Construction of Configuration Space Kähler Geometry from Symmetry Principles

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Contents

Abstract

There are three separate approaches to the challenge of constructing WCW Kähler geometry and spinor structure. The first one relies on a direct guess of the Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach relies on the construction of spinor structure assuming that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for the second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of spinor structure.

In this chapter the construction of Kähler form and metric based on symmetries is discussed. The basic vision is that WCW can be regarded as the space of generalized Feynman diagrams with lines thickned to light-like 3-surfaces and vertices identified as partonic 2-surfaces. In zero energy ontology the strong form of General Coordinate Invariance (GCI) strongly suggests effective 2-dimensionality and the basic objects are taken to be pairs partonic 2-surfaces X^2 at opposite light-like boundaries of causal diamonds (CDs). This has however turned out to be too strong formulation for effective 2-dimensionality string world sheets carrying induced spinor fields are also present.

The hypothesis is that WCW can be regarded as a union of infinite-dimensional symmetric spaces G/H labeled by zero modes having an interpretation as classical, non-quantum fluctuating variables. A crucial role is played by the metric 2-dimensionality of the light-cone boundary δM^4_+ and of light-like 3-surfaces implying a generalization of conformal invariance. The group G acting as isometries of WCW is tentatively identified as the symplectic group of $\delta M^4_+ \times CP_2$. H corresponds to sub-group acting as diffeomorphisms at preferred 3-surface, which can be taken to correspond to maximum of Kähler function.

In zero energy ontology (ZEO) 3-surface corresponds to a pair of space-like 3-surfaces at the opposide boundaries of causal diamond (CD) and thus to a more or less unique extremal of Kähler action. The interpretation would be in terms of holography. One can also consider the inclusion of the light-like 3-surfaces at which the signature of the induced metric changes to the 3-surface so that it would become connected.

An explicit construction for the Hamiltonians of WCW isometry algebra as so called flux Hamiltonians using Haltonians of light-cone boundary is proposed and also the elements of Kähler form can be constructed in terms of these. Explicit expressions for WCW flux Hamiltonians as functionals of complex coordinates of the Cartesian product of the infinite-dimensional symmetric spaces having as points the partonic 2-surfaces defining the ends of the the light 3-surface (line of generalized Feynman diagram) are proposed.

This construction suffers from some rather obvious defects. Effective 2-dimensionality is realized in too strong sense, only covariantly constant right-handed neutrino is involved, and WCW Hamiltonians do not directly reflect the dynamics of Kähler action. The construction however generalizes in very straightforward manner to a construction free of these problems. This however requires the understanding of the dynamics of preferred extremals and Kähler-Dirac action.

1 Introduction

The most general expectation is that configuration space ("world of classical worlds" (WCW)) can be regarded as a union of coset spaces which are infinite-dimensional symmetric spaces with Kähler structure: $C(H) = \cup_i G/H(i)$. Index i labels 3-topology and zero modes. The group G, which can depend on 3-surface, can be identified as a subgroup of diffeomorphisms of $\delta M^4_+ \times CP_2$ and H must contain as its subgroup a group, whose action reduces to $Diff(X^3)$ so that these transformations leave 3-surface invariant.

In zero energy ontology (ZEO) 3-surface corresponds to a pair of space-like 3-surfaces at the opposide boundaries of causal diamond (CD) and thus to a more or less unique extremal of Kähler action. The interpretation would be in terms of holography. One can also consider the inclusion of the light-like 3-surfaces at which the signature of the induced metric changes to the 3-surface so that it would become connected.

The task is to identify plausible candidate for G and H and to show that the tangent space of the WCW allows Kähler structure, in other words that the Lie-algebras of G and $H(i)$ allow complexification. One must also identify the zero modes and construct integration measure for the functional integral in these degrees of freedom. Besides this one must deduce information about the explicit form of WCW metric from symmetry considerations combined with the hypothesis that Kähler function is Kähler action for a preferred extremal of Kähler action. One must of course understand what "preferred" means.

1.1 General Coordinate Invariance And Generalized Quantum Gravitational Holography

The basic motivation for the construction of WCW geometry is the vision that physics reduces to the geometry of classical spinor fields in the infinite-dimensional WCW of 3-surfaces of $M^4_+ \times CP_2$ or of $M^4 \times CP_2$. Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that WCW possesses Kähler geometry. Kähler geometry is coded into Kähler function.

The original belief was that the four-dimensional general coordinate invariance of Kähler function reduces the construction of the geometry to that for the boundary of configuration space consisting of 3-surfaces on $\delta M^4_+ \times CP_2$, the moment of big bang. The proposal was that Kähler function $K(Y^3)$ could be defined as a preferred extremal of so called Kähler action for the unique space-time surface $X^4(Y^3)$ going through given 3-surface Y^3 at $\delta M^4_+ \times CP_2$. For Diff^4 transforms of Y^3 at $X^4(Y^3)$ Kähler function would have the same value so that Diff^4 invariance and degeneracy would be the outcome. The proposal was that the preferred extremals are absolute minima of Kähler action.

This picture turned out to be too simple.

1. I have already described the recent view about light-like 3-surfaces as generalized Feynman diagrams and space-time surfaces as preferred extremals of Kähler action and will not repeat what has been said. Note that the inclusion of space-like ends at boundaries of CD gives analog of Wilson loop.

2. It has also become obvious that the gigantic symmetries associated with $\delta M^4_\pm \times CP_2 \subset CD \times CP_2$ manifest themselves as the properties of propagators and vertices. Cosmological considerations, Poincare invariance, and the new view about energy favor the decomposition of the WCW to a union of configuration spaces assignable to causal diamonds CDs defined as intersections of future and past directed light-cones. The minimum assumption is that CDs label the sectors of CH: the nice feature of this option is that the considerations of this chapter restricted to $\delta M^4_+ \times CP_2$ generalize almost trivially. This option is beautiful because the center of mass degrees of freedom associated with the different sectors of CH would correspond to M^4 itself and its Cartesian powers.

The definition of the Kähler function requires that the many-to-one correspondence $X^3 \rightarrow X^4(X^3)$ must be replaced by a bijective correspondence in the sense that X^3_l as light-like 3-surface is unique among all its Diff^4 translates. This also allows physically preferred "gauge fixing" allowing to get rid of the mathematical complications due to Diff4 degeneracy. The internal geometry of the space-time sheet must define the preferred 3-surface X^3_l.

The realization of this vision means a considerable mathematical challenge. The effective metric 2-dimensionality of 3-dimensional light-like surfaces X^3_l of M^4 implies generalized conformal and symplectic symmetries allowing to generalize quantum gravitational holography from light like boundary so that the complexities due to the non-determinism can be taken into account properly.

1.2 Light Like 3-D Causal Determinants And Effective 2-Dimensionality

The light like 3-surfaces X^3_l of space-time surface appear as 3-D causal determinants. Basic examples are boundaries and elementary particle horizons (parton orbits) at which Minkowskian signature of the induced metric transforms to Euclidian one. This brings in a second conformal symmetry related to the metric 2-dimensionality of the 3-D light-like 3-surface. This symmetry is analogous to TGD counterpart of the Kac Moody symmetry of string models and seems to be associated with quantum criticality implying non-uniqueness of the space-time surface with given space-like ends at boundaries of CD. Critical deformations would be Kac-Moody type transformation preserving the light-likeness of the parton orbits. The challenge is to understand the relationship of this symmetry to WCW geometry and the interaction between the two conformal symmetries.

1. Field-particle duality is realized. Light-like 3-surfaces X_l^3 -generalized Feynman diagrams - correspond to the particle aspect of field-particle duality whereas the physics in the interior of space-time surface $X^4(X_l^3)$ would correspond to the field aspect. Generalized Feynman diagrams in 4-D sense could be identified as regions of space-time surface having Euclidian signature.

2. One could also say that light-like 3-surfaces X_l^3 and the space-like 3-surfaces X^3 in the intersections of $X^4(X_l^3) \cap CD \times CP_2$ where the causal diamond CD is defined as the intersections of future and past directed light-cones provide dual descriptions.

3. Generalized coset construction implies that the differences of super-symplectic and Super Kac-Moody type Super Virasoro generators annihilated physical states. This construction in turn led to the realization that WCW for fixed values of zero modes - in particular the values of the induced Kähler form of $\delta M^4_\pm \times CP_2$ - allows identification as a coset space obtained by dividing the symplectic group of $\delta M^4_\pm \times CP_2$ with Kac-Moody group, whose generators vanish at $X^2 = X_l^3 \times \delta M^4_\pm \times CP_2$. One can say that quantum fluctuating degrees of freedom in a very concrete sense correspond to the local variant of $S^2 \times CP_2$.

The analog of conformal invariance in the light-like direction of X_l^3 and in the light-like radial direction of $\delta M^4_\pm$ suggests that the data at either X^3 or X_l^3 should be enough to determine WCW geometry. This implies that the relevant data is contained to their intersection X^2 at least for finite regions of X^3. This is the case if the deformations of X_l^3 not affecting X^2 and preserving light likeness corresponding to zero modes or gauge degrees of freedom and induce deformations of X^3 also acting as zero modes. The outcome is effective 2-dimensionality. One must be however cautious in order to not make over-statements. The reduction to 2-D theory in global sense would trivialize the theory and the reduction to 2-D theory must takes places for finite region of X^3 only so one has in well defined sense three-dimensionality in discrete sense. A more precise formulation of this vision is in terms of hierarchy of CDs containing CDs containing.... The introduction of sub-CD:s brings in improved measurement resolution and means also that effective 2-dimensionality is realized in the scale of sub-CD only.

Experience has however taught to be extremely cautious: it could also be that in ZEO the unions of the space-like 3-surfaces at the ends of CD and of the light-like partonic orbits at which the signature of the induced metric changes are the basic objects analogous to Wilson loops. In this case the notion of effective 2-dimensionality is not so clear. Also in this case the Kac-Moody type symmetry preserving the light-likeness of partonic orbits could reduce the additional degrees of freedom to a finite number of conformal equivalence classes of partonic orbits for given pair of 3-surfaces.

One cannot over-emphasize the importance of the effective 2-dimensionality. It indeed simplifies dramatically the earlier formulas for WCW metric involving 3-dimensional integrals over $X^3 \subset M^4_+ \times CP_2$ reducing now to 2-dimensional integrals. Note that X^3 is determined by preferred extremal property of $X^4(X_l^3)$ once X_l^3 is fixed and one can hope that this mapping is one-to-one.

1.3 Magic Properties Of Light Cone Boundary And Isometries Of WCW

The special conformal, metric and symplectic properties of the light cone of four-dimensional Minkowski space: δM^4_+, the boundary of four-dimensional light cone is metrically 2-dimensional(!) sphere allowing infinite-dimensional group of conformal transformations and isometries(!) as well as Kähler structure. Kähler structure is not unique: possible Kähler structures of light cone boundary are parameterized by Lobatchevski space $SO(3,1)/SO(3)$. The requirement that the isotropy group $SO(3)$ of S^2 corresponds to the isotropy group of the unique classical 3-momentum assigned to $X^4(Y^3)$ defined as a preferred extremum of Kähler action, fixes the choice of the complex structure uniquely. Therefore group theoretical approach and the approach based on Kähler action complement each other.

1. The allowance of an infinite-dimensional group of isometries isomorphic to the group of conformal transformations of 2-sphere is completely unique feature of the 4-dimensional light

cone boundary. Even more, in case of $\delta M^4_+ \times CP_2$ the isometry group of δM^4_+ becomes localized with respect to CP_2! Furthermore, the Kähler structure of δM^4_+ defines also symplectic structure.

Hence any function of $\delta M^4_+ \times CP_2$ would serve as a Hamiltonian transformation acting in both CP_2 and δM^4_+ degrees of freedom. These transformations obviously differ from ordinary local gauge transformations. This group leaves the symplectic form of $\delta M^4_+ \times CP_2$, defined as the sum of light cone and CP_2 symplectic forms, invariant. The group of symplectic transformations of $\delta M^4_+ \times CP_2$ is a good candidate for the isometry group of the WCW.

2. The approximate symplectic invariance of Kähler action is broken only by gravitational effects and is exact for vacuum extremals. If Kähler function were exactly invariant under the symplectic transformations of CP_2, CP_2 symplectic transformations wiykd correspond to zero modes having zero norm in the Kähler metric of WCW. This does not make sense since symplectic transformations of $\delta M^4 \times CP_2$ actually parameterize the quantum fluctuation degrees of freedom.

3. The groups G and H, and thus WCW itself, should inherit the complex structure of the light cone boundary. The diffeomorphims of M^4 act as dynamical symmetries of vacuum extremals. The radial Virasoro localized with respect to $S^2 \times CP_2$ could in turn act in zero modes perhaps inducing conformal transformations: note that these transformations lead out from the symmetric space associated with given values of zero modes.

1.4 Symplectic Transformations Of $\Delta M^4_+ \times CP_2$ As Isometries Of WCW

The symplectic transformations of $\delta M^4_+ \times CP_2$ are excellent candidates for inducing symplectic transformations of the WCW acting as isometries. There are however deep differences with respect to the Kac Moody algebras.

1. The conformal algebra of the WCW is gigantic when compared with the Virasoro + Kac Moody algebras of string models as is clear from the fact that the Lie-algebra generator of a symplectic transformation of $\delta M^4_+ \times CP_2$ corresponding to a Hamiltonian which is product of functions defined in δM^4_+ and CP_2 is sum of generator of δM^4_+-local symplectic transformation of CP_2 and CP_2-local symplectic transformations of δM^4_+. This means also that the notion of local gauge transformation generalizes.

2. The physical interpretation is also quite different: the relevant quantum numbers label the unitary representations of Lorentz group and color group, and the four-momentum labeling the states of Kac Moody representations is not present. Physical states carrying no energy and momentum at quantum level are predicted. The appearance of a new kind of angular momentum not assignable to elementary particles might shed some light to the longstanding problem of baryonic spin (quarks are not responsible for the entire spin of proton). The possibility of a new kind of color might have implications even in macroscopic length scales.

3. The central extension induced from the natural central extension associated with $\delta M^4_+ \times CP_2$ Poisson brackets is anti-symmetric with respect to the generators of the symplectic algebra rather than symmetric as in the case of Kac Moody algebras associated with loop spaces. At first this seems to mean a dramatic difference. For instance, in the case of CP_2 symplectic transformations localized with respect to δM^4_+ the central extension would vanish for Cartan algebra, which means a profound physical difference. For $\delta M^4_+ \times CP_2$ symplectic algebra a generalization of the Kac Moody type structure however emerges naturally.

The point is that δM^4_+-local CP_2 symplectic transformations are accompanied by CP_2 local δM^4_+ symplectic transformations. Therefore the Poisson bracket of two δM^4_+ local CP_2 Hamiltonians involves a term analogous to a central extension term symmetric with respect to CP_2 Hamiltonians, and resulting from the δM^4_+ bracket of functions multiplying the Hamiltonians. This additional term could give the entire bracket of the WCW Hamiltonians at the maximum of the Kähler function where one expects that CP_2 Hamiltonians vanish and have a form essentially identical with Kac Moody central extension because it is indeed symmetric with respect to indices of the symplectic group.

The most natural option is that symplectic and Kac-Moody algebras together generate the isometry algebra and that the corresponding transformations leaving invariant the partonic 2-surfaces and their 4-D tangent space data act as gauge transformations and affect only zero modes.

1.5 Does The Symmetric Space Property Reduce To Coset Construction For Super Virasoro Algebras?

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ . \quad [t, t] \subset h \ . \quad [h, t] \subset t \ . \tag{1.1}$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough.

WCW geometry allows two super-conformal symmetries assignable the coset space decomposition G/H for a sector of WCW with fixed values of zero moes. One can assign to the tangent space algebras g resp. h of G resp. H analogous to Kac-Moody algebras super Virasoro algebras and construct super-conformal representation as a coset representation meaning that the differences of super Virasoro generators annihilate the physical states. This obviously generalizes Goddard-Olive-Kent construction [A6].

The identification of the two algeras is not a mechanical task and has involved a lot of trial and erroring. The algebra g should be be spanned by the generators of super-symplectic algebra of light-cone boundary and by the Kac-Moody algebra acting on light-like orbits of partonic 2-surfaces. The sub-algebra h should be spanned by generators which vanish for a preferred point of WCW analogous to origin of $CP_2 = SU(3)/U(2)$. Now this point would correspond to maximum or minimum of Kähler function (no saddle points are allowed if the WCW metric has definite signature). In hindsight it is obvious that the generators of both symplectic and Kac-Moody algebras are needed to generate g and h: already the effective 2-dimensionality meaning that 4-D tangent space data of partonic surface matters requires this.

The maxima of Kähler function could correspond to this kind of points (pairs formed by 3-surfaces at different ends of CD in ZEO) and could play also an essential role in the integration over WCW by generalizing the Gaussian integration of free quantum field theories. It took quite a long time to realize that Kähler function must be identified as Kähler action for the Euclidian region of preferred extremal. Kähler action for Minkowskian regions gives imaginary contribution to the action exponential and has interpretation in terms of Morse function. This part of Kähler action can have and is expected to have saddle points and to define Hessian with signature which is not positive definite.

1.6 What Effective 2-Dimensionality And Holography Really Mean?

Concerning the interpretation of Kac-Moody algebra there are some poorly understood points, which directly relate to what one means with holography.

1. Holography suggests that light-like 3-surfaces with fixed ends give rise to same WCW metric and the deformations of these surfaces by Kac-Moody algebra correspond to zero modes just like the interior degrees of freedom for space-like 3-surface do. The same would be true for space-like 3-surfaces at the ends of space-time surface with respect to symplectic transformations.

2. The non-trivial action of Kac-Moody algebra in the interior of X_l^3 together with effective 2-dimensionality and holography would encourage the interpretation of Kac-Moody symmetries acting trivially at X^2 as gauge symmetries. Light-like 3-surfaces having fixed partonic 2-surfaces at their ends would be equivalent physically and effective 2-dimensionality and holography would be realized modulo gauge transformations. As a matter fact, the action on WCW metric would be a change of zero modes so that one could identify it as analog of conformal scaling. The action of symplectic transformations vanishing in the interior of space-like 3-surface at the end of space-time surface affects only zero modes.

1.7 For The Reader

Few words about the representation of ideas are in order. For a long time the books about TGD served as kind of lab note books - a bottom-up representation providing kind of a ladder making clear the evolution of ideas. This led gradually to a rather chaotic situation in which it was difficult for me to control the internal consistency and for the possible reader to distinguish between the big ideas and ad hoc guesses, most of them related to the detailed realization of big visions. Therefore I have made now and the decision to clean up a lot of the ad hoc stuff. In this process I have also changed the representation so that it is more top-down and tries to achieve over-all views.

There are several visions about what TGD is and I have worked hardly to achieve a fusion of these visions. Hence simple linear representation in which reader climbs to a tree of wisdom is impossible. I must summarize overall view from the beginning and refer to the results deduced in chapters towards the end of the book and also to ideas discussed in other books. For instance, the construction of WCW ("world of classical worlds" (WCW)) spinor structure discussed in chapters [K19] provides the understanding necessary to make the construction of configuration space geometry more detailed. Also number theoretical vision discussed in another book [K12] is necessary. Somehow it seems that a graphic representation emphasizing visually the big picture should be needed to make the representation more comprehensible.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [?].

2 How To Generalize The Construction Of WCW Geometry To Take Into Account The Classical Non-Determinism?

If the embedding space were $H_+ = M^4_+ \times CP_2$ and if Kähler action were deterministic, the construction of WCW geometry reduces to $\delta M^4_+ \times CP_2$. Thus in this limit quantum holography principle [B2, B4] would be satisfied also in TGD framework and actually reduce to the general coordinate invariance. The classical non-determinism of Kähler action however means that this construction is not quite enough and the challenge is to generalize the construction.

2.1 Quantum Holography In The Sense Of Quantum GravityTheories

In string theory context quantum holography is more or less synonymous with Maldacena conjecture Maldacena which (very roughly) states that string theory in Anti-de-Sitter space AdS is equivalent with a conformal field theory at the boundary of AdS. In purely quantum gravitational context [B2] , quantum holography principle states that quantum gravitational interactions at high energy limit in AdS can be described using a topological field theory reducing to a conformal (and non-gravitational) field theory defined at the *time like* boundary of the AdS. Thus the time like boundary plays the role of a dynamical hologram containing all information about correlation functions of $d + 1$ dimensional theory. This reduction also conforms with the fact that black hole entropy is proportional to the horizon area rather than the volume inside horizon.

Holography principle reduces to general coordinate invariance in TGD. If the action principle assigning space-time surface to a given 3-surface X^3 at light cone boundary were completely deterministic, four-dimensional general coordinate invariance would reduce the construction of the configuration geometry for the space of 3-surfaces in $M^4_+ \times CP_2$ to the construction of the geometry at the boundary of WCW consisting of 3-surfaces in $\delta M^4_+ \times CP_2$ (moment of big bang). Also the quantum theory would reduce to the boundary of the future light cone.

