Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

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Contents

Abstract

Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. Category theory might provide the desired systematic approach to fuse together the bundles of general ideas related to the construction of quantum TGD proper. Category theory might also have natural applications in the general theory of consciousness and the theory of cognitive representations.

1. The ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences could be expressed elegantly using the language of the category theory. Quantum classical correspondence might allow a mathematical formulation in terms of structure respecting functors mapping the categories associated with the three kinds of existences to each other. Basic vision is following.
   (a) Self hierarchy would have a functorial map to the hierarchy of space-time sheets and also WCW spinor fields reflect it. Thus the self referentiality of conscious experience would have a functorial formulation (it is possible to be conscious about what one *was* conscious).
   (b) The inherent logic for category defined by Heyting algebra must be modified in TGD context. Set theoretic inclusion would be replaced with the topological condensation, which can occur simultaneously to several space-time sheets.
   (c) The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the WCW geometry and realizes the cosmologies within cosmologies scenario.
   (d) In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.
   (e) The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity). The duality would allow to construct new preferred extremals of Kähler action.
2. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience.
3. Categories possess inherent generalized logic based on set theoretic inclusion which in TGD framework is naturally replaced with topological condensation: the outcome is quantum variants for the notions of sieve, topos, and logic. This suggests the possibility of geometrizing the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through three-valued logic. Also the right-wrong logic of moral rules and beautiful-ugly logic of aesthetics seem to be too naive and might be replaced with a more general quantum logic.

1 Introduction

Goro Kato has proposed an ontology of consciousness relying on category theory [A10, A16]. Physicist friendly summary of the basic concepts of category theory can be found in [A12]) whereas the books [A7, A11] provide more mathematically oriented representations. Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. To mention only one example, C. J. Isham [A12] has proposed that topos theory could provide a new approach to quantum gravity in which space-time points would be replaced by regions of space-time and that category theory could geometrize and dynamicize even logic by replacing the standard Boolean logic with a dynamical logic dictated by the structure of the fundamental category purely geometrically [A18].

Although I am an innocent novice in this field and know nothing about the horrible technicalities of the field, I have a strong gut feeling that category theory might provide the desired systematic approach to quantum TGD proper, the general theory of consciousness, and the theory of cognitive representations [K12].

1.1 Category Theory As A Purely Technical Tool

Category theory could help to disentangle the enormous technical complexities of the quantum TGD and to organize the existing bundle of ideas into a coherent conceptual framework. The construction of the geometry of the configuration space ("world of classical worlds") [K9, K6]. of classical configuration space spinor fields [K19]. and of S-matrix [K4] using a generalization of the quantum holography principle are especially natural applications. Category theory might also help in formulating the new TGD inspired view about number system as a structure obtained by "gluing together" real and p-adic number fields and TGD as a quantum theory based on this generalized notion of number [K16, K17, K15].

1.2 Category Theory Based Formulation Of The Ontology Of TGD Universe

It is interesting to find whether also the ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences [?] could be expressed elegantly using the language of the category theory.

There are indeed natural and non-trivial categories involved with many-sheeted space-time and the geometry of the configuration space ("the world of classical worlds"); with configuration space spinor fields; and with the notions of quantum jump, self and self hierarchy. Functors between these categories could express more precisely the quantum classical correspondences and self-referentiality of quantum states allowing them to express information about quantum jump sequence.

1. Self hierarchy has a structure of category and corresponds functorially to the hierarchical structure of the many-sheeted space-time.

2. Quantum jump sequence has a structure of category and corresponds functorially to the category formed by a sequence of maximally deterministic regions of space-time sheet. Even the quantum jump could have space-time correlates made possible by the generalization of the Boolean logic to what might be space-time correlate of quantum logic and allowing to identify space-time correlate for the notion of quantum superposition.

3. The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

4. In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.

5. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K17]. The duality would allow to construct new preferred extremals of Kähler action.

1.3 Other Applications

One can imagine also other applications.

1. Categories posses inherent logic [A18] based on the notion of sieves relying on the notion of presheaf which generalizes Boolean logic based on inclusion. In TGD framework inclusion is naturally replaced by topological condensation and this leads to a two-valued logic realizing space-time correlate of quantum logic based on the notions of quantum sieve and quantum topos.

   This suggests the possibility to geometrize the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through logic in which the law of excluded middle is not true. Interestingly, the p-adic logic of cognition is naturally

2-valued whereas the real number based logic of sensory experience allows excluded middle (is the person at the door in or out, in and out, or neither in nor out?). The quantum logic naturally associated with spinors (in the "world of classical worlds") is consistent with the logic based on quantum sieves.

2. Simple Boolean logic of right and wrong does not seem to be ideal for understanding moral rules. Same applies to the beauty-ugly logic of aesthetic experience. The logic based on quantum sieves would perhaps provide a more flexible framework.

3. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience. Here the new elements associated with the ontology of space-time due to the generalization of number concept would be central. Category theory could be also helpful in the modelling of conscious communications, in particular the telepathic communications based on sharing of mental images involving the same mechanism which makes possible space-time correlates of quantum logic and quantum superposition.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L2]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L3].

2 What Categories Are?

In the following the basic notions of category theory are introduced and the notion of presheaf and category induced logic are discussed.

2.1 Basic Concepts

Categories [A7, A11, A12] are roughly collections of objects A, B, C... and morphisms $f(A \rightarrow B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Topological/linear spaces form a category with continuous/linear maps acting as morphisms. Also algebraic structures of a given type form a category: morphisms are now homomorphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can can be very general: for instance, partial ordering $a \leq b$ can define morphism $f(A \rightarrow B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Group representation is example of this kind of a functor: now group action in group is mapped to a linear action at the level of the representations. Commuting square is an easy visual manner to understand the basic properties of a functor, see **Fig. 1**.

The product $C = AB$ for objects of categories is defined by the requirement that there are projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \rightarrow A)$ and $g(D \rightarrow B)$ there exist morphism $h(D \rightarrow C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically (see **Fig. 1**) this corresponds to a square diagram in which pairs A, B and C, D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC.

Examples of product categories are Cartesian products of topological and linear spaces, of differentiable manifolds, groups, etc. Also tensor products of linear spaces satisfies these axioms. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

2.2 Presheaf As A Generalization For The Notion Of Set

Presheafs can be regarded as a generalization for the notion of set. Presheaf is a functor X that assigns to any object of a category **C** an object in the category **Set** (category of sets) and maps

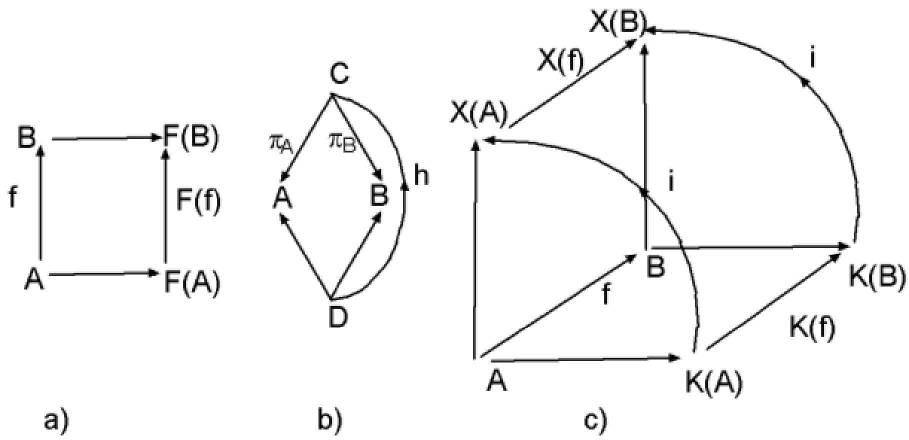

Figure 1: Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".)

morphisms to morphisms (maps between sets for **C**). In order to have a category of presheafs, also morphisms between presheafs are needed. These morphisms are called natural transformations $N : X(A) \rightarrow Y(A)$ between the images $X(A)$ and $Y(A)$ of object A of **C**. They are assumed to obey the commutativity property $N(B)X(f) = Y(f)N(A)$ which is best visualized as a commutative square diagram. Set theoretic inclusion $i : X(A) \subset Y(A)$ is obviously a natural transformation.

An easy manner to understand and remember this definition is commuting diagram consisting of two pages of book with arrows of natural transformation connecting the corners of the pages: see **Fig. ??**.

As noticed, presheafs are generalizations of sets and a generalization for the notion of subset to a sub-object of presheaf is needed and this leads to the notion of topos [A18, A12]. In the classical set theory a subset of given sets X can be characterized by a mapping from set X to the set $\Omega = \{true, false\}$ of Boolean statements. Ω itself belongs to the category **C**. This idea generalizes to sub-objects whose objects are collections of sets: Ω is only replaced with its Cartesian power. It can be shown that in the case of presheafs associated with category **C** the sub-object classifier Ω can be replaced with a more general algebra, so called Heyting algebra [A18, A12] possessing the same basic operations as Boolean algebra (and, or, implication arrow, and negation) but is not in general equivalent with any Boolean algebra. What is important is that this generalized logic is inherent to the category **C** so that many-valued logic ceases to be an ad hoc construct in category theory.

In the theory of presheafs sub-object classifier Ω, which belongs to **Set**, is defined as a particular presheaf. Ω is defined by the structure of category **C** itself so that one has a geometrization of the notion of logic implied by the properties of category. The notion of sieve is essential here. A sieve for an object A of category **C** is defined as a collection of arrows $f(A \rightarrow ...)$ with the property that if $f(A \rightarrow B)$ is an arrow in sieve and if $g(B \rightarrow C)$ is any arrow then $gf(A \rightarrow C)$ belongs to sieve.

In the case that morphism corresponds to a set theoretic inclusion the sieve is just either empty set or the set of all sets of category containing set A so that there are only two sieves corresponding to Boolean logic. In the case of a poset (partially ordered set) sieves are sets for which all elements are larger than some element.

2.3 Generalized Logic Defined By Category

The presheaf $\Omega : \mathbf{C} \rightarrow \mathbf{Set}$ defining sub-object classifier and a generalization of Boolean logic is defined as the map assigning to a given object A the set of all sieves on A. The generalization of maps $X \rightarrow \Omega$ defining subsets is based on the notion of sub-object K. K is sub-object of presheaf X in the category of presheaves if there exist natural transformation $i : K \rightarrow X$ such that for each A one has $K(A) \subset X(A)$ (so that sub-object property is reduced to subset property).

The generalization of the map $X \to \Omega$ defining subset is achieved as follows. Let K be a sub-object of X. Then there is an associated characteristic arrow $\chi^K : X \to \Omega$ generalizing the characteristic Boolean valued map defining subset, whose components $\chi^K_A : X(A) \to \Omega(A)$ in **C** is defined as

$$\chi^K_A(x) = \{f(A \to B)|X(f)(x) \in K(B)\} \ .$$

By using the diagrammatic representation of **Fig. 1** for the natural transformation i defining sub-object, it is not difficult to see that by the basic properties of the presheaf K $\chi^K_A(x)$ is a sieve. When morphisms f are inclusions in category Set, only two sheaves corresponding to all sets containing X and empty sheaf result. Thus binary valued maps are replaced with sieve-valued maps and sieves take the role of possible truth values. What is also new that truths and logic are in principle context dependent since each object A of **C** serves as a context and defines its own collection of sieves.

The generalization for the notion of point of set X exists also and corresponds to a selection of single element γ_A in the set $X(A)$ for each A object of **C**. This selection must be consistent with the action of morphisms $f(A \to B)$ in the sense that the matching condition $X(f)(\gamma_A) = \gamma_B$ is satisfied. It can happen that category of presheaves has no points at all since the matching condition need not be satisfied globally.

It turns out that TGD based notion of subsystem leads naturally to what might be called quantal versions of topos, presheaves, sieves and logic.

3 More Precise Characterization Of The Basic Categories And Possible Applications

In the following the categories associated with self and quantum jump are discussed in more precise manner and applications to communications and cognition are considered.

3.1 Intuitive Picture About The Category Formed By The Geometric Correlates Of Selves

Space-time surface $X^4(X^3)$ decomposes into regions obeying either real or p-adic topology and each region of this kind corresponds to an unentangled subsystem or self lasting at least one quantum jump. By the localization in the zero modes these decompositions are equivalent for all 3-surfaces X^3 in the quantum superposition defined by the prepared WCW spinor fields resulting in quantum jumps. There is a hierarchy of selves since selves can contain sub-selves. The entire space-time surface $X^4(X^3)$ represents the highest level of the self hierarchy.

