tips of the CD, the smaller the momenta are and the better is the IR resolution.

4. The proposal has been that one has a hierarchy of discrete size scales for the CDs. Momentum conservation gives a constraint on the positions of quarks at the ends of propagator lines in M^8 mapped to a constraint for their images in H: the sum of image points in H is however not vanishing since inversion is not a linear map.

5. QFT intuition would suggest that at the level of M^8 the scattering diagrams decompose to sub-diagrams for which momentum conservation is separately satisfied. If two such sub-diagrams A and B have common momenta, they correspond to internal lines of the diagram involving local propagator D_p, whose non-local counterpart at the level of H connects the image point to corresponding point of all copies of B.

   The usual integral over the endpoint of the propagator line $D(x,y)$ at space-time level should correspond to a sum in which the H image of B is shifted in M^4. Introduction of a large number of copies of H image of the sub-diagram looks however extremely ugly and challenges the idea of starting from the QFT picture.

   What comes in mind is that all momenta allowed by cognitive representation and summing up to zero define the scattering amplitude as a kind of super-vertex and that Yanigian approach allows this construction.

5.2 How do the algebraic geometry in M^8 and the sub-manifold geometry in H relate?

Space-time surfaces in H have also Euclidian regions - in particular wormhole contacts - with induced metric having Euclidian signature due to the large CP_2 contribution to the induced metric. They are separated from Minkowskian regions by a light-like 3-surfaces identifiable as partonic orbits at which the induced metric becomes degenerate.

1. The possible M^8 counterparts of these regions are expected to have Euclidian signature of the number theoretic metric defined by complexified octonion inner product, which must be real in these regions so that the coordinates for the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ are either imaginary or real. This allows several signatures.

2. The first guess is that the energy p^0 assignable to I_1 becomes imaginary. This gives tachyonic p^2. The second guess is that all components of 3-momentum $\{iI_3, iI_5, iI_7\}$ become imaginary meaning that the length of 3-momentum becomes imaginary.

3. One cannot exclude the other signatures, for instance the situation in which 1 or 2 components of the 3-momentum become imaginary. Hence the transition could occur in 3 steps as $(1,-1,-1-,1) \rightarrow (1,1,-1,-1) \rightarrow (1,1,1,-1) \rightarrow (1,1,1,1)$. The values of $p^2 \equiv Re(p_c^2)$ would be non-negative and also their images in $M^4 \subset H$ would be inside future light-cone. This could relate to the fact that all these signatures are possible inthe twistor Grassmannian approach.

4. These regions belong to the complex mass shell $p_c^2 = r_n = m_0^2 = r_n$ appearing as a root to the co-associativity condition $X = 0$. This gives the conditions

$$\begin{aligned} Re(p_c) \cdot Im(p_c^2) &= Im(r_n) \ , \\ Re(p_c^2) \equiv p^2 &= Im(p_c^2) + m_n^2 \ , \\ m_n^2 &\equiv Re(r_n) \geq 0 \ . \end{aligned} \tag{5.1}$$

   Consider first the case $(1,1,1,1)$.

1. The components of p_c are either real or imaginary. Using the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$ the components of p_c are real in the Minkowskian region and imaginary in the totally time-like Euclidian region. One has for the totally time-like momentum $p = (p_0, iIm(p_3))$ in the canonical basis.

   This would give

$$Re(p_c^2) \equiv p^2 = p_0^2 = -Im(p_3)^2 + m_n^2 \ . \tag{5.2}$$

   The number theoretic metric is Euclidian and totally time-like but one has $p^2 \geq 0$ in the range $[m_0^2, 0]$. This region is a natural counterpart for an Euclidian space-time region in H. The region $p^2 \geq m_0^2$ has Minkowskian signature and counterpart for Minkowskian regions in H. The region $0 \leq p^2 < m_0^2$ is a natural candidate for an Euclidian region in M^4.

   Remark: A possible objection is that Euclidian regions in O_c are totally time-like and totally space-like in H.

2. The image of these regions under the map $Re(p^k) \rightarrow M^k$ under inversion plus octonionic conjugation defined as $p^k \rightarrow \hbar_{eff}\overline{p^k}/p^2$ (to be discussed in more detail in the sequel) consists of points M^k in the future light-cone of $M^4 \subset H$. The image of the real Euclidian region of O_c with $p^2 \in [0, m_0^2)$ is mapped to the region $M^k M_k < \hbar_{eff}^2/m_0^2$ of $M^4 \subset H$.

3. The contribution of CP_2 metric to the induced metric is space-like so that it can become Euclidian. This would naturally occur in the image of a totally time-like Euclidian region and this region would correspond to small scales $M^k M_k < \hbar_{eff}^2/m_0^2$. The change of the signature should take place at the orbits of partonic 2-surfaces and the argument does not say anything about this. The boundary of between the two regions corresponds to momenta $p = (p_0, 0)$ which is is a time-like line perhaps identifiable as the analog of the light-like geodesic defining the M^4 projection of CP_2 type extremal, which is an idealized solution to actual field equations.

This transition does not explain the M^8 counterpart of the 3-D light-like partonic orbit to which the light-light geodesic thickens in the real situation?

The above argument works also for the other signatures of co-associative real sub-spaces and provides additional insights. Besides the Minkowskian signature, 3 different situations with signatures $(1,1,1,1)$, $(1,-1,1,1)$, and $(1,-1-1,1)$ with non-space-like momentum squared are possible.

The following formulas list the signatures, the expressions of real momentum squared, and dimension D of the transition transition $Im(p_c^2) = 0$ as generalization of partonic orbit and the possible identification of the transition region.

$$\begin{array}{lll}
\textbf{Signature} & p^2, & D \\
(+,-,-,+): & (p^0)^2 - (p^1)^2 - (p^2)^2 = -Im(p^3)^2 + m_n^2 & 3\ , \\
\textbf{Identification} & \text{partonic orbit} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,-,+,+): & (p^0)^2 - (p^1)^2 = -Im(p^2)^2 - Im(p^3)^2 + m_n^2\ , & 2\ , \\
\textbf{Identification} & \text{string world sheet} & . \\
 & & \\
\textbf{Signature} & p^2 & D \\
(+,+,+,+): & (p^0)^2 = -Im(p^1)^2 - Im(p^2)^2 - Im(p^3)^2 + m_n^2\ , & 1\ . \\
\textbf{Identification} & \text{string boundary} & .
\end{array} \tag{5.3}$$

Since the map of the co-associative normal space to CP_2 does not depend on the signature, $M^8 - H$ duality is well defined for all these signatures. One can ask whether a single transition creates partonic orbit, two transitions a string world sheet and 3 transitions ends of string world sheet inside partonic orbit or even outside it.

5.3 Quantization of octonionic spinors

There are questions related to the quantization of octonionic spinors.

1. Co-associative gamma matrices identified as octonion units are associative with respect to their octonionic product so that matrix representation is possible. Do second quantized octonionic spinors in M^8 make sense? Is it enough to second quantize them in M^4 as induced octonionic spinors? Are the anti-commutators of oscillator operators Kronecker deltas or delta functions in which case divergence difficulties might be encountered? This is not needed since the momentum space propagators can be identified as those for E^8_c restricted to X^4_r as a subspace with real octonion norm.

   The propagators are just massless Dirac propagators for the choice of M^4 for which light-like M^8 momentum reduces to M^4 momentum. Could one formulate the scattering amplitudes using only massless inverse propagators as in the twistor Grassmannian approach?This does not seem to be the case.

2. Could the counterpart of quark propagator as inverse propagator in M^8 as the idea about defining momentum space integrals as residue integrals would suggest? This would allow on-mass-shell propagation like in twistor diagrams and would conform with the idea that inversion relates M^8 and H descriptions. This is suggested by the fact that no integration over intermediate virtual momenta appears in the graphs defined by the algebraic points of the pre-images of the partonic 2-surfaces X^2_r .

How to identify external quarks? Note that bosons would consist of correlated quark-antiquark pairs with the propagator obtained as a convolution of quark propagators. The correlation would be present for the external states and possibly also for the states in the diagram and produced by topologically.

1. The polynomial P and the $P = 0$ surface with 6-D real projection X^6_r is not affected by octonion automorphisms. Quarks with different states of motion would correspond to the same P but to different choices of M^4 as co-associative subspace for M^8_c. P could be seen as defining a class of scattering diagrams. P determines the vertices.

2. The space-time surface associated with a quark carrying given 4-momentum should be obtainable by a Lorenz transformation in $SO(3,1) \subset G_{2,c}$ to give it light-like M^4 so that complexified octonionic automorphisms would generate 3-surfaces representing particles. If $M^4 \subset M^8$ and thus the CD associated with the quark is chosen suitably, the quark is massless. Any incoming particle would be massless in this frame.

   Lorentz invariance however requires a common Lorentz frame provided by the CD. The momentum of a quark in CD would be obtained by $G_{2,c}$ transformation. In the frame of CD the external quark momenta arriving to the interior of CD at vertices associated with $X^3_r \cap Y^3_r$ are time-like. Momentum conservation would hold in this frame. The difference between massive constituent quarks and massless current quarks could be understood as reflecting M^8 picture.

To sum up, the resulting picture is similar to that at the level of H these diagrammatic structures would be mapped to H by momentum inversion. Quantum classical correspondence would be very detailed providing both configuration space and momentum space pictures.

5.4 Does $M^8 - H$ duality relate momentum space and space-time representations of scattering amplitudes?

It would seem that the construction of the scattering amplitudes is possible also at the level of M^8 [L22]. M^8 picture would provide momentum representation of scattering diagrams whereas H picture would provide the space-time representation.

Consider first a possible generalization of QFT picture involving propagators and vertices.

1. At first it seems that it is not possible to talk about propagation at the level of momentum space: in positive energy ontology nothing propagates in momentum space if the propagator

is a free propagator D_p! In ZEO this is not quite so. One can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD): one has momentum space propagation from p to $-p$! The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. The momentum space propagator D_p would actually result from the pairing of creation creation operators with the opposite values of p and the notation $D(p, -p)$ would be more appropriate.