The classical non-determinism of Kähler action however implies that quantum holography in this strong form fails. This is very desirable from the point of view of both physics and consciousness theory. Classical determinism would also mean that time would be lost in TGD as it is lost in GRT. Classical non-determinism is also absolutely essential for quantum consciousness and makes possible conscious experiences with contents localized into finite time interval despite the fact that quantum jumps occur between WCW spinor fields defining what I have used to call quantum histories. Classical non-determinism makes it also possible to generalize quantum-classical correspondence in the sense that classical non-determinism at the space-time level provides correlate for quantum non-determinism. The failure of classical determinism is a difficult challenge for the construction of WCW geometry. One might however hope that the notion of quantum holography generalizes.

2.2 How Does The Classical Determinism Fail In TGD?

One might hope that determinism in a generalized sense might be achieved by generalizing the notion of 3-surface by allowing unions of space-like 3-surfaces with time like separations with very strong but not complete correlations between the space-like 3-surfaces. In this case the non-determinism would mean that the 3-surfaces Y^3 at light cone boundary correspond to at most enumerable number of preferred extremals $X^4(Y^3)$ of Kähler action so that one would get finite or at most enumerably infinite number of replicas of a given WCW region and the construction would still reduce to the light cone boundary.

1. This is probably quite too simplistic view. Any 4-surface which has CP_2 projection which belongs to so called Lagrange manifold of CP_2 having by definition vanishing induced Kähler form is vacuum extremal. Thus there is an infinite variety of 6-dimensional sub-manifolds of H for which all extremals of Kähler action are vacua.

2. CP_2 type vacuum extremals are different since they possess non-vanishing Kähler form and Kähler action. They are identifiable as classical counterparts of elementary particles have M^4_+ projection which is a random light like curve (this in fact gives rise to conformal invariance identifiable as counterpart of quaternion conformal invariance). Thus there are good reasons to suspect that classical non-determinism might destroy the dream about complete reduction to the light cone boundary.

3. The wormhole contacts connecting different space-time sheets together can be seen as pieces of CP_2 type extremals and one expects that the non-determinism is still there and that the metrically 2-dimensional elementary particle horizons (light like 3-surfaces of H surrounding wormhole contacts and having time-like M^4_+ projection) might be a crucial element in the understanding of quantum TGD. The non-determinism of CP_2 type extremals is absolutely crucial for the ordinary elementary particle physics. It seems that the conformal symmetries responsible for the ordinary elementary particle quantum numbers acting in these degrees of freedom do not contribute to the WCW metric line element.

The treatment of the non-determinism in a framework in which the prediction of time evolution is seen as initial value problem, seems to be difficult. Also the notion of WCW becomes a messy concept. ZEO changes the situation completely. Light-like 3-surfaces become representations of generalized Feynman diagrams and brings in the notion of finite time resolution. One obtains a direct connection with the concepts of quantum field theory with path integral with cutoff replaced with a sum over various preferred extremals with cutoff in time resolution.

2.3 The Notions Of Embedding Space, 3-Surface, And Configuration Space

The notions of embedding space, 3-surface (and 4-surface), and configuration space ("world of classical worlds", WCW) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

2.3.1 The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K16, K17, K15] .

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of ZEO [K19, K4] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K11] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K6] led to a further generalization of the notion of embedding space - at least as a convenient auxiliary structure. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW.

   It seems that the covering of embedding space is only a convenient auxiliary structure. The space-time surfaces in the n-fold covering correspond to the n conformal equivalence classes of space-time surfaces connecting fixed 3-surfaces at the ends of CD: the space-time surfaces are branched at their ends. The situation can be interpreted at the level of WCW in several ways. There is single 3-surface at both ends but by non-determinism there are n space-time branches of the space-time surface connecting them so that the Kähler action is multiplied by factor n. If one forgets the presence of the n branches completely, one can say that one has $h_{eff} = n \times h$ giving $1/\alpha_K = n/\alpha_K(n = 1)$ and scaling ofKähler action. One can also imagine that the 3-surfaces at the ends of CD are actually surfaces in the n-fold covering space consisting of n identical copies so that Kähler action is multiplied by n. One could also include the light-like partonic orbits to the 3-surface so that 3-surfaces would not have boundaries: in this case the n-fold degeneracy would come out very naturally.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [K13] .

2.3.2 The notion of 3-surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of General Coordinate Invariance since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces and their 4-D tangent spaces. It is however essential that information about normal space of the 2-surface is needed.

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further complication relates to the hierarchy of Planck constants. At "microscopic" level this means that there number of conformal equivalence classes of space-time surfaces connecting the 3-surfaces at boundaries of CD matters and this information is coded by the value of $h_{eff} = n \times h$. One can divide WCW to sectors corresponding to different values of h_{eff} and conformal symmetry breakings connect these sectors: the transition $n_1 \rightarrow n_2$ such that n_1 divides n_2 occurs spontaneously since it reduces the quantum criticality by transforming super-generators acting as gauge symmetries to dynamical ones.

2.3.3 The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of ZEO (with motivation coming from the non-determinism of Kähler action) it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW.

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW. The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW (WCW) is a union of WCWs associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.

2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub-WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!

4. Generalized coset construction and coset space structure have very deep physical meaning since they realize Equivalence Principle at quantum level. Contrary to the original belief, this construction does not provide a realization of Equivalence Principle at quantum level. The proper realization of EP at quantum level seems to be based on the identification of classical Noether charges in Cartan algebra with the eigenvalues of their quantum counterparts assignable to Kähler-Dirac action. At classical level EP follows at GRT limit obtained by lumping many-sheeted space-time to M^4 with effective metric satisfying Einstein's equations as a reflection of the underlying Poincare invariance.

5. Now it has become clear that EP in the sense of quantum classical correspondence allows a concrete realization for the fermion lines defined by the light-like boundaries of string world sheets at light-like orbits of partonic 2-surfaces. Fermion lines are always light-like or space-like locally. Kähler-Dirac equation reducing to its algebraic counterpart with light-like 8-momentum defined by the tangent of the boundary curve. 8-D light-likeness means the possibility of massivation in M^4 sense and gravitational mass is defined in an obvious manner. The M^4-part of 8-momentum is by quantum classical correspondence equal to the 4-momentum assignable to the incoming fermion. EP generalizes also to CP_2 degrees of freedom and relates $SO(4)$ acting as symmetries of Eucldian part of 8-momentum to color $SU(3)$. $SO(4)$ can be assigned to hadrons and $SU(3)$ to quarks and gluons.

   The 8-momentum is light-like with respect to the effective metric defined by K-D gamma matrices. Is it also light-like with respect to the induced metric and proportional to the tangent vector of the fermion line? If this is not the case, the boundary curve is locally space-like in the induced metric. Could this relate to the still poorly understand question how the necessariy tachyonic ground state conformal weight of super-conformal representations needed in padic mass calculations [K8] emerges? Could it be that "empty" lines carrying no fermion number are tachyonic with respect to the induced metric?

2.4 The Treatment Of Non-Determinism Of Kähler Action In Zero Energy Ontology

The non-determinism of Kähler action means that the reduction of the construction of WCW geometry to the light cone boundary fails. Besides degeneracy of the preferred extrema of Kähler action, the non-determinism should manifest itself as a presence of causal determinants also other than light cone boundary.

One can imagine two kinds of causal determinants.

1. Elementary particle horizons and light-like boundaries $X^3_l \subset X^4$ of 4-surfaces representing wormhole throats act as causal determinants for the space-time dynamics defined by Kähler action. The boundary values of this dynamics have been already considered.

2. At embedding space level causal determinants correspond to light like CD forming a fractal hierarchy of CDs within CDs. These causal determinants determine the dynamics of zero energy states having interpretation as pairs of initial and final states in standard quantum theory.

The manner to treat the classical non-determinism would be roughly following.

1. The replacement of space-like 3-surface X^3 with X_l^3 transforms initial value problem for X^3 to a boundary value problem for X_l^3. In principle one can also use the surfaces $X^3 \subset \delta CD \times CP_2$ if X_l^3 fixes $X^4(X_l^3)$ and thus X^3 uniquely. For years an important question was whether both X^3 and X_l^3 contribute separately to WCW geometry or whether they provide descriptions, which are in some sense dual.

2. Only Super-Kac-Moody type conformal algebra makes sense in the interior of X_l^3. In the 2-D intersections of X_l^3 with the boundary of causal diamond (CD) defined as intersection of future and past directed light-cones super-symplectic algebra makes sense. This implies effective two-dimensionality which is broken by the non-determinism represented using the hierarchy of CDs meaning that the data from these 2-D surfaces and their normal spaces at boundaries of CDs in various scales determine the WCW metric.

3. An important question has been whether Kac-Moody and super-symplectic algebras provide descriptions which are dual in some sense. At the level of Super-Virasoro algebras duality seems to be satisfied in the sense of generalized coset construction meaning that the differences of Super Virasoro generators of super-symplectic and super Kac-Moody algebras annihilate physical states. Among other things this means that four-momenta assignable to the two Super Virasoro representations are identical. T he interpretation is in terms of a generalization of Equivalence Principle [K19, K4] . This gives also a justification for p-adic thermodynamics applying only to Super Kac-Moody algebra.

4. Light-like 3-surfaces can be regarded also as generalized Feynman diagrams. The finite length resolution mean means also a cutoff in the number of generalized Feynman diagrams and this number remains always finite if the light-like 3-surfaces identifiable as maxima of Kähler function correspond to the diagrams. The finiteness of this number is also essential for number theoretic universality since it guarantees that the elements of M-matrix are algebraic numbers if momenta and other quantum numbers have this property. The introduction of new sub-CDs means also introduction of zero energy states in corresponding time scale.

5. The notion of finite measurement resolution expressed in terms of hierarchy of CDs within CDs is important for the treatment of classical non-determinism. In a given resolution the non-determinism of Kähler action remains invisible below the time scale assigned to the smallest CDs. One could also say that complete non-determinism characterized in terms path integral with cutoff is replaced in TGD framework with the partial failure of classical non-determinism leading to generalized Feynman diagrams. This gives rise to discrete coupling constant evolution and avoids the mathematical ill-definedness and infinities plaguing path integral formalism since the functional integral over 3-surfaces is well defined.

2.5 Category Theory And WCW Geometry

Due the effects caused by the classical non-determinism even classical TGD universes are very far from simple Cartesian clockworks, and the understanding of the general structure of WCW is a formidable challenge. Category theory is a branch of mathematics which is basically a theory about universal aspects of mathematical structures. Thus category theoretical thinking might help in disentangling the complexities of WCW geometry and the basic ideas of category theory are discussed in this spirit and as an innocent layman. It indeed turns out that the approach makes highly non-trivial predictions.

In ZEO the effects of non-determinism are taken into account in terms of causal diamonds forming a hierarchical fractal structure. One must allow also the unions of CDs, CDs within CDs, and probably also overlapping of CDs, and there are good reasons to expert that CDs and corresponding algebraic structures could define categories. If one does not allow overlapping CDs then set theoretic inclusion map defines a natural arrow. If one allows both unions and intersections then CDs would form a structure analogous to the set of open sets used in set theoretic topology. One could indeed see CDs (or rather their Cartesian products with CP_2) as analogs of open sets in Minkowskian signature.

So called ribbon categories seem to be tailor made for the formulation of quantum TGD and allow to build bridge to topological and conformal field theories. This discussion based on standard ontology. In [K2] rather detailed category theoretical constructions are discussed. Important role is played by the notion of operad operad,operads : this structure can be assigned with both generalized Feynman diagrams and with the hierarchy of symplectic fusion algebras realizing symplectic analogs of the fusion rules of conformal field theories.

3 Identification Of The Symmetries And Coset Space Structure Of WCW

In this section the identification of the isometry group of the configuration ("world of classical worlds" or briefly WCW) will be discussed at general level.

3.1 Reduction To The Light Cone Boundary

The reduction to the light cone boundary would occur exactly if Kähler action were strictly deterministic. This is not the case but it is possible to generalize the construction at light cone boundary to the general case if causal diamonds define the basic structural units of the WCW .

3.1.1 Old argument

The identification of WCW follows as a consequence of 4-dimensional Diff invariance. The right question to ask is the following one. How could one coordinatize the physical(!) vibrational degrees of freedom for 3-surfaces in Diff^4 invariant manner: coordinates should have same values for all Diff^4 related 3-surfaces belonging to the orbit of X^3? The answer is following:

1. Fix some 3-surface (call it Y^3) on the orbit of X^3 in Diff^4 invariant manner.

2. Use as WCW coordinates of X^3 and all its diffeomorphs the coordinates parameterizing small deformations of Y^3. This kind of replacement is physically acceptable since metrically the WCW is equivalent with $Map/Diff^4$.

3. Require that the fixing procedure is Lorentz invariant, where Lorentz transformations in question leave light M^4_+ invariant and thus act as isometries.

The simplest choice of Y^3 is the intersection of the orbit of 3-surface (X^4) with the set $\delta M^4_+ \times CP_2$, where δM^4_+ denotes the boundary of the light cone (moment of big bang):

$$Y^3 \quad = \quad X^4 \cap \delta M^4_+ \times CP_2 \tag{3.1}$$

Lorentz invariance allows also the choice $X \times CP_2$, where X corresponds to the hyperboloid $a = \sqrt{(m^0)^2 - r_M^2} = constant$ but only the proposed choice ($a = 0$) leads to a natural complexification in M^4 degrees of freedom. This choice is also cosmologically very natural and completely analogous to the quantum gravitational holography of string theories.

WCW has a fiber space structure. Base space consists of 3-surfaces $Y^3 \subset \delta M^4_+ \times CP_2$ and fiber consists of 3-surfaces on the orbit of Y^3, which are Diff^4 equivalent with Y^3. The distance between the surfaces in the fiber is vanishing in WCW metric. An elegant manner to avoid difficulties caused by Diff^4 degeneracy in WCW integration is to *define* integration measure as integral over the reduced WCW consisting of 3-surfaces Y^3 at the light cone boundary.

Situation is however quite not so simple. The vacuum degeneracy of Kähler action suggests strongly classical non-determinism so that there are several, possibly, infinite number of preferred extremals $X^4(Y^3)$ associated with given Y^3 on light cone boundary. This implies additional degeneracy.

One might hope that the reduced WCW could be replaced by its covering space so that given Y^3 corresponds to several points of the covering space and WCW has many-sheeted structure. Obviously the copies of Y^3 have identical geometric properties. WCW integral would decompose into a sum of integrals over different sheets of the reduced WCW . Note that WCW spinor fields are in general different on different sheets of the reduced WCW .

Even this is probably not enough: it is quite possible that all light like surfaces of M^4 possessing Hamilton Jacobi structure (and thus interpretable as light fronts) are involved with the construction of the WCW geometry. Because of their metric two-dimensionality the proposed construction should generalize. This would mean that WCW geometry has also local laboratory scale aspects and that the general ideas might allow testing.

3.1.2 New version of the argument

The above summary was the basic argument for two decades ago. A more elegant formulation would in terms of light-like 3-surfaces connecting the boundaries of causal diamond taken as basic geometric objects and identified as generalized Feynman diagrams so that they are singular as manifolds at the vertices.

If both formulations are required to be correct, the only conclusion is that effective 2-dimensionality must hold true in the scale of given CD. In other words, the intersection $X^2 = X^3_l \cap X^3$ at the boundary of CD is effectively the basic dynamical unit. The failure of strict non-determinism however forces to introduce entire hierarchy of CDs responsible also for coupling constant evolution defined in terms of the measurement resolution identified as the size of the smallest CD present.

3.2 WCW As A Union Of Symmetric Spaces

In finite-dimensional context globally symmetric spaces are of form G/H and connection and curvature are independent of the metric, provided it is left invariant under G. The hope is that same holds true in infinite-dimensional context. The most one can hope of obtaining is the decomposition $C(H) = \cup_i G/H_i$ over orbits of G. One could allow also symmetry breaking in the sense that G and H depend on the orbit: $C(H) = \cup_i G_i/H_i$ but it seems that G can be chosen to be same for all orbits. What is essential is that these groups are infinite-dimensional. The basic properties of the coset space decomposition give very strong constraints on the group H, which certainly contains the subgroup of G, whose action reduces to diffeomorphisms of X^3.

3.2.1 Consequences of the decomposition

If the decomposition to a union of coset spaces indeed occurs, the consequences for the calculability of the theory are enormous since it suffices to find metric and curvature tensor for single representative 3-surface on a given orbit (contravariant form of metric gives propagator in perturbative calculation of matrix elements as functional integrals over the WCW). The representative surface can be chosen to correspond to the maximum of Kähler function on a given orbit and one obtains perturbation theory around this maximum (Kähler function is not isometry invariant).

The task is to identify the infinite-dimensional groups G and H and to understand the zero mode structure of the WCW . Almost twenty (seven according to long held belief!) years after the discovery of the candidate for the Kähler function defining the metric, it became finally clear that these identifications follow quite nicely from $Diff^4$ invariance and $Diff^4$ degeneracy as well as special properties of the Kähler action.

The guess (not the first one!) would be following. G corresponds to the symplectic transformations of $\delta M^4_\pm \times CP_2$ leaving the induced Kähler form invariant. If G acts as isometries the values of Kähler form at partonic 2-surfaces (remember effective 2-dimensionality realized in simplistic manner) are zero modes and WCW allows slicing to symplectic orbits of the partonic 2-surface with fixed induced Kähler form. Quantum fluctuating degrees of freedom would correspond to symplectic group and to the fluctuations of the induced metric. The group H dividing G would act as diffeomorphisms at the preferred 3-surface X^3 and leaving X^3 itself invariant. Therefore the identification of g and h would be in terms of tangent space algebra of WCW sector realized as coset space G/H.

3.2.2 Coset space structure of WCW and Equivalence Principle

The realization of WCW sectors with fixed values of zero modes as symmetric spaces G/H (analogous to $CP_2 = SU(3)/U(2)$) suggests that one can assign super-Virasoro algebras with G. What the two algebras g and h are is however difficult question. The following vision is only one of the many (the latest one).

1. Symplectic algebra g generates isometries and h is identified as algebra, whose generators generate diffeormorphisms at preferred X^3.

2. The original long-held belief was that the Super Kac-Moody symmetry corresponds to local embedding space isometries for light-like 3-surfaces X_l^3, which might be boundaries of X^4 (probably not: it seems that boundary conditions cannot be satisfied so that space-time surfaces must consists of regions defining at least double coverings of M^4) and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry would be identifiable as the counterpart of the Kac Moody symmetry of string models.

   It has turned out that one can assume Kac-Moody algebra to be sub-algebra of symplectic algebra consisting of the symplectic isometries of embedding space. This Super Kac-Moody algebra is generated by super-currents assignable to the modes of induced spinor fields other than right-handed neutrino and localized at string world sheets. The entire symplectic algebra would correspond to the modes of right-handed neutrino and the entire algebra one would be direct sum of these two algebras so that the number of tensor factors would be indeed 5. The beauty of this option is that localization would be for both algebras inherent and with respect to the light-like coordinate of light-cone boundary rather than forced by hand.

3. p-Adic mass calculations require that symplectic and Kac-Moody algebras together generate the entire algebra. In this situation strong form of holography implies that transformations located to the interior of space-like 3-surface and light-like partonic orbit define zero modes and act like gauge symmetries. The physically non-trivial transformations correspond to transformations acting non-trivially at partonic 2-surfaces. g corresponds to the algebra generated by these transformations and for preferred 3-surface - identified as (say) maximum of Kähler function - h corresponds to the elements of this algebra generating diffeomorphisms of X^3. Super-conformal representation has five tensor factors corresponding to color algebra, two factors from electroweak $U(2)$, one factor from transversal M^4 translations and one factor from symplectic algebra (note that also Hamiltonians which are products of δM^4_+ and CP_2 Hamiltonians are possible.

Equivalence Principle (EP) has been a longstanding problem for TGD although the recent stringy view about graviton mediated scattering makes it can be argued to reduce to a tautology. I have considered several explanations for EP and coset representation has been one of them.

1. Coset representation associated with the super Virasoro algebra is defined by the condition that the differences of super Virasoro generators for g and h annihilate the physical. The original proposal for the realization of EP was that this condition implies that the four-momenta associated with g and h are identical and identifiable as inertial and gravitational four-momenta. Translations however lead out from CD boundary and cannot leave 3-surface invariant. Hence the Virasoro generators for h should not carry four-momentum. Therefore EP cannot be understood in terms of coset representations.

2. The equivalence of classical Noether momentum associated with Kähler action with eigenvalues of the corresponding quantal momentum for Kähler-Dirac action certainly realizes quantum classical correspondence (QCC) EP could correspond to QCC.

3. A further option is that EP reduces to the identification of the four momenta for Super Virasoro representations assignable to space-like and light-like 3-surfaces and therefore become part of strong form of holography in turn implied by strong form of GCI! It seems that this option is the most plausible one found hitherto.

