Appendix A: Quantum Groups and Related Structures

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February 2, 2024
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Contents

1   Introduction

The great dream of a physicist believing in reductionism and TGD would be a formalism generalizing Feynman diagrams allowing any graduate student to compute the predictions of the theory. TGD has forced myself to give up naive reductionism but I believe that TGD allows generalization of Feynman diagram in such a way that one gets rid of the infinities plaguing practically all existing theories. The purpose of this chapter is to develop general vision about how this might be achieved. The vision is based on generalization of mathematical structures discovered in the construction of topological quantum field theories (TQFT), conformal field theories (CQFT). In particular, the notions of Hopf algebras and quantum groups, and categories are central. The following gives a very concise summary of the basic ideas.

In introduced the original version of this chapter in extended form long time ago having also remarkably long title. I ended up with diagrams which had category theoretical meaning and had huge symmetries generalizing the duality symmetry of the hadronic string model and allowing to reduce their number dramatically leaving only tree diagrams still having symmetries. These diagrams were however not Feynman diagrams and this made me scared. After the emergence of twistor Grassmann approach and twistor diagrams the situation changed completely and it might be that I was on the correct track after all. This chapter is what was left from my adventure.

1. Feynman diagrams as generalized braid diagrams

The first key idea is that generalized Feynman diagrams with diagrams analogous to knot and link diagrams in the sense that diagrams involving loops are equivalent with tree diagrams. This would be a generalization of duality symmetry of string models.

TGD itself provides general arguments supporting same idea. The identification of preferred extremal of Kähler action (perhaps absolute minimum in Eucdlian space-time regions) as a four-dimensional Feynman diagram characterizing particle reaction means that there is only single Feynman diagram instead of functional integral over 4-surfaces: this diagram is expected to be minimal one. S-matrix element as a representation of a path defining continuation of configuration space spinor field between different sectors of it corresponding different 3-topologies leads also to the conclusion that all continuations and corresponding Feynman diagrams are equivalent. Universe as a compute metaphor idea allowing quite concrete realization by generalization of what is meant by space-time point leads to the view that generalized Feynman diagrams characterize equivalent computations.

2.   Coupling constant evolution from infinite number of critical values of Kähler coupling strength

The basic objection that this vision does not allow to understand coupling constant evolution involving loops in an essential way can be circumvented.

Quantum criticality requires that Kähler coupling constant α_K is analogous to critical temperature (so that the loops for configuration space integration vanish). The hypothesis motivated by the enormous vacuum degeneracy of Kähler action is that gauge couplings have an infinite number of possible values labelled by p-adic length scales and probably also by the fractal dimensions of effective tensor factors defined hierarchy of II_1 factors (so called Beraha numbers).

The dependence on p-adic length scale L_p corresponds to the usual renormalization group evolution whereas the latter dependence would correspond to angular resolution and finite-dimensional extensions of p-adic number fields R_p. Finite resolution and renormalization group evolution are forced by the algebraic continuation of rational number based physics to real and p-adic number fields since p-adic and real notions of distance between rational points differ dramatically.

TGD suggests discrete p-adic coupling constant evolution in which coupling constants are renormalization group invariants for the evolution associated with given p-adic prime p. This would mean vanishing of loops obtained also in $\mathcal{N} = 4$ SUSY allowing twistorialization. Gauge couplings could depend on prime p characterizing the p-adic length scale. The p-adic prime and therefore also the length scale and coupling constants characterizing the dynamics for given CD would vary wildly as function of integer characterizing CD size scale. This could mean that the CDs whose size scales are related by multiplication of small integer are close to each other. They would be near to each other in logarithmic sense and logarithms indeed appear in running coupling constants. This "prediction" is of course subject to criticism.

The discrete p-adic coupling constant evolution should relate to the continuous RG evolution of QFTs. This requires understanding of how this space-time corresponds to the many-sheeted

space-time of TGD. GRT space-time as effective space-time obtained by replacing the many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of space-time sheets from Minkowski metric. Also gauge potentials of standard model would correspond classically to superpositions of induced gauge potentials over space-time sheets. Gravitational constant, cosmological constant, and various gauge couplings emerge as predictions. Ordinary continuous coupling constant evolution would follow only at GRT-QFT limit.

3. Equivalence of loop diagrams with tree diagrams from the axioms of generalized ribbon category

A further was that Hopf algebra related structures and appropriately generalized ribbon categories could provide a concrete realization of this picture. Generalized Feynman diagrams which are identified as braid diagrams with strands running in both directions of time and containing besides braid operations also boxes representing algebra morphisms with more than one incoming and outgoing strands describing particle reactions (3-particle vertex should be enough). In particular, fusion of 2-particles and decay of particle to two would correspond to generalizations of algebra product μ and co-product Δ to morphisms of the category defined by the super-symplectic algebras associated with 3-surfaces with various topologies and conformal structures. The basic axioms for this structure generalizing Hopf algebra axioms state that diagrams with self energy loops, vertex corrections, and box diagrams are equivalent with tree diagrams.

To sum up, I could not develop this project further - mainly by the lack of needed mathematical knowhow, and this chapter is more like an Appendix. It might however be that twistorial approach could allow to concretize the idea about equivalence of loop diagrams with tree diagrams for generalization of Feynman diagrams. What is certain, that the notion of coupling constant evolution aat the level of many-sheeted space-time, should be very simple in TGD- maybe trivial for a given algebraic extension of given p-adic number field. Only at the QFT limit the lumping of sheets to single one is expected to induce complexities.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

2   Hopf Algebras And Ribbon Categories As Basic Structures

In this section the basic notions related to Hopf algebras and categories are discussed from TGD point of view. Examples are left to appendix. The new element is the graphical representation of the axioms leading to the idea about the equivalent of loop diagrams and tree diagrams based on general algebraic axioms.

2.1   Hopf Algebras And Ribbon Categories Very Briefly

An algebraic formulation generalizing braided Hopf algebras and related structures to what might be called quantum category would involve the replacement of the co-product of Hopf algebras with morphism of quantum category having as its objects the Clifford algebras associated with WCW spinor structure for various 3-topologies. The corresponding Fock spaces would would define algebra modules and the objects of the category would consists of pairs of algebras and corresponding modules. The underlying primary structure would be second quantized free induced spinor fields associated with 3-surfaces with various 3-topologies and generalized conformal structures.

1. Bi-algebras

Bi-algebras have two algebraic operations. Besides ordinary multiplication $\mu : H \otimes H \to H$ there is also co-multiplication $\Delta : H \to H \otimes H$. Algebra satisfies the associativity axiom (Ass): $a(bc) = (ab)c$, or more formally, $\mu(id \otimes \mu) = \mu(\mu \otimes id)$, and the unit axiom (Un) stating that there is morphism $\eta : k \to A$ mapping the unit of A to the unit of field k. Commutativity axiom (Co) $ab = ba$ translates to $\mu \otimes \tau \equiv \mu^{op} = \mu$, where τ permutes factors in tensor product $A \otimes A$.

Δ satisfies mirror images of these axioms. Co-associativity axiom (Coass) reads as $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, co-unit axiom (Coun) states existence of morphism $\epsilon : k \to C$ mapping the unit of A

to that of k, and co-commutativity (Coco) reads as $\tau \circ \Delta \equiv \Delta^{op} = \Delta$. For a bi-algebra H also additional axioms are satisfied: in particular, Δ (μ) acts as algebra (bi-algebra) morphism. When represented graphically, this constraint states that a box diagram is equivalent to a tree diagram as will be found and served as the stimulus for the idea that loop diagrams might be equivalent with tree diagrams.

Left and right algebra modules and algebra representations are defined in an obvious manner and satisfy associativity and unit axioms. A left co-module corresponds a pair (V, Δ_V) where the co-action $\Delta_N : V \to A \otimes V$ satisfies co-associativity and co-unit axioms. Right co-module is defined in an analogous manner.

Particle fusion $A \otimes B \to C$ corresponds to μ: $A \otimes B \to C = AB$. Co-multiplication Δ corresponds time reversal $C \to A \otimes B$ of this process, which is kind of a time-reversal for multiplication. The generalization would mean that μ and Δ become morphisms $\mu : B \otimes C \to A$ and $\Delta : A \to B \otimes C$, where A, B, C are objects of the quantum category. They could be either representations of same algebra or even different algebras.

2. Drinfeld's quantum double

Drinfeld's quantum double [A3] is a braided Hopf algebra obtained by combining Hopf algebra $(H, \mu, \Delta, \eta, \epsilon, S, R)$ and its dual $H^\star$ to a larger Hopf algebra known as quasi-triangular Hopf algebra satisfying $\Delta = R\Delta^{op}R^{-1}$, where $\Delta^{op}(a)$ is obtained by permuting the two tensor factors. Duality means existence of a scalar product and the two algebras correspond to Hermitian conjugates of each other.

In TGD framework the physical states associated with these algebras have opposite energies since in TGD framework antimatter (or matter depending on the phase of matter) corresponds to negative energy states. The states of the Universe would correspond to states with vanishing conserved quantum numbers, and in concordance with crossing symmetry, particle reactions could be interpreted as transitions generating zero energy states from vacuum.

The notion of duality [A3] is needed to define an inner product and S-matrix. Essentially Dirac's bra-ket formalism is in question. The so called evaluation map $ev : V \otimes V^\star \to k$ defined as $ev(v^i \otimes v_j) = \langle v^i, v_j \rangle = \delta_{ij}$ defines an inner product in any Hopf algebra module. The inverse of this map is the linear map $k \to V$ defined by $\delta_v(1) = v_i \otimes v^i$. For a tensor category with unit I, field k is replaced with unit I, and left duality these maps are replaced with maps $b_V : I \to V \otimes V^\star$ and $d_V = V \otimes V^\star \to I$. Right duality is defined in an analogous manner. The map d_V assigns to a given zero energy state S-matrix element. Algebra morphism property $b_V(ab) = b_V(a)b_V(b)$ would mean that the outcome is essentially the counterpart of free field theory Feynman diagram. This diagram is convoluted with the S-matrix element coded to the entanglement coefficients between positive and negative energy particles of zero energy state.

3. Ribbon algebras and ribbon categories

The so called ribbon algebra [A3] is obtained by replacing one-dimensional strands with ribbons and adding to the algebra the so called twist operation θ acting as a morphism in algebra and in any algebra module. Twist allows to introduce the notion of trace, in particular quantum trace.

The thickening of one-dimensional strands to 2-dimensional ribbons is especially natural in TGD framework, and corresponds to a replacement of points of time=constant section of 4-surface with one-dimensional curves along which the S-matrix defined by R-matrix is constant. Ribbon category is defined in an obvious manner. There is also a more general definition of ribbon category with objects identified as representations of a given algebra and allowing morphisms with arbitrary number of incoming and outgoing strands having interpretation as many-particle vertices in TGD framework. The notion of quantum category defined as a generalization of a ribbon category involving the generalization of algebra product and co-product as morphisms between different objects of the category and allowing objects to correspond different algebras might catch the essentials of the physics of TGD Universe.

2.2   Algebras, Co-Algebras, Bi-Algebras,And Related Structures

It is useful to formulate the notions of algebra, co-algebra, bi-algebra, and Hopf algebra in order to understand how they might help in attempt to formulate more precisely the view about what generalized Feynman diagrams could mean. Since I am a novice in the field of quantum groups,

the definitions to be represented are more or less as such from the book "Quantum Groups" of Christian Kassel [A3] with some material (such as the construction of Drinfeld double) taken from [A6]. What is new is a graphical representation of algebra axioms and the proposal that algebra and co-algebra operations have interpretation in terms of generalized Feynman diagrams.

In the following considerations the notation id_k for the isomorphism $k \to k \otimes k$ defined by $x \to x \otimes x$ and its inverse will be used.

2.2.1   Algebras

Algebra can be defined as a triple (A, μ, η), where A is a vector space over field k and $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps satisfying the following axioms (Ass) and (Un).

(Ass): The square

$$0[2](10, 30)[O1]$$

$A \otimes A \otimes A$ $(50, 30)[02]A \otimes A$ $(10, 15)[O3]A \otimes A$ $(50, 15)[O4]A$ $(25, 30)(45, 30)\mu \otimes id$ $(25, 15)(45, 15)\mu$ $(10, 30)(10, 15)id \otimes \mu$ $(50, 30)(50, 15)\mu$ (2.1)

commutes.

