Identification of the Configuration Space Kähler Function

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Contents

Abstract

There are two basic approaches to quantum TGD. The first approach, which is discussed in this chapter, is a generalization of Einstein's geometrization program of physics to an infinite-dimensional context. Second approach is based on the identification of physics as a generalized number theory. The first approach relies on the vision of quantum physics as infinite-dimensional Kähler geometry for the "world of classical worlds" (WCW) identified as the space of 3-surfaces in in certain 8-dimensional space.

There are three separate manners to meet the challenge of constructing WCW Kähler geometry and spinor structure. The first approach relies on direct guess of Kähler function. Second approach relies on the construction of Kähler form and metric utilizing the huge symmetries of the geometry needed to guarantee the mathematical existence of Riemann connection. The third approach relies on the construction of spinor structure based on the hypothesis that complexified WCW gamma matrices are representable as linear combinations of fermionic oscillator operator for second quantized free spinor fields at space-time surface and on the geometrization of super-conformal symmetries in terms of WCW spinor structure.

In this chapter the proposal for Kähler function based on the requirement of 4-dimensional General Coordinate Invariance implying that its definition must assign to a given 3-surface a unique space-time surface. Quantum classical correspondence requires that this surface is a preferred extremal of some some general coordinate invariant action, and so called Kähler action is a unique candidate in this respect. The preferred extremal has in positive energy ontology interpretation as an analog of Bohr orbit so that classical physics becomes and exact part of WCW geometry and therefore also quantum physics. In zero energy ontology (ZEO) it is not clear whether this interpretation can be preserved except for maxima of Kähler function.

The basic challenge is the explicit identification of WCW Kähler function K. Two assumptions lead to the identification of K as a sum of Chern-Simons type terms associated with the ends of causal diamond and with the light-like wormhole throats at which the signature of the induced metric changes. The first assumption is the weak form of electric magnetic duality. Second assumption is that the Kähler current for preferred extremals satisfies the condition $j_K \wedge dj_K = 0$ implying that the flow parameter of the flow lines of j_K defines a global space-time coordinate. This would mean that the vision about reduction to almost topological QFT would be realized.

Second challenge is the understanding of the space-time correlates of quantum criticality. Electric-magnetic duality helps considerably here. The realization that the hierarchy of Planck constant realized in terms of coverings of the embedding space follows from basic quantum TGD leads to a further understanding. The extreme non-linearity of canonical momentum densities as functions of time derivatives of the embedding space coordinates implies that the correspondence between these two variables is not 1-1 so that it is natural to introduce coverings of $CD \times CP_2$. This leads also to a precise geometric characterization of the criticality of the preferred extremals. Sub-algebra of conformal symmetries consisting of generators for which conformal weight is integer multiple of given integer n is conjectured to act as critical deformations, that there are n conformal equivalence classes of extremals and that n defines the effective value of Planck constant $h_{eff} = n \times h$.

1 Introduction

The topics of this chapter are the purely geometric aspects of the vision about physics as an infinite-dimensional Kähler geometry of the "world of classical worlds", with " classical world" identified either as light-like 3-D surface of the unique Bohr orbit like 4-surface traversing through it. The non-determinism of Kähler action forces to generalize the notion of 3-surface so that unions of space-like surfaces with time like separations must be allowed. Zero energy ontology allows to formulate this picture elegantly in terms of causal diamonds defined as intersections of future and past directed light-cones. Also a a geometric realization of coupling constant evolution and finite measurement resolution emerges.

There are two separate but closely related tasks involved.

1. Provide WCW with Kähler geometry which is consistent with 4-dimensional general coordinate invariance so that the metric is Diff^4 degenerate. General coordinate invariance implies that the definition of metric must assign to a given light-like 3-surface X^3 a 4-surface as a kind of Bohr orbit $X^4(X^3)$.

2. Provide WCW with a spinor structure. The great idea is to identify WCW gamma matrices in terms of super algebra generators expressible using second quantized fermionic oscillator operators for induced free spinor fields at the space-time surface assignable to a given 3-surface. The isometry generators and contractions of Killing vectors with gamma matrices would thus form a generalization of Super Kac-Moody algebra.

In this chapter a summary about basic ideas related to the construction of the Kähler geometry of infinite-dimensional configuration of 3-surfaces (more or less-equivalently, the corresponding 4-surfaces defining generalized Bohr orbits) or "world of classical worlds" (WCW).

1.1 The Quantum States Of Universe As Modes Of Classical Spinor Field In The "World Of Classical Worlds"

The vision behind the construction of WCW geometry is that physics reduces to the geometry of classical spinor fields in the infinite-dimensional WCW of 3-surfaces of $M^4_+ \times CP_2$ or $M^4 \times CP_2$, where M^4 and M^4_+ denote Minkowski space and its light cone respectively. This WCW might be called the "world of classical worlds".

Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that WCW possesses Kähler geometry. One of the basic features of the Kähler geometry is that it is solely determined by the so called. which defines both the J and the components of the g in complex coordinates via the general formulas [A3]

$$\begin{aligned} J &= i\partial_k\partial_{\bar{l}}K dz^k \wedge d\bar{z}^l \ . \\ ds^2 &= 2\partial_k\partial_{\bar{l}}K dz^k d\bar{z}^l \ . \end{aligned} \tag{1.1}$$

Kähler form is covariantly constant two-form and can be regarded as a representation of imaginary unit in the tangent space of the WCW

$$J_{mr}J^{rn} = -g_m^{\ n} \ . \tag{1.2}$$

As a consequence Kähler form defines also symplectic structure in WCW.

1.2 WCW Kähler Metric From Kähler Function

The task of finding Kähler geometry for the WCW reduces to that of finding Kähler function and identifying the complexification. The main constraints on the Kähler function result from the requirement of Diff^4 symmetry and degeneracy. requires that the definition of the Kähler function assigns to a given 3-surface X^3, which in Zero Energy Ontology is union of 3-surfaces at the opposite boundaries of causal diamond CD, a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with X^3. The natural guess is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of CP_2 coordinates.

Absolute minimization was the first guess for how to fix $X^4(X^3)$ uniquely. It has however become clear that this option might well imply that Kähler is negative and infinite for the entire Universe so that the vacuum functional would be identically vanishing. This condition can make sense only inside wormhole contacts with Euclidian metric and positive definite Kähler action.

Quantum criticality of TGD Universe suggests the appropriate principle to be the criticality, that is vanishing of the second variation of Kähler action. This principle now follows from the conservation of Noether currents the Kähler-Dirac action. This formulation is still rather abstract and if spinors are localized to string world sheets, it it is not satisfactory. A further step in progress was the realization that preferred extremals could carry vanishing super-conformal Noether charges for sub-algebras whose generators have conformal weight vanishing modulo n with n identified in terms of effective Planck constant $h_{eff}/h = n$.

If Kähler action would define a strictly deterministic variational principle, Diff^4 degeneracy and general coordinate invariance would be achieved by restricting the consideration to 3-surfaces Y^3 at the boundary of M^4_+ and by defining Kähler function for 3-surfaces X^3 at $X^4(Y^3)$ and diffeo-related to Y^3 as $K(X^3) = K(Y^3)$. The classical non-determinism of the Kähler action however introduces complications. As a matter fact, the hierarchy of Planck constants has nice interpretation in terms of non-determinism: the space-time sheets connecting the 3-surface at the ends of CD form n conformal equivalence classes. This would correspond to the non-determinism of quantum criticality accompanied by generalized conformal invariance

1.3 WCW Kähler Metric From Symmetries

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan [A1] [A1] has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that WCW is a union of symmetric spaces labelled by zero modes not appearing in the line element as differentials. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and embedding space which is a product of four-dimensional Minkowski space or its future light cone with CP_2.

The detailed formulas for the matrix elements of the Kähler metric however remain educated guesses so that this approach is not entirely satisfactory.

1.4 WCW Kähler Metric As Anti-commutators Of Super-Symplectic Super Noether Charges

The third approach identifies the Kähler metric of WCW as anti-commutators of WCW gamma matrices. This is not yet enough to get concrete expressions but the identification of WCW gamma matrices as Noether super-charges for super-symplectic algebra assignable to the boundary of WCW changes the situation. One also obtains a direct connection with elementary particle physics.

The super charges are linear in the mode of induced spinor field and second quantized spinor field itself, and involve the infinitesimal action of symplectic generator on the spinor field. One can fix fermionic anti-commutation relations by second quantization of the induced spinor fields (as a matter fact, here one can still consider two options). Hence one obtains explicit expressions for the matrix elements of WCW metric.

If the induced spinor fields are localized at string world sheets - as the well-definedness of em charge and number theoretic arguments suggest - one obtains an expression for the matrix elements of the metric in terms of 1-D integrals over strings connecting partonic 2-surfaces. If spinors are localized to string world sheets also in the interior of CP_2, the integral is over a closed circle and could have a representation analogous to a residue integral so that algebraic continuation to p-adic number fields might become straightforward.

The matrix elements of WCW metric are labelled by the conformal weights of spinor modes, those of symplectic vector fields for light-like CD boundaries and by labels for the irreducible representations of $SO(3)$ acting on light-cone boundary $\delta M^4_\pm = R_+ \times S^2$ and of $SU(3)$ acting in CP_2. The dependence on spinor modes and their conformal weights could not be guessed in the approach based on symmetries only. The presence of two rather than only one conformal weights distinguishes the metric from that for loop spaces [A1] and reflects the effective 2-dimensionality. The metric codes a rather scarce information about 3-surfaces. This is in accordance with the notion of finite measurement resolution. By increasing the number of partonic 2-surfaces and string world sheets the amount of information coded - measurement resolution - increases. Fermionic quantum state gives information about 3-geometry. The alternative expression for WCW metric in terms of Kähler function means analog of AdS/CFT duality: Kähler metric can be expressed either in terms of Kähler action associated with the Euclidian wormhole contacts defining Kähler function or in terms of the fermionic oscillator operators at string world sheets connecting partonic 2-surfaces.

1.5 What Principle Selects The Preferred Extremals?

In positive energy ontology space-time surfaces should be analogous to Bohr orbits in order to make possible possible realization of general coordinate invariance. The first guess was that absolute minimization of Kähler action might be the principle selecting preferred extremals. One can criticize the assumption that extremals correspond to the absolute minima of Kähler action for entire spacetime surface, as too strong since Kähler action from Minkowskian regions is proportional to imaginary unit and corresponds to ordinary QFT action defining a phase factor of vacuum functional. Furthermore, the notion of absolute minimization does not make sense in p-adic context unless one manages to reduce it to purely algebraic conditions. Absolute minimization could however make sense for Euclidian space-time regions defining the lines of generalized Feynman diagras, where Kähler action has definite sign. Kähler function is indeed the Kähler action for these regions.

What is needed is the association of a unique space-time surface to a given 3-surface defined as union of 3-surfaces at opposite boundaries of CD. One can imagine many way to achieve this. "Unique" is too much to demand: for the proposal unique space-time surface is replaced with finite number of conformal gauge equivalence classes of space-time surfaces. In any case, it is better to talk just about preferred extremals of Kähler action and accept as the fact that there are several proposals for what this notion could mean.

