# Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?

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**Abstract**

TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of Kähler action. So called $M^8-H$ duality is one of these approaches. The beauty of $M^8-H$ duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

1. I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

2. It will be shown how $M^8-H$ duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in $M^8$ would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable $o_c$ decomposing as $o_c = q_c^1 + q_c^2 I^4$ and projected to a Minkowskian sub-space $M^8$ of complexified $O$. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

   One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c\overline{q_c}$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and $M^8-H$ correspondence could generalize.

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates $z_i^k$, $k = 1,2$, of $O$ vanishes that is critical as function of quaternionic components $z_1^k$ or $z_2^k$ associated with $q_1$ and $q_2$ in the decomposition $o = q_1 + q_2 I_4$, call this component $X_i$. In the generic case this gives 3-D surface.

   In this generic case $M^8-H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to $H$, and only determines the boundary conditions of the

dynamics in $H$ determined by the twistor lift of Kähler action. $M^8 - H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial $P$ so that the criticality conditions do not reduce the dimension: $X_i$ would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components $X_i$. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of $X_i$ conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in $H$ in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8 - H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8 - H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8 - H$ duality determines boundary conditions.

3. This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

Also a sketchy proposal for the description of interactions is discussed.

1. $IM(P_1P_2) = 0$ is satisfied for $IM(P_1) = 0$ and $IM(P_2) = 0$ since $IM(o_1o_2)$ is linear in $IM(o_i)$ and one obtains union of space-time varieties. $RE(P_1P_2) = 0$ cannot be satisfied in this manner since $RE(o_1o_2)$ is not linear in $RE(o_i)$ so that the two varieties interact and this interaction could give rise to a wormhole contact connecting the two space-time varieties.

2. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii $t_n$ given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety $X^4$. The intersections $X^2 = X^4 \cap S^6(t_n)$ is tentatively identified as partonic 2-varieties defining topological interaction vertices.

   The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties $X^2$ are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

3. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

   The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(\prod P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

4. The possibility of super-octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would

reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

Gromov-Witten (G-W) invariants belong to the realm of quantum enumerative geometry briefly discussed in [L12]. They count numbers of points in the intersection of varieties ("branes") with quantum intersection identified as the existence of "string world sheet(s)" intersecting the branes. Also octonionic geometry gives rise to brane like objects. G-W invariants are rational numbers but it is proposed that they could be integers in TGD framework.

Riemann-Roch theorem (RR) and its generalization Atyiah-Singer index theorem (AS) relate dimensions of various kinds of moduli spaces to topological invariants. The possible generalizations of RR and AS to octonionic framework and the implications of $M^8-H$ duality for the possible generalizations are discussed. The adelic hierarchy of extensions of rationals and criticality conditions make the moduli spaces discrete so that one expects kind of particle in box type quantization selecting discrete points of moduli spaces about the dimension.

# 1 Introduction

There are good reasons to hope that TGD is integrable theory in some sense. Classical physics is an exact part of quantum physics in TGD and during years I have ended up with several proposals for the general solution of classical field equations (classical TGD is an exact part of quantum TGD).

## 1.1 Various approaches to classical TGD

### 1.1.1 World of classical worlds

The first approach is based on the geometry of the "world of classical worlds" (WCW) [K7, K3, L32].

1. The study of classical field equations led rather early to the realization that preferred extremals for the twistor lift of Kähler action with Minkowskian signature of induced metric define a slicing of space-time surfaces defined by 2-D string world sheets and partonic two-surfaces locally orthogonal to them. The interpretation is in terms of position dependent light-like momentum vector and polarization vector defining the local decompositions $M^2(x) \times E^2(x)$ of tangent space integrating to a foliation by partonic 2-surfaces and string world sheets. I christened this structure Hamilton-Jacobi structure. Its Euclidian counterpart is complex structure in Euclidian regions of space-time surface.

2. The formulation of quantum TGD in terms of spinor fields in WCW [K17] leads to the conclusion that WCW must have Kähler geometry [K7, K3] and has it only if it has maximal group of isometries identified as symplectic transformations of $\delta M^4_\pm \times CP_2$, where $\delta M^4_\pm$ denotes light cone boundary two which upper/lower boundary of causal diamond (CD) belongs. Symplectic Lie algebra extends naturally to supersymplectic algebra (SSA).

3. Space-time surfaces would be preferred extremals of twistor lift of Kähler action [K32] and the conditions realizing strong form of holography (SH) would state that sub-algebra of SSA isomorphic with it and its commutator with SSA give rise to vanishing Noether charges and these charges annihilate physical states or create zero norm states from them. One should solve these conditions.

4. The dynamics involves also fermions. Induced spinor fields are located inside space-time surface but for some yet not completely understood reason only the information about spinor at 2-D string world sheets is needed in the construction of scattering amplitudes. This dynamics would be 2-dimensional. The construction of twistor amplitudes even suggests that it is 1-dimensional in the sense that 1-D light-like curves at light-like partonic orbits defining boundaries of Minkowskian and Euclidian regions determines the scattering amplitudes. String world sheets are however needed only as correlates for entanglement between fermions at different partonic orbits.

   The 2-D character of fermionic dynamics conforms with the strong form of holography (SH) but how the string world sheets and partonic 2-surfaces are selected from Hamilton-Jacobi slicing? Electromagnetic neutrality could select string worlds sheets but one can actually always find a gauge in which the induced classical electroweak field at these surfaces is purely electromagnetic.

### 1.1.2 Twistor lift of TGD

Second approach to preferred extremals is based on TGD version [K28, K26, K25, K32] of twistor Grassmann approach [B1, B4, B3].

1. The twistor lift of TGD leads to a proposal that space-time surfaces can be represented as sections in their 6-D twistor spaces identified as twistor bundles in the product $T(H) = T(M^4) \times T(CP_2)$ of 6-D twistor spaces of $M^4$ and $CP_2$. Twistor structure would be induced from $T(H)$. Kähler action can be lifted to the level of twistor spaces only for $M^4 \times CP_2$ since only for these spaces twistor space allows Kähler structure [A3]. Twistors were originally introduced by Penrose with the motivation that one could apply algebraic geometry in Minkowskian signature. The bundle property is extremely powerful and should be consistent with the algebraic geometrization at the level of $M^8_c$. The challenge is to formulate the twistor lift at the level of $M^8$.

2. The twistor lift of Kähler action contains also volume term. Field equations have two kinds of solutions. For the solutions of first kind the dynamics of volume term and Käction are coupled and the interpretation is in terms of interaction regions. Solutions of second kind are minimal surfaces and extremals of both Kähler action and volume term, whose dynamics decouple completely and all coupling constants disappear from the dynamics. These extremals are natural candidates for external particles. For these solutions at least the field equations reduce to the existence of Hamilton-Jacobi structure. The completely universal dynamics of these regions suggests interpretation in terms of maximal quantum criticality characterized by the extension of the usual conformal invariance to its quaternionic analog.

3. A connection with zero energy ontology (ZEO) emerges. Causal diamond (CD, intersection of future and past directed light-cones of $M^4$ with points replaced by $CP_2$) would naturally determine the interaction region to which external particles enter through its 2 future and past boundaries. But where does ZEO emerge?

### 1.1.3 $M^8 - H$ **duality**

The third approach is based on number theoretic vision [K14, K15, K13, K21].

1. $M^8-H$ duality [K15, K21, K22] means that one can see space-times as 4-surfaces in either $M^8$ or $H = M^4 \times CP_2$. One could speak "number theoretical compactification" having however nothing to do with stringy version of compactification, which is dynamical. $M^8 - H$ duality suggests that space-time surfaces in $H = M^4 \times CP_2$ are images of space-time surfaces in $M^8$ or actually of $M^8$ projections of complexified space-time surfaces in $M^8_c$ identified as space of complexified octonions. These space-time surfaces could contain the integrated distributions of string world sheets and partonic 2-surfaces mentioned in the previous item. Space-time surfaces must have associative tangent or normal space for $M^8 - H$ correspondence to exist.

2. The fascinating possibility mentioned already earlier is that in $M^8$ these surfaces could correspond to zero loci for real or imaginary parts of real analytic octonionic polynomials $P(o) = RE(P) + IM(P)I_4$, $I_4$ an octonionic imaginary unit orthogonal to quaternionic ones. The condition $IM(P) = 0$ ($RE(P) = 0$) would give associative (co-associative) space-time surface. In the simplest case these functions would be polynomials so that one would have algebraic geometry for algebraically 4-D complex surfaces in 8-D complex space.

   **Remark**: The naive guess that space-time surfaces reduce to quaternionic curves in quaternionic plane fails due to the non-commutativity of quaternions meaning that one has $P(o) = P(q_1, q_2, \overline{q}_1, \overline{q}_2)$ rather than $P(o) = P(q_1, q_2)$.

   **Remark**: Why not rational functions expressible as ratios $R = P_1/P_2$ of octonionic polynomials? It has become clear that one can develop physical arguments in favor of this option.

The zero loci for $IM(P_i)$ would represent space-time varieties. Zero loci for $RE(P_1/P_2) = 0$ and $RE(P_1/P_2) = \infty$ would represent their interaction presumably realized as wormhole contacts connecting these varieties. In the sequel most considerations are for polynomials: the replacement of polynomials with rational functions does not introduce big differences and its discussed in the section "Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view" of [L13].

3. The objection against this proposal is obvious. $M^8 - H$ correspondence cannot hold true since the dynamics defined by octonionic polynomials in $M^8$ contains no coupling constants whereas the dynamics of twistor lift of Kähler action depends on coupling constants in the generic space-time region. However, for space-time surfaces representing external particles entering inside CD at its boundaries this is however not the case! They could satisfy $M^8 - H$ correspondence!

   This suggests that inside CDs the space-time surfaces are not associative/co-associative in $M^8$ so that $M^8 - H$ correspondence cannot map them to $H$ and the twistor lifted Kähler action and SH take care of the dynamics. External particles are associative and quantum critical and $M^8 - H$ correspondence makes sense. The quantum criticality and associativity at the boundaries of CD poses extremely powerful conditions and allows to satisfy infinite number of vanishing conditions for SSA charges.

   It has later turned out [L22] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

4. This picture is consistent with the the explicit formulation of the associativity conditions $Re(P) = 0$ and $IM(P) = 0$ for varieties. The calculations shows that associativity can be realized either by posing a condition making them 3-dimensional except, when the situation is critical in the sense that the 4-D variety is analogous to a double root of polynomial: now however the polynomial corresponds to prime polynomial decomposing to product of polynomials in the extension of rationals such that the product contains higher powers of the factors. One has ramification at the level of polynomial primes so that the criticality condition does not bring anything new but need not make the situation associative. At most 3 conditions need to be applied to guarantee associativity and they might leave the space-time surface 4-D.

5. The coordinates of $M^4$ as octonionic roots $x + iy$ of the 4 real polynomials need not be real. Space-time surface must have $M^4_c$ projection, which reduces to $M^4$. There are two options.

   (a) The original proposal was that the *projection* from $M^8_c$ to real $M^4$ (for which $M^1$ coordinate is real and $E^3$ coordinates are imaginary with respect to $i$!) defines the real space-time surface mappable by $M^8 - H$ duality to $CP_2$. One can howeerver critize the allowance of a nonvanishing imaginary part of space-time surface in $M^4_c$.

   (b) A more stringent condition is that the roots of the 4 vanishing polynomials as coordinates of $M^4_c$ belong automatically to $M^4$ so that $m^0$ would be real root and $m^k$, $k = 1, ..., 3$ imaginary with respect to $i \to -i$. $M^8_c$ coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality.*

   This could allow to identify CDs in very elegant manner: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

6. This octonionic view as also lower-dimensional quaternionic counterpart. In this case one considers 2-D commutative/co-commutative surfaces tentatively identifiable as string world sheets and partonic 2-surfaces. Why not all 2-surfaces appearing in the Hamilton-Jacobi

slicing are not selected? The above mechanism would work also now. The commutativity conditions reduce in the generic case give 1-D curve as a solution. The interpretation would be as orbit of point like particle at 3-D partonic orbit appearing in the construction of twistorial amplitudes. In critical situation one would obtains string world sheet serving as a correlate for entanglement between point like particles at its ends: one would have quantum critical bound state.

I have considered also other attempts to define what quaternion structure could mean.

1. One could also consider the possibility that the tangent spaces of space-time surfaces in $H$ are associative or co-associative [K21]. This is not necessary although it seems that this might be the case for the known extremals. If this holds true, one can construct further preferred extremals by functional composition by generalization of $M^8-H$ correspondence to $H-H$ correspondence.

2. I have considered also the possibility of quaternion analyticity in the sense of generalization of Cauchy-Riemann equations, which tell that left- or right quaternionic differentiation makes sense [L6]. It however seems that this approach is not promising. The conditions are quite too restrictive and bring nothing essentially new. Octonion/quaternion analyticity in the above mentioned sense does not require the analogs of Cauchy-Riemann conditions.

## 1.2 Could one identify space-time surfaces as zero loci for octonionic polynomials with real coefficients?

The identification of space-time surfaces as zero loci of real or imaginary part of octonionic polynomial has several extremely nice features.

1. Octonionic polynomial is an algebraic continuation of a real valued polynomial on real line so that the situation is effectively 1-dimensional! Once the degree of polynomial is known, the value of polynomial at finite number of points are needed to determine it and cognitive representation could give this information! This would strengthen the view strong form of holography (SH) - this conforms with the fact that states in conformal field theory are determined by 1-D data.

2. One can add, sum, multiply, and functionally compose these polynomials provided they correspond to the same quaternionic moduli labelled by $CP_2$ points and share same time-line containing the origin of quaternionic and octonionic coordinates and real octonions (or actually their complexification by commuting imaginary unit). Classical space-time surfaces - classical worlds - would form an associative and commutative algebra. This algebra induces an analog of group algebra since these operations can be lifted to the level of functions defined in this algebra. These functions form a basic building brick of WCW spinor fields defining quantum states.