(Un): The diagram

$$0[2](10, 30)[O1]$$

$k \otimes A$ $(50, 30)[02]A \otimes A$ $(90, 30)[O3]A \otimes k$ $(50, 10)[O4]A$ $(20, 30)(45, 30)\eta \otimes id$ $(85, 30)(65, 30)id \otimes \eta$ $(10, 27)(50, 10) \cong$ $(50, 27)(50, 10)\mu$ $(90, 27)(50, 10) \cong$ (2.2)

commutes. Note that η imbeds field k to A.

(Comm) If algebra is commutative, the triangle

$$0[2](10, 30)[O1]$$

$A \otimes A$ $(50, 30)[02]A \otimes A$ $(30, 10)[O3]A$ $(20, 30)(45, 30)\tau_{A,A}$ $(10, 27)(30, 10)\mu$ $(50, 27)(30, 10)\mu$ (2.3)

commutes. Here $\tau_{A,A}$ is the flip switching the factors: $\tau_{A,A}(a \otimes a') = a' \otimes a$.

A morphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map $A \to A'$ such that

$$\mu' \circ (f \otimes f) = f \circ \mu, \text{ and } f \circ \eta = \eta' \ .$$

A graphical representation of the algebra axioms is obtained by assigning to the field k a dashed line to be referred as a vacuum line in the sequel and to A a full line, to η a vertex $\times$ at which k-line changes to A-line. The product μ can be represented as 3-particle vertex in which algebra lines fuse together. The three axioms (Ass), (Un) and (Comm) can are expressed graphically in figure **??**.

Note that associativity axiom implies that two tree diagrams not equivalent as Feynman diagrams are equivalent in the algebraic sense.

2.2.2   Co-algebras

The definition of co-algebra is obtained by systematically reversing the directions of arrows in the previous diagrams.

A co-algebra is a triple(C, Δ, ϵ), where C is a vector space over field k and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are linear maps satisfying the following axioms (Coass) and (Coun).

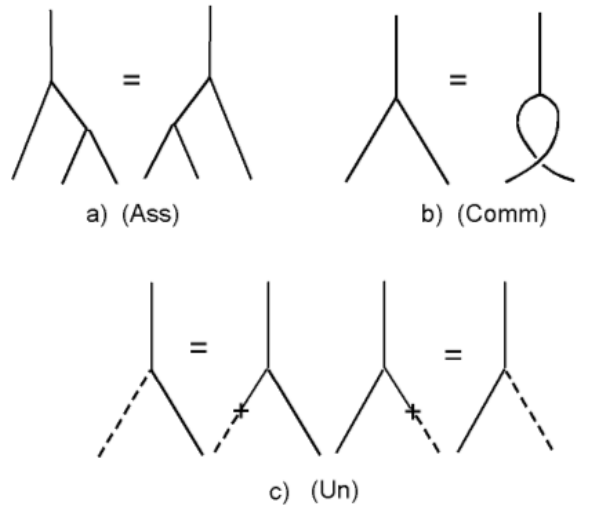(Coass): The square

$$0[2](10, 30)[O1]$$

Figure 1: Graphical representation for the axioms of algebra. a) $a(bc) = (ab)c$, b) $ab = ba$, c) $ka = \mu(\eta(k), a)$ and $ak = \mu(a, \eta(k))$.

C(60, 30)[02]C⊗C (10, 15)[O3]$C \otimes C$ (60, 15)[O4]$C \otimes C \otimes C$ (27, 30)(45, 30)Δ (27, 15)(45, 15)$\Delta \otimes id$
(10, 30)(10, 15)Δ (60, 30)(60, 15)$id \otimes \Delta$ (2.4)

commutes.

(Coun): The diagram

$$0[2](10, 30)[O1]$$

k⊗C (50, 30)[02]$C \otimes C$ (90, 30)[O3]$C \otimes k$ (50, 10)[O4]C (45, 30)(20, 30)$\epsilon \otimes id$ (65, 30)(85, 30)$id \otimes \epsilon$
(50, 10)(10, 27)$\cong$ (50, 10)(50, 27)Δ (50, 10)(90, 27)$\cong$ (2.5)

commutes. The map Δ is called co-product or co-multiplication whereas ϵ is called the counit. The commutative diagram state that the co-product is co-associative and that co-unit commutes with co-product.

(Cocomm) If co-algebra is commutative, the triangle

$$0[2](30, 30)[O1]$$

C(10, 10)[O2]C⊗C (50, 10)[03]$C \otimes C$ (30, 30)(10, 10)Δ (30, 30)(50, 10)Δ (20, 10)(45, 10)$\tau_{C,C}$ (2.6)

commutes. Here $\tau_{C,C}$ is the flip switching the factors: $\tau_{C,C}(c \otimes c') = c' \otimes c$.

A morphism of co-algebras $f : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ is a linear map $C \to C'$ such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f \ , \ \text{ and } \ \epsilon = \epsilon' \circ f \ .$$

It is straightforward to define notions like co-ideal and co-factor algebra by starting from the notions of ideal and factor algebra. A very useful notation is Sweedler's sigma notation for $\Delta(x)$, $x \in C$ as element of $C \otimes C$:

$$\Delta(x) = \sum_i x'_i \otimes x''_i \equiv \sum_{\{x\}} x' \otimes x'' \ .$$

Also co-algebra axioms allow graphical representation. One assigns to ϵ a vertex $\times$ at which C-line changes to k-line: the interpretation is as an absorption of a particle by vacuum. The co-product Δ can be represented as 3-particle vertex in which C-line decays to two C-lines. The graphical representation of the three axioms (Coass), (Coun), and (Cocomm) is related to the representation of algebra axioms by "time reversal", that is turning the diagrams for the algebra axioms upside down (see figure **??**).
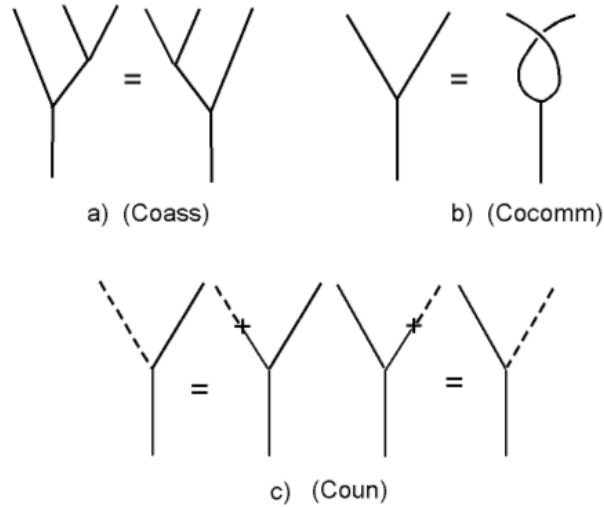
a) (Coass)   b) (Cocomm)

c) (Coun)

Figure 2: Graphical representation for the axioms of co-algebra is obtained by turning the representation for algebra axioms upside down. a) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$, b) $\Delta = \Delta^{op}$, c) $(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$.

2.2.3   Bi-algebras

Consider next a vector space H equipped simultaneously with an algebra structure (H, μ, η) and a co-algebra structure (H, Δ, ϵ). There are some compatibility conditions between these two structures. $H \otimes H$ can be given the induced structures of a tensor product of algebras and of co-algebras.

The following two statements are equivalent.

1. The maps μ and η are morphisms of co-algebras. For μ this means that the diagrams

$$0[2](10, 40)[O1]$$

H$\otimes$H $(90, 40)[02]H$ $(10, 15)[O3](H \otimes H) \otimes (H \otimes H)$ $(90, 15)[O4]H \otimes H$ $(40, 40)(60, 40)\mu$ $(40, 15)(60, 15)\mu \otimes \mu$ $(10, 40)(10, 15)(id \otimes \tau \otimes id) \otimes (\Delta \otimes \Delta)$ $(90, 40)(90, 15)\Delta$ (2.7)

and

$$0[2](10, 40)[O1]$$

H$\otimes$H $(60, 40)[02]k \otimes k$ $(10, 15)[O3]H$ $(60, 15)[O4]k$ $(20, 40)(50, 40)\epsilon \otimes \epsilon$ $(20, 15)(50, 15)\epsilon$ $(10, 40)(10, 15)\mu$ $(60, 40)(60, 15)id$ (2.8)

commute. For η this means that the diagrams

$$0[2](10, 40)[O1]$$

k(60, 40)[02]H(10, 15)[O3]k⊗k (60, 15)[O4]$H \otimes H$ (20, 40)(50, 40)η (20, 15)(50, 15)$\eta \otimes \eta$ (10, 40)(10, 15)id (60, 40)(60, 15)Δ (80, 40)[O1]k (120, 40)[02]H (100, 15)[O4]k (80, 40)(120, 40)η (80, 40)(100, 15)id (120, 40)(100, 15)ϵ (2.9)

commute.

2. The maps Δ and ϵ are morphisms of algebras.
   For Δ this means that diagrams

$$0[2](10, 40)[O1]$$

H⊗H (90, 40)[02]$(H \otimes H) \otimes (H \otimes H)$ (10, 15)[O3]H (90, 15)[O4]$H \otimes H$ (40, 40)(60, 40)$\Delta \otimes \Delta$ (40, 15)(60, 15)Δ (10, 40)(10, 15)μ (90, 40)(90, 15)$(\mu \otimes \mu)(id \otimes \tau \otimes id)$ (2.10)

and

$$0[2](10, 40)[O1]$$

k(60, 40)[02]H(10, 15)[O3]k⊗k (60, 15)[O4]$H \otimes H$ (20, 40)(50, 40)η (20, 15)(50, 15)$\eta \otimes \eta$ (10, 40)(10, 15)id (60, 40)(60, 15)Δ (2.11)

commute.

For ϵ this means that the diagrams

$$0[2](10, 40)[O1]$$

H⊗H (60, 40)[02]$k \otimes k$ (10, 15)[O3]H (60, 15)[O4]k (20, 40)(50, 40)$\epsilon \otimes \epsilon$ (20, 15)(50, 15)ϵ (10, 40)(10, 15)μ (60, 40)(60, 15)id (80, 40)[O1]k (120, 40)[02]H (100, 15)[O4]k (80, 40)(120, 40)η (80, 40)(100, 15)id (120, 40)(100, 15)ϵ (2.12)

commute. The proof of the theorem involves the comparison of the commutative diagrams expressing both statements to see that they are equivalent.

The theorem inspires the following definition.

<u>Definition</u>: A bi-algebra is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$, where (H, μ, η) is an algebra and (H, Δ, ϵ) is co-algebra satisfying the mutually equivalent conditions of the previous theorem. A morphisms of bi-algebras is a morphism for the underlying algebra and bi-algebra structures.

An element $x \in H$ is known as primitive if one has $\Delta(x) = 1 \otimes x + x \otimes 1$ and have $\epsilon(x) = 0$. The subspace of primitive elements is closed with respect to the commutator $[x, y] = xy - yx$. Note that for primitive elements $\mu \circ \Delta = 2id_H$ holds true so that $\mu/2$ acts as the left inverse of Δ.

Given a vector space V, there exists a unique bi-algebra structure on the tensor algebra $T(V)$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\epsilon(v) = 0$ for any element v of V. By the symmetry of Δ this bi-algebra structure is co-commutative and corresponds to the "classical limit". Also the Grassmann algebra associated with V allows bi-algebra structure defined in the same manner.

Figure **??** provides a representation for the axioms of bi-algebra stating that Δ and ϵ act as algebra morphisms of algebra and or equivalent that μ and η act as co-algebra morphisms. The axiom stating that Δ (μ) is algebra (co-algebra) morphism implies that scattering diagrams differing by a box loop are equivalent. The statement that μ is co-algebra morphism reads $(id \otimes \mu \otimes id)(\Delta \otimes \Delta) = \Delta \circ \mu$ whereas the mirror statement $\Delta(ab) = \Delta(a)\Delta(b)$ for Δ reads as $\Delta \circ \mu = \mu(\Delta \otimes \Delta)$ and gives rise to the same graph.
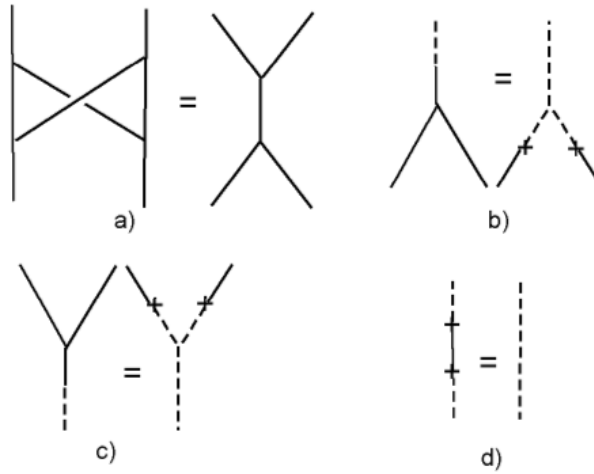
Figure 3: Graphical representation for the conditions guaranteeing that μ and η (Δ and ϵ) act as homomorphisms of co-algebra (algebra). a)$(id \otimes \mu \otimes id)(\Delta \otimes \Delta) = \Delta \circ \mu$, b) $\epsilon \circ \mu = id \circ (\epsilon \otimes \epsilon)$, c) $\Delta \circ \eta = \mu \otimes \circ id_k$, d) $\epsilon \circ \eta = id_k$.