3.2.3 WCW isometries as a subgroup of $Diff(\delta M^4_+ \times CP_2)$

The reduction to light cone boundary leads to the identification of the isometry group as some subgroup of for the group G for the diffeomorphisms of $\delta M^4_+ \times CP_2$. These diffeomorphisms indeed act in a natural manner in δCH, the space of 3-surfaces in $\delta M^4_+ \times CP_2$. WCW is expected to decompose to a union of the coset spaces G/H_i, where H_i corresponds to some subgroup of G containing the transformations of G acting as diffeomorphisms for given X^3. Geometrically the vector fields acting as diffeomorphisms of X^3 are tangential to the 3-surface. H_i could depend on the topology of X^3 and since G does not change the topology of 3-surface each 3-topology defines separate orbit of G. Therefore, the union involves sum over all topologies of X^3 plus possibly other "zero modes". Different topologies are naturally glued together since singular 3-surfaces intermediate between two 3-topologies correspond to points common to the two sectors with different topologies.

3.2.4 Isometries of WCW geometry as symplectic transformations of $\delta M^4_+ \times CP_2$

During last decade I have considered several candidates for the group G of isometries of WCW as the sub-algebra of the subalgebra of $Diff(\delta M^4_+ \times CP_2)$. To begin with let us write the general decomposition of $diff(\delta M^4_+ \times CP_2)$:

$$diff(\delta M^4_+ \times CP_2) = S(CP_2) \times diff(\delta M^4_+) \oplus S(\delta M^4_+) \times diff(CP_2) \ . \qquad (3.2)$$

Here $S(X)$ denotes the scalar function basis of space X. This Lie-algebra is the direct sum of light cone diffeomorphisms made local with respect to CP_2 and CP_2 diffeomorphisms made local with respect to light cone boundary.

The idea that entire diffeomorphism group would act as isometries looks unrealistic since the theory should be more or less equivalent with topological field theory in this case. Consider now the various candidates for G.

1. The fact that symplectic transformations of CP_2 and M^4_+ diffeomorphisms are dynamical symmetries of the vacuum extremals suggests the possibility that the diffeomorphisms of the light cone boundary and symplectic transformations of CP_2 could leave Kähler function invariant and thus correspond to zero modes. The symplectic transformations of CP_2 localized with respect to light cone boundary acting as symplectic transformations of CP_2 have interpretation as local color transformations and are a good candidate for the isometries. The fact that local color transformations are not even approximate symmetries of Kähler action is not a problem: if they were exact symmetries, Kähler function would be invariant and zero modes would be in question.

2. CP_2 local conformal transformations of the light cone boundary act as isometries of δM^4_+. Besides this there is a huge group of the symplectic symmetries of $\delta M^4_+ \times CP_2$ if light cone boundary is provided with the symplectic structure. Both groups must be considered as candidates for groups of isometries. $\delta M^4_+ \times CP_2$ option exploits fully the special properties of $\delta M^4_+ \times CP_2$, and one can develop simple argument demonstrating that $\delta M^4_+ \times CP_2$ symplectic invariance is the correct option. Also the construction of WCW gamma matrices as super-symplectic charges supports $\delta M^4_+ \times CP_2$ option.

3.2.5 WCW as a union of symmetric spaces

The idea about symmetric space is extremely beautiful but it took a long time and several false alarms before the time was ripe for identifying the precise form of the Cartan decomposition $g = t + h$ satisfying the defining conditions

$$g = t + h \ , \quad [t,t] \subset h \ , \quad [h,t] \subset t \ . \qquad (3.3)$$

The ultimate solution of the puzzle turned out to be amazingly simple and came only after quantum TGD was understood well enough. $[t,t] \subset h$ condition is highly nontrivial and equivalent with the existence of involution. Inversion in the light-like radial coordinate of δM^4 is a natural guess for this involution and induces complex conjugation in super-conformal algebras mapping positive and negative conformal weights to each other.

WCW geometry allows two super-conformal symmetries. The first one corresponds to super-symplectic transformations acting at the level of embedding space. The second one corresponds to super Kac-Moody symmetry. The original identification of Kac-Moody was in terms of deformations of light-like 3-surfaces respecting their light-likeness. This not wrong as such: also entire symplectic algebra can be assigned with light-like surfaces and the theory can be constructed using also these conformal algebras. This identification however makes it very difficult to see how Kac-Moody could act as isometry: in particular, the localization with respect to internal coordinates of 3-surface produces technical problems since symplectic algebra is localized with respect to the light-like radial coordinate of light-cone boundary.

The more plausible identification is as the sub-algebra of symplectic algebra realized as isometries of δCD so that localization is inherent and in terms of the radial light-like coordinate of light-like boundary [K14]. This identification is made possible by the wisdom gained from the solutions of the Kähler-Dirac equations predicting the localization of its modes (except right-handed neutrino) to string world sheets.

1. g would thus correspond to a direct sum of super-symplectic algebra and super Kac-Moody algebra defined by its isometry sub-algebra but represented in different manner (this is absolutely essential!). More concretely, neutrino modes defined super Hamiltonians associated with the super symplectic algebra and other modes of induced spinor field the super Hamiltonians associated with the super Kac-Moody algebra. The maxima of Kähler function could be chosen as natural candidates for the preferred points and could play also an essential role in WCW integration by generalizing the Gaussian integration of free quantum field theories.

2. These super-conformal algebra representations form a direct sum. p-Adic mass calculations require five super-conformal tensor factors and the number of tensor factors would be indeed this.

3. This algebra has as sub-algebra the algebra for which generators leave 3-surface invariant - in other words, induce its diffeomorphism. Quantum states correspond to the coset representations for entire algebra and this algebra so that differences of the corresponding super-Virasoro generators annihilate physical states. This obviously generalizes Goddard-Olive-Kent construction [A6]. It seems now clear that coset representation does not imply EP: the four-momentum simply does not appear in the representation of the isotropy sub-algebra since translations lead out of CD boundary.

To minimize confusions it must be emphasized that only the contribution of the symplectic algebra realized in terms of single right-handed neutrino mode is discussed in this chapter and the WCW Hamiltonians have 2-dimensional representation. Also the direct connection with the dynamics of Kähler action is lacking. A more realistic construction [K14] uses 3-dimensional representations of Hamiltonians and requires all modes of right-handed neutrino for symplectic algebra and the modes of induced spinor field carrying electroweak quantum numbers in the case of Kac-Moody algebra.

4 Complexification

A necessary prerequisite for the Kähler geometry is the complexification of the tangent space in vibrational degrees of freedom. What this means in recent context is non-trivial.

4.1 Why Complexification Is Needed?

The Minkowskian signature of M^4 metric seems however to represent an insurmountable obstacle for the complexification of M^4 type vibrational degrees of freedom. On the other hand, complexification seems to have deep roots in the actual physical reality.

1. In the perturbative quantization of gauge fields one associates to each gauge field excitation polarization vector e and massless four-momentum vector p ($p^2 = 0$, $p \cdot e = 0$). These vectors define the decomposition of the tangent space of M^4: $M^4 = M^2 \times E^2$, where M^2 type polarizations correspond to zero norm states and E^2 type polarizations correspond to physical states with non-vanishing norm. Same type of decomposition occurs also in the linearized theory of gravitation. The crucial feature is that E^2 allows complexification! The general conclusion is that the modes of massless, linear, boson fields define always complexification of M^4 (or its tangent space) by effectively reducing it to E^2. Also in string models similar situation is encountered. For a string in D-dimensional space only D-2 transversal Euclidian degrees of freedom are physical.

2. Since symplectically extended isometry generators are expected to create physical states in TGD approach same kind of physical complexification should take place for them, too: this indeed takes place in string models in critical dimension. Somehow one should be able to associate polarization vector and massless four momentum vector to the deformations of a given 3-surface so that these vectors define the decomposition $M^4 = M^2 \times E^2$ for each mode. Configuration space metric should be degenerate: the norm of M^2 deformations should vanish as opposed to the norm of E^2 deformations.

Consider now the implications of this requirement.

1. In order to associate four-momentum and polarization (or at least the decomposition $M^4 = M^2 \times E^2$) to the deformations of the 3-surface one should have field equations, which determine the time development of the 3-surface uniquely. Furthermore, the time development for small deformations should be such that it makes sense to associate four momentum and polarization or at least the decomposition $M^4 = M^2 \times E^2$ to the deformations in suitable basis.

   The solution to this problem is afforded by the proposed definition of the Kähler function. The definition of the Kähler function indeed associates to a given 3-surface a unique four-surface as the preferred extremal of the Kähler action. Therefore one can associate a unique time development to the deformations of the surface X^3 and if TGD describes the observed world this time development should describe the evolution of photon, gluon, graviton, etc. states and so we can hope that tangent space complexification could be defined.

2. We have found that M^2 part of the deformation should have zero norm. In particular, the time like vibrational modes have zero norm in WCW metric. This is true if Kähler function is not only $Diff^3$ invariant but also Diff^4 invariant in the sense that Kähler function has same value for all 3-surfaces belonging to the orbit of X^3 and related to X^3 by diffeomorphism of X^4. This is indeed the case.

3. Even this is not enough. One expects the presence of massive modes having also longitudinal polarization and for these states the number of physical vibrational degrees of freedom is 3 so that complexification seems to be impossible by odd dimension.

The reduction to the light cone boundary implied by $Diff^4$ invariance makes possible to identify the complexification. Crucial role is played by the special properties of the boundary of 4-dimensional light cone, which is metrically two-sphere and thus allows generalized complex and Kähler structure.

4.2 The Metric, Conformal And Symplectic Structures Of The Light Cone Boundary

The special metric properties of the light cone boundary play a crucial role in the complexification. The point is that the boundary of the light cone has degenerate metric: although light cone boundary is topologically 3-dimensional it is metrically 2-dimensional: effectively sphere. In standard spherical Minkowski coordinates light cone boundary is defined by the equation $r_M = m^0$ and induced metric reads

$$ds^2 = -r_M^2 d\Omega^2 = -r_M^2 dz d\bar{z}/(1+z\bar{z})^2 \ , \tag{4.1}$$

and has Euclidian signature. Since S^2 allows complexification and thus also Kähler structure (and as a by-product also symplectic structure) there are good hopes of obtaining just the required type of complexification in non-degenerate M^4 degrees of freedom: WCW would effectively inherit its Kähler structure from $S^2 \times CP_2$.

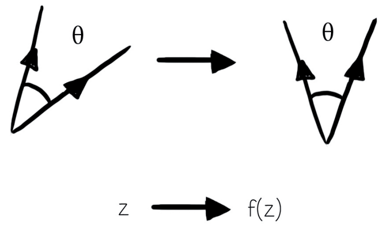

Figure 1: Conformal symmetry preserves angles in complex plane

By its effective two-dimensionality the boundary of the four-dimensional light cone has infinite-dimensional group of (local) conformal transformations. Using complex coordinate z for S^2 the general local conformal transformation reads (see **Fig. 1**)

$$\begin{aligned} r &\rightarrow f(r_M, z, \bar{z}) \ , \\ z &\rightarrow g(z) \ , \end{aligned} \tag{4.2}$$

where f is an arbitrary real function and g is an arbitrary analytic function with a finite number of poles. The infinitesimal generators of this group span an algebra, call it C, analogous to Virasoro algebra. This algebra is semidirect sum of two algebras L and R given by

$$\begin{aligned} C &= L \oplus R \ , \\ [L, R] \subset R \ , \end{aligned} \tag{4.3}$$

where L denotes standard Virasoro algebra of the two- sphere generated by the generators

$$L_n = z^{n+1} d/dz \tag{4.4}$$

and R denotes the algebra generated by the vector fields

$$R_n = f_n(z, \bar{z}, r_M)\partial_{r_M} \ , \tag{4.5}$$

where $f(z, \bar{z}, r_M)$ forms complete real scalar function basis for light cone boundary. The vector fields of R have the special property that they have vanishing norm in M^4 metric.

This modification of conformal group implies that the Virasoro generator L_0 becomes $L_0 = zd/dz - r_M d/dr_M$ so that the scaling momentum becomes the difference $n-m$ or S^2 and radial scaling momenta. One could achieve conformal invariance by requiring that S^2 and radial scaling quantum numbers compensate each other.

Of crucial importance is that light cone boundary allows infinite dimensional group of isometries! An arbitrary conformal transformation $z \rightarrow f(z)$ induces to the metric a conformal factor given by $|df/dz|^2$. The compensating radial scaling $r_M \rightarrow r_M/|df/dz|$ compensates this factor so that the line element remains invariant.

The Kähler structure of light cone boundary defines automatically symplectic structure. The symplectic form is degenerate and just the area form of S^2 given by

$$J = r_M^2 sin(\theta) d\theta \wedge d\phi ,$$

in standard spherical coordinates, there is infinite-dimensional group of symplectic transformations leaving the symplectic form of the light cone boundary (that is S^2) invariant. These transformations are local with respect to the radial coordinate r_M. The symplectic and Kähler structures of light cone boundary are not unique: different structures are labeled by the coset space $SO(3,1)/SO(3)$. One can however associate with a given 3-surface Y^3 a unique structure by requiring that the corresponding subgroup $SO(3)$ of Lorentz group acts as the isotropy group of the conserved classical four-momentum assigned to Y^3 by the preferred extremal property.

In the case of $\delta M^4_+ \times CP_2$ both the conformal transformations, isometries and symplectic transformations of the light cone boundary can be made local also with respect to CP_2. The idea that the infinite-dimensional algebra of symplectic transformations of $\delta M^4_+ \times CP_2$ act as isometries of WCW and that radial vector fields having zero norm in the metric of light cone boundary possess zero norm also in WCW metric, looks extremely attractive.

In the case of $\delta M^4_+ \times CP_2$ one could combine the symplectic and Kähler structures of δM^4_+ and CP_2 to single symplectic/Kähler structure. The symplectic transformations leaving this symplectic structure invariant would be generated by the function algebra of $\delta M^4_+ \times CP_2$ such that a arbitrary function serves as a Hamiltonian of a symplectic transformation. This group serves as a candidate for the isometry group of WCW . An alternative identification for the isometry algebra is as symplectic symmetries of CP_2 localized with respect to the light cone boundary. Hamiltonians would be also now elements of the function algebra of $\delta M^4_+ \times CP_2$ but their Poisson brackets would be defined using only CP_2 symplectic form.

The problem is to decide which option is correct. There is a simple argument fixing the latter option. The symplecticly imbedded CP_2 would be left invariant under δM^4_+ local symplectic transformations of CP_2. This seems strange. Under symplectic algebra of $\delta M^4_+ \times CP_2$ also symplecticly imbedded CP_2 is deformed and this sounds more realistic. The isometry algebra is therefore assumed to be the group $can(\delta M^4_+ \times CP_2)$ generated by the scalar function basis $S(\delta M^4_+ \times CP_2) = S(\delta M^4_+) \times S(CP_2)$ of the light cone boundary using the Poisson brackets to be discussed in more detail later.

There are some no-go theorems associated with higher-dimensional Abelian extensions [A7], and although the contexts are quite different, it is interesting to consider the recent situation in light of these theorems.

1. Conformal invariance is an essentially 2-dimensional notion. Light cone boundary is however metrically and conformally 2-sphere, and therefore the conformal algebra is effectively that associated with the 2-sphere. In the same manner, the quaternion conformal algebra associated with the metrically 2-dimensional elementary particle horizons surrounding wormhole contacts allows the usual Kac Moody algebra and actually also contributes to the WCW metric.

2. In dimensions $D > 2$ Abelian extensions of the gauge algebra are extensions by an infinite-dimensional Abelian group rather than central extensions by the group $U(1)$. This result has an analog at the level of WCW geometry. The extension associated with the symplectic algebra of CP_2 localized with respect to the light cone boundary is analogous a symplectic extension defined by Poisson bracket $\{p, q\} = 1$. The central extension is the function space associated with δM^4_+ and indeed infinite-dimensional if only only CP_2 symplectic structure induces the Poisson bracket but one-dimensional if $\delta M^4_+ \times CP_2$ Poisson bracket induces the extension. In the latter case the symmetries fix the metric completely at the point corresponding to the origin of symmetric space (presumably the maximum of Kähler function for given values of zero modes).

3. $D > 2$ extensions possess no unitary faithful representations (satisfying certain well motivated physical constraints) [A7]. It might be that the degeneracy of the WCW metric is the analog for the loss of faithful representations.

4.3 Complexification And The Special Properties Of The Light Cone Boundary

In case of Kähler metric G and H Lie-algebras must allow complexification so that the isometries can act as holomorphic transformations. Since G and H can be regarded as subalgebras of the vector fields of $\delta M^4_+ \times CP_2$, they inherit in a natural manner the complex structure of the light cone boundary.

There are two candidates for WCW complexification. The simplest, and also the correct, alternative is that complexification is induced by natural complexification of vector field basis on $\delta M^4_+ \times CP_2$. In CP_2 degrees of freedom there is natural complexification

$$\xi \rightarrow \bar{\xi} \ .$$

In δM^4_+ degrees of freedom this could involve the transformation

$$z \rightarrow \bar{z}$$

and certainly involves complex conjugation for complex scalar function basis in the radial direction:

$$f(r_M) \rightarrow \overline{f(r_M)} \ ,$$

which turns out to play same role as the function basis of circle in the Kähler geometry of loop groups [A2].

The requirement that the functions are eigen functions of radial scalings favors functions $(r_M/r_0)^k$, where k is in general a complex number. The function can be expressed as a product of real power of r_M and logarithmic plane wave. It turns out that the radial complexification alternative is the correct manner to obtain Kähler structure. The reason is that symplectic transformations leave the value of r_M invariant. Radial Virasoro invariance plays crucial role in making the complexification possible.

One could consider also a second alternative assumed in the earlier formulation of the WCW geometry. The close analogy with string models and conformal field theories suggests that for Virasoro generators the complexification must reduce to the hermitian conjugation of the conformal field theories: $L_n \rightarrow L_{-n} = L_n^\dagger$. Clearly this complexification is induced from the transformation $z \rightarrow \frac{1}{z}$ and differs from the complexification induced by complex conjugation $z \rightarrow \bar{z}$. The basis would be polynomial in z and $\bar{z}$. Since radial algebra could be also seen as Virasoro algebra localized with respect to $S^2 \times CP_2$ one could consider the possibility that also in radial direction the inversion $r_M \rightarrow \frac{1}{r_M}$ is involved.

In fact, the complexification changing the signs of radial conformal weights is induced from inversion $r_M/r_0 \rightarrow r_0/r_M$. This transformation is also an excellent candidate for the involution necessary for obtaining the structure of symmetric space implying among other things the covariant constancy of the curvature tensor, which is of special importance in infinite-D context.

The essential prerequisite for the Kähler structure is that both G and H allow same complexification so that the isometries in question can be regarded as holomorphic transformations. In finite-dimensional case this essentially what is needed since metric can be constructed by parallel translation along the orbit of G from H-invariant Kähler metric at a representative point. The requirement of H-invariance forces the radial complexification based on complex powers r_M^k: radial complexification works since symplectic transformations leave r_M invariant.

Some comments on the properties of the proposed complexification are in order.

1. The proposed complexification, which is analogous to the choice of gauge in gauge theories is not Lorentz invariant unless one can fix the coordinates of the light cone boundary apart from $SO(3)$ rotation not affecting the value of the radial coordinate r_M (if the imaginary part of k in r_M^k is always non-vanishing). This is possible as will be explained later.

2. It turns out that the function basis of light-cone boundary multiplying CP_2 Hamiltonians corresponds to unitary representations of the Lorentz group at light cone boundary so that the Lorentz invariance is rather manifest.

3. There is a nice connection with the proposed physical interpretation of the complexification. At the moment of the big bang all particles move with the velocity of light and therefore behave as massless particles. To a given point of the light cone boundary one can associate a unique direction of massless four-momentum by semiclassical considerations: at the point $m^k = (m^0, m^i)$ momentum is proportional to the vector $(m^0, -m^i)$. Since the particles are massless only two polarization vectors are possible and these correspond to the tangent vectors to the sphere $m^0 = r_M$. Of course, one must always fix polarizations at some point of tangent space but since massless polarization vectors are not physical this doesn't imply difficulties: different choices correspond to different gauges.

4. Complexification in the proposed manner is not possible except in the case of four-dimensional Minkowski space. Non-zero norm deformations correspond to vector fields of the light cone boundary acting on the sphere S^{D-2} and the decomposition to $(1,0)$ and $(0,1)$ parts is possible only when the sphere in question is two-dimensional since other spheres do allow neither complexification nor Kähler structure.

4.4 How To Fix The Complex And Symplectic Structures In A Lorentz Invariant Manner?

One can assign to light-cone boundary a symplectic structure since it reduces effectively to S^2. The possible symplectic structures of δM^4_+ are parameterized by the coset space $SO(3,1)/SO(3))$, where H is the isotropy group $SO(3)$ of a time like vector. Complexification also fixes the choice of the spherical coordinates apart from rotations around the quantization axis of angular momentum.