This structure defines in a natural manner a category. Objects are all possible sub-selves contained in the self hierarchy: sub-self is set consisting of lower level sub-selves, which in turn have a further decomposition to sub-selves, etc... The naïve expectation is that geometrically sub-self belongs to a self as a subset and this defines an inclusion map acting as a natural morphism in this category. This expectation is not quite correct. More natural morphisms are the arrows telling that self as a set of sub-selves contains sub-self as an element. These arrows define a structure analogous to a composite of hierarchy trees.

To be more precise, for a single space-time surface $X^4(X^3)$ this hierarchy corresponds to a subjective time slice of the self hierarchy defined by a single quantum jump. The sequence of hierarchies associated with a sequence of quantum jumps is a natural geometric correlate for the self hierarchy. This means that the objects are now sequences of submoments of consciousness. Sequences are not arbitrary. Self must survive its lifetime although sub-selves at various levels can disappear and reappear (generation and disappearance of mental images). Geometrically this means typically a phase transition transforming real or p_1-adic to p_2-adic space-time region with same topology as the environment. Also sub-selves can fuse to single sub-self. The constraints on self sequences must be such that it takes these processes into account. Note that these constraints emerge naturally from the fact that quantum jumps sequences define the sequences of surfaces $X^4(X^3)$.

By the rich anatomy of the quantum jump there is large number of quantum jumps leading from a given initial quantum history to a given final quantum history. One could envisage quantum jump also as a discrete path in the space of WCW spinor fields leading from the initial state to the final state. In particular, for given self there is an infinite number of closed elementary paths leading from the initial quantum history back to the initial quantum history and these paths in principle give all possible conscious information about a given quantum history/idea: kind of self morphisms are in question (analogous to, say, group automorhisms). Information about point of space is obtained only by moving around and coming back to the point, that is by studying the surroundings of the point. Self in turn can be seen as a composite of elementary paths defined by the quantum jumps. Selves can define arbitrarily complex composite closed paths giving information about a given quantum history.

3.2 Categories Related To Self And Quantum Jump

3.2.1 The categories defined by moments of consciousness and the notion of self

Since quantum jump involves state reduction and the sequence of self measurement reducing all entanglement except bound state entanglement, it defines a hierarchy of unentangled subsystems allowing interpretation as objects of a category. Arrows correspond to subsystem-system relationship and the two subsystems resulting in self measurement to the system. What subsystem corresponds mathematically is however not at all trivial and the naïve description as a tensor factor does not work. Rather, a definition relying on the notion of p-adic length scale cutoff identified as a fundamental aspect of nature and consciousness is needed.

It is not clear what the statement that self corresponds to a subsystem which remains unentangled in subsequent quantum jump means concretely since subsystem can certainly change in some limits. What is clear that bound state entanglement between selves means a loss of consciousness. Category theory suggests that there should exist a functor between categories defined by two subsequent moments of consciousness. This functor maps submoments of consciousness to submoments of consciousness and arrows to arrows. Two subsequent submoments of consciousness belong to same sub-self is the functor maps the first one to the latter one. Thus category theory would play essential role in the precise definition of the notion of self.

The sequences of moments of consciousness form a larger category containing sub-selves as sequences of unentangled subsystems mapped to each other by functor arrows functoring subsequent quantum jumps to each other.

What might then be the ultimate characterizer of the self-identity? The theory of infinite primes suggests that space-time surface decomposes into regions labelled by finite p-adic primes. These primes must label also real regions rather than only p-adic ones. A p-adic space-time region characterized by prime p can transform to a real one or vice versa in quantum jump if the sizes of real and p-adic regions are characterized by the p-adic length scale L_p (or n-ary p-adic length scale $L_p(n)$. One can also consider the possibility that real region is accompanied by a p-adic region characterized by a definite prime p and providing a cognitive self-representation of the real region.

If this view is correct, the p-adic prime characterizing a given real or p-adic space-time sheet could be one characterizer of the self-identity. Self identity is lost in bound state entanglement with another space-time sheet (at least when a space-time sheet with smaller value of the p-adic prime joins by flux tube to a one with a higher value of the p-adic prime). Self identity is also lost if a space-time sheet characterized by a given p-adic prime disappears in quantum jump.

3.2.2 The category associated with quantum jump sequences

There are several similarities between the ontologies and epistemologies of TGD and of category theory. Conscious experience is always determined by the discrete paths in the space of configuration space spinor fields defined by a quantum jump connecting two quantum histories (states) and is never determined by single quantum history as such (quantum states are unconscious). Also category theory is about relations between objects, not about objects directly: self-morphisms give information about the object of category (in case of group composite paths would correspond to products of group automorphisms). Analogously closed paths determined by quantum jump sequences give information about single quantum history. The point is however that it is impossible to have direct knowledge about the quantum histories: they are not conscious.

One can indeed define a natural category, call it **QSelf**, applying to this situation. The objects of the category **QSelf** are initial quantum histories of quantum jumps and correspond to prepared quantum states. The discrete path defining quantum jump can be regarded as an elementary morphism. Selves are composites of elementary morphisms of the initial quantum history defined by quantum jumps: one can characterize the morphisms by the number of the elementary morphisms in the product. Trivial self contains no quantum jumps and corresponds to the identity morphism, null path. Thus the collection of all possible sequences of quantum jumps, that is collections of selves allows a description in terms of category theory although the category in question is not a subcategory of the category **Set**.

Category **QSelf** does not possess terminal and initial elements (for terminal (initial) element T there is exactly one arrow $A \rightarrow T$ ($T \rightarrow A$) for every A: now there are always many paths between quantum histories involved).

3.3 Communications In TGD Framework

Goro Kato identifies communications between conscious entities as natural maps between them whereas in TGD natural maps bind submoments of consciousness to selves. In TGD framework quantum measurement and the sharing of mental images are the basic candidates for communications. The problem is that the identification of communications as sharing of mental images is not consistent with the naïve view about subsystem as a tensor factor. Many-sheeted space-time however forces length scale dependent notion of subsystem at space-time level and this saves the situation.

3.3.1 What communications are?

Communication is essentially generation of desired mental images/sub-selves in receiver. Communication between selves need not be directly conscious: in this case communication would generate mental images at some lower level of self hierarchy of receiver: for instance generate large number of sub-sub-selves of similar type. This is like communications between organizations. Communication can be also vertical: self can generate somehow sub-self in some sub-sub....sub-self or sub-sub...sub-self can generate sub-self of self somehow. This is communication from boss to the lower levels organization or vice versa.

These communications should have direct topological counterparts. For instance, the communication between selves could correspond to an exchange of mental image represented as a space-time region of different topology inside sender self space-time sheet. The sender self would simply throw this space-time region to a receiver self like a ball. This mechanism applies also to vertical communications since the ball could be also thrown from a boss to sub...sub-self at some lower level of hierarchy and vice versa.

The sequence of space-time surfaces provides a direct topological counterpart for communication as throwing balls representing sub-selves. Quantum jump sequence contains space-time surfaces in which the regions corresponding to receiver and sender selves are connected by a flux tube (perhaps massless extremal) representing classically the communication: during the communication the receiver and sender would form single self. The cartoon vision about rays connecting the eyes of communication persons would make sense quite concretely.

More refined means of communication would generate sub-selves of desired type directly at the end of receiver. In this case it is not so obvious how the sequence $X^(X^3)$ of space-time surfaces could represent communication. Of course, one can question whether communication is really what happens in this kind of situation. For instance, sender can affect the environment of receiver to be such that receiver gets irritated (computer virus is good manner to achieve this!) but one can wonder whether this is real communication.

3.3.2 Communication as quantum measurement?

Quantum measurement generates one-one map between the states of the entangled systems resulting in quantum measurement. Both state function reduction and self measurement give rise to this kind of map. This map could perhaps be interpreted as quantum communication between unentangled subsystems resulting in quantum measurement. For the state reduction process the space-time correlates are the values of zero modes. For state preparation the space-time correlates should correspond to classical spinor field modes correlating for the two subsystems generated in self measurement.

3.3.3 Communication as sharing of mental images

It has become clear that the sharing of mental images induced by quantum entanglement of sub-selves of two separate selves represents genuine conscious communication which is analogous telepathy and provides general mechanism of remote mental interactions making possible even molecular recognition mechanisms.

1. The sharing of mental images is not possible unless one assumes that self hierarchy is defined by using the notion of length scale resolution defined by p-adic length scale. The notion of scale of resolution is indeed fundamental for all quantum field theories (renormalization group invariance) for all quantum field theories and without it the practical modelling of physics would not be possible. The notion reflects directly the length scale resolution of conscious experience. For a given sub-self of self the resolution is given by the p-adic length scale associated with the sub-self space-time sheet.

2. Length scale resolution emerges naturally from the fact that sub-self space-time sheets having Minkowskian signature of metric are separated from the one representing self by wormhole contacts with Euclidian signature of metric. The signature of the induced metric changes from Minkowskian signature to Euclidian signature at "elementary particle horizons" surrounding the throats of the wormhole contacts and having degenerate induced metric. Elementary particle horizons are thus metrically two-dimensional light like surfaces analogous to the boundary of the light cone and allow conformal invariance. Elementary particle horizons act as causal horizons. Topologically condensed space-time sheets are analogous to black hole interiors and due to the lack of the causal connectedness the standard description of sub-selves as tensor factors of the state space corresponding to self is not appropriate.

   Hence systems correspond, not to the space-time sheets plus entire hierarchy of space-time sheets condensed to it, but rather, to space-time sheets with holes resulting when the space-time sheets representing subsystems are spliced off along the elementary particle horizons around wormhole contacts. This does not mean that all information about subsystem is lost: subsystem space-time sheet is only replaced by the elementary particle horizon. In analogy with the description of the black hole, some parameters (mass, charges, ...) characterizing the classical fields created by the sub-self space-time sheet characterize sub-self.

   One can say that the state space of the system contains "holes". There is a hierarchy of state spaces labelled by p-adic primes defining length scale resolutions. This picture resolves a longstanding puzzle relating to the interpretation of the fact that particle is characterize by both classical and quantum charges. Particle cannot couple simultaneously to both and this is achieved if quantum charge is associated with the lowest level description of the particle as CP_2 extremal and classical charges to its description at higher levels of hierarchy.

3. The immediate implication indeed is that it is possible to have a situation in which two selves are unentangled although their sub-selves (mental images) are entangled. This corresponds to the fusion and sharing of mental images. The sharing of the mental images means that union of disjoint hierarchy trees with levels labelled by p-adic primes p is replaced by a union of hierarchy threes with horizontal lines connecting subsystems at the same level of hierarchy. Thus the classical correspondence defines a category of presheaves with both vertical arrows replaced by sub-self-self relationship, horizontal arrows representing sharing of mental images, and natural maps representing binding of submoments of consciousness to selves.

3.3.4 Comparison with Goro Kato's approach

It is of interest to compare Goro Kato's approach with TGD approach. The following correspondence suggests itself.

1. In TGD each quantum jumps defines a category analogous to the Goro Kato's category of open sets of some topological space but set theoretic inclusion replaced by topological condensation. The category defined by a moment of consciousness is dynamical whereas the category of open sets is non-dynamical.

2. The assignment of a 3-surface acting as a causal determinant to each unentangled subsystem defined by a moment of consciousness defines a unique "quantum presheaf" which is the counterpart of the presheaf in Goro Kato's theory. The conscious entity of Kato's theory corresponds to the classical correlate for a moment of consciousness.

3. Natural maps between the causal determinants correspond to the space-time correlates for the functor arrows defining the threads connecting submoments of consciousness to selves. In Goro Kato's theory natural maps are interpreted as communications between conscious entities. The sharing of mental images by quantum entanglement between subsystems of unentangled systems defines horizontal bi-directional arrows between subsystems associated with same moment of consciousness and is counterpart of communication in TGD framework. It replaces the union of disjoint hierarchy trees associated with various unentangled subsystems with hierarchy trees having horizontal connections defining the bi-directional arrows. The sharing of mental images is not possible if subsystem is identified as a tensor factor and thus without taking into account length scale resolution.

3.4 Cognizing About Cognition

There are close connections with basic facts about cognition.

1. Categorization means classification and abstraction of common features in the class formed by the objects of a category. Already quantum jump defines category with hierarchical structure and can be regarded as consciously experienced analysis in which totally entangled entire universe $U\Psi_i$ decomposes to a product of maximally unentangled subsystems. The sub-selves of self are like elements of set and are experienced as separate objects whereas sub-sub-selves of sub-self self experiences as an average: they belong to a class or category formed by the sub-self. This kind of averaging occurs also for the contributions of quantum jumps to conscious experience of self.