2.2.4   Hopf algebras

Given an algebra (A, μ, η) and co-algebra (C, Δ, ϵ), one can define a bilinear map, the convolution on the vector space $Hom(C, A)$ of linear maps from C to A. By definition, if f and g are such linear maps, then the convolution $f \star g$ is the composition of the maps

$$0[2](10, 10)$$

C$(40, 10)[02]$C$\otimes$C $(70, 10)[O3]A \otimes A$ $(100, 10)[O4]A$ $(25, 10)(20, 10)\Delta$ $(57, 10)(50, 10)f \otimes g$ $(85, 10)(80, 10)\mu$ (2.13)

Using Sweedler's sigma notion one has

$$f \star g(x) = \sum_{\{x\}} f(x')g(x'') \ . \tag{2.14}$$

It can be shown that the triple $(Hom(C, A), \star, \Delta, \eta \circ \epsilon)$ is an algebra and that the map $\Lambda_{C,A} : A \otimes C^\star \to Hom(C, A)$ defined as

$$\Lambda_{C,A}(a \otimes \gamma)(c) = \gamma(c)a$$

is a morphism of algebras, where $C^\star$ is the dual of the finite-dimensional co-algebra C.

For $A = C$ the result gives a mathematical justification for the crossing symmetry inspired re-interpretation of the unitary S-matrix interpreted usually as an element of $Hom(A, A)$ as a state generated by element of $A \otimes A^\star$ from the vacuum $|vac\rangle = |vac_A\rangle \otimes |vac_{A^\star}\rangle$. This corresponds to the interpretation of the reaction $a_i|vac_A\rangle \to a_f|vac_A\rangle$ as a transition creating state $a_i \otimes a_f^\star|vac\rangle$ with vanishing conserved quantum numbers from vacuum.

With these prerequisites one can introduce the notion of Hopf algebra. Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bi-algebra. An endomorphism S of H is called an antipode for the bi-algebra H if

$$S \star id_H = id_H \star S = \eta \circ \epsilon \ .$$

A Hopf algebra is a bi-algebra with an antipode. A morphism of a Hopf algebra is a morphism between the underlying bi-algebras commuting with the antipodes.

The graphical representation of the antipode axiom is given in the figure below.
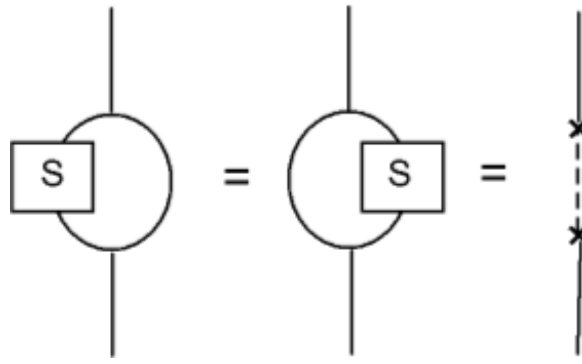
Figure 4: Graphical representation of antipode axiom $S \star id_H = id_H \star S = \eta \circ \epsilon$.

The notion of scalar product central for physical applications boils down to the notion of duality. Duality between Hopf algebras U and H means the existence of a morphism $x \to \Psi(x)$: $H \to U^\star$ defined by a bilinear form $\langle u, x \rangle = \Psi(x)(u)$ on $U \times H$, which is a bi-algebra morphism. This means that the conditions

$$\langle uv, x \rangle = \langle u \otimes v, \Delta(x) \rangle \quad , \quad \langle u, xy \rangle = \langle \Delta(u), x \otimes y \rangle \quad ,$$

$$\langle 1, x \rangle = \epsilon(x) \quad , \qquad\qquad \langle u, 1 \rangle = \epsilon(u) \quad , \tag{2.15}$$

$$\langle S(u), x \rangle = \langle u, S(x) \rangle$$

are satisfied. The first condition on multiplication and co-multiplication, when expressed graphically, states that the decay $x \to u \otimes v$ can be regarded as time reversal for the fusion of $u \otimes v \to x$. Second condition has analogous interpretation.
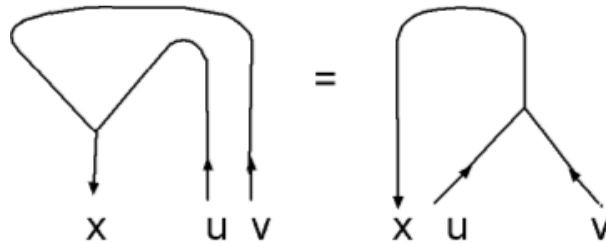

Figure 5: Graphical representation of the duality condition $\langle uv, x \rangle = \langle u \otimes v, \Delta(x) \rangle$.

2.2.5   Modules and comodules

Left and right algebra modules and algebra representations are defined in an obvious manner and satisfy associativity and unit axioms having diagrammatic representation similar to that for corresponding algebra axioms.

A left co-module corresponds a pair (V, Δ_V), where the co-action Δ_N: $V \to C \otimes V$ satisfies co-associativity axiom $(id_C \otimes \Delta_N) \circ \Delta_N = (\Delta \otimes id_N) \circ \Delta_N$ and co-unit axiom $(\epsilon \otimes id) \circ \Delta_N = id_N$. A right co-module is defined in an analogous manner. It is convenient to introduce Sweedlers's notation for Δ_N as $\Delta_N = \sum_{\{c\}} x_C \otimes x_N$.

One can define module and comodule morphisms and tensor product of modules and co-modules in a rather obvious manner. The module N could be also algebra, call it A, in which case μ_A and η_A are assumed to act as H-comodule morphisms.

The standard example is quantum plane $A = M(2)_q$ is the free algebra generated variables x, y subject to relations $yx = qxy$ and having coefficients in k. The action of Δ_A reads as

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \ .$$

Δ_A defines algebra morphism from A to $SL_e(2)_q \otimes A$: $\Delta_a(yx) = \Delta_A(y)\Delta_A(x) = q\Delta_A(x)\Delta_A(y) = \Delta(qxy)$.

2.2.6   Braided bi-algebras

$\Delta^{op} = \tau_{H,H} \circ \Delta$ defines the opposite co-algebra H^{op} of H. A braided bi-algebra $(H, \mu, \eta, \Delta, \epsilon)$ is called quasi-co-commutative (or quasi-triangular) if there exists an element R of algebra $H \otimes H$ such that for all $x \in H$ one has

$$\Delta^{op} = R\Delta R^{-1} \ .$$

One can express R in the form

$$R = \sum_i s_i \otimes t_i \ .$$

It is convenient to denote by R_{ij} the R matrix acting in i^{th} and j^{th} tensor factors of nth tensor power of H. More precisely, R_{ij} can be defined as an operator acting in an n-fold tensor power of H by the formula $R_{ij} = y^{(1)} \otimes y^{(2)} \otimes ... \otimes y^{(p)}$, $p \leq n$, $y^{(k_i)} = s_i$ and $y^{(k_j)} = t_j$, $y^{(k)} = 1$ otherwise. For instance, one has $R_{13} = \sum_i s_i \otimes 1 \otimes t_i$.

With these prerequisites one can define a braided bi-algebra as a quasi-commutative bi-algebra $(H, \mu, \eta, \Delta, \epsilon, S, S^{-1}, R)$ as an algebra with a preferred element $R \in H \otimes H$ satisfying the two relations

$$\begin{aligned} (\Delta \otimes id_H)(R) &= R_{13}R_{23} \ , \\ (id_H \otimes \Delta)(R) &= R_{13}R_{12} \ . \end{aligned}$$

$$(2.16)$$

Braided bi-algebras, known also as quasi-triangular bi-algebras, are central in the theory of quantum groups, R-matrices, and braid groups. By a direct calculations one can verify the following relations.

1. Yang-Baxter equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ , \tag{2.17}$$

    and the relation

$$(\epsilon \otimes id_H)(R) = 1 \tag{2.18}$$

    hold true.

2. Since H has an invertible antipode S, one has

$$\begin{aligned} (S \otimes id_H)(R) &= R^{-1} = (id_H \otimes S^{-1})(R) \ , \\ (S \otimes S)(R) &= R \ . \end{aligned} \tag{2.19}$$

The graphical representation of the Yang-Baxter equation in terms of the relations of braid group generators is given in the figure **??**.
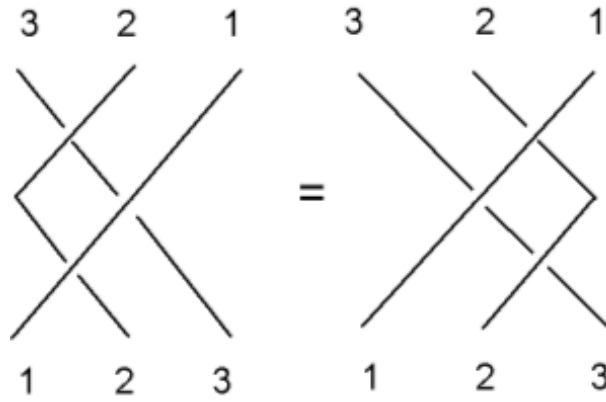
Figure 6: Graphical representation of Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

2.2.7   Ribbon algebras

Let H be a braided Hopf algebra with a universal matrix $R = \sum_i s_i \otimes t_i$ and set $u = \sum_i S(t_i)s_i$. It can be shown that u is invertible with the inverse $u^{-1} = \sum s_i S^2(t_i)$ and that $uS(u) = S(u)u$ is central element in H. Furthermore, one has $\epsilon(u) = 1$ and $\Delta(u) = (R_{21}R)^{-1}(u \otimes u)$, and the antipode is given for any $x \in H$ by $S^2(x) = uxu^{-1}$.

Ribbon algebra has besides $R \in H \otimes H$ also a second preferred element called θ. A braided Hopf algebra is called ribbon algebra if there exists a central element θ of H satisfying the relations

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta) \ , \quad \epsilon(\theta) = 1 \ , \quad S(\theta) = \theta \ . \tag{2.20}$$

It can be shown that θ^2 acts like $S(u)u$ on any finite-dimensional module [A3].

2.2.8   Drinfeld's quantum double

Drinfeld's quantum double construction allows to build a quasi-triangular Hopf algebra by starting from any Hopf algebra H and its dual $H^\star$, which exists in a finite-dimensional case always, and as a vector space is isomorphic with H. Besides duality normal ordering is second ingredient of the construction. Physically the generators of the algebra and its dual correspond to creation and annihilation operator type operators. Drinfeld's quantum double construction is represented in a very general manner in [A3]. A construction easier to understand by a physicist is discussed in [A6]. For this reason this representation is summarized here although the style differs from the representation of [A3] followed in the other parts of appendices.

Consider first what is known.

1. Duality means the existence of basis $\{e_a\}$ for H and $\{e^a\}$ for $H^\star$ and inner product (or evaluation as it is called in [A3]) $ev : H^\star \otimes H \to k$ defined as $ev(e^a e_b) \equiv \langle e^a, e_b \rangle = \delta^a_b$ and its inverse $\delta : k \to H^\star \otimes H$ defined by $\delta(1) = e^a e_a$. One can extend the inner product to an inner product in the tensor product $(H^\star \otimes H^\star) \otimes (H \otimes H)$ in an obvious manner.