3. One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD [L2]. Also zero zero energy ontology (ZEO) could be forced by the failure of number field property for quaternions at light-cone boundaries. It indeed turns out that light-cone boundary emerges quite generally as singular zero locus of polynomials $P(o)$ containing no linear part: this is essentially due to the non-commutativity of the octonionic units. Also the emergence of CDs can be understood. At this surface the region with $RE(P)=0$ can transform to $IM(P)=0$ region. In Euclidian signature this singularity corresponds to single point. A natural conjecture is that also the light-like orbits of partonic 2-surfaces correspond to this kind of singularities for non-trivial Hamilton-Jacobi structures.

4. The reduction to algebraic geometry would mean enormous boost to the vision about cognition with cognitive representations identified as generalized rational points common to reals rationals and various p-adic number fields defining the adele for given extension of rationals.

Hamilton-Jacobi structure would result automatically from the decomposition of quaternions to real and imaginary parts which would be now complex numbers.

5. Also a connection with infinite primes is suggestive [K15]. The light-like partonic orbits, partonic 2-surfaces at their ends, and points at the corners of string world sheets might be interpreted in terms of singularities of varying rank and the analog of catastrophe theory emerges.

The great challenge is to prove rigorously that these approaches - or at least some of them - are indeed equivalent. Also it remains to be proven that the zero loci of real/imaginary parts of octonionic polynomials with real coefficients are associative or co-associative. I shall restrict the considerations of this article mostly to $M^8-H$ duality. The strategy is simple: try to remember all previous objections against $M^8-H$ duality and invent new ones since this is the best manner to make real progress.

## 1.3 Topics to be discussed

### 1.3.1 Key notions and ideas of algebraic geometry

Before going of octonionic algebraic geometry, I will discuss basic notions of algebraic geometry such as algebraic variety (see `http://tinyurl.com/hl6sjmz`), - surface (see `http://tinyurl.com/y8d5wsmj`), and - curve (see `http://tinyurl.com/nt6tkey`), rational point of variety central for TGD view about cognitive representation, elliptic curves (see `http://tinyurl.com/lovksny`) and - surfaces (see `http://tinyurl.com/yc33a6dg`), and rational points (see `http://tinyurl.com/ybbnnysu`) and potentially rational varieties (see `http://tinyurl.com/yablk4xt`). Also the notion of Zariski topology (see `http://tinyurl.com/h5pv4vk`) and Kodaira dimension (see `http://tinyurl.com/yadoj2ut`) are discussed briefly. I am not a mathematician. What hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

Much of algebraic geometry is counting numbers of say rational points or of varieties satisfying some conditions. One can also count dimensions of moduli spaces. Hence the basic notions and methods of enumerative geometry are discussed. There is also a discussion of Gromow-Witten invariants and Riemann-Roch theorem having Atyiah-Singer index theorem as a generalization. These notions will be applied in the second part of the article [L13].

### 1.3.2 $M^8-H$ duality

$M^8-H$ duality [K22, K15, K21] would reduce classical TGD to the algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces. Space-time surfaces in $M^8$ would be algebraic varieties identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable $o$ decomposing as $o = q_c^1 + q_c^2 I_4$ and projected to a Minkowskian sub-space $M^8$ of $o$. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces in $M^8$ would form commutative and associative algebra with addition, product and functional composition.

As already noticed, the associativity conditions do not allow 4-D solutions except for criticality so that $M^8-H$ correspondence can hold true only in these space-time regions and one has these nice features at the level of $M^8$. In critical regions $M^8-H$ correspondence is true and these features have $H$ counterparts

The basic problem is to understand the map mediating $M^8-H$ duality mapping the point $(m,e)$ of $M^8 = M_0^4 \times E^4$ to a point $(m,s)$ of $M_0^4 \times CP_2$, where $M_0^4$ point is obtained as a projection to a suitably chosen $M_0^4 \subset M^8$ and $CP_2$ point parameterizes the tangent space as quaternionic sub-space containing preferred $M_0^2(x) \subset M^4(x)$. This map involves slightly non-local information and could allow to understand why the preferred extremals at the level of $H$ are determined by partial differential equations rather than algebraic equations. Also the generalization to the level of twistor lift is briefly touched.

### 1.3.3 Challenges of the octonionic algebraic geometry

The construction and interpretation of the octonionic geometry involves several challenges.

1. The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity essential for the emergence of causal diamonds (CDs).

   This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

   In fact, all algebras obtained by Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding imaginary unit repeatedly to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and a $M^8-H$ correspondence could generalize (maybe even TGD!).

2. It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i = RE(Y)^i + IM(Y)^i I_4$ of the gradient of $RE(P) = Y = 0$ with respect to the complex coordinates $z_i^k$, $k = 1, 2$, of $O$ vanishes that is critical as function of quaternionic components $z_1^k$ or $z_2^k$ associated with $q_1$ and $q_2$ in the decomposition $o = q_1 + q_2 I_4$, call this component $X_i$. In the generic case this gives 3-D surface.

   In this generic case $M^8-H$ duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to $H$, and only determines the boundary conditions of the dynamics in $H$ determined by the twistor lift of Kähler action. $M^8-H$ duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

   One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial $P$ so that the criticality conditions do not reduce the dimension: $X_i$ would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components $X_i$. Space-time surface would be analogous to a polynomial with a multiple root.

   Various components of octonion polynomial $P$ of degree $n$ are polynomials of same degree. Could criticality reduces to the degeneracy of roots for some component polynomials? Could $P$ as a polynomial of real variable have degenerate roots?

   The criticality of $X_i$ conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory [A2] emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of Kähler action in $H$ in regions, where Kähler action and volume term decouple and dynamics does not depend on coupling constants.

   One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by $M^8-H$ duality to preferred critical extremals for the twistor lift of Kähler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of Kähler action and volume term: these represent external particles. $M^8-H$ duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which Kähler action and volume term couple and coupling constants make themselves visible in the dynamics. $M^8-H$ duality determines boundary conditions.

3. This picture generalizes also to the level of complex/co-complex surfaces associated with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for

bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

4. The super variant of the octonionic geometry relying on octonionic triality makes sense and the geometry of the space-time variety correlates with fermion and antifermion numbers assigned with it. This new view about super-geometry involving also automatic SUSY breaking at the level of space-time geometry.

### 1.3.4 Description of interactions

Also a sketchy proposal for the description of interactions is discussed.

1. $IM(P_1P_2) = 0$ is satisfied for $IM(P_1) = 0$ and $IM(P_2) = 0$ since $IM(o_1o_2)$ is linear in $IM(o_i)$ and one obtains union of space-time varieties. $RE(P_1P_2) = 0$ cannot be satisfied in this manner since $RE(o_1o_2)$ is not linear in $RE(o_i)$ so that the two varieties interact and this interaction could give rise to a wormhole contact connecting the two space-time varieties.

2. The surprise that $RE(P) = 0$ and $IM(P) = 0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii $t_n$ given by the roots of the real $P(t)$, whose octonionic extension defines the space-time variety $X^4$. The intersections $X^2 = X^4 \cap S^6(t_n)$ are tentatively identified as partonic 2-varieties defining topological interaction vertices. $S^6$ and therefore also $X^2$ are doubly critical, $S^6$ is also singular surface.

   The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties $X^2$ are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

3. CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

   The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis the condition $IM(\prod P_i) = 0$ reduces to $IM(P_i) = 0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i) = 0$ does not reduce to $RE(\prod P_i) = 0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

4. The possibility of super-octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

   If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic. Indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N} = 4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the appropriate extension of rationals.

### 1.3.5 About the analogs of Gromow-Witten invariants and branes in TGD

Gromov-Witten (G-W) invariants belong to the realm of quantum enumerative geometry briefly discussed in [L12]. They count numbers of points in the intersection of varieties ("branes") with quantum intersection identified as the existence of "string world sheet(s)" intersecting the branes. Also octonionic geometry gives rise to brane like objects. G-W invariants are rational numbers but it is proposed that they could be integers in TGD framework.

Riemann-Roch theorem (RR) and its generalization Atyiah-Singer index theorem (AS) relate dimensions of various kinds of moduli spaces to topological invariants. The possible generalizations of RR and AS to octonionic framework and the implications of $M^8-H$ duality for the possible generalizations are discussed. The adelic hierarchy of extensions of rationals and criticality conditions make the moduli spaces discrete so that one expects kind of particle in box type quantization selecting discrete points of moduli spaces about the dimension.

The discussion of RR as also the notion of infinite primes and infinite rationals as counterparts of zero energy states suggests that rational functions $R = P_1/P_2$ could be more appropriate than mere polynomials. The construction of space-time varieties would not be modified in essential manner: one would have zero loci of $IM(P_i)$ identifiable as space-time sheets and zero- and $\infty$-loci of $RE(P_1/P_2)$ naturally identifiable as wormhole contacts connecting the space-time sheets.

### 1.3.6 Miscellaneous topics

As I started writing this article I had in mind cognitive representations. My hope was that $M^8-H$ duality could help to improve my understanding about them. It indeed did so and I have therefore included two sections strictly speaking do not represent the central topic of the article.

1. Cognitive representations are identified as sets of rational points for algebraic varieties with "active" points containing fermion. The representations are discussed at both $M^8$- and $H$ level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [L14, L15, L5, L9].

   In TGD the reason would be simple: associativity and quantum criticality are satisfied in the generic case only at lower dimensional selected varieties: 3-surfaces at the ends of space-time surface and partonic orbits and also at string world sheets and fermion lines. For external particles these properties hold true in 4-D sense and cognitive representation could be 4-D-perhaps because rational points (in extension of rationals) form a dense set in these cases. This indeed conforms with the fact that we can solve free field theories!

2. Some aspects related to homology charge (Kähler magnetic charge) and genus-generation correspondence are discussed. Both topological quantum numbers are central in the proposed model of elementary particles and it is interesting to see whether the picture is internally consistent and how algebraic variety property affects the situation. Also possible problems related to $h_{eff}/h = n$ hierarchy [K5, K19] [L14] realized in terms of $n$-fold coverings of space-time surfaces are discussed from this perspective.

The easiest manner to kill $M^8-H$ duality in the form it is represented here is to prove that 4-D zero loci for imaginary/real parts of octonionic polynomials with real coefficients can never be associative/co-associative being always 3-D. I hope that some professional mathematician would bother to check this.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); Strong Form of GCI (SGCI); Quantum Criticality (QC); Strong Form of Holography (SH); World of Classical Worlds (WCW); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Causal Diamond (CD); Number Theoretical Universality (NTU) are the most often occurring acronyms.

# 2 Some basic notions, ideas, results, and conjectures of algebraic geometry

In this section I will summarize very briefly the basic notions of algebraic geometry needed in the sequel.

## 2.1 Algebraic varieties, curves and surfaces

The basic notion of algebraic geometry is algebraic variety.

1. One considers affine space $A^n$ with $n$ coordinates $x^1, ..., x^n$ having values in a number field $K$ usually assumed to be algebraically closed (note that affine space has no preferred origin like linear space). Algebraic variety is defined as a solution of one or more algebraic equations stating the vanishing of polynomials of $n$ variables: $P^i(x^1, ..., x^n) = 0$, $i = 1, ..., r \leq n$. One can restrict the coefficients of polynomials to p-adic number field or or its extension to an extension of rationals. One talks about polynomials on $k \subset K$.

2. The basic condition is that the variety is not a union of disjoint varieties. This for instance happens, when the polynomial $P(x^1, .., x^n)$ defining co-dimension 1 manifold is product of polynomials $P = \prod_r P_r$. Algebraic variety need not be a manifold meaning that it can have singular points. For instance, for co-dimension 1 variety the Jacobian matrix $\partial P/\partial x^i$ of the polynomial can vanish at singularity.

3. One can define projective varieties (see `http://tinyurl.com/ybsqvy3r`) in projective space $P^n$ having coordinatization in terms of $n+1$ homogenous coordinates $(x^1, ..., x^{n+1})$ in $K$ with points differing by an overall scaling identified. Projective variety is defined as zero locus of homogenous polynomials of $n + 1$ coordinates so that solutions remain solutions under the overall scaling of all coordinates. By identifying the points related by scaling one obtains a surface in $P^n$. Grassmannian of linear space $V^n$ (not affine space!) is a projective spaces defined as space of $k$-planes of $V^n$. These spaces are encountered in twistor Grassmannian approach to scattering amplitudes.

For polynomials of single variable one obtains just the roots of $P_n(x) = 0$ in an algebraic extension assignable to the polynomial. For several variables one can in principle proceed step by step by solving variable $x^1$ as algebraic function of others from $P_1(x^1, ..., x^n) = 0$ , proceed to solve $x^2$ from $P_2(x^1(x^2, ...), x^2, ...) = 0$ as as algebraic function of the remaining variables, and so one. The algebraic functions involved get increasingly complex but in some exceptional situations the solution has parametric representation in terms of *rational* rather than algebraic functions of parameters $t^k$. For co-dimension $d_c > 1$ case the intersection of surfaces $P^i = 0$ need not be complete and the tangent spaces of the hyper-surfaces $P^i = 0$ need not intersect transversally in the generic case. Therefore $d_c > 1$ case is not gained so much attention as $d_c = 1$ case.

A more advanced treatment relies on ring theory by assigning to polynomials a ring as the ring of polynomials in the space involved divided by the ring of polynomials vanishing at zero loci of polynomials $P^i$.

1. The notion of ideal is central and determined as a subring invariant under the multiplication by elements of ring. Prime ideal generalizes the notion of prime and one can say that the notion of integer generalizes to that of ideal. One can also define the notion of fractional ideal.

2. Zariski topology (see `http://tinyurl.com/h5pv4vk`) replacing the topology based on real norm is second highly advanced notion. The closed sets in this topology are algebraic varieties of various dimensions. Since the complement of any algebraic variety is open set this topology and open also in the ordinary real topology, this topology is considerable rougher than the ordinary than the ordinary topology.

Some remarks from the point of view of TGD are in order.

1. In the scenario inspired by $M^8 - H$ duality one has co-dimension 4 surfaces in 8-D complex space. Octonionicity of polynomials however implies huge symmetries since the polynomial is determined by single real polynomial of real variable, whose values at finite number of points determined the polynomial.