The selection of some preferred symplectic structure in an ad hoc manner breaks manifest Lorentz invariance but is possible if physical theory remains Lorentz invariant. The more natural possibility is that 3-surface Y^3 itself fixes in some natural manner the choice of the symplectic structure so that there is unique subgroup $SO(3)$ of $SO(3,1)$ associated with Y^3. If WCW Kähler function corresponds to a preferred extremal of Kähler action, this is indeed the case. One can associate unique conserved four-momentum $P^k(Y^3)$ to the preferred extremal $X^4(Y^3)$ of the Kähler action and the requirement that the rotation group $SO(3)$ leaving the symplectic structure invariant leaves also $P^k(Y^3)$ invariant, fixes the symplectic structure associated with Y^3 uniquely.

Therefore WCW decomposes into a union of symplectic spaces labeled by $SO(3,1)/SO(3)$ isomorphic to $a = constant$ hyperboloid of light cone. The direction of the classical angular momentum vector $w^k = \epsilon^{klmn}P_l J_{mn}$ determined by the classical angular momentum tensor of associated with Y^3 fixes one coordinate axis and one can require that $SO(2)$ subgroup of $SO(3)$ acting as rotation around this coordinate axis acts as phase transformation of the complex coordinate z of S^2. Other rotations act as nonlinear holomorphic transformations respecting the complex structure.

Clearly, the coordinates are uniquely fixed modulo $SO(2)$ rotation acting as phase multiplication in this case. If $P^k(Y^3)$ is light like, one can only require that the rotation group $SO(2)$ serving as the isotropy group of 3-momentum belongs to the group $SO(3)$ characterizing the symplectic structure and it seems that symplectic structure cannot be uniquely fixed without additional constraints in this case. Probably this has no practical consequences since the 3-surfaces considered have actually infinite size and 4-momentum is most probably time like for them. Note however that the direction of 3-momentum defines unique axis such that $SO(2)$ rotations around this axis are represented as phase multiplication.

Similar almost unique frame exists also in CP_2 degrees of freedom and corresponds to the complex coordinates transforming linearly under $U(2)$ acting as isotropy group of the Lie-algebra element defined by classical color charges Q_a of Y^3. One can fix unique Cartan subgroup of $U(2)$ by noticing that $SU(3)$ allows completely symmetric structure constants d_{abc} such that $R_a = d_a^{\ bc} Q_b Q_c$ defines Lie-algebra element commuting with Q_a. This means that R_a and Q_a span in generic case $U(1) \times U(1)$ Cartan subalgebra and there are unique complex coordinates for which this subgroup acts as phase multiplications. The space of nonequivalent frames is isomorphic with $CP(2)$ so that one can say that cm degrees of freedom correspond to Cartesian product of $SO(3,1)/SO(3)$ hyperboloid and CP_2 whereas coordinate choices correspond to the Cartesian product of $SO(3,1)/SO(2)$ and $SU(3)/U(1) \times U(1)$.

Symplectic transformations leave the value of δM^4_+ radial coordinate r_M invariant and this implies large number of additional zero modes characterizing the size and shape of the 3-surface. Besides this Kähler magnetic fluxes through the $r_M = constant$ sections of X^3 as a function of r_M provide additional invariants, which are functions rather than numbers. The Fourier components for the magnetic fluxes provide infinite number of symplectic invariants. The presence of these zero modes imply that 3-surfaces behave much like classical objects in the sense that neither their shape nor form nor classical Kähler magnetic fields, are subject to Gaussian fluctuations. Of course, quantum superpositions of 3-surfaces with different values of these invariants are possible.

There are reasons to expect that at least certain infinitesimal symplectic transformations correspond to zero modes of the Kähler metric (symplectic transformations act as dynamical symmetries of the vacuum extremals of the Kähler action). If this is indeed the case, one can ask whether it is possible to identify an integration measure for them.

If one can associate symplectic structure with zero modes, the symplectic structure defines integration measure in a standard manner (for 2n-dimensional symplectic manifold the integration measure is just the n-fold wedge power $J \wedge J ... \wedge J$ of the symplectic form J). Unfortunately, in infinite-dimensional context this is not enough since divergence free functional integral analogous to a Gaussian integral is needed and it seems that it is not possible to integrate in zero modes and that this relates in a deep manner to state function reduction. If all symplectic transformations of $\delta M^4_+ \times CP_2$ are represented as symplectic transformations of the configuration space, then the existence of symplectic structure decomposing into Kähler (and symplectic) structure in complexified degrees of freedom and symplectic (but not Kähler) structure in zero modes, is an automatic consequence.

4.5 The General Structure Of The Isometry Algebra

There are three options for the isometry algebra of WCW .

1. Isometry algebra as the algebra of CP_2 symplectic transformations leaving invariant the symplectic form of CP_2 localized with respect to δM^4_+.

2. Certainly the WCW metric in δM^4_+ must be non-trivial and actually given by the magnetic flux Hamiltonians defining symplectic invariants. Furthermore, the super-symplectic generators constructed from quarks automatically give as anti-commutators this part of the WCW metric. One could interpret these symplectic invariants as WCW Hamiltonians for δM^4_+ symplectic transformations obtained when CP_2 Hamiltonian is constant.

3. Isometry algebra consists of $\delta M^4_+ \times CP_2$ symplectic transformations. In this case a local color transformation involves necessarily a local S^2 transformation. Unfortunately, it is difficult to decide at this stage which of these options is correct.

The eigen states of the rotation generator and Lorentz boost in the same direction defining a unitary representation of the Lorentz group at light cone boundary define the most natural function basis for the light cone boundary. The elements of this bases have also well defined scaling quantum numbers and define also a unitary representation of the conformal algebra. The product of the basic functions is very simple in this basis since various quantum numbers are additive.

Spherical harmonics of S^2 provide an alternative function basis for the light cone boundary:

$$H^m_{jk} \equiv Y_{jm}(\theta,\phi) r_M^k \quad . \tag{4.6}$$

One can criticize this basis for not having nice properties under Lorentz group.

The product of basis functions is given by Glebch-Gordan coefficients for symmetrized tensor product of two representation of the rotation group. Poisson bracket in turn reduces to the Glebch-Gordans of anti-symmetrized tensor product. The quantum numbers m and k are additive. The basis is eigen-function basis for the imaginary part of the Virasoro generator L_0 generating rotations around quantization axis of angular momentum. In fact, only the imaginary part of the Virasoro generator $L_0 = zd/dz = \rho\partial_\rho - \frac{2}{2}\partial_\phi$ has global single valued Hamiltonian, whereas the corresponding representation for the transformation induced by the real part of L_0, with a compensating radial scaling added, cannot be realized as a global symplectic transformation.

The Poisson bracket of two functions $H^m_{j_1k_1}$ and $H^m_{j_2k_2}$ can be calculated and is of the general form

$$\{H^{m_1}_{j_1k}, H^{m_2}_{j_2k_2}\} \equiv C(j_1m_1j_2m_2|j, m_1+m_2)_A H^{m_1+m_2}_{j,k_1+k_2} \quad . \tag{4.7}$$

The coefficients are Glebch-Gordan coefficients for the anti-symmetrized tensor product for the representations of the rotation group.

The isometries of the light cone boundary correspond to conformal transformations accompanied by a local radial scaling compensating the conformal factor coming from the conformal transformations having parametric dependence of radial variable and CP_2 coordinates. It seems however that isometries cannot in general be realized as symplectic transformations. The first difficulty is that symplectic transformations cannot affect the value of the radial coordinate. For rotation algebra the representation as symplectic transformations is however possible.

In CP_2 degrees of freedom scalar function basis having definite color transformation properties is desirable. Scalar function basis can be obtained as the algebra generated by the Hamiltonians of color transformations by multiplication. The elements of basis can be typically expressed as monomials of color Hamiltonians H^A_c

$$H^A_D = \sum_{\{B_j\}} C^A_{DB_1B_2....B_N} \prod_{B_i} H^{B_i}_c \quad , \tag{4.8}$$

where summation over all index combinations $\{B_i\}$ is understood. The coefficients $C^A_{DB_1B_2....B_N}$ are Glebch-Gordan coefficients for completely symmetric N: th power $8 \otimes 8... \otimes 8$ of octet representations. The representation is not unique since $\sum_A H^A_c H^A_c = 1$ holds true. One can however find for each representation D some minimum value of N.

The product of Hamiltonians $H^{D_1}_A$ and $H^B_{D_2}$ can be decomposed by Glebch-Gordan coefficients of the symmetrized representation $(D_1 \otimes D_2)_S$ as

$$H^A_{D_1} H^B_{D_2} = C^{ABD}_{D_1D_2DC}(S) H^C_D \quad , \tag{4.9}$$

where $'S'$ indicates that the symmetrized representation is in question. In the similar manner one can decompose the Poisson bracket of two Hamiltonians

$$\{H^A_{D_1}, H^B_{D_2}\} = C^{ABD}_{D_1D_2DC}(A) H^C_D \quad . \tag{4.10}$$

Here $'A'$ indicates that Glebch-Gordan coefficients for the anti-symmetrized tensor product of the representations D_1 and D_2 are in question.

One can express the infinitesimal generators of CP_2 symplectic transformations in terms of the color isometry generators J^B_c using the expansion of the Hamiltonian in terms of the monomials of color Hamiltonians:

$$\begin{aligned} j^A_{DN} &= F^A_{DB} J^B_c \quad , \\ F^A_{DB} &= N \sum_{\{B_j\}} C^A_{DB_1B_2...B_{N-1}B} \prod_j H^{B_j}_c \quad , \end{aligned} \tag{4.11}$$

where summation over all possible $\{B_j\}$: s appears. Therefore, the interpretation as a color group localized with respect to CP_2 coordinates is valid in the same sense as the interpretation of space-time diffeomorphism group as local Poincare group. Thus one can say that TGD color is localized with respect to the entire $\delta M^4_+ \times CP_2$.

A convenient basis for the Hamiltonians of $\delta M^4_+ \times CP_2$ is given by the functions

$$H^{mA}_{jkD} = H^m_{jk} H^A_D \quad .$$

The symplectic transformation generated by H^{mA}_{jkD} acts both in M^4 and CP_2 degrees of freedom and the corresponding vector field is given by

$$J^r = H^A_D J^{rl}(\delta M^4_+)\partial_l H^m_{jk} + H^m_{jk} J^{rl}(CP_2)\partial_l H^A_D \ . \tag{4.12}$$

The general form for their Poisson bracket is:

$$\begin{aligned} \{H^{m_1A_1}_{j_1k_1D_1}, H^{m_2A_2}_{j_2k_2D_2}\} &= H^{A_1}_{D_1}H^{A_2}_{D_2}\{H^{m_1}_{j_1k_1}, H^{m_2}_{j_2k_2}\} + H^{m_1}_{j_1k_1}H^{m_2}_{j_2k_2}\{H^{A_1}_{D_1}, H^{A_2}_{D_2}\} \\ &= \left[C^{A_1A_2A}_{D_1D_2D}(S)C(j_1m_1j_2m_2|jm)_A + C^{A_1A_2A}_{D_1D_2D}(A)C(j_1m_1j_2m_2|jm)_S\right] H^{mA}_{j,k_1+k_2,D} \ . \end{aligned} \tag{4.13}$$

What is essential that radial "momenta" and angular momentum are additive in δM^4_+ degrees of freedom and color quantum numbers are additive in CP_2 degrees of freedom.

4.6 Representation Of Lorentz Group And Conformal Symmetries At Light Cone Boundary

A guess deserving testing is that the representations of the Lorentz group at light cone boundary might provide natural building blocks for the construction of the WCW Hamiltonians. In the following the explicit representation of the Lorentz algebra at light cone boundary is deduced, and a function basis giving rise to the representations of Lorentz group and having very simple properties under modified Poisson bracket of δM^4_+ is constructed.

4.6.1 Explicit representation of Lorentz algebra

It is useful to write the explicit expressions of Lorentz generators using complex coordinates for S^2. The expression for the $SU(2)$ generators of the Lorentz group are

$$\begin{aligned} J_x &= (z^2-1)d/dz + c.c. = L_1 - L_{-1} + c.c. \ , \\ J_y &= (iz^2+1)d/dz + c.c. = iL_1 + iL_{-1} + c.c. \ , \\ J_z &= iz\frac{d}{dz} + c.c. = iL_z + c.c. \ . \end{aligned} \tag{4.14}$$

The expressions for the generators of Lorentz boosts can be derived easily. The boost in m^3 direction corresponds to an infinitesimal transformation

$$\begin{aligned} \delta m^3 &= -\varepsilon r_M \ , \\ \delta r_M &= -\varepsilon m^3 = -\varepsilon\sqrt{r_M^2 - (m^1)^2 - (m^2)^2} \ . \end{aligned} \tag{4.15}$$

The relationship between complex coordinates of S^2 and M^4 coordinates m^k is given by stereographic projection

$$\begin{aligned} z &= \frac{(m^1+im^2)}{(r_M - \sqrt{r_M^2-(m^1)^2-(m^2)^2})} \\ &= \frac{sin(\theta)(cos\phi + isin\phi)}{(1-cos\theta)} \ , \\ cot(\theta/2) &= \rho = \sqrt{z\bar{z}} \ , \\ tan(\phi) &= \frac{m^2}{m^1} \ . \end{aligned} \tag{4.16}$$

This implies that the change in z coordinate doesn't depend at all on r_M and is of the following form

$$\delta z \quad = \quad -\frac{\varepsilon}{2}(1+\frac{z(z+\bar{z})}{2})(1+z\bar{z}) \ . \tag{4.17}$$

The infinitesimal generator for the boosts in z-direction is therefore of the following form

$$L_z \quad = \quad [\frac{2z\bar{z}}{(1+z\bar{z})}-1]r_M\frac{\partial}{\partial_{r_M}}-iJ_z \ . \tag{4.18}$$

Generators of L_x and L_y are most conveniently obtained as commutators of $[L_z, J_y]$ and $[L_z, J_x]$. For L_y one obtains the following expression:

$$L_y \quad = \quad 2\frac{(z\bar{z}(z+\bar{z})+i(z-\bar{z}))}{(1+z\bar{z})^2}r_M\frac{\partial}{\partial_{r_M}}-iJ_y \ , \tag{4.19}$$

For L_x one obtains analogous expressions. All Lorentz boosts are of the form $L_i = -iJ_i + \textit{local radial scaling}$ and of zeroth degree in radial variable so that their action on the general generator $X^{klm} \propto z^k\bar{z}^l r_M^m$ doesn't change the value of the label m being a mere local scaling transformation in radial direction. If radial scalings correspond to zero norm isometries this representation is metrically equivalent with the representations of Lorentz boosts as Möbius transformations.

4.6.2 Representations of the Lorentz group reduced with respect to $SO(3)$

The ordinary harmonics of S^2 define in a natural manner infinite series of representation functions transformed to each other in Lorentz transformations. The inner product defined by the integration measure $r_M^2 d\Omega dr_M/r_M$ remains invariant under Lorentz boosts since the scaling of r_M induced by the Lorentz boost compensates for the conformal scaling of $d\Omega$ induced by a Lorentz transformation represented as a Möbius transformation. Thus unitary representations of Lorentz group are in question.

The unitary main series representations of the Lorentz group are characterized by half-integer m and imaginary number $k_2 = i\rho$, where ρ is any real number [A5]. A natural guess is that $m = 0$ holds true for all representations realizable at the light cone boundary and that radial waves are of form r_M^k, $k = k_1 + ik_2 = -1 + i\rho$ and thus eigen states of the radial scaling so that the action of Lorentz boosts is simple in the angular momentum basis. The inner product in radial degrees of freedom reduces to that for ordinary plane waves when $log(r_M)$ is taken as a new integration variable. The complexification is well-defined for non-vanishing values of ρ.

It is also possible to have non-unitary representations of the Lorentz group and the realization of the symmetric space structure suggests that one must have $k = k_1 + ik_2$, k_1 half-integer. For these representations unitarity fails because the inner product in the radial degrees of freedom is non-unitary. A possible physical interpretation consistent with the general ideas about conformal invariance is that the representations $k = -1 + i\rho$ correspond to the unitary ground state representations and $k = -1 + n/2 + i\rho$, $n = \pm 1, \pm 2, ...$, to non-unitary representations. The general view about conformal invariance suggests that physical states constructed as tensor products satisfy the condition $\sum_i n_i = 0$ completely analogous to Virasoro conditions.

4.6.3 Representations of the Lorentz group with $E^2 \times SO(2)$ as isotropy group

One can construct representations of Lorentz group and conformal symmetries at the light cone boundary. Since $SL(2, C)$ is the group generated by the generators L_0 and $L_\pm$ of the conformal algebra, it is clear that infinite-dimensional representations of Lorentz group can be also regarded as representations of the conformal algebra. One can require that the basis corresponds to eigen functions of the rotation generator J_z and corresponding boost generator L_z. For functions which do not depend on r_M these generators are completely analogous to the generators L_0 generating scalings and iL_0 generating rotations. Also the generator of radial scalings appears in the formulas and one must consider the possibility that it corresponds to the generator L_0.

In order to construct scalar function eigen basis of L_z and J_z, one can start from the expressions

$$
\begin{aligned}
L_3 &\equiv i(L_z + L_{\bar{z}}) = 2i[\frac{2z\bar{z}}{(1+z\bar{z})} - 1]r_M \frac{\partial}{\partial_{r_M}} + i\rho\partial_\rho \quad , \\
J_3 &\equiv iL_z - iL_{\bar{z}} = i\partial_\phi \quad . \qquad (4.20)
\end{aligned}
$$

If the eigen functions do not depend on r_M, one obtains the usual basis z^n of Virasoro algebra, which however is not normalizable basis. The eigenfunctions of the generators L_3, J_3 and $L_0 = ir_M d/dr_M$ satisfying

$$
\begin{aligned}
J_3 f_{m,n,k} &= m f_{m,n,k} \quad , \\
L_3 f_{m,n,k} &= n f_{m,n,k} \quad , \\
L_0 f_{m,n,k} &= k f_{m,n,k} \quad . \qquad (4.21)
\end{aligned}
$$

are given by

$$
f_{m,n,k} = e^{im\phi} \frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \quad . \qquad (4.22)
$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$
n_1 - k_1 \geq 0
$$

is required by regularity at the origin of S^2 The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{(1+\rho^2)^2} d\rho d\phi$) implies

$$
n_1 < 3k_1 + 2 \quad .
$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n-k)$. This suggests that the condition

$$
|m| \leq 2(n-k) \qquad (4.23)
$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3,1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra.

The representations of the Lorentz group are characterized by one half-integer valued and one complex parameter. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

The nice feature of the function basis is that various quantum numbers are additive under multiplication:

$$
f(m_a, n_a, k_a) \times f(m_b, n_b, k_b) = f(m_a + m_b, n_a + n_b, k_a + k_b) \quad .
$$

These properties allow to cast the Poisson brackets of the symplectic algebra of WCW into an elegant form.

The Poisson brackets for the δM^4_+ Hamiltonians defined by f_{mnk} can be written using the expression $J^{\rho\phi} = (1+\rho^2)/\rho$ as

$$
\begin{aligned}
\{f_{m_a,n_a,k_a}, f_{m_b,n_b,k_b}\} &= i\left[(n_a - k_a)m_b - (n_b - k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-2,k_a+k_b} \\
&+ 2i\left[(2-k_a)m_b - (2-k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-1,k_a+k_b-1} \quad . \qquad (4.24)
\end{aligned}
$$

4.6.4 Can one find unitary light-like representations of Lorentz group?

It is interesting to compare the representations in question to the unitary representations Gelfand.

1. The unitary representations discussed in [A5] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labeled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labeled by three different quantum numbers.

2. The representations of [A5] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labeling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ formal unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves for $k_1 = -1$. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteeing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity is possible only for the function basis consisting of spherical harmonics.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, WCW spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Hermitian form for light cone Hamiltonians involves also the integration over S^2 degrees of freedom and the non-unitarity of the inner product reflects itself as non-orthogonality of the eigen function basis. Introducing the variable $u = \rho^2 + 1$ as a new integration variable, one can express the inner product in the form

$$\begin{aligned} \langle m_a, n_a, k_a | m_b, n_b, k_b \rangle &= \pi\delta(k_{2a} - k_{2b}) \times \delta_{m_1, m_2} \times I \ , \\ I &= \int_1^\infty f(u) du \ , \\ f(u) &= \frac{(u-1)^{\frac{(N-K)+i\Delta}{2}}}{u^{K+2}} \ . \end{aligned} \quad (4.25)$$

The integrand has cut from $u = 1$ to infinity along real axis. The first thing to observe is that for $N = K$ the exponent of the integral reduces to very simple form and integral exists only for $K = k_{1a} + k_{1b} > -1$. For $k_{1i} = -1/2$ the integral diverges.

The discontinuity of the integrand due to the cut at the real axis is proportional to the integrand and given by

$$\begin{aligned} f(u) - f(e^{i2\pi}u) &= \left[1 - e^{-\pi\Delta}\right] f(u) \ , \\ \Delta &= n_{1a} - k_{1a} - n_{1b} + k_{1b} \ . \end{aligned} \quad (4.26)$$

This means that one can transform the integral to an integral around the cut. This integral can in turn completed to an integral over closed loop by adding the circle at infinity to the integration path. The integrand has $K+1$-fold pole at $u = 0$.