2. In QFT interaction vertices are local in space-time but non-local in momentum space. The n-vertex conserves the total momentum. Therefore one should just select points of M^8 and they are indeed selected by cognitive representation and assign scattering amplitude to this set of points. To each point one could assign momentum space propagator of quark in M^8_c but it would not represent propagation! The vertex would be a multilocal entity defined by the vertices defining the masses involved at light cone boundary and mass shells.

   The challenge would be to identify these vertices as poly-local entities. In the QFT picture there would be a set of n-vertices with some momenta common. What could this mean now? One would have subset sets of momenta summing up to zero as vertices. If two subsets have a common momentum this would correspond to a propagator line connecting them. Should one decompose the points of cognitive representation so that it represents momentum space variant of Feynman graph? How unique this decomposition is and do this kind of decompositions exist unless one poses the condition that the total momenta associated with opposite boundaries sum up to zero as done in ZEO. A given n-vertex in the decomposition means the presence of sub-CDs for which the external momenta sum up to zero. This poses very tight constraints on the cognitive representation, and one can wonder they can be satisfied if the cognitive representation is finite as it is in the generic case.

3. Note that for given a polynomial P allowing only points in cognitive representation, one would *not* have momentum space integrations as in QFT: they could however come from integrations over the polynomial coefficients and would correspond to integration of WCW. In adelic picture one allows only rational coefficients for the polynomials. This strongly suggests that the twistor Grasmmannian picture [**?**, **?**, **?**, **?**] in which residue integral in the momentum space gives as residues inverse quark propagators at the poles. M^8 picture would represent the end result of this integration and only on mass shell quarks would be involved. One could even challenge the picture based on propagators and vertices and start from Yangian algebra based on the generalization of local symmetries to multilocal symmetries [A5, A7] [**?**] [L3].

4. In the case of H restriction of the second quantized free quark field of H to space-time surface defines the propagators. In the recent case one would have a second quantized octonionic spinor field in M^8. The allowed modes of H spinor field are just the co-associative modes for fixed selection of M^4 analogous to momentum space spinors and restricted to Y^3_r. One could speak of wave functions at Y^3_r, which is very natural since they correspond to mass shells.

   The induced spinor field would have massless part corresponding to wave functions at the M^4 light-cone boundary and part corresponding to X^3 at which the modes would have definite mass. $P = 0$ would select a discrete set of masses. Could second quantization have the standard meaning in terms of anti-commutation relations posed on a free M^8 spinor field. In the case of M^8_c one avoids normal ordering problems since there is no Dirac action. The anti-commutators however have singularities of type 7-D delta function. The anti-commutators of oscillator operators at the same point are the problem. If only a single quark oscillator operator at a given point of M^8 is allowed since there is no local action in coordinate space with the interaction part producing the usual troubles.

5. Could one perform a second quantization for E^8 spinor field using free Dirac action? Could one restrict the expansion of the spinor field to co-associative space-time surfaces giving oscillator operators at the points of cognitive representation with the additional restriction to the pre-image of given partonic 2-surface, whose identification was already considered. Scattering amplitudes would involve n-vertices consisting of momenta summing up to zero and connected to opposite incoming momenta at the opposite sides of the HCs with the same tip in M^8. Scattering amplitude would decompose to sub-diagrams defining a cluster

decomposition, and would correspond to sub-CDs. The simplest option is that there are no internal propagator lines. The vanishing of the total momenta poses stringent conditions on the points of cognitive representation.

Normal ordering divergences can however produce problems for this option in the case of bosonic charges bilear in oscillator operators. At the level of H the solution came from a bilocal modified Dirac action leading to bilocal expressions for conserved charges. Now Yangian symmetry suggests a different approach: local vertices in momentum space can involve only commuting oscillator operators.

Indeed, in ZEO one can regard annihilation operators as creation operators for the fermionic vacuum associated with the opposite HC of CD (or sub-CD). The expressions of bosonic charges would be indeed bi-local with annihilation and creation operators associated with the mirror paired points in the two HCs of CD forming pairs. As already noticed, also the momentum space propagator $D_p = D(p, -p)$ would be also a bi-local object.

6. This is not enough yet. If there is only a single quark at given momentum, genuine particle creation is not possible and the particle reactions are only re-arrangements of quarks but already allowing formation of bosons as bound states of quarks and antiquarks. Genuine particle creation demands local composites of several quarks at the same point p having interpretation as a state with collinear momenta summing up to p and able to decay to states with the total momentum p. This suggests the analog of SUSY proposed in [L14]. Also Yangian approach is highly suggestive.

   To sum up, momentum conservation together with the assumption of finite cognitive representations is the basic obstacle requiring new thinking.

5.5 Is the decomposition to propagators and vertices needed?

One can challenge the QFT inspired picture.

1. As already noticed, the relationship $P_1(t) = P(2T - t)$ makes it possible to satisfy this condition at least for the entire set of momenta. This does not yet allow non-trivial interactions without posing additional conditions on the momentum spectrum. This does not look nice. One can ask whether there is a kind of natural selection leading to polynomials defining space-time surfaces allowing cognitive representations with vertex decompositions and polynomials $P(t)$ and $P_r(t)$ without this symmetry? This idea looks ugly. Or could evolution start from simplest surfaces allowing 4 vertices and lead to an engineering of more complex scattering diagrams from these?

2. The map of momentum space propagators regarded as completely local objects in M^8 to H propagators is second ugly feature. The beauty and simplicity of the original picture would be lost by introducing copies of sub-diagrams mapped to the various translations in H.

3. The Noether charges of the Dirac action in H fail to give rise to 4-fermion vertex operator. The theory would be naturally just free field theory if one assumes cognitive representations.

The first heretic question is whether the propagators are really needed at the level of momentum space. This seems to be the case.

1. In ZEO the propagators pair creation and operators with opposite 4-momenta assignable to the opposite HCs of CD having conjugate fermionic vacua (Dirac sea of negative energy fermions and Dirac sea of positive energy fermions) so that momentum space propagators $D(p, -p)$ are non-local objects. The propagators would connect positive and negative energy fermions at the opposite HCs and this should be essential in the formulation of scattering amplitudes. They cannot be avoided.

2. The propagators would result from the contractions of fermion oscillator operators giving a 7-D delta function at origin in continuum theory. This catastrophe is avoided in the number theoretic picture. Since one allows only points with M^8 coordinates in an extension of rationals, one can assume Kronecker delta type anti-commutators. Besides cognitive

representations, this would reflect the profound difference between momentum space and space-time.

This would also mean that the earlier picture about the TGD analog of SUSY based on local composites of oscillator operators [L14] makes sense at the level of M^8. The composites could be however local only for oscillator operators associated with the HC of CD. With the same restriction they could be local also in the$H$ picture.

What about vertices? Could Yangian algebra give directly the scattering amplitudes? This would simplify dramatically the $M^8 - H$ duality for transition amplitudes. For this option the $P_1(t) = P(2T - t)$ option required by continuity would be ideal.

1. Without vertices the theory would be a free field theory. The propagators would connect opposite momenta in opposite HCs of CD. Vertices are necessary and they should be associated with sub-CDs. Unless sub-CDs can have different numbers of positive and negative energy quarks at the opposite HCs, the total quark number is the same in the initial and final states if quarks and antiquarks associated with bosons as bound states of fermion and antiquark are counted. This option would require minimally 4-quark vertex having 2 fermions of opposite energies at the two hemi-spheres of the CD. A more general option looks more plausible. One obtains non-trivial scattering amplitudes by contracting fermions assigned to the boundary P (F) past (future) HC of CD to the past (future) boundary P_{sub} (F_{sub}) of a sub-CD. Sub-CD and CD must have an opposite arrow of time to get the signs of energies correctly.

   Sub-CDs would thus make particle creation and non-trivial scattering possible. There could be an arbitrary number of sub-CDs and they should be assignable to the pre-images of the partonic 2-surfaces X^2_r if the earlier picture is correct. The precise identification of the partonic 2-surfaces is still unclear as also the question whether light-like orbits of partonic 2-surfaces meet along their ends in the vertices.

2. As in the case of H, one could assign the analogs of n-vertices at pre-images of partonic 2-surfaces at X^2_r representing the momenta of massive modes of the octonionic Dirac equation and belonging to the cognitive representations. The idea is to use generators of super-Yangian algebra to be discussed later which are both bosonic and fermionic. The simplest construction would assign these generators to the vertices as points in cognitive representation.

   An important point is that Yangian symmetry would be a local symmetry at the level of momentum space and correspond to non-local symmetry at the level of space-time rather than vice versa as usually. The conserved currents would be local composites of quark oscillator operators with same momentum just as they are in QFTs at space-time level representing parallelly propagating quarks and antiquarks.

   The simplest but not necessary assumption is that they are linear and bilinear in oscillator operators associated with the same point of M^8 and thus carrying 8-momenta assignable to the modes of E^8 spinor field and restricted to the co-associative 4-surface. Their number of local composites is finite and corresponds to the number 8 of different states of 8-spinors of given chirality.

   Also a higher number of quarks is possible, and this was indeed suggested in [L14]. The proposal was that instance leptons would correspond to local composites of 3 quarks. The TGD based view about color allows this. These states would be analogous to the monomials of theta parameters in the expansion of super-field. The H picture allows milder assumptions: leptonic quarks reside at partonic 2-surface at different points but this is not necessary.

3. Instead of super-symplectic generators one has $G_{2,c}$ as the complexified automorphism group. Also the Galois group of the extension acts as an automorphism group and is proposed to have a central role in quantum TGD with applications to quantum biology [L2, L21]. As found, $G_{2,c}$ acts as an analog of gauge or Kac-Moody group. Yangian has analogous structure but the analogs of conformal weights are non-negative.