2. In TGD the extension of rationals can be assumed to contain also powers for some root of $e$ since in p-adic context this gives rise to a finite-dimensional extensions due to the fact that $e^p$ is ordinary p-adic number. Also a restriction to a finite field are possible and restriction of rational coefficients to their modulo $p$ counterparts reduces the polynomial to polynomial in finite field. This reduction is used as a technical tool. In the case of Diophantine equations (see `http://tinyurl.com/nt6tkey` and `http://tinyurl.com/y8hm4zce`) the coefficients are restricted to be integers.

3. In adelic TGD [L15] [L14] the number fields involved are reals and extensions of p-adic numbers. The coefficient field for the coefficients of polynomials would be naturally extension of rationals or extension of p-adics induced by it. The coefficients of polynomials serve as coordinates of adelic WCW. p-Adic numbers are not algebraically closed and one must assume an extension of p-adic numbers from that for the coefficients one to allow maximal number of roots.

   This suggests an evolutionary process [L17] extending the number field for the coefficients of polynomials. Arbitrary root of polynomial for given extension can be realized only if the original extension is extended further. But this allows polynomial coefficients in this new extension: WCW is now larger. Now one has however roots in even larger extension so that the unavoidable outcome is number theoretic evolution as increase of complexity.

4. What is so remarkable is that octonionic polynomials with rational coefficients could be determined by their values at finite set of points for a polynomial of real argument once the order of polynomial is fixed. Real coordinate corresponds to preferred time axis naturally. A cognitive representation consisting of finite number of rational points could fix the entire space-time surface! This would extend ordinary holography to its discrete variant!

5. Algebraic variety is rather simple object as compared to the solutions of partial differential equations encountered in physics - say those for minimal surfaces. Now one must fix boundary values or initial values at $n-1$-dimensional surface to fix the solution. For integrable theories the situation can change. In TGD SH suggests that the classical solutions are determined by data at 2-surfaces, which together with conformal invariance could reduce the data to one-dimensional data specified by a polynomial. $M^8 - H$ correspondence allows to consider this option seriously.

6. $M^8 - H$ duality suggests that space-time surfaces are co-dimension $d_c = 4$ algebraic curves in $M^8$. Could space-time surfaces define closed sets for the analog of Zariski topology? Could string world sheets and partonic 2-surfaces do the same inside space-time surfaces? An interesting question is whether this generalizes also to the level of imbedding space $H$ and could perhaps define a topology rougher than real topology in better accord with the notion of finite measurement resolution.

## 2.2 About algebraic curves and surfaces

The realization $M^8 - H$ correspondence to be considered allows to understand space-time surfaces as 4-D complex algebraic surfaces $X^4_c$ in the space $o$ of complexified octonions projected to real sub-space of $O^c$ with Minkowskian signature. Due to the non-commutativity of quaternions, the reduction of space-time surfaces to curves in quaternionic plane is not possible. Despite this it is instructive to start from the algebraic geometry of curves and surfaces.

### 2.2.1 Degree and genus of the algebraic curve

Algebraic curve is defined as zero locus of a polynomial $P(x^1, x^2, ..., x^n)$ with $x^n$ in some - preferably algebraically closed - number field $K$ and coefficients in some number field $k \subset K$. In adelic physics $K$ corresponds to real or complex numbers and $k$ to the extension of rationals defining adeles. In p-adic sectors $k$ corresponds to tje extension of p-adic numbers induced by $k$. In general roots belong to an extension of $k$.

Degree, genus, and Euler characteristic are the basic characterizers of algebraic curve.

1. The degree $d$ of algebraic curve corresponds to the highest power for the variables appearing in the polynomial. One can also define multi-degree in an obvious manner. A useful geometric interpretation for the degree is that line intersects curve (also complex) of degree $d$ in at most $d$ points as is clear from the fact that the equation of curve reduces the equation for curve to an equation for the roots of $d$:th order polynomial of single variable.

2. Also the genus $g$ of the curve (see `http://tinyurl.com/ybm3wfue`) is important characteristic. One can distinguish between topological genus, geometric genus and arithmetic genus. For curves these notions are equivalent. The connection between genus and degree $d$ of non-singular algebraic curve is very useful:

$$g = \frac{(d-1)(d-2)}{2} . \tag{2.1}$$

   Spherical topology for complex curves corresponds to $n = 1$ and $n = 2$.

   A more general formula reads as:

$$g = \frac{(d-1)(d-2)}{2} + \frac{n_s}{2} . \tag{2.2}$$

   Here $n_s$ is the number of holes of the curve behaving like holes and increasing the genus.

3. Euler characteristic (for Euler characteristic see `http://tinyurl.com/pp52zd4`) is a homological invariant making sense in arbitrary dimension and also for manifolds. Homological definition based on simplicial homology relies on counting of simplexes of various dimension. The definition in terms of dimensions of homology groups $H_n$ is given by

$$\chi = b_0 - b_1 + b_2 ... + (-1)^n b_n , \tag{2.3}$$

   where $b_k$ is the dimension of $k$:th homology group (see `http://tinyurl.com/j48ojys`).

The following gives the engineering rules for obtaining Euler characteristic of the surface obtained from simpler building blocks. Note that algebraic variety property is not essential here.

1. Euler characteristic is homotopy invariant so that it does not change one adds homologically trivial space such as $E^n$ as a Cartesian factor.

2. $\chi$ is additive under disjoint union. Inclusion-exclusion principle states that if $M$ and $N$ intersect, one has $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$.

3. Euler characteristic for the connected sum $A\#B$ of $n$-dimensional manifolds obtained by drilling balls $B^n$ from summands, giving opposite orientation to the boundaries of the hole, and connecting with cylinder $D \times S^{n-1}$ is given by $\chi(A)+\chi(B)-\chi(S^{n-1})$. One has $\chi(S^2) = 2$ and $\chi(D^2) = 1$.

4. The Euler characteristic for product $M \times N$ is $\chi(M) \times \chi(N)$.

5. The Euler characteristic for $N$-fold covering space $M_n$ is $N \times \chi(M)$ with a correction term coming from the singularities of the covering (ramified covering space).

6. For a fibration $M \rightarrow B$ with fiber $S$, which differs from fiber bundle in that the fibers are only homeomorphic, one has $\chi(M) = \chi(B) \times \chi(S)$.

Euler characteristic and the genus of 2-surface (or complex) curve are related by the equation

$$\chi \ = \ 2(1-g) \ . \tag{2.4}$$

having values $2, 0, -2, .....$ If the 2-surface has $n_s$ holes (punctures), one has

$$\chi = 2(1-g) - n_s \ . \tag{2.5}$$

Punctures must be distinguished from singularities at which some sheets of covering meet at single point.

A formal generalization of the definition of genus for varieties in terms of Euler characteristic makes sense.

$$g \ = \ -\frac{\chi}{2} + 1 - \frac{n_s}{2} \ . \tag{2.6}$$

Disk has genus 1/2 and drilling of $n$ holes increases genus by $n/2$. Pair of holes gives same contribution to $g$ and the cylinder connecting the holes. Note that for complex curves the definition of puncture is obvious. For real curves the puncture would mean missing point of the curve.

The latter definitions of genus can be identified in terms of Euler characteristic also for higher-dimensional varieties. For curves these notions are equivalent if there are no singularities. For algebraic curves $g$ is same for the real and complex variants of the curve in $RP_1$ and $CP_1$ respectively.

### 2.2.2 Elliptic curves and elliptic surfaces

Elliptic curves (see `http://tinyurl.com/lovksny`) are cubic curves with no singularities (cusps or self-intersections) having representation of form $y^2 - x^3 - ax - b = 0$. These singularities can occur only at special values of parameters (($a = 0, b = 0$). Since the degree equals to $d = 3$, elliptic curve has genus $g = 1$.

Elliptic curves allow a group of Abelian symmetries generated by a finite number of generators. The emergence of abelian group structure can be intuitively understood as follows.

1. Given line intersects the curve of degree 3 in at most 3 points. Let $P$ and $Q$ be two of these points. Then there can be also a third intersection point $R$ and by the $Z^2$ symmetry changing the sign of $y$ also the reflection of $R$ - identify it as $-R$ - belongs to the curve. Define the sum of $P + Q$ to be $-R$.

   The actual proof is slightly more complicated since the number of intersection points for the line with curve can be also 2 or 1. By writing explicit expressions for the coordinates $x_R$ and $y_R$, one can also find that they are indeed rational if the points $P$ and $Q$ are rational. If the elliptic curve as single rational point it has infinite number of them.

2. The generators with finite order give rise to torsion. The rank of generators of infinite order is called rank and conjectured to be arbitrarily large (see `http://tinyurl.com/lovksny`) . Therefore elliptic curve is an Abelian group and one talks about Abelian variety. If elliptic curve contains a rational point it contains entire lattice of rational points obtained as shifts of this point.

**Remark:** Complex elliptic curves are 2-surfaces in complex projective plane $CP_2$ and therefore highly interesting from TGD point of view. $g = 1$ partonic 2-surfaces would in TGD framework correspond to second generation fermions [K2]. Abelian varieties define a generalization of elliptic curves to higher dimensions and simplest space-time surfaces allowing also large cognitive representations could correspond to such.

Elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) are fibrations with an algebraic curve as base space and elliptic curve as fiber (fibration is more general notion than fiber space since the fibers are only homeomorphic). The singular fibers failing to be elliptic curves have been classified by Kodaira.

## 2.3 The notion of rational point and its generalization

The notion of algebraic integer (see http://tinyurl.com/y8z389a7) makes sense for any number field as a root of a monic polynomial (polynomial with integer coefficients with coefficient of highest power equal to unity). The field of fractions for given number field consists of ratios of algebraic integers. The same is true for the notion of prime. The more precise definition forces to replace integers and primes with ideals.

Rational varieties are expressible as maps defined by rational functions with rational coefficients in some extension of $Q$ and contain infinite number of rational points. If the variety is not rational, one can ask whether it could allow a dense set of rational points with rational number replaced with the ratio of algebraic integers for some extension of $Q$. This leads to the idea of potentially rational point, and one can classify algebraic varieties according to whether they are potentially rational or not. The variety is potentially rational if it allows a parameteric representation using rational functions. Otherwise the parametric representation involves algebraic functions such as roots of rational functions.

The interpretation in terms of cognition would be that large enough extension makes the situation "cognitively easy" since cognitive representations involving fermions at the rational points and defining discretizations of the algebraic variety could be arbitrary large. The simpler the surface is cognitively, the large the number of rational points or potentially rational points is.

Complexity of algebraic varieties is measured by Kodaira dimension $d_K$ (see http://tinyurl.com/yadoj2ut). The spectrum for this dimension varies in the range $(-\infty, 0, 1, 2, ...d)$, where $d$ is the algebraic dimension of the variety. Maximal value equals to the ordinary topological dimension $d$ and corresponds to maximal complexity: in this case the set of rational points is finite. Minimal Kodaira dimension is $d_K = -\infty$: in this case the set of rational points is infinite. Rational surfaces are maximally simple and this corresponds to the existence of parametric representations using only rational functions.

### 2.3.1 Rational points for algebraic curves

The sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see http://tinyurl.com/y9oq37ce) states that a curve over $Q$ with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere $CP_1$ in $CP_2$ has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A5, A8]).

   $g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in $CP_2$ with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in $CP_2$ (see http://tinyurl.com/lovksny) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

   $g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and $CP_2$ as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in $CP_2$ (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially dense set of rational points.

**Remark**: In TGD framework algebraic varieties could be zero loci of octonionic polynomials and have algebraic dimension 4 so that the classification for algebraic curves does not help. Octonion analyticity must bring in symmetries which simplify the situation.

### 2.3.2 Enriques-Kodaira classification

The tables of (see `http://tinyurl.com/ydelr4np`) give an overall view about the Enriques-Kodaira classification of algebraic curves, surfaces, and varieties in terms of Kodaira dimension (see `http://tinyurl.com/yadoj2ut`).

1. For instance, general curves ($g \geq 2$) have $d_K = 1$, elliptic curves ($g = 1$) have $d_K = 0$ and projective line ($g = 0$) has $d_K = -\infty$. $CP_1 \subset CP_2$ is a rational curve so that rational points are dense. Elliptic curves allow infinite number or rational points forming an Abelian group if they containing single rational point and are therefore cognitively easy.

2. Algebraic varieties (with real dimension $d_R = 4$ in complex case) with $d_K = 2$ are surfaces of general type, elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) have $d_K = 1$, surfaces with attribute abelian, hyper-elliptic, K3, and Enriques, have $d_K = 0$.

   **Remark:** All real 2-surfaces are hyper-elliptic for $g \leq 2$, in other words allow $Z_2$ as global conformal symmetry. Genus-generation correspondence [K2] for fermions allows to assign to the 3 lowest generations of fermions hyper-elliptic partonic 2-surfaces with genus $g = 0, 1, 2$. These surfaces would have $d_K = 0$ and be rather simple as real surfaces in Kodaira classification. Could one regard them as $M^4$ projection of complex hyper-elliptic surfaces of real dimension $d_R = 4$? $d_K = -\infty$ holds true for rational surfaces and ruled surfaces, which allow straight line through any point are maximally simple. In complex case the line would be $CP_1$.

3. The Wikipedia article gives also a table about the classification of algebraic 3-folds. Real algebraic 3-surfaces might well occur in TGD framework. The twistor space for space-time surface might allow realization as complex 3-fold and since it has $S^2$ has fiber, it would naturally correspond to an uni-ruled surface with $d_K = -\infty$. The table shows that one can build higher dimensional algebraic varieties with $d_K < d$ from lower-dimensional ones as fiber-space like structures, which based or fiber having $d_K < d$. 3-D Abelian varieties and Calabi-Yau 3-folds are complex manifolds with $d_K = 0$, which cannot be engineered in this manner.

4. Space-time surfaces would be surfaces of algebraic dimension 4. Wikipedia tables do not give direct information about this case but one can make guesses on basis of the three tables. Octonionic polynomials are analytic continuations of real polynomials of real variable, which must mean a huge simplification, which also favor cognitive representability. The best that one might have infinite sets of rational points. The examples about extremals of Kähler action does not however favor this wish.