Under these conditions one obtains

$$\begin{aligned} I &= \frac{2\pi i}{1 - e^{-\pi\Delta}} \times R \times (R-1) .... \times (R-K-1) \times (-1)^{\frac{N-K}{2} - K - 1} \ , \\ R &\equiv \frac{N-K}{2} + i\Delta \ . \end{aligned} \quad (4.27)$$

This expression is non-vanishing for $\Delta \neq 0$. Thus it is not possible to satisfy orthogonality conditions without the un-physical $n = k, k_1 = 1/2$ constraint. The result is finite for $K > -1$ so that $k_1 > -1/2$ must be satisfied and if one allows only half-integers in the spectrum, one must have $k_1 \geq 0$, which is very natural if real conformal weights which are half integers are allowed.

4.7 How The Complex Eigenvalues Of The Radial Scaling OperatorRelate To Symplectic Conformal Weights?

Complexified Hamiltonians can be chosen to be eigenmodes of the radial scaling operator $r_M d/dr_M$, and the first guess was that the correct interpretation is as conformal weights. The problem is however that the eigenvalues are complex. Second problem is that general arguments are not enough to fix the spectrum of eigenvalues. There should be a direct connection to the dynamics defined by Kähler action and the Kähler-Dirac action defined by it.

The construction of WCW spinor structure in terms of second quantized induced spinor fields [K19] leads to the conclusion that the modes of induced spinor fields must be restricted at surfaces with 2-D CP_2 projection to guarantee vanishing W fields and well-defined em charge for them. In the generic case these surfaces are 2-D string world sheets (or possibly also partonic 2-surfaces) and in the non-generic case can be chosen to be such. The modes are labeled by generalized conformal weights assignable to complex or hypercomplex string coordinate. Conformal weights are expected to be integers from the experience with string models.

It is an open question whether these conformal weights are independent of the symplectic formal weights or not but on can consider also the possibility that they are dependent. Note hovewer that string coordinate is not reducible to the light-like radial coordinate in the generic case and one can imagine situations in which r_M is constant although string coordinate varies. Dependency would be achieved if the Hamiltonians are generalized eigen modes of $D = \gamma^x d/dx$, $x = log(r/r_0)$, satisfying $DH = \lambda\gamma^x H$ and thus of form $exp(\lambda x) = (r/r_0)^\lambda$ with the same spectrum of eigenvalues λ as associated with the Kähler-Dirac operator. That $log(r/r_0)$ naturally corresponds to the coordinate u assignable to the generalized eigen modes of Kähler-Dirac operator supports this interpretation.

The recent view is that the two conformal weights are independent. The conformal weights associated with the modes of Kähler-Dirac operator localized at string world sheets by the condition that the electromagnetic charge is well-defined for the modes (classical induced W field must vanish at string world sheets). The conformal weights of spinor modes would be integer valued as in string models. About super-symplectic conformal weights associated one cannot say this.

This revives the forgotten TGD inspired conjecture that the conformal weights associated with the generators (in the technical sense of the word) of the super-symplectic algebra are given by the negatives of the zeros of Riemann Zeta $h = -1/2 + iy_i$. Note that these conformal weights have negative real part having interpretation in terms of tachyonic ground state needed in p-adic mass calculations [K8]. The spectrum of conformal weights would be of form $h = n/2 + \sum_i n_i y_i$. This would conform with the association of Riemann Zeta to critical systems. From the identification of mass squared as conformal weight, the total conformal weights for the physical states should have vanishing imaginary part be therefore non-negative integers. This would give rise to what might be called conformal confinement.

5 Magnetic And Electric Representations Of WCW Hamiltonians

Symmetry considerations lead to the hypothesis that WCW Hamiltonians are apart from a factor depending on symplectic invariants equal to magnetic flux Hamiltonians. On the other hand, the hypothesis that Kähler function corresponds to a preferred extremal of Kähler action leads to the hypothesis that WCW Hamiltonians corresponds to classical charges associated with the Hamiltonians of the light cone boundary. These charges are very much like electric charges. The requirement that two approaches are equivalent leads to the hypothesis that magnetic and electric Hamiltonians are identical apart from a factor depending on isometry invariants. At the level of CP_2 corresponding duality corresponds to the self-duality of Kähler form stating that the magnetic and electric parts of Kähler form are identical.

5.1 Radial Symplectic Invariants

All $\delta M^4_+ \times CP_2$ symplectic transformations leave invariant the value of the radial coordinate r_M. Therefore the radial coordinate r_M of X^3 regarded as a function of $S^2 \times CP_2$ coordinates serves as height function. The number, type, ordering and values for the extrema for this height function in the interior and boundary components are isometry invariants. These invariants characterize not only the topology but also the size and shape of the 3-surface. The result implies that WCW metric indeed differentiates between 3-surfaces with the size of Planck length and with the size of galaxy. The characterization of these invariants reduces to Morse theory. The extrema correspond to topology changes for the two-dimensional (one-dimensional) $r_M = constant$ section of 3-surface (boundary of 3-surface). The height functions of sphere and torus serve as a good illustrations of the situation. A good example about non-topological extrema is provided by a sphere with two horns.

There are additional symplectic invariants. The "magnetic fluxes" associated with the δM^4_+ symplectic form

$$J_{S^2} = r_M^2 sin(\theta) d\theta \wedge d\phi$$

over any $X^2 \subset X^3$ are symplectic invariants. In particular, the integrals over $r_M = constant$ sections (assuming them to be 2-dimensional) are symplectic invariants. They give simply the solid angle $\Omega(r_M)$ spanned by $r_M = constant$ section and thus $r_M^2 \Omega(r_M)$ characterizes transversal geometric size of the 3-surface. A convenient manner to discretize these invariants is to consider the Fourier components of these invariants in radial logarithmic plane wave basis discussed earlier:

$$\Omega(k) \quad = \quad \int_{r_{min}}^{r_{max}} (r_M/r_{max})^k \Omega(r_M) \frac{dr_M}{r_M} \quad , \quad k = k_1 + ik_2 \quad , \quad perk_1 \geq 0 \quad . \tag{5.1}$$

One must take into account that for each section in which the topology of $r_M = constant$ section remains constant one must associate invariants with separate components of the two-dimensional section. For a given value of r_M, r_M constant section contains several components (to visualize the situation consider torus as an example).

Also the quantities

$$\Omega^+(X^2) = \int_{X^2} |J| \equiv \int |\epsilon^{\alpha\beta} J_{\alpha\beta}| \sqrt{g_2} d^2x$$

are symplectic invariants and provide additional geometric information about 3-surface. These fluxes are non-vanishing also for closed surfaces and give information about the geometry of the boundary components of 3-surface (signed fluxes vanish for boundary components unless they enclose the tip of the light cone).

Since zero norm generators remain invariant under complexification, their contribution to the Kähler metric vanishes. It is not at all obvious whether WCW integration measure in these degrees of freedom exists at all. A localization in zero modes occurring in each quantum jump seems a more plausible and under suitable additional assumption it would have interpretation as a state function reduction. In string model similar situation is encountered; besides the functional integral determined by string action, one has integral over the moduli space.

If the effective 2-dimensionality implied by the strong form of general coordinate invariance discussed in the introduction is accepted, there is no need to integrate over the variable r_M and just the fluxes over the 2-surfaces X^2_i identified as intersections of light like 3-D causal determinants with X^3 contain the data relevant for the construction of the WCW geometry. Also the symplectic invariants associated with these surfaces are enough.

5.2 Kähler Magnetic Invariants

The Kähler magnetic fluxes defined both the normal component of the Kähler magnetic field and by its absolute value

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \tag{5.2}$$

over suitably defined 2-surfaces are invariants under both Lorentz isometries and the symplectic transformations of CP_2 and can be calculated once X^3 is given.

For a closed surface $Q_m(X^2)$ vanishes unless the homology equivalence class of the surface is nontrivial in CP_2 degrees of freedom. In this case the flux is quantized. $Q_M^+(X^2)$ is non-vanishing for closed surfaces, too. Signed magnetic fluxes over non-closed surfaces depend on the boundary of X^2 only:

$$\begin{array}{l} \int_{X^2} J = \int_{\delta X^2} A \ . \\ J = dA \ . \end{array}$$

Un-signed fluxes can be written as sum of similar contributions over the boundaries of regions of X^2 in which the sign of J remains fixed.

$$\begin{aligned} Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\ Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ , \end{aligned} \tag{5.3}$$

There are also symplectic invariants, which are Lorentz covariants and defined as

$$\begin{aligned} Q_m(K,X^2) &= \int_{X^2} f_K J_{CP_2} \ , \\ Q_m^+(K,X^2) &= \int_{X^2} f_K |J_{CP_2}| \ , \\ f_{K\equiv(s,n,k)} &= e^{is\phi}\times\frac{\rho^{n-k}}{(1+\rho^2)^k}\times(\frac{r_M}{r_0})^k \end{aligned} \tag{5.4}$$

These symplectic invariants transform like an infinite-dimensional unitary representation of Lorentz group.

There must exist some minimal number of symplectically non-equivalent 2-surfaces of X^3, and the magnetic fluxes over the representatives these surfaces give thus good candidates for zero modes.

1. If effective 2-dimensionality is accepted, the surfaces X^2_i defined by the intersections of light like 3-D causal determinants X^3_l and X^3 provide a natural identification for these 2-surfaces.

2. Without effective 2-dimensionality the situation is more complex. Since symplectic transformations leave r_M invariant, a natural set of 2-surfaces X^2 appearing in the definition of fluxes are separate pieces for $r_M = constant$ sections of 3-surface. For a generic 3-surface, these surfaces are 2-dimensional and there is continuum of them so that discrete Fourier transforms of these invariants are needed. One must however notice that $r_M = constant$ surfaces could be be 3-dimensional in which case the notion of flux is not well-defined.

5.3 Isometry Invariants And Spin Glass Analogy

The presence of isometry invariants implies coset space decomposition $\cup_i G/H$. This means that quantum states are characterized, not only by the vacuum functional, which is just the exponential $exp(K)$ of Kähler function (Gaussian in lowest approximation) but also by a wave function in vacuum modes. Therefore the functional integral over the WCW decomposes into an integral over zero modes for approximately Gaussian functionals determined by $exp(K)$. The weights for the various vacuum mode contributions are given by the probability density associated with the zero modes. The integration over the zero modes is a highly problematic notion and it could be eliminated if a localization in the zero modes occurs in quantum jumps. The localization would correspond to a state function reduction and zero modes would be effectively classical variables correlated in one-one manner with the quantum numbers associated with the quantum fluctuating degrees of freedom.

For a given orbit K depends on zero modes and thus one has mathematical similarity with spin glass phase for which one has probability distribution for Hamiltonians appearing in the partition function $exp(-H/T)$. In fact, since TGD Universe is also critical, exact similarity requires that also the temperature is critical for various contributions to the average partition function of spin glass phase. The characterization of isometry invariants and zero modes of the Kähler metric provides a precise characterization for how TGD Universe is quantum analog of spin glass.

The spin glass analogy has been the basic starting point in the construction of p-adic field theory limit of TGD. The ultra-metric topology for the free energy minima of spin glass phase motivates the hypothesis that effective quantum average space-time possesses ultra-metric topology. This approach leads to excellent predictions for elementary particle masses and predicts even new branches of physics [K9, K18]. As a matter fact, an entire fractal hierarchy of copies of standard physics is predicted.

5.4 Magnetic Flux Representation Of The Symplectic Algebra

Accepting the strong form of general coordinate invariance implying effective two-dimensionality WCW Hamiltonians correspond to the fluxes associated with various 2-surfaces X_i^2 defined by the intersections of light-like light-like 3-surfaces $X_{l,i}^3$ with X^3 at the boundaries of CD considered. Bearing in mind that zero energy ontology is the correct approach, one can restrict the consideration on fluxes at $\delta M_+^4 \times CP_2$ One must also remember that if the proposed symmetries hold true, it is in principle choose any partonic 2-surface in the conjectured slicing of the Minkowskian space-time sheet to partonic 2-surfaces parametrized by the points of stringy world sheets.vA physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K7] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

5.4.1 Generalized magnetic fluxes

Isometry invariants are just special case of the fluxes defining natural coordinate variables for WCW . Symplectic transformations of CP_2 act as $U(1)$ gauge transformations on the Kähler potential of CP_2 (similar conclusion holds at the level of $\delta M_+^4 \times CP_2$).

One can generalize these transformations to local symplectic transformations by allowing the Hamiltonians to be products of the CP_2 Hamiltonians with the real and imaginary parts of the functions $f_{m,n,k}$ (see Eq. 4.22) defining the Lorentz covariant function basis H_A, $A \equiv (a, m, n, k)$ at the light cone boundary: $H_A = H_a \times f(m, n, k)$, where a labels the Hamiltonians of CP_2.

One can associate to any Hamiltonian H^A of this kind both signed and unsigned magnetic flux via the following formulas:

$$\begin{aligned} Q_m(H_A|X^2) &= \int_{X^2} H_A J \ , \\ Q_m^+(H_A|X^2) &= \int_{X^2} H_A|J| \ . \end{aligned} \tag{5.5}$$

Here X^2 corresponds to any surface X_i^2 resulting as intersection of X^3 with $X_{l,i}^3$. Both signed and unsigned magnetic fluxes and their superpositions

$$Q_m^{\alpha,\beta}(H_A|X^2) = \alpha Q_m(H_A|X^2) + \beta Q_m^+(H_A|X^2) \ , \ A \equiv (a,s,n,k) \tag{5.6}$$

provide representations of Hamiltonians. Note that symplectic invariants $Q_m^{\alpha,\beta}$ correspond to $H^A = 1$ and $H^A = f_{s,n,k}$. $H^A = 1$ can be regarded as a natural central term for the Poisson bracket algebra. Therefore, the isometry invariance of Kähler magnetic and electric gauge fluxes follows as a natural consequence.

The obvious question concerns about the correct values of the parameters α and β. One possibility is that the flux is an unsigned flux so that one has $\alpha = 0$. This option is favored by the construction of the WCW spinor structure involving the construction of the fermionic super charges anti-commuting to WCW Hamiltonians: super charges contain the square root of flux, which must be therefore unsigned. Second possibility is that magnetic fluxes are signed fluxes so that β vanishes.

One can define also the electric counterparts of the flux Hamiltonians by replacing J in the defining formulas with its dual $*J$

$$*J_{\alpha\beta} = \epsilon_{\alpha\beta}^{\ \ \gamma\delta} J_{\gamma\delta}.$$

For $H_A = 1$ these fluxes reduce to ordinary Kähler electric fluxes. These fluxes are however not symplectic covariants since the definition of the dual involves the induced metric, which is not symplectic invariant. The electric gauge fluxes for Hamiltonians in various representations of the color group ought to be important in the description of hadrons, not only as string like objects, but quite generally. These degrees of freedom would be identifiable as non-perturbative degrees of freedom involving genuinely classical Kähler field whereas quarks and gluons would correspond to the perturbative degrees of freedom, that is the interactions between CP_2 type extremals.

5.4.2 Poisson brackets

From the symplectic invariance of the radial component of Kähler magnetic field it follows that the Lie-derivative of the flux $Q_m^{\alpha,\beta}(H_A)$ with respect to the vector field $X(H_B)$ is given by

$$X(H_B) \cdot Q_m^{\alpha,\beta}(H_A) = Q_m^{\alpha,\beta}(\{H_B, H_A\}) \ . \tag{5.7}$$

The transformation properties of $Q_m^{\alpha,\beta}(H_A)$ are very nice if the basis for H_B transforms according to appropriate irreducible representation of color group and rotation group. This in turn implies that the fluxes $Q_m^{\alpha,\beta}(H_A)$ as functionals of 3-surface on given orbit provide a representation for the Hamiltonian as a functional of 3-surface. For a given surface X^3, the Poisson bracket for the two fluxes $Q_m^{\alpha,\beta}(H_A)$ and $Q_m^{\alpha,\beta}(H_B)$ can be defined as

$$\begin{aligned} \{Q_m^{\alpha,\beta}(H_A), Q_m^{\alpha,\beta}(H_B)\} &\equiv X(H_B) \cdot Q_m^{\alpha,\beta}(H_A) \\ &= Q_m^{\alpha,\beta}(\{H_A, H_B\}) = Q_m^{\alpha,\beta}(\{H_A, H_B\}) \ . \end{aligned} \tag{5.8}$$

The study of WCW gamma matrices identifiable as symplectic super charges demonstrates that the supercharges associated with the radial deformations vanish identically so that radial deformations correspond to zero norm degrees of freedom as one might indeed expect on physical grounds. The reason is that super generators involve the invariants $j^{ak}\gamma_k$ which vanish by $\gamma_{r_M} = 0$.

The natural central extension associated with the symplectic group of CP_2 ($\{p,q\} = 1!$) induces a central extension of this algebra. The central extension term resulting from $\{H_A, H_B\}$ when CP_2 Hamiltonians have $\{p,q\} = 1$ equals to the symplectic invariant $Q^{\alpha,\beta}_m(f(m_a+m_b, n_a+n_b, k_a+k_b))$ on the right hand side. This extension is however anti-symmetric in symplectic degrees of freedom rather than in loop space degrees of freedom and therefore does not lead to the standard Kac Moody type algebra.

Quite generally, the Virasoro and Kac Moody algebras of string models are replaced in TGD context by much larger symmetry algebras. Kac Moody algebra corresponds to the deformations of light-like 3-surfaces respecting their light-likeness and leaving partonic 2-surfaces at δCD intact and are highly relevant to the elementary particle physics. This algebra allows a representation in terms of X^3_l local Hamiltonians generating isometries of $\delta M^4_\pm \times CP_2$. Hamiltonian representation is essential for super-symmetrization since fermionic super charges anti-commute to Hamiltonians rather than vector fields: this is one of the deep differences between TGD and string models. Kac-Moody algebra does not contribute to WCW metric since by definition the generators vanish at partonic 2-surfaces. This is essential for the coset space property.

A completely new algebra is the CP_2 symplectic algebra localized with respect to the light cone boundary and relevant to the configuration space geometry. This extends to $S^2 \times CP_2$ -or rather $\delta M^4_\pm \times CP_2$ symplectic algebra and this gives the strongest predictions concerning WCW metric. The local radial Virasoro localized with respect to $S^2 \times CP_2$ acts in zero modes and has automatically vanishing norm with respect to WCW metric defined by super charges.

5.5 Symplectic Transformations Of $\Delta M^4_\pm \times CP_2$ As Isometries And Electric-Magnetic Duality

According to the construction of Kähler metric, symplectic transformations of $\delta M^4_\pm \times CP_2$ act as isometries whereas radial Virasoro algebra localized with respect to CP_2 has zero norm in the WCW metric.

Hamiltonians can be organized into light like unitary representations of $so(3,1) \times su(3)$ and the symmetry condition $Zg(X,Y) = 0$ requires that the component of the metric is $so(3,1) \times su(3)$ invariant and this condition is satisfied if the component of metric between two different representations D_1 and D_2 of $so(3,1)\times su(3)$ is proportional to Glebch-Gordan coefficient $C_{D_1 D_2, D_S}$ between $D_1 \otimes D_2$ and singlet representation D_S. In particular, metric has components only between states having identical $so(3,1) \times su(3)$ quantum numbers.

Magnetic representation of WCW Hamiltonians means the action of the symplectic transformations of the light cone boundary as WCW isometries is an intrinsic property of the light cone boundary. If electric-magnetic duality holds true, the preferred extremal property only determines the conformal factor of the metric depending on zero modes. This is precisely as it should be if the group theoretical construction works. Hence it should be possible by a direct calculation check whether the metric defined by the magnetic flux Hamiltonians as half Poisson brackets in complex coordinates is invariant under isometries. Symplectic invariance of the metric means that matrix elements of the metric are left translates of the metric along geodesic lines starting from the origin of coordinates, which now naturally corresponds to the preferred extremal of Kähler action. Since metric derives from symplectic form this means that the matrix elements of symplectic form given by Poisson brackets of Hamiltonians must be left translates of their values at origin along geodesic line. The matrix elements in question are given by flux Hamiltonians and since symplectic transforms of flux Hamiltonian is flux Hamiltonian for the symplectic transform of Hamiltonian, it seems that the conditions are satisfied.

5.6 Quantum Counterparts Of The Symplectic Hamiltonians

The matrix elements of WCW Kähler metric can be expressed in terms of anti-commutators of WCW gamma matrices identified as super-symplectic super-charges, which might be called super-Hamiltonians. It is these operators which are the most relevant from the point of view of quantum TGD.

The generalization for the definition WCW super-Hamiltonians defining WCW gamma matrices is discussed in detail in [K14] feeds in the wisdom gained about preferred extremals of Kähler action and solutions of the Kähler-Dirac action: in particular, about their localization at string worlds sheets (right handed neutrino could be an exception). Second quantized Noether charges in turn define representation of WCW Hamiltonians as operators.