1. For instance, one can consider the identification of space-time surface as associative (co-associative) sub-manifold meaning that tangent space of space-time surface can be regarded as associative (co-associative) sub-manifold of complexified octonions defining tangent space of embedding space. One way to define "associative sub-manifold" is by introducing octonionic representation of embedding space gamma matrices identified as tangent space vectors. It must be also assumed that the tangent space contains a preferred commutative (co-commutative) sub-space at each point and defining an integrable distribution having identification as string world sheet (also slicing of space-time sheet by string world sheets can be considered). Associativity and commutativity would define the basic dynamical principle. A closely related approach is based on so called Hamilton-Jacobi structure [K1] defining also this kind of slicing and the approaches could be equivalent.

2. In zero energy ontology (ZEO) 3-surfaces become pairs of space-like 3-surfaces at the boundaries of causal diamond (CD). Even the light-like partonic orbits could be included to give the analog of Wilson loop. In absence of non-determinism of Kähler action this forces to ask whether the attribute "preferred" is un-necessary. There are however excellent reasons to expect that there is an infinite gauge degeneracy assignable to quantum criticality and represented in terms of Kac-Moody type transformations of partonic orbits respecting their light-likeness and giving rise to the degeneracy behind hierarchy of Planck constants $h_{eff} = n \times h$. n would give the number of conformal equivalence classes of space-time surfaces with same ends. In given measurement resolution one might however hope that the "preferred" could be dropped away.

   The already mentioned vanishing of Noether charges for sub-algebras of conformal algebras with conformal weights coming as multiples of n at the ends of space-time surface would be a concrete realization of this picture.

3. The construction of quantum TGD in terms of the Kähler- Dirac action associated with Kähler action led to a possible answer to the question about the principle selecting preferred extremals. The Noether currents associated with Kähler-Dirac action are conserved if second variations of Kähler action vanish. This is nothing but space-time correlate for quantum criticality and it is amusing that I failed to realize this for so long time. A further very important result is that in generic case the modes of induced spinor field are localized at 2-D surfaces from the condition that em charge is well-defined quantum number (W fields must vanish and also Z^0 field above weak scale in order to avoid large parity breaking effects). The criticality conditions are however rather complicated and it seems that the vanishing of the symplectic Noether charges is the practical way to formulate what "preferred" does mean.

In this chapter I will first consider the basic properties of the WCW, briefly discuss the various approaches to the geometrization of the WCW, and introduce the alternative strategies for the construction of Kähler metric based on a direct guess of Kähler function, on the group theoretical approach assuming that WCW can be regarded as a union of symmetric spaces, and on the identification of Kähler metric as anti-commutators of gamma matrices identified as Noether super charges for the symplectic algebra. After these preliminaries a definition of the Kähler function is proposed and various physical and mathematical motivations behind the proposed definition are discussed. The key feature of the Kähler action is classical non-determinism, and various implications of the classical non-determinism are discussed.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L1].

2 WCW

The view about configuration space ("world of classical worlds", WCW) has developed considerably during the last two decades. Here only the recent view is summarized in order to not load reader with unessential details.

2.1 Basic Notions

The notions of embedding space, 3-surface (and 4-surface), and WCW or "world of classical worlds" (WCW), are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$ (see **Figs.** `http://tgdtheory.fi/appfigures/Hoo.jpg`, `http://tgdtheory.fi/appfigures/cp2.jpg`, `http://tgdtheory.fi/appfigures/Hoo.futurepast`, `http://tgdtheory.fi/appfigures/penrose.jpg`, which are also in the appendix of this book), and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize GCI. During years these notions have however evolved considerably.

2.1.1 The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K11, K12, K10].

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book. As matter fact, this gluing idea generalizes to the level of WCW .

2. With the discovery of zero energy ontology [K14, K2] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K13] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K4] led to a further generalization of the notion of embedding space. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and possibly also factor spaces of CD and CP_2 to form a book like structure. There are good physical and mathematical arguments suggesting that only the singular coverings should be allowed [K10]. The particles at different pages of

this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW .

2.1.2 The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial and the receont view is an outcome of a long and tedious process involving many hastily done mis-interpretations.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence implied by GCI. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the GCI in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. Light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces. Therefore it seems that one must choose between light-like and space-like 3-surfaces or assume generalized GCI requiring that equivalently either space-like 3-surfaces or light-like 3-surfaces at the ends of CDs can be identified as the fundamental geometric objects. General GCI requires that the basic objects correspond to the partonic 2-surfaces identified as intersections of these 3-surfaces plus common 4-D tangent space distribution.

   At the level of WCW metric this suggests that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. Since the information about normal space of the 2-surface is needed one has only effective 2-dimensionality. Weak form of self-duality [K3] however implies that the normal data (flux Hamiltonians associated with Kähler electric field) reduces to magnetic flux Hamiltonians. This is essential for conformal symmetries and also simplifies the construction enormously.

   It however turned out that this picture is too simplistic. It turned out that the solutions of the Kähler-Dirac equation are localized at 2-D string world sheets, and this led to a generalization of the formulation of WCW geometry: given point of partonic 2-surface is effectively replaced with a string emanating from it and connecting it to another partonic 2-surface. Hence the formulation becomes 3-dimensional but thanks to super-conformal symmetries acting like gauge symmetries one obtains effective 2-dimensionality albeit in weaker sense [K9].

3. At some stage came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

4. A further but inessential complication relates to the hierarchy of Planck constants forcing to generalize the notion of embedding space and also to the fact that for non-standard values of Planck constant there is symmetry breaking due to preferred plane M^2 preferred homologically trivial geodesic sphere of CP_2 having interpretation as geometric correlate for the selection of quantization axis. For given sector of CH this means union over choices of this kind.

The basic vision forced by the generalization of GCI has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits.

Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

The study of the Kähler-Dirac equation led to the realization that classical field equations for Kähler action can be seen as consistency conditions for the Kähler-Dirac action and led to the identification of preferred extremals in terms of criticality. This identification which follows naturally also from quantum criticality.

1. The condition that electromagnetic charge is well-defined for the modes of Kähler-Dirac operator implies that in the generic case the modes are restricted to 2-D surfaces (string world sheets or possibly also partonic 2-surfaces) with vanishing W fields [K14]. Above weak scale at least one can also assume that Z^0 field vanishes. Also for space-time surfaces with 2-D CP_2 projection (cosmic strongs would be examples) the localization is expected to be possible. This localization is possible only for Kähler action and the set of these 2-surfaces is discrete except for the latter case. The stringy form of conformal invariance allows to solve Kähler-Dirac equation just like in string models and the solutions are labelled by integer valued conformal weights.

2. The next step of progress was the realization that the requirement that the conservation of the Noether currents associated with the Kähler-Dirac equation requires that the second variation of the Kähler action vanishes. In strongest form this condition would be satisfied for all variations and in weak sense only for those defining dynamical symmetries. The interpretation is as a space-time correlate for quantum criticality and the vacuum degeneracy of Kähler action makes the criticality plausible.

   The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig. ??** in the appendix of this book).

   Weak form of electric-magnetic duality gives a precise formulation for how Kähler coupling strength is visible in the properties of preferred extremals. A generalization of the ideas of the catastrophe theory to infinite-dimensional context results. These conditions make sense also in p-adic context and have a number theoretical universal form.

The notion of number theoretical compactication led to important progress in the understanding of the preferred extremals and the conjectures were consistent with what is known about the known extremals.

1. The conclusion was that one can assign to the 4-D tangent space $T(X^4(X_l^3)) \subset M^8$ a subspace $M^2(x) \subset M^4$ having interpretation as the plane of non-physical polarizations. This in the case that the induced metric has Minkowskian signature. If not, and if co-hyper-quaternionic surface is in question, similar assigned should be possible in normal space. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW . This is as it must be since complexification does not make sense in M^2 degrees of freedom.

2. In number theoretical framework $M^2(x)$ has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. The condition $M^2(x) \subset T(X^4(X_l^3)))$ in principle fixes the tangent space at X_l^3, and one has good hopes that the boundary value problem is well-defined and could fix $X^4(X^3)$ at least partially as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2(x) \subset M^4$ plays also other important roles.

3. At the level of H the counterpart for the choice of $M^2(x)$ seems to be following. Suppose that $X^4(X_l^3)$ has Minkowskian signature. One can assign to each point of the M^4 projection $P_{M^4}(X^4(X_l^3))$ a sub-space $M^2(x) \subset M^4$ and its complement $E^2(x)$, and the distributions of these planes are integrable and define what I have called Hamilton-Jacobi coordinates

which can be assigned to the known extremals of Kähler with Minkowskian signature. This decomposition allows to slice space-time surfaces by string world sheets and their 2-D partonic duals. Also a slicing to 1-D light-like surfaces and their 3-D light-like duals Y_l^3 parallel to X_l^3 follows under certain conditions on the induced metric of $X^4(X_l^3)$. This decomposition exists for known extremals and has played key role in the recent developments. Physically it means that 4-surface (3-surface) reduces effectively to 3-D (2-D) surface and thus holography at space-time level. A physically attractive realization of the slicings of space-time surface by 3-surfaces and string world sheets is discussed in [K6] by starting from the observation that TGD could define a natural realization of braids, braid cobordisms, and 2-knots.

4. The weakest form of number theoretic compactification [K12] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

If one takes M^-H duality seriously, one must conclude that one can choose any partonic 2-surface in the slicing of X^4 as a representative. This means gauge invariance reflect in the definition of Kähler function as $U(1)$ gauge transformation $K \rightarrow K + f + \overline{f}$ having no effect on Kähler metric and Kähler form.

Although the details of this vision might change it can be defended by its ability to fuse together all great visions about quantum TGD. In the sequel the considerations are restricted to 3-surfaces in $M^4_\pm \times CP_2$. The basic outcome is that Kähler metric is expressible using the data at partonic 2-surfaces $X^2 \subset \delta M^4_+ \times CP_2$. The generalization to the actual physical situation requires the replacement of $X^2 \subset \delta M^4_+ \times CP_2$ with unions of partonic 2-surfaces located at light-like boundaries of CDs and sub-CDs.

The notions of space-time sheet and many-sheeted space-time are basic pieces of TGD inspired phenomenology (see **Fig. ??** in the appendix of this book). Originally the space-time sheet was understood to have a boundary as "sheet" strongly suggests. It has however become clear that genuine boundaries are not allowed. Rather, space-time sheet is typically double (at least) covering of M^4. The light-like 3-surfaces separating space-time regions with Euclidian and Minkowskian signature are however very much like boundaries and define what I call generalized Feynman diagrams. A fascinating possibility is that every material object is accompanied by an Euclidian region representing the interior of the object and serving as TGD analog for blackhole like object. Space-time sheets suffer topological condensation (gluing by wormhole contacts or topological sum in more mathematical jargon) at larger space-time sheets. Space-time sheets form a length scale hierarchy. Quantitative formulation is in terms of p-adic length scale hypothesis and hierarchy of Planck constants proposed to explain dark matter as phases of ordinary matter.