2. The notions of category theory might be useful in an attempt to construct a theory of cognitive structures since cognition is indeed to high degree classification and abstraction process. The sub-selves of a real self indeed have p-adic space-time sheets as geometric correlates and thus correspond to cognitive sub-selves, thoughts. A meditative experience of empty mind means in case of real self the total absence of thoughts.

3. Predicate logic provides a formalization of the natural language and relies heavily on the notion of n-ary relation. Binary relations $R(a, b)$ corresponds formally to the subset of the product set $A \times B$. For instance, statements like "A does something to B" can be expressed as a binary relation, particular kind of arrow and morphism ($A \leq B$ is a standard example). For sub-selves this relation would correspond to a dynamical evolution at space-time level modelling the interaction between A and B. The dynamical path defined by a sequence of quantum jumps is able to describe this kind of relationships too at level of conscious experience. For instance, "A touches B" would involve the temporary fusion of sub-selves A and B to sub-self C.

4 Logic And Category Theory

Category theory allows naturally more general than Boolean logics inherent to the notion of topos associated with any category. Basic question is whether the ordinary notion of topos algebra based on set theoretic inclusion or the notion of quantum topos based on topological condensation is physically appropriate. Starting from the quasi-Boolean algebra of open sets one ends up to the conclusion that quantum logic is more natural. Also WCW spinor fields lead naturally to the notion of quantum logic.

4.1 Is The Logic Of Conscious Experience Based On Set Theoretic Inclusion Or Topological Condensation?

The algebra of open sets with intersections and unions and complement defined as the interior of the complement defines a modification of Boolean algebra having the peculiar feature that the points at the boundary of the closure of open set cannot be said to belong to neither interior of open set or of its complement. There are two options concerning the interpretation.

1. 3-valued logic could be in question. It is however not possible to understand this three-valuedness if one defines the quasi-Boolean algebra of open sets as Heyting algebra. The resulting logic is two-valued and the points at boundaries of the closure do not correspond neither to the statement or its negation. In p-adic context the situation changes since p-adic open sets are also closed so that the logic is strictly Boolean. That our ordinary cognitive mind is Boolean provides a further good reason for why cognition is p-adic.

2. These points at the boundary of the closure belong to both interior and exterior in which case a two-valued "quantum logic" allowing superposition of opposite truth values is in question. The situation is indeed exactly the same as in the case of space-time sheet having wormhole contacts to several space-time sheets.

The quantum logic brings in mind Zen consciousness [J2] (which I became fascinated of while reading Hofstadter's book "Gödel, Escher, Bach" [A9]) and one can wonder whether selves having real space-time sheets as geometric correlates and able to live simultaneously in many parallel worlds correspond to Zen consciousness and Zen logic. Zen logic would be also logic of sensory experience whereas cognition would obey strictly Boolean logic.

The causal determinants associated with space-time sheets correspond to light like 3-surfaces which could elementary particle horizons or space-time boundaries and possibly also 3-surfaces separating two maximal deterministic regions of a space-time sheet. These surfaces act as 3-dimensional quantum holograms and have the strange Zen property that they are neither space-like nor time-like so that they represent both the state and the process. In the TGD based model for topological quantum computation (TQC) light-like boundaries code for the computation so that TQC program code would be equivalent with the running program [K1].

4.2 Do WCW Spinor Fields Define Quantum Logic And Quantum Topos

I have proposed already earlier that WCW spinor fields define what might be called quantum logic. One can wonder whether WCW spinor s could also naturally define what might be called quantum topos since the category underlying topos defines the logic appropriate to the topos. This question remains unanswered in the following: I just describe the line of though generalizing ordinary Boolean logic.

4.2.1 Finite-dimensional spinors define quantum logic

Spinors at a point of an $2N$-dimensional space span 2^N-dimensional space and spinor basis is in one-one correspondence with Boolean algebra with N different truth values (N bits). 2N=2-dimensional case is simple: Spin up spinor= true and spin-dow spinor=false. The spinors for $2N$-dimensional space are obtained as an N-fold tensor product of 2-dimensional spinors (spin up, spin down): just like in the case of Cartesian power of Ω.

Boolean spinors in a given basis are eigen states for a set N mutually commuting sigma matrices providing a representation for the tangent space group acting as rotations. Boolean spinors define N Boolean statements in the set Ω^N so that one can in a natural manner assign a set with a Boolean spinor. In the real case this group is $SO(2N)$ and reduces to $SU(N)$ for Kähler manifolds. For pseudo-euclidian metric some non-compact variant of the tangent space group is involved. The selections of N mutually commuting generators are labelled by the flag-manifold $SO(2N)/SO(2)^N$ in real context and by the flag-manifold $U(N)/U(1)^N$ in the complex case. The selection of these generators defines a collection of N 2-dimensional linear subspaces of the tangent space.

Spinors are in general complex superpositions of spinor basis which can be taken as the product spinors. The quantum measurement of N spins representing the Cartan algebra of $SO(2N)$

$(SU(N))$ leads to a state representing a definite Boolean statement. This suggests that quantum jumps as moments of consciousness quite generally make universe classical, not only in geometric but also in logical sense. This is indeed what the state preparation process for WCW spinor field seems to do.

4.2.2 Quantum logic for finite-dimensional spinor fields

One can generalize the idea of the spinor logic also to the case of spinor fields. For a given choice of the local spinor basis (which is unique only modular local gauge rotation) spinor field assigns to each point of finite-dimensional space a quantum superposition of Boolean statements decomposing into product of N statements.

Also now one can ask whether it is possible to find a gauge in which each point corresponds to definite Boolean statement and is thus an eigen state of a maximal number of mutually commuting rotation generators Σ_{ij}. This is not trivial if one requires that Dirac equation is satisfied. In the case of flat space this is certainly true and constant spinors multiplied by functions which solve d'Alembert equation provide a global basis.

The solutions of Dirac equation in a curved finite-dimensional space do not usually possess a definite spin direction globally since spinor curvature means the presence of magnetic spin-flipping interaction and since there need not exist a global gauge transformation leading to an eigen state of the local Cartan algebra everywhere. What might happen is that the local gauge transformation becomes singular at some point: for instance, the direction of spin would be radial around given point and become ill defined at the point. This is much like the singularities for vector fields on sphere. The spinor field having this kind of singularity should vanish at singularity but the local gauge rotation rotating spin in same direction everywhere is necessarily ill-defined at the singularity.

In fact, this can be expressed using the language of category theory. The category in question corresponds to a presheaf which assigns to the points of the base space the fiber space of the spinor bundle. The presence of singularity means that there are no global section for this presheaf, that is a continuous choice of a non-vanishing spinor at each point of the base space. The so called Kochen-Specker theorem discussed in [A12] is closely related to a completely analogous phenomenon involving non-existence of global sections and thus non-existence of a global truth value.

Thus in case of curved spaces is not necessarily possible to have spinor field basis representing globally Boolean statements and only the notion of locally Boolean logic makes sense. Indeed, one can select the basis to be eigen state of maximal set of mutually commuting rotation generators in single point of the compact space. Any such choice does.

4.2.3 Quantum logic and quantum topos defined by the prepared WCW spinor fields

The prepared WCW spinor fields occurring as initial and final states of quantum jumps are the natural candidates for defining quantum logic. The outcomes of the quantum jumps resulting in the state preparation process are maximally unentangled states and are as close to Boolean states as possible.

WCW spinors correspond to fermionic Fock states created by infinite number of fermionic (leptonic and quarklike) creation and annihilation operators. The spin degeneracy is replaced by the double-fold degeneracy associated with a given fermion mode: given state either contains fermion or not and these two states represent true and false now. If WCW were flat, the Fock state basis with definite fermion and anti-fermion numbers in each mode would be in one-one correspondence with Boolean algebra.

Situation is however not so simple. Finite-dimensional curved space is replaced with the fiber degrees of freedom of WCW in which the metric is non-vanishing. The precise analogy with the finite-dimensional case suggests that if the curvature form of the WCW spinor connection is nontrivial, it is impossible to diagonalize even the prepared maximally unentangled WCW spinor fields Ψ_i in the entire fiber of WCW (quantum fluctuating degrees of freedom) for given values of the zero modes. Local singularities at which the spin quantum numbers of the diagonalized but vanishing WCW spinor field become ill-defined are possible also now.

In the infinite-dimensional context the presence of the fermion-anti-fermion pairs in the state means that it does not represent a definite Boolean statement unless one defines a more general basis of WCW spinor s for which pairs are present in the states of the state basis: this generalization is indeed possible. The sigma matrices of the WCW appearing in the spinor connection term of the Dirac operator of WCW indeed create fermion-fermion pairs. What is decisive, is not the absence of fermion-anti-fermion pairs, but the possibility that the spinor field basis cannot be reduced to eigen states of the local Cartan algebra in fiber degrees of freedom globally.

Also for bound states of fermions (say leptons and quarks) it is impossible to reduce the state to a definite Boolean statement even locally. This would suggest that fermionic logic does not reduce to a completely Boolean logic even in the case of the prepared states.

Thus WCW spinor fields could have interpretation in terms of non-Boolean quantum logic possessing Boolean logics only as sub-logics and define what might be called quantum topos. Instead of Ω^N-valued maps the values for the maps are complex valued quantum superpositions of truth values in Ω^N.

An objection against the notion of quantum logic is that Boolean algebra operationsandOR do not preserve fermion number so that quantum jump sequences leading from the product state defined by operands to the state representing the result of operation are therefore not possible. One manner to circumvent the objection is to consider the sub-algebra spanned by fermion and anti-fermion pairs for given mode so that fermion number conservation is not a problem. The objection can be also circumvented for pairs of space-time sheets with opposite time orientations and thus opposite signs of energies for particles. One can construct the algebra in question as pairs of many fermion states consisting of positive energy fermion and negative energy anti-fermion so that all states have vanishing fermion number and logical operations become possible. Pairs of MEs with opposite time orientations are excellent candidates for carries of these fermion-anti-fermion pairs.

4.2.4 Quantum classical correspondence and quantum logic

The intuitive idea is that the global Boolean statements correspond to sections of Z^2 bundle. Möbius band is a prototype example here. The failure of a global statement would reduce to the non-existence of global section so that true would transforms to false as one goes around full 2π rotation.

One can ask whether fermionic quantum realization of Boolean logic could have space-time counterpart in terms of Z_2 fiber bundle structure. This would give some hopes of having some connection between category theoretical and fermionic realizations of logic. The following argument stimulated by email discussion with Diego Lucio Rapoport suggests that this might be the case.

1. The hierarchy of Planck constants realized using the notion of generalized embedding space involves only groups $Z_{n_a} \times Z_{n_b}$, $n_a, n_b \neq 2$ if one takes Jones inclusions as starting point. There is however no obvious reason for excluding the values $n_a = 2$ and $n_b = 2$ and the question concerns physical interpretation. Even if one allows only $n_i \geq 3$ one can ask for the physical interpretation for the factorization $Z_{2n} = Z_2 \times Z_n$. Could it perhaps relate to a space-time correlates for Boolean two-valuedness?

2. An important implication of fiber bundle structure is that the partonic 2-surfaces have $Z_{n_a} \times Z_{n_b} = Z_{n_a n_b}$ as a group of conformal symmetries. I have proposed that n_a or n_b is even for fermions so that Z_2 acts as a conformal symmetry of the partonic 2-surface. Both n_a and n_b would be odd for truly elementary bosons. Note that this hypothesis makes sense also for $n_i \geq 3$.

3. Z_2 conformal symmetry for fermions would imply that all partonic 2-surfaces associated with fermions are hyper-elliptic. As a consequence elementary particle vacuum functionals defined in modular degrees of freedom would vanish for fermions for genus $g > 2$ so that only three fermion families would be possible in accordance with experimental facts. Since gauge bosons and Higgs correspond to pairs of partonic 2-surfaces (the throats of the wormhole contact) one has 9 gauge boson states labelled by the pairs (g_1, g_2) which can be grouped to SU(3) singlet and octet. Singlet corresponds to ordinary gauge bosons.

super-symplectic bosons are truly elementary bosons in the sense that they do not consist of fermion-anti-fermion pairs. For them both n_a and n_b should be odd if the correspondence is taken seriously and all genera would be possible. The super-conformal partners of these bosons have the quantum numbers of right handed neutrino. Since both spin directions are possible, one can ask whether Boolean Z_2 must be present also now. This need not be the case, ν_R generates only super-symmetries and does not define a family of fermionic oscillator operators. The electro-weak spin of ν_R is frozen and it does not couple at all to electro-weak intersections. Perhaps (only) odd values of n_i are possible in this case.