4. The identification of the analogs of the poly-local vertex operators as produces of charges generators associated with FHC anbd PHC is the basic challenge. They should consist of quark creation operators (annihilation operators being associated as creation operators at

the opposite HC) and be generators of infinitesimal symmetries which in number theoretic physics would correspond instead of isometries of WCW to the octonionic automorphism group G_2 complexified to $G_{2,c}$ containing also the generators of $SO(4) \subset G_2$ and thus also those of Lorentz group $SO(1,3) \subset G_{2,c}$.

The construction Noether charges of E^8 second quantized spinor field at momentum space representation gives bilinear expressions in creation and annihilation operators associated with opposite 3-momenta and would have a single fermion in a given HC. This is not enough: there should be at least 4 fermions.

What strongly suggests itself are Yangian algebras [A5] [L3] having poly-local generators and considered already earlier and appearing in the twistor Grassmannian approach [**?**, **?**]. The sums of various quantum numbers would vanish for the vertex operators. These algebras are quantum algebras and the construction of n-vertices could involve co-algebra operation. What is new as compared to Lie algebras is that Yangian algebras are quantum algebras having co-algebra structure allowing to construct n-local generators representing scattering amplitudes. It might be possible replace oscillator operators with the quantum group counterparts.

5.6 Does the condition that momenta belong to cognitive representations make scattering amplitudes trivial?

Yangian symmetry is associated with 2-D integrable QFTs which tend to be physically rather uninteresting. The scattering is in the forward direction and only phase shifts are induced. There is no particle creation. If the relationship $P_1(t) = P(2T - t)$ is applied the momentum spectra for FHC and PHC differ only by the sign. If all momenta are involved and the cognitive representations are finite, the situation would be the same! Also the existence of cluster compositions involving summations of subsets of momenta to zero is implausible. Something seems to go wrong!

The basic reason for the problem is the assumption that the momenta belong to cognitive representations assumed to be finite as they indeed are in the generic case. But are they finite in the recent situation involving symmetries?

1. The assumption that all possible momenta allowed by cognitive representation are involved, allows only forward scattering unless there are several subsets of momenta associated with either HC such that the momenta sum-up to the same total momentum. This would allow the change of the particle number. The subsets S_i with same total momentum p_{tot} in the final state could save as final states of subsets S_j with the same total momentum p in the initial state. What could be the number theoretical origin of this degeneracy?

2. In the generic case the cognitive representation contains only a finite set of points (Fermat theorem, in which one considers rational roots of $x^n + y^n = z^n$, $n > 2$ is a basic example of this) . There are however special cases in which this is not true. In particular, M^4 and its geodesic sub-manifolds provide a good example: all points in the extension of rationals are allowed in M^4 coordinates (note that there are preferred coordinates in the number theoretic context).

   The recent situation is indeed highly symmetric due to the Lorentz invariance of space-time surfaces as roots reducing the equations to ordinary algebraic equations for a single complex variable. $X = 0$ condition gives as a result $a_c^2 = constant$ complex hyperboloid with a real mass hyperboloid as a real projection. $a_c^2 = r_n$ is in the extension of rationals as a root of n:th order polynomial. One has the condition $Re(m^2)^2 - Im(m^2) = Re(r_n)$ giving X_r^4 a slicing by real mass hyperboloids. If $Im(m)$ and the spatial part of $Re(m)$ belongs to the extension, one has for real time coordinate $t = \sqrt{r_M^2 + Im(m^2) + r_n}$. If $r_M^2 + Im(m)^2 + r_n$ is a square in the extension also t belongs to the extension. Cognitive representation would contain an infinite number of points and the it would be possible to have non-trivial cluster decompositions. Scattering amplitude would be a sum over different choices of the momenta of the external particles satisfying momentum conservation condition.

   As found, the intersection of X_r^4 and X_r^6 is either empty or X_r^4 belongs to X_r^6, Cognitive representations would have an infinite number of points also now by the previous argument. Partonic 2-surfaces at X_r^3 would be replaced with 3-D surfaces in X_r^4 in this situation and would contain a large number of roots. The partonic 2-surfaces would be still present and correspond to the intersections of incoming space-time surfaces of quarks inside X_r^6. These surfaces would also contain the vertices.

3. Could number theoretic evolution gradually select space-time surfaces for which the number theoretic dynamics involving massive quarks is possible? First would be generic polynomials for which X_r^3 would be empty and only massless quarks arriving at the light-cone boundary would be possible. After that surfaces allowing non-empty X_r^3 and massive quarks would appear. There is a strong resemblance with the view about cosmological evolution starting from massless phases and proceeding as a sequence of symmetry breakings causing particle massivation. Now the massivation would not be caused by Higgs like fields but have purely number theoretic interpretation and conform with the p-adic mass calculations [K8].

   Also a cognitive explosion would occur since these space-time surfaces would be cognitively superior after the emergence of massive quarks. If this picture has something to do with reality, the space-time surfaces contributing to the scattering amplitudes would be very special and interactions could be seen as a kind of number theoretical resonance phenomenon.

4. Even is not enough to obtain genuine particle reaction instead of re-arrangements: one must have also local composites of collinear quarks at the same momentum p identifiable as the sum of parallel momenta discussed in [L14]. This kind of situation is also encountered for on-mass-shell vertices in twistor Grassmannian approach. The local composites could decay to local composites with a smaller number of quarks but respecting momentum conservation. Here the representations of Yangian algebra would come in rescue.

5.7 Momentum conservation and on-mass-shell conditions for cognitive representations

Momentum conservation and on-mass shell-conditions are very powerful for cognitive representations, which in the generic case are finite. At mass shells the cognitive representations consist of momenta in the extension of rationals satisfying the condition $p^2 = Re(r_n)$, r_n a complex root of X, which is polynomial of degree n in p^2 defined by the odd part of P. If $\sqrt{Re(r_n)}$ does not belong to the extension defined by P, it can be extended to contain also $\sqrt{Re(r_n)}$.

For Pythagorean triangles in the field of rationals, mass shell condition gives for the momentum components in extension an equation analogous to the equation $k^2 + l^2 = m^2$, which can be most easily solved by noticing that the equation has rotation group $SO(2)$ consisting of rational rotation matrices as symmetries. The solutions are of form $(k = r^2 - s^2, l = 2rs, m = r^2 + s^2)$. By $SO(2)$ invariance, one can choose the coordinate frame so that one has $(k, l) = (r^2 + s^2, 0)$. By applying to this root a rational rotation with $cos(\phi) = (r^2 - s^2)/(r^2 + s^2), sin(\phi) = 2rs/(r^2 + s^2)$ to obtain the general solution $(k = r^2 - s^2, l = 2rs, n = r^2 + s^2)$. The expressions for k and l can be permuted, which means replacing ϕ with $\phi - pi/2$. For a more general case $k^2 + l^2 = n$ one can replace n with $\sqrt{n}$ so that one has an extension of rationals.

For the hyperbolic variants of Pythagorean triangles, one has $k^2 - l^2 = m^2$ or equivalently $l^2 + m^2 = k^2$ giving a Pythagorean triangle. The solution is $k = r^2 + s^2, l = r^2 - s^2, m^2 = 2rs$. The expressions for l and ma can be permuted. Rotation is replaced with 2-D Lorentz boost $cosh(\eta) = (r^2 + s^2)/(r^2 - s^2)$ and $sinh(\eta) = 2rs/(r^2 - s^2)$ with rational matrix elements.

Consider now the 4-D case.

1. The algebra behind the solution depends in no manner on the number field considered and makes sense even for the non-commutative case if m and n commute. Hence one can apply the Pythagorean recipe also in 4-D case to the extension of rationals defined by P by adding to it $\sqrt{r_n}$.

2. Assume that a Lorentz frame can be chosen to be the rest frame in which one has $p = (E = \sqrt{Rer_n}, 0)$ (this might not be possible always). As in the Pythagorean case, there must be a consistency condition. Now it would be of form $E = \sqrt{r_n} = p_0^2 - p_1^2 - p_2^2 - p_3^2$ in the extension defined by $\sqrt{r_n}$. It is not clear whether this condition can be solved for all choices of momentum components in the extension or assuming that algebraic integers of extension are in question. One can also consider an option in which one has algebraic integer divided

by some integer N. p-Adic considerations would suggest that prime powers $N = p^k$ might be interesting.

The solutions $\sqrt{r_n} = p_1^2 - p_2^2$ represent a special case. The general solution is obtained by making Lorenz transformation with a matrix with elements in the discrete subgroup of Lorentz group with matrix elements in the extension of rationals.

3. The solutions would define a discretization of the mass shell (3-D hyperbolic space) defined as the orbit of the infinite discrete subgroup of $SO(1,3)$ considered - perhaps the subgroup of $SL(2,C)$ with matrix elements identified as algebraic integers.

If the entire subgroup of SL(2,C) with matrix elements in the extension of rationals is realized, the situation would correspond effectively to a continuous momentum spectrum for infinite cognitive representations. The quantization of momenta is however physically a more realistic option.

1. An interesting situation corresponds to momenta with the same time component, in which case the group would be a discrete subgroup of $SO(3)$. The finite discrete symmetry subgroups act as symmetries of Platonic solids and polygons forming the ADE hierarchy associated to the inclusions of hyperfinite factors of type II_1 and proposed to provide description of finite measurement resolution in TGD framework.

2. The scattering would be analogous to diffraction and only to the directions specified by the vertices of the Platonic solid. Platonic solids, in particular, icosahedron appear also in TGD inspired quantum biology [L1, L19], and also in Nature. Could their origin be traced to $M^8 - H$ duality mapping the Platonic momentum solids to H by inversion?

A more general situation would correspond to the restriction to a discrete non-compact subgroup $\Gamma \subset SL(2,C)$ with matrix elements in the extension of rationals. $Sl(2,C)$ has a representation as Möbius transformations of upper half-plane H^2 of complex plane acting as conformal transformations whereas the action in H^3 is as isometries. The Möbius transformation acting as isometries of H^2 corresponds to $SL(2,Z)$ having also various interesting subgroups, in particular congruence subgroups.