2.1.3 The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the basis question is "M^4_+ or M^4?" and that this question had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world

of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW .

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW . The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW (WCW) is a union of WCW s associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. It must be however emphasized that Kähler function depends on partonic 2-surfaces at both ends of space-time surface so that WCW is topologically Cartesian product of corresponding symmetric spaces. WCW metric must therefore have parts corresponding to the partonic 2-surfaces (free part) and also an interaction term depending on the partonic 2-surface at the opposite ends of the light-like 3-surface. The conclusion is that geometrization reduces to that for single like of generalized Feynman diagram containing partonic 2-surfaces at its ends. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case corresponding to a line of generalized Feynman diagram. One can also deduce the free part of the metric by restricting the consideration to partonic 2-surfaces at single end of generalized Feynman diagram.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.

2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub- WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras (symmetries respecting light-likeness of light-like 3-surfaces). WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with Kac-Moody group. Equivalently, the local coset space $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!

2.2 Constraints On WCW Geometry

The constraints on the WCW result both from the infinite dimension of WCW and from physically motivated symmetry requirements. There are three basic physical requirements on the WCW geometry: namely four-dimensional GCI in strong form, Kähler property and the decomposition of WCW into a union $\cup_i G/H_i$ of symmetric spaces G/H_i, each coset space allowing G-invariant metric such that G is subgroup of some "universal group" having natural action on 3-surfaces. Together with the infinite dimensionality of WCW these requirements pose extremely strong constraints on WCW geometry. In the following we shall consider these requirements in more detail.

2.2.1 Diff^4 invariance and Diff^4 degeneracy

Diff^4 plays fundamental role as the gauge group of General Relativity. In string models $Diff^2$ invariance ($Diff^2$ acts on the orbit of the string) plays central role in making possible the elimination of the time like and longitudinal vibrational degrees of freedom of string. Also in the present case the elimination of the tachyons (time like oscillatory modes of 3-surface) is a physical necessity and Diff^4 invariance provides an obvious manner to do the job.

In the standard path l integral formulation the realization of Diff^4 invariance is an easy task at the formal level. The problem is however that path integral over four-surfaces is plagued by divergences and doesn't make sense. In the present case WCW consists of 3-surfaces and only $Diff^3$ emerges automatically as the group of re-parameterizations of 3-surface. Obviously one should somehow define the action of Diff^4 in the space of 3-surfaces. Whatever the action of Diff^4 is it must leave the WCW metric invariant. Furthermore, the elimination of tachyons is expected to be possible only provided the time like deformations of the 3-surface correspond to zero norm vector fields of WCW so that 3-surface and its Diff^4 image have zero distance. The conclusion is that WCW metric should be both Diff^4 invariant and Diff^4 degenerate.

The problem is how to define the action of Diff^4 in $C(H)$. Obviously the only manner to achieve Diff^4 invariance is to require that the very definition of the WCW metric somehow associates a unique space time surface to a given 3-surface for Diff^4 to act on. The obvious physical interpretation of this space time surface is as "classical space time" so that "Classical Physics" would be contained in WCW geometry. In fact, this space-time surface is analogous to Bohr orbit so that semiclassical quantization rules become an exact part of the quantum theory. It is this requirement, which has turned out to be decisive concerning the understanding of the WCW geometry.

2.2.2 Decomposition of WCW into a union of symmetric spaces G/H

The extremely beautiful theory of finite-dimensional symmetric spaces constructed by Elie Cartan suggests that WCW should possess decomposition into a union of coset spaces $CH = \cup_i G/H_i$ such that the metric inside each coset space G/H_i is left invariant under the infinite dimensional isometry group G. The metric equivalence of surfaces inside each coset space G/H_i does not mean that 3-surfaces inside G/H_i are physically equivalent. The reason is that the vacuum functional is exponent of Kähler action which is not isometry invariant so that the 3-surfaces, which correspond to maxima of Kähler function for a given orbit, are in a preferred position physically. For instance, one can imagine of calculating functional integral around this maximum perturbatively. Symmetric space property actually allows also much more powerful non-perturbative approach based on harmonic analysis [K14]. The sum of over i means actually integration over the zero modes of the metric (zero modes correspond to coordinates not appearing as coordinate differentials in the metric tensor).

The coset space G/H is a symmetric space only under very special Lie-algebraic conditions. Denoting the decomposition of the Lie-algebra g of G to the direct sum of H Lie-algebra h and its complement t by $g = h \oplus t$, one has

$$[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ .$$

This decomposition turn out to play crucial role in guaranteeing that G indeed acts as isometries and that the metric is Ricci flat.

The four-dimensional $Diff$ invariance indeed suggests to a beautiful solution of the problem of identifying G. The point is that any 3-surface X^3 is $Diff^4$ equivalent to the intersection of $X^4(X^3)$ with the light cone boundary. This in turn implies that 3-surfaces in the space $\delta H = \delta M^4_+ \times CP_2$ should be all what is needed to construct WCW geometry. The group G can be identified as some subgroup of diffeomorphisms of δH and H_i contains that subgroup of G, which acts as diffeomorphisms of the 3-surface X^3. Since G preserves topology, WCW must decompose into union $\cup_i G/H_i$, where i labels 3-topologies and various zero modes of the metric. For instance, the elements of the Lie-algebra of G invariant under WCW complexification correspond to zero modes.

The reduction to the light cone boundary, identifiable as the moment of big bang, looks perhaps odd at first. In fact, it turns out that the classical non-determinism of Kähler action does not allow the complete reduction to the light cone boundary: physically this is a highly desirable implication but means a considerable mathematical challenge.

2.2.3 Kähler property

Kähler property implies that the tangent space of the configuration space allows complexification and that thereexists a covariantly constant two-form J_{kl}, which can be regarded as a representation of the imaginary unit in the tangent space of the WCW :

$$J_k^{\ r} J_{rl} = -G_{kl} \quad . \tag{2.1}$$

There are several physical and mathematical reasons suggesting that WCW metric should possess Kähler property in some generalized sense.

1. The deepest motivation comes from the need to geometrize hermitian conjugation which is basic mathematical operation of quantum theory.

2. Kähler property turns out to be a necessary prerequisite for defining divergence free WCW integration. We will leave the demonstration of this fact later although the argument as such is completely general.

3. Kähler property very probably implies an infinite-dimensional isometry loop groups $Map(S^1, G)$ [A1] shows that loop group allows only

   Riemann connection and this metric allows local G as its isometries!

   To see this consider the construction of Riemannian connection for $Map(X^3, H)$. The defining formula for the connection is given by the expression

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y,Z) + Y(Z,X) - Z(X,Y) \\ &+ ([X,Y],Z) + ([Z,X],Y) - ([Y,Z],X) \end{aligned} \tag{2.2}$$

   X, Y, Z are smooth vector fields in $Map(X^3, G)$. This formula defines $\nabla_X Y$ uniquely provided the tangent space of Map is complete with respect to Riemann metric. In the finite-dimensional case completeness means that the inverse of the covariant metric tensor exists so that one can solve the components of connection from the conditions stating the covariant constancy of the metric. In the case of the loop spaces with Kähler metric this is however not the case.

   Now the symmetry comes into the game: if X, Y, Z are left (local gauge) invariant vector fields defined by the Lie-algebra of local G then the first three terms drop away since the scalar products of left invariant vector fields are constants. The expression for the covariant derivative is given by

$$\nabla_X Y = (Ad_X Y - Ad^*_X Y - Ad^*_Y X)/2 \tag{2.3}$$

   where Ad^*_X is the adjoint of Ad_X with respect to the metric of the loop space.

   At this point it is important to realize that Freed's argument does not force the isometry group of WCW to be $Map(X^3, M^4 \times SU(3))$! Any symmetry group, whose Lie algebra is complete with respect to the WCW metric (in the sense that any tangent space vector is expressible as superposition of isometry generators modulo a zero norm tangent vector) is an acceptable alternative.

   The Kähler property of the metric is quite essential in one-dimensional case in that it leads to the requirement of left invariance as a mathematical consistency condition and we expect that dimension three makes no exception in this respect. In 3-dimensional case the degeneracy of the metric turns out to be even larger than in 1-dimensional case due to the four-dimensional Diff degeneracy. So we expect that the metric ought to possess some infinite-dimensional isometry group and that the above formula generalizes also to the 3-dimensional case and to the case of local coset space. Note that in M^4 degrees of freedom $Map(X^3, M^4)$ invariance would imply the flatness of the metric in M^4 degrees of freedom.

The physical implications of the above purely mathematical conjecture should not be underestimated. For example, one natural looking manner to construct physical theory would be based on the idea that configuration space geometry is dynamical and this approach is followed in the attempts to construct string theories [B1]. Various physical considerations (in particular the need to obtain oscillator operator algebra) seem to imply that WCW geometry is necessarily Kähler. The above result however states that WCW Kähler geometry cannot be dynamical quantity and is dictated solely by the requirement of internal consistency. This result is extremely nice since it has been already found that the definition of the WCW metric must somehow associate a unique classical space time and "classical physics" to a given 3-surface: uniqueness of the geometry implies the uniqueness of the "classical physics".

4. The choice of the embedding space becomes highly unique. In fact, the requirement that WCW is not only symmetric space but also (contact) Kähler manifold inheriting its (degenerate) Kähler structure from the embedding space suggests that spaces, which are products of four-dimensional Minkowski space with complex projective spaces CP_n, are perhaps the only possible candidates for H. The reason for the unique position of the four-dimensional Minkowski space turns out to be that the boundary of the light cone of D-dimensional Minkowski space is metrically a sphere S^{D-2} despite its topological dimension $D-1$: for $D = 4$ one obtains two-sphere allowing Kähler structure and infinite parameter group of conformal symmetries!

5. It seems possible to understand the basic mathematical structures appearing in string model in terms of the Kähler geometry rather nicely.

   (a) The projective representations of the infinite-dimensional isometry group (not necessarily Map!) correspond to the ordinary representations of the corresponding centrally extended group [A4]. The representations of Kac Moody group indeed play central role in string models [B5, B3] and WCW approach would explain their occurrence, not as a result of some quantization procedure, but as a consequence of symmetry of the underlying geometric structure.

   (b) The bosonic oscillator operators of string models would correspond to centrally extended Lie-algebra generators of the isometry group acting on spinor fields of the WCW .

   (c) The "fermionic" fields (Ramond fields, Schwartz, Green) should correspond to gamma matrices of the WCW . Fermionic oscillator operators would correspond simply to contractions of isometry generators j_A^k with complexified gamma matrices of WCW

$$\begin{aligned} \Gamma_A^{\pm} &= j_A^k \Gamma_k^{\pm} \\ \Gamma_k^{\pm} &= (\Gamma^k \pm J_l^k \Gamma^l)/\sqrt{2} \end{aligned} \tag{2.4}$$

   (J_l^k is the Kähler form of WCW) and would create various spin excitations of WCW spinor field. $\Gamma_k^{\pm}$ are the complexified gamma matrices, complexification made possible by the Kähler structure of the WCW .