4. If fermionic Boolean logic has a space-time correlate, one can wonder whether the fermionic Z_2 conformal symmetry might correspond to a space-time correlate for the Boolean true-false dichotomy. If the partonic 2-surface contains points which are fixed points of Z_2 symmetry, there exists no everywhere non-vanishing sections. Furthermore, induced spinor fields should vanish at the fixed points of Z_2 symmetry since they correspond to singular orbifold points so that one could not actually have a situation in which true and false are true simultaneously. Global sections could however fail to exist since CP_2 spinor bundle is non-trivial.

4.3 Category Theory And The Modelling Of Aesthetic And Ethical Judgements

Consciousness theory should allow to model model the logics of ethics and aesthetics. Evolution (representable as p-adic evolution in TGD framework) is regarded as something positive and is a good candidate for defining universal ethics in TGD framework. Good deeds are such that they support this evolution occurring in statistical sense in any case. Moral provides a practical model for what good deeds are and moral right-wrong statements are analogous to logical statements. Often however the two-valued right-wrong logic seems to be too simplistic in case of moral statements. Same applies to aesthetic judgements. A possible application of the generalized logics defined by the inherent structure of categories relates to the understanding of the dilemmas associated with the moral and aesthetic rules.

As already found, quantum versions of sieves provide a formal generalization of Boolean truth values as a characteristic of a given category. Generalized moral rules could perhaps be seen as sieve valued statements about deeds. Deeds are either right or wrong in what might be called Boolean moral code. One can also consider Zen moral in which some deeds can be said to be right and wrong simultaneously. Some deeds could also be such that there simply exists no globally consistent moral rule: this would correspond to the non-existence of what is called global section assigning to each object of the category consisting of the pairs formed by a moral agents and given deed) a sieve simultaneously.

5 Platonism, Constructivism, And Quantum Platonism

During years I have been trying to understand how Category Theory and Set Theory relate to quantum TGD inspired view about fundamentals of mathematics and the outcome section is added to this chapter several years after its first writing. I hope that reader does not experience too unpleasant discontinuity. I managed to clarify my thoughts about what these theories are by reading the article Structuralism, Category Theory and Philosophy of Mathematics by Richard Stefanik [A19]. Blog discussions and email correspondence with Sampo Vesterinen have been very stimulating and inspired the attempt to represent TGD based vision about the unification of mathematics, physics, and consciousness theory in a more systematic manner.

Before continuing I want to summarize the basic ideas behind TGD vision. One cannot understand mathematics without understanding mathematical consciousness. Mathematical consciousness and its evolution must have direct quantum physical correlates and by quantum classical correspondence these correlates must appear also at space-time level. Quantum physics must allow to realize number as a conscious experience analogous to a sensory quale. In TGD based ontology there is no need to postulate physical world behind the quantum states as mathematical entities (theory is the reality). Hence number cannot be any physical object, but can be identified as a quantum state or its label and its number theoretical anatomy is revealed by the conscious experiences induced by the number theoretic variants of particle reactions. Mathematical systems and their axiomatics are dynamical evolving systems and physics is number theoretically universal selecting rationals and their extensions in a special role as numbers, which can can be regarded elements of several number fields simultaneously.

5.1 Platonism And Structuralism

There are basically two philosophies of mathematics.

1. Platonism assumes that mathematical objects and structures have independent existence. Natural numbers would be the most fundamental objects of this kind. For instance, each natural number has its own number-theoretical anatomy decomposing into a product of prime numbers defining the elementary particles of Platonia. For quantum physicist this vision is attractive, and even more so if one accepts that elementary particles are labelled by primes (as I do)! The problematic aspects of this vision relate to the physical realization of the Platonia. Neither Minkowski space-time nor its curved variants understood in the sense of set theory have no room for Platonia and physical laws (as we know them) do not seem to allow the realization of all imaginable internally consistent mathematical structures.

2. Structuralist believes that the properties of natural numbers result from their relations to other natural numbers so that it is not possible to speak about number theoretical anatomy in the Platonic sense. Numbers as such are structureless and their relationships to other numbers provide them with their apparent structure. According to [A19] structuralism is however not enough for the purposes of number theory: in combinatorics it is much more natural to use intensional definition for integers by providing them with inherent properties such as decomposition into primes. I am not competent to take any strong attitudes on this statement but my physicist's intuition tells that numbers have number theoretic anatomy and that this anatomy can be only revealed by the morphisms or something more general which must have physical counterparts. I would like to regard numbers are analogous to bound states of elementary particles. Just as the decays of bound states reveal their inner structure, the generalizations of morphisms would reveal to the mathematician the inherent number theoretic anatomy of integers.

5.2 Structuralism

Set theory and category theory represent two basic variants of structuralism and before continuing I want to clarify to myself the basic ideas of structuralism: the reader can skip this section if it looks too boring.

5.2.1 Set theory

Structuralism has many variants. In set theory [A4] the elements of set are treated as structureless points and sets with the same cardinality are equivalent. In number theory additional structure must be introduced. In the case of natural numbers one introduces the notion of successor and induction axiom and defines the basic arithmetic operations using these. Set theoretic realization is not unique. For instance, one can start from empty set Φ identified as 0, identify 1 as $\{\Phi\}$, 2 as $\{0,1\}$ and so on. One can also identify 0 as Φ, 1 as $\{0\}$, 2 as $\{\{0\}\}$, ... For both physicist and consciousness theorist these formal definitions look rather weird.

The non-uniqueness of the identification of natural numbers as a set could be seen as a problem. The structuralist's approach is based on an extensional definition meaning that two objects are regarded as identical if one cannot find any property distinguishing them: object is a representative for the equivalence class of similar objects. This brings in mind gauge fixing to the mind of physicists.

5.2.2 Category theory

Category theory [A1] represents a second form of structuralism. Category theorist does not worry about the ontological problems and dreams that all properties of objects could be reduced to the arrows and formally one could identify even objects as identity morphisms (looks like a trick to me). The great idea is that functors between categories respecting the structure defined by morphisms provide information about categories. Second basic concept is natural transformation which maps functors to functors in a structure preserving manner. Also functors define a category so that one can construct endless hierarchy of categories. This approach has enormous unifying power since functors and natural maps systemize the process of generalization. There is no doubt that category theory forms a huge piece of mathematics but I find difficult to believe that arrows can catch all of it.

The notion of category can be extended to that of n-category. In the blog post "First edge of the cube" (see `http://tinyurl.com/yydjavv8`) I have proposed a geometric realization of this hierarchy in which one defines 1-morphisms by parallel translations, 2-morphisms by parallel translations of parallel translations, and so on. In infinite-dimensional space this hierarchy would be infinite. Abstractions about abstractions about.., thoughts about thoughts about, statements about statements about..., is the basic idea behind this interpretation. Also the hierarchy of logics of various orders corresponds to this hierarchy. This encourages to see category theoretic thinking as being analogous to higher level self reflection which must be distinguished from the direct sensory experience.

In the case of natural numbers category theoretician would identify successor function as the arrow binding natural numbers to an infinitely long string with 0 as its end. If this approach would work, the properties of numbers would reflect the properties of the successor function.

5.3 The View About Mathematics Inspired By TGD And TGD Inspired Theory Of Consciousness

TGD based view might be called quantum Platonism. It is inspired by the requirement that both quantum states and quantum jumps between them are able to represent number theory and that all quantum notions have also space-time correlates so that Platonia should in some sense exist also at the level of space-time. Here I provide a brief summary of this view as it is now.

5.3.1 Physics is fixed from the uniqueness of infinite-D existence and number theoretic universality

1. The basic philosophy of quantum TGD relies on the geometrization of physics in terms of infinite-dimensional Kähler geometry of WCW , whose uniqueness is forced by the mere mathematical existence. Space-time dimension and embedding space $H = M^4 \times CP_2$ are fixed among other things by this condition and allow interpretation in terms of classical number fields. Physical states correspond to WCW spinor fields with WCW spinor s having interpretation as Fock states. Rather remarkably, WCW Clifford algebra defines standard representation of so called hyper finite factor of II_1, perhaps the most fascinating von Neumann algebra.

2. Number theoretic universality states that all number fields are in a democratic position. This vision can be realized by requiring generalization of notions of embedding space by gluing together real and p-adic variants of embedding space along common algebraic numbers. All algebraic extensions of p-adic numbers are allowed. Real and p-adic space-time sheets intersect along common algebraics. The identification of the p-adic space-time sheets as correlates of cognition and intentionality explains why cognitive representations at space-time level are always discrete. Only space-time points belonging to an algebraic extension of rationals associated contribute to the data defining S-matrix. These points define what I call number theoretic braids. The interpretation in of algebraic discreteness terms of a physical realization of axiom of choice is highly suggestive. The axiom of choice would be dynamical and evolving quantum jump by quantum jump as the algebraic complexity of quantum states increases.

5.3.2 Holy trinity of existence

In TGD framework one would have 3-levelled ontology numbers should have representations at all these levels [L1].

1. Subjective existence as a sequence of quantum jumps giving conscious sensory representations for numbers and various geometric structures would be the first level.

2. Quantum states would correspond to Platonia of mathematical ideas and mathematician- or if one is unwilling to use this practical illusion- conscious experiences about mathematic ideas, would be in quantum jumps. The quantum jumps between quantum states respecting the symmetries characterizing the mathematical structure would provide conscious information about the mathematical ideas not directly accessible to conscious experience. Mathematician would live in Plato's cave. There is no need to assume any independent physical reality behind quantum states as mathematical entities since quantum jumps between these states give rise to conscious experience. Theory-reality dualism disappears since the theory is reality or more poetically: painting is the landscape.

3. The third level of ontology would be represented by classical physics at the space-time level essential for quantum measurement theory. By quantum classical correspondence space-time physics would be like a written language providing symbolic representations for both quantum states and changes of them (by the failure of complete classical determinism of the fundamental variational principle). This would involve both real and p-adic space-time sheets corresponding to sensory and cognitive representations of mathematical concepts. This representation makes possible the feedback analogous to formulas written by mathematician crucial for the ability of becoming conscious about what one was conscious of and the dynamical character of this process allows to explain the self-referentiality of consciousness without paradox.

This ontology releases a deep Platonistic sigh of relief. Since there are no physical objects, there is no need to reduce mathematical notions to objects of the physical world. There are only quantum states identified as mathematical entities labelled naturally by integer valued quantum numbers; conscious experiences, which must represent sensations giving information about the number theoretical anatomy of a given quantum number; and space-time surfaces providing space-time correlates for quantum physics and therefore also for number theory and mathematical structures in general.

5.3.3 Factorization of integers as a direct sensory perception?

Both physicist and consciousness theorist would argue that the set theoretic construction of natural numbers could not be farther away from how we experience integers. Personally I feel that neither structuralist's approach nor Platonism as it is understood usually are enough. Mathematics is a conscious activity and this suggests that quantum theory of consciousness must be included if one wants to build more satisfactory view about fundamentals of mathematics.

Oliver Sack's book *The man who mistook his wife for a hat* [J1] (see also [K13]) contains fascinating stories about those aspects of brain and consciousness which are more or less mysterious from the view point of neuroscience. Sacks tells in his book also a story about twins who were classified as idiots but had amazing number theoretical abilities. I feel that this story reveals something very important about the real character of mathematical consciousness.

The twins had absolutely no idea about mathematical concepts such as the notion of primeness but they could factorize huge numbers and tell whether they are primes. Their eyes rolled wildly during the process and suddenly their face started to glow of happiness and they reported a discovery of a factor. One could not avoid the feeling that they quite concretely saw the factorization process. The failure to detect the factorization served for them as the definition of primeness. For them the factorization was not a process based on some rules but a direct sensory perception.

The simplest explanation for the abilities of twins would in terms of a model of integers represented as string like structures consisting of identical basic units. This string can decay to strings. If string containing n units decaying into $m > 1$ identical pieces is not perceived, the conclusion is that a prime is in question. It could also be that decay to units smaller than 2 was forbidden in this dynamics. The necessary connection between written representations of numbers and representative strings is easy to build as associations.

This kind theory might help to understand marvellous feats of mathematicians like Ramanujan who represents a diametrical opposite of Groethendienck as a mathematician (when Groethendienck was asked to give an example about prime, he mentioned 57 which became known as Groethendienck prime!).