1. Subgroups Γ of the modular group $SL(2,Z)$ define tessellations (analogs of orindary lattices in a curved space) of both H^2 and H^4. The fundamental domain [A1] (`https://cutt.ly/ahBrtT5`) of the tessellation defined by $\Gamma \subset SL(2,C)$ contains exactly one point at from each orbit of Γ. The fundamental domain is analogous to lattice cell for an Euclidian 3-D lattice.

   Γ must be small enough since the orbits would be otherwise dense just like rationals are a dense sub-set of reals. In the case of rationals this leaves into consideration tje modular subgroup $SL(2,Z)$ or its subgroups. In the recent situation an extension of the modular group allowing matrix elements to be algebraic integers of the extension is natural. Physically this would correspond to the quantization of momentum components as algebraic integers. The tessellation in M^8 and its image in H would correspond to reciprocal lattice and lattice in condensed matter physics.

2. So called uniform honeycombs [A3, A2, A4] (see `https://cutt.ly/xhBwTph`, `https://cutt.ly/lhBwPRc`, and `https://cutt.ly/OhBwUOO`) in H^3 assignable to $SL(2,Z)$ can be regarded as polygons in 4-D space and H^3 takes the roles of sphere S^2 for platonic solids for which the tessellation defined by faces is finite.

   The four regular compact honeycombs in H^3 for which the faces and vertex figures (the faces meeting the vertex) are finite are of special interest physically. In the Schönflies notation characterizing polytopes (tessellations are infinite variants of them) they are labelled by (p,q,r), where p is the number of vertices of face, q is the number of faces meeting at vertex, and s is the number of cells meeting at edge.

   The regular compact honeycombs are listed by (5,3,4), (4,3,5), (3,5,3), (5,3,5). For Platonic solids (5,3) characterizes dodecahedron, (4,3) cube, and (3,5) for icosahedron so that these Platonic solids serve as basic building bricks of these tessellations. Rather remarkably, icosahedral symmetries central in the TGD based model of genetic code [L1, L19], characterize cells for 3 uniform honeycombs.

Consider now the momentum conservation conditions explicitly assuming momenta to be algebraic integers. It is natural to restrict the momenta to algebraic integers in the extension of rationals defined by the polynomial P. This allows linearization of the constraints from momentum conservation quite generally.

Pythagorean case allows to guess what happens in 4-D case.

1. One can start from momentum conservation in the Pythagorean case having interpretation in terms of complex integers $p = (r + is)^2 = r^2 - s^2 + 2irs$. The momenta in the complex plane are squares of complex integers $z = r + is$ obtained by map $z \to w = z^2$ and complex integers. One picks up in the w-plane integer momenta for the incoming and outgoing states satisfying the conservation conditions $\sum_i P_{out,i} = \sum_k P_{in,k}$: what is nice is that the conditions are linear in w-plane. After this one checks whether the inverse images $\sqrt{P_{out,i}}$ and $\sqrt{P_{in,i}}$ are also complex integers.

2. To get some idea about constraints, one can check what CM system for a 2-particle system means (it is not obvious whether it is always possible to find a CM system: one could have massive particles which cannot form a rest system). One must have opposite spatial momenta for $P_1 = (r_1+is_1)^2$ and $P_2 = (r_2+is_2)^2$. This gives $r_{s1} = r_2 s_2$. The products $r_i s_i$ correspond to different compositions of the same integer N to factors. The values of $r_i^2 + si^2$ are different.

3. In hyperbolic case one obtains the same conditions since the roles of $r^2 - s^2$ and $r^2 + s^2$ in the conditions are changed so that $r^2 - s^2$ corresponds now to mass mass mass and differs for different decomposition of N to factors. The linearization of the conservation conditions generalizes also to the algebraic extensions of rationals with integers replaced by algebraic integers.

The generalization to the 4-D case is possible in terms of octonions.

1. Replace complex numbers by quaternions $q = q_0 + \overline{q}$. The square of quaternion is $q^2 = q_0^2 - \overline{q} \cdot \overline{q} + 2iq_0\overline{q}$. Allowed momenta for given mass correspond to points in q^2-plane. Conservation conditions in the q^2 plane are linear and satisfied by quaternionic integers, which are squares. So that in the q^2 plane the allowed momenta form an integer lattice and the identification as a square selects a subset of this lattice. This generalizes also to the algebraic integers in the extension of rationals.

2. What about the co-associative case corresponding to the canonical basis $\{I_1, iI_3, iI_5, iI_7\}$? Momenta would be as co-associative octonion o but o^2 is a quaternion in the plane defined by $\{I_0, iI_2, iI_4, iI_6\}$. o representable in terms of a complexified quaternion $q = q_0 + i\overline{q}$ as $o = I_4 q$ and the in general complex values norm squared is give by $o\overline{o}$ with conjugation of octonionic imaginary units but not i: this gives Minkowskian norm squared. This reduces the situation to the quaternionic case.

3. In this case the CM system for two-particle case corresponds to the conditions $q_{1,0}\overline{q}_1 = q_{2,0}\overline{q}_2$ implying that q_1 and q_2 have opposite directions and $q_{1,0}|\overline{q}_1| = q_{2,0}|\overline{q}_2|$. The ratio of the lengths of the momenta is integer. Now the squares $q_{i,0}|\overline{q}_i|^2$, $i = 1, 2$ are factorizations of the same integer N. Masses are in general different.

4. The situation generalizes also to complexified quaternions - the interpretation of the imaginary part of momentum might be in terms of a decay width - and even to general octonions since associativity is not involved with the conditions.

5.8 Further objections

The view about scattering amplitudes has developed rather painfully by objections creating little shocks. The representation of scattering amplitudes is based on quark oscillator operator algebra. This raises two further objections.

The non-vanishing contractions of the oscillator operators are necessary for obtaining non-trivial scattering amplitudes but is this condition possible to satisfy.

1. One of the basic deviations of TGD from quantum field theories (QFTs) is the hypothesis that all elementary particles, in particular bosons, can be described as bound states of fermions, perhaps only quarks. In TGD framework the exchange of boson in QFT would mean an emission of a virtual quark pair and its subsequent absorption. In ZEO in its basic form this seems to be impossible.

2. If scattering corresponds to algebra morphism mapping products to products of co-products - the number of quarks in say future HC is higher than in the past HC as required. But how to obtain non-vanishing scattering amplitudes? There should be non-vanishing counterparts of propagators between points of FHC but this is not possible if only creation operators are present in a given HC as ZEO requires. All particle reactions would be re-arrangements of quarks and antiquarks to elementary fermions and bosons (OZI rule of the hadronic string model: https://en.wikipedia.org/wiki/OZI_rule). The emission of virtual or real bosons requires the creation of quark antiquark pairs and seems to be in conflict with the OZI rule.

3. It would be natural to assign to quarks and bosons constructed as their bound states non-trivial inner product in a given HC of CD. Is this possible if the counterparts of annihilation operators act as creation operators in the opposite HC? Can one assign inner product to a given boundary of CD by assuming that hermitian conjugates of quark oscillator operators act in the dual Hilbert space of the quark Fock space? Could this dual Hilbert space relate to the Drinfeld's double?

How could one avoid the OZI rule?

1. Is it enough to also allow annihilation operators in given HC? Bosonic $G_{2,c}$ generators could involve them. The decay of boson to quark pair would still correspond to re-arrangement but one would have inner product for states at given HC. The creation of bosons would still be a problem. Needless to say, this option is not attractive.

2. A more plausible solution for this problem is suggested by the phenomenological picture in which quarks at the level of H are assigned with partonic 2-surfaces and their orbits, string world sheets, and their boundaries at the orbits of partonic 2-surfaces. By the discussion in the beginning of this section, these surfaces could correspond at the level of M^8 to space-time regions of complexified space-time surface with real number theoretic metric having signature (+,+,-,-), (+,+,+,-), (+,+,+,+) having 2,3, or 4 time-like dimensions. They would allow non-negative values of mass squared and would be separated from the region of Minkowskian signature by a transition region space-time region with dimension $D \in \{3, 2, 1\}$ mapped to CP_2.

   In these regions one would have 1, 2, or 3 additional energy like momentum components $p_i = E_i$. E_i. Could the change of sign for E_i transform creation operator to annihilation operator as would look natural. This would give bosonic states with a non-vanishing norm and also genuine boson creation. What forces to take this rather radical proposal seriously that it conforms with the phenomenological picture.

   In this region one could have a non-trivial causal diamond CD with signature (+,+,-,-), (+,+,+,-). For the signature (+,+,+,+) CD reduces to a point with a vanishing four-momentum and would correspond to CP_2 type extremals (wormhole contacts). Elementary fermions and bosons would consist of quarks in regions with signature (+,+,-,-) and (+,+,+,-). It would seem that the freedom to select signature in twistorial amplitude is not mere luxury but has very deep physical content.

One can invent a further objection. Suppose that the above proposal makes sense and allows to assign propagators to a given HC. Does Yangian co-product allow a construction of zero energy states giving rise to scattering amplitudes, which typically have a larger number of particles in the future HC (FHC) than in past HC (PHC) and represent a genuine creation of quark pairs?

1. One can add to the PHC quarks and bosons one-by-one by forming the product super $G(2,c)$ generators assignable to the added particles. To the FHC one would add the product of co-products of these super $G(2,c)$ generators (co-product of product is product of co-products as an algebra morphism).

2. By the basic formula of co-product each addition would correspond to a superposition of two states in FHC. The first state would be the particle itself having suffered a forward scattering. Second state would involve 2 generators of super $G_{2,c}$ at different momenta summing up to that for the initial state, and represent a scattering $q \rightarrow q + b$ for a quark in the initial state and scattering $b \rightarrow 2b$, $b \rightarrow 2b$, or $b \rightarrow 2q$ for a boson in the initial state.

   Number theoretic momentum conservation assuming momenta to be algebraic integers should allow processes in which quark oscillator operators are contracted between the states in FHC and PHC or between quarks in the FHC.

3. Now comes the objection. Suppose that the state in PC consists of fundamental quarks. Also the FC containing the product of co-products of quarks must contain these quarks with the same momenta. But momentum conservation does not allow anything else in FC! The stability of quarks is a desirable property in QFTs but something goes wrong! How to solve the problem?