This suggests that some generalization of the so called Super Kac Moody algebra of string models [B5, B3] should be regarded as a spectrum generating algebra for the solutions of field equations in configuration space.

Although the Kähler structure seems to be physically well motivated there is a rather heavy counter argument against the whole idea. Kähler structure necessitates complex structure in the tangent space of WCW . In CP_2 degrees of freedom no obvious problems of principle are expected: WCW should inherit in some sense the complex structure of CP_2.

In Minkowski degrees of freedom the signature of the Minkowski metric seems to pose a serious obstacle for complexification: somehow one should get rid of two degrees of freedom so that only two Euclidian degrees of freedom remain. An analogous difficulty is encountered in quantum field theories: only two of the four possible polarizations of gauge boson correspond to physical degrees of freedom: mathematically the wrong polarizations correspond to zero norm states and transverse states span a complex Hilbert space with Euclidian metric. Also in string model analogous situation occurs: in case of D-dimensional Minkowski space only $D-2$ transversal degrees of freedom are physical. The solution to the problem seems therefore obvious: WCW metric must be degenerate so that each vibrational mode spans effectively a 2-dimensional Euclidian plane allowing complexification.

We shall find that the definition of Kähler function to be proposed indeed provides a solution to this problem and also to the problems listed before.

1. The definition of the metric doesn't differentiate between 1- and N-particle sectors, avoids spin statistics difficulty and has the physically appealing property that one can associate to each 3-surface a unique classical space time: classical physics is described by the geometry of WCW and d the geometry of WCW is determined uniquely by the requirement of mathematical consistency.

2. Complexification is possible only provided the dimension of the Minkowski space equals to four and is due to the effective 3-dimensionality of light-cone boundary.

3. It is possible to identify a unique candidate for the necessary infinite-dimensional isometry group G. G is subgroup of the diffeomorphism group of $\delta M^4_+ \times CP_2$. Essential role is played by the fact that the boundary of the four-dimensional light cone, which, despite being topologically 3-dimensional, is metrically two-dimensional Euclidian sphere, and therefore allows infinite-parameter groups of isometries as well as conformal and symplectic symmetries and also Kähler structure unlike the higher-dimensional light cone boundaries. Therefore WCW metric is Kähler only in the case of four-dimensional Minkowski space and allows symplectic $U(1)$ central extension without conflict with the no-go theorems about higher dimensional central extensions.

   The study of the vacuum degeneracy of Kähler function defined by Kähler action forces to conclude that the isometry group must consist of the symplectic transformations of $\delta H = \delta M^4_+ \times CP_2$. The corresponding Lie algebra can be regarded as a loop algebra associated with the symplectic group of $S^2 \times CP_2$, where S^2 is $r_M = constant$ sphere of light cone boundary. Thus the finite-dimensional group G defining loop group in case of string models extends to an infinite-dimensional group in TGD context. This group has a monstrous size. The radial Virasoro localized with respect to $S^2 \times CP_2$ defines naturally complexification for both G and H. The general form of the Kähler metric deduced on basis of this symmetry has same qualitative properties as that deduced from Kähler function identified as preferred extremal of Kähler action. Also the zero modes, among them isometry invariants, can be identified.

4. The construction of the WCW spinor structure is based on the identification of the WCW gamma matrices as linear superpositions of the oscillator operators associated with the second quantized induced spinor fields. The extension of the symplectic invariance to super symplectic invariance fixes the anti-commutation relations of the induced spinor fields, and WCW gamma matrices correspond directly to the super generators. Physics as number theory vision suggests strongly that WCW geometry exists for 8-dimensional embedding space only and that the choice $M^4_+ \times CP_2$ for the embedding space is the only possible one.

3 Identification Of The Kähler Function

There are three approaches to the construction of the WCW geometry: a direct physics based guess of the Kähler function, a group theoretic approach based on the hypothesis that CH can be regarded as a union of symmetric spaces, and the approach based on the construction of WCW spinor structure first by second quantization of induced spinor fields. Here the first approach is discussed.

3.1 Definition Of Kähler Function

Consider first the basic definitions related to Kähler metric and Kähler function.

3.1.1 Kähler metric in terms of Kähler function

Quite generally, Kähler function K defines Kähler metric in complex coordinates via the following formula

$$J_{k\bar{l}} = ig_{k\bar{l}} = i\partial_k\partial_{\bar{l}}K \ . \qquad (3.1)$$

Kähler function is defined only modulo a real part of holomorphic function so that one has the gauge symmetry

$$K \rightarrow K + f + \overline{f} \ . \qquad (3.2)$$

Let X^3 be a given 3-surface and let X^4 be any four-surface containing X^3 as a sub-manifold: $X^4 \supset X^3$. The 4-surface X^4 possesses in general boundary. If the 3-surface X^3 has nonempty boundary δX^3 then the boundary of X^3 belongs to the boundary of X^4: $\delta X^3 \subset \delta X^4$.

3.1.2 Induced Kähler form and its physical interpretation

Induced Kähler form defines a Maxwell field and it is important to characterize precisely its relationship to the gauge fields as they are defined in gauge theories. Kähler form J is related to the corresponding Maxwell field F via the formula

$$J = xF \ , \ x = \frac{g_K}{\hbar} \ . \qquad (3.3)$$

Similar relationship holds true also for the other induced gauge fields. The inverse proportionality of J to $\hbar$ does not matter in the ordinary gauge theory context where one routinely choses units by putting $\hbar = 1$ but becomes very important when one considers a hierarchy of Planck constants [K4].

Unless one has $J = (g_K/\hbar_0)$, where $\hbar_0$ corresponds to the ordinary value of Planck constant, $\alpha_K = g_K^2/4\pi\hbar$ together the large Planck constant means weaker interactions and convergence of the functional integral defined by the exponent of Kähler function and one can argue that the convergence of the functional integral is what forces the hierarchy of Planck constants. This is in accordance with the vision that Mother Nature likes theoreticians and takes care that the perturbation theory works by making a phase transition increasing the value of the Planck constant in the situation when perturbation theory fails. This leads to a replacement of the M^4 (or more precisely, causal diamond CD) and CP_2 factors of the embedding space ($CD \times CP_2$) with its $r = \hbar/\hbar_0$-fold singular covering (one can consider also singular factor spaces). If the components of the space-time surfaces at the sheets of the covering are identical, one can interpret r-fold value of Kähler action as a sum of r identical contributions from the sheets of the covering with ordinary value of Planck constant and forget the presence of the covering. Physical states are however different even in the case that one assumes that sheets carry identical quantum states and anyonic phase could correspond to this kind of phase [K8].

3.1.3 Kähler action

One can associate to Kähler form Maxwell action and also Chern-Simons anomaly term proportional to $\int_{X^4} J \wedge J$ in well known manner. Chern Simons term is purely topological term and well defined for orientable 4-manifolds, only. Since there is no deep reason for excluding non-orientable space-time surfaces it seems reasonable to drop Chern Simons term from consideration. Therefore Kähler action $S_K(X^4)$ can be defined as

$$S_K(X^4) = k_1 \int_{X^4;X^3 \subset X^4} J \wedge (*J) \ . \qquad (3.4)$$

The sign of the square root of the metric determinant, appearing implicitly in the formula, is defined in such a way that the action density is negative for the Euclidian signature of the induced metric and such that for a Minkowskian signature of the induced metric Kähler electric field gives a negative contribution to the action density.

The notational convention

$$k_1 \equiv \frac{1}{16\pi\alpha_K} , \tag{3.5}$$

where α_K will be referred as Kähler coupling strength will be used in the sequel. If the preferred extremals minimize/maximize [K12] the absolute value of the action in each region where action density has a definite sign, the value of α_K can depend on space-time sheet.

3.1.4 Kähler function

One can define the Kähler function in the following manner. Consider first the case $H = M^4_+ \times CP_2$ and neglect for a moment the non-determinism of Kähler action. Let X^3 be a 3-surface at the light-cone boundary $\delta M^4_+ \times CP_2$. Define the value $K(X^3)$ of Kähler function K as the value of the Kähler action for some preferred extremal in the set of four-surfaces containing X^3 as a sub-manifold:

$$K(X^3) = K(X^4_{pref}) , \quad X^4_{pref} \subset \{X^4 | X^3 \subset X^4\} . \tag{3.6}$$

The most plausible identification of preferred extremals is in terms of quantum criticality in the sense that the preferred extremals allow an infinite number of deformations for which the second variation of Kähler action vanishes. Combined with the weak form of electric-magnetic duality forcing appearance of Kähler coupling strength in the boundary conditions at partonic 2-surfaces this condition might be enough to fix preferred extremals completely.

The precise formulation of Quantum TGD has developed rather slowly. Only quite recently- 33 years after the birth of TGD - I have been forced to reconsider the question whether the precise identification of Kähler function. Should Kähler function actually correspond to the Kähler action for the space-time regions with Euclidian signature having interpretation as generalized Feynman graphs? If so what would be the interpretation for the Minkowskian contribution?

1. If one accepts just the formal definition for the square root of the metric determinant, Minkowskian regions would naturally give an imaginary contribution to the exponent defining the vacuum functional. The presence of the phase factor would give a close connection with the path integral approach of quantum field theories and the exponent of Kähler function would make the functional integral well-defined.

2. The weak form of electric magnetic duality would reduce the contributions to Chern-Simons terms from opposite sides of wormhole throats with degenerate four-metric with a constraint term guaranteeing the duality.

The motivation for this reconsideration came from the applications of ideas of Floer homology to TGD framework [K7]: the Minkowskian contribution to Kähler action for preferred extremals would define Morse function providing information about WCW homology. Both Kähler and Morse would find place in TGD based world order.

One of the nasty questions about the interpretation of Kähler action relates to the square root of the metric determinant. If one proceeds completely straightforwardly, the only reason conclusion is that the square root is imaginary in Minkowskian space-time regions so that Kähler action would be complex. The Euclidian contribution would have a natural interpretation as positive definite Kähler function but how should one interpret the imaginary Minkowskian contribution? Certainly the path integral approach to quantum field theories supports its presence. For some mysterious reason I was able to forget this nasty question and serious consideration of the obvious answer to it. Only when I worked between possibile connections between TGD and Floer homology [K7] I realized that the Minkowskian contribution is an excellent candidate for Morse function whose critical points give information about WCW homology. This would fit nicely with the vision about TGD as almost topological QFT.

Euclidian regions would guarantee the convergence of the functional integral and one would have a mathematically well-defined theory. Minkowskian contribution would give the quantal interference effects and stationary phase approximation. The analog of Floer homology would represent quantum superpositions of critical points identifiable as ground states defined by the extrema of Kähler action for Minkowskian regions. Perturbative approach to quantum TGD would rely on functional integrals around the extrema of Kähler function. One would have maxima also for the Kähler function but only in the zero modes not contributing to the WCW metric.