Bombieri-Lang conjecture (see `http://tinyurl.com/y887yn5b`) states that, for any variety $X$ of general type over a number field $k$, the set of $k$-rational points of $X$ fails to be Zariski dense (see `http://tinyurl.com/jm9fh74`) in $X$. This means that , the $k$-rational points are contained in a *finite* union of lower-dimensional sub-varieties of $X$. In dimension 1, this is exactly Faltings theorem, since a curve is of general type if and only if it has $g \geq 2$. The conjecture of Vojta (see `http://tinyurl.com/y9sttuu4`) states that varieties of general type cannot be potentially dense. As will be found, these conjectures might be highly relevant for TGD.

# 3 About enumerative algebraic geometry

Algebraic geometry is something very different from Riemann geometry, Kähler geometry, or sub-manifold geometry based on local notions. Sub-manifolds are replaced with sub-varieties defined as zero loci for polynomials with coefficients in the field of rationals or extension of rationals. Partial differential equations are replaced with algebraic ones. One can generalize algebraic geometry to any number field.

String theorists have worked with algebraic geometry with motivation coming from various moduli spaces emerging in string theory. The moduli spaces for closed and open strings possibly in presence of branes are involved. Also Calabi-Yau compacticication leads to algebraic geometry, and topological string theories of type A and B involve also moduli spaces and enumerative algebraic geometry.

In TGD the motivation for enumerative algebraic geometry comes from several sources.

1. Twistor lift of TGD lifts space-time surfaces to their 6-D twistor spaces representable as surfaces in the product of 6-D twistor spaces of $M^4$ and $CP_2$ and possessing Kähler structure - this makes these spaces completely unique and strongly suggests the role of algebraic geometry, in particular in the generalization of twistor Grassmannian approach [L13].

2. There are three threads in number theoretic vision: p-adic numbers and adelics, classical number fields, and infinite primes. Adelic physics [L15] as physics of sensory experience and cognition unifies real physics and various p-adic physics in the adele characterized by an extension of rationals inducing those of p-adic number fields. This leads to algebraic geometry and counting of points with imbedding space coordinates in the extension of rationals and defining a discrete cognitive representation. The core of the scattering amplitude would be defined by this cognitive representation identifiable in terms of points appearing as arguments of n-point function in QFT picture [L11].

3. $M^8 - M^4 \times CP_2$ duality is the analog of the rather adhoc spontaneous compactification in string models but would be non-dynamical and thus allow to avoid landscape catastrophe. Classical physics would reduce to octonionic algebraic geometry at the level of complexified octonions with several special features due to non-commutativity and non-associativity: space-time could be seen as 4-surface in the complexification of of octonions. The commuting imaginary unit would make possible the realization of algebraic extensions of rationals.

   The moduli space for the varieties is discrete if the coefficients of the polynomials are in the extension of rationals. If one poses additional conditions such as associativity of 4-surfaces, the moduli space is further reduced by the resulting criticality conditions realizing quantum criticality at the fundamental level raising hopes about extremely simple formulation of scattering amplitudes at the level of $M^8$ [L13].

   Also complex and co-complex sub-manifolds of associative space-time surface are important and would realize strong form of holography (SH). For non-associative regions of space-time surface it might not be possible to define complex and co-complex surfaces in unique manner since the basic $M^2 \subset M^4$ local flag structure is missing. String world sheets and partonic 2-surfaces and their moduli spaces are indeed in key role and the topology of partonic surfaces plays a key role in understanding of family replication phenomenon in TGD [L11].

In this framework one cannot avoid enumerative algebraic geometry.

1. One might want to know the number of points of sub-variety belonging to the number field defining the coefficients of the polynomials. This problem is very relevant in $M^8$ formulation of TGD, where these points are carriers of sparticles. In TGD based vision about cognition [L15] they define cognitive representations as points of space-time surface, whose $M^8$ coordinates can be thought of as belonging to both real number field and to extensions of various p-adic number fields induced by the extension of rationals. If these cognitive representations define the vertices of analogs of twistor Grassmann diagrams in which sparticle lines meet, one would have number theoretically universal adelic formulation of scattering amplitudes and a deep connection between fundamental physics and cognition.

2. Second kind of problem involves a set algebraic surfaces represented as zero loci for polynomials - lines and circles in the simplest situations. One must find the number of algebraic surfaces intersecting or touching the surfaces in this set. Here the notion of incidence is central. Point can be incident on line or two lines (being their intersection), line on plane, etc.. This leads to the notion of Grassmannians and flag-manifolds.

Moduli spaces parameterizing sub-varieties of given kind - lines, circles, algebraic curves of given degree, are central for the more advanced formulation of algebraic geometry. These moduli spaces emerge also in the formulation of TGD. The moduli space of conformal equivalence classes of partonic 2-surfaces is one example involved with the explanation of family replication phenomenon [K2]. One can assign moduli spaces also to octonion and quaternion structures in $M^8$ (or equivalently with the complexification of $E^8$). One can identify $CP_2$ as a moduli space of quaternionic sub-spaces of octonions containing preferred complex sub-space.

One cannot avoid these moduli spaces in the formulation of the scattering amplitudes and this leads to $M^8 - H$ duality. The hard core of the calculation should however reduce to the understanding of the algebraic geometry of 4-surfaces in octonionic space. Clearly, $M^8$ picture seems to provide the simplest formulation of the number theoretic vision.

## 3.1 Some examples about enumerative algebraic geometry

Some examples give an idea about what enumerative algebraic geometry (see `http://tinyurl.com/y7yzt67b`) is.

1. Consider 4 lines in 3-D space. What is the number of lines intersecting these 4 lines [A13] (see `http://tinyurl.com/ycrbr5aj`). One could deduce the number of lines and lines by writing the explicit equations for the lines with each line characterized by 2+3=5 parameters specifying direction $t$ vector and arbitarily chosen point $x_0$ on the line. 2+3=5 parameters characterize each sought-for line.

   For intersection points $x_i$ of sought for line with $i$:th one has $x_i = x_0 + k_i t_0$, $i = 1, ..., 4$ for the sought for line with direction $t_0$. At the intersection points at the 4 lines one has $x_i = x_{0i} + K_i t_i$ with fixed directions $t_i$. Combining the two equations for each line one has $4 \times 3 = 12$ equations and 3+4+2 parameters for the sought for line plus 4 parameters $K_i$ for the four lines. This gives 13 unknown parameters corresponding to $x_0, k_i, K_i$. One would have one parameter set of solutions: something goes wrong.

   One has however projective invariance: one can shift $x_0$ along the line by $x_0 \rightarrow x_0 - at$, $k_i \rightarrow k_i + a$ and using this freedom assume for instance $k_1 = 0$. This reduces the number of parameters to 12 and one has finite number of solutions in the generic case. Actually the number is 2 in the generic case but can be infinite in some special cases. The challenge is to deduce the number of the solutions by geometric arguments.Below Schubert's argument proving that the number of solutions is 2 will be discussed.

   The idea of enumerative geometry is to do this using general geometric arguments allowing to deform the problem topologically to a simpler one in which case the number of solutions is obvious which in the most abstract formulation become topological.

2. Apollonius can be seen as founder of enumerative algebraic geometry. Apollonian circles (see `http://tinyurl.com/ycvxe688`) represent second example. One has 3 circles in plane. What is the number of circles tangential to all these 3 circles. Wikipedia link represents the geometric solution of the problem. The number of circles is 8 in the generic case but there are exceptional cases.

3. In Steiner's conic problem (see `http://tinyurl.com/yahshsjo`) one have 5 conical sections (circles, cones, ellipsoids, hyperbole) in plane. How many different conics tangential to the conics there exist? This problem is rather difficult and the thumb rules of enumerative geometry (dimension counting, Bezout's rule, Schubert calculus) fail. This is a problem in projective geometry where one is forced to introduce moduli space for conics tangential to given conic. This space is algebraic sub-variety of all conics in plane which is 5-D projective space. One must be able to deduce the number of points in the intersection of these sub-varieties so that the original problem in 2-D plane is replaced with a problem in moduli space.

## 3.2 About methods of algebraic enumerative geometry

A brief summary about methods of algebraic geometry is in order to give some idea about what is involved (see `http://tinyurl.com/y7yzt67b`).

1. Dimension counting is the simplest method. If two geometric objects of $n$-D space have dimensions $k$ and $l$, there intersection is $n - k - l$-dimensional for $n - k - l \geq 0$ or empty in the generic case. For $k + l = n$ one obtains discrete set of intersection points.

2. Bezouts theorem is a more advanced method. Consider for instance, curves in plane defined by the curves polynomials $x = P^m(y)$ and $x = P^n(y)$ of degrees $k = m$ and $k = n$. The number $N$ of intersection points in the generic case is bounded above by $N = m \times n$ (in this case all roots are real). One can understand this by noticing that one has $m$ roots $y_k$ or given $x$ giving rise to a $m$-branched graph of function $y = f(x)$. The number of intersections for the graphs of the two polynomials is at most $m \times n$. If one has curve in plane represented by polynomial equation $P^{m,n}(x, y) = 0$, one can also estimate immediately the minimal multi-degree $(m, n)$ for this polynomials.

3. Schubert calculus `http://tinyurl.com/y766ddw2`) is a more advanced but not completely rigorous method of enumerative geometry [A13] (see `http://tinyurl.com/ycrbr5aj`).

   Schubert's vision was that the number of intersection points is stable against deformations in the generic case. This is not quite true always but in exceptional cases one can say that two separate solutions degenerate to single one, just like roots of polynomial can do for suitable values of coefficients.

   For instance, Schubert's solution to the already mentioned problem of finding a line intersecting 4 lines in generic position relies on this assumption. The idea is to deform the situation so that one has two intersecting pairs of lines. One solution to the problem is a line going through the intersection points for line pairs. Second solution is obtained as intersection of the planes. It can happen that planes are parallel in which case this does not work.

   Schubert calculus it applies to linear sub-varieties but can be generalized also to non-linear varieties. The notion of incidence allowing a general formulation for intersection and tangentiality (touching) is central. This leads to the notions of flag, flag manifold, and Schubert variety as sub-variety of Grassmannian.

   Flag is a hierarchy of incident subspaces $A_0 \subset A_1 \subset A_2 ... \subset A_n$ with the property that the dimension $d_i \leq n$ of $A_i$ satisfies $d_i \geq i$. As a special case this notion leads to the notion of Grassmannian $G(k, n)$ consisting of $k$-planes in $n$-dimensional space: in this case $A_0$ corresponds to $k$-planes and $A_2$ to space $A_n$. More general flag manifolds are moduli spaces and sub-varieties of Grassmannian providing a solution to some conditions. Flag varieties as sub-varieties of Grassmannians are Schubert varieties (see `http://tinyurl.com/y7ehcrzg`). They are also examples of singular varieties. More general Grassmannians are obtained as coset spaces of $G/P$, where $G$ is algebraic group and $P$ is parabolic sub-group of $G$.

   **Remark**: $CP_2$ corresponds to the space of complex lines in $C^3$. $CP_2$ can be also understood as the space of quaternionic planes in octonionic 8-space containing fixed 2-plane so that also now one has flag. String world sheets inside space-time surfaces define curved flags with 2-D and 4-D tangent spaces defining an integrable distribution of local flags.

4. Cohomology combined with Poincare duality allows a rigorous formulation of Schubert calculus. Schubert's idea about possibility to deform the generic position corresponds to homotopy invariance, when the degeneracies of the solutions are taken into account. Homology and cohomology become basic tools and the so called cup product for cohomology together with Poincare duality and Künneth formula for the cohomology of Cartesian product in terms of cohomologies of factors allows to deduce intersection numbers algebraically. Schubert cells define a basis for the homology of Grassmannian containing only even-dimensional generators.

   Grassmannians play a key role in twistor Grassmannian approach as auxiliary manifolds. In particular, the singularities of the integrand of the scattering amplitude defined as a multiple residue integral over $G(k, n)$ define a hierarchy of Schubert cells. The so called positive Grassmannian [B2] defines a subset of singularities appearing in the scattering amplitudes of $\mathcal{N} = 4$ SUSY. This hierarchy and its $CP_2$ counterpart are expected also in TGD framework.

   **Remark**: Schubert's vision might be relevant for the notion of conscious intelligence. Could problem solving involve the transformation of a problem to a simple critical problem, which is easy but for which some solutions can become degenerate? The transformation of general position for 4 lines to a pair of intersecting lines would be example of this. One can wonder whether quantum criticality could help problem solving by finding critical cases.

5. Moduli spaces of curves and varieties provide the most refined methods. Flag manifolds define basic examples of moduli spaces. Quantum cohomology represents even more refined conceptualization: the varieties (branes in M-theory terminology) are said to be connected or intersect if each of them has a common point with the same pseudo-holomorphic variety ("string world sheet"). Pseudo-holomorphy - which could have minimal surface property as counterpart - implies that the connecting 2-surface is not arbitrary.

   Quantum intersection for the "string world sheet" and "brane" is possible also when it is not stable classically (the co-dimension of brane is smaller than 2). Even in the case that it possible classically quantum intersection makes possible kind of "telepathic" quantum contact mediated by the "string world sheet" naturally involved with the description of quantum entanglement in TGD framework.

## 3.3 Gromow-Witten invariants

Gromow-Witten invariants repreent example of so called quantum invariants natural in string models and M-theory. They provide new invariants in algebraic and symplectic geometry.

### 3.3.1 Formal definition

Consider first the definition of Gromow-Witten (G-W) invariants (see `http://tinyurl.com/y9b5vbcw`). G-W invariants are rational number valued topological invariants useful in algebraic and symplectic geometry. These quantum invariants give information about these geometries not provided by classical invariants. Despite being rational numbers in the general case G-W invariants in some sense give the number of string world sheets connecting given branes.

1. One considers collection of $n$ surfaces ("branes") with even dimensions in some symplectic manifold $X$ of dimension $D = 2k$ (say Kähler manifold) and pseudo-holomorphic curves ("string world sheets") $X^2$, which have the property that they connect these $n$ surfaces in the sense that they intersect the "branes" in the marked points $x_i$, $i = 1, .., n$.