   Also now phenomenological picture comes to the rescue and tells that elementary particles - as opposed to fundamental fermions - are composites of fundamental fermions assignable to flux tubes like structures involving 2 wormhole contacts. In particular, quarks as elementary particles would involve quark at either throat of the first wormhole contact and quark-antiquark pair associated with the second wormhole contact. The state would correspond to a quantum superposition of different multilocal momentum configurations defining multi-local states at M^8 level. The momentum conservation constraint could be satisfied without trivializing the scattering amplitudes since the contractions could occur between different components of the superposition - this would be essential.

   Note also that at H level there can be several quarks at a given wormhole throat defining a multilocal state in M^8: one could have a superposition of these states with different momenta and again different components of the wave function could contract. By Uncertainty Principle the almost locality in H would correspond to strong non-locality in M^8. This could be seen as an approximate variant of the TGD variant of H variant of SUSY considered in [L14].

   Could the TGD variant of SUSY proposed in [L14] but realized at the level of momentum space help to circumvent the objection? Suppose that the SUSY multiplet in M^8 can be created by a local algebraic product possessing a co-product delocalizing the local product of oscillator operators at point p in PC and therefore represents the decay of the local composite to factors with momenta at p_1 and $p - p_1$ in FC. This would not help to circumvent the objection. Non-locality and wave functions in momentum space is needed.

6 Symmetries in M^8 picture

6.1 Standard model symmetries

Can one understand standard model symmetries in M^8 picture?

1. $SU(3) \subset G_2$ would respect a given choice of time axis as preferred co-associative set of imaginary units ($I_2 \subset \{I_2, iI_3, iIb, iI_7\}$ for the canonical choice). The labels would therefore correspond to the group $SU(3)$. $SU(3)_c$ would be analogous to the local color gauge group in the sense that the element of local $SU(3)_c$ would generate a complecofied space-time surface from the flat and real M^4. The real part of pure $SU(3)_c$ gauge potential would not however reduce to pure $SU(3)$ gauge potential. Could the vertex factors be simply generators of $SU(3)$ or $SU(3)_c$?

2. What about electroweak quantum numbers in M^8 picture? Octonionic spinors have spin and isospin as quantum numbers and can be mapped to H spinors. Bosons would be bound states of quarks and antiquarks at both sides.

   How could electroweak interactions emerge at the level of M^8? At the level of H an analogous problem is met: spinor connection gives only electroweak spinor connection but color symmetries are isometries and become manifest via color partial waves. Classical color gauge potentials can be identified as projections of color isometry generators to the space-time surface.

   Could electroweak gauge symmetries at the level of M^8 be assigned with the subgroup $U(2) \subset SU(3)$ of $CP_2 = SU(3)/U(2)$ indeed playing the role of gauge group? There is a large number of space-time surfaces mapped to the same surface in H and related by a local $U(2)$ transformation. If this transformation acted on the octonionic spinor basis, it would be a gauge transformation but this is not the case: constant octonion basis serves as a gauge fixing. Also the space-time surface in M^8 changes but preserves its "algebraic shape".

6.2 How the Yangian symmetry could emerge in TGD?

Yangian symmetry [A5, A7] appears in completely 2-D systems. The article [?] (`https://arxiv.org/pdf/1606.02947.pdf`) gives a representation which is easy to understand by a physicist like me whereas the Wikipedia article remains completely incomprehensible to me.

Yangian symmetry is associated with 2-D QFTs which tend to be physically rather uninteresting. The scattering is in forward direction and only phase shifts are induced. There is no particle creation. Yangian symmetry appears in 4-D super gauge theories [?] and in the twistor approach to scattering amplitudes [?, ?, ?, ?]. I have tried to understand the role of Yangian symmetry in TGD [L3].

6.2.1 Yangian symmetry from octonionic automorphisms

An attractive idea is that the Yangian algebra having co-algebra structure could allow to construct poly-local conserved charges and that these could define vertex operators in M^8.

1. Yangian symmetry appears in 2-D systems only. In TGD framework strings world sheets could be these systems as co-commutative 2-surfaces of co-associative space-time surface.

2. What is required is that there exists a conserved current which can be also regarded as a flat connection. In TGD the flat connection would a connection for $G_{2,c}$ or its subgroup associated with the map taking standard co-associative sub-space of O_c for which the number theoretic norm squared is real and has Minkowski signature (M^4 defined by the canonical choice $\{I_2, iI_3, iI_5, iI_7\}$.

   The recent picture about the solution of co-associativity conditions fixes the subgroup of G_2 to $SU(3)$. X^4 corresponds to element g of the local $SU(3)$ acting on preferred $M^4 \subset M^8_c$ with the additional condition that the 4-surface $X^4 \subset M^8$ is invariant under $U(2) \subset SU(3)$ so that each point of X^4 corresponds to a CP_2 point. At the mas shells as roots of a polynomial P, g reduces to unity and the 4-D tangent space is parallel to the preferred M^4 on which g acts.

   One can induce this flat connection to string world sheet and holomorphy of g at this surface would guarantee the conservation of the current given by $j_{0)} = g^{-1}dg$.

3. Under these conditions the integral of the time component of current along a space-like curve at string world sheets with varying end point is well-defined and the current

$$j_{1)}(x) = \epsilon\mu\nu j_{0),\nu}(x) - \frac{1}{2}[j^{\mu}_{0)}(x,t), \int^{x} j^{0}_{0)}(t,y)dy]$$

   is conserved. This is called the current at first level. Note that the currents have values in the Lie algebra considered. It is essential that the integration volume is 1-D and its boundary is characterized by a value of single coordinate x.

4. One can continue the construction by replacing j_0 with j_1 in the above formula and one obtains an infinite hierarchy of conserved currents $j_{n)}$ defined by the formula

$$j_{n+1)}(x) \quad = \quad \epsilon\mu\nu j_{n),\nu}(x) - \frac{1}{2}[j^{\mu}_{n)}(x,t), \int^{x} j^{0}_{n)}(t,y)dy] \tag{6.1}$$

   The corresponding conserved charges Q_n define the generators of Yangian algebra.

5. 2-D metric appears in the formulas. In the TGD framework one does not have Riemann metric - only the number theoretic metric which is real only at real space-time surfaces already discussed. Is the (effective) 2-dimensionality and holomorphy enough to avoid the possible problems? Holomorphy makes sense also number theoretically and implies that the metric disappears from the formulas for currents. Also current conservation reduces to the statement of that current is equivalent to complex differential form.

6. Conserved charges would however require a 1-D integral and number theory does not favor this. The solution of the problem comes from the observation that one can construct a slicing of string world sheet to time-like curves as Hamiltonian orbits with Hamiltonian belonging to the Yangian algebra and defined by the conserved current by standard formula $j^{\alpha} = J^{\alpha\beta}\partial_{\beta}H$ in terms of Kähler form defined by the 2-D Kähler metric of string world sheet. This generalizes to Minkowskian signature and also makes sense for partonic 2-surfaces. Hamiltonians become the classical conserved charges constant along the Hamiltonian orbit. This gives an infinite hierarchy of conserved Hamiltonian charges in involution. Hamiltonian can be any combination of the Hamiltonians in the hierarchy and labelled by a non-negative integer and the label of $G_{2,c}$ generator. This is just what integrability implied by Yangian algebra means. Co-associativity and co-commutativity would be the deeper number theoretic principles implying the Yangian symmetry.

7. Could one formulate this argument in dimension $D = 4$? Could one consider instead of local current the integral of conserved currents over 2-D surfaces labelled by single coordinate x for a given value of t? If the space-time surface in M^8 (analog of Fermi sphere) allows a slicing by orthogonal strings sheets and partonic 2-surfaces, one might consider the fluxes of the currents $g^{-1}dg$ over the 2-D partonic 2-surfaces labelled by string coordinates (t, x) as effectively 2-D currents, whose integrals over x would give the conserved charge. Induced metric should disappear from the expressions so that fluxes of holomorphic differential forms over partonic 2-surface at (t, x) should be in question. Whether this works is not clear.

One should interpret the above picture at the level of momentum space instead of ordinary space-time. The roles of momentum space and space-time are changed. At this point, one can proceed by making questions.

1. One should find a representation for the algebra of the Hamiltonians associated with $g(x)$ defining the space-time surface. The charges are associated with the slicings of string world sheets or partonic 2-surfaces by the orbits of Hamiltonian dynamics defined by a combination of conserved currents so that current conservation becomes charge conservation. These charges are labelled by the coordinate x characterizing the slices defined by the Hamiltonian orbits and from these one can construct a non-local basis discrete basis using Fourier transform.

2. What the quantization of these classical charges - perhaps using fermionic oscillator operators in ZEO picture for which the local commutators vanish - could mean (only the anti-commutators of creation operators associated with the opposite half-cones of CD with opposite momenta are non-vanishing)? Do the Yangian charges involve only creation operators of either type with the same 8-momentum as locality at M^8 level suggests? Locality is natural l since these Yangian charges are analogous to charges constructed from local currents at space-time level.

3. Could the Yangian currents give rise to poly-local charges assignable to the set of vertices in a cognitive representation and labelled by momenta? Could the level n somehow correspond to the number n of the vertices and could the co-product Δ generate the charges? What does the tensor product appearing in the co-product really mean: do the sector correspond to different total quark numbers for the generators? Is it a purely local operation in M^8 producing higher monomials of creation operators with the same momentum label or is superposition over Hamiltonian slices by Fourier transform possibly involved ?

6.2.2 How to construct quantum charges

One should construct quantum charges. In the TGD framework the quantization of $g(x)$ is not an attractive idea. Could one represent the charges associated with g it in terms of quark oscillator operators induced from the second quantized E^8 spinors so that propagators would emerge in the second quantization? Analogs of Kac Moody representations but with a non-negative spectrum of conformal weights would be in question. Also super-symplectic algebra would have this property making the formulation of the analogs of gauge conditions possible, and realizing finite measurement resolution in terms of hierarchy of inclusions of hyper-finite factors of type II_1 [K15, K5]. The Yangian algebra for $G_{2,c}$ or its subgroup could be the counterpart for these symmetries at the level of H.