There is a further question related to almost topological QFT character of TGD. Should one assume that the reduction to Chern-Simons terms occurs for the preferred extremals in *both* Minkowskian and Euclidian regions or only in Minkowskian regions?

1. All arguments for this have been represented for Minkowskian regions [K14] involve local light-like momentum direction which does not make sense in the Euclidian regions. This does not however kill the argument: one can have non-trivial solutions of Laplacian equation in the region of CP_2 bounded by wormhole throats: for CP_2 itself only covariantly constant right-handed neutrino represents this kind of solution and at the same time supersymmetry. In the general case solutions of Laplacian represent broken super-symmetries and should be in one-one correspondences with the solutions of the Kähler-Dirac equation. The interpretation for the counterparts of momentum and polarization would be in terms of classical representation of color quantum numbers.

2. If the reduction occurs in Euclidian regions, it gives in the case of CP_2 two 3-D terms corresponding to two 3-D gluing regions for three coordinate patches needed to define coordinates and spinor connection for CP_2 so that one would have two Chern-Simons terms. I have earlier claimed that without any other contributions the first term would be identical with that from Minkowskian region apart from imaginary unit and different coefficient. This statement is wrong since the space-like parts of the corresponding 3-surfaces are discjoint for Euclidian and Minkowskian regions.

3. There is also an argument stating that Dirac determinant for Chern-Simons Dirac action equals to Kähler function, which would be lost if Euclidian regions would not obey holography. The argument obviously generalizes and applies to both Morse and Kähler function which are definitely not proportional to each other.

3.1.5 CP breaking and ground state degeneracy

The Minkowskian contribution of Kähler action is imaginary due to the negativity of the metric determinant and gives a phase factor to vacuum functional reducing to Chern-Simons terms at wormhole throats. Ground state degeneracy due to the possibility of having both signs for Minkowskian contribution to the exponent of vacuum functional provides a general view about the description of CP breaking in TGD framework.

1. In TGD framework path integral is replaced by inner product involving integral over WCV. The vacuum functional and its conjugate are associated with the states in the inner product so that the phases of vacuum functionals cancel if only one sign for the phase is allowed. Minkowskian contribution would have no physical significance. This of course cannot be the case. The ground state is actually degenerate corresponding to the phase factor and its complex conjugate since $\sqrt{g}$ can have two signs in Minkowskian regions. Therefore the inner products between states associated with the two ground states define 2×2 matrix and non-diagonal elements contain interference terms due to the presence of the phase factor. At the limit of full CP_2 type vacuum extremal the two ground states would reduce to each other and the determinant of the matrix would vanish.

2. A small mixing of the two ground states would give rise to CP breaking and the first principle description of CP breaking in systems like $K - \overline{K}$ and of CKM matrix should reduce to this mixing. K^0 mesons would be CP even and odd states in the first approximation and correspond to the sum and difference of the ground states. Small mixing would be present having exponential sensitivity to the actions of CP_2 type extremals representing wormhole

throats. This might allow to understand qualitatively why the mixing is about 50 times larger than expected for B^0 mesons.

3. There is a strong temptation to assign the two ground states with two possible arrows of geometric time. At the level of M-matrix the two arrows would correspond to state preparation at either upper or lower boundary of CD. Do long- and shortlived neutral K mesons correspond to almost fifty-fifty orthogonal superpositions for the two arrow of geometric time or almost completely to a fixed arrow of time induced by environment? Is the dominant part of the arrow same for both or is it opposite for long and short-lived neutral measons? Different lifetimes would suggest that the arrow must be the same and apart from small leakage that induced by environment. CP breaking would be induced by the fact that CP is performed only K^0 but not for the environment in the construction of states. One can probably imagine also alternative interpretations.

3.2 The Values Of The Kähler Coupling Strength?

Since the vacuum functional of the theory turns out to be essentially the exponent $exp(K)$ of the Kähler function, the dynamics depends on the normalization of the Kähler function. Since the Theory of Everything should be unique it would be highly desirable to find arguments fixing the normalization or equivalently the possible values of the Kähler coupling strength α_K.

3.2.1 Quantization of α_K follow from Dirac quantization in WCW?

The quantization of Kähler form of WCW could result in the following manner. It will be found that Abelian extension of the isometry group results by coupling spinors of WCW to a multiple of Kähler potential. This means that Kähler potential plays role of gauge connection so that Kähler form must be integer valued by Dirac quantization condition for magnetic charge. So, if Kähler form is co-homologically nontrivial the value of α_K is quantized.

3.2.2 Quantization from criticality of TGD Universe?

Mathematically α_K is analogous to temperature and this suggests that α_K is analogous to critical temperature and therefore quantized. This analogy suggests also a physical motivation for the unique value or value spectrum of α_K. Below the critical temperature critical systems suffer something analogous to spontaneous magnetization. At the critical point critical systems are characterized by long range correlations and arbitrarily large volumes of magnetized and non-magnetized phases are present. Spontaneous magnetization might correspond to the generation of Kähler magnetic fields: the most probable 3-surfaces are Kähler magnetized for subcritical values of α_K. At the critical values of α_K the most probable 3-surfaces contain regions dominated by either Kähler electric and or Kähler magnetic fields: by the compactness of CP_2 these regions have in general outer boundaries.

This suggests that 3-space has hierarchical, fractal like structure: 3-surfaces with all sizes (and with outer boundaries) are possible and they have suffered topological condensation on each other. Therefore the critical value of α_K allows the richest possible topological structure for the most probable 3-space. In fact, this hierarchical structure is in accordance with the basic ideas about renormalization group invariance. This hypothesis has highly nontrivial consequences even at the level of ordinary condensed matter physics.

Unfortunately, the exact definition of renormalization group concept is not at all obvious. There is however a much more general but more or less equivalent manner to formulate the condition fixing the value of α_K. Vacuum functional $exp(K)$ is analogous to the exponent $exp(-H/T)$ appearing in the definition of the partition function of a statistical system and S-matrix elements and other interesting physical quantities are integrals of type $\langle O\rangle = \int exp(K)O\sqrt{G}dV$ and therefore analogous to the thermal averages of various observables. α_K is completely analogous to temperature. The critical points of a statistical system correspond to critical temperatures T_c for which the partition function is non-analytic function of $T-T_c$ and according RGE hypothesis critical systems correspond to fixed points of renormalization group evolution. Therefore, a mathematically more precise manner to fix the value of α_K is to require that some integrals of type $\langle O\rangle$ (not necessary S-matrix elements) become non-analytic at $1/\alpha_K - 1/\alpha_K^c$.

Renormalization group invariance is closely related with criticality. The self duality of the Kähler form and Weyl tensor of CP_2 indeed suggest RG invariance. The point is that in $N = 1$ super-symmetric field theories duality transformation relates the strong coupling limit for ordinary particles with the weak coupling limit for magnetic monopoles and vice versa. If the theory is self dual these limits must be identical so that action and coupling strength must be RG invariant quantities. The geometric realization of the duality transformation is easy to guess in the standard complex coordinates ξ_1, ξ_2 of CP_2 (see Appendix of the book). In these coordinates the metric and Kähler form are invariant under the permutation $\xi_1 \leftrightarrow \xi_2$ having Jacobian -1.

Consistency requires that the fundamental particles of the theory are equivalent with magnetic monopoles. The deformations of so called CP_2 type vacuum extremals indeed serve as building bricks of a elementary particles. The vacuum extremals are are isometric embeddings of CP_2 and can be regarded as monopoles. Elementary particle corresponds to a pair of wormhole contacts and monopole flux runs between the throats of of the two contacts at the two space-time sheets and through the contacts between space-time sheets. The magnetic flux however flows in internal degrees of freedom (possible by nontrivial homology of CP_2) so that no long range $1/r^2$ magnetic field is created. The magnetic contribution to Kähler action is positive and this suggests that ordinary magnetic monopoles are not stable, since they do not minimize Kähler action: a cautious conclusion in accordance with the experimental evidence is that TGD does not predict magnetic monopoles. It must be emphasized that the prediction of monopoles of practically all gauge theories and string theories and follows from the existence of a conserved electromagnetic charge.

3.2.3 Does α_K have spectrum?

The assumption about single critical value of α_K is probably too strong.

1. The hierarchy of Planck constants which would result from non-determinism of Kähler action implying n conformal equivalences of space-time surface connecting 3-surfaces at the boundaries of causal diamond CD would predict effective spectrum of α_K as $\alpha_K = g_K^2/4\pi\hbar_{eff}$, $\hbar_{eff}/h = n$. The analogs of critical temperatures would have accumulation point at zero temperature.

2. p-Adic length scale hierarchy together with the immense vacuum degeneracy of the Kähler action leads to ask whether different p-adic length scales correspond to different critical values of α_K, and that ordinary coupling constant evolution is replaced by a piecewise constant evolution induced by that for α_K.

3.3 What Conditions Characterize The Preferred Extremals?

The basic vision forced by the generalization of General Coordinate Invariance has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action and are thus analogous to Bohr orbits. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

In positive energy ontology space-time surfaces should be analogous to Bohr orbits in order to make possible possible realization of general coordinate invariance. The first guess was that absolute minimization of Kähler action might be the principle selecting preferred extremals. One can criticize the assumption that extremals correspond to the absolute minima of Kähler action for entire space-time surface as too strong since the Kähler action from Minkowskian regions is proportional to imaginary unit and corresponds to ordinary QFT action defining a phase factor of vacuum functional. Absolute minimization could however make sense for Euclidian space-time regions defining the lines of generalized Feynman diagras, where Kähler action has definite sign. Kähler function is indeed the Kähler action for these regions. Furthermore, the notion of absolute minimization does not make sense in p-adic context unless one manages to reduce it to purely algebraic conditions.

3.3.1 Is preferred extremal property needed at all in ZEO?

It is good to start with a critical question. Could it be that the notion of preferred extremal might be un-necessary in ZEO (ZEO)? The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is unique.

Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and same Kähler action and same conserved quantities. One must be however cautious: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal gauge invariance is associated with 2-D criticality and is expected to be present also now. and this is the recent view.

One can of course ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated - this the starting point in ZEO. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations might be present and correspond to the Bohr orbit property, space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics. This indeed seems to be the correct conclusion.

3.3.2 How to identify preferred extremals?

What is needed is the association of a unique space-time surface to a given 3-surface defined as union of 3-surfaces at opposite boundaries of CD. One can imagine many ways to achieve this. "Unique" is too much to demand: for the proposal unique space-time surface is replaced with finite number of conformal gauge equivalence classes of space-time surfaces. In any case, it is better to talk just about preferred extremals of Kähler action and accept as the fact that there are several proposals for what this notion could mean.