   "Connect" does not reduce to intersection in topologically stable sense since connecting is possible also for branes with dimension smaller than $D - 2$. One allows all surfaces that $X^2$ that intersects the $n$ surfaces at marked points if they are pseudo-holomorphic even if the basic dimension rule is not satisfied. In 4-dimensional case this does not seem to have implications since partonic 2-surfaces satisfy automatically the dimension rule. The $n$ branes intersect or touch in quantum sense: there is no concrete intersection but intersection with the mediation of "string world sheet".

2. Pseudo-holomorphy means that the Jacobian $df$ of the imbedding map $f : X^2 \rightarrow X$ commutes with the symplectic structures $j$ *resp.* $J$ of $X^2$ *resp.* $X$: i.e. one has $df(jT) = Jdf(T)$ for any tangent vector $T$ at given point of $X^2$. For $X^2 = X = C$ this gives Cauchy-Riemann conditions.

   In the symplectic case $X^2$ is characterized topologically by its genus $g$ and homology class $A$ as surface of $X$. In algebraic geometry context the degree $d$ of the polynomial defining $X^2$ replaces $A$. In TGD $X^2$ corresponds to string world sheet having also boundary. $X^2$ has also $n$ marked points $x_1, ..., x_n$ corresponding to intersections with the $n$ surfaces.

3. G-W invariant $GW_{g,n}^{X,A}$ gives the number of pseudo-holomorphic 2-surfaces $X^2$ connecting $n$ given surfaces in $X$ - each at single marked point. In TGD these surfaces would be partonic 2-surfaces and marked points would be carriers of sparticles.

The explicit definition of G-W invariant is rather hard to understand by a layman like me. I however try to express the basic idea on basis of Wikipedia definition (see `http://tinyurl.com/y9b5vbcw`). I apologize for my primitive understanding of higher algebraic geometry. The article of Vakil [L4] (see `http://tinyurl.com/ybobccub`) discusses the notion of G-W invariant in detail.

1. The situation is conformally invariant meaning that one considers only the conformal equivalence classes for the marked pseudo-holomorphic curves $X^2$ parameterized by the points of so called Deligne-Mumford moduli space $\overline{M}_{g,n}$ of curves of genus $g$ with $n$ marked points (see

http://tinyurl.com/yaq8n6dp): note that these curves are just abstract objects without no imbedding as surface to $X$ assumed. $\overline{M}_{g,n}$ has *complex* dimension

$$d_0 = 3(g-1) + n \ .$$

$n$ corresponds complex dimensions assignable to the marked points and $3(g-1)$ correspond to the complex moduli in absence of marked points. This space appears in TGD framework in the construction of elementary particle vacuum functionals [K2].

2. Since these curves must be represented as surfaces in $X$ one must introduces the moduli space $\overline{M}_{g,n}(X,A)$ of their maps $f$ to $X$ with given homology equivalence class. The elements in this space are of form $(C, x_1, .., x_n, f)$ where $C$ is one particular representative of $A$.

3. The complex dimension $d$ of $\overline{M}_{g,n}(X,A)$ can be calculated. One has

$$d = d_0 + c_1^X(A) + (g-1)k \ .$$

Here $c_1^X(A)$ is the first Chern class defining element of second cohomology of $X$ evaluated for $A$. For Calabi-Yau manifolds one has $c_1 = 0$. The contribution $(g-1)k$ to the dimension vanishing for torus topology should have some simple explanation.

4. One defines so called evaluation map $ev$ from $\overline{M}_{g,n}(X,A) \to Y$, $Y = \overline{M}_{g,n} \times X^n$ in terms of stabilization $st(C, x_1, ..., x_n) \in \overline{M}_{g,n}(X,A)$ of $C$ (I understand that stabilization means that the automophism group of the stabilized surface defined by $f$ is finite [A12] (see http://tinyurl.com/y8r44uhl). I am not quite sure what the finiteness of the automorphism group means. One might however think that conformal transformations must be in question. One has

$$ev(C, x_1, .., x_n, f) = (st(C, x_1, .., x_n), f(x_1), ...f(x_n)) \ .$$

Evaluation map assigns to the concrete realization of string world sheet with marked points the abstract curve $st(C, x_1, .., x_n)$ and points $(f(x_i), ..., f(x_n)) \in X^n$ possibly interpretable as positions $f(x_i)$ of $n$ particles. One could say that one has many particle system with particles represented by surfaces of $X_i$ of $X$ connected by $X^2$ - string world sheet - mediating interaction between $X_i$ via the intersection points.

5. Evaluation map takes the fundamental class of $\overline{M}_{g,n}(X,A)$ in $H_d(\overline{M}_{g,n}(X,A))$ to an element of homology group $H_d(Y)$. This homology equivalence class defines G-W invariant, which is rational valued in the general case.

6. One can make this more concrete by considering homology equivalence class $\beta$ in $\overline{M}_{g,n}$ and homology equivalence classes $\alpha_i$, $i = 1, ..., n$ represented by the surfaces $X_i$. The co-dimensions of these $n+1$ homology equivalence classes must sum up to $d$. The homologies of $\overline{M}_{g,n}$ and $Y = \overline{M}_{g,n} \times X^n$ induce homology of $Y$ by Künneth formula (see http://tinyurl.com/yd9ttlfr) implying that $Y$ has class of $H_d(Y)$ given by the product $\beta \cdot \alpha_1 ... \cdot \alpha_n$.

   One can identify the value of $GW_{g,n}^{X,A}$ for a given class $\beta \cdot \alpha_1 ... \cdot \alpha_n$ as the coefficients in its expansion as sum of all elements in $H_d(Y)$. This coefficient is the value of its intersection product of $GW_{g,n}^{X,A}$ with the product $\beta \cdot \alpha 1 ... \cdot \alpha_n$ and gives element of $H_0(Q)$, which is rational number.

7. There are two non-classical features. Classically intersection must be topologically stable. This would require $\alpha_i$ to have codimension 2 but all even co-dimensions are allowed. That the value for the number of connecting string world sheets is rational number does not conform with the classical geometric intuition. The Wikipedia explanation is that the orbifold singularities for the space $\overline{M}_{g,n}(X,A)$ of stable maps are responsible for rational number.

### 3.3.2 Application to string theory

Topological string theories give a physical realization of this picture. Here the review article *Instantons, Topological Strings, and Enumerative Geometry* of Szabo [A12] (see `http://tinyurl.com/y8r44uhl`) is very helpful.

1. In M-theory framework and for topological string models of type A and B the physical interpretation for the varieties associated with $\alpha_i$ would be as branes of various dimensions needed to satisfy Dirichlet boundary conditions for strings.

2. In topological string theories one considers sigma model with target space $X$, which can be rather general. The symplectic or complex structure of $X$ is however essential. $X$ is forced to be 3-D (in complex sense) Calabi-Yau manifold by consistency of quantum theory. Interestingly, the super twistor space $CP(3|4)$ is super Calabi-Yau manifold although $CP_3$ is not and must therefore have trivial first Chern class $c_1$ appearing in the formula for the dimension $d$ above. I must admit that I do not understand why this is the case.

   Closed topological strings have no marked points and one has $n = 0$. Open topological strings world sheets meet $n$ branes at points $x_i$, where they satisfy Dirichlet boundary conditions. Branes an be identified as even-dimensional Lagrangian sub-manifolds with vanishing induced symplectic form.

3. For topological closed string theories of type A one considers holomorphically imbedded curves in $X$ characterized by genus $g$ and homology class $A$: one speaks of world sheet instantons. $A = \sum n_i S_i$ is sum over the generating classes $S_i$ with integer coefficients. For given $g$ and $A$ one has analog of product of non-interacting systems at temperatures $1/t_i$ assignable to the homology classes $S_i$ with energies identifiable as $n_i$. One can assign Boltzmann weight labelled by $(g, A)$ as $Q^\beta = \prod_i Q_i^{n_i}$, $Q_i = exp(-t_i)$.

   One can construct partition function for the entire system as sum over Boltzmann weights with degeneracy factors telling the number of world sheet instantons with given $(g, A)$. One can calculate free energy as sum $\sum N_{g,\beta} Q^\beta$ over contributions labelled by $(g, A)$. The coefficients $N_{g,\beta}$ count the rational valued degeneracies of the world sheet instantons of given type and reduce to G-W invariants $GW_{g,0}^{X,A}$.

   **Remark**: If one allows powers of a root $e^{-1/n}$, $t = n$, in the extension of rationals or replace $e^{-t}$ with $p^n$, partition functions make sense also in the p-adic context.

4. For topological open string theories of type A one has also branes. Homology equivalence classes are relative to the brane configuration. The coefficients $N_{g,\beta}$ are given by $GW_{g,n}^{X,A}$ for a given configuration of branes: the above described general formulas correspond to these.

5. For topological string theories of type B, string world sheets reduce to single point and thus correspond to constant solutions to the field equations of sigma model. Quantum intersection reduces to ordinary intersection and one has $x_1 = x_2 ... = x_n$. G-W invariants involve only classical cohomology and give for $n = 2$ the number of common points for two surfaces in $X$ with dimension $d_1$ and $d_2 = n - d$. The duality between topological string theories of type A and B related to the mirror symmetry supports the idea that one could generalize the calculation of these invariants in theories B to theories A. It is not clear whether this option as any analog in TGD.

The so called Witten conjecture (see `http://tinyurl.com/yccahv3q`) proved by Kontsevich states that the partition function in one formulation of stringy quantum gravity and having as coefficients of free energy G-W invariants of the target space is same as the partition function in second formulation and expressible in terms of so called tau function associated with KdV hierarchy. This leads to non-trivial identities. Witten conjecture actually follows from the invariance of partition function with respect to half Virasoro algebra and Virasoro conjecture (see `http://tinyurl.com/y7xcc9hm`) stating just this generalizes Witten's conjecture.

## 3.4 Riemann-Roch theorem

Riemann-Roch theorem (RR) is also part of enumerative geometry albeit more abstract. Instead of counting of numbers of points, one counts dimensions of various function spaces associated with Riemann surfaces. RR provides information about the dimensions for the spaces of meromorphic functions and 1-forms with prescribed zeros and poles.

### 3.4.1 Basic notions

Riemann surface is the basic notion. Riemann surface is orientable is characterized by its genus $g$ and number of holes/punctures in it. Riemann surface can also possess marked points, which seem to be equivalent with punctures. The moduli space of these complex curves is parameterized by a moduli space $\overline{M}_{g,n}$ of curves of genus $g$ with $n$ marked points (see `http://tinyurl.com/yaq8n6dp`) (see `http://tinyurl.com/yaq8n6dp`).

Dolbeault cohomology (see `http://tinyurl.com/y7cvs5sx`) generalizes the notion of differential form so that it has has well-defined degrees with respect to complex coordinates and their conjugates: one can write in general complex manifold this kind of form as

$$\omega = \omega_{i_1 i_2 .. i_n, j_1 j_2 ... j_n} dz^{i_1} \wedge dz^{i_2} ... dz^{i_n} d\overline{z}^{j_1} \wedge d\overline{z}^{j_2} ... dz^{j_n} \quad .$$

The ordinary exterior derivative $d$ is replaced with its holomorphic counterpart $\partial$ and its conjugate. One can construct the counterparts of cohomology groups (Hodge theory) $H^{p,q} = H^{q,p}$. Betti numbers as numbers $h_{i,j}$ defining the dimensions of the cohomology groups forms of degrees $i$ and $j$ with respect to $dz^i$ and $d\overline{z}^j$. One can define the holomorphic Euler's characteristic as $\chi_C = h_{0,0} - h_{01} = 1 - g$ whereas orinary Euler characteristic is $\chi_R = h_{0,0} - (h_{01} + h_{10}) + h_{1,1} = 2(1 - g)$.

One considers meromorphic functions having poles and zeros as the only singularities (points at which the map does not preserve angles): rational functions provide the basic example. Riemann zeta provides example of meromorphic function not reducing to rational function. Holomorphic functions have only zeros and entire functions have neither zeros nor poles. If analytic functions on Riemann surfaces can be interpreted as maps of compact Riemann surface to itself rather than to complex plane, meromorphy reduces to holomorphy since the point $\infty$ belongs to the Riemann surface.

The elements of free group of divisors are defined as formal sums of integers associated with the points $P$ of Riemann surface. Divisors $D = \sum_P n(P)$, where $(P)$ is integer, are analogous to integer valued "wave functions" on Riemann surface. The number of points with $n(P) \neq 0$ is countable. The degree of divisor is obtained as the ordinary sum $deg(D)$ of the integers defining the divisor.

Although divisors can be seen as purely formal objects, they are in practice associated to both meromorphic functions and 1-forms. The divisor of a meromorphic function is known as principal divisor. Meromorphic functions and 1-forms differing by a multiplication with meromorphic function are regarded as linearly equivalent - in other words, one can add to a given divisor a divisor of a meromorphic function without changing its equivalence class. It can be shown that all divisors associated with meromorphic 1-forms linearly equivalent and one can talk about canonical divisor. Note that $deg(D)$ is linear invariant since the degree of globally meromorphic function is zero.

The motivation for the divisors is following. Consider the space of meromorphic functions $h$ with the property that the degrees of poles associated with the poles of these functions are not higher than given integers $n(P)$. In other words, one has $\langle h(P) \rangle + D(P) \geq 0$ for all points $P$ ($\langle h \rangle$ is the divisor of $h$). For $D(P) > 0$ the pole has degree not higher than $D(P)$. For non-positive $D(P)$ the function has zero of order $D(P)$ at least.

### 3.4.2 Formulation of RR theorem

With these prerequisites it is possibly to formulate RR (for Wikipedia article see `http://tinyurl.com/mdmbcx6`). The Wikipedia article is somewhat confusing and a more precise description of RR can be found in the article "Riemann-Roch theorem" by Vera Talovikova [A14] (see `http://tinyurl.com/ktww7ks`).

Let $l(D)$ be the dimension of the space of meromorphic functions with principal divisor $D$ or 1-forms linearly equivalent with canonical divisor $K$. Then the equality

$$l(D) - l(K - D) = deg(D) - g + 1 \quad (3.1)$$

is true for both meromorphic functions and canonical divisors. For $D = K$ one obtains using $l(0) = 1$

$$l(K) = deg(K) - g + 2 \quad (3.2)$$

giving the dimension of the space of canonical divisors. $l(K) > 0$ in general is not in conflict with the fact that canonical divisors are linearly equivalent. $deg(K) = 2g - 2$ in the above formula gives $l(K) = g$.