The basic formulas generalize as such: the only modification is that the super-Hamiltonian of $\delta M^4_\pm \times CP_2$ at given point of partonic 2-surface is replaced with the Noether super charge associated with the Hamiltonian obtained by integrating the 1-D super current over string emanating from partonic 2-surface. Right handed neutrino spinor is replaced with any mode of the Kähler-Dirac operator localized at string world sheet in the case of Kac-Moody sub-algebra of super-symplectic algebra corresponding to symplectic isometries at light-cone boundary and CP_2. The original proposal involved only the contractions with covariantly constant right- handed neutrino spinor mode but now one can allow contractions with all spinor modes - both quark like and leptonic ones. One obtains entire super-symplectic algebra and the direct sum of these algebras is used to construct physical states. This step is analogous to the replacement of point like particle with string.

The resulting super Hamiltonians define WCW gamma matrices. They are labelled by two conformal weights. The first one is the conformal weight associated with the light-like coordinate of $\delta M^4_\pm \times CP_2$. Second conformal weight is associated with the spinor mode and the coordinate along stringy curve and corresponds to the usual stringy conformal weight. The symplectic conformal weight can be more general - I have proposed its spectrum to be generated by the zeros of Riemann zeta. The total conformal weight of a physical state would be non-negative real integer meaning conformal confinement. Symplectic conformal symmetry can be assumed to be broken: an entire hierarchy of breakings is obtained corresponding to hierarchies of sub-algebra of the symplectic algebra isomorphic with it quantum criticalities, Planck constants, and dark matter. Breaking means that only the sub-algebra of super-symplectic algebra isomorphic to it corresponds vanishing elements of the WCW metric: in Hilbert space picture these gauge degrees of freedom correspond to zero norm states.

The presence of two conformal weights is in accordance with the idea that a generalization of conformal invariance to 4-D situation is in question. If Yangian extension of conformal symmetries is possible and would bring an additional integer n telling the degree of multi-locality of Yangian generators defined as the number of strings at which the generator acts (the original not proposal was as the number of partonic 2-surfaces). For super-symplectic algebra the degree of multi-locality equals to $n = 1$. Measurement resolution increases with n. This is also visible in the properties of space-time surfaces since string world sheets and possibly also partonic 2-surfaces and their light-like orbits provide the holographic data - kind of skeleton - determining space-time surface associated with them.

6 General Expressions For The Symplectic And Kähler Forms

One can derive general expressions for symplectic and Kähler forms as well as Kähler metric of WCW . The fact that these expressions involve only first variation of the Kähler action implies huge simplification of the basic formulas. Duality hypothesis leads to further simplifications of the formulas.

6.1 Closedness Requirement

The fluxes of Kähler magnetic and electric fields for the Hamiltonians of $\delta M^4_+ \times CP_2$ suggest a general representation for the components of the symplectic form of the WCW . The basic requirement is that Kähler form satisfies the defining condition

$$X \cdot J(Y, Z) + J([X, Y], Z) + J(X, [Y, Z]) = 0 , \tag{6.1}$$

where X, Y, Z are now vector fields associated with Hamiltonian functions defining WCW coordinates.

6.2 Matrix Elements Of The Symplectic Form As Poisson Brackets

Quite generally, the matrix element of $J(X(H_A), X(H_B))$ between vector fields $X(H_A))$ and $X(H_B))$ defined by the Hamiltonians H_A and H_B of $\delta M^4_+ \times CP_2$ isometries is expressible as

Poisson bracket

$$J^{AB} = J(X(H_A), X(H_B)) = \{H_A, H_B\} \ . \tag{6.2}$$

J^{AB} denotes contravariant components of the symplectic form in coordinates given by a subset of Hamiltonians. The magnetic flux Hamiltonians $Q_m^{\alpha,\beta}(H_{A,k})$ of Eq. 5.5 provide an explicit representation for the Hamiltonians at the level of WCW so that the components of the symplectic form of the WCW are expressible as classical charges for the Poisson brackets of the Hamiltonians of the light cone boundary:

$$J(X(H_A), X(H_B)) = Q_m^{\alpha,\beta}(\{H_A, H_B\}) \ . \tag{6.3}$$

Recall that the superscript α, β refers the coefficients of J and $|J|$ in the superposition of these Kähler magnetic fluxes. Note that $Q_m^{\alpha,\beta}$ contains unspecified conformal factor depending on symplectic invariants characterizing Y^3 and is unspecified superposition of signed and unsigned magnetic fluxes.

This representation does not carry information about the tangent space of space-time surface at the partonic 2-surface, which motivates the proposal that also electric fluxes are present and proportional to magnetic fluxes with a factor K, which is symplectic invariant so that commutators of flux Hamiltonians come out correctly. This would give

$$Q_m^{\alpha,\beta}(H_A)_{em} = Q_e^{\alpha,\beta}(H_A) + Q_m^{\alpha,\beta}(H_A) = (1+K)Q_m^{\alpha,\beta}(H_A) \ . \tag{6.4}$$

Since Kähler form relates to the standard field tensor by a factor $e/\hbar$, flux Hamiltonians are dimensionless so that commutators do not involve $\hbar$. The commutators would come as

$$Q_{em}^{\alpha,\beta}(\{H_A, H_B\}) \rightarrow (1+K)Q_m^{\alpha,\beta}(\{H_A, H_B\}) \ . \tag{6.5}$$

The factor $1+K$ plays the same role as Planck constant in the commutators.

WCW Hamiltonians vanish for the extrema of the Kähler function as variational derivatives of the Kähler action. Hence Hamiltonians are good candidates for the coordinates appearing as coordinates in the perturbative functional integral around extrema (with maxima giving dominating contribution). It is clear that WCW coordinates around a given extremum include only those Hamiltonians, which vanish at extremum (that is those Hamiltonians which span the tangent space of G/H) In Darboux coordinates the Poisson brackets reduce to the symplectic form

$$\begin{aligned} \{P^I, Q^J\} &= J^{IJ} = J_I \delta^{I,J} \ . \\ J_I &= 1 \ . \end{aligned} \tag{6.6}$$

It is not clear whether Darboux coordinates with $J_I = 1$ are possible in the recent case: probably the unit matrix on right hand side of the defining equation is replaced with a diagonal matrix depending on symplectic invariants so that one has $J_I \neq 1$. The integration measure is given by the symplectic volume element given by the determinant of the matrix defined by the Poisson brackets of the Hamiltonians appearing as coordinates. The value of the symplectic volume element is given by the matrix formed by the Poisson brackets of the Hamiltonians and reduces to the product

$$Vol = \prod_I J_I$$

in generalized Darboux coordinates.

Kähler potential (that is gauge potential associated with Kähler form) can be written in Darboux coordinates as

$$A = \sum_I J_I P_I dQ^I \ . \tag{6.7}$$

6.3 General Expressions For Kähler Form, Kähler Metric And Kähler Function

The expressions of Kähler form and Kähler metric in complex coordinates can obtained by transforming the contravariant form of the symplectic form from symplectic coordinates provided by Hamiltonians to complex coordinates:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \partial_{H^A} Z^i \partial_{H^B} \bar{Z}^j J^{AB} \quad , \tag{6.8}$$

where J^{AB} is given by the classical Kahler charge for the light cone Hamiltonian $\{H^A, H^B\}$. Complex coordinates correspond to linear coordinates of the complexified Lie-algebra providing exponentiation of the isometry algebra via exponential mapping. What one must know is the precise relationship between allowed complex coordinates and Hamiltonian coordinates: this relationship is in principle calculable. In Darboux coordinates the expressions become even simpler:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \sum_I J(I)(\partial_{P^i} Z^i \partial_{Q^I} \bar{Z}^j - \partial_{Q^I} Z^i \partial_{P^I} \bar{Z}^j) \quad . \tag{6.9}$$

Kähler function can be formally integrated from the relationship

$$\begin{aligned} A_{Z^i} &= i\partial_{Z^i} K \quad , \\ A_{\bar{Z}^i} &= -i\partial_{Z^i} K \quad . \end{aligned} \tag{6.10}$$

holding true in complex coordinates. Kähler function is obtained formally as integral

$$K \quad = \quad \int_0^Z (A_{Z^i} dZ^i - A_{\bar{Z}^i} d\bar{Z}^i) \quad . \tag{6.11}$$

6.4 $Diff(X^3)$ Invariance And Degeneracy And Conformal Invariances Of The Symplectic Form

$J(X(H_A), X(H_B))$ defines symplectic form for the coset space G/H only if it is $Diff(X^3)$ degenerate. This means that the symplectic form $J(X(H_A), X(H_B))$ vanishes whenever Hamiltonian H_A or H_B is such that it generates diffeomorphism of the 3-surface X^3. If effective 2-dimensionality holds true, $J(X(H_A), X(H_B))$ vanishes if H_A or H_B generates two-dimensional diffeomorphism $d(H_A)$ at the surface X_i^2.

One can always write

$$J(X(H_A), X(H_B)) = X(H_A)Q(H_B|X_i^2) \quad .$$

If H_A generates diffeomorphism, the action of $X(H_A)$ reduces to the action of the vector field X_A of some X_i^2-diffeomorphism. Since $Q(H_B|r_M)$ is manifestly invariant under the diffemorphisms of X^2, the result is vanishing:

$$X_A Q(H_B|X_i^2) = 0 \quad ,$$

so that $Diff^2$ invariance is achieved.

The radial diffeomorphisms possibly generated by the radial Virasoro algebra do not produce trouble. The change of the flux integrand X under the infinitesimal transformation $r_M \to r_M + \epsilon r_M^n$ is given by $r_M^n dX/dr_M$. Replacing r_M with $r_M^{-n+1}/(-n+1)$ as variable, the integrand reduces to a total divergence dX/du the integral of which vanishes over the closed 2-surface X_i^2. Hence radial Virasoro generators having zero norm annihilate all matrix elements of the symplectic form. The induced metric of X_i^2 induces a unique conformal structure and since the conformal transformations of X_i^2 can be interpreted as a mere coordinate changes, they leave the flux integrals invariant.

6.5 Complexification And Explicit Form Of The Metric And Kähler Form

The identification of the Kähler form and Kähler metric in symplectic degrees of freedom follows trivially from the identification of the symplectic form and definition of complexification. The requirement that Hamiltonians are eigen states of angular momentum (and possibly Lorentz boost generator), isospin and hypercharge implies physically natural complexification. In order to fix the complexification completely one must introduce some convention fixing which states correspond to "positive" frequencies and which to "negative frequencies" and which to zero frequencies that is to decompose the generators of the symplectic algebra to three sets Can_+, Can_- and Can_0. One must distinguish between Can_0 and zero modes, which are not considered here at all. For instance, CP_2 Hamiltonians correspond to zero modes.

The natural complexification relies on the imaginary part of the radial conformal weight whereas the real part defines the $g = t + h$ decomposition naturally. The wave vector associated with the radial logarithmic plane wave corresponds to the angular momentum quantum number associated with a wave in S^1 in the case of Kac Moody algebra. One can imagine three options.

1. It is quite possible that the spectrum of k_2 does not contain $k_2 = 0$ at all so that the sector Can_0 could be empty. This complexification is physically very natural since it is manifestly invariant under $SU(3)$ and $SO(3)$ defining the preferred spherical coordinates. The choice of $SO(3)$ is unique if the classical four-momentum associated with the 3-surface is time like so that there are no problems with Lorentz invariance.

2. If $k_2 = 0$ is possible one could have

$$\begin{aligned} Can_+ &= \{H^a_{m,n,k=k1_+ik_2}, k_2 > 0\} \ , \\ Can_- &= \{H^a_{m,n,k}, k_2 < 0\} \ , \\ Can_0 &= \{H^a_{m,n,k}, k_2 = 0\} \ . \end{aligned} \tag{6.12}$$

3. If it is possible to $n_2 \neq 0$ for $k_2 = 0$, one could define the decomposition as

$$\begin{aligned} Can_+ &= \{H^a_{m,n,k}, k_2 > 0 \ \text{ or } \ k_2 = 0, n_2 > 0\} \ , \\ Can_- &= \{H^a_{m,n,k}, k_2 < 0 \ \text{ or} k_2 = 0, n_2 < 0\} \ , \\ Can_0 &= \{H^a_{m,n,k}, k_2 = n_2 = 0\} \ . \end{aligned} \tag{6.13}$$

   In this case the complexification is unique and Lorentz invariance guaranteed if one can fix the $SO(2)$ subgroup uniquely. The quantization axis of angular momentum could be chosen to be the direction of the classical angular momentum associated with the 3-surface in its rest system.

The only thing needed to get Kähler form and Kähler metric is to write the half Poisson bracket defined by Eq. 6.15

$$\begin{aligned} J_f(X(H_A), X(H_B)) &= 2Im\left(iQ_f(\{H_A, H_B\}_{-+})\right) \ , \\ G_f(X(H_A), X(H_B)) &= 2Re\left(iQ_f(\{H_A, H_B\}_{-+})\right) \ . \end{aligned} \tag{6.14}$$

Symplectic form, and thus also Kähler form and Kähler metric, could contain a conformal factor depending on the isometry invariants characterizing the size and shape of the 3-surface. At this stage one cannot say much about the functional form of this factor.

6.6 Comparison Of CP_2 Kähler Geometry With Configuration Space Geometry

The explicit discussion of the role of $g = t + h$ decomposition of the tangent space of WCW provides deep insights to the metric of the symmetric space. There are indeed many questions to be answered. To what point of WCW (that is 3-surface) the proposed $g = t + h$ decomposition corresponds to? Can one derive the components of the metric and Kähler form from the Poisson brackets of complexified Hamiltonians? Can one characterize the point in question in terms of the properties of WCW Hamiltonians? Does the central extension of WCW reduce to the symplectic central extension of the symplectic algebra or can one consider also other options?

6.6.1 Cartan decomposition for CP_2

A good manner to gain understanding is to consider the CP_2 metric and Kähler form at the origin of complex coordinates for which the sub-algebra $h = u(2)$ defines the Cartan decomposition.

1. $g = t + h$ decomposition depends on the point of the symmetric space in general. In case of CP_2 $u(2)$ sub-algebra transforms as $g \circ u(2) \circ g^{-1}$ when the point s is replaced by gsg^{-1}. This is expected to hold true also in case of WCW (unless it is flat) so that the task is to identify the point of WCW at which the proposed decomposition holds true.

2. The Killing vector fields of h sub-algebra vanish at the origin of CP_2 in complex coordinates. The corresponding Hamiltonians need not vanish but their Poisson brackets must vanish. It is possible to add suitable constants to the Hamiltonians in order to guarantee that they vanish at origin.

3. It is convenient to introduce complex coordinates and decompose isometry generators to holomorphic components $J^a_+ = j^{ak}\partial_k$ and $j^a_- = j^{a\bar{k}}\partial_{\bar{k}}$. One can introduce what might be called half Poisson bracket and half inner product defined as

$$\begin{aligned} \{H^a, H^b\}_{-+} &\equiv \partial_{\bar{k}} H^a J^{\bar{k}l} \partial_l H^b \\ &= j^{ak} J_{k\bar{l}} j^{b\bar{l}} = -i(j^a_+, j^b_-) \ . \end{aligned} \tag{6.15}$$

   One can express Poisson bracket of Hamiltonians and the inner product of the corresponding Killing vector fields in terms of real and imaginary parts of the half Poisson bracket:

$$\begin{aligned} \{H^a, H^b\} &= 2Im\left(i\{H^a, H^b\}_{-+}\right) \ , \\ (j^a, j^b) &= 2Re\left(i(j^a_+, j^b_-)\right) = 2Re\left(i\{H^a, H^b\}_{-+}\right) \ . \end{aligned} \tag{6.16}$$

   What this means that Hamiltonians and their half brackets code all information about metric and Kähler form. Obviously this is of utmost importance in the case of the WCW metric whose symplectic structure and central extension are derived from those of CP_2.

Consider now the properties of the metric and Kähler form at the origin.

1. The relations satisfied by the half Poisson brackets can be written symbolically as

$$\begin{aligned} & \{h, h\}_{-+} = 0 \ , \\ & Re\left(i\{h, t\}_{-+}\right) = 0 \ , \quad Im\left(i\{h, t\}_{-+}\right) = 0 \ , \\ & Re\left(i\{t, t\}_{-+}\right) \neq 0 \ , \quad Im\left(i\{t, t\}_{-+}\right) \neq 0 \ . \end{aligned} \tag{6.17}$$

2. The first two conditions state that h vector fields have vanishing inner products at the origin. The first condition states also that the Hamiltonians for the commutator algebra $[h,h] = SU(2)$ vanish at origin whereas the Hamiltonian for $U(1)$ algebra corresponding to the color hyper charge need not vanish although it can be made vanishing. The third condition implies that the Hamiltonians of t vanish at origin.

3. The last two conditions state that the Kähler metric and form are non-vanishing between the elements of t. Since the Poisson brackets of t Hamiltonians are Hamiltonians of h, the only possibility is that $\{t,t\}$ Poisson brackets reduce to a non-vanishing $U(1)$ Hamiltonian at the origin or that the bracket at the origin is due to the symplectic central extension. The requirement that all Hamiltonians vanish at origin is very attractive aesthetically and forces to interpret $\{t,t\}$ brackets at origin as being due to a symplectic central extension. For instance, for S^2 the requirement that Hamiltonians vanish at origin would mean the replacement of the Hamiltonian $H = cos(\theta)$ representing a rotation around z-axis with $H_3 = cos(\theta) - 1$ so that the Poisson bracket of the generators H_1 and H_2 can be interpreted as a central extension term.

4. The conditions for the Hamiltonians of $u(2)$ sub-algebra state that their variations with respect to g vanish at origin. Thus $u(2)$ Hamiltonians have extremum value at origin.

5. Also the Kähler function of CP_2 has extremum at the origin. This suggests that in the case of the WCW the counterpart of the origin corresponds to the maximum of the Kähler function.

6.6.2 Cartan algebra decomposition at the level of WCW

The discussion of the properties of CP_2 Kähler metric at origin provides valuable guide lines in an attempt to understand what happens at the level of WCW . The use of the half bracket for WCW Hamiltonians in turn allows to calculate the matrix elements of the WCW metric and Kähler form explicitly in terms of the magnetic or electric flux Hamiltonians.

The earlier construction was rather tricky and formula-rich and not very convincing physically. Cartan decomposition had to be assigned with something and in lack of anything better it was assigned with Super Virasoro algebra, which indeed allows this kind of decompositions but without any strong physical justification.

It must be however emphasized that holography implying effective 2-dimensionality of 3-surfaces in some length scale resolution is absolutely essential for this construction since it allows to effectively reduce Kac-Moody generators associated with X_l^3 to $X^2 = X_l^3 \cap \delta M_\pm^4 \times CP_2$. In the similar manner super-symplectic generators can be dimensionally reduced to X^2. Number theoretical compactification forces the dimensional reduction and the known extremals are consistent with it [K1]. The construction of WCW spinor structure and metric in terms of the second quantized spinor fields [K19] relies to this picture as also the recent view about M-matrix [K3].

In this framework the coset space decomposition becomes trivial.

1. The algebra g is labeled by color quantum numbers of CP_2 Hamiltonians and by the label (m,n,k) labeling the function basis of the light cone boundary. Also a localization with respect to X^2 is needed. This is a new element as compared to the original view.

2. Super Kac-Moody algebra is labeled by color octet Hamiltonians and function basis of X^2. Since Lie-algebra action does not lead out of irreps, this means that Cartan algebra decomposition is satisfied.

6.7 Comparison With Loop Groups

It is useful to compare the recent approach to the geometrization of the loop groups consisting of maps from circle to Lie group G [A2], which served as the inspirer of the WCW geometry approach but later turned out to not apply as such in TGD framework.

In the case of loop groups the tangent space T corresponds to the local Lie-algebra $T(k,A) = exp(ik\phi)T_A$, where T_A generates the finite-dimensional Lie-algebra g and ϕ denotes the angle variable of circle; k is integer. The complexification of the tangent space corresponds to the decomposition

$$T = \{X(k > 0, A)\} \oplus \{X(k < 0, A)\} \oplus \{X(k = 0, A)\} = T_+ \oplus T_- \oplus T_0$$

of the tangent space. Metric corresponds to the central extension of the loop algebra to Kac Moody algebra and the Kähler form is given by

$$J(X(k_1 < 0, A), X(k_2 > 0, B)) = k_2\delta(k_1 + k_2)\delta(A, B) \ .$$

In present case the finite dimensional Lie algebra g is replaced with the Lie-algebra of the symplectic transformations of $\delta M^4_+ \times CP_2$ centrally extended using symplectic extension. The scalar function basis on circle is replaced with the function basis on an interval of length Δr_M with periodic boundary conditions; effectively one has circle also now.

The basic difference is that one can consider two kinds of central extensions now.