1. For instance, one can consider the identification of space-time surface as associative (co-associative) sub-manifold meaning that tangent space of space-time surface can be regarded as associative (co-associative) sub-manifold of complexified octonions defining tangent space of embedding space. One manner to define "associative sub-manifold" is by introducing octonionic representation of embedding space gamma matrices identified as tangent space vectors. It must be also assumed that the tangent space contains a preferred commutative (co-commutative) sub-space at each point and defining an integrable distribution having identification as string world sheet (also slicing of space-time sheet by string world sheets can be considered). Associativity and commutativity would define the basic dynamical principle. A closely related approach is based on so called Hamilton-Jacobi structure [K1] defining also this kind of slicing and the approaches could be equivalent.

2. In ZEO 3-surfaces become pairs of space-like 3-surfaces at the boundaries of causal diamond (CD). Even the light-like partonic orbits could be included to give the analog of Wilson loop. In absence of non-determinism of Kähler action this forces to ask whether the attribute "preferred" is un-necessary. There are however excellent reasons to expect that there is an infinite gauge degeneracy assignable to quantum criticality and represented in terms of Kac-Moody type transformations of partonic orbits respecting their light-likeness and giving rise to the degeneracy behind hierarchy of Planck constants $h_{eff} = n \times h$. n would give the number of conformal equivalence classes of space-time surfaces with same ends. In given measurement resolution one might however hope that the "preferred" could be dropped away.

   The vanishing of Noether charges for sub-algebras of conformal algebras with conformal weights coming as multiples of n at the ends of space-time surface would be a concrete realization of this picture and looks the most feasible option at this moment since it is direct classical correlated for broken super-conformal gauge invariance at quantum level.

3. The construction of quantum TGD in terms of the Kähler-Dirac action associated with Kähler action suggested a possible answer to the question about the principle selecting preferred extremals. The Noether currents associated with Kähler-Dirac action are conserved if second variations of Kähler action vanish. This is nothing but space-time correlate for quantum criticality and it is amusing that I failed to realize this for so long time. A further very important result is that in generic case the modes of induced spinor field are localized at 2-D

surfaces from the condition that em charge is well-defined quantum number (W fields must vanish and also Z^0 field above weak scale in order to avoid large parity breaking effects).

The localization at string world sheets means that quantum criticality as definition of "preferred" works only if there selection of string world sheets, partonic 2-surfaces, and their light-like orbits fixes the space-time surface completely. The generalization of AdS/CFT correspondence (or strong form of holography) suggests that this is indeed the case. The criticality conditions are however rather complicated and it seems that the vanishing of the symplectic Noether charges is the practical manner to formulate what "preferred" does mean.

3.4 Why Non-Local Kähler Function?

Kähler function is non-local functional of 3-surface. Non-locality of the Kähler function seems to be at odds with basic assumptions of local quantum field theories. Why this rather radical departure from the basic assumptions of local quantum field theory? The answer is shortly given: WCW integration appears in the definition of the inner product for WCW spinor fields and this inner product must be free from perturbative divergences. Consider now the argument more closely.

In the case of finite-dimensional symmetric space with Kähler structure the representations of the isometry group necessitate the modification of the integration measure defining the inner product so that the integration measure becomes proportional to the exponent $exp(K)$ of the Kähler function [B2]. The generalization to infinite-dimensional case is obvious. Also the requirement of Kac-Moody symmetry leads to the presence of this kind of vacuum functional as will be found later. The exponent is in fact uniquely fixed by finiteness requirement. WCW integral is of the following form

$$\int \bar{S}_1 exp(K) S_1 \sqrt{g} dX \quad . \tag{3.7}$$

One can develop perturbation theory using local complex coordinates around a given 3-surface in the following manner. The $(1,1)$-part of the second variation of the Kähler function defines the metric and therefore propagator as contravariant metric and the remaining $(2,0)-$ and $(0,2)$-parts of the second variation are treated perturbatively. The most natural choice for the 3-surface are obviously the 3-surfaces, which correspond to extrema of the Kähler function.

When perturbation theory is developed around the 3-surface one obtains two ill-defined determinants.

1. The Gaussian determinant coming from the exponent, which is just the inverse square root for the matrix defined by the metric defining $(1,1)$-part of the second variation of the Kähler function in local coordinates.

2. The metric determinant. The matrix representing covariant metric is however same as the matrix appearing in Gaussian determinant by the defining property of the Kähler metric: in local complex coordinates the matrix defined by second derivatives is of type $(1,1)$. Therefore these two ill defined determinants (recall the presence of Diff degeneracy) cancel each other exactly for a unique choice of the vacuum functional!

Of course, the cancellation of the determinants is not enough. For an arbitrary local action one encounters the standard perturbative divergences. Since most local actions (Chern-Simons term is perhaps an exception [B7]) for induced geometric quantities are extremely nonlinear there is no hope of obtaining a finite theory. For non-local action the situation is however completely different. There are no local interaction vertices and therefore no products of delta functions in perturbation theory.

A further nice feature of the perturbation theory is that the propagator for small deformations is nothing but the contravariant metric of WCW . Also the various vertices of the theory are closely related to the metric of WCW since they are determined by the Kähler function so that perturbation theory would have a beautiful geometric interpretation. Furthermore, since four-dimensional Diff degeneracy implies that the propagator doesn't couple to un-physical modes.

It should be noticed that divergence cancellation arguments do not necessarily exclude Chern Simons term from vacuum functional defined as imaginary exponent of $exp(ik_2 \int_{X^4} J \wedge J)$. The term is not well defined for non-orientable space-time surfaces and one must assume that k_2 vanishes for these surfaces. The presence of this term might provide first principle explanation for CP breaking. If k_2 is integer multiple of $1/(8\pi)$ Chern Simons term gives trivial contribution for closed space-time surfaces since instanton number is in question. By adding a suitable boundary term of form $exp(ik_3 \int_{\delta X^3} J \wedge A)$ it is possible to guarantee that the exponent is integer valued for 4-surfaces with boundary, too.

There are two arguments suggesting that local Chern Simons term would not introduce divergences. First, 3-dimensional Chern Simons term for ordinary Abelian gauge field is known to define a divergence free field theory [B7]. The term doesn't depend at all on the induced metric and therefore contains no dimensional parameters (CP_2 radius) and its expansion in terms of CP_2 coordinate variables is of the form allowed by renormalizable field theory in the sense that only quartic terms appear. This is seen by noticing that there always exist symplectic coordinates, where the expression of the Kähler potential is of the form

$$A = \sum_k P_k dQ^k \ . \tag{3.8}$$

The expression for Chern-Simons term in these coordinates is given by

$$k_2 \int_{X^3} \sum_{k,l} P_l dP_k \wedge dQ^k \wedge dQ^l \ , \tag{3.9}$$

and clearly quartic CP_2 coordinates. A further nice property of the Chern Simons term is that this term is invariant under symplectic transformations of CP_2, which are realized as $U(1)$ gauge transformation for the Kähler potential.

The expressibility of WCW Kähler metric as anti-commutators of super-symplectic Noether super-charges localized at 2-D string world sheets inspires an even stronger conjecture about Kähler action. The super-symmetry between Kähler-Dirac action and Kähler action suggests that Kähler action is expressible as sum of string world sheet areas in the effective metric defined by the anti-commutators of K-D gamma matrices. This would conform with the strong form of holography in turn implies by strong form of General Coordinate Invariance, and could be seen as analog of AdS/CFT correspondence, which as such is not enough in TGD possessing super-conformal symmetries, which are gigantic as compared to those of super string models.

4 Some Properties Of Kähler Action

In this section some properties of Kähler action and Kähler function are discussed in light of experienced gained during about 15 years after the introduction of the notion.

4.1 Vacuum Degeneracy And Some Of Its Implications

The vacuum degeneracy is perhaps the most characteristic feature of the Kähler action. Although it is not associated with the preferred extremals of Kähler action, there are good reasons to expect that it has deep consequences concerning the structure of the theory.

4.1.1 Vacuum degeneracy of the Kähler action

The basic reason for choosing Kähler action is its enormous vacuum degeneracy, which makes long range interactions possible (the well known problem of the membrane theories is the absence of massless particles [B6]). The Kähler form of CP_2 defines symplectic structure and any 4-surface for which CP_2 projection is so called Lagrangian manifold (at most two dimensional manifold with vanishing induced Kähler form), is vacuum extremal due to the vanishing of the induced Kähler form. More explicitly, in the local coordinates, where the vector potential A associated with the Kähler form reads as $A = \sum_k P_k dQ^k$. Lagrangian manifolds are expressible locally in the following form

$$P_k = \partial_k f(Q^i) \ . \tag{4.1}$$

where the function f is arbitrary. Notice that for the general YM action surfaces with one-dimensional CP_2 projection are vacuum extremals but for Kähler action one obtains additional degeneracy.

There is also a second kind of vacuum degeneracy, which is relevant to the elementary particle physics. The so called CP_2 type vacuum extremals are warped embeddings X^4 of CP_2 to H such that Minkowski coordinates are functions of a single CP_2 coordinate, and the one-dimensional projection of X^4 is random light like curve. These extremals have a non-vanishing action but vanishing Poincare charges. Their small deformations are identified as space-time counterparts of fermions and their super partners. Wormhole throats identified as pieces of these extremals are identified as bosons and their super partners.

The conditions stating light likeness are equivalent with the Virasoro conditions of string models and this actually led to the eventualo realization that conformal invariance is a basic symmetry of TGD and that WCW can be regarded as a union of symmetric spaces with isometry groups having identification as symplectic and Kac-Moody type groups assignable to the partonic 2-surfaces.

4.1.2 Approximate symplectic invariance

Vacuum extremals have diffeomorphisms of M^4_+ and M^4_+ local symplectic transformations as symmetries. For non-vacuum extremals these symmetries leave induced Kähler form invariant and only induced metric breaks these symmetries. Symplectic transformations of CP_2 act on the Maxwell field defined by the induced Kähler form in the same manner as ordinary $U(1)$ gauge symmetries. They are however not gauge symmetries since gauge invariance is still present. In fact, the construction of WCW geometry relies on the assumption that symplectic transformations of $\delta M^4_+ \times CP_2$ which infinitesimally correspond to combinations of M^4_+ local CP_2 symplectic and CP_2-local M^4_+ symplectic transformations act as isometries of WCW . In zero energy ontology these transformations act simultaneously on all partonic 2-surfaces characterizing the space-time sheet representing a generalized Feynman diagram inside CD.

The fact that CP_2 symplectic transformations do not act as genuine gauge transformations means that $U(1)$ gauge invariance is effectively broken. This has non-trivial implications. The field equations allow purely geometric vacuum 4-currents not possible in Maxwell's electrodynamics [K1]. For the known extremals (massless extremals) they are light-like and a possible interpretation is in terms of Bose-Einstein condensates of collinear massless bosons.