The following proposal for the construction for the charges and super-charges of Yangian algebra in terms of quark oscillator operators is the first attempt.

1. One knows the Lie-algebra part of Yangian from the Poisson brackets of Hamiltonians associated with string world sheet slicing and possibly also for a similar slicing for partonic 2-surfaces. One should construct a representation in terms of quark ocillator operators in ZEO framework for both Lie-algebra generators and their super-counterparts. Also co-product should be needed.

2. The oscillator operators of E^8 spinor field located at the points of X^4 are available. The charges must be local and describe states with non-linear quarks and antiquarks.

   One must construct conserved charges as currents associated with the Hamiltonian orbits. Bosonic currents are bilinear in quark and antiquark oscillator operators and their super counterparts linear in quark or antiquark oscillator operators.

3. Since the system is 2-D one can formally assume in Euclidian signature (partonic 2-surface) Kähler metric $g^{z\overline{z}}$ and Kähler form $J^{z\overline{z}} = igz\overline{z}$, which is antisymmetric and real in real coordinates ($J^{kl} = -J^{lk}$) knowing that they actually disappear from the formulas. One can also define gamma matrices $\Gamma_\alpha = \gamma_k \partial_\alpha p^k$ as projections of embedding space gamma matrices to the string world sheet. In the case of string world sheet one can introduce light-like coordinates (u, v) as analogous of complex coordinates and the only non-vanishing component of the metric is g^{uv}.

4. The claim is that the time components J^u_n the bosonic currents

$$J^\alpha_n = b^\dagger_p \overline{v}(p) \Gamma^\alpha H_n u(p) a^\dagger \tag{6.2}$$

   at the Hamiltonian curves with time coordinate t define conserved charges ($\alpha \in \{u, v\}$ at the string world sheet).

   Remark: v_p corresponds to momentum $-p$ for the corresponding plane wave in the Fourier expansion of quark field but the physical momentum is p and the point of M^8 that this state corresponds.

   Therefore one should have

$$\frac{J^u_n}{du} = 0 \tag{6.3}$$

   One can check by a direct calculation what additional conditions are possibly required by this condition.

5. The first point is that H_n is constant if $v = constant$ coordinate line is a Hamiltonian orbit. Also oscillator operators creating fermions and antifermions are constant. The derivative of $u(p)$ is

$$\frac{du(p)}{du} = \frac{\partial u(p)}{\partial_{p^k}} \frac{dp^k}{du} \quad .$$

. u_p is expressible as $u_p = Du_a$, where D is a massless Dirac operator in M^8 and u_a is a constant 8-D quark spinor with fixed chirality. D is sum of M^4- and E^4 parts and M^4 part is given by $D(M^4) = \gamma^k p_k$ so that one has $dp^k/dt = \gamma_r dp^r/dt$.

This gives

$$\frac{d(\Gamma^u H_n u(p))}{du} = g^{uv}\gamma_k \partial_v p^k \frac{du(p)}{du} = g^{uv}\partial_u p \cdot \partial_v p \quad .$$

If the tangent curves of u and v are orthogonal in the induced metric and $v = 0$ constant lines are Hamiltonian orbits the bosonic charges are conserved.

One can perform a similar calculation for $d\frac{d\overline{v}(p)}{du}$ and the result is vanishing.

One must also have $dg^{uv}/du = 0$. This should reduce to the covariant constancy of g^{uv}. If the square root of the metric determinant for string world sheet is included it cancels g^{uv}.

6. From the bosonic charges one construct corresponding fermionic super charges by replacing the fermionic or anti-quark oscillator operator part with a constant spinor.

The simplest option is that partonic 2-surfaces contain these operators at points of cognitive representation. One can ask whether co-product could forces local operators having a higher quark number. What is clear that this number is limited to the number $n = 0$ of spin degrees of $n = 8$.

1. The commutators of bosonic and fermionic charges are fermionic charges and co-product would in this case be a superposition of tensor products of bosonic and fermionic charges, whose commutator gives bosonic charge. Now however the bosonic and fermionic charges commute in the same half-cone of CD. Does this mean that the tensor product in question must be tensor product for the upper and lower half-cones of CD?

   For instance, in the fermionic case one would obtain superposition over pairs of fermions at say lower half-cone and bosons at the upper half-cone. The momenta would be opposite meaning that a local bosonic generator would have total momentum $2p$ at point p and fermionic generator at opposite cone would have momentum $-p$. The commutator would have momentum p as required. In this manner one could create bosons in either half-cone.

2. One can also assign to the bosonic generators a co-product as a pair of bosonic generators in opposite half-cones commuting to the bosonic generator. Assume that bosonic generator is at lower half-cone. Co-product must have a local composite of 4 oscillator operators in the lower half-cone and composite of 2 oscillator operators in the upper half-cone. Their anti-commutator contracts two pairs and leaves an operator of desired form. It therefore seems.

   Statistics allows only generators with a finite number of oscillator operators corresponding to 8 spin indices, which suggests an interpretation in terms of the proposed SUSY [L14]. The roots of P are many-sheeted coverings of M^4 and this means that there are several 8-momenta with the same M^4 projection. This degree of freedom corresponds to Galois degrees of freedom.

3. Only momenta in cognitive representation are allowed and momentum is conserved. The products of generators appearing in the sum defining the co-product of a given generator T, which is a local composite of quarks, would commute or anti-commute to T, and their momenta would sum-up to the momentum associated with T. The co-product would be poly-local and receive contributions from the points of the cognitive representation. Also other quantum numbers are conserved.

6.2.3 About the physical picture behind Yangian and definition of co-product

The physical picture behind the definition of Yangian in the TGD framework differs from that adopted by Drinfeld, who has proposed - besides a general definition of the notion of quantum algebra - also a definition of Yangian. In the Appendix Drinfeld's definition is discussed in detail: this discussion appears almost as such in [L3].

1. Drinfeld proposes a definition in terms of a representation in terms of generators of a free algebra to which one poses relations [?]. Yangian can be seen as an analog of Kac-Moody algebra but with generators labelled by integer $n \geq 0$ as an analog of non-negative conformal weight. Also super-symplectic algebra has this property and its Yangianization is highly suggestive. The generators of Yangian as algebra are elements J_n^A, $n \geq 0$, with $n = 0$ and $n = 1$. Elements J_0^A define the Lie algebra and elements J_1^A transform like Lie-algebra elements so that commutators at this level are fixed.

   Remark: I have normally used generator as synonym for the element of Lie algebra: I hope that this does not cause confusion

   The challenge is to construct higher level generators J_n^A. Their commutators with JA^0 with J^An are fixed and also the higher level commutators can be guessed from the additivity of n and the transformation properties of generators J_n^A. The commutators are very similar to those for Kac-Moody algebra. In the TGD picture the representation as Hamiltonians fixes these commutation relations as being induced by a Poisson bracket. The Lie-algebra part of Yangian can be therefore expressed explicitly.

2. The challenge is to understand the co-product Δ. The first thing to notice is that Δ is a Lie algebra homomorphism so that one has $\Delta(XY) = \Delta(X)\Delta(Y)$ plus formulas expressing linearity. The intuitive picture is that Δ adds a tensor factor and is a kind of time reversal of the product conserving total charges and the total value of the weight n. Already this gives a good overall view about the general structure of the co-commutation relations.

   The multiplication of generators by the unit element Id of algebra gives the generator itself so that $\Delta(J_A)$ should involve part $Id \otimes J^A \oplus J^A \otimes Id$. Generators are indeed additive in the ordinary tensor product for Lie-algebra generators - for instance, rotation generators are sums of those for the two systems. However, one speaks of interaction energy: could the notion of "interaction quantum numbers" make sense quite generally. Could this notion provide some insights to proton spin puzzle [?] meaning that quark spins do not seem to contribute considerably to proton spin? A possible TGD based explanation is in terms of angular momentum associated with the color magnetic flux tubes [K9], and the formulation of this notion at M^8 level could rely on the notion of "interaction angular momentum".

   The time reversal rule applied to $[J_A^m, J_B^n] \propto f_{ABC}J_C^{m+n}$ suggests that $\Delta(T_A^n)$ contains a term proportional to $f_{CBA}J_C^m \otimes J_B^{n-m}$. This would suggest that co-product as a time reversal involves also in the case of J_A^0 the term $k_1 f_{CBA} J_C^0 \otimes J_B^0$, where k_1 as an analog of interaction energy.

   Drinfeld's proposal does not involve this term in accordance with Drinfeld's intuition that co-product represents a deformation of Lie-algebra proportional to a parameter denoted by $\hbar$, which need not (and cannot!) actually correspond to $\hbar$. This view could be also defended by the fact that J_0^A do not create physical states but only measures the quantum numbers generated by J_A^n, $n > 0$. TGD suggests interpretation as the analog of the interaction energy.

3. In Drinfeld's proposal, the Lie-algebra commutator is taken to be $[J_A^0, J_B^0] = kf_{ABC}J_C^0$, $k = 1$. Usually one thinks that generators have the dimension of $\hbar$ so that dimensional consistency requires $k = \hbar$. It seems that Drinfeld puts $\hbar = 1$ and the $\hbar$ appearing in the co-product has nothing to do with the actual $\hbar$.

   The conservation of dimension applied to the co-product would give $k_1 = 1/\hbar$! What could be the interpretation? The scattering amplitudes in QFTs are expanded in powers of gauge coupling strengths $\alpha = g^2/4\pi\hbar$. In ZEO co-product would be essential for obtaining non-trivial scattering amplitudes and the expansion in terms of $1/\hbar$ would emerge automatically from the corrections involving co-products - in path integral formalism this expansion emerges from propagors

This view would also conform with the vision that Mother Nature loves her theoreticians. The increase of $h_{eff}/h_0 = n$ as dimension of extension of rationals would be Mother Nature's way to make perturbation theory convergent [K4]. The increase of the degree of P defining the space-time surface increases the algebraic complexity of the space-time surface but reduces the value of α as a compensation.