2. The product (co-product) in H ($H^\star$) coincides with the co-product (product) in $H^\star$ (H) in the sense that one has

$$\begin{aligned} \langle e^c, e_a e_b \rangle = m^c_{ab} = \langle \Delta(e^c), e_b \otimes e_a \rangle \ , \\ \langle e^a e^b, e_c \rangle = \mu^{ab}_c = \langle e^a \otimes e^b, \Delta(e_c) \rangle \ , \end{aligned} \tag{2.21}$$

These equations are quite general expressions for the duality expressed graphically in figure ??.

3. The antipodes S for H and $H^\star$ can be represented as matrices

$$S_H(e_a) = S_a{}^b e_b \ , \quad S_{H^\star}(e^a) = (S^{-1})^a{}_b e^b \ . \tag{2.22}$$

The task is to construct algebra product μ and co-algebra product Δ, unit η and co-unit ϵ, antipode, and R-matrix R for for $H \otimes H^\star$. The natural basis for $H \otimes H^\star$ consists of $e_a \otimes e^b$.

1. Co-product Δ is simply the product of co-products

$$\Delta(e_a e^b) = \Delta(e_a)\Delta(e^b) = m^b_{vu}\mu^{cd}_a e_c e^u \otimes e_d e^v \ . \tag{2.23}$$

2. Product μ involves normal ordering prescription allowing to transform products $e^a e_b$ (elements of $H^\star \otimes H$) to combinations of basis elements $e_a e^b$ (elements of $H \otimes H^\star$. This map must be consistent with the requirement that co-product acts as an algebra morphism. Drinfeld's normal ordering prescription, or rather a map $c_{H^\star,H}\colon H^\star \otimes H \to H \otimes H^\star$ is given by

$$c_{H^\star,H}(e^a e_b) = R^{ac}_{bd} e_c e^d \ , \quad R^{ac}_{bd} = m^x_{kd} m^a_{xu} \mu^{vy}_b \mu^{ck}_y (S^{-1})^u_v e_c e^d \ . \tag{2.24}$$

The details of the formula are far from being obvious: the axioms of tensor category with duality to be discussed later might allow to relate $R_{H^\star,H}$ to $R_{H,H}$ and this might help to understand the origin of the expression. Normal ordering map can be interpreted as braid operation exchanging H and $H^\star$ and the matrix defining the map could be regarded as R-matrix $R_{H \otimes H^\star}$.

3. The universal R-matrix is given by

$$R \ = \ (e_a \otimes id_{H^\star}) \otimes (id_H \otimes e^a) \ , \tag{2.25}$$

where the summation convention is applied. One can show that $R\Delta = \Delta^{op}R$ by a direct calculation.

4. The antipode $S_{H \otimes H^\star}$ follows from the product of antipodes for H and $H^\star$ using the fact that antipode is antihomomorphism using the normal ordering prescription

$$S_{H \otimes H^\star}(e_a e^b) = c_{H^\star,H}(S(e^b)S(e_a)) \ . \tag{2.26}$$

2.2.9   Quasi-Hopf algebras and Drinfeld associator

Braided Hopf algebras are quasi-commutative in the sense that one has $\Delta^{op} = R\Delta R^{-1}$. Also the strict co-associativity can be given up and this means that one has

$$(\Delta \otimes id)\Delta = \Phi(id \otimes \Delta)\Phi^{-1} \ , \tag{2.27}$$

where $\Phi \in H \otimes H \otimes H$ is known as Drinfeld's associator and appears in the of conformal fields theories. If the resulting structure satisfies also the so called Pentagon Axiom (to be discussed later, see Eq. 2.3.2 and figure **??**), it is called quasi-Hopf algebra. Pentagon Axiom boils down to the condition

$$(id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi) = (id \otimes \Phi)(id \otimes id \otimes \Delta)(\Phi)(\Phi \otimes id) \ . \tag{2.28}$$

The Yang-Baxter equation for quasi-Hopf algebra reads as

$$R_{12}\Phi_{312}R_{13}\Phi_{1322}^{-1}R_{23}\Phi_{123} = \Phi_{321}R_{23}\Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi_{123} \ \ . \tag{2.29}$$

The left-hand side arises from a sequence of transformations

$$0[2]$$

(10, 30)(12)3 (33, 30)1(23) (56, 30)1(32) (79, 30)(31)2 (102, 30)3(12) (125, 30)3(12) (148, 30)3(21)

(12, 30)(32, 30)Φ_{123} (34, 30)(54, 30)R_{23} (58, 30)(77, 30)Φ_{132}^{-1} (81, 30)(100, 30)R_{13} (104, 30)(123, 30)Φ_{312} (127, 30)(146, 30)R_{12} .

The right-hand side arises from the sequence

$$0[2]$$

(10, 30)(12)3 (33, 30)(21)3 (56, 30)2(13) (79, 30)2(31) (102, 30)(23)1 (125, 30)(32)1 (148, 30)3(21)

(12, 30)(32, 30)R_{12} (34, 30)(54, 30)Φ_{213} (58, 30)(77, 30)R_{13} (81, 30)(100, 30)Φ_{231}^{-1} (104, 30)(123, 30)R_{23} (127, 30)(146, 30)Φ_{321} .

One can produce new quasi-Hopf algebras by gauge (or twist) transformations using invertible element $\Omega \in H \otimes H$ called twist operator

$$\begin{aligned}
\Delta(a) &\rightarrow \Omega\Delta(a)\Omega^{-1} \ , \\
\Phi &\rightarrow \Omega_{23}(id \otimes \Delta)(\Omega)\Phi(\Delta \otimes id)(\Omega^{-1})\Omega_{12}^{-1} \ , \\
R &\rightarrow \Omega R\Omega^{-1} \ .
\end{aligned} \tag{2.32}$$

Quasi-Hopf algebras appear in conformal field theories and correspond quantum universal enveloping algebras divided by their centralizer. Consider as an example the R-matrix R^{j_1,j_2} relating $j_1 \otimes j_2$ and $j_2 \otimes j_1$ representations $\Delta^{j_1,j_2}(a)$ and $\Delta^{j_2,j_1}(a)$ of the co-product Δ of $U(sl(2))_q$. $\Delta^{j,j}(a)$ commutes with R^{jj} for all elements of the quantum group. The action of $g_i = qR^{jj}$ acting in i^{th} and $(i+1)^{th}$ tensor factors extends to the representation $(V_j)^{\times n}$ in an obvious manner. From the Yang-Baxter equation it follows that the operators g_i define a representation of braid group B_n:

$$\begin{aligned}
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \ , \\
g_i g_j &= g_j g_i \ , \ \text{for } |j - k| \geq 2 \ .
\end{aligned} \tag{2.33}$$

Under certain conditions the braid group generators generate the whole centralizer C_q^n for the representation of quantum group. For instance, this occurs for $j = 1/2$. In this case the additional condition

$$g_i^2 = (q^2 - 1)g_i + q^2 \times 1 \ , \tag{2.34}$$

so that the centralizer is isomorphic with the Hecke algebra $H_n(q)$, which can be regarded as a q-deformation of permutation group S_n.

The result generalizes. In Wess-Zumino-Witten model based on group G the relevant algebraic structure is $U(G_q)/C^n(q)$. This is quasi-Hopf algebra and the so called Drinfeld associator characterizes the quasi-associativity.

2.3 Tensor Categories

Hopf algebras and related structures do not seem to be quite enough in order to formulate elegantly the construction of S-matrix in TGD framework. A more general structure known as a braided tensor category with left duality and twist operation making the category to a ribbon category is needed. The algebra product μ and co-product Δ must be generalized so that they appear as morphisms $\mu_{A \otimes B \to C}$ and $\Delta_{A \to B \otimes C}$: this gives hopes of describing 3-vertices algebraically. It is not clear whether one can assume single underlying algebra so that objects would correspond to different representations of this algebra or whether one allow even non-isomorphic algebras.

In the tensor category the tensor products of objects and corresponding morphisms belong to the category. In a braided category the objects $U \otimes V$ and $V \otimes U$ are related by a braiding morphism. The notion of braided tensor category appears naturally in topological and conformal quantum field theories and seems to be an appropriate tool also in TGD context. The basic category theoretical notions are discussed in [A3] and I have already earlier considered category theory as a possible tool in the construction of quantum TGD and TGD inspired theory of consciousness [K1].

In braided tensor categories one introduces the braiding morphism $c_{V,W} : V \otimes W \to W \otimes V$, which is closely related to R-matrix. In categories allowing duality arrows with both directions are allowed ad diagrams analogous to pair creation from vacuum are possible. In ribbon categories one introduces also the twist operation θ_V as a morphism of object and the Θ_W satisfies the axiom: $\theta_{V \otimes W} = (\theta_V \otimes \theta_W)c_{W,V}c_{V,W}$. One can also introduce morphisms with arbitrary number of incoming lines and outgoing lines and visualize them as boxes, coupons. Isotopy principle, originally related to link and knot diagrams provides a powerful tool allowing to interpret the basic axioms of ribbon categories in terms of isotopy invariance of the diagrams and to invent theorems by just isotoping.

2.3.1 Categories, functors, natural transformations

Categories [A3, A2, A7, A8] are roughly collections of objects A, B, C... and morphisms $f(A \to B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Examples of categories are open sets of some topological spaces with continuous maps between them acting as morphisms, linear spaces with linear maps between them acting as morphisms, groups with group homomorphisms taking the role of morphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can be very general: for instance, partial ordering $a \leq b$ can define a morphism $f(A \to B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Functor $F : \mathcal{C} \to \mathcal{D}$ commutes also with the maps s and b assigning to a morphism $f : V \to W$ its source $s(f) = V$ and target $b(f) = W$.

A natural transformation between functors F and G from $\mathcal{C} \to \mathcal{C}'$ is a family of morphisms $\eta(V) : F(V) \to G(V)$ in $\mathcal{C}'$ indexed by objects V of $\mathcal{C}$ such that for any morphisms $f : V \to W$ in $\mathcal{C}$, the square

$$0[2](10, 30)[O1]$$

F(V)(50, 30)[02]G(V)(10, 15)[O3]F(W)(50, 15)[O4]G(W)(25, 30)(45, 30)$\eta(V)$ (25, 15)(45, 15)$\eta(W)$ (10, 30)(10, 15)$F(f)$ (50, 30)(50, 15)$G(f)$ (2.35)

commutes.

The functor $F : \mathcal{C} \to \mathcal{D}$ is said to be equivalence of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such and natural isomorphisms

$$\eta : id_{\mathcal{D}} \to FG \quad \text{and} \quad \theta : GF \to id_{\mathcal{C}} FG \ .$$

The notion of adjoint functor is a more general notion than equivalence of categories. In this case η and θ are natural transformations but not necessary natural isomorphisms in such a way that the composite maps

$$0[2](10, 30)$$

$\mathrm{F}(V)(60, 30)(\mathrm{FGF})(V)(110, 30)\mathrm{F}(V)(10, 10)\mathrm{G}(W)(60, 10)(\mathrm{GFG})(W)(110, 10)\mathrm{G}(W)(20, 30)(40, 30)\eta(F(V))$
$(80, 30)(100, 30)F(\theta(V))\ (20, 10)(40, 10)G(\eta(W))\ (80, 10)(100, 10)\theta(G(W)))\ (2.36)$

are identify morphisms for all objects V in $\mathcal{C}$ and W in $\mathcal{D}$.

The product $C = AB$ for objects of categories is defined by the requirement that there exist projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \to A)$ and $g(D \to B)$ there exist morphism $h(D \to C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically this corresponds to a square diagram in which pairs A, B and C, D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC. Examples of product categories are Cartesian products of topological spaces, linear spaces, differentiable manifolds, groups, etc. The tensor products of linear spaces and algebras provides an especially interesting example of product in the recent case. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

2.3.2   Tensor categories

Let $\mathcal{C}$ be a category. Tensor product $\otimes$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ if

1. there is an object $V \otimes W$ associated with any pair (V, W) of objects of $\mathcal{C}$

2. there is an morphism $f \otimes g$ associated with any pair (f, g) of morphisms of $\mathcal{C}$ such that
   $s(f \otimes g) = s(f) \otimes s(g)$ and $b(f \otimes g) = b(f) \otimes b(g)$,

3. if f' and g' are morphisms such that $s(f') = b(f)$ and $s(g') = b(g)$ then
   $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$,

4. $id_{V \otimes W} = id_{W \otimes V}$.

   Any functor with these properties is called tensor product. The tensor product of vector spaces provides the most familiar example of a tensor product functor.