4.1.3 Spin glass degeneracy

Vacuum degeneracy means that all surfaces belonging to $M^4_+ \times Y^2$, Y^2 any Lagrangian sub-manifold of CP_2 are vacua irrespective of the topology and that symplectic transformations of CP_2 generate new surfaces Y^2. If preferred extremals are obtained as small deformations of vacuum extremals (for which the criticality is maximal), one expects therefore enormous ground state degeneracy, which could be seen as 4-dimensional counterpart of the spin glass degeneracy. This degeneracy corresponds to the hypothesis that WCW is a union of symmetric spaces labeled by zero modes which do not appear at the line-element of the WCW metric.

Zero modes define what might be called the counterpart of spin glass energy landscape and the maxima Kähler function as a function of zero modes define a discrete set which might be called reduced configuration space. Spin glass degeneracy turns out to be crucial element for understanding how macro-temporal quantum coherence emerges in TGD framework. One of the basic ideas about p-adicization is that the maxima of Kähler function define the TGD counterpart of spin glass energy landscape [K11, K5]. The hierarchy of discretizations of the symmetric spaces corresponding to a hierarchy of measurement resolutions [K14] could allow an identification in terms of a hierarchy spin glass energy landscapes so that the algebraic points of the WCW would correspond to the maxima of Kähler function. The hierarchical structure would be due to the failure of strict non-determinism of Kähler action allowing in zero energy ontology to add endlessly details to the space-time sheets representing zero energy states in shorter scale.

4.1.4 Generalized quantum gravitational holography

The original naïve belief was that the construction of the configuration space geometry reduces to $\delta H = \delta M^4_+ \times CP_2$. An analogous idea in string model context became later known as quantum gravitational holography. The basic implication of the vacuum degeneracy is classical non-determinism, which is expected to reflect itself as the properties of the Kähler function and WCW geometry. Obviously classical non-determinism challenges the notion of quantum gravitational holography.

The hope was that a generalization of the notion of 3-surface is enough to get rid of the degeneracy and save quantum gravitational holography in its simplest form. This would mean that one just replaces space-like 3-surfaces with "association sequences" consisting of sequences of space-like 3-surfaces with time like separations as causal determinants. This would mean that the absolute minima of Kähler function would become degenerate: same space-like 3-surface at δH would correspond to several association sequences with the same value of Kähler function.

The life turned out to be more complex than this. CP_2 type extremals have Euclidian signature of the induced metric and therefore CP_2 type extremals glued to space-time sheet with Minkowskian signature of the induced metric are surrounded by light like surfaces X^3_l, which might be called elementary particle horizons. The non-determinism of the CP_2 type extremals suggests strongly that also elementary particle horizons behave non-deterministically and must be regarded as causal determinants having time like projection in M^4_+. Pieces of CP_2 type extremals are good candidates for the wormhole contacts connecting a space-time sheet to a larger space-time sheet and are also surrounded by an elementary particle horizons and non-determinism is also now present. That this non-determinism would allow the proposed simple description seems highly implausible.

Zero energy ontology realized in terms of a hierarchy of CDs seems to provide the most plausible treatment of the non-determinism and has indeed led to a breakthrough in the construction and understanding of quantum TGD. At the level of generalized Feynman diagrams sub-CDs containing zero energy states represent a hierarchy of radiative corrections so that the classical determinism is direct correlate for the quantum non-determinism. Determinism makes sense only when one has specified the length scale of measurement resolution. One can always add a CD containing a vacuum extremal to get a new zero energy state and a preferred extremal containing more details.

4.1.5 Classical non-determinism saves the notion of time

Although classical non-determinism represents a formidable mathematical challenge it is a must for several reasons. Quantum classical correspondence, which has become a basic guide line in the development of TGD, states that all quantum phenomena have classical space-time correlates. This is not new as far as properties of quantum states are considered. What is new that also quantum jumps and quantum jump sequences which define conscious existence in TGD Universe, should have classical space-time correlates: somewhat like written language is correlate for the contents of consciousness of the writer. Classical non-determinism indeed makes this possible. Classical non-determinism makes also possible the realization of statistical ensembles as ensembles formed by strictly deterministic pieces of the space-time sheet so that even thermodynamics has space-time representations. Space-time surface can thus be seen as symbolic representations for the quantum existence.

In canonically quantized general relativity the loss of time is fundamental problem. If quantum gravitational holography would work in the most strict sense, time would be lost also in TGD since all relevant information about quantum states would be determined by the moment of big bang. More precisely, geometro-temporal localization for the contents of conscious experience would not be possible. Classical non-determinism together with quantum-classical correspondence however suggests that it is possible to have quantum jumps in which non-determinism is concentrated in space-time region so that also conscious experience contains information about this region only.

4.2 Four-Dimensional General Coordinate Invariance

The proposed definition of the Kähler function is consistent with GCI and implies also 4-dimensional Diff degeneracy of the Kähler metric. Zero energy ontology inspires strengthening of the GCI in the sense that space-like 3-surfaces at the boundaries of CD are physically equivalent with the light-like 3-surfaces connecting the ends. This implies that basic geometric objects are partonic 2-surfaces at the boundaries of CDs identified as the intersections of these two kinds of surfaces.

Besides this the distribution of 4-D tangent planes at partonic 2-surfaces would code for physics so that one would have only effective 2-dimensionality. The failure of the non-determinism of Kähler action in the standard sense of the word affects the situation also and one must allow a fractal hierarchy of CDs inside CDs having interpretation in terms of radiative corrections.

4.2.1 Resolution of tachyon difficulty and absence of Diff anomalies

In TGD as in string models the tachyon difficulty is potentially present: unless the time like vibrational excitations possess zero norm they contribute tachyonic term to the mass squared operator of Super Kac Moody algebra. This difficulty is familiar already from string models [B5, B3].

The degeneracy of the metric with respect to the time like vibrational excitations guarantees that time like excitations do not contribute to the mass squared operator so that mass spectrum is tachyon free. It also implies the decoupling of the tachyons from physical states: the propagator of the theory corresponds essentially to the inverse of the Kähler metric and therefore decouples from time like vibrational excitations. The experience with string model suggests that if metric is degenerate with respect to diffeomorphisms of $X^4(X^3)$ there are indeed good hopes that time like excitations possess vanishing norm with respect to WCW metric.

The four-dimensional Diff invariance of the Kähler function implies that Diff invariance is guaranteed in the strong sense since the scalar product of two Diff vector fields given by the matrix associated with $(1,1)$ part of the second variation of the Kähler action vanishes identically. This property gives hopes of obtaining theory, which is free from Diff anomalies: in fact loop space metric is not Diff degenerate and this might be the underlying reason to the problems encountered in string models [B5, B3].

4.2.2 Complexification of WCW

Strong form of GCI plays a fundamental role in the complexification of WCW . GCI in strong form reduces the basic building brick of WCW to the pairs of partonic 2-surfaces and their 4-D tangent space data associated with ends of light-like 3-surface at light-like boundaries of CD. At boths end the embedding space is effectively reduces to $\delta M^4_+ \times CP_2$ (forgetting the complications due to non-determinism of Kähler action). Light cone boundary in turn is metrically 2-dimensional Euclidian sphere allowing infinite-dimensional group of conformal symmetries and Kähler structure. Therefore one can say that in certain sense configuration space metric inherits the Kähler structure of $S^2 \times CP_2$. This mechanism works in case of four-dimensional Minkowski space only: higher-dimensional spheres do not possess even Kähler structure. In fact, it turns out that the quantum fluctuating degrees of freedom can be regarded in well-defined sense as a local variant of $S^2 \times CP_2$ and thus as an infinite-dimensional analog of symmetric space as the considerations of [K3] demonstrate.

The details of the complexification were understood only after the construction of WCW geometry and spinor structure in terms of second quantized induced spinor fields [K14]. This also allows to make detailed statements about complexification [K3].

4.2.3 Contravariant metric and Diff^4 degeneracy

Diff degeneracy implies that the definition of the contravariant metric, which corresponds to the propagator associated to small deformations of minimizing surface is not quite straightforward. We believe that this problem is only technical. Certainly this problem is not new, being encountered in both GRT and gauge theories [B8, B4]. In TGD a solution of the problem is provided by the existence of infinite-dimensional isometry group. If the generators of this group form a complete set in the sense that any vector of the tangent space is expressible as as sum of these generators plus some zero norm vector fields then one can restrict the consideration to this subspace and in this subspace the matrix $g(X,Y)$ defined by the components of the metric tensor indeed indeed possesses well defined inverse $g^{-1}(X,Y)$. This procedure is analogous to gauge fixing conditions in gauge theories and coordinate fixing conditions in General Relativity.

It has turned that the representability of WCW as a union of symmetric spaces makes possible an approach to WCW integration based on harmonic analysis replacing the perturbative approach based on perturbative functional integral. This approach allows also a p-adic variant and leads an effective discretization in terms of discrete variants of WCW for which the points of symmetric space consist of algebraic points. There is an infinite number of these discretizations [K11] and the interpretation is in terms of finite measurement resolution. This gives a connection with the p-adicization program, infinite primes, inclusions of hyper-finite factors as representation of the finite measurement resolution, and the hierarchy of Planck constants [K10] so that various approaches to quantum TGD converge nicely.

4.2.4 General Coordinate Invariance and WCW spinor fields

GCI applies also at the level of quantum states. WCW spinor fields are Diff^4 invariant. This in fact fixes not only classical but also quantum dynamics completely. The point is that the values of the WCW spinor fields must be essentially same for all Diff^4 related 3-surfaces at the orbit X^4 associated with a given 3-surface. This would mean that the time development of Diff^4 invariant configuration spinor field is completely determined by its initial value at the moment of the big bang!

This is of course a naïve over statement. The non-determinism of Kähler action and zero energy ontology force to take the causal diamond (CD) defined by the intersection of future and past directed light-cones as the basic structural unit of WCW , and there is fractal hierarchy of CDs within CDs so that the above statement makes sense only for giving CD in measurement resolution neglecting the presence of smaller CDs. Strong form of GCI also implies factorization of WCW spinor fields into a sum of products associated with various partonic 2-surfaces. In particular, one obtains time-like entanglement between positive and negative energy parts of zero energy states and entanglement coefficients define what can be identified as M-matrix expressible as a "complex square root" of density matrix and reducing to a product of positive definite diagonal square root of density matrix and unitary S-matrix. The collection of orthonormal M-matrices in turn define unitary U-matrix between zero energy states. M-matrix is the basic object measured in particle physics laboratory.

4.3 WCW Geometry, Generalized Catastrophe Theory, And Phase Transitions

The definition of WCW geometry has nice catastrophe theoretic interpretation. To understand the connection consider first the definition of the ordinary catastrophe theory [A2].

1. In catastrophe theory one considers extrema of the potential function depending on dynamical variables x as function of external parameters c. The basic space decomposes locally into cartesian product $E = C \times X$ of control variables c, appearing as parameters in potential function $V(c, x)$ and of state variables x appearing as dynamical variables. Equilibrium states of the system correspond to the extrema of the potential $V(x, c)$ with respect to the variables x and in the absence of symmetries they form a sub-manifold of M with dimension equal to that of the parameter space C. In some regions of C there are several extrema of potential function and the extremum value of x as a function of c is multi-valued. These regions of $C \times X$ are referred to as catastrophes. The simplest example is cusp catastrophe (see **Fig. ??**) with two control parameters and one state variable.