4. Drinfeld gives the definition of Yangian in terms of relations for the generating elements with weight $n = 0$ and $n = 1$. From these one can construct the generators by applying Δ repeatedly. Explicit commutation relations are easier to understand by a physicist like me, and I do not know whether the really nasty looking representation relations - Drinfeld himself calls "horrible" [?] - are the only manner to define the algebra. In the TGD framework the definition based on the idea about co-product as a strict time reversal of product would mean deviation in the $n = 0$ sector giving rise to an interaction term having natural interpretation as analog of interaction energy.

5. Drinfeld proposes also what is known as Drinfeld's double [A8] (see `http://tinyurl.com/y7tpshkp`) as a fusion of two Hopf algebras and allowing to see product and co-product as duals of each other. The algebra involves slight breaking of associativity characterized by Drinfeld's associator. ZEO suggests [K7] that the members of Drinfeld's double correspond to algebra and co-algebra located at the opposite half-cones and there are two different options. Time reversal occurring in "big" state functions reductions (BSFRs) would transform the members to each other and change the roles of algebra and co-algebra (fusion would become decay).

In the TGD framework there is also an additional degree of freedom related to the momenta in cognitive representation, which could be regarded also as a label of generators. The idea that commutators and co-commutators respect conservation of momentum allows the fixing of the general form of Δ. Co-product of a generator at momentum p ina given half-cone would be in the opposite half-cone and involve sum over all momentum pairs of generators at p_1 and p_2 with the constraint $p_1 + p_2 + p = 0$.

Summation does not make sense for momenta in the entire extension of rationals. The situation changes if the momenta are algebraic integers for the extension of rationals considered: quarks would be particles in a number theoretic box. In the generic case, very few terms - if any - would appear in the sum but for space-time surfaces as roots of octonionic polynomials this is not the case. The co-products would as such define the basic building bricks of the scattering amplitudes obtained as vacuum expectation reducing the pairs of fermions in opposite half-cones to propagators.

7 Appendix: Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

7.1 Yang-Baxter equation (YBE)

YBE has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A7]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [?] (see `http://tinyurl.com/mcvvcqp`) and in conformal field theories and models for anyons. Yangian symmetry appears also in the twistor Grassmann approach to scattering amplitudes [?, ?] and thus involves YBE. At the same time new invariants for links were discovered and a new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [?] (see `http://tinyurl.com/l4z6zyr`, where one can also find a list of references).

YBE was first discovered by McGuire (1964) and 3 years later by Yang in a quantum mechanical many-body problem involving a delta function potential $\sum_{i<j}\delta(x_i - x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as a building brick 2-particle scattering matrix - R-matrix. YBE emerged for the R-matrix as a consistency condition for factorization. Baxter discovered in 1972 a solution of the eight vertex model in terms of YBE. Zamolodchikov pointed out that the algebraic mechanism behind factorization of 2-D QFTs is the same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed a quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. The interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of `http://tinyurl.com/kyw6xu6`). In 2-D quantum field theories and in models for basic gate in topological quantum computation the R-matrix is unitary. R-matrix can be regarded as an endomorphism mapping $V_1 \otimes V_2$ to $V_2 \otimes V_1$ representing permutation of the particles.

7.1.1 YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v) = R_{12}(v)R_{13}(u+v)R_{23}(u) \tag{7.1}$$

having interpretation as associativity condition for quantum algebras.

At the limit $u, v \to \infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braiding operation replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on the basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V \otimes V$ the condition states that $R(0)$ is proportional to the permutation matrix P for the factors.

7.1.2 General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued to meromorphic functions in the complex plane with poles forming an Abelian group. R-matrices can be classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have a pole at origin and elliptic solutions have a lattice of poles. In [?] (see `http://tinyurl.com/l4z6zyr`) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [?] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.

3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anythingto do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of a Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be as interesting as Yangian: in this case co-product Δ does not seem to have a natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$. For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.

5. The article of Jimbo discusses also a fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs the R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

7.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \to A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \to C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Also ϵ satisfies associativity condition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [**?**] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n+1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n + 1 \geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretations improve the understanding of classical correspondence (QCC).

7.2.1 Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian for $\mathcal{N}=4$ SUSY [?], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

However, it must be emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC}J_0^B \times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \times J_1^A + f^A_{BC}J_0^B \times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n=0$ and $n=1$. The first half of these relations discussed in very clear manner in [?] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f^{AB}_C J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f^{AB}_C J^{(1)C} \quad . \tag{7.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right] + \left[J^{(1)B},\left[J^{(1)C},\ J^A\right]\right] + \left[J^{(1)C},\left[J^{(1)A},\ J^B\right]\right] \\
&\qquad = \frac{1}{24} f^{ADK}f^{BEL}f^{CFM}f_{KLM}\{J_D,J_E,J_F\} \quad , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\ \left[J^C,J^{(1)D}\right]\right] + \left[\left[J^{(1)C},J^{(1)D}\right],\ \left[J^A,J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24}(f^{AGL}f^{BEM}f^{CD}_K \\
&\qquad +f^{CGL}f^{DEM}f^{AB}_K)f^{KFN}f_{LMN}\{J_G,J_E,J_F\} \quad .
\end{aligned} \tag{7.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A,B,C\}$ denotes the symmetrized product of three generators.

The right hand side often has coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant and as noticed in the main text has dimension of $1/\hbar$. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A} = J^A$ the first Serre relation reduces to Jacobi identity and second to the antisymmetry of the Lie bracket. The right hand side involved completely symmetrized trilinears $\{J_D, J_E, J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra, whose elements are labeled by a non-negative integer n. The generators obtained in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purpose of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exist also for continuum one-dimensional index).

Under certain consistency conditions, a discrete one-dimensional lattice provides a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} \quad = \quad f^A_{BC}\sum_{i<j} J_i^B J_j^C \quad . \tag{7.4}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N}=4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned}\Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C\end{aligned} \tag{7.5}$$

Δ allows to imbed Lie algebra into the tensor product in a non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

7.2.2 Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N}=4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [**?**].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can involve besides the unit operator also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrized tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the following form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes the identity matrix and $PU(n|m)$ and $SU(n|m)$ are the same. This does not happen for $n = m$: this is an important delicacy since this case corresponds to $\mathcal{N}=4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as a new physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$
\begin{aligned}
J_C^{(1)} &= g_{CC'}J^{(1)C'} = g_{CC'}f^{C'}_{AB}\sum_{i<j}J_i^AJ_j^B \\
&= g_{CC'}f^{C'}_{AB}g^{AA'}g^{BB'}\sum_{i<j}J^i_{A'}J^j_{B'} \quad .
\end{aligned}
\tag{7.6}
$$

Here $g_{AB} = Str(J_AJ_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

8 Conclusions

$M^8 - H$ duality plays a crucial role in quantum TGD and this motivated a critical study of the basic assumptions involved.

8.1 Co-associativity is the only viable option

The notion of associativity of the tangent or normal space as a number theoretical counterpart of a variational principle. This is not enough in order to have $M^8 - H$ duality. The first guess was that the tangent space is associative and contains a commutative 2-D sub-manifold to guarantee $M^8 - H$ duality.

1. The cold shower came as I learned that 4-D associative sub-manifolds of quaternion spaces are geodesic manifolds and thus trivial. Co-associativity is however possible since any distribution of associative normal spaces integrates to a sub-manifold. Typically these sub-manifolds are minimal surfaces, which conforms with the physical intuitions. Therefore the surface X^4_r given by holography should be co-associative. By the same argument space-time surface contains string world sheets and partonic 2-surfaces as co-complex surfaces.

2. $X = Re_Q(o) = 0$ and $Y = Im_Q(P) = 0$ allow M^4 and its complement as associative/co-associative subspaces of O_c. The roots $P = 0$ for the complexified octonionic polynomials satisfy two conditions $X = 0$ and $Y = 0$.

   Surprisingly, universal solutions are obtained as brane-like entities X^6_c with real dimension 12, having real projection X^6_r ("real" means that the number theoretic complex valued octonion norm squared is real valued).

   Equally surprisingly, the non-universal solutions to the conditions to $X = 0$ correspond complex mass shells with real dimension 6 rather than 8. The solutions to $X = Y = 0$ correspond to common roots of the two polynomials involved and are also 6-D complex mass shells.

   The reason for the completely unexpected behavior is that the equations $X = 0$ and $Y = 0$ are reduced by Lorentz invariance to equations for the ordinary roots of polynomials for the complexified mass squared type variable. The intersection is empty unless X and Y have a common root and X^4_r belongs to X^6_r for a common root.

How to associate to the polynomial P a real 4-surface satisfying the conditions making $M8-H$-duality?

1. P would fix complex mass shells in terms of its roots but not the 4-surfaces, contrary to the original expectations. The fact that the 3-D mass shells belong to the same M^4 and also their tangent spaces are parallel to M^4 together with rationality conditions for local $SU(3)$ element suggests number theoretical holography.

2. The key observation is that G_2 as the automorphism group of octonions respects the co-associativity of the 4-D real sub-basis of octonions. Therefore a local G_2 gauge transformation applied to a 4-D co-associative sub-space $M^c \subset O_c$ gives a co-associative four-surface as a real projection. Also octonion analyticity allows G_2 gauge transformation. If X^4 is the image

M^4 by a local $SU(3)$ element such that it also remains invariant under $SU(2)$ at each point, one obtains automatically M^8-H duality.

The image of X^4 under M^8-H duality depends on g so that gauge invariance is not in question. The plausible interpretation in case of $SU(3)$ is in terms of Kac-Moody - or even Yangian symmetry. Note that at QFT limit the gauge potentials defined at H level as projections of Killing vector fields of $SU(3)$ are replaced by their sums over parallel space-time sheets to give gauge fields as the space-time sheets are approximated with a single region of Minkowski space.