$l(K) = 0$ for $g = 0$ case looks strange: one should actually make notational distinction between dimensions of spaces of meromorphic functions and one-forms (this is done in the article of Talivakova). The explanation is that $l(K)$ here is not the dimension of the space of canonical 1-forms but that of the holomorphic functions with the divisor of $K$. The canonical form $K$ for the sphere has second order pole at $\infty$ so that one cannot have meromorphic forms holomorphic outside $P$.

Riemann's inequality

$$l(D) \geq deg(D) - g + 1 \quad (3.3)$$

follows from $l(K - D) \geq 0$, which can be seen as a correction term to the formula

$$l(D) = deg(D) - g + 1 \ . \quad (3.4)$$

In what sense this is true, becomes clear from what follows.

### 3.4.3 The dimension of the space meromorphic functions corresponding to given divisor

The simplest divisor associated with meromorphic function involves only one point. Multiplying a function, which is non-vanishing and finite at $P$ by $(z - z(P))^{-n}$ gives a pole of order $n$ (zero has negative order in this sense). One is interested on the dimension $l(nP)$ of the space $nP$ of meromorphic functions and RR allows to deduce information about $l(nP)$. One is interested on the behavior of $l(nP)$ as function of genus $g$ of Riemann surface (more general situation would allow also punctures). For $n = 0$ one has entire function without poles and zeros. Only constant function is possible: $l(0) = 1$.

First some general observations. $K$ has degree $deg(K) = 2g - 2$, which gives $l(K) = g$. For $n = deg(D) > deg(K) = 2g - 2$ the correction term vanishes since $deg(K - D)$ becomes negative, and one has $l(D) = deg(D) - g + 1$. This gives $l(n = 2g - 1) = g$. Therefore $n \in \{2g - 1, 2g, ...\}$ corresponds to $l(nP) \in \{g, g + 1, ...\}$. $n < 2g - 2$ corresponds to $l(nP) = 1$. What about the range $n \in \{2, ..., 2g - 2\}$? Note that $2g - 2$ is the negative of the Euler character of Riemann surface.

1. $g = 0$ case. $K$ on sphere. dz canonical 1-form on Riemann sphere covered by two complex coordinate patches. $z \rightarrow w = 1/z$ relates the coordinates. There is second order pole at infinity ($dw = -dz/z^2$). One has therefore $deg(K) = -2$ for sphere in accordance with the general formula $deg(K) = 2g - 2$. The formula $l(nP) = deg(D) + 1$ holds for all $n$ and there is no correction term now. One as $l(nP) = n + 1$ .

2. $g = 1$ case.

   One has $deg(K) = 2g - 2 = 0$ for torus reflecting the fact that the canonical form $\omega = dz$ has no poles or zeros (torus is obtained by identifying the cells of a periodic lattice in complex plane). Correction term vanishes since it would have negative degree for all $n$ and one has $l(nP) \in \{1, 1, 2, 3, ...\}$.

3. $g = 2$ case.

   For $n = deg(D) \geq 2 \times 2 - 1 = 3$ gives $l(D) = n - 1$ giving for $n \geq 3$ $l(nP) \in \{2, 3, ...\}$. What about $n = g = 2$? For generic points one has $l(2) = 1$. There are 6 points at which one has $l(D) = 2$ so that there is additional meromorphic function having pole of order 2 at this kind of point. These points are fixed points under $Z_2$ defining hyper-ellipticity. Note that $g \leq 2$ Riemann surfaces are always hyper-elliptic in the sense that it allows $Z_2$ as conformal symmetry (see `http://tinyurl.com/y9sdu4o3`).

   One has therefore $l(nP) \in \{1, 1, 1, 2, ..\}$ for a generic point and $l(nP) \in \{1, 1, 2, 2......\}$ for 6 points fixed under $Z_2$. An interesting question is whether this phenomenon could have physical interpretation in TGD framework.

4. $g > 2$ case.

   For $g > 2$ . $l(nP)$ in the range $\{2, 2g - 2\}$ can depend on point and even on the conformal moduli. There are more special points in which correction term differs from that in the generic case. $g = 3$ illustrates the situation. $n \in \{1, 1, 1, 1, 1, 2, ...\}$ is obtained for a generic point. At special points and for $n < 3$ there are also other options for $l(nP)$. Also the dependence of $l(nP)$ on moduli emerges for $g \geq 3$. The natural guess layman is that these points are fixed points of conformal symmetries. Also now hyper-elliptic surfaces allowing projective $Z_2$ covering are special. In the general case hyper-ellipticity is not possible.

In TGD framework Weierstrass points(see `http://tinyurl.com/y9wehsml`) are of special interest physically.

1. My layman guess is that special points known as Weierstrass points (see `http://tinyurl.com/y9wehsml`) correspond to singularities for projective coverings for which conformal symmetries permute the sheets of the covering. Several points coincide for the covering since a sub-group of conformal symmetries would act trivially on the Weierstrass point.

   Note that for $g > 2$ $Z_2$ covering is not possible except for hyper-elliptic surfaces, and one can wonder whether this relates to the experimental absence fo $g > 2$ fermion families [K2]. Second interesting point is that elementary particles indeed correspond to double sheeted structures from the condition that monopole fluxes flow along closed flux tubes (there are no free magnetic monopoles).

2. There is an obvious analogy with the coverings associated with the cognitive representation defined by the points of space-time surface with coordinates in an extension of rationals [L15, L11] [L14]. Fixed points for a sub-group of Galois group generate singularities at which sheets touch each other. These singular points are physically the most interesting and could carry sparticles. The action of discrete conformal groups restricted to cognitive representation could be represented as the action of Galois group on points of cognitive representation. Cognitive representation would indeed represent!

   Remarkably, if the tangent spaces are not parallel for the touching sheets, these points are mapped to several points in $H$ in $M^8 - H$ correspondence. If this picture is correct, the hyper-elliptic symmetry $Z_2$ of genera $g \leq 2$ could give rise to this kind of exceptional singularities for $g \geq 2$.

   What is worrying that there are two views about twistorial amplitudes. One view relying on the notion of octonionic super-space $M^8$ [L11] is analogous to that of SUSYs: sparticles can be seen as completely local composites of fermions. Second view relies on imbedding space $M^4 \times CP_2$ [K32] and on the identification sparticles as non-local many-fermion states at partonic 2-surfaces. These two views could be actually equivalent by $M^8 - H$ duality.

3. When these singular points are present at partonic 2-surfaces at boundaries of CD and at vertices, the topology of partonic 2-surface is in well-defined sense between $g$ and $g+1$ external particles: one has criticality. The polynomials representing external particles indeed satisfy criticality conditions guaranteeing associativity or co-associativity (quantum criticality of TGD Universe is the basic postulate of quantum TGD). At partonic orbits the touching pieces of partonic 2-surface could separate ($g$) or fuse ($g + 1$). Could this topological mixing give rise to CKM mixing of fermions [K2, K9, K11]?

#### 3.4.4 RR for algebraic varieties and bundles

RR can be generalized to algebraic varieties (see http://tinyurl.com/y9asz4qg). In complex case the real dimension is four so that this generalization is interesting from TGD point of view and will be considered later. The generalization involves rather advanced mathematics such as the notion of sheaf (see http://tinyurl.com/nudhxo6). Zeros and poles appearing in the divisor are for complex surfaces replaced with 2-D varieties so that the generalization is far from trivial.

The following is brief summary based on Wikipedia article.

1. Genus $g$ is replaced with algebraic genus and $deg(D)$ plus correction term is replaced with the intersection number (see http://tinyurl.com/y7dcffb6) for $D$ and $D-K$, where $K$ is the canonical divisor associated with 2-forms, which is also unique apart from linear equivalence Points of divisor are replaced with 2-varieties.

2. The generalization to complex surfaces (with real dimension equal to 4) reads as

$$\chi(D) = \chi(0) + \frac{1}{2} D \cdot (D-K) \ . \tag{3.5}$$

   $\chi(D)$ is holomorphic Euler characteristic associated with the divisor. $\chi(0)$ is defined as $\chi(0) = h_{0,0} - h_{0,1} + h_{0,2}$, where $h_{i,j}$ are Betti numbers for holomorphic forms. '$\cdot$' denotes intersection product in cohomology made possibly by Poincare duality. $K$ is canonical two-form which is a section of determinant bundle having unique divisor (their is linear equivalence due to the possibility to multiply with meromorphic function.

   One has $\chi(0) = 1 + p_a$, where $p_a$ is arithmetic genus. Noether's formula gives

$$\chi(0) = \frac{c_1^2 + c_2}{12} = \frac{K \cdot K + e}{12} \ . \tag{3.6}$$

   $c_1^2$ is Chern number and $e = c_2$ is topological Euler characteristic.

   Clearly the information given by $\chi(D)$ is about Dolbeault homology. For comparison note that RR for curves states $l(D) - l(K-D) = \chi(D) = \chi(0) + deg(D)$.

RR can be also generalized so that it applies to vector bundles. Ordinary RR can be interpreted as applying to a bundle for which the fiber is point. This requires the notion of the inverse bundle defined as a bundle with the property that its direct sum (Whitney sum) with the bundle itself is trivial bundle. One ends up with various characteristic classes, which represent homologically non-trivial forms in the base spaces characterizing the bundle. For instance, the generalizations of RR give information about the dimensions of the spaces of sections of the vector bundle.

Atyiah-Singer index theorem (see http://tinyurl.com/k6daqco) deals with so called elliptic operators in compact manifolds and represents a generalization important in recent theoretical physics, in particular gauge theories and string models. The theorem relates analytical index - typically characterizing the dimension for the spectrum of solutions of elliptic operator to a topological index. Elliptic operator is assigned with small perturbations for a given solution of field equations. Perturbations of a given solution of say Yang-Mills equations is a representative example.

# 4 Does $M^8 - H$ duality allow to use the machinery of algebraic geometry?

The machinery of algebraic geometry is extremely powerful. In particular, the number theoretical universality of algebraic geometry implies that same equations make sense for all number fields: this is just what adelic physics [L15] [L14] demands. Therefore it would be extremely nice if one could somehow use this machinery also in TGD framework as it is used in string models. How this could be achieved? There are several guide lines.

1. Twistor lift of TGD [K28, K26, K25, K32] is now a rather well-established idea although a lot of work remains to be done with the details. Twistors were originally introduced in order to be able to use this machinery and involves complexification of Minkowski space $M^4$ to $M_c^4$ as an auxiliary tool. Complexification in sufficiently general sense seems to be a necessary auxiliary tool but it cannot be a trick (like Wick rotation) but something fundamental and here complexification at the level of $M^8$ is suggestive. In the sequel I will used $M^4$ for $M_c^4$ and $M^8$ for $M_c^8$ unless it is necessary to emphasize that $M_c^8$ is in question. The essential point is that the Euclidian metric is complexified and it reduces to a real metric in various sub-spaces defining besides Euclidian space also Minkowski spaces with varying signature when the complex coordinates are real or imaginary.

2. If $M^8 - H$ duality holds true, one can solve field equations in $M^8 = M^4 \times E^8$ by assuming that either the tangent space or normal space of the space-time surface $X^4$ is associative (quaternionic) at each point and contains preferred $M^2$ in its tangent space. $M^2$ could depend on $x$ but $M^2(x)$:s should integrate to a 2-surface. This allows to map space-time surface $M^8$ to a surface in $M^4 \times CP_2$ since tangent spaces are parameterized by points of $CP_2$ and $CP_2$ takes the role of moduli space. The image of tangent space as point of $CP_2$ is same irrespective of whether one has quaternions or complexified quaternions ($H_c$).

   It came a surprise that associativity/co-associativity is possible only if the space-time surface is critical in the sense that some gradients of 8 complex components of the octonionic polynomial $P$ vanish without posing them as additional conditions reducing thus the dimension of the space-time surface. This occurs when the coefficients of $P$ satisfy additional conditions. One obtains associative/co-associative space-time regions and regions without either property and they correspond nicely to two solution types for the twistor lift of Kähler action.

3. Contrary to the original expectations, $M^4 \subset M_c^8$ must be identified as co-associative (co-quaternionic) subspace so that $E^4$ is the associative/quaternionic sub-space. This allows to have light-cone boundary as the counterpart of point-like singularity in ordinary algebraic geometry and also allows to understand the emergence of CDs and ZEO.

**Remark**: A useful convention to be used in the sequel. $RE(o)$ and $IM(o)$ denote the real and imaginary parts of the octionion in the decomposition $o = RE(o) + IM(o)I_4$ and $Re(o)$ and $Im(o)$ its real number valued and purely imaginary parts in the usual decomposition.

The problems related to the signature of $M^4$ have been a longstanding head-ache of $M^8$ duality.

1. The intuitive vision has been that the problems can be solved by replacing $M^8$ with its complexification $M_c^8$ identifiable as complexified octonions $o$. This requires introduction of imaginary unit $i$ commuting with the octonionic units $E^k \leftrightarrow (1, I_1, ..., I_7)$. The real octonionic components are thus replaced with ordinary complex numbers $z_i = x_i + iy_i$.

2. Importantly, complex conjugation $o \to \overline{o}$ changes only the sign of $I_i$ but *not!* that of $i$ so that the octonionic inner product $(o_1, o_2) = o_1\overline{o}_2 = o_1^k o_2^l \delta_{k,l}$ becomes complex valued. Norm is equal to $O\overline{O} = \sum_i z_i^2$. Both norm and inner product are in general complex valued and real valued only in sub-spaces in which octonionic coordinates are real or imaginary. Sub-spaces have all possible signatures of metric. These sub-spaces are not closed under multiplication and this has been an obstacle in the earlier attempts based on the notion of octonion analyticity. This argument applies also to quaternions and one obtains signatures $(1,1,1,1)$, $(1,1,1,-1)$, $(1,1,-1,-1)$, and $(1,-1,-1,-1)$. Why just the usual Minkowskian signature $(1,-1,-1,-1)$ is physical, should be understood.