The lesson would be that one very fundamental representation of integers would be, not as objects, but conscious experiences. Primeness would be like the quale of redness. This of course does not exclude also other representations.

5.3.4 Experience of integers in TGD inspired quantum theory of consciousness

In quantum physics integers appear very naturally as quantum numbers. In quantal axiomatization or interpretation of mathematics same should hold true.

1. In TGD inspired theory of consciousness [L1] quantum jump is identified as a moment of consciousness. There is actually an entire fractal hierarchy of quantum jumps consisting of quantum jumps and this correlates directly with the corresponding hierarchy of physical states and dark matter hierarchy. This means that the experience of integer should be reducible to a certain kind of quantum jump. The possible changes of state in the quantum jump would characterize the sensory representation of integer.

2. The quantum state as such does not give conscious information about the number theoretic anatomy of the integer labelling it: the change of the quantum state is required. The above geometric model translated to quantum case would suggest that integer represents a multiplicatively conserved quantum number. Decays of this this state into states labelled by integers n_i such that one has $n = \prod_i n_i$ would provide the fundamental conscious representation for the number theoretic anatomy of the integer. At the level of sensory perception based the space-time correlates a string-like bound state of basic particles representing n=1.

3. This picture is consistent with the Platonist view about integers represented as structured objects, now labels of quantum states. It would also conform with the view of category theorist in the sense that the arrows of category theorist replaced with quantum jumps are necessary to gain conscious information about the structure of the integer.

5.3.5 Infinite primes and arithmetic consciousness

Infinite primes [K15] were the first mathematical fruit of TGD inspired theory of consciousness and the inspiration for writing this posting came from the observation that the infinite primes at the lowest level of hierarchy provide a representation of algebraic numbers as Fock states of a super-symmetric arithmetic QFT so that it becomes possible to realize quantum jumps revealing the number theoretic anatomy of integers, rationals, and perhaps even that of algebraic numbers.

1. Infinite primes have a representation as Fock states of super-symmetric arithmetic QFT and at the lowest level of hierarchy they provide representations for primes, integers, rationals and algebraic numbers in the sense that at the lowest level of hierarchy of second quantizations the simplest infinite primes are naturally mapped to rationals whereas more complex infinite primes having interpretation as bound states can be mapped to algebraic numbers. Conscious experience of number can be assigned to the quantum jumps between these quantum states revealing information about the number theoretic anatomy of the number represented. It would be wrong to say that rationals only label these states: rather, these states represent rationals and since primes label the particles of these states.

2. More concretely, the conservation of number theoretic energy defined by the logarithm of the rational assignable with the Fock state implies that the allowed decays of the state to a product of infinite integers are such that the rational can decompose only into a product of rationals. These decays could provide for the above discussed fundamental realization of multiplicative aspects of arithmetic consciousness. Also additive aspects are represented since the exponents k in the powers p^k appearing in the decomposition are conserved so that only the partitions $k = \sum_i k_i$ are representable. Thus both product decompositions and partitions, the basic operations of number theorist, are represented.

3. The higher levels of the hierarchy represent a hierarchy of abstractions about abstractions bringing strongly in mind the hierarchy of n-categories and various similar constructions including n: th order logic. It also seems that the n+1: th level of hierarchy provides a quantum representation for the n: th level. Ordinary primes, integers, rationals, and algebraic numbers would be the lowest level, -the initial object- of the hierarchy representing nothing at low level. Higher levels could be reduced to them by the analog of category theoretic reductionism in the sense that there is arrow between n: th and n+1: th level representing the second quantization at this level. On can also say that these levels represent higher reflective level of mathematical consciousness and the fundamental sensory perception corresponds the lowest level.

4. Infinite primes have also space-time correlates. The decomposition of particle into partons can be interpreted as a infinite prime and this gives geometric representations of infinite primes and also rationals. The finite primes appearing in the decomposition of infinite prime correspond to bosonic or fermionic partonic 2-surfaces. Many-sheeted space-time provides a representation for the hierarchy of second quantizations: one physical prediction is that many particle bound state associated with space-time sheet behaves exactly like a boson or fermion. Nuclear string model is one concrete application of this idea: it replaces nucleon reductionism with reductionism occurs first to strings consisting of $A \leq 4$ nuclei and which in turn are strings consisting of nucleons. A further more speculative representation of infinite rationals as space-time surfaces is based on their mapping to rational functions.

5.3.6 Number theoretic Brahman=Atman identity

The notion of infinite primes leads to the notion of algebraic holography in which space-time points possess infinitely rich number-theoretic anatomy. This anatomy would be due to the existence of infinite number of real units defined as ratios of infinite integers which reduce to unit in the real sense and various p-adic senses. This anatomy is not visible in real physics but can contribute directly to mathematical consciousness [K15].

The anatomies of single space-time point could represent the entire world of classical worlds and quantum states of universe: the number theoretic anatomy is of course not visible in the structure of these these states. Therefore the basic building brick of mathematics - point- would become the Platonia able to represent all of the mathematics consistent with the laws of quantum physics. Space-time points would evolve, becoming more and more complex quantum jump by quantum jump. WCW and quantum states would be represented by the anatomies of space-time points. Some space-time points are more "civilized" than others so that space-time decomposes into "civilizations" at different levels of mathematical evolution.

Paths between space-time points represent processes analogous to parallel translations affecting the structure of the point and one can also define n-parallel translations up to $n = 4$ at level of space-time and $n = 8$ at level of embedding space. At level of world of classical worlds whose points are representable as number theoretical anatomies arbitrary high values of n can be realized.

It is fair to say that the number theoretical anatomy of the space-time point makes it possible self-reference loop to close so that structured points are able to represent the physics of associated with with the structures constructed from structureless points. Hence one can speak about algebraic holography or number theoretic Brahman=Atman identity.

5.3.7 Finite measurement resolution, Jones inclusions, and number theoretic braids

In the history of physics and mathematics the realization of various limitations have been the royal road to a deeper understanding (Uncertainty Principle, Gödel's theorem). The precision of quantum measurement, sensory perception, and cognition are always finite. In standard quantum measurement theory this limitation is not taken into account but forms a corner stone of TGD based vision about quantum physics and of mathematics too as I want to argue in the following.

The finite resolutions has representation both at classical and quantum level.

1. At the level of quantum states finite resolution is represented in terms of Jones inclusions N subset M of hyper-finite factors of type II_1 (HFFs) [K8]. N represents measurement resolution in the sense that the states related by the action of N cannot be distinguished

in the measurement considered. Complex rays are replaced by N rays. This brings in non-commutativity via quantum groups [K2]. Non-commutativity in TGD Universe would be therefore due to a finite measurement resolution rather than something exotic emerging in the Planck length scale. Same applies to p-adic physics: p-adic space-time sheets have literally infinite size in real topology!

2. At the space-time level discretization implied by the number theoretic universality could be seen as being due to the finite resolution with common algebraic points of real and p-adic variant of the partonic 3-surface chosen as representatives for regions of the surface. The solutions of Kähler-Dirac equation are characterized by the prime in question so that the preferred prime makes itself visible at the level of quantum dynamics and characterizes the p-adic length scale fixing the values of coupling constants. Discretization could be also understood as effective non-commutativity of embedding space points due to the finite resolution implying that second quantized spinor fields anti-commute only at a discrete set of points rather than along stringy curve.

In this framework it is easy to imagine physical representations of number theoretical and other mathematical structures.

1. Every compact group corresponds to a hierarchy of Jones inclusions corresponding to various representations for the quantum variants of the group labelled by roots of unity. I would be surprised if non-compact groups would not allow similar representation since HFF can be regarded as infinite tensor power of n-dimensional complex matrix algebra for any value of n. Somewhat paradoxically, the finite measurement resolution would make possible to represent Lie group theory physically [K8].

2. There is a strong temptation to identify the Galois groups of algebraic numbers as the infinite permutation group S_∞ consisting of permutations of finite number of objects, whose projective representations give rise to an infinite braid group B_∞. The group algebras of these groups are HFFs besides the representation provided by the spinors of the world of classical worlds having physical identification as fermionic Fock states. Therefore physical states would provide a direct representation also for the more abstract features of number theory [K10].

3. Number theoretical braids crucial for the construction of S-matrix provide naturally representations for the Galois groups G associated with the algebraic extensions of rationals as diagonal embeddings $G \times G \times ....$ to the completion of S_∞ representable also as the action on the completion of spinors in the world of classical worlds so that the core of number theory would be represented physically [K10]. At the space-time level number theoretic braid having G as symmetries would represent the G. These representations are analogous to global gauge transformations. The elements of S_∞ are analogous to local gauge transformations having a natural identification as a universal number theoretical gauge symmetry group leaving physical states invariant.

5.3.8 Hierarchy of Planck constants and the generalization of embedding space

Jones inclusions inspire a further generalization of the notion of embedding space obtained by gluing together copies of the embedding space H regarded as coverings $H \rightarrow H/G_a \times G_b$. In the simplest scenario $G_a \times G_b$ leaves invariant the choice of quantization axis and thus this hierarchy provides embedding space correlate for the choice of quantization axes inducing these correlates also at space-time level and at the level of world of classical worlds [K8].

Dark matter hierarchy is identified in terms of different sectors of H glued together along common points of base spaces and thus forming a book like structure. For the simplest option elementary particles proper correspond to maximally quantum critical systems in the intersection of all pages. The field bodies of elementary particles are in the interiors of the pages of this "book".

One can assign to Jones inclusions quantum phase $q = exp(i2\pi/n)$ and the groups Z_n acts as exact symmetries both at level of M^4 and CP_2. In the case of M^4 this means that space-time sheets have exact Z_n rotational symmetry. This suggests that the algebraic numbers q^m could have geometric representation at the level of sensory perception as Z_n symmetric objects. We need not be conscious of this representation in the ordinary wake-up consciousness dominated by sensory perception of ordinary matter with $q = 1$. This would make possible the idea about transcendentals like π, which do not appear in any finite-dimensional extension of even p-adic numbers (p-adic numbers allow finite-dimensional extension by since e^p is ordinary p-adic number). Quantum jumps in which state suffers an action of the generating element of Z_n could also provide a sensory realization of these groups and numbers $exp(i2\pi/n)$.

Planck constant is identified as the ratio n_a/n_b of integers associated with M^4 and CP_2 degrees of freedom so that a representation of rationals emerge again. The so called ruler and compass rationals whose definition involves only a repeated square root operation applied on rationals are cognitively the simplest ones and should appear first in the evolution of mathematical consciousness. The successful [K7] quantum model for EEG is only one of the applications providing support for their preferred role. Other applications are to Bohr quantization of planetary orbits interpreted as being induced by the presence of macroscopically quantum coherent dark matter [K14].

5.4 Farey Sequences, Riemann Hypothesis, Tangles, And TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *"Platonia as the best possible world in the sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

5.4.1 Farey sequences

Some basic facts about Farey sequences [A2] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \quad (5.1)$$

Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p + 1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p + 1)$ to the sequence.

The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the embedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [K5] would naturally relate to the Farey sequence.

5.4.2 Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \quad .$$

In other words, $d_{n,N}$ is the difference between the n: th term of the N: th Farey sequence, and the n: th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\begin{aligned} \sum_{n=1,\ldots,|F_N|} |d_{n,N}| &= O(N^r) \text{ for any } r > 1/2 \ , \\ \sum_{n=1,\ldots,|F_N|} d_{n,N}^2 &= O(N^r) \text{ for any } r > 1 \ . \end{aligned} \tag{5.2}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

5.4.3 Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [K5, K3].

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [K5].

5.4.4 Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$: th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

5.4.5 Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [A17] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2, Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles $[0]$, $[\infty]$, $\pm[1]$, $\pm 1/[1]$, $\pm[2]$, $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M+N-2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M+N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N > 2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N >> 2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N > 2$ case.

1. Could the commutativity of tangle product allow to characterize the $N > 2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N > 2$. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N: th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the

non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a,b]^T \rightarrow a/b$ from a rational 2-spinor $[a,b]^T$ to which SL(2(N-1), Z) acts. Equivalence means that the columns $[a,b]^T$ and $[c,d]^T$ combine to form element of SL(2, Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N-1$ $2(N-1)$-component projective column-spinors $[a_i^1, a_i^2, .., a_i^{2(N-1)}]^T$, $i = 1, ...N-1$ so that only the ratios $a_i^k/a_i^{2(N-1)} \leq 1$ matter? Could equivalence for them mean that the $N-1$ spinors combine to form $N-1+N-1$ columns of $SL(2(N-1), Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc = \pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g, Z) so that $N-1$ would be analogous to g and $1 \leq N \geq 3$- braids would correspond to $g \leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2,Q)$ labelled by N (the generator $\tau \rightarrow \tau + 2$ of modular group is replaced with $\tau \rightarrow \tau + 2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N-1), Q)$. Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N: th root of unity instead of phase U satisfying $U^2 = 1$?