1. Central extension is most naturally induced by the natural central extension ($\{p, q\} = 1$) defined by Poisson bracket. This extension is anti-symmetric with respect to the generators of the symplectic group: in the case of the Kac Moody central extension it is symmetric with respect to the group G. The symplectic transformations of CP_2 might correspond to non-zero modes also because they are not exact symmetries of Kähler action. The situation is however rather delicate since $k = 0$ light cone harmonic has a diverging norm due to the radial integration unless one poses both lower and upper radial cutoffs although the matrix elements would be still well defined for typical 3-surfaces. For Kac Moody group $U(1)$ transformations correspond to the zero modes. Light cone function algebra can be regarded as a local $U(1)$ algebra defining central extension in the case that only CP_2 symplectic transformations local with respect to δM^4_+ act as isometries: for Kac Moody algebra the central extension corresponds to an ordinary $U(1)$ algebra. In the case that entire light cone symplectic algebra defines the isometries the central extension reduces to a $U(1)$ central extension.

6.8 Symmetric Space Property Implies Ricci Flatness And Isometric Action Of Symplectic Transformations

The basic structure of symmetric spaces is summarized by the following structural equations

$$\begin{array}{l} g = h + t \ , \\ [h, h] \subset h \ , \quad [h, t] \subset t \ , \quad [t, t] \subset h \ . \end{array} \tag{6.18}$$

In present case the equations imply that all commutators of the Lie-algebra generators of $Can(\neq 0)$ having non-vanishing integer valued radial quantum number n_2, possess zero norm. This condition is extremely strong and guarantees isometric action of $Can(\delta M^4_+ \times CP_2)$ as well as Ricci flatness of the WCW metric.

The requirement $[t, t] \subset h$ and $[h, t] \subset t$ are satisfied if the generators of the isometry algebra possess generalized parity P such that the generators in t have parity $P = -1$ and the generators belonging to h have parity $P = +1$. Conformal weight n must somehow define this parity. The first possibility to come into mind is that odd values of n correspond to $P = -1$ and even values to $P = 1$. Since n is additive in commutation, this would automatically imply $h \oplus t$ decomposition with the required properties. This assumption looks however somewhat artificial. TGD however forces a generalization of Super Algebras and N-S and Ramond type algebras can be combined to a larger algebra containing also Virasoro and Kac Moody generators labeled by half-odd integers. This suggests strongly that isometry generators are labeled by half integer conformal weight and that half-odd integer conformal weight corresponds to parity $P = -1$ whereas integer conformal weight corresponds to parity $P = 1$. Coset space would structure would state conformal invariance of the theory since super-symplectic generators with integer weight would correspond to zero modes.

Quite generally, the requirement that the metric is invariant under the flow generated by vector field X leads together with the covariant constancy of the metric to the Killing conditions

$$X \cdot g(Y, Z) \quad = \quad 0 = g([X, Y], Z) + g(Y, [X, Z]) \ . \tag{6.19}$$

If the commutators of the complexified generators in $Can(\neq 0)$ have zero norm then the two terms on the right hand side of Eq. (6.19) vanish separately. This is true if the conditions

$$Q_m^{\alpha,\beta}(\{H^A,\{H^B,H^C\}\}) = 0 \ , \tag{6.20}$$

are satisfied for all triplets of Hamiltonians in $Can_{\neq 0}$. These conditions follow automatically from the $[t,t] \subset h$ property and guarantee also Ricci flatness as will be found later.

It must be emphasized that for Kähler metric defined by purely magnetic fluxes, one cannot pose the conditions of Eq. (6.20) as consistency conditions on the initial values of the time derivatives of embedding space coordinates whereas in general case this is possible. If the consistency conditions are satisfied for a single surface on the orbit of symplectic group then they are satisfied on the entire orbit. Clearly, isometry and Ricci flatness requirements and the requirement of time reversal invariance might well force Kähler electric alternative.

7 Ricci Flatness And Divergence Cancelation

Divergence cancelation in WCW integration requires Ricci flatness and in this section the arguments in favor of Ricci flatness are discussed in detail.

7.1 Inner Product From Divergence Cancelation

Forgetting the delicacies related to the non-determinism of the Kähler action, the inner product is given by integrating the usual Fock space inner product defined at each point of WCW over the reduced WCW containing only the 3-surfaces Y^3 belonging to $\delta H = \delta M^4_+ \times CP_2$ ("light-cone boundary") using the exponent $exp(K)$ as a weight factor:

$$\begin{aligned} \langle \Psi_1|\Psi_2\rangle &= \int \overline{\Psi}_1(Y^3)\Psi_2(Y^3)exp(K)\sqrt{G}dY^3 \ , \\ \overline{\Psi_1}(Y^3)\Psi_2(Y^3) &\equiv \langle \Psi_1(Y^3)|\Psi_2(Y^3)\rangle_{Fock} \ . \end{aligned} \tag{7.1}$$

The degeneracy for the preferred extremals of Kähler action implies additional summation over the degenerate extremals associated with Y^3. The restriction of the integration on light cone boundary is Diff^4 invariant procedure and resolves in elegant manner the problems related to the integration over Diff^4 degrees of freedom. A variant of the inner product is obtained dropping the bosonic vacuum functional $exp(K)$ from the definition of the inner product and by assuming that it is included into the spinor fields themselves. Probably it is just a matter of taste how the necessary bosonic vacuum functional is included into the inner product: what is essential that the vacuum functional $exp(K)$ is somehow present in the inner product.

The unitarity of the inner product follows from the unitary of the Fock space inner product and from the unitarity of the standard L^2 inner product defined by WCW integration in the set of the L^2 integrable scalar functions. It could well occur that $Diff^4$ invariance implies the reduction of WCW integration to $C(\delta H)$.

Consider next the bosonic integration in more detail. The exponent of the Kähler function appears in the inner product also in the context of the finite dimensional group representations. For the representations of the non-compact groups (say $SL(2,R)$) in coset spaces (now $SL(2,R)/U(1)$ endowed with Kähler metric) the exponent of Kähler function is necessary in order to get square integrable representations [B3]. The scalar product for two complex valued representation functions is defined as

$$(f,g) = \int \overline{f}gexp(nK)\sqrt{g}dV \ . \tag{7.2}$$

By unitarity, the exponent is an integer multiple of the Kähler function. In the present case only the possibility $n=1$ is realized if one requires a complete cancelation of the determinants. In finite dimensional case this corresponds to the restriction to single unitary representation of the group in question.

The sign of the action appearing in the exponent is of decisive importance in order to make theory stable. The point is that the theory must be well defined at the limit of infinitely large system. Minimization of action is expected to imply that the action of infinitely large system is bound from above: the generation of electric Kähler fields gives negative contributions to the action. This implies that at the limit of the infinite system the average action per volume is non-positive. For systems having negative average density of action vacuum functional $exp(K)$ vanishes so that only configurations with vanishing average action per volume have significant probability. On the other hand, the choice $exp(-K)$ would make theory unstable: probability amplitude would be infinite for all configurations having negative average action per volume. In the fourth part of the book it will be shown that the requirement that average Kähler action per volume cancels has important cosmological consequences.

Consider now the divergence cancelation in the bosonic integration. One can develop the Kähler function as a Taylor series around maximum of Kähler function and use the contravariant Kähler metric as a propagator. Gaussian and metric determinants cancel each other for a unique vacuum functional. Ricci flatness guarantees that metric determinant is constant in complex coordinates so that one avoids divergences coming from it. The non-locality of the Kähler function as a functional of the 3-surface serves as an additional regulating mechanism: if $K(X^3)$ were a local functional of X^3 one would encounter divergences in the perturbative expansion.

The requirement that quantum jump corresponds to a quantum measurement in the sense of quantum field theories implies that quantum jump involves localization in zero modes. Localization in the zero modes implies automatically p-adic evolution since the decomposition of the WCW into sectors D_P labeled by the infinite primes P is determined by the corresponding decomposition in zero modes. Localization in zero modes would suggest that the calculation of the physical predictions does not involve integration over zero modes: this would dramatically simplify the calculational apparatus of the theory. Probably this simplification occurs at the level of practical calculations if U-matrix separates into a product of matrices associated with zero modes and fiber degrees of freedom.

One must also calculate the predictions for the ratios of the rates of quantum transitions to different values of zero modes and here one cannot actually avoid integrals over zero modes. To achieve this one is forced to define the transition probabilities for quantum jumps involving a localization in zero modes as

$$P(x,\alpha \rightarrow y,\beta) = \sum_{r,s} |S(r,\alpha \rightarrow s,\beta)|^2 |\Psi_r(x)|^2 |\Psi_s(y)|^2 \ ,$$

where x and y correspond to the zero mode coordinates and r and s label a complete state functional basis in zero modes and $S(r,m \rightarrow s,n)$ involves integration over zero modes. In fact, only in this manner the notion of the localization in the zero modes makes mathematically sense at the level of S-matrix. In this case also unitarity conditions are well-defined. In zero modes state function basis can be freely constructed so that divergence difficulties could be avoided. An open question is whether this construction is indeed possible.

Some comments about the actual evaluation of the bosonic functional integral are in order.

1. Since WCW metric is degenerate and the bosonic propagator is essentially the contravariant metric, bosonic integration is expected to reduce to an integration over the zero modes. For instance, isometry invariants are variables of this kind. These modes are analogous to the parameters describing the conformal equivalence class of the orbit of the string in string models.

2. α_K is a natural small expansion parameter in WCW integration. It should be noticed that α_K, when defined by the criticality condition, could also depend on the coordinates parameterizing the zero modes.

3. Semiclassical approximation, which means the expansion of the functional integral as a sum over the extrema of the Kähler function, is a natural approach to the calculation of the bosonic integral. Symmetric space property suggests that for the given values of the zero

modes there is only single extremum and corresponds to the maximum of the Kähler function. There are theorems (Duistermaat-Hecke theorem) stating that semiclassical approximation is exact for certain systems (for example for integrable systems [A3]). Symmetric space property suggests that Kähler function might possess the properties guaranteeing the exactness of the semiclassical approximation. This would mean that the calculation of the integral $\int exp(K)\sqrt{G}dY^3$ and even more complex integrals involving WCW spinor fields would be completely analogous to a Gaussian integration of free quantum field theory. This kind of reduction actually occurs in string models and is consistent with the criticality of the Kähler coupling constant suggesting that all loop integrals contributing to the renormalization of the Kähler action should vanish. Also the condition that WCW integrals are continuable to p-adic number fields requires this kind of reduction.

7.2 Why Ricci Flatness

It has been already found that the requirement of divergence cancelation poses extremely strong constraints on the metric of the WCW. The results obtained hitherto are the following.

1. If the vacuum functional is the exponent of Kähler function one gets rid of the divergences resulting from the Gaussian determinants and metric determinants: determinants cancel each other.

2. The non-locality of the Kähler action gives good hopes of obtaining divergence free perturbation theory.

The following arguments show that Ricci flatness of the metric is a highly desirable property.

1. Dirac operator should be a well defined operator. In particular its square should be well defined. The problem is that the square of Dirac operator contains curvature scalar, which need not be finite since it is obtained via two infinite-dimensional trace operations from the curvature tensor. In case of loop spaces [A2] the Kähler property implies that even Ricci tensor is only conditionally convergent. In fact, loop spaces with Kähler metric are Einstein spaces (Ricci tensor is proportional to metric) and Ricci scalar is infinite.

   In 3-dimensional case situation is even worse since the trace operation involves 3 summation indices instead of one! The conclusion is that Ricci tensor had better to vanish in vibrational degrees of freedom.

2. For Ricci flat metric the determinant of the metric is constant in geodesic complex coordinates as is seen from the expression for Ricci tensor [A4]

$$R_{k\bar{l}} = \partial_k \partial_{\bar{l}} ln(det(g)) \qquad (7.3)$$

   in Kähler metric. This obviously simplifies considerably functional integration over WCW: one obtains just the standard perturbative field theory in the sense that metric determinant gives no contributions to the functional integration.

3. The constancy of the metric determinant results not only in calculational simplifications: it also eliminates divergences. This is seen by expanding the determinant as a functional Taylor series with respect to the coordinates of WCW. In local complex coordinates the first term in the expansion of the metric determinant is determined by Ricci tensor

$$\delta\sqrt{g} \propto R_{k\bar{l}} z^k \bar{z}^l \quad . \qquad (7.4)$$

   In WCW integration using standard rules of Gaussian integration this term gives a contribution proportional to the contraction of the propagator with Ricci tensor. But since the propagator is just the contravariant metric one obtains Ricci scalar as result. So, in order to avoid divergences, Ricci scalar must be finite: this is certainly guaranteed if Ricci tensor vanishes.

4. The following group theoretic argument suggests that Ricci tensor either vanishes or is divergent. The holonomy group of the WCW is a subgroup of $U(n=\infty)$ ($D=2n$ is the dimension of the Kähler manifold) by Kähler property and Ricci flatness is guaranteed if the $U(1)$ factor is absent from the holonomy group. In fact Ricci tensor is proportional to the trace of the $U(1)$ generator and since this generator corresponds to an infinite dimensional unit matrix the trace diverges: therefore given element of the Ricci tensor is either infinite or vanishes. Therefore the vanishing of the Ricci tensor seems to be a mathematical necessity. This naïve argument doesn't hold true in the case of loop spaces, for which Kähler metric with finite non-vanishing Ricci tensor exists [A2] . Note however that also in this case the sum defining Ricci tensor is only conditionally convergent.

There are indeed good hopes that Ricci tensor vanishes. By the previous argument the vanishing of the Ricci tensor is equivalent with the absence of divergences in WCW integration. That divergences are absent is suggested by the non-locality of the Kähler function as a functional of 3-surface: the divergences of local field theories result from the locality of interaction vertices. Ricci flatness in vibrational degrees of freedom is not only necessary mathematically. It is also appealing physically: one can regard Ricci flat WCW as a vacuum solution of Einstein's equations $G^{\alpha\beta}=0$.

7.3 Ricci Flatness And Hyper Kähler Property

Ricci flatness property is guaranteed if WCW geometry is Hyper Kähler [A8, A1] (there exists 3 covariantly constant antisymmetric tensor fields, which can be regarded as representations of quaternionic imaginary units). Hyper Kähler property guarantees Ricci flatness because the contractions of the curvature tensor appearing in the components of the Ricci tensor transform to traces over Lie algebra generators, which are $SU(n)$ generators instead of $U(n)$ generators so that the traces vanish. In the case of the loop spaces left invariance implies that Ricci tensor in the vibrational degrees is a multiple of the metric tensor so that Ricci scalar has an infinite value. This is basically due to the fact that Kac-Moody algebra has $U(1)$ central extension.

Consider now the arguments in favor of Ricci flatness of the WCW.

1. The symplectic algebra of δM^4_+ takes effectively the role of the $U(1)$ extension of the loop algebra. More concretely, the $SO(2)$ group of the rotation group $SO(3)$ takes the role of $U(1)$ algebra. Since volume preserving transformations are in question, the traces of the symplectic generators vanish identically and in finite-dimensional this should be enough for Ricci flatness even if Hyper Kähler property is not achieved.

2. The comparison with CP_2 allows to link Ricci flatness with conformal invariance. The elements of the Ricci tensor are expressible in terms of traces of the generators of the holonomy group $U(2)$ at the origin of CP_2, and since $U(1)$ generator is non-vanishing at origin, the Ricci tensor is non-vanishing. In recent case the origin of CP_2 is replaced with the maximum of Kähler function and holonomy group corresponds to super-symplectic generators labelled by integer valued real parts k_1 of the conformal weights $k=k_1+i\rho$. If generators with $k_1=n$ vanish at the maximum of the Kähler function, the curvature scalar should vanish at the maximum and by the symmetric space property everywhere. These conditions correspond to Virasoro conditions in super string models.

   A possible source of difficulties are the generators having $k_1=0$ and resulting as commutators of generators with opposite real parts of the conformal weights. It might be possible to assume that only the conformal weights $k=k_1+i\rho$, $k_1=0,1,...$ are possible since it is the imaginary part of the conformal weight which defines the complexification in the recent case. This would mean that the commutators involve only positive values of k_1.

3. In the infinite-dimensional case the Ricci tensor involves also terms which are non-vanishing even when the holonomy algebra does not contain $U(1)$ factor. It will be found that symmetric space property guarantees Ricci flatness even in this case and the reason is essentially the vanishing of the generators having $k_1=n$ at the maximum of Kähler function.

There are also arguments in favor of the Hyper Kähler property.

1. The dimensions of the embedding space and space-time are 8 and 4 respectively so that the dimension of WCW in vibrational modes is indeed multiple of four as required by Hyper Kähler property. Hyper Kähler property requires a quaternionic structure in the tangent space of WCW. Since any direction on the sphere S^2 defined by the linear combinations of quaternionic imaginary units with unit norm defines a particular complexification physically, Hyper Kähler property means the possibility to perform complexification in S^2-fold ways.

2. S^2-fold degeneracy is indeed associated with the definition of the complex structure of WCW. First of all, the direction of the quantization axis for the spherical harmonics or for the eigen states of Lorentz Cartan algebra at δM^4_+ can be chosen in S^2-fold ways. Quaternion conformal invariance means Hyper Kähler property almost by definition and the S^2-fold degeneracy for the complexification is obvious in this case.

If these naïve arguments survive a more critical inspection, the conclusion would be that the effective 2-dimensionality of light like 3-surfaces implying generalized conformal and symplectic symmetries would also imply Hyper Kähler property of WCW and make the theory well-defined mathematically. This obviously fixes the dimension of space-time surfaces as well as the dimension of Minkowski space factor of the embedding space.

In the sequel we shall show that Ricci flatness is guaranteed provided that the holonomy group of WCW is isomorphic to some subgroup of $SU(n=\infty)$ instead of $U(n=\infty)$ (n is the complex dimension of WCW) implied by the Kähler property of the metric. We also derive an expression for the Ricci tensor in terms of the structure constants of the isometry algebra and WCW metric. The expression for the Ricci tensor is formally identical with that obtained by Freed for loop spaces: the only difference is that the structure constants of the finite-dimensional group are replaced with the group $Can(\delta H)$. Also the arguments in favor of Hyper Kähler property are discussed in more detail.

7.4 The Conditions Guaranteeing Ricci Flatness

In the case of Kähler geometry Ricci flatness condition can be characterized purely Lie-algebraically: the holonomy group of the Riemann connection, which in general is subgroup of $U(n)$ for Kähler manifold of complex dimension n, must be subgroup of $SU(n)$ so that the Lie-algebra of this group consists of traceless matrices. This condition is easy to derive using complex coordinates. Ricci tensor is given by the following expression in complex vielbein basis

$$R^{A\bar{B}} = R^{A\bar{C}B}_{\ \ \ \ \bar{C}} \ , \tag{7.5}$$

where the latter summation is only over the antiholomorphic indices $\bar{C}$. Using the cyclic identities

$$\sum_{cycl\ \bar{C}B\bar{D}} R^{A\bar{C}B\bar{D}} = 0 \ , \tag{7.6}$$

the expression for Ricci tensor reduces to the form

$$R^{A\bar{B}} = R^{A\bar{B}C}_{\ \ \ \ C} \ , \tag{7.7}$$

where the summation is only over the holomorphic indices C. This expression can be regarded as a trace of the curvature tensor in the holonomy algebra of the Riemann connection. The trace is taken over holomorphic indices only: the traces over holomorphic and anti-holomorphic indices cancel each other by the antisymmetry of the curvature tensor. For Kähler manifold holonomy algebra is subalgebra of $U(n)$, when the complex dimension of manifold is n and Ricci tensor vanishes if and only if the holonomy Lie-algebra consists of traceless matrices, or equivalently: holonomy group is subgroup of $SU(n)$. This condition is expected to generalize also to the infinite-dimensional case.

We shall now show that if WCW metric is Kähler and possesses infinite-dimensional isometry algebra with the property that its generators form a complete basis for the tangent space (every tangent vector is expressible as a superposition of the isometry generators plus zero norm vector) it is possible to derive a representation for the Ricci tensor in terms of the structure constants of the isometry algebra and of the components of the metric and its inverse in the basis formed by the isometry generators and that Ricci tensor vanishes identically for the proposed complexification of the WCW provided the generators $\{H_{A,m\neq 0}, H_{B,n\neq 0}\}$ correspond to zero norm vector fields of WCW.

The general definition of the curvature tensor as an operator acting on vector fields reads

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \ . \tag{7.8}$$

If the vector fields considered are isometry generators the covariant derivative operator is given by the expression

$$\begin{aligned} \nabla_X Y &= (Ad_X Y - Ad^*_X Y - Ad^*_Y X)/2 \ , \\ (Ad^*_X Y, Z) &= (Y, Ad_X Z) \ , \end{aligned} \tag{7.9}$$

where $Ad_X Y = [X,Y]$ and Ad^*_X denotes the adjoint of Ad_X with respect to WCW metric.