   The convention consistent with that used in TGD corresponds to a negative length squared for space-like vectors and positive for time-like vectors. This gives $m = (o^0, io^1, ..., io^7)$ with real $o^k$. The projection $M_c^8 \to M^8$ defines the projection of $X_c^4 \subset M_c^8$ to $X^4 \subset M^8$ serving as the pre-image of $X^4 \subset M^8$ in $M^8 - H$ correspondence.

3. $o$ is not field anymore as is clear from the fact that $1/o = \overline{o}/o\overline{o}$ is formally infinite in Minkowskian sub-spaces, when octonion defines a light-like vector. $o$ (and $H_c$) remains however a ring so that sum and products are well-defined but division can lead to problems unless one stays inside 7+7-dimensional light-cone with $Re(o\overline{o}) > 0$ ($Re(q\overline{q}) > 0$).

Although the number field structure is lost, one can still define polynomials needed to define algebraic varieties by requiring their simultaneous vanishing and rational functions make sense inside the light-cone. Also rational functions can be defined but poles are replaced with light-cones in Minkowskian section. Algebraic geometry would thus be forced by the complexification of octonions. This looks to me highly non-trivial! The extension of zeros and poles to light-cones making propagation possible could be a good reason for why Minkowskian signature is physical. Interestingly, the allowed octonionic momenta are light-like quaternions [K32].

4. An interesting question is whether ZEO and the emergence of CDs relates to the failure of field property. It seems now clear that CDs must be assigned even with elementary particles. I have asked whether they could form an analog for the covering of manifold by coordinate patches (in TGD inspired theory of consciousness CDs would be correlates for perceptive fields for conscious entities assignable to CDs [L17]). These observations encourage to ask whether the tips of CD should correspond to a pair formed by two poles/two zeros or by pole and zero assignable to positive and negative energy states.

   It turns out that the space-time surfaces in the interior of CD would naturally correspond to non-associative surfaces and only their 3-D boundaries would have associative 4-D tangent spaces allowing mapping to $H$ by $M^8$-duality, which is enough by holography.

5. The relationship between light-like 3-surface bounding Minkowskian and Euclidian space-time regions and light-like boundaries of CDs is interesting. Could also the partonic orbits be understood a singularities of octonionic polynomials with $IM(P) = RE(P) = 0$?

## 4.1 What does one really mean with $M^8 - H$ duality?

The original proposal was that $M^8$ duality should map the associative tangent/normal planes of associative/co-associative space-time surface containing preferred $M^2$, call it $M_0^2$, to $CP_2$: the map read as $(m, e) \in M^4 \times E^4 \to (m, s) \in M^4 \times CP_2$. Eventually it became clear that the choice of $M^2$ can depend on position with $M^2(x)$ forming an integrable distribution to $CP_2$: this would define what I have called Hamilton-Jacobi structures [K22]. String like objects have minimal surface as $M^4$ projection for almost any general coordinate invariant action, and internal consistency requires that $M^2(x)$ integrate to a minimal surface. The details are however not understood well enough.

1. $M^4$ coordinate would correspond simply to projection to a fixed $M_0^4$ in the decomposition $M^8 = M_0^4 \times E_0^4$. One can however challenge this interpretation. How $M_0^4$ is chosen? Is it possible to choose it uniquely? Could also $M^4$ coordinates represent moduli analogous to $CP_2$ coordinates? What about ZEO?

   There is an elegant general manner to formulate the choice of $M_0^4$ at the level of $M^8$. The complexified quaternionic sub-spaces of $M_c^8$ ($M^8$) are parameterized by moduli space defining the quaternionic moduli. The moduli space in question is $CP_2$. The choice of $M_0^4$ corresponds to fixing of the quaternionic moduli by fixing a point of $CP_2$.

   **Warning**: Note that one should be very careful in distinguishing between quaternionic as sub-spaces of $M^8$ and as the tangent space $M^8$ of given point of $M^8$.

2. One can ask whether there could be a connection with ZEO, where CDs play a key role. Indeed, the complexified Minkowski inner product means that the complexified octonions (quaternions) inside $M_c^8$ ($M_c^4$) have inverse only inside 7-D (4-D) complexified light-cone and this would motivate the restriction of space-time surfaces inside future or past light-cone or both but not yet force CD.

   If one allows rational functions and even meromorphic functions of octonionic or quaternionic variable, one could consider the possibility of restricting the space-time surface defined as their zeros to a maximally sized region containing no poles.

3. Consider complexified quaternions $H_c$. Poles $(q\overline{q})^{-n}$, $n \geq 1$ would correspond $M^4$ light-cone boundaries since $q\overline{q} = 0$ at them. Also zeros $q\overline{q} = 0$, for $n \geq 1$ correspond to light-like boundaries. Could one have two poles with with time-like distance defining CD or a pair of pole and zero?

There is also a possible connection with the notion of infinite primes [K13]. The notion of infinite prime leads to the proposal that rationals defined as ratios of infinite integers but having unit real norm (and also p-adic norms) could correspond pairs of positive and negative energy states with identical total quantum numbers and located at opposite boundaries of CD. Infinite rationals can be mapped to rational functions. Could positive energy states correspond to the numerators with zeros at second boundary of CD and negative energy states to denominators with zeros at opposite boundary of CD?

#### 4.1.1 Is the choice of the pair $(M_0^2, M_0^4)$ consistent with the properties of known extremals in $H$

It should be made clear that the notion of associativity/co-associativity (quaternionicity/co-quaternionicity) of the tangent/normal space need not make sense at the level of $H$. I shall however study this working hypothesis in the sequel.

The choice of the pair $(M_0^2, M_0^4)$ means choosing preferred co-commutative (commutative) subspace $M_0^2$ of $M^8$ defining a subspace of fixed co-quaternionic (quaternionic) sub-space $M_0^4 \subset M^8$.

**Remark**: $M^4$ should indeed be the co-associative/co-quaternionic subspace of $M^8$ if the argument about emergence of CDs is accepted and if $M^8 - H$ correspondence maps associative to associative and co-associative to co-associative.

$M_0^4$ in turn contains preferred $M_0^2$ defining co-commutative (hyper-complex) structure. Both $M_0^2$ and $M_0^4$ are needed in order to label the choice by $CP_2$ point (that is as a point of Grassmannian).

Is the projection to a fixed factor $M_0^4 \subset M_0^4 \times E^4$ as a choice of co-quaternionic moduli consistent with what we know about the extremals of twistor lift of Kähler action in $H$? How could one fix $M_0^4$ from the data about the extremal in $H$? One can make similar equations about the choice of $M_0^2$ as a fixed co-complex moduli characterized by a unit quaternion. Note that this choice is expected to relate closely to the twistor structure and Kähler structure.

It is best to check the proposal for the known extremals in $H$ [K22]. Consider first $CP_2$ type extremals for which $M^4$ projection is a piece of light-like geodesic.

1. The $CP_2$ projection for the image of $X^4 \subset M^8$ differs from single point only if the tangent space isomorphic to $M^4$ and parameterized by $CP_2$ point varies. Consider $CP_2$ type extremals for the twistor lift of Kähler action [**?**]n $H$ having light-like geodesic as $M^4$ projection as an example. The light-like geodesic defines a light-like vector in the tangent space of $CP_2$ type extremal. This light-like vector together with its dual spans fixed $M^2$, which however does not belong to the tangent space so that associative surface would not be in question.

   What about co-associativity or associativity (the latter is favored by above argument)? This property should hold true for the pre-image of $CP_2$ type extremal in $M^8$ but I am not able to say anything about this. It is questionable to require this property at the level $H$ but one can of course look what it would give.

   What about associativity for $CP_2$ tangent space? The normal space of $CP_2$ type extremal is 3-D (!) since the only the light-like tangent vector of the geodesic and 2 vectors orthogonal to it are orthogonal to $CP_2$ tangent vectors. For Euclidian signature this would mean that tangent space is 5-D and cannot be associative but now the tangent space is 4-D. Can one still say that tangent space is associative. The co-associativity of the tangent space makes sense trivially. Can one conclude that $CP_2$ is co-associative.

   The associativity for $CP_2$ tangent space might make sense since the tangent space is 4-D. The light-like vector $k$ defines $M_0^2$. The associativity conditions involving two tangent space vectors of $CP_2$ and the light-like vector $k$ contracted with the corresponding octonion components. The contributions from the components of $k$ to the associator should cancel each other. Since one can change the relative sign of the components of $k$, this mechanism does not seem to work for all components. Hence associativity cannot hold true. Neither does $M_0^2$ belong to the normal space since $k$ and its dual are not orthogonal.

   Could one conclude that $CP_2$ type extremal is co-associative in accordance with the original belief thanks to the light-like projection to $M^4$? This does not conform with what the singularity considerations for the octonionic polynomials would suggest. Or is it simply not correct to try to apply associativity at the level of $H$. Or does $M^8 - H$ correspondence map associative tangent spaces to co-associative ones?

2. The normal space $M^4$ of $CP_2$ type extremal have all orientations characterized by its $CP_2$ projection. The normal space must contain the $M^2_0$ determined by the tangent of the light-like geodesic and this is indeed the case. Note that $CP_2$ type extremals cannot have entire $CP_2$ as $CP_2$ projection: they necessarily have hole at either end, which would be naturally be at the boundary of CD.

   $CP_2$ type extremals seem to be consistent with $M^8 - H$ correspondence. It however seems that one cannot fix the choice of $M^4_0$ uniquely in terms of the properties of the extremal. There is a moduli space for $M^4_0$:s defined by $CP_2$ and obviously codes for moduli for quaternion structures in octonionic space. The distributions of $M^2(x)$ (minimal surfaces) would code for quaternion structures (decomposition of octonionic coordinates to quaternionic coordinates in turn decomposing to pairs of complex coordinates).

Consider next the associativity condition for cosmic strings in $X^2 \times Y^2 \subset M^4 \times CP_2$. Now $CP_2$ projection is 2-D complex surfaces and $M^4$ projection is minimal surface. Situation is clearly associative. How unique the choice of $M^4_0$ is now?

1. Now $M^2(x)$ depends on position but $M^2(x)$:s define an integrable distribution defining string orbit $X^2$ as a minimal surface. $M^4_0$ must contain all surfaces $M^2(x)$, which would fix $M^4_0$ to a high degree for complex enough cosmic strings.

2. Each point of $X^2$ corresponds to the same partonic surface $Y^2 \subset CP_2$ labelling the tangent spaces for its pre-image in $M^8$. All the tangent surfaces $M^2(x) \times E^2(y)$ for $X^2 \times Y^2 \subset M^8$ share only $M^2(x) \subset M^4_0$. $M^4_0$ must contain all tangent spaces $M^2(x)$ and the inverse image of $Y^2 \subset CP_2$ must belong to the orthogonal complement $E^4$ of $M^4_0$. This is completely analogous with the condition $X^2 = X^2 \times Y^2 \subset M^4 \times CP_2$.

   Consider a decomposition $M^8 = M^4_0 \times E^4$, $M^4_0 = M^2_0 \times E^2_0$. If the inverse image of $Y^2$ at point $x$ belongs to $E^4$, the $M^4_0$ projection belongs to $M^4_0$ also in $M^8$. If this does not pose any condition on the tangent spaces assignable to the points of $Y^2$ defining points of $CP_2$, there are no problems. What could happen that the tangent spaces assignable to $Y^2$ could force the projection of the inverse image of $Y^2$ to intersect $M^4_0$.

One should also understand massless extremals (MEs). How to choose $M^4_0$ in this case?

1. MEs are given as zeros of arbitrary functions of $CP_2$ coordinates and 2 $M^4$ coordinates $u$ and $v$ representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant and define $M^4_0 = M^2_0 \times E^2_0$ decomposition everywhere so that $M^4_0$ is uniquely defined. Same applies also when the directions are not constant. In the general case light-like direction would define the local tangent plane of string world sheet and local polarization plane. Since the dimension of $M^4$ projection is 4 there seems to be no problems involved.

2. Tangent plane of $X^4$ is parameterized by $CP_2$ coordinates depending on 2 coordinates $u$ and $v$. The surface $X^4 \subset M^8$ must be graph for a map $M^4_0 \to E^4$ so that a 2-parameter deformation of $M^4_0$ as tangent plane is in question. The distribution of tangent planes of $X^4 \subset M^8$ is 2-D as is also the $CP_2$ projection in $H$.

To sum up, the original vision about the associativity properties of the known extremals at level of $H$ survives. On the other hand, CDs emerge if $M^4$ corresponds to the co-associative part of $O$. Does this mean that $M^8 - H$ correspondence maps associative to co-associative by multiplying the quaternionic tangent space in $M^8$ by $I_4$ to get that in $H$ and vice versa or that the notions of associative and co-associative do not make sense at the level of $H$? This does not affect the correspondence since the same $CP_2$ point parametrizes both associative tangent space and its complement.

### 4.1.2 Space-time surfaces as co-dimension 4 algebraic varieties defined by the vanishing of real or imaginary part of octonionic polynomial?

If the theory intended to be a theory of everything, the solution ansatz for the field equations defining space-time surfaces should be ambitious enough: nothing less than a general solution of field equations should be in question.

1. One cannot exclude the possibility that all analytic functions of complexified octonionic variable with real Taylor or even Laurent coefficients. These would would a commutative and associative algebra. Space-time surfaces would be identified as their zero loci. This option is however number theoretically attractive and can also leads to problems with adelic physics. Since Taylor series at rational point need not anymore give a rational value.

2. Polynomials of complexified octonion variable $o$ with real coefficients define the simplest option but also rational functions formed as ratios of this kind of polynomials must be considered. Polynomials form a non-associative ring allowing sum, product, and functional decomposition as basic operations. If the coefficients $o_n$ of polynomials are complex numbers $o_n = a_n + ib_n$, $a_n, b_n$ real, where $i$ refers to the commutative imaginary unit complexifying the octonions, the ring is associative. It is essential to allow only powers $o^n$ (or $(o - o_0))^n$ with $o_0 = a_0 + ib_0$, $a_0, b_0$ real numbers). Physically this means that a preferred time axis is fixed. This time axis could connect the tips of CD in ZEO.