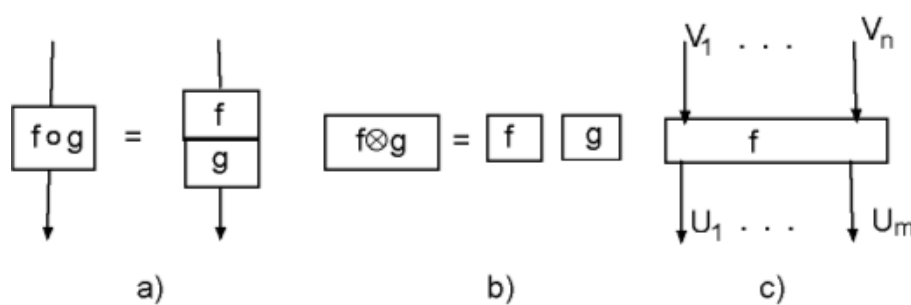   In figure **??** the general rules for graphical representations of morphisms are given.

Figure 7: The graphical representation of morphisms. a) $g \circ f \colon V \to W$, b) $f \otimes g$, c) $f \colon U_1 \otimes ... \otimes U_m \to V_1 \otimes .... \otimes V_n$.

An associativity constraint for the tensor product is a natural isomorphism

$$a : \otimes(\otimes \times id) \to \otimes(id \times \otimes) \ \ .$$

On basis of general definition of natural isomorphisms (see Eq. 2.3.1) one can conclude that for any triple (U, V, W) of objects of $\mathcal{C}$ there exists an isomorphism

$$0[2](10, 40)[O1]$$

$(U \otimes V) \otimes W$ $(80, 40)[02]U \otimes (V \otimes W)$ $(10, 10)[O3](U' \otimes V') \otimes W'$ $(100, 10)[O4]U' \otimes (V' \otimes W')$ $(30, 40)(45, 40)a_{U,V,W}$ $(30, 10)(55, 10)a_{U',V\ prime,W'}$ $(10, 40)(10, 10)(f \otimes g) \otimes h$ $(80, 40)(80, 10)f \otimes (g \otimes h)$ (2.37)

Associativity constraints satisfies Pentagon Axiom [A3] if the following diagrams commutes.

$$0[2](10, 70)[O1]$$

$U \otimes (V \otimes W) \otimes X$ $(100, 70)[02]((U \otimes V) \otimes W)) \otimes X$ $(10, 10)[O3]U \otimes ((V \otimes W) \otimes X)$ $(100, 10)[O4]U \otimes (V \otimes (W \otimes X))$ $(100, 40)(U \otimes V) \otimes (W \otimes X)$ $(70, 70)(50, 70)a_{U,V,W} \otimes id_X$ $(50, 10)(70, 10)id_U \otimes a_{V,W,X}$ $(10, 70)(10, 10)a_{U,V \otimes W,X}$ $(100, 70)(100, 40)a_{U \otimes V,W,X}$ $(100, 40)(100, 10)a_{U,V,W \otimes X}$ (2.38)

Pentagon axiom has been already mentioned while discussing the definition of quasi-Hopf algebras. In figure **??** are graphical illustrations of associativity morphism $a(U, V, W)$, Triangle Axiom, and Pentagon Axiom are given.

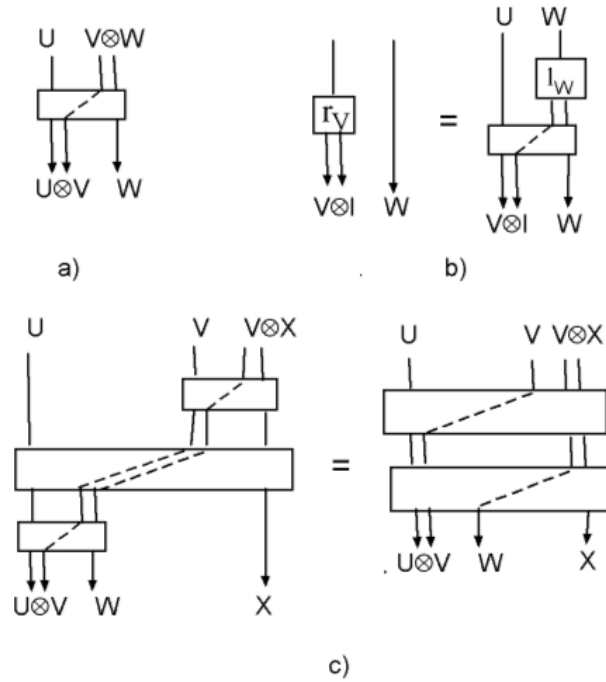**Figure 8:** Graphical representations of a) the associativity isomorphism $a_{U,V,W}$, b) Triangle Axiom, c) Pentagon Axiom.

Assume that an object I is fixed in the category. A left unit constraint with respect to I is a natural isomorphism

$$l : \otimes (I \times id) \to id$$

By Eq. 2.3.1 this means that for any object V of C there exists an isomorphism

$$l_V : I \otimes V \to V \tag{2.39}$$

such that

$$0[2](10,30)[O1]$$

I$\otimes$V (50, 30)[02]V (10, 15)[O3]I $\otimes$ V' (50, 15)[O4]V' (25, 30)(45, 30)l_V (25, 15)(45, 15)$l_{V'}$ (10, 30)(10, 15)$id_I \otimes f$ (50, 30)(50, 15)f (2.40)

The right unit constraint $r : \otimes(id \times I) \to id$ can be defined in a completely analogous manner.

Given an associativity constraint a, and left and right unit constraints l, r with respect to an object I, one can say that the Triangle Axiom is satisfies if the triangle

$$0[2](10,30)[O1]$$

(V$\otimes$I)$\otimes$W (70, 30)[02]V $\otimes$ (I $\otimes$ W) (40, 10)[O3]V $\otimes$ W (30, 30)(50, 30)$a_{V,I,W}$ (10, 27)(40, 12)$r_V \otimes id_W$ (70, 27)(40, 12)$id_W \otimes l_W$ (2.41)

commutes (see figure **??**).

These ingredients lead allow to define tensor category $(\mathcal{C}, I, a, l, r)$ as a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying associativity constraint a, left unit constraint l and right unit constraint r with respect to I, such that Pentagon Axiom and Triangle Axiom are satisfied.

The definition of a tensor functor $F : \mathcal{C} \to \mathcal{D}$ involves also additional isomorphisms. $\phi_0 : I \to F(I)$ satisfies commutative diagrams involving right and left unit constraints l and r. The family of isomorphisms

$$\phi_2(U, V) : F(U) \otimes F(V) \to F(U \otimes V)$$

satisfies a commutative diagram stating that ϕ_2 commutes with associativity constraints. The interested reader can consult [A3] for details. One can also define the notions of natural tensor transformation, natural tensor isomorphism, and tensor equivalence between tensor categories by applying the general category theoretical tools.

Keeping track of associativity isomorphisms is obviously a rather heavy burden. Fortunately, it can be shown that one can assign to a tensor category $\mathcal{C}$ a strictly associative (or briefly, strict) tensor category which is tensor equivalent of $\mathcal{C}$.

2.3.3   Braided tensor categories

Braided tensor categories satisfy also commutativity constraint c besides associativity constraint a. Denote by $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the flip functor defined by $\tau(V, W) = (W, V)$. Commutativity constraint is a natural isomorphism

$$c : \otimes \to \otimes\tau \ .$$

This means that for any pair (V, W) of objects there exists isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V$$

such that the square

$$0[2](10,30)[O1]$$

V$\otimes$W (60, 30)[02]W $\otimes$ V (10, 15)[O3]V'$\otimes$W' (60, 15)[O4]W'$\otimes$V' (25, 30)(45, 30)$c_{V,W}$ (25, 15)(45, 15)$c_{V',W'}$ (10, 30)(10, 15)$f \otimes g$ (60, 30)(60, 15)$g \otimes f$ (2.42)

commutes.

The commutativity constraint satisfies Hexagon Axiom if the two hexagonal diagrams

(H1)

$$0[2](10, 70)$$

$\mathrm{U} \otimes (V \otimes W)$ (100, 70)$(V \otimes W) \otimes U$ (10, 40)$(U \otimes V) \otimes W$ (100, 40)$V \otimes (W \otimes U)$ (10, 10)$(V \otimes U) \otimes W$ (100, 10)$V \otimes (U \otimes W)$ (30, 70)(80, 70)$c_{U,V \otimes W}$ (30, 10)(80, 10)$a_{V,U,W}$ (10, 40)(20, 70)$a_{U,V,W}$ (90, 70)(100, 40)$a_{V,W,U}$ (10, 40)(20, 10)$c_{U,V} \otimes id_W$ (90, 10)(100, 40)$id_V \otimes c_{U,W}$ (2.43)

and (H2)

$$0[2](10, 70)$$

$(\mathrm{U} \otimes V) \otimes W$ (100, 70)$W \otimes (U \otimes V)$ (10, 40)$U \otimes (V \otimes W)$ (100, 40)$(W \otimes U) \otimes V$ (10, 10)$U \otimes (W \otimes V)$ (100, 10)$(U \otimes W) \otimes V$ (30, 70)(80, 70)$c_{U \otimes V, W}$ (30, 10)(80, 10)$a_{U,W,V}^{-1}$ (10, 40)(20, 70)$a_{U,V,W}^{-1}$ (90, 70)(100, 40)$a_{W,U,V}^{-1}$ (10, 40)(20, 10)$id_U \otimes c_{V,W}$ (90, 10)(100, 40)$c_{U,W} \otimes id_V$ (2.44)

commute.

The braiding operation $c_{V,W}$ and the association operation $a(U,V,W)$, and pentagon and hexagon axioms are illustrated in the figure **??** below.
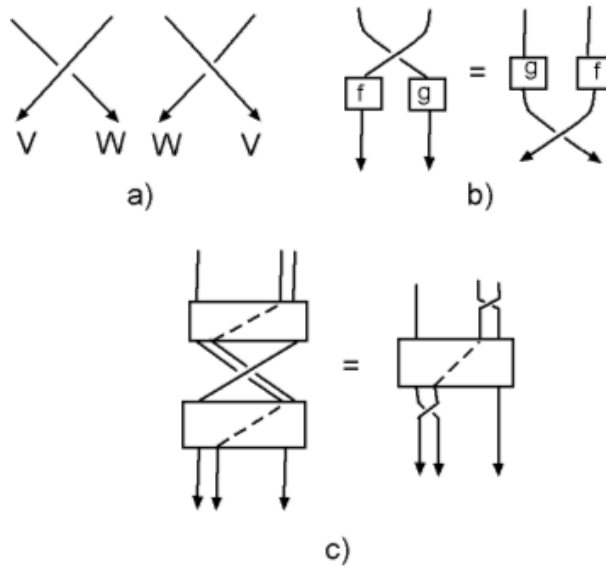
Figure 9: Graphical representations a) of the braiding morphism $c_{V,W}$ and its inverse $c_{V,W}^{-1}$, b) of naturality of $c_{V,W}$, c) of First Hexagon Axiom.

2.3.4   Duality and tensor categories

The notion of a dual of the finite-dimensional vector space as a space of linear maps from V to field k can lifted to a concept applying for tensor category. A strict (strictly associative) tensor category $(\mathcal{C}, \otimes, I)$ with unit object I is said to possess left duality if for each object V of $\mathcal{C}$ there exists an object $V^\star$ and morphisms

$$b_V : I \to V \otimes V^\star \quad \text{and} \quad d_V : V^\star \otimes V \to I$$

such that

$$(id \otimes d_V)(b_V \otimes id_V) = id_V \quad and \quad (d_V \otimes id_{V^\star})(id_{V^\star} \otimes b_V) = id_{V^\star} \quad . \tag{2.45}$$

One can define the transpose of f in terms of b_V and d_V. The idea how this is achieved is obvious from **Fig. ??**.

$$f^\star = (d_V \otimes id_{U^\star})(id_{V^\star} \otimes f \otimes id_{U^\star})(id_{V^\star} \otimes b_U) \ . \tag{2.46}$$

Also the braiding operation $c_{V^\star,W}$ can be expressed in terms of $c_{V,W}^{-1}$, b_V and d_V by using the isotopy of **Fig. 10**:

$$c_{V^\star,W} = (d_V \otimes id_{W \otimes V^\star})(id_{V^\star} \otimes c_{V,W}^{-1} \otimes id_{V^\star})(id_{V^\star \otimes W} \otimes b_V) \ . \tag{2.47}$$

Drinfeld quantum double can be regarded as a tensor product of Hopf algebra and its dual and in this case one can introduce morphisms $ev_H : H \otimes H^\star \to k$ defined as $e^i \otimes e_j \to \delta_j^i$ defining inner product and its inverse $\delta : k \to H \otimes H$ defined as $1 \to e^i e_i$, where summation over i is understood. For categories these morphisms are generalized to morphism d_V from objects V of category to unit object I and b_V from I to object of category. The elements of H and $H^\star$ are described as strands with opposite directions, whereas d_V and b_V correspond to annihilation and creation of strand–anti-strand pair as show in **Fig. 10**.
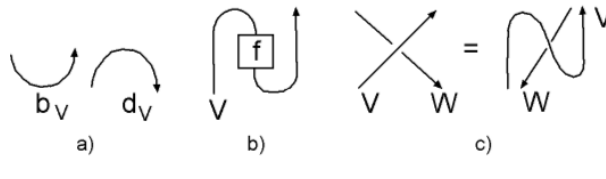

Figure 10: Graphical representations a) of the morphisms b_V and d_V, b) of the transpose $f^\star$, c) of braiding operation $c_{V^\star,W}$ expressed in terms of $c_{V,W}$.