5.4.6 How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N: th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

6 Quantum Quandaries

John Baez's [A14] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories

(TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

6.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

6.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups [K2] too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

6.3 Tqft As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

6.4 The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

6.4.1 Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A8] only the exotic diffeo-structures modify the situation in 4-D case.

6.4.2 Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

6.4.3 Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of $2 \rightarrow 2$ reaction open string is pinched to a point at vertex. $1 \rightarrow 2$ vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

6.4.4 Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

6.4.5 Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. There is U-matrix acting in zero energy states. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K11].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics.

7 How To Represent Algebraic Numbers As Geometric Objects?

Physics blogs are also interesting because they allow to get some grasp about very different styles of thinking of a mathematician and physicist. For mathematician it is very important that the result is obtained by a strict use of axioms and deduction rules. Physicist is a cognitive opportunist: it does not matter how the result is obtained by moving along axiomatically allowed paths or not, and the new result is often more like a discovery of a new axiom and physicist is ever-grateful for Gödel for giving justification for what sometimes admittedly degenerates to a creative hand-waving. For physicist ideas form a kind of bio-sphere and the fate of the individual idea depends on its ability to survive, which is determined by its ability to become generalized, its consistency with other ideas, and ability to interact with other ideas to produce new ideas.

7.1 Can One Define Complex Numbers As Cardinalities Of Sets?

During few days before writing this we have had in Kea's blog a little bit of discussion inspired by the problem related to the categorification of basic number theoretical structures. I have learned that sum and product are natural operations for the objects of category. For instance, one can define sum as in terms of union of sets or direct sum of vector spaces and product as Cartesian product of sets and tensor product of vector spaces: rigs [A6] are example of categories for which natural numbers define sum and product.

Subtraction and division are however problematic operations. Negative numbers and inverses of integers do not have a realization as a number of elements for any set or as dimension of vector space. The naïve physicist inside me asks immediately: why not go from statics to dynamics and take operations (arrows with direction) as objects: couldn't this allow to define subtraction and division? Is the problem that the axiomatization of group theory requires something which purest categorification does not give? Or aren't the numbers representable in terms of operations of finite groups not enough? In any case cyclic groups would allow to realize roots of unity as operations (Z_2 would give -1).

One could also wonder why the algebraic numbers might not somehow result via the representations of permutation group of infinite number of elements containing all finite groups and thus Galois groups of algebraic extensions as subgroups? Why not take the elements of this group as objects of the basic category and continue by building group algebra and hyper-finite factors of type II_1 isomorphic to spinors of world of classical worlds, and so on.

After having written the first half of the section, I learned that something similar to the transition from statics to dynamics is actually carried out but by manner which is by many orders of magnitudes more refined than the proposal above and that I had never been able to imagine. The article *Objects of categories as complex numbers* of Marcelo Fiore and Tom Leinster [A6] describes a fascinating idea summarized also by John Baez [A5] about how one can assign to the objects of a category complex numbers as roots of a polynomial $Z = P(Z)$ defining an isomorphism of object. Z is the element of a category called rig, which differs from ring in that integers are replaced with natural numbers. One can replace Z with a complex number $|Z|$ defined as a root of polynomial. $|Z|$ is interpreted formally as the cardinality of the object. It is essential to have natural numbers and thus only product and sum are defined. This means a restriction: for instance, only complex algebraic numbers associated with polynomials having natural numbers as coefficients are obtained. Something is still missing.

Note that this correspondence assumes the existence of complex numbers and one cannot say that complex numbers are categorified. Maybe basic number fields must be left outside categorification. One can however require that all of them have a concrete set theoretic representation rather than only formal interpretation as cardinality so that one still encounters the problem how to represent algebraic complex number as a concrete cardinality of a set.

7.2 In What Sense A Set Can Have Cardinality -1?

The discussion in Kea's blog led me to ask what the situation is in the case of p-adic numbers. Could it be possible to represent the negative and inverse of p-adic integer, and in fact any p-adic number, as a geometric object? In other words, does a set with -1 or $1/n$ or even $\sqrt{-1}$ elements exist? If this were in some sense true for all p-adic number fields, then all this wisdom combined together might provide something analogous to the adelic representation for the norm of a rational number as product of its p-adic norms. As will be found, alternative interpretations of complex algebraic numbers as p-adic numbers representing cardinalities of p-adic fractals emerge. The fractal defines the manner how one must do an infinite sum to get an infinite real number but finite p-adic number.

Of course, this representation might not help to define p-adics or reals categorically but might help to understand how p-adic cognitive representations defined as subsets for rational intersections of real and p-adic space-time sheets could represent p-adic number as the number of points of p-adic fractal having infinite number of points in real sense but finite in the p-adic sense. This would also give a fundamental cognitive role for p-adic fractals as cognitive representations of numbers.

7.2.1 How to construct a set with -1 elements?

The basic observation is that p-adic -1 has the representation

$$-1 = (p-1)/(1-p) = (p-1)(1+p+p^2+p^3....)$$

As a real number this number is infinite or -1 but as a p-adic number the series converges and has p-adic norm equal to 1. One can also map this number to a real number by canonical identification taking the powers of p to their inverses: one obtains p in this particular case. As a matter fact, any rational with p-adic norm equal to 1 has similar power series representation.

The idea would be to represent a given p-adic number as the infinite number of points (in real sense) of a p-adic fractal such that p-adic topology is natural for this fractal. This kind of fractals can be constructed in a simple manner: from this more below. This construction allows to represent any p-adic number as a fractal and code the arithmetic operations to geometric operations for these fractals.

These representations - interpreted as cognitive representations defined by intersections of real and p-adic space-time sheets - are in practice approximate if real space-time sheets are assumed to have a finite size: this is due to the finite p-adic cutoff implied by this assumption and the meaning a finite resolution. One can however say that the p-adic space-time itself could by its necessarily infinite size represent the *idea* of given p-adic number faithfully.

This representation applies also to the p-adic counterparts of algebraic numbers in case that they exist. For instance, roughly one half of p-adic numbers have square root as ordinary p-adic number and quite generally algebraic operations on p-adic numbers can give rise to p-adic numbers so that also these could have set theoretic representation. For $p\ mod\ 4 = 1$ also $\sqrt(-1)$ exists: for instance, for $p = 5$: $2^2 = 4 = -1\ mod\ 5$ guarantees this so that also imaginary unit and complex numbers would have a fractal representation. Also many transcendentals possess this kind of representation. For instance $exp(xp)$ exists as a p-adic number if x has p-adic norm not larger than 1: also $log(1 + xp)$ does so.

Hence a quite impressive repertoire of p-adic counterparts of real numbers would have representation as a p-adic fractal for some values of p. Adelic vision would suggest that combining these representations one might be able to represent quite a many real numbers. In the case of π I do not find any obvious p-adic representation (for instance $sin(\pi/6) = 1/2$ does not help since the p-adic variant of the Taylor expansion of $\pi/6 = arcsin(1/2)$ does not converge p-adically for any value of p). It might be that there are very many transcendentals not allowing fractal representation for any value of p.

7.2.2 Conditions on the fractal representations of p-adic numbers

Consider now the construction of the fractal representations in terms of rational intersections of real real and p-adic space-time sheets. The question is what conditions are natural for this representation if it corresponds to a cognitive representation is realized in the rational intersection of real and p-adic space-time sheets obeying same algebraic equations.

1. Pinary cutoff is the analog of the decimal cutoff but is obtained by dropping away high positive rather than negative powers of p to get a finite real number: example of pinary cutoff is $-1 = (p-1)(1+p+p^2+...) \rightarrow (p-1)(1+p+p^2)$. This cutoff must reduce to a fractal cutoff meaning a finite resolution due to a finite size for the real space-time sheet. In the real sense the p-adic fractal cutoff means not forgetting details below some scale but cutting out all above some length scale. Physical analog would be forgetting all frequencies below some cutoff frequency in Fourier expansion.

   The motivation comes from the fact that TGD inspired consciousness assigns to a given biological body there is associated a field body or magnetic body containing dark matter with large $\hbar$ and quantum controlling the behavior of biological body and so strongly identifying with it so as to belief that this all ends up to a biological death. This field body has an onion like fractal structure and a size of at least order of light-life. Of course, also larger onion layers could be present and would represent those levels of cognitive consciousness not depending on the sensory input on biological body: some altered states of consciousness could relate to these levels. In any case, the larger the magnetic body, the better the numerical skills of the p-adic mathematician.

2. Lowest pinary digits of $x = x_0 + x_1p + x_2p^2 + ...,\ x_n \leq p$ must have the most reliable representation since they are the most significant ones. The representation must be also highly redundant to guarantee reliability. This requires repetitions and periodicity. This is guaranteed if the representation is hologram like with segments of length p^n with digit x_n represented again and again in all segments of length p^m, $m > n$.

3. The TGD based physical constraint is that the representation must be realizable in terms of induced classical fields assignable to the field body hierarchy of an intelligent system interested in artistic expression of p-adic numbers using its own field body as instrument. As a matter, sensory and cognitive representations are realized at field body in TGD Universe and EEG is in a fundamental role in building this representation. By p-adic fractality fractal

wavelets are the most natural candidate. The fundamental wavelet should represent the p different pinary digits and its scaled up variants would correspond to various powers of p so that the representation would reduce to a Fourier expansion of a classical field.

7.2.3 Concrete representation

Consider now a concrete candidate for a representation satisfying these constraints.

1. Consider a p-adic number

$$y = p^{n_0}x, \quad x = \sum x_n p^n \quad , \quad n \geq n_0 = 0 \quad .$$

   If one has a representation for a p-adic unit x the representation of is by a purely geometric fractal scaling of the representation by p^n. Hence one can restrict the consideration to p-adic units.

2. To construct the representation take a real line starting from origin and divide it into segments with lengths $1, p, p^2, ...$ In TGD framework this scalings come actually as powers of $p^{1/2}$ but this is just a technical detail.

3. It is natural to realize the representation in terms of periodic field patterns. One can use wavelets with fractal spectrum $p^n\lambda_0$ of “wavelet lengths”, where λ_0 is the fundamental wavelength. Fundamental wavelet should have p different patterns correspond to the p values of pinary digit as its structures. Periodicity guarantees the hologram like character enabling to pick n: th digit by studying the field pattern in scale p^n anywhere inside the field body.

4. Periodicity guarantees also that the intersections of p-adic and real space-time sheets can represent the values of pinary digits. For instance, wavelets could be such that in a given p-adic scale the number of rational points in the intersection of the real and p-adic space-time sheet equals to x_n. This would give in the limit of an infinite pinary expansion a set theoretic realization of any p-adic number in which each pinary digit x_n corresponds to infinite copies of a set with x_n elements and fractal cutoff due to the finite size of real space-time sheet would bring in a finite precision. Note however that p-adic space-time sheet necessarily has an infinite size and it is only real world realization of the representation which has finite accuracy.

5. A concrete realization for this object would be as an infinite tree with $x_n + 1 \leq p$ branches in each node at level n ($x_n + 1$ is needed in order to avoid the splitting tree at $x_n = 0$). In 2-adic case -1 would be represented by an infinite pinary tree. Negative powers of p correspond to the of the tree extending to a finite depth in ground.

7.3 Generalization Of The Notion Of Rig By Replacing Naturals With P-Adic Integers

Previous considerations do not relate directly to category theoretical problem of assigning complex numbers to objects. It however turns out that p-adic approach allows to generalize the proposal of [A6] by replacing natural numbers with p-adic integers in the definition of rig so that any algebraic complex number can define cardinality of an object of category allowing multiplication and sum and that these complex numbers can be replaced with p-adic numbers if they make sense as such so that previous arguments provide a concrete geometric representation of the cardinality. The road to the realization this simple generalization required a visit to the John Baez's Weekly Finds (Week 102) [A5].

The outcome was the realization that the notion of rig used to categorify the subset of algebraic numbers obtained as roots of polynomials with *natural number* valued coefficients generalizes trivially by replacing natural numbers by *p-adic integers*. As a consequence one obtains beautiful p-adicization of the generating function F(x) of structure as a function which converges p-adically for any rational $x = q$ for which it has prime p as a positive power divisor.