In the sequel we shall assume that the vector fields in question belong to the basis formed by the isometry generators. The matrix representation of Ad_X in terms of the structure constants $C_{X,Y:Z}$ of the isometry algebra is given by the expression

$$\begin{aligned} Ad^m_{Xn} &= C_{X,Y:Z}\hat{Y}_n Z^m \ , \\ [X,Y] &= C_{X,Y:Z}Z \ , \\ \hat{Y} &= g^{-1}(Y,V)V \ , \end{aligned} \tag{7.10}$$

where the summation takes place over the repeated indices and $\hat{Y}$ denotes the dual vector field of Y with respect to the WCW metric. From its definition one obtains for Ad^*_X the matrix representation

$$\begin{aligned} Ad^{*m}_{Xn} &= C_{X,Y:Z}\hat{Y}^m Z_n \ , \\ Ad^*_X Y &= C_{X,U:V}g(Y,U)g^{-1}(V,W)W = g(Y,U)g^{-1}([X,U],W)W \ , \end{aligned} \tag{7.11}$$

where the summation takes place over the repeated indices.

Using the representations of ∇_X in terms of Ad_X and its adjoint and the representations of Ad_X and Ad^*_X in terms of the structure constants and some obvious identities (such as $C_{[X,Y],Z:V} = C_{X,Y:U}C_{U,Z:V}$) one can by a straightforward but tedious calculation derive a more detailed expression for the curvature tensor and Ricci tensor. Straightforward calculation of the Ricci tensor has however turned to be very tedious even in the case of the diagonal metric and in the following we shall use a more convenient representation [A2] of the curvature tensor applying in case of the Kähler geometry.

The expression of the curvature tensor is given in terms of the so called Toeplitz operators T_X defined as linear operators in the "positive energy part" G_+ of the isometry algebra spanned by the $(1,0)$ parts of the isometry generators. In present case the positive and negative energy parts and cm part of the algebra can be defined just as in the case of loop spaces:

$$\begin{aligned} G_+ &= \{H^{Ak}|k>0\} \ , \\ G_- &= \{H^{Ak}|k<0\} \ , \\ G_0 &= \{H^{Ak}|k=0\} \ . \end{aligned} \tag{7.12}$$

Here H^{Ak} denote the Hamiltonians generating the symplectic transformations of δH. The positive energy generators with non-vanishing norm have positive radial scaling dimension: $k \geq 0$, which corresponds to the imaginary part of the scaling momentum $K = k_1 + i\rho$ associated with the factors $(r_M/r_0)^K$. A priori the spectrum of ρ is continuous but it is quite possible that the spectrum of ρ

is discrete and $\rho = 0$ does not appear at all in the spectrum in the sense that the flux Hamiltonians associated with $\rho = 0$ elements vanish for the maximum of Kähler function which can be taken to be the point where the calculations are done.

T_X differs from Ad_X in that the negative energy part of $Ad_X Y = [X, Y]$ is dropped away:

$$\begin{aligned} T_X : G_+ &\rightarrow G_+ \ , \\ Y &\rightarrow [X, Y]_+ \ . \end{aligned} \tag{7.13}$$

Here "+" denotes the projection to "positive energy" part of the algebra. Using Toeplitz operators one can associate to various isometry generators linear operators $\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ acting on G_+:

$$\begin{aligned} \Phi(X_0) &= T_{X_0} \ , \ X_0 \varepsilon G_0 \ , \\ \Phi(X_-) &= T_{X_-} \ , \ X_- \varepsilon G_- \ , \\ \Phi(X_+) &= -T^*_{X_-} \ , \ X_+ \varepsilon G_+ \ . \end{aligned} \tag{7.14}$$

Here "*" denotes hermitian conjugate in the diagonalized metric: the explicit representation $\Phi(X_+)$ is given by the expression [A2]

$$\begin{aligned} \Phi(X_+) &= D^{-1} T_{X_-} D \ , \\ D X_+ &= d(X) X_- \ , \\ d(X) &= g(X_-, X_+) \ . \end{aligned} \tag{7.15}$$

Here $d(X)$ is just the diagonal element of metric assumed to be diagonal in the basis used. denotes the conformal factor associated with the metric.

The representations for the action of ,$\Phi(X_0)$, $\Phi(X_-)$ and $\Phi(X_+)$ in terms of metric and structure constants of the isometry algebra are in the case of the diagonal metric given by the expressions

$$\begin{aligned} \Phi(X_0) Y_+ &= C_{X_0, Y_+ : U_+} U_+ \ , \\ \Phi(X_-) Y_+ &= C_{X_-, Y_+ : U_+} U_+ \ , \\ \Phi(X_+) Y_+ &= \frac{d(Y)}{d(U)} C_{X_-, Y_- : U_-} U_+ \ . \end{aligned} \tag{7.16}$$

The expression for the action of the curvature tensor in positive energy part G_+ of the isometry algebra in terms of the these operators is given as [A2] :

$$R(X, Y) Z_+ = \{[\Phi(X), \Phi(Y)] - \Phi([X, Y])\} Z_+ \ . \tag{7.17}$$

The calculation of the Ricci tensor is based on the observation that for Kähler manifolds Ricci tensor is a tensor of type $(1, 1)$, and therefore it is possible to calculate Ricci tensor as the trace of the curvature tensor with respect to indices associated with G_+.

$$Ricci(X_+, Y_-) = (\hat{Z}_+, R(X_+, Y_-) Z_+) \equiv Trace(R(X_+, Y_-)) \ , \tag{7.18}$$

where the summation over Z_+ generators is performed.

Using the explicit representations of the operators Φ one obtains the following explicit expression for the Ricci tensor

$$\begin{aligned} Ricci(X_+, Y_-) &= Trace\{[D^{-1} T_{X_+} D, T_{Y_-}] - T_{[X_+, Y_-]_{|G_0 + G_-}} \\ &- D^{-1} T_{[X_+, Y_-]_{|G_+}} D\} \ . \end{aligned} \tag{7.19}$$

This expression is identical to that encountered in case of loop spaces and the following arguments are repetition of those applying in the case of loop spaces.

The second term in the Ricci tensor is the only term present in the finite-dimensional case. This term vanishes if the Lie-algebra in question consists of traceless matrices. Since symplectic transformations are volume-preserving the traces of Lie-algebra generators vanish so that this term is absent. The last term gives a non-vanishing contribution to the trace for the same reason.

The first term is quadratic in structure constants and does not vanish in case of loop spaces. It can be written explicitly using the explicit representations of the various operators appearing in the formula:

$$\begin{aligned} Trace\{[D^{-1}T_{X_-}D, T_{Y_-}]\} &= \sum_{Z_+,U_+} [C_{X_-,U_-:Z_-} C_{Y_-,Z_+:U_+} \frac{d(U)}{d(Z)} \\ &- C_{X_-,Z_-:U_-} C_{Y_-,U_+:Z_+} \frac{d(Z)}{d(U)}] \ . \end{aligned} \tag{7.20}$$

Each term is antisymmetric under the exchange of U and Z and one might fail to conclude that the sum vanishes identically. This is not the case. By the diagonality of the metric with respect to radial quantum number, one has $m(X_-) = m(Y_-)$ for the non-vanishing elements of the Ricci tensor. Furthermore, one has $m(U) = m(Z) - m(Y)$, which eliminates summation over $m(U)$ in the first term and summation over $m(Z)$ in the second term. Note however, that summation over other labels related to symplectic algebra are present.

By performing the change $U \rightarrow Z$ in the second term one can combine the sums together and as a result one has finite sum

$$\begin{aligned} \sum_{0<m(Z)<m(X)} [C_{X_-,U_-:Z_-} C_{Y_-,Z_+:U_+} \frac{d(U)}{d(Z)} &= C \sum_{0<m(Z)<m(X)} \frac{m(X)}{m(Z)-m(X)} \ , \\ C &= \sum_{Z,U} C_{X,U:Z} C_{Y,Z:U} \frac{d_0(U)}{d_0(Z)} \ . \end{aligned} \tag{7.21}$$

Here the dependence of $d(X) = |m(X)| d_0(X)$ on $m(X)$ is factored out; $d_0(X)$ does not depend on k_X. The dependence on $m(X)$ in the resulting expression factorizes out, and one obtains just the purely group theoretic term C, which should vanish for the space to be Ricci flat.

The sum is quadratic in structure constants and can be visualized as a loop sum. It is instructive to write the sum in terms of the metric in the symplectic degrees of freedom to see the geometry behind the Ricci flatness:

$$C = \sum_{Z,U} g([Y,Z],U) g^{-1}([X,U],Z) \ . \tag{7.22}$$

Each term of this sum involves a commutator of two generators with a non-vanishing norm. Since tangent space complexification is inherited from the local coset space, the non-vanishing commutators in complexified basis are always between generators in $Can_{\neq 0}$; that is they do not not belong to rigid $su(2) \times su(3)$.

The condition guaranteeing Ricci flatness at the maximum of Kähler function and thus everywhere is simple. All elements of type $[X_{\neq 0}, Y_{\neq 0}]$ vanish or have vanishing norm. In case of CP_2 Kähler geometry this would correspond to the vanishing of the $U(2)$ generators at the origin of CP_2 (note that the holonomy group is $U(2)$ in case of CP_2). At least formally stronger condition is that the algebra generated by elements of this type, the commutator algebra associated with $Can_{\neq 0}$, consist of elements of zero norm. Already the (possibly) weaker condition implies that adjoint map $Ad_{X\neq 0}$ and its hermitian adjoint $Ad^*_{X_{\neq 0}}$ create zero norm states. Since isometry conditions involve also adjoint action the condition also implies that $Can_{\neq 0}$ acts as isometries. More concrete form for the condition is that all flux factors involving double Poisson bracket and three generators in $Can_{\neq 0}$ vanish:

$$Q_e(\{H_A,\{H_B,H_C\}\}) = 0 \ , \ \text{for } H_A,H_B,H_C \text{ in } Can_{\neq 0} \ . \quad (7.23)$$

The vanishing of fluxes involving two Poisson brackets and three Hamiltonians guarantees isometry invariance and Ricci flatness and, as found in [K5] , is implied by the $[t,t] \subset h$ property of the Lie-algebra of coset space G/H having symmetric space structure.

The conclusion is that the mere existence of the proposed isometry group (guaranteed by the symmetric space property) implies the vanishing of the Ricci tensor and vacuum Einstein equations. The existence of the infinite parameter isometry group in turn follows basically from the condition guaranteeing the existence of the Riemann connection. Therefore vacuum Einstein equations seem to arise, not only as a consequence of a physically motivated variational principle but as a mathematical consistency condition in infinite dimensional Kähler geometry. The flux representation seems to provide elegant manner to formulate and solve these conditions and isometry invariance implies Ricci flatness.

7.5 Is WCW Metric Hyper Kähler?

The requirement that WCW integral integration is divergence free implies that WCW metric is Ricci flat. The so called Hyper-Kähler metrics [A8, A1] , [B5] are particularly nice representatives of Ricci flat metrics. In the following the basic properties of Hyper-Kähler metrics are briefly described and the problem whether Hyper Kähler property could realized in case of $M^4_+ \times CP_2$ is considered.

7.5.1 Hyper-Kähler property

Hyper-Kähler metric is a generalization of the Kähler metric. For Kähler metric metric tensor and Kähler form correspond to the complex numbers 1 and i and therefore define complex structure in the tangent space of the manifold. For Hyper Kähler metric tangent space allows three closed Kähler forms I,J,K, which with respect to the multiplication obey the algebra of quaternionic imaginary units and have square equal to - 1, which corresponds to the metric of Hyper Kähler space.

$$I^2 = J^2 = K^2 = -1 \ \ IJ = -JI = K, \text{ etc. } . \quad (7.24)$$

To define Kähler structure one must choose one of the Kähler forms or any linear combination of I,J and K with unit norm. The group $SO(3)$ rotates different Kähler structures to each other playing thus the role of quaternion automorphisms. This group acts also as coordinate transformations in Hyper Kähler manifold but in general fails to act as isometries.

If K is chosen to define complex structure then K is tensor of type $(1,1)$ in complex coordinates, I and J being tensors of type $(2,0)+(0,2)$. The forms $I+iJ$ and $I-iJ$ are holomorphic and anti-holomorphic forms of type $(2,0)$ and $(0,2)$ respectively and defined standard step operators I_+ and I_- of $SU(2)$ algebra. The holonomy group of Hyper-Kähler metric is always $Sp(k)$, $k \leq dimM/4$, the group of $k \times k$ unitary matrices with quaternionic entries. This group is indeed subgroup of SU(2k), so that its generators are traceless and Hyper Kähler metric is therefore Ricci flat.

Hyper Kähler metrics have been encountered in the context of 3-dimensional super symmetric sigma models: a necessary prerequisite for obtaining $N=4$ super-symmetric sigma model is that target space allows Hyper Kähler metric [B5, B1] . In particular, it has been found that Hyper Kähler property is decisive for the divergence cancelation.

Hyper-Kähler metrics arise also in monopole and instanton physics [A1] . The moduli spaces for monopoles have Hyper Kähler property. This suggests that Hyper Kähler property is characteristic for the configuration (or moduli) spaces of 4-dimensional Yang Mills types systems. Since YM action appears in the definition of WCW metric there are hopes that also in present case the metric possesses Hyper-Kähler property.

CP_2 allows what might be called almost Hyper-Kähler structure known as quaternionion structure. This means that the Weil tensor of CP_2 consists of three components in one-one correspondence with components of iso-spin and only one of them- the one corresponding to Kähler form-is covariantly constant. The physical interpretation is in terms of electroweak symmetry breaking selecting one isospin direction as a favored direction.

7.5.2 Does the "almost" Hyper-Kähler structure of CP_2 lift to a genuine Hyper-Kähler structure in WCW?

The Hyper-Kähler property of WCW metric does not seem to be in conflict with the general structure of TGD.

1. In string models the dimension of the "space-time" is two and Weyl invariance and complex structures play a decisive role in the theory. In present case the dimension of the space-time is four and one therefore might hope that quaternions play a similar role. Indeed, Weyl invariance implies YM action in dimension 4 and as already mentioned moduli spaces of instantons and monopoles enjoy the Hyper Kähler property.

2. Also the dimension of the embedding space is important. The dimension of Hyper Kähler manifold must be multiple of 4. The dimension of WCW is indeed infinite multiple of 8: each vibrational mode giving one "8".

3. The complexification of the WCW in symplectic degrees of freedom is inherited from $S^2 \times CP_2$ and CP_2 Kähler form defines the symplectic form of WCW. The point is that CP_2 Weyl tensor has 3 covariantly constant components, having as their square metric apart from sign. One of them is Kähler form, which is closed whereas the other two are non-closed forms and therefore fail to define Kähler structure. The group $SU(2)$ of electro-weak isospin rotations rotate these forms to each other. It would not be too surprising if one could identify WCW counterparts of these forms as representations of quaternionic units at the level of WCW. The failure of the Hyper Kähler property at the level of CP_2 geometry is due to the electro-weak symmetry breaking and physical intuition (in particular, p-adic mass calculations [K10]) suggests that electro-weak symmetry might not be broken at the level of WCW geometry).

A possible topological obstruction for the Hyper Kähler property is related to the cohomology of WCW: the three Kähler forms must be co-homologically trivial as is clear from the following argument. If any of 3 quaternionic 2-form is cohomologically nontrivial then by $SO(3)$ symmetry rotating Kähler forms to each other all must be co-homologically nontrivial. On the other hand, electro-weak isospin rotation leads to a linear combination of 3 Kähler forms and the flux associated with this form is in general not integer valued. The point is however that Kähler form forms only the $(1,1)$ part of the symplectic form and must be co-homologically trivial whereas the zero mode part is same for all complexifications and can be co-homologically nontrivial. The co-homological non-triviality of the zero mode part of the symplectic form is indeed a nice feature since it fixes the normalization of the Kähler function apart from a multiplicative integer. On the other hand the hypothesis that Kähler coupling strength is analogous to critical temperature provides a dynamical (and perhaps equivalent) manner to fix the normalization of the Kähler function.

Since the properties of the WCW metric are inherited from $M^4_+ \times CP_2$ then also the Hyper Kähler property should be understandable in terms of the embedding space geometry. In particular, the complex structure in CP_2 vibrational degrees of freedom is inherited from CP_2. Hyper Kähler property implies the existence of a continuum (sphere S^2) of complex structures: any linear superposition of 3 independent Kähler forms defines a respectable complex structure. Therefore also CP_2 should have this continuum of complex structures and this is certainly not the case.

Indeed, if we had instead of CP_2 Hyper Kähler manifold with 3 covariantly constant 2-forms then it would be easy to understand the Hyper Kähler structure of WCW. Given the Kähler structure of WCW would be obtained by replacing induced Kähler electric and magnetic fields in the definition of flux factors $Q(H_{A,m})$ with the appropriate component of the induced Weyl tensor. CP_2 indeed manages to be very nearly Hyper Kähler manifold!

How CP_2 fails to be Hyper Kähler manifold can be seen in the following manner. The Weyl tensor of CP_2 allows three independent components, which are self dual as 2-forms and rotated to each other by vielbein rotations.

$$\begin{aligned} W_{03} &= W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ W_{01} &= W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ W_{02} &= W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \end{aligned} \tag{7.25}$$

The component I_3 is just the Kähler form of CP_2. Remaining components are covariantly constant only with respect to spinor connection and not closed forms so that they cannot be interpreted as Maxwell fields. Their squares equal however apart from sign with the metric of CP_2, when appropriate normalization factor is used. If these forms were covariantly constant Kähler action defined by any linear superposition of these forms would indeed define Kähler structure in WCW and the group $SO(3)$ would rotate these forms to each other. The projections of the components of the Weyl tensor on 3-surface define 3 vector fields as their duals and only one of these vector fields (Kähler magnetic field) is divergenceless. One might regard these 3 vector fields as counter parts of quaternion units associated with the broken Hyper Kähler structure, that is quaternion structure. The interpretation in terms of electro-weak symmetry breaking is obvious.

One cannot exclude the possibility that the symplectic invariance of the induced Kähler electric field implies that the electric parts of the other two components of induced Weyl tensor are symplectic invariants. This is the minimum requirement. What is however obvious is that the magnetic parts cannot be closed forms for arbitrary 3-surfaces at light cone boundary. One counter example is enough and CP_2 type extremals seem to provide this counter example: the components of the induced Weyl tensor are just the same as they are for CP_2 and clearly not symplectically invariant.

Thus it seems that WCW could allow Hyper Kähler structure broken by electro-weak interactions but it cannot be inherited from CP_2. An open question is whether it allows genuine quaternionic structure. Good prospects for obtaining quaternionic structure are provided by the quaternionic counterpart QP_2 of CP_2, which is 8-dimensional and has coset space structure $QP_2 = Sp(3)/Sp(2) \times Sp(1)$. This choice does not seem to be consistent with the symmetries of the standard model. Note however that the over all symmetry group is obtained by replacing complex numbers with quaternions on the matrix representation of the standard model group.

7.5.3 Could different complexifications for M^4_+ and light like surfaces induce Hyper Kähler structure for WCW?

Quaternionic structure means also the existence of a family of complex structures parameterized by a sphere S^2. The complex structure of the WCW is inherited from the complex structure of some light like surface.

In the case of the light cone boundary δM^4_+ the complex structure corresponds to the choice of quantization axis of angular momentum for the sphere $r_M = constant$ so that the coordinates orthogonal to the quantization axis define a complex coordinate: the sphere S^2 parameterizes these choices. Thus there is a temptation to identify the choice of quantization axis with a particular imaginary unit and Hyper Kähler structure would directly relate to the properties rotation group. This would bring an additional item to the list of miraculous properties of light like surfaces of 4-dimensional space-times.

This might relate to the fact that WCW geometry is not determined by the symplectic algebra of CP_2 localized with respect to the light cone boundary as one might first expect but consists of $M^4_+ \times CP_2$ Hamiltonians so that infinitesimal symplectic transformation of CP_2 involves always also M^4_+-symplectic transformation. M^4_+ Hamiltonians are defined by a function basis generated as products of the Hamiltonians H_3 and $H_1 \pm iH_2$ generating rotations with respect to three orthogonal axes, and two of these Hamiltonians are complexified.

Also the light like 3-surfaces X^3_l associated with quaternion conformal invariance are determined by some 2-surface X^2 and the choice of complex coordinates and if X^2 is sphere the choices are labelled by S^2. In this case, the presence of quaternion conformal structure would be almost obvious since it is possible to choose some complex coordinate in several ways and the choices are labelled by S^2. The choice of the complex coordinate in turn fixes 2-surface X^2 as a surface for which the remaining coordinates are constant. X^2 need not however be located at the elementary particle horizon unless one poses additional constraint. One might hope that different choices of X^2 resulting in this manner correspond to all possible different selections of the complex structure and that this choice could fix uniquely the conformal equivalence class of X^2 appearing as argument in elementary particle vacuum functionals. If X^2 has a more complex topology the identification is not so clear but since conformal algebra SL(2,C) containing algebra of rotation group is involved, one might argue that the choice of quantization axis also now involves S^2 degeneracy. If these arguments are correct one could conclude that Hyper Kähler structure is implicitly involved and guarantees Ricci flatness of the WCW metric.

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