2.3.5 Ribbon categories

According to the definition of [A3] ribbon category is a strict braided tensor category $(\mathcal{C}, \otimes, I)$ with a left duality with a family of natural morphisms $\theta_V : V \to V$ indexed by the objects V of $\mathcal{C}$ satisfying the conditions

$$\begin{aligned} \theta_{V \otimes W} &= \theta_V \otimes \theta_W c_{W,V} c_{V,W} \ , \\ \theta_{V^\star} &= (\theta_V)^\star \end{aligned} \tag{2.48}$$

for all objects V, W of $\mathcal{C}$. The naturality of twist means for for any morphisms $f : V \to W$ one has $\theta_W f = f \theta_V$. The graphical representation for the axioms and is in **Fig. 11**.
The existence of the twist operation provides $\mathcal{C}$ with right duality necessary in order to define trace (see **Fig. ??**).

$$\begin{aligned} d_V' &= (id_{V^\star} \otimes \theta_V) c_{V,V^\star} b_V \ , \\ b_V' &= d_V c_{V,V^\star} (\theta_V \otimes id_{V^\star}) \ . \end{aligned} \tag{2.49}$$

One can define quantum trace for any endomorphisms f of ribbon category:

$$tr_q(f) = d_V'(f \otimes id_{V^\star}) b_V = d_V c_{V,V^\star} (\theta_V f \otimes id_{V^\star}) b_V \ . \tag{2.50}$$

Again the graphical representation is the best manner to understand the definition, see **Fig. ??**. Quantum trace has the basic properties of trace: $tr_q(fg) = tr_q(gf)$, $tr_q(f \otimes g) = tr_q(f) tr_q(g)$, $tr_q(f) = tr_q(f^\star)$. The proof of these properties is easiest using isotropy principle.
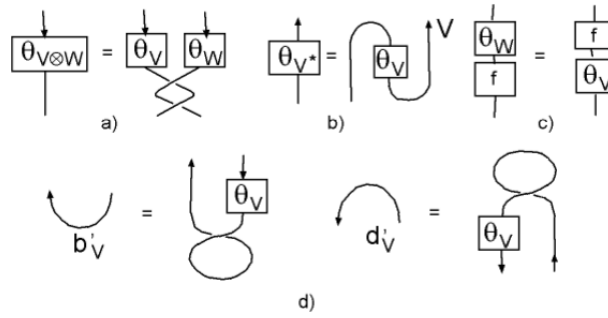
Figure 11: Graphical representations a) of $\theta_{V \otimes W} = \theta_V \otimes \theta_W c_{W,V} c_{V,W}$, b) of $\theta_{V^\star} = (\theta_V)^\star$, c) of $\theta_W f = f \theta_V$, d) of right duality for a ribbon category.
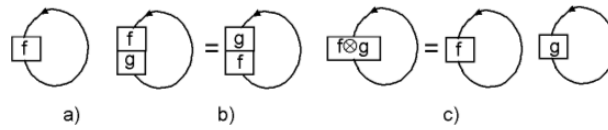

Figure 12: Graphical representations of a) $tr_q(f)$, b) of $tr_q(fg) = tr_q(gf)$, c) of $tr(f \otimes g) = tr(f)tr(g)$.

The quantum dimension of an object V of ribbon category can be defined as the quantum trace for the identity morphism of V: $dim_q(V) = tr_q(id_V) = d'_V b_V$. Quantum dimension is represented as a vacuum bubble. Quantum dimension satisfies the conditions $dim_q(V \otimes W) = dim_q(V)dim_q(W)$ and $dim_q(V) = dim_q(V^\star)$.

A more general definition of ribbon category inspired by the considerations of [A6] is obtained by allowing the generalization of morphisms μ and Δ so that they become morphisms $\mu_{A \otimes B \to C}$ and $\Delta_{C \to A \otimes B}$ of ribbon category. Graphically the general morphism with arbitrary number of incoming outgoing strands can be represented as a box or "coupon". An important special case of ribbon categories consists of modules over braided Hopf algebras allowing ribbon algebra structure.

3 Axiomatic Approach To S-Matrix Based On The Notion Of Quantum Category

This section can be regarded as an attempt of a physicists with some good intuitions and intentions but rather poor algebraic skills to formulate basic axioms about S-matrix in terms of what might be called quantum category. The basic result is an interpretation for the equivalence of loop diagrams with tree diagrams as a consequence of basic algebra and co-algebra axioms generalized to the level of tensor category. The notion of quantum category emerges naturally as a generalization of ribbon category, when algebra product and co-algebra product are interpreted as morphisms between different objects of the ribbon category.

The general picture suggest that the operations Δ and μ generalized to algebra homomorphisms $A \to B \otimes C$ and $A \otimes B \to C$ in a tensor category whose objects are either representations of an algebra or even algebras might provide an appropriate mathematical tool for saying something interesting about S-matrix in TGD Universe. These algebras need not necessarily be bi-algebras. In the following it is demonstrated that the equivalence of loop diagrams to tree diagrams follows from suitably generalized bi-algebra axioms. Also the interpretation of various morphisms involved with Hopf algebra structure is discussed.

3.1   Δ And μ And The Axioms Eliminating Loops

The first task is to find a physical interpretation for the basic algebraic operations and how the basic algebra axioms might allow to eliminate loops. The physical interpretation of morphisms Δ and μ as algebra or category morphisms has been already discussed. As already found, the condition that Δ (μ) acts as an algebra (co-algebra) morphism leads to a condition stating that a box graph for 2-particle scattering is equivalent with tree graph. It is interesting to identify the corresponding conditions in the case of self energy loops and vertex corrections.

The condition

$$\mu_{B\otimes C\to A} \circ \Delta_{A\to B\otimes C} \;\; = \;\; K \times id_A \;\; , \tag{3.1}$$

where K is a numerical factor, is a natural additional condition stating that a line with a self energy loop is equivalent with a line without the loop. The condition is illustrate in figure **??**. For the co-commutative tensor algebra $T(V)$ of vector space with $\Delta(x) = 1 \otimes x + x \otimes 1$ one would have $K = 2$ for the generators of $T(V)$. For a product of n generators one has $K = 2^n$.
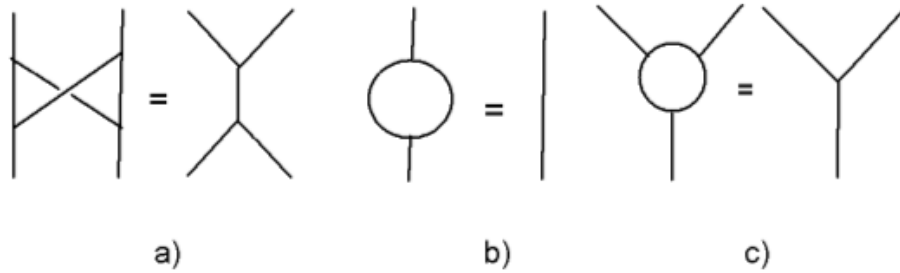

Figure 13: Graphical representations for the conditions a) $(id \otimes \mu \otimes id)(\Delta \otimes \Delta) = \Delta \circ \mu$, b) $\mu_{B\otimes C\to A} \circ \Delta_{A\to B\otimes C} = K \times id_A$, and c) $(\mu \otimes id) \circ (\Delta \otimes id) \circ \Delta = K \times \Delta$.

The condition $\Delta_{A\to B\otimes C} \circ \mu_{B\otimes C\to A} = K \times id_A$ cannot hold true since multiplication is not an irreversible process. If this were the case one could reduce tree diagrams to collections of free propagator lines.

In quantum field theories also vertex corrections are a source of divergences. The requirement that the graph representing a vertex correction is equivalent with a simple tree graph representing a decay gives an additional algebraic condition. For bi-algebras the condition would read

$$(\mu \otimes id) \circ (\Delta \otimes id) \circ \Delta = K\Delta \;\; , \tag{3.2}$$

where K is a simple multiplicative factor. In fact, for the co-commutative tensor algebra $T(V)$ of vector space the left hand side would be $3 \times \Delta(x)$ giving $K = 3$ for generators $T(V)$. The condition is illustrated in figure **??**.

Using the standard formulas of appendix for quantum groups one finds that in the case of $U_q(sl(2))$ the condition $\mu \circ \Delta(X) = K_X X$, K_X constant, is not true in general. Rather, one has $\mu \circ \Delta(X) = X K_X(q^{H/2} + q^{-H/2}, q^{1/2}, q^{-1/2})$. The action on the vacuum state is however proportional to that of X, being given by $K_X(2,1,1)X$. The function K_X for a given X can be deduced from $\mu \circ \Delta(X_\pm) = q^{H/2}X_\pm + X_\pm q^{-H/2} = X_\pm(q^{\pm 1/2} + q^{H/2} + q^{-H/2})$. The eigen states of Cartan algebra generators are expected to be eigen states of $\mu \circ \Delta$ also in the case of a general quantum group. $\mu \circ \Delta$ is analogous to a single particle operator like kinetic energy and its action on multi-particle state is a sum over all tensor factors with $\mu \circ \Delta$ applied to each of them. For eigen states of $\mu \circ \Delta$ the projective equivalence of loop diagrams with tree diagrams would make sense.

Since self energy loops, vertex corrections, and box diagrams represent the basic divergences of renormalizable quantum field theories, these axioms raise the hope that the basic infinities of

quantum field theories could be eliminated by the basic axioms for the morphisms of quantum category.

There are also morphisms related to the topology changes in which the 3-surface remains connected. For instance, processes in which the number of boundary components can change could be of special relevance if the family replication phenomenon reduces to the boundary topology. Also 3-topology can change. The experience with topological quantum field theories [A5], stimulates the hope that the braid group representations of the topological invariants of 3-topology might be of help in the construction of S-matrix.

The equivalence of loop diagrams with tree diagrams must have algebraic formulation using the language of standard quantum field theory. In the third section it was indeed found that thanks to the presence of the emission of vacuons, the equivalence of loop diagrams with tree diagrams corresponds to the vanishing of loop corrections in the standard quantum field theory framework. Furthermore, the non-cocommutative Hopf algebra of Feynman diagrams discussed in [A4] becomes co-commutative when the loop corrections vanish so that TGD program indeed has an elegant algebraic formulation also in the standard framework.

3.2   The Physical Interpretation Of Non-Trivial Braiding And Quasi-Associativity

The exchange of the tensor factors by braiding could also correspond to a physically non-trivial but unitary operation as it indeed does in anyon physics [D2, D1]. What would differentiate between elementary particles and anyons would be the non-triviality of the super-canonical and Super Kac-Moody conformal central extensions which have the same origin (addition of a multiplication by a multiple of the Hamiltonian of a canonical transformation to the action of isometry generator). The proposed interpretation of braiding acting in the complex plane in which the conformal weights of the elements of the super-canonical algebra represent punctures justifies the non-triviality. Hexagon Axioms would state that two generalized Feynman diagrams involving exchanges, dissociations and re-associations are equivalent.