Effectively this generalization means the replacement of natural numbers as coefficients of the polynomial defining the rig with all rationals, also negative, and *all* complex algebraic numbers find a category theoretical representation as "cardinalities". These cardinalities have a dual interpretation as p-adic integers which in general correspond to infinite real numbers but are mappable to real numbers by canonical identification and have a geometric representation as fractals.

7.3.1 Mapping of objects to complex numbers and the notion of rig

The idea of rig approach is to categorify the notion of cardinality in such a way that one obtains a *subset* of algebraic complex numbers as cardinalities in the category-theoretical sense. One can assign to an object a polynomial with coefficients, which are *natural numbers* and the condition $Z = P(Z)$ says that $P(Z)$ acts as an isomorphism of the object. One can interpret the equation also in terms of complex numbers. Hence the object is mapped to a complex number Z defining a root of the polynomial interpreted as an ordinary polynomial: it does not matter which root is chosen. The complex number Z is interpreted as the "cardinality" of the object but I do not really understand the motivation for this. The deep further result is that also more general polynomial equations $R(|Z|) = Q(|Z|)$ satisfied by the generalized cardinality Z imply $R(Z) = Q(Z)$ as isomorphism.

I try to reproduce what looks the most essential in the explanation of John Baez and relate it to my own ideas but take this as my talk to myself and visit This Week's Finds [A5], one of the many classics of Baez, to learn of this fascinating idea.

1. Baez considers first the ways of putting a given structure to n-element set. The set of these structures is denoted by F_n and the number of them by $|F_n|$. The generating function $|F|(x) = \sum_n |F_n|x^n$ packs all this information to a single function.

   For instance, if the structure is binary tree, this function is given by $T(x) = \sum_n C_{n-1}x^n$, where C_{n-1} are Catalan numbers and n¿0 holds true. One can show that T satisfies the formula

   $$T = X + T^2 \ ,$$

   since any binary tree is either trivial or decomposes to a product of binary trees, where two trees emanate from the root. One can solve this second order polynomial equation and the power expansion gives the generating function.

2. The great insight is that one can also work directly with structures. For instance, by starting from the isomorphism $T = 1 + T^2$ applying to an object with cardinality 1 and substituting T^2 with $(1 + T^2)^2$ repeatedly, one can deduce the amazing formula $T^7(1) = T(1)$ mentioned by Kea, and this identity can be interpreted as an isomorphism of binary trees.

3. This result can be generalized using the notion of rig category [A6]. In rig category one can add and multiply but negatives are not defined as in the case of ring. The lack of subtraction and division is still the problem and as I suggested in previous posting p-adic integers might resolve the problem.

   Whenever Z is object of a rig category, one can equip it with an isomorphism $Z = P(Z)$ where $P(Z)$ is polynomial with *natural numbers* as coefficients and one can assign to object "cardinality" as any root of the equation $Z = P(Z)$. Note that set with n elements corresponds to $P(|Z|) = n$. Thus subset of algebraic complex numbers receive formal identification as cardinalities of sets. Furthermore, if the cardinality satisfies another equation $Q(|Z|) = R(|Z|)$ such that neither polynomial is constant, then one can construct an isomorphism $Q(Z) = R(Z)$. Isomorphisms correspond to equations!

4. This is indeed nice that there is something which is not so beautiful as it could be: why should we restrict ourselves to *natural numbers* as coefficients of $P(Z)$? Could it be possible to replace them with integers to obtain *all complex algebraic numbers* as cardinalities? Could it be possible to replace natural numbers by p-adic integers?

7.3.2 p-Adic rigs and Golden Object as p-adic fractal

The notions of generating function and rig generalize to the p-adic context.

1. The generating function $F(x)$ defining isomorphism Z in the rig formulation converges p-adically for any p-adic number containing p as a factor so that the idea that all structures have p-adic counterparts is natural. In the real context the generating function typically diverges and must be defined by analytic continuation. Hence one might even argue that p-adic numbers are more natural in the description of structures assignable to finite sets than reals.

2. For rig one considers only polynomials $P(Z)$ (Z corresponds to the generating function F) with coefficients which are natural numbers. Any p-adic integer can be however interpreted as a non-negative integer: natural number if it is finite and "super-natural" number if it is infinite. Hence can generalize the notion of rig by replacing natural numbers by p-adic integers. The rig formalism would thus generalize to arbitrary polynomials with integer valued coefficients so that all complex algebraic numbers could appear as cardinalities of category theoretical objects. Even rational coefficients are allowed. This is highly natural number theoretically.

3. For instance, in the case of binary trees the solutions to the isomorphism condition $T = p+T^2$ giving $T = [1 \pm (1-4p)^{1/2}]/2$ and T would be complex number $[p \pm (1-4p)^{1/2}]/2$. $T(p)$ can be interpreted also as a p-adic number by performing power expansion of square root in case that the p-adic square root exists: this super-natural number can be mapped to a real number by the canonical identification and one obtains also the set theoretic representations of the category theoretical object $T(p)$ as a p-adic fractal. This interpretation of cardinality is much more natural than the purely formal interpretation as a complex number. This argument applies completely generally. The case $x = 1$ discussed by Baez gives $T = [1 \pm (-3)^{1/2}]/2$ allows p-adic representation if $-3 == p-3$ is square mod p. This is the case for $p = 7$ for instance.

4. John Baez [A5] poses also the question about the category theoretic realization of "Golden Object", his big dream. In this case one would have $Z = G = -1 + G^2 = P(Z)$. The polynomial on the right hand side does not conform with the notion of rig since -1 is not a natural number. If one allows p-adic rigs, $x = -1$ can be interpreted as a p-adic integer $(p-1)(1+p+...)$, positive and infinite and "super-natural", actually largest possible p-adic integer in a well defined sense.

   A further condition is that Golden Mean converges as a p-adic number: this requires that $\sqrt{5}$ must exist as a p-adic number: $(5 = 1+4)^{1/2}$ certainly converges as power series for $p = 2$ so that Golden Object exists 2-adically. By using [A3] of Euler, one finds that 5 is square mod p only if p is square mod 5. To decide whether given p is Golden it is enough to look whether p mod 5 is 1 or 4. For instance, $p = 11, 19, 29, 31$ ($=M_5$) are Golden. Mersennes M_k, $k = 3, 7, 127$ and Fermat primes are not Golden. One representation of Golden Object as p-adic fractal is the p-adic series expansion of $[1/2 \pm 5^{1/2}]/2$ representable geometrically as a binary tree such that there are $0 \leq x_n + 1 \leq p$ branches at each node at height n if n: th p-adic coefficient is x_n. The "cognitive" p-adic representation in terms of wavelet spectrum of classical fields is discussed in the previous posting.

5. It would be interesting to know how quantum dimensions of quantum groups assignable to Jones inclusions [K18, K8, K2] relate to the generalized cardinalities. The root of unity property of quantum phase ($q^{n+1} = q$) suggests $Q = Q^{n+1} = P(Q)$ as the relevant isomorphism. For Jones inclusions the cardinality $q = exp(i2\pi/n)$ would not be however equal to quantum dimension $D(n) = 4cos^2(\pi/n)$.

7.3.3 Is there a connection with infinite integers?

Infinite primes [K15] correspond to Fock states of a super-symmetric arithmetic quantum field theory and there is entire infinite hierarchy of them corresponding to repeated second quantization. Also infinite primes and rationals make sense. Besides free Fock states spectrum contains at each level also what might be identified as bound states. All these states can be mapped to polynomials. Since the roots of polynomials represent complex algebraic numbers and as they seem to characterize objects of categories, there are reasons to expect that infinite rationals might allow also interpretation in terms of say rig categories or their generalization. Also the possibility to identify space-time coordinate as isomorphism of a category might be highly interesting concerning the interpretation of quantum classical correspondence.

8 Gerbes And TGD

The notion of gerbes has gained much attention during last years in theoretical physics and there is an abundant gerbe-related literature in hep-th archives. Personally I learned about gerbes from the excellent article of Jouko Mickelson [A15] (Jouko was my opponent in PhD dissertation for more than two decades ago: so the time flows!).

I have already applied the notion of bundle gerbe in TGD framework in the construction of the Dirac determinant which I have proposed to define the Kähler function for the WCW (see [K19]). The insights provided by the general results about bundle gerbes discussed in [A15] led, not only to a justification for the hypothesis that Dirac determinant exists for the Kähler-Dirac action, but also to an elegant solution of the conceptual problems related to the construction of Dirac determinant in the presence of chiral symmetry. Furthermore, on basis of the special properties of the Kähler-Dirac operator there are good reasons to hope that the determinant exists even without zeta function regularization. The construction also leads to the conclusion that the space-time sheets serving as causal determinants must be geodesic sub-manifolds (presumably light like boundary components or "elementary particle horizons"). Quantum gravitational holography is realized since the exponent of Kähler function is expressible as a Dirac determinant determined by the local data at causal determinants and there would be no need to find absolute minima of Kähler action explicitly.

In the sequel the emergence of 2-gerbes at the space-time level in TGD framework is discussed and shown to lead to a geometric interpretation of the somewhat mysterious cocycle conditions for a wide class of gerbes generated via the $\wedge d$ products of connections associated with 0-gerbes. The resulting conjecture is that gerbes form a graded-commutative Grassmman algebra like structure generated by -1- and 0-gerbes. 2-gerbes provide also a beautiful topological characterization of space-time sheets as structures carrying Chern-Simons charges at boundary components and the 2-gerbe variant of Bohm-Aharonov effect occurs for perhaps the most interesting asymptotic solutions of field equations especially relevant for anyonics systems, quantum Hall effect, and living matter [K1].

8.1 What Gerbes Roughly Are?

Very roughly and differential geometrically, gerbes can be regarded as a generalization of connection. Instead of connection 1-form (0-gerbe) one considers a connection $n+1$-form defining n-gerbe. The curvature of n-gerbe is closed $n+2$-form and its integral defines an analog of magnetic charge. The notion of holonomy generalizes: instead of integrating n-gerbe connection over curve one integrates its connection form over n+1-dimensional closed surface and can transform it to the analog of magnetic flux.

There are some puzzling features associated with gerbes. Ordinary $U(1)$-bundles are defined in terms of open sets U_α with gauge transformations $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ defined in $U_\alpha \cap U_\beta$ relating the connection forms in the patch U_β to that in patch U_α. The 3-cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \tag{8.1}$$

makes it possible to glue the patches to a bundle structure.

In the case of 1-gerbes the transition functions are replaced with the transition functions $g_{\alpha\beta\gamma} = g_{\gamma\beta\alpha}^{-1}$ defined in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and 3-cocycle must be replaced with 4-cocycle:

$$g_{\alpha\beta\gamma} g_{\beta\gamma\delta} g_{\gamma\delta\alpha} g_{\delta\alpha\beta} = 1 \ . \tag{8.2}$$

The generalizations of these conditions to n-gerbes is obvious.

In the case of 2-intersections one can build a bundle structure naturally but in the case of 3-intersections this is not possible. Hence the geometric interpretation of the higher gerbes is far from obvious. One possible interpretation of non-trivial 1-gerbe is as an obstruction for lifting projective bundles with fiber space CP_n to vector bundles with fiber space C^{n+1} [A15]. This involves the lifting of the holomorphic transition functions g_α defined in the projective linear group $PGL(n+1,C)$ to $GL(n+1,C)$. When the 3-cocycle condition for the lifted transition functions $\overline{g}_{\alpha\beta}$ fails it can be replaced with 4-cocycle and one obtains 1-gerbe.

8.2 How Do 2-Gerbes Emerge In TGD?

Gerbes seem to be interesting also from the point of view of TGD, and TGD approach allows a geometric interpretation of the cocycle conditions for a rather wide class of gerbes.

Recall that the Kähler form J of CP_2 defines a non-trivial magnetically charged and self-dual $U(1)$-connection A. The Chern-Simons form $\omega = A\wedge J = A\wedge dA$ having CP_2 Abelian instanton density $J\wedge J$ as its curvature form and can thus be regarded as a 3-connection form of a 2-gerbe. This 2-gerbe is induced by 0-gerbe.

The coordinate patches U_α are same as for $U(1)$ connection. In the transition between patches A and ω transform as

$$\begin{aligned} A &\rightarrow A+d\phi \ , \\ \omega &\rightarrow \omega + dA_2 \ , \\ A_2 &= \phi\wedge J \ . \end{aligned} \tag{8.3}$$

The transformation formula is induced by the transformation formula for $U(1)$ bundle. Somewhat mysteriously, there is no need to define anything in the intersections of U_α in the recent case.