The study of octonionic Dirac equation shows that the solutions correspond to momenta at mass shells $m^2 = r_n$ obtained as roots of the polynomial P and that co-associativity is an essential for the octonionic Dirac equation. This conforms with the reduction of everything to algebraic conditions at the level of M^8.

8.2 Construction of the momentum space counter parts of scattering amplitudes in M^8

The construction of scattering amplitudes in M^8 was the main topic of this article. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. 0

1. The fact that $SU(3)$ gauge transformation with boundary conditions defined by the mass shells as roots of polynomial P defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differing from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anti-commutators since the cognitive representation is discrete.

2. The main objection is that the scattering amplitudes are trivial if quark momenta belong to cognitive representations, which are finite in the generic case. This would be the case also in 2-D integrable theories. The objection can be circumvented. First, the huge symmetries imply that cognitive representations can contain a very large - even an infinite - number of points. At partonic 2-surface this number could reduce to finite. Equally importantly, local composites of quark oscillation operators with collinear quark momenta are possible and would be realized in terms of representations of Yangian algebra for $G_{2,c}$ serving as the counterpart for super-symplectic and Kac-Moody algebras at the level of H.

3. ZEO leads to a concrete proposal for the construction of zero energy states - equivalently scattering amplitudes - by using a representation of Yangian algebra realized in terms of positive and negative energy quarks in opposite half-cones. Co-product plays a key role in the construction. Also the proposed local composites of quarks proposed in [L14] make sense.

4. Momentum conservation conditions and mass shell conditions combined with the requirement that the momenta are algebraic integers in the extension of rationals determined by the polynomial P look rather difficult to solve. These conditions however linearize in the sense that one can express the allowed momenta as squares of integer quaternions.

Also the construction of scattering amplitudes in M^8 is considered. ZEO and the interpretation of M^8 as a momentum space analogous to the interior of the Fermi sphere give powerful constraints on the scattering amplitudes. The fact that $G_{2,c}$ gauge transformation defines space-time surface and the corresponding gauge field vanishes plus the fact that at string world sheets the gauge potential defines a conserved current by holomorphy strongly suggest Yangian symmetry differering from Kac-Moody symmetry in that the analogs of conformal weights are non-negative. This leads to a proposal for how vertex operators can be constructed in terms of co-product using fermionic oscillator operators but with Kronecker delta anticommutators since the cognitive representation is discrete.

The main objection is that the scattering amplitudes are trivial if quark momenta belong to cognitive representations, which are finite in the generic case. This would be the case also in 2-D integrable theories. The objection can be circumvented. First, the huge symmetries imply that cognitive representations can contain a very large - even an infinite - number of points. At partonic 2-surface this number could reduce to finite. Equally importantly, local composites of quark oscillation operators with collinear quark momenta are possible and would be realized in terms of representations of Yangian algebra for $G_{2,c}$ serving as the counterpart for super-symplectic and Kac-Moody algebras at the level of H.

ZEO leads to a concrete proposal for the construction of zero energy states - equivalently scattering amplitudes - by using a representation of Yangian algebra realized in terms of positive and negative energy quarks in opposite half-cones. Co-product plays a key role in the construction. Also the proposed local composites of quarks proposed in [L14] make sense.

Momentum conservation conditions and mass shell conditions combined with the requirement that the momenta are algebraic integers in the extension of rationals determined by the polynomial P look rather difficult to solve. These conditions however linearize in the sense that one can express the allowed momenta as squares of integer quaternions.

REFERENCES

Mathematics

[A1] Fundamental domain. Available at: https://en.wikipedia.org/wiki/Fundamental_domain.

[A2] Honeycomb geometry. Available at: https://en.wikipedia.org/wiki/Honeycomb_(geometry).

[A3] List of regular polytopes and compounds. Available at: https://en.wikipedia.org/wiki/List_of_regular_polytopes_and_compounds.

[A4] Uniform honeycomes in hyperbolic space. Available at: https://en.wikipedia.org/wiki/Uniform_honeycombs_in_hyperbolic_space.

[A5] Yangian symmetry. Available at: https://en.wikipedia.org/wiki/Yangian.

[A6] Urbano F Barros MM, Chen B-Y. Quaternion CR-submanifolds of quaternion manifolds. *Kodai Mathematical Journal*, 4(3):399–417, 2020. Available at: https://www.researchgate.net/publication/38325032_Quaternion_CR-submanifolds_of_quaternion_manifolds.

[A7] MacKay NJ. Introduction to Yangian symmetry in integrable field theory, 2009. Available at: https://wenku.baidu.com/view/01e4ebdbad51f01dc281f180.html.

[A8] Hajac PM. On and around the Drinfeld double. Available at: https://www.impan.pl/~burgunde/WSBC09/Ddouble_Hajac.pdf.

Books related to TGD

[K1] Pitkänen M. About the Nottale's formula for h_{gr} and the possibility that Planck length l_P and CP_2 length R are related. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/vzerovariableG.pdf, 2023.

[K2] Pitkänen M. Breakthrough in understanding of $M^8 - H$ duality. Available at: https://tgdtheory.fi/pdfpool/M8H.pdf., 2023.

[K3] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K4] Pitkänen M. Does TGD Predict a Spectrum of Planck Constants? In *Dark Matter and TGD:* *https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/Planck, 2023.

[K5] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Topological Geometrodynamics: Overview: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdoverview2*. Available at: https://tgdtheory.fi/pdfpool/vNeumannnew, 2023.

[K6] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/kahler.pdf., 2023.

[K7] Pitkänen M. Is Non-Associative Physics and Language Possible Only in Many-Sheeted Space-Time? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/braidparse.pdf, 2023.

[K8] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K9] Pitkänen M. New Physics Predicted by TGD: Part I. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys1.pdf, 2023.

[K10] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K11] Pitkänen M. Some questions related to the twistor lift of TGD. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistquestions.pdf, 2023.

[K12] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K13] Pitkänen M. The classical part of the twistor story. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistorstory.pdf, 2023.

[K14] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

[K15] Pitkänen M. Was von Neumann Right After All? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/vNeumann.pdf, 2023.

[K16] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

[K17] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory. In *Topological Geometrodynamics: Overview: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdoverview2*. Available at: https://tgdtheory.fi/pdfpool/kahlersm, 2023.

Articles about TGD

[L1] Pitkänen M. Geometric theory of harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[L2] Pitkänen M. About $h_{eff}/h = n$ as the number of sheets of space-time surface as Galois covering. Available at: https://tgdtheory.fi/public_html/articles/Galoisext.pdf., 2017.

[L3] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? Available at: https://tgdtheory.fi/public_html/articles/Yangianagain.pdf., 2017.

[L4] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: https://tgdtheory.fi/public_html/articles/ratpoints1.pdf., 2017.

[L5] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: https://tgdtheory.fi/public_html/articles/ratpoints2.pdf., 2017.

[L6] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part III. Available at: https://tgdtheory.fi/public_html/articles/ratpoints3.pdf., 2017.

[L7] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L8] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L9] Pitkänen M. About the physical interpretation of the velocity parameter in the formula for the gravitational Planck constant . Available at: https://tgdtheory.fi/public_html/articles/vzero.pdf., 2018.

[L10] Pitkänen M. $M^8 - H$ duality and consciousness. Available at: https://tgdtheory.fi/public_html/articles/M8Hconsc.pdf., 2019.

[L11] Pitkänen M. $M^8 - H$ duality and the two manners to describe particles. Available at: https://tgdtheory.fi/public_html/articles/susysupertwistor.pdf., 2019.

[L12] Pitkänen M. New results related to $M^8 - H$ duality. Available at: https://tgdtheory.fi/public_html/articles/M8Hduality.pdf., 2019.

[L13] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L14] Pitkänen M. SUSY in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/susyTGD.pdf., 2019.

[L15] Pitkänen M. Twistor lift of TGD and WCW geometry. Available at: https://tgdtheory.fi/public_html/articles/wcwnewest.pdf., 2019.

[L16] Pitkänen M. Twistors in TGD. Available at: https://tgdtheory.fi/public_html/articles/twistorTGD.pdf., 2019.

[L17] Pitkänen M. Could ZEO provide a new approach to the quantization of fermions? Available at: https://tgdtheory.fi/public_html/articles/secondquant.pdf., 2020.

[L18] Pitkänen M. Fermionic variant of $M^8 - H$ duality. Available at: https://tgdtheory.fi/public_html/articles/M8Hfermion.pdf., 2020.

[L19] Pitkänen M. How to compose beautiful music of light in bio-harmony? Research Gate: https://www.researchgate.net/publication/344623253_How_to_compose_beautiful_music_of_light_in_bio-harmony., 2020.

[L20] Pitkänen M. Summary of Topological Geometrodynamics. Research Gate: https://www.researchgate.net/publication/343601105_Summary_of_Topological_Geometrodynamics., 2020.

[L21] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: https://tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[L22] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory. Available at: https://tgdtheory.fi/public_html/articles/kahlersmhort.pdf., 2020.

[L23] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory (short version). Available at: https://tgdtheory.fi/public_html/articles/kahlersm.pdf., 2020.

[L24] Pitkänen M. Can TGD Predict the Value of Newton's Constant? *Pre-Space-Time Journal*, 11(5), 2020. See also https://tgtheory.fi/public_html/articles/Gagain.pdf.

[L25] Pitkänen M. The Dynamics of State Function Reductions as Quantum Measurement Cascades. *Pre-Space-Time Journal*, 11(2), 2020. See also https://tgtheory.fi/public_html/articles/SSFRGalois.pdf.

[L26] Pitkänen M. Negentropy Maximization Principle and Second Law. Available at: https://tgdtheory.fi/public_html/articles/nmpsecondlaw.pdf., 2021.

[L27] Pitkänen M. Neutrinos and TGD. https://tgdtheory.fi/public_html/articles/TGDneutrino.pdf., 2021.

[L28] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L29] Pitkänen M. Time reversal and the anomalies of rotating magnetic systems. Available at: https://tgdtheory.fi/public_html/articles/freereverse.pdf., 2021.

[L30] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.