2. In catastrophe regions the actual equilibrium state must be selected by some additional physical requirement. If system obeys flow dynamics defined by first order differential equations the catastrophic jumps take place along the folds of the cusp catastrophe (delay rule). On the other hand, the Maxwell rule obeyed by thermodynamic phase transitions states that the equilibrium state corresponds to the absolute minimum of the potential function and the state of system changes in discontinuous manner along the Maxwell line in the middle between the folds of the cusp (see **Fig. 1**).

3. As far as discontinuous behavior is considered, fold catastrophe is the basic catastrophe: all catastrophes contain folds as there "satellites" and one aim of the catastrophe theory is to derive all possible ways for the stable organization of folds into higher catastrophes. The fundamental result of the catastrophe theory is that for dimensions d of C smaller than 5 there are only 7 basic catastrophes and polynomial potential functions provide a canonical

representation for the catastrophes: fold catastrophe corresponds to third order polynomial (in fold the two real roots become a pair of complex conjugate roots), cusp to fourth order polynomial, etc.

Consider now the TGD counterpart of this. TGD allows allows two kinds of catastrophe theories.

1. The first one is related to Kähler action as a local functional of 4-surface. The nature of this catastrophe theory depends on what one means with the preferred extremals.

2. Second catastrophe theory corresponds to Kähler function a non-local functional of 3-surface. The maxima of the vacuum functional defined as the exponent of Kähler function define what might called effective space-times, and discontinuous jumps changing the values of the parameters characterizing the maxima are possible.

Consider first the option based on Kähler action.

1. Potential function corresponds to Kähler action restricted to the solutions of Euler Lagrange equations. Catastrophe surface corresponds to the four-surfaces found by extremizing Kähler action with respect to the variables of X (time derivatives of coordinates of C specifying X^3 in H_a) keeping the variables of C specifying 3-surface X^3 fixed. Preferred extremal property is analogous to the Bohr quantization since canonical momenta cannot be chosen freely as in the ordinary initial value problems of the classical physics. Preferred extremals are by definition at criticality. Behavior variables correspond to the deformations of the 4-surface keeping partonic 2-surfaces and 3-D tangent space data fixed and preserving extremal property. Control variables would correspond to these data.

2. At criticality the rank of the infinite-dimensional matrix defined by the second functional derivatives of the Kähler action is reduced. Catastrophes form a hierarchy characterized by the reduction of the rank of this matrix and Thom's catastrophe theory generalizes to infinite-dimensional context. Criticality in this sense would be one aspect of quantum criticality having also other aspects. No discrete jumps would occur and system would only move along the critical surface becoming more or less critical.

3. There can exist however several critical extremals assignable to a given partonic 2-surface but have nothing to do with the catastrophes as defined in Thom's approach. In presence of degeneracy one should be able to choose one of the critical extremals or replace this kind of regions of WCW by their multiple coverings so that single partonic 2-surface is replaced with its multiple copy. The degeneracy of the preferred extremals could be actually a deeper reason for the hierarchy of Planck constants involving in its most plausible version n-fold singular coverings of CD and CP_2. This interpretation is very satisfactory since the generalization of the embedding space and hierarchy of Planck constants would follow naturally from quantum criticality rather than as separate hypothesis.

4. The existence of the catastrophes is implied by the vacuum degeneracy of the Kähler action. For example, for pieces of Minkowski space in $M^4_+ \times CP_2$ the second variation of the Kähler action vanishes identically and only the fourth variation is non-vanishing: these 4-surfaces are analogous to the tip of the cusp catastrophe. There are also space-time surfaces for which the second variation is non-vanishing but degenerate and a hierarchy of subsets in the space of extremal 4-surfaces with decreasing degeneracy of the second variation defines the boundaries of the projection of the catastrophe surface to the space of 3-surfaces. The space-times for which second variation is degenerate contain as subset the critical and initial value sensitive preferred extremal space-times.

Consider next the catastrophe theory defined by Kähler function.

1. In this case the most obvious identification for the behavior variables would be in terms of the space of all 3-surfaces in $CD \times CP_2$ - and if one believes in holography and zero energy ontology - the 2-surfaces assignable the boundaries of causal diamonds (CDs).

2. The natural control variables are zero modes whereas behavior variables would correspond to quantum fluctuating degrees of freedom contributing to the WCW metric. The induced Kähler form at partonic 2-surface would define infinitude of purely classical control variables. There is also a correlation between zero modes identified as degrees of freedom assignable to the interior of 3-surface and quantum fluctuating degrees of freedom assigned to the partonic 2-surfaces. This is nothing but holography and effective 2-dimensionality justifying the basic assumption of quantum measurement theory about the correspondence between classical and quantum variables. The absence of several maxima implies also the presence of saddle surfaces at which the rank of the matrix defined by the second derivatives is reduced. This could lead to a non-positive definite metric. It seems that it is possible to have maxima of Kähler function without losing positive definiteness of the metric since metric is defined as (1, 1)-type derivatives with respect to complex coordinates. In case of CP_2 however Kähler function has single degenerate maximum corresponding to the homologically trivial geodesic sphere at $r = \infty$. It might happen that also in the case of infinite-D symmetric space finite maxima are impossible.

3. The criticality of Kähler function would be analogous to thermodynamical criticality and to the criticality in the sense of catastrophe theory. In this case Maxwell's rule is possible and even plausible since quantum jump replaces the dynamics defined by a continuous flow.

Cusp catastrophe provides a simple concretization of the situation for the criticality of Kähler action (as distinguished from that for Kähler function).

1. The set M of the critical 4-surfaces corresponds to the V-shaped boundary of the 2-D cusp catastrophe in 3-D space to plane. In general case it forms codimension one set in WCW . In TGD Universe physical system would reside at this line or its generalization to higher dimensional catastrophes. For the criticality associated with Kähler action the transitions would be smooth transitions between different criticalities characterized by the rank defined above: in the case of cusp (see **Fig. 1**) from the tip of cusp to the vertex of cusp or vice versa. Evolution could mean a gradual increase of criticality in this sense. If preferred extremals are not unique, cusp catastrophe does not provide any analogy. The strong form of criticality would mean that the system would be always "at the tip of cusp" in metaphoric sense. Vacuum extremals are maximally critical in trivial sense, and the deformations of vacuum extremals could define the hierarchy of criticalities.

2. For the criticality of Kähler action Maxwell's rule stating that discontinuous jumps occur along the middle line of the cusp is in conflict with catastrophe theory predicting that jumps occurs along at criticality. For the criticality of Kähler function - if allowed at all by symmetric space property - Maxwell's rule can hold true but cannot be regarded as a fundamental law. It is of course known that phase transitions can occur in different ways (super heating and super cooling).

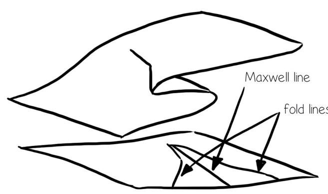

Figure 1: Cusp catastrophe

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. Conformal symmetry would be naturally associated with the super-symplectic algebra of $\delta M^4_\pm$ for which the light-like radial coordinate plays the role of complex coordinate z for ordinary 2-D conformal symmetry. At criticality the symplectic subalgebra represented as gauge symmetries would change to its isomorphic subalgebra or which versa and having conformal weights are multiples of integer n. One would have fractal hierarchies of sub-algebras characterized by integers $n_{i+1} = \prod_{k<i+1} m_k$.

In each transition to lower criticality the gauge sub-algebra of the symplectic algebra would become a sub-algebra of the original one. These transitions would occur spontaneously. The transitions in the reverse direction would not take place spontaneously. The proposal is that these phase transitions take place in both directions in living matter and that the phase transitions reducing criticality require metabolic energy.

The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg` or **Fig. ??** in the appendix of this book). The hierarchy of Planck constants in turn is identified as dark phases of matter [K4].

REFERENCES

Mathematics

[A1] Freed DS. The Geometry of Loop Groups, 1985.

[A2] Zeeman EC. *Catastrophe Theory.* Addison-Wessley Publishing Company, 1977.

[A3] Hanson J Eguchi T, Gilkey B. *Phys Rep*, 66, 1980.

[A4] Olive D Goddard P. Kac-Moody and Virasoro algebras in relation to Quantum Physics.

Theoretical Physics

[B1] Rajeev SG Bowick MM. The holomorphic geometry of closed bosonic string theory and $Diff(S^1)/S^1$. *Nucl Phys B*, 293, 1987.

[B2] Witten E. Coadjoint orbits of the Virasoro Group, 1987. PUPT-1061 preprint.

[B3] Witten E Green MB, Schwartz JH. *Superstring Theory.* Cambridge University Press, Cambridge, 1987.

[B4] Zuber J-B Itzykson C. *Quantum Field Theory.* Mc Graw-Hill, New York, 1980.

[B5] Schwartz JH. *Super strings. The first 15 years of Superstring Theory.* World Scientific, 1985.

[B6] Stelle KS Pope CN. The Witten index for the supermembrane, 1988.

[B7] Jackiw R. In *Gauge Theories of Eighties, Conference Proceedings, Äkäslompolo, Finland*, Lecture Notes in Physics. Springer Verlag, 1983.

[B8] Christensen SM. *Quantum Theory of Gravity.* Adam Hilger Ltd, 1984.

Condensed Matter Physics

Books related to TGD

[K1] Pitkänen M. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: `https://tgdtheory.fi/pdfpool/class.pdf`, 2023.

[K2] Pitkänen M. Construction of Quantum Theory: Symmetries. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/quthe.pdf, 2023.

[K3] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K4] Pitkänen M. Does TGD Predict a Spectrum of Planck Constants? In *Dark Matter and TGD:* *https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/Planck, 2023.

[K5] Pitkänen M. Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/mblocks.pdf, 2023.

[K6] Pitkänen M. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/knotstgd.pdf, 2023.

[K7] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infmotives.pdf, 2023.

[K8] Pitkänen M. Quantum Hall effect and Hierarchy of Planck Constants. In *TGD and Condensed Matter.* *https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/anyontgd.pdf, 2023.

[K9] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K10] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K11] Pitkänen M. TGD as a Generalized Number Theory: p-Adicization Program. In *Quantum Physics as Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visiona.pdf, 2023.

[K12] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionb.pdf, 2023.

[K13] Pitkänen M. TGD View about Coupling Constant Evolution. In *Quantum TGD: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdquantum2.html*. Available at: https://tgdtheory.fi/pdfpool/ccevolution.pdf, 2023.

[K14] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

Articles about TGD

[L1] Pitkänen M. CMAP representations about TGD, and TGD inspired theory of consciousness and quantum biology. Available at: https://www.tgdtheory.fi/tgdglossary.pdf., 2014.