An interesting question is whether the association $(A, B) \rightarrow (A \otimes B)$ could be interpreted as a formation of bound state entanglement between A and B. A possible space-time correlate for association is topological condensation of A and B to the same space-time sheet. Association would be trivial if all particles are at same space-time sheet X^4 but non-trivial if some subset of particles condense at an intermediate space-time sheet Y^4 condensing in turn at X^4.

Be as it may, association isomorphisms $a_{A,B,C}$ would state that the state space obtained by binding A with bound bound states $(B \otimes C)$ is unitarily related with the state space obtained by binding $(A \otimes B)$ bound states with C. With this interpretation Pentagon axiom would state that two generalized Feynman diagrams depicted in figure **??** leading from initial to final to final state by dissociation and re-association are equivalent.

3.3   Generalizing The Notion Of Bi-Algebra Structures At The Level Of WCW

WCW of 3-surfaces decomposes into sectors corresponding to different 3-topologies. Also other signatures might be involved and I have proposed that the sectors are characterized by the collection of p-adic primes labelling space-time sheets of the 3-surface and that a given space-time surface could be characterized by an infinite prime or integer. The general problem is to continue various geometric structures from a given sector A of WCW to other sector B.

An especially interesting special case corresponds to a continuation from 1-particle sector to two-particle sector or vice versa and corresponds to TGD variant of 3-vertex. All these continuations involve the embedding of a structure associated with the sector A to a structure associated with sector B. For the continuation from 1-particle sector to 2-particle sector the map is analogous to co-algebra homomorphism Δ. For the reverse continuation it is analogous to the algebra product μ. Now however one does not have maps $\Delta : A \rightarrow A \otimes A$ and $\mu : A \otimes A \rightarrow A$ but $\Delta : A \rightarrow B \otimes C$ and $\mu : B \otimes C \rightarrow A$ unless the algebras are isomorphic. $\mu \circ \Delta = id$ should hold true as an additional condition but $\Delta \circ \mu = id$ cannot hold true since product maps many pairs to the same element.

3.3.1 Continuation of the WCW spinor structure

The basic example of a structure to be continued is configuration space spinor structure. WCW spinor fields in different sectors should be related to each other. The isometry generators and gamma matrices of WCW span a super-canonical algebra. The continuation requires that the super algebra basis of different sectors are related. Also vacua must be related. Isometry generators correspond to bosonic generators of the super-canonical algebra. There is also a natural extension of the super-canonical algebra defined by the Poisson structure of WCW .

   This view suggests that in the first approximation one could see the construction of S-matrix as following process.

1. Incoming/outgoing states correspond to positive/negative energy states localized to the sectors of WCW with fixed 3-topologies.

2. In order to construct an S-matrix matrix element between two states localized in sectors A and B, one must continue the state localized in A to B or vice versa and calculate overlap. The continuation involves a sequence of morphisms mapping various structures between sectors. In particular, topological transformations describing particle decay and fusion are possible so that the analogs of product μ and co-product Δ are involved. The construction of three-manifold topological invariants [A5] in topological quantum field theories provides concrete ideas about how to proceed.

3. The S-matrix element describing a particular transition can be expressed as any path leading from the sector A to B or vice versa. There is a huge symmetry very much analogous to the independence of the final result of the analytic continuation on the path chosen since generalized Feynman graphs allow all moves changing intermediate topologies so that initial and final 3-topologies are same. Generalized conformal invariance probably also poses restrictions on possible paths of continuation. In the path integral approach one would have simply sum over all these equivalent paths and thus encounter the fundamental difficulties related to the infinite-dimensional integration.

4. Quantum classical correspondence suggests that the continuation operation has a space-time correlate. That is, the preferred extremal of Kähler action going through the initial and final 3-sheets defines a sequence of transitions changing the topology of 3-sheet. The localization to a particular sector of course selects particular preferred extremal. There are two possible interpretations. Either the continuation from A is not possible to all possible sectors but only to those with 3-topologies appearing in X^4, or the preferred extremal represents some kind of minimal continuation involving minimum amount of calculational labor.

5. Quantum classical correspondence and the possibility to represent the rows of S-matrix as zero energy quantum states suggests that the paths for continuation can be also represented at the space-time level, perhaps in terms of braided flux tubes connecting two light like 3-surfaces representing the initial and final states of particle reaction. Since light like 3-surfaces are metrically two-dimensional and allow conformal invariance, this suggests a connection with braid diagrams in the sense that it should be possible to regard the paths connecting sectors of WCW consisting of unions of disjoint 3-surfaces (corresponding interacting 4-surfaces are connected) as generalized braids for which also decay and fusion for the strands of braid are possible. Quantum algebra structure and effective metric 2-dimensionality of the light like 3-surfaces suggests different braidings for flux tubes connecting boundaries of 3-surfaces define non-equivalent 3-surfaces.

3.3.2 Co-multiplication and second quantized induced spinor fields

At the microscopic level the construction of S-matrix reduces to understanding what happens for the classical spinor fields in a vertex, which corresponds to an incoming 3-surface A decaying to two outgoing 3-surfaces B and C. At the classical level incoming spinor field A develops into a spinor fields B and C expressible as linear combinations of appropriate spinor basis. At quantum level one must understand how the Fock space defined by the incoming spinor fields of A is mapped

to the tensor product of Fock spaces of B and C. The idea about the possible importance of co-algebras came with the realization that this mapping is obviously is very much like a co-product. Co-algebras and bi-algebras possessing both algebra and co-algebra structure indeed suggest a general approach giving hopes of understanding how Feynman diagrammatics generalizes to TGD framework.

The first guess is that fermionic oscillator operators are mapped by the embedding Δ to a superposition of operators $a_{Bn}^\dagger \otimes Id_C$ and $Id_B \otimes a_{Cn}^\dagger$ with obvious formulas for Hermitian conjugates. Δ induces the mapping of higher Fock states and the construction of S-matrix should reduce to the construction of this map.

Δ is analogous to the definition for co-product operation although there is also an obvious difference due to the fact that Δ imbeds algebra A to $B \otimes C$ rather than to $A \otimes A$. Only in the case that the algebras are isomorphic, the situation reduces to that for Hopf algebras. Category theoretical approach however allows to consider a more general situation in which Δ is a morphism in the category of Fock algebras associated with 3-surfaces.

Δ preserves fermion number and should respect Fock algebra structure, in particular commute with the anti-commutation relations of fermionic oscillator operators. The basis of fermionic oscillator operators would naturally correspond to fermionic super-canonical generators in turn defining WCW gamma matrices.

Since any leg can be regarded as incoming leg, strong consistency conditions result on the coefficients in the expression

$$\Delta(a_{An}^\dagger) \quad = \quad C(A,B)_n{}^m a_{Bm}^\dagger \otimes Id_C + C(A,C)_n{}^m Id_B \otimes a_{Cm}^\dagger \tag{3.3}$$

by forming the cyclic permutations in A, B, C. This option corresponds to the co-commutative situation and quantum group structure. If identity matrices are replaced with something more general, co-product becomes non-cocommutative.

3.4 Ribbon Category As A Fundamental Structure?

There exists a generalization of the braided tensor category inspired by the axiomatic approach to topological quantum field theories which seems to almost catch the proposed mathematical requirements. This category is also called ribbon [A1] [A9] but in more general sense than it is defined in [A3].

One adds to the tangle diagrams (braid diagrams with both directions of strands and possibility of strand–anti-strand annihilation) also "coupons", which are boxes representing morphisms with arbitrary numbers of incoming and outgoing strands. As a special case 3-particle vertices are obtained. The strands correspond to representations of a fixed Hopf algebra H.

In the recent case it would seem safest to postulate that strands correspond to algebras, which can be different because of the potential dependence of the details of Fock algebra on 3-topology and other properties of 3-surface. For instance, WCW metric defined by anti-commutators of the gamma matrices is degenerate for vacuum extremals so that the infinite Clifford algebra is definitely "smaller" than for surfaces with $D \geq 3$-dimensional CP_2 projection.

One might feel that the full ribbon algebra is an un-necessary luxury since only 3-particle vertices are needed since higher vertices describing decays of 3-surfaces can be decomposed to 3-vertices in the generic case. On the other hand, many-sheeted space-time and p-adic fractality suggest that coupons with arbitrary number of incoming and outgoing strands are needed in order to obtain the p-adic hierarchy of length scale dependent theories.

The situation would be the same as in the effective quantum field theories involving arbitrarily high vertices and would require what might be called universal algebra allowing n-ary multiplications and co-multiplications rather than only binary ones. Also strands within strands hierarchy is strongly suggestive and would require a fractal generalization of the ribbon algebra. Note that associativity and commutativity conditions for morphisms which more than three incoming and outgoing lines would force to generalize the notion of R-matrix and would bring in conditions stating that more complex loop diagrams are equivalent with tree diagrams.

3.5 Minimal Models And TGD

Quaternion conformal invariance with non-vanishing c and k for anyons is highly attractive option and minimal super-conformal field theories attractive candidate since they describe critical systems and TGD Universe is indeed a quantum critical system.

3.5.1 Rational conformal field theories and TGD

The highest weight representations of Virasoro algebra are known as Verma modules containing besides the ground state with conformal weight Δ the states generated by Virasoro generators L_n, $n \geq 0$. For some values of Δ Verma module contains states with conformal weight $\Delta + l$ annihilated by Virasoro generators L_n, $n \geq 1$. In this case the number of primary fields is reduced since Virasoro algebra acts as a gauge algebra. The conformal weights Δ of the Verma modules allowing null states are given by the Kac formula

$$
\begin{aligned}
\Delta_{mm'} &= \Delta_0 + \frac{1}{4}(\alpha_+ m + \alpha_- m')^2 \ , \ \ m, m' \in \{1, 2...\} \ , & (3.4) \\
\Delta_0 &= \frac{1}{24}(c-1) \ , \\
\alpha_\pm &= \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \ . & (3.5)
\end{aligned}
$$

The descendants $\prod_{n \geq 1} L_n^{k_n} |\Delta\rangle$ annihilated by L_n, $n > 0$, have conformal weights at level $l = \sum_n n k_n = mm'$.

In the general case the operator products of primary fields satisfying these conditions form an algebra spanned by infinitely many primary fields. The situation changes if the central charge c satisfies the condition

$$
c = 1 - \frac{6(p'-p)^2}{pp'} \ , \tag{3.6}
$$

where p and p' are mutually prime positive integers satisfying $p < p'$. In this case the Kac weights are rational

$$
\Delta_{m,m'} = \frac{(mp' - m'p)^2 - (p'-p)^2)}{4pp'} \ , \ \ 0 < m < p \ , \ \ 0 < m' < p' \ . \tag{3.7}
$$

Obviously, the number of primary fields is finite. This option does not seem to be realistic in TGD framework were super-conformal invariance is realized.

For $N = 1$ super-conformal invariance the unitary representations have central extension and conformal weights given by

$$
\begin{aligned}
c &= \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right) \ , \\
\Delta_{p,q}(NS) &= \frac{[(m+2)p) - mq)]^2 - 4}{8m(m+2)} \ , \ \ 0 \leq p \leq m \ , \ \ 1 \leq q \leq m+2 \ .
\end{aligned}
\tag{3.8}
$$

For Ramond representations the conformal weights are

$$
\Delta_{p,q}(R) = \Delta(NS) + \frac{1}{16} \ . \tag{3.9}
$$

The states with vanishing conformal weights correspond to light elementary particles and the states with $p = q$ have vanishing conformal weight in NS sector. Also this option is non-realistic since in TGD framework super-generators carry fermion number so that G cannot be a Hermitian operator.

$N = 2$ super-conformal algebra is the most interesting one from TGD point of view since it involves also a bosonic $U(1)$ charge identifiable as fermion number and $G^{\pm}(z)$ indeed carry $U(1)$ charge[1]. Hence one has $N = 2$ super-conformal algebra is generated by the energy momentum tensor $T(z)$, $U(1)$ current $J(z)$, and super generators $G^{\pm}(z)$. $U(1)$ current would correspond to fermion number and super generators would involve contraction of covariantly constant neutrino spinor with second quantized induced spinor field. The further facts that $N = 2$ algebra is associated naturally with Kähler geometry, that the partition functions associated with $N = 2$ super-conformal representations are modular invariant, and that $N = 2$ algebra defines so called chiral ring defining a topological quantum field theory [A6], lend further support for the belief that $N = 2$ super-conformal algebra acts in super-canonical degrees of freedom.