The connection form of the 2-gerbe can be regarded as a second $\wedge d$ power of Kähler connection:

$$A_3 \equiv A\wedge dA \ . \tag{8.4}$$

The generalization of this observation allows to develop a different view about n-gerbes generated as $\wedge d$ products of 0-gerbes.

8.2.1 The hierarchy of gerbes generated by 0-gerbes

Consider a collection of $U(1)$ connections $A^{i)}$. They generate entire hierarchy of gerbe-connections via the $\wedge d$ product

$$A_3 = A^{1)}\wedge dA^{2)} \tag{8.5}$$

defining 2-gerbe having a closed curvature 4-form

$$F_4 = dA^{1)}\wedge dA^{2)} \ . \tag{8.6}$$

$\wedge d$ product is commutative apart from a gauge transformation and the curvature forms of $A^{1)}\wedge dA^{2)}$ and $A^{2)}\wedge dA^{1)}$ are the same.

Quite generally, the connections A_m of $m-1$ gerbe and A_n of $n-1$-gerbe define $m+n+1$ connection form and the closed curvature form of $m+n$-gerbe as

$$\begin{aligned} A_{m+n+1} &= A^{1)}_m\wedge dA^{2)}_n \ , \\ F_{m+n+2} &= dA^{1)}_m\wedge dA^{2)}_n \ . \end{aligned} \tag{8.7}$$

The sequence of gerbes extends up to $n = D - 2$, where D is the dimension of the underlying manifold. These gerbes are not the most general ones since one starts from 0-gerbes. One can of course start from $n > 0$-gerbes too.

The generalization of the $\wedge d$ product to the non-Abelian situation is not obvious. The problems stem from the that the Lie-algebra valued connection forms $A^{1)}$ and $A^{2)}$ appearing in the covariant version $D = d + A$ do not commute.

8.3 How To Understand The Replacement Of 3-Cycles With N-Cycles?

If n-gerbes are generated from 0-gerbes it is possible to understand how the intersections of the open sets emerge. Consider the product of 0-gerbes as the simplest possible case. The crucial observation is that the coverings U_α for $A^{1)}$ and V_β for $A^{2)}$ need not be same (for CP_2 this was the case). One can form a new covering consisting of sets $U_\alpha \cap V_{\alpha_1}$. Just by increasing the index range one can replace V with U and one has covering by $U_\alpha \cap U_{\alpha_1} \equiv U_{\alpha\alpha_1}$.

The transition functions are defined in the intersections $U_{\alpha\alpha_1} \cap U_{\beta\beta_1} \equiv U_{\alpha\alpha_1\beta\beta_1}$ and cocycle conditions must be formulated using instead of intersections $U_{\alpha\beta\gamma}$ the intersections $U_{\alpha\alpha_1\beta\beta_1\gamma\gamma_1}$. Hence the transition functions can be written as $g_{\alpha\alpha_1\beta\beta_1}$ and the 3-cocycle are replaced with 5-cocycle conditions since the minimal co-cycle corresponds to a sequence of 6 steps instead of 4:

$$U_{\alpha\alpha_1\beta\beta_1} \rightarrow U_{\alpha_1\beta\beta_1\gamma} \rightarrow U_{\beta\beta_1\gamma\gamma_1} \rightarrow U_{\beta_1\gamma\gamma_1\alpha} \rightarrow U_{\gamma\gamma_1\alpha\alpha_1} \quad .$$

The emergence of higher co-cycles is thus forced by the modification of the bundle covering necessary when gerbe is formed as a product of lower gerbes. The conjecture is that any even gerbe is expressible as a product of 0-gerbes.

An interesting application of the product structure is at the level of WCW ("world of classical worlds"). The Kähler form of WCW defines a connection 1-form and this generates infinite hierarchy of connection $2n + 1$-forms associated with $2n$-gerbes.

8.4 Gerbes As Graded-Commutative Algebra: Can One Express All Gerbes As Products Of -1 And 0-Gerbes?

If one starts from, say 1-gerbes, the previous argument providing a geometric understanding of gerbes is not applicable as such. One might however hope that it is possible to represent the connection 2-form of any 1-gerbe as a $\wedge d$ product of a connection 0-form ϕ of "-1" -gerbe and connection 1-form A of 0-gerbe:

$$A_2 = \phi dA \equiv A \wedge d\phi \quad ,$$

with different coverings for ϕ and A. The interpretation as an obstruction for the modification of the underlying bundle structure is consistent with this interpretation.

The notion of -1-gerbe is not well-defined unless one can define the notion of -1 form precisely. The simplest possibility that 0-form transforms trivially in the change of patch is not consistent. One could identify contravariant n-tensors as $-n$-forms and d for them as divergence and d^2 as the antisymmetrized double divergence giving zero. ϕ would change in a gauge transformation by a divergence of a vector field. The integral of a divergence over closed M vanishes identically so that if the integral of ϕ over M is non-vanishing it corresponds to a non-trivial 0-connection. This interpretation of course requires the introduction of metric.

The requirement that the minimal intersections of the patches for 1-gerbes are of form $U_{\alpha\beta\gamma}$ would be achieved if the intersections patches can be restricted to the intersections $U_{\alpha\beta\gamma}$ defined by $U_\alpha \cap V_\gamma$ and $U_\beta \cap V_\gamma$ (instead of $U_\beta \cap V_\delta$), where the patches V_γ would be most naturally associated with -1-gerbe. It is not clear why one could make this restriction. The general conjecture is that any gerbe decomposes into a multiple $\wedge d$ product of -1 and 0-gerbes just like integers decompose into primes. The $\wedge d$ product of two odd gerbes is anti-commutative so that there is also an analogy with the decomposition of the physical state into fermions and bosons, and gerbes for a graded-commutative super-algebra generalizing the Grassmann algebra of manifold to a Grassmann algebra of gerbe structures for manifold.

8.5 The Physical Interpretation Of 2-Gerbes In TGD Framework

2-gerbes could provide some insight to how to characterize the topological structure of the many-sheeted space-time.

1. The cohomology group H^4 is obviously crucial in characterizing 2-gerbe. In TGD framework many-sheetedness means that different space-time sheets with induced metric having Minkowski signature are separated by elementary particle horizons which are light like 3-surfaces at which the induced metric becomes degenerate. Also the time orientation of the space-time sheet can change at these surfaces since the determinant of the induced metric vanishes.

   This justifies the term elementary particle horizon and also the idea that one should treat different space-time sheets as generating independent direct summands in the homology group of the space-time surface: as if the space-time sheets not connected by join along boundaries bonds were disjoint. Thus the homology group H^4 and 2-gerbes defining instanton numbers would become important topological characteristics of the many-sheeted space-time.

2. The asymptotic behavior of the general solutions of field equations can be classified by the dimension D of the CP_2 projection of the space-time sheet. For $D = 4$ the instanton density defining the curvature form of 2-gerbe is non-vanishing and instanton number defines a topological charge. Also the values of the Chern-Simons invariants associated with the boundary components of the space-time sheet define topological quantum numbers characterizing the space-time sheet and their sum equals to the instanton charge. CP_2 type extremals represent a basic example of this kind of situation. From the physical view point $D = 4$ asymptotic solutions correspond to what might be regarded chaotic phase for the flow lines of the Kähler magnetic field. Kähler current vanishes so that empty space Maxwell's equations are satisfied.

3. For $D = 3$ situation is more subtle when boundaries are present so that the higher-dimensional analog of Aharonov-Bohm effect becomes possible. In this case instanton density vanishes but the Chern-Simons invariants associated with the boundary components can be non-vanishing. Their sum obviously vanishes. The space-time sheet can be said to be a neutral C-S multipole. Separate space-time sheets can become connected by flux tubes in a quantum jump replacing a space-time surface with a new one. This means that the cohomology group H^4 as well as instanton charges and C-S charges of the system change.

Concerning the asymptotic dynamics of the Kähler magnetic field, $D = 3$ phase corresponds to an extremely complex but highly organized phase serving as an excellent candidate for the modelling of living matter. Both the TGD based description of anyons and quantum Hall effect and the model for topological quantum computation based on the braiding of magnetic flux tubes rely heavily on the properties $D = 3$ phase [K1].

The non-vanishing of the C-S form implies that the flow lines of the Kähler magnetic are highly entangled and have as an analog mixing hydrodynamical flow. In particular, one cannot define non-trivial order parameters, say phase factors, which would be constant along the lines. The interpretation in terms of broken super-conductivity suggests itself. Kähler current can be non-vanishing so that there is no counterpart for this phase at the level of Maxwell's equations.

9 Appendix: Category Theory And Construction Of S-Matrix

The construction of WCW geometry, spinor structure and of S-matrix involve difficult technical and conceptual problems and category theory might be of help here. As already found, the application of category theory to the construction of WCW geometry allows to understand how the arrow of psychological time emerges.

The construction of the S-matrix involves several difficult conceptual and technical problems in which category theory might help. The incoming states of the theory are what might be called free states and are constructed as products of the WCW spinor fields. One can effectively regard them as being defined in the Cartesian power of WCW divided by an appropriate permutation group. Interacting states in turn are defined in the WCW .

Cartesian power of WCW of 3-surfaces is however in geometrical sense more or less identical with WCW since the disjoint union of N 3-surfaces is itself a 3-surface in WCW . Actually it differs from WCW itself only in that the 3-surfaces of many particle state can intersect each other and if one allows this, one has paradoxical self-referential identification $CH = \overline{CH^2}/S_2 = ... = \overline{CH^N}/S_N...$, where over-line signifies that intersecting 3-surfaces have been dropped from the product.

Note that arbitrarily small deformation can remove the intersections between 3-surfaces and four-dimensional general coordinate invariance allows always to use non-intersecting representatives. In case of the spinor structure of the Cartesian power this identification means that the tensor powers SCH^N of the WCW spinor structure are in some sense identical with the spinor structure SCH of the WCW . Certainly the oscillator operators of the tensor factors must be assumed to be mutually anti-commuting.

The identities $CH = \overline{CH^2}/S_2 = ..$ and corresponding identities $SCH = SCH^2 = ...$ for the space SCH of WCW spinor fields might imply very deep constraints on S-matrix. What comes into mind are counterparts for the Schwinger-Dyson equations of perturbative quantum field theory providing defining equations for the n-point functions of the theory [A13]. The isomorphism between SCH^2 and SCH is actually what is needed to calculate the S-matrix elements. Category theory might help to understand at a general level what these self-referential and somewhat paradoxical looking identities really imply and perhaps even develop TGD counterparts of Schwinger-Dyson equations.

There is also the issue of bound states. The interacting states contain also bound states not belonging to the space of free states and category theory might help also here. It would seem that the state space must be constructed by taking into account also the bound states as additional "free" states in the decomposition of states to product states.

A category naturally involved with the construction of the S-matrix (or U-matrix) is the space of preferred extremals of the Kähler action which might be called interacting category. The symplectic transformations acting as isometries of the configuration space geometry act naturally as the morphisms of this category. The group $Diff^4$ of general coordinate transformations in turn acts as gauge symmetries.

S-matrix relates free and interacting states and is induced by the classical long range interactions induced by the criticality of the preferred extremals in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes S-matrix elements are essentially Glebch-Gordan coefficients relating the states in the tensor power of the interacting super-symplectic representation with the interacting super-symplectic representation itself. More concretely, N-particle free states can be seen as WCW spinor fields in CH^N obtained as tensor products of ordinary WCW spinor fields. Free states correspond classically to the unions of space-time surfaces associated with the 3-surfaces representing incoming particles whereas interacting states correspond classically to the space-time surfaces associated with the unions of the 3-surfaces defining incoming states. These two states define what might be called free and interacting categories with canonical transformations acting as morphisms.

The classical interaction is represented by a functor $S : \overline{CH^N}/S_N \to CH$ mapping the classical free many particle states, that is objects of the product category defined by $\overline{CH^N}/S_N$ to the interacting category CH. This functor assigns to the union $\cup_i X^4(X^3_i)$ of the absolute minima $X^4(X^3_i)$ of Kähler action associated with the incoming, free states X^3_i the preferred extreal $X^4(\cup X^3_i)$ associated with the union of 3-surfaces representing the outgoing interacting state. At quantum level this functor maps the state space SCH^N associated with $\cup_i X^4(X^3_i)$ to SCH in a unitary manner. An important constraint on S-matrix is that it acts effectively as a flow in zero modes correlating the quantum numbers in fiber degrees of freedom in one-to-one manner with the values of zero modes so that quantum jump $U\Psi_i \to \Psi_0...$ gives rise to a quantum measurement.

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