The values of c and conformal weights for $N = 2$ super-conformal field theories are given by

$$
\begin{aligned}
c &= \frac{3k}{k+2} \ , \\
\Delta_{l,m}(NS) &= \frac{l(l+2) - m^2}{4(k+2)} \ , \quad l = 0, 1, ..., k \ , \\
q_m &= \frac{m}{k+2} \ , \quad m = -l, -l+2, ..., l-2, l \ .
\end{aligned}
\tag{3.10}
$$

q_m is the fractional value of the $U(1)$ charge, which would now correspond to a fractional fermion number. For $k = 1$ one would have $q = 0, 1/3, -1/3$, which brings in mind anyons. $\Delta_{l=0,m=0} = 0$ state would correspond to a massless state with a vanishing fermion number. Note that $SU(2)_k$ Wess-Zumino model has the same value of c but different conformal weights. More information about conformal algebras can be found from the appendix of [A6].

For Ramond representation $L_0 - c/24$ or equivalently G_0 must annihilate the massless states. This occurs for $\Delta = c/24$ giving the condition $k = 2 \left[l(l+2) - m^2 \right]$ (note that k must be even and that $(k, l, m) = (4, 1, 1)$ is the simplest non-trivial solution to the condition). Note the appearance of a fractional vacuum fermion number $q_{vac} = \pm c/12 = \pm k/4(k+2)$. I have proposed that NS and Ramond algebras could combine to a larger algebra containing also lepto-quark type generators.

Quaternion conformal invariance [K2] encourages to consider the possibility of super-symmetrizing also spin and electro-weak spin of fermions. In this case the conformal algebra would extend to a direct sum of Ramond and NS $N = 8$ algebras associated with quarks and leptons. This algebra in turn extends to a larger algebra if lepto-quark generators acting as half odd-integer Virasoro generators are allowed. The algebra would contain spin and electro-weak spin as fermionic indices. Poincare and color Kac-Moody generators would act as symplectically extended isometry generators on WCW Hamiltonians expressible in terms of Hamiltonians of $X_l^3 \times CP_2$. Electro-weak and color Kac-Moody currents have conformal weight $h = 1$ whereas T and G have conformal weights $h = 2$ and $h = 3/2$.

The experience with $N = 4$ super-conformal invariance suggests that the extended algebra requires the inclusion of also second quantized induced spinor fields with $h = 1/2$ and their superpartners with $h = 0$ and realized as fermion-anti-fermion bilinears. Since G and Ψ are labelled by 2×4 spinor indices, super-partners would correspond to $2 \times (3+1) = 8$ massless electro-weak gauge boson states with polarization included. Their inclusion would make the theory highly predictive since induced spinor and electro-weak fields are the fundamental fields in TGD.

In TGD framework both quark and lepton numbers correspond to NS and Ramond type representations, which in conformal field theories can be assigned to the topologies of complex plane and cylinder. This would suggest that a given 3-surface allows either NS or Ramond representation and is either leptonic or quark like but one must be very cautious with this kind of conclusion. Interestingly, NS and Ramond type representations allow a symmetry acting as a spectral flow in the indices of the generators and transforming NS and Ramond type representations continuously to each other [A6]. The flow acts as

---

[1] I realized that TGD super-conformal algebra corresponds to $N = 2$ algebra while writing this and proposed it earlier as a generalization of super-conformal algebra!

$$L_n \to L_n + \alpha J_n + \frac{c}{6}\alpha^2 \delta_{n,0}$$
$$J_n \to J_n + \frac{c}{3}\alpha \delta_{n,0} \ ,$$
$$G_n^\pm \to G_{n\pm\alpha}^\pm \ . \tag{3.11}$$

The choice $\alpha = \pm 1/2$ transforms NS representation to Ramond representation. The idea that leptons could be transformed to quarks in a continuous manner does not sound attractive in TGD framework. Note that the action of Super Kac-Moody Virasoro algebra in the space of super-canonical conformal weights can be interpreted as a spectral flow.

3.5.2 Co-product for Super Kac-Moody and Super Virasoro algebras

By the previous considerations the quantized conformal weights z_1, z_2, z_3 of super-canonical generators defining punctures of 2-surface should correspond to line punctures of 3-surface. One cannot avoid the thought that these line punctures should meet at single point so that three-vertex would have also quantum field theoretical interpretation.

Each point z_k corresponds to its own Virasoro algebra $V_k = \{L_n^{z_k)}\}$ and Kac-Moody algebra $J_k = \{J_n^{z_k)}\}$ defined by Laurent series of $T(z)$ and $J(z)$ at z_k. Also super-generators are involved. To minimize notational labor denote by $X_n^{z_k)}$, $k = 1, 2, 3$ the generators in question.

The co-algebra product for Super-Virasoro and Super-Kac-Moody involves in the case of fusion $A_1 \otimes A_2 \to A_3$ a co-algebra product assigning to the generators $X_n^{z_3)}$ direct sum of generators of $X_k^{z_1)}$ and $X_l^{z_2)}$. The most straightforward approach is to express the generators $X_n^{z_3)}$ in terms of generators $X_k^{z_1)}$ and $X_l^{z_2)}$. This is achieved by using the expressions for generators as residy integrals of energy momentum tensor and Kac Moody currents. For Virasoro generators this is carried out explicitly in [A6]. The resulting co-product conserves the value of central extension whereas for the naïve co-product this would not be the case. Obviously, the geometric co-product does not conserve conformal weight.

4 Some Examples Of Bi-Algebras And Quantum Groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

4.1 Hecke Algebra And Temperley-Lieb Algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_n e_{n+1} &= e_n e_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{4.1}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and embedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M} : \mathcal{N} < 4$ is given by the relations

$$
\begin{aligned}
e_{n+1}e_n en + 1 &= e_{n+1} \\
e_n e_{n+1} e_n &= e_n \ , \\
e_n^2 &= t e_n \ , \quad , t = -\sqrt{\mathcal{M} : \mathcal{N}} = -2cos(\pi/n) \ , n = 3, 4, ...
\end{aligned}
\tag{4.2}
$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$
\begin{aligned}
e_{n+1}e_n en + 1 &= t e_{n+1} \\
e_n e_{n+1} e_n &= t e_n \ , \\
e_n^2 &= e_n = e_n^* \ , \quad , t = -\sqrt{\mathcal{M} : \mathcal{N}} = -2cos(\pi/n) \ , n = 3, 4, ...
\end{aligned}
\tag{4.3}
$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

4.2 Simplest Bi-Algebras

Let $k(x_1, .., x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1, ..., x_n), A)$ of algebra homomorphisms $k(x_1, ..., x_n) \to A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1, ...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x], A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a, b, c, d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a, b, c, d)$ to $M_2^{\otimes 2} = k(a', a'', b', b'', c', c'', d', d'')$ to $k(a, b, c, d)$ in matrix form as

$$
\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \ .
$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL_e(2)$ and $SL_e(2)$ provide standard examples about bi-algebras. $SL_e(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a, b, c, d)$ spanned by the generators a, b, c, d by the ideal $det - 1 = ad - bc - 1 = 0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .
$$

Δ, counit ϵ, and antipode S can be written in case of $SL_e(2)$ as

$$
\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,
$$

$$
\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .
$$

$$
S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ .
$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a, b, c, d of the algebra. For instance, one has $\Delta(a) = a \to a \otimes a + b \otimes c$. The resulting algebra is both commutative and co-commutative.

$SL_e(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a, b, c, d by the relations

$$
\begin{aligned}
ba &= qab \ , & db &= qbd \ , \\
ca &= qac \ , & dc &= QCD \ , \\
bc &= cb \ , & ad - da &= (q^{-1} - 1)bc \ ,
\end{aligned}
$$

and the relation

$$
det_q = ad - q^{-1}bc = 1
$$

stating that the quantum determinant of $SL_e(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL_e(2)$. Antipode S is defined by

$$
S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \ .
$$

The relations above guarantee that it defines quantum inverse of A. For q an nth root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

4.3 Quantum Group $U_Q(Sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL_e(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$
[X_+, X_-] = H \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{4.4}
$$

$U_q(sl(2))$ allows co-algebra structure given by

$$
\Delta(J) = J \otimes 1 + 1 \otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_\pm, H \ ,
$$
$$
S(1) = 1 \ , \qquad\qquad \epsilon(1) = 1 \ . \tag{4.5}
$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \to H^\star$ reads as

$$
\begin{aligned}
a &\to a^\star \ , & ab &= (ab)^\star = b^\star a^\star \ , \\
1 &\to 1 \ , & H &\to H^\star = hH \ , & X_+ &\to (X_+)^\star = hX_- \ .
\end{aligned} \tag{4.6}
$$

The commutation relations of $U_q(sl(2))$ read as

$$
[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{4.7}
$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$
\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm \ ,
$$
$$
S(X_\pm) = -q^{\pm 1}X_\pm \ . \tag{4.8}
$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H \otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \quad . \tag{4.9}$$

When q is m: th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [A6].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \to \Psi(x)$ $M_q(2) \to U_q^\star$ defined by a bilinear form $\langle u, x \rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\langle uv, x \rangle = \langle u \otimes v, \Delta(x) \rangle \quad , \quad \langle u, xy \rangle = \langle \Delta(u), x \otimes y \rangle \quad ,$$

$$\langle 1, x \rangle = \epsilon(x) \quad , \qquad\qquad \langle u, 1 \rangle = \epsilon(u)$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \quad ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \to U_q^\star$.

4.4   General Semisimple Quantum Group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [A6]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:

1. A is indecomposable, that is does not reduce to a direct sum of matrices.

2. $a_{ij} \leq 0$ holds true for $i < j$.

3. $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.

A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.
The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{aligned} k_i k_j &= k_j k_i \quad , & k_i e_j &= q_i^{a_{ij}} e_j k_i \quad , \\ k_i f_j &= q_i^{-a_{ij}} e_j k_i \quad , & e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \quad , \end{aligned} \tag{4.10}$$

and so called Serre relations

$$\sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \ \ i \neq j \ \ ,$$
$$\sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \ \ , \ \ i \neq j \ \ . \tag{4.11}$$

Here $q_i = q^{D_i}$ where one has $D_i a_{ij} = a_{ij} D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) \ = \ k_i \otimes k_i \ \ , \tag{4.12}$$
$$\Delta(e_i) \ = \ e_i \otimes k_i + 1 \otimes e_i \ \ , \tag{4.13}$$
$$\Delta(f_i) \ = \ f_i \otimes 1 + k_i^{-1} \otimes 1 \ \ . \tag{4.14}$$
$$\tag{4.15}$$

The action of antipode S is defined as

$$S(e_i) = -e_i k_i^{-1} \ \ , \ \ S(f_i) = -k_i f_i \ \ , \ \ S(k_i) = -k_i^{-1} \ \ . \tag{4.16}$$

4.5   Quantum Affine Algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [A6].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l + 1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1) \times (l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0: th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL_e(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l + 1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 4.11 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{G}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0} e_i \ \ , \ \ [d, f_i] = \delta_{i0} f_i, \ \ [d, k_i] = 0 \ \ . \tag{4.17}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{G}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L_e(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \quad , \tag{4.18}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \quad . \tag{4.19}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L_e(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L_e(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \quad , \tag{4.20}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\Psi(a, b) = -\Psi(b, a) \quad ,$$
$$\Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) = 0 \quad . \tag{4.21}$$

The two-cocycle defines the central extension of loop algebra $L_e(\mathcal{G}_l)$ to Kac Moody algebra $L_e(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,] + \Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$
\begin{aligned}
J_n^x &= x \otimes t^n \quad , \\
[J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \quad .
\end{aligned} \tag{4.22}
$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$
\begin{aligned}
J_n^x|v\rangle &= 0, \quad n > 0 \quad , \\
c|v\rangle &= k|v\rangle \quad .
\end{aligned} \tag{4.23}
$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \to U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$
\begin{aligned}
D_t(e_i) &= t^{\delta_{i0}}e_i \quad , \quad D_t(f_i) = t^{\delta_{i0}}f_i \quad , \\
D_t(k_i) &= k_i \qquad\qquad D_t(d) = d \quad ,
\end{aligned} \tag{4.24}
$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \quad , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \quad , \tag{4.25}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{4.26}$$

and satisfies the equations

$$\begin{aligned}
&\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\[4pt]
&(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\[4pt]
&(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\[4pt]
&\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ .
\end{aligned} \tag{4.27}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations $e_i, f_i, k_i, i > 0$.

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