   One can write

$$P(o) = \sum_k p_k o^k \equiv RE(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) + IM(P)(q_1, q_2, \overline{q_1}, \overline{q}_2) \times I_4 \quad , p_k \ \text{real} \quad , \tag{4.1}$$

   where the notations

$$o = q_1 + q_2 I_4 \quad , \quad q_i = z_i^1 + z_i^2 I_2 \quad , \quad \overline{q}_i = z_i^1 - z_i^2 I_2 \quad , z_i^j = x_i^j + i y_i^j \tag{4.2}$$

   Note that the conjugation does *not* change the sign of $i$. Due to the non-commutativity of octonions $P^i$ as functions of quaternions are in general *not* analytic in the sense that they would be polynomials of $q_i$ with real coefficients! They are however analytic functions of $z_i$. The real and imaginary parts of $x_i^j$ correspond to Minkowskian and Euclidian signatures.

   In adelic physics coefficients $o_n$ of the octonionic polynomials define WCW coordinates and should be rational numbers or rationals in the extension of rationals defining the adele. The polynomials form an associative algebra since associativity holds for powers $o^n$ multiplied by real number. Thus complex analyticity crucial in algebraic geometry would be a key element of adelic physics.

3. If the preferred extremals correspond to the associative algebra formed by these polynomials, one could construct a completely general solution of the field equations as zero loci of their real or imaginary parts and build up of new solutions using algebra operation sum, product, and functional decomposition. One could identify space-time regions as associative or co-associative algebraic varieties in terms of these polynomials and they would form an algebra.

The motivation for this dream comes from 2-D electrostatics, where conducting surfaces correspond to curves at which the real part $u$ or imaginary part $v$ of analytic function $w = f(z) = u + iv$ vanishes. In electrostatics curves form families with curves orthogonal to each other locally and the map $w = u + iv \rightarrow v - iu$ defines a duality in which curves of constant potential and the curves defining their normal vectors are mapped to each other.

1. The generalization to the recent situation would be vanishing of the imaginary part $IM(P)$ or real part $RE(P)$ of the octonionic polynomial, where real and imaginary parts are defined

via $o = q_c^1 + q_c^2 I_4$. One can consider also the possibility that imaginary or real part has constant value $c$ are restricted to be rational so that one can regard the constant value set also as zero set for a polynomial with constant shift. Note that the rationals could be also complexified by addition of $i$. One would have

$$RE(P)(z_i^k) \quad \text{or} \quad IM(P)(z_i^k) = c \ , \quad c = c_0 \ \text{rational} \ . \tag{4.3}$$

$c_0$ must be real. These two options should correspond to the situations in which tangent space or normal space is associative (associativity/co-associativity). Complexified space-time surfaces $X_c^4$ corresponding to different constant values $c$ of imaginary or real part (with respect to $i$) would define foliations of $M_c^8$ by locally orthogonal 4-dimensional surfaces in $M_c^8$ such that normal space for surface $X_c^4$ would be tangent space for its co-surface.

CDs and ZEO emerges naturally if the $IM(o)$ corresponds to co-quaternionic part of octonion.

2. It must be noticed that one has moduli space for the quaternionic structures even when $M_0^4$ is fixed. The simplest choices of complexified quaternionic space $H_c = M_{c,0}^4$ containing preferred complex plane $M_{c,0}^2$ and its orthogonal complement are parameterized by $CP_2$. More complex choices are characterized by the choice of distribution of $M^2(x)$ integrable to (presumably minimal) 2-surface in $M^4$. Also the choice of the origin matters as found and one has preferred coordinates. Also the 8-D Lorentz boosts give rise to further quaternionic moduli. The physically interesting question concerns the interpretation of space-time surfaces with different moduli. For instance, under which conditions they can interact?

The proposal has several extremely nice features.

1. Single real valued polynomial of real coordinate extended to octonionic polynomial and fixed by real coefficients in extension of rationals would determine space-time surfaces.

2. The notion of analyticity needed in concrete equations is just the ordinary complex analyticity forced by the octonionic complexification: there is no need for the application to have left- or right quaternion analyticity since quaternionic derivatives are not needed. Algebraically one has the most obvious guess for the counterpart of real analyticity for polynomials generalized to octonionic framework and there is no need for the quaternionic generalization of Cauchy-Riemann equations [A15, A6] [A15, A6] (`http://tinyurl.com/yb8l34b5`) plagued by the problems with the definition of differentiation in non-commutative and non-associative context. There would be no problems with non-associativity and non-commutativity thanks to commutativity of complex coordinates with octonionic units.

3. The vanishing of the real or imaginary part gives rise to 4 conditions for 8 complex coordinates $z_1^k$ and $z_2^k$ allowing to solve $z_2^k$ as algebraic functions $z_2^k = f^k(z_1^l)$ or vice versa. The conditions would reduce to algebraic geometry in complex co-dimension $d_c = 4$ and all methods and concepts of algebraic geometry can be used! Algebraic geometry would become part of TGD as it is part of M-theory too.

## 4.2 Is the associativity of tangent-/normal spaces really achieved?

The non-trivial challenge is to prove that the tangent/normal spaces are indeed associative for the two options. The surfaces $X_c^4$ are indeed associative/co-associative if one considers the *internal* geometry since points are in $M_c^4$ or its orthogonal complement.

One should however prove that $X_c^4$ are also associative *as sub-manifolds* of $O$ and therefore have quaternionic tangent space or normal space at each point parameterized by a point of $CP_2$ in the case that tangent space containing position dependent $M_c^2$, which integrate to what might be called a 2-D complexified string world sheet inside $M_c^4$.

1. The first thing to notice that associativity and quaternionicity need not be identical concepts. Any surface with complex dimension $d < 4$ in $O$ is associative and any surface with dimension

$d > 4$ co-associative. Quaternionic and co-quaternionic surfaces are 4-D by definition. One can of course ask whether one should consider a generalization of brane hierarchy of M-theory also in TGD context and allow associativity in its most general sense. In fact, the study of singularity of $o^2$ shows that 6 and 5-dimensional surfaces are allowed for which the only interpretation would be as co-associative spaces. This exceptional situation is due to the additional symmetries increasing the dimension of the zero locus.

2. One has clearly quaternionicity at the level of $o$ obtained by putting $Y = 0$ and at the level of the tangent space for the resulting surface. The tangent space should be quaternionic. The Jacobian of the map defined by $P$ is such that it takes fixed quaternionic subspace $H_c \to M^4_{0,c}$ of $O$ to a quaternionic tangent space of $X^4$. The Jacobian applied to the vectors of $H_c$ gives the octonionic tangent vectors and they should span a quaternionic sub-space.

3. The notion of quaternionic surface is rigorous. $M^8 - H$ correspondence could be actually interpreted in terms of the construction of quaternionic surface in $M^8$. One has 4-D integrable distribution of quaternionic planes in $O$ with given quaternion structure labelled by points of $CP_2$ and has representation at the level of $H$ as space-time surface and should be preferred extremals. These quaternion planes should integrate to a slicing by 4-surfaces and their duals. One obtains this slicing by fixing the values 4 of the suitably defined octonionic coordinates $P^i$, $i = 1,..,8$, to a real constants depending on the surface of the slicing. This gives a space-time surfaces for which tangent space-spaces or normal spaces are quaternionic.

   The first guess for these coordinates $P^i$ be as real or imaginary parts of real polynomials $P(o)$. But how to prove and understand this?

Could the following argument be more than wishful thinking?

1. In complex case an analytic function $w(z) = u + iv$ of $z = x + iy$ mediates a map between complex planes $Z$ and $W$. One can interpret the imaginary unit appearing in $w$ locally as a tangent vector along $u = constant$ coordinate line.

2. One can interpret also octonionic polynomials with real coefficients as mediating a map from octonionic plane $O$ to second octonionic plane, call if $W$. The decomposition $P = P^{1)} + P^{2)} I_4$ would have interpretation in terms of coordinates of $W$ with coordinate lines representing quaternions and co-quaternions.

3. This would suggests that the quaternionic coordinate lines in $W$ can be identified as coordinate curves in $O$ - that space-time surfaces - which are quaternionic/co-quaternionic surfaces for $P^1 = constant/P^2 = constant$ lines. One would have a representation of the same thing in two spaces, and if sameness includes also quaternionicity/co-quaternionicity as attributes, then also associativity and co-associativity should hold true.

The most reasonable approach is based on generality. Associativity/quaternionicity means a slicing of octonion space by orthogonal quaternionic and co-quaternionic 4-D surfaces defined by constant value surfaces of octonionic polynomial with real coefficients. This slicing should make sense also for quaternions: one should have a slicing by complex and co-complex (commutative/co-commutative) surfaces and in TGD string world sheets and partonic 2-surfaces assignable to Hamilton-Jacobi structure would define this kind of slicing. In the case of complex numbers one has a slicing in terms of constant value curves for real and imaginary parts of analytic function and Cauchy-Riemann equations should define the property and co-property. The first guess that the tangent space of the curve is real or imaginary is wrong.

### 4.2.1 Could associativity and commutativity conditions be seen as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "Whatever it is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: $k$-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of division algebras by assuming only algebra property by using Cayley-Dickson construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the structure already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

### 4.2.2 Complex curves in real plane cannot have real tangent space

Going from octonions to quaternions to complex numbers, could constant value curves of real and imaginary parts of ordinary analytic function in complex plane make sense? The curves $u = 0$ and $v = 0$ of functions $f(z) = u + iv$, $z = x + iy$ define a slicing of plane by orthogonal curves completely analogous to its octonionic and quaternionic variants. Can one say that the tangent vectors for these curves are real/imaginary? For $u = 0$ these curves have tangent $\partial_x u + i\partial_y u$, which is not real unless one has $f(z) = k(x + iy)$, $k$ real.

Reality condition is clearly too strong. In fact, it is the well-ordering of the points of the 1-dimensional curve, which is the property in question and lost for complex numbers and regained at $u = 0$ and $v = 0$ curves. The reasonable interpretation is in terms of hierarchy of conditions multilinear in the gradients of coordinates proposed above and linear Cauchy-Riemann conditions is the only option in the case of complex plane. What is special in this curves that the tangent vectors define flows which by Cauchy-Riemann conditions are divergenceless and irrotational locally.

Pessimistic would conclude that since the conjecture fails except for linear polynomials in complex case, it fails also in the case of quaternions and octonions. For quaternionic polynomial $q^2$ the conditions are however satisfied and it turns out that the resulting conditions make sense also in the general case. Optimistic would argue that reality condition is not analogous to commutativity and associativity so that this example tells nothing. Less enthusiastic optimist might admit that the reality condition is a natural generalization to complex case but that the conjecture might be true only for a restricted set of polynomials - in complex case of for $f(z) = kz$, $k$ real. In quaternionic and octonionic case but hopefully for a larger set of polynomials with real coefficients, maybe even all polynomials with real coefficients.

### 4.2.3 Associativity and commmutativity conditions as a generalization of Cauchy-Rieman conditions?

Quaternionicity in the octonionic case, complexity in quaternionic case, and what-ever-it-is in complex case should be seen as a 3-levelled hierarchy of geometric conditions satisfied by polynomial maps with real coefficients for polynomials in case of octonions and quaternions. Of course, also Taylor and even Laurent series might be considered. The "whatever-it-is" cannot be nothing but Cauchy-Riemann conditions defining complex analyticity for complex maps.

The hierarchy looks obvious. In the case of Cauchy-Riemann conditions one has commutative and associative structure and Cauchy-Riemann conditions are linear in the partial derivatives. In the case of commutative sub-manifolds of quaternionic space the conditions are quadratic in the partial derivatives. In the case of associative sub-manifolds of octonionic space the conditions are trilinear in partial derivatives. One would have nothing but a generalization of Cauchy-Riemann equations to multilinear equations in dimensions $D = 2^k$, $k = 1, 2, 3$: $k$-linearity with $k = 1, 2, 3$!

One can continue the hierarchy of number fields by assuming only algebra property by adding additional imaginary units as done in Cayley-Hamilton construction (see `http://tinyurl.com/ybuyla2k`) by adding repeatedly a non-commuting imaginary unit to the algebra already obtained and thus doubling the dimension of the algebra each time. Polynomials with real coefficients should still define an associative and commutative algebra if the proposal is to make sense. All these algebras are indeed power associative: one has $x^m x^n = x^{m+n}$. For instance, sedenions define 16-D algebra. Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions? Could this hierarchy corresponds to a hierarchy of analyticities satisfying generalized Cauchy-Riemann conditions?

One would have also a nice physical interpretation: in the case of quaternions one would have "quaternionic conformal invariance" as conformal invariances inside string world sheets and partonic 2-surfaces in a nice agreement with basic vision about TGD. At the level of octonions would have "quaternionic conformal invariance" inside space-time surfaces and their duals. What selects the preferred commutative or co-commutative surfaces is of course an interesting problem. Is a gauge choice in question? Are these surfaces selected by some special property such as singular character? Or does one have wave function in the set of these surfaces for a given space-time surface?

#### 4.2.4 Could quaternionic polynomials define complex and co-complex surfaces in $H_c$?

What about complex and co-complex (commutative/co-commutative) surfaces in the space of quaternions? One would have a slicing of the quaternionic space by pairs of complex and co-complex surfaces and would have natural identification as quaternion/Hamilton-Jacobi structure and relate to the decomposition of space-time to string world sheets and partonic 2-surfaces. Now the condition of associativity would be replaced with commutativity.

1. In the quaternionic case the tangent vectors of the 2-D complex sub-variety would be commuting. Can this be the case for the zero loci real polynomials $P(q)$ with $IM(P) = 0$ or $RE(P) = 0$? In this case the commutativity condition is that the tangent vectors have imaginary parts (as quaternions) proportional to each other but can have different real parts. The vanishing of cross product is the condition now and involves only two vectors whereas associativity condition involves 3 vectors and is more difficult.

2. The tangent vectors of a commutative 2-surface commute: $[t^1, t^2] = 0$. The commutator reduces to the vanishing of the cross product for the imaginary parts:

$$Im(t^1) \times Im(t^2) = 0 \ . \tag{4.4}$$

3. Expressing $z_1^i$ as a function of $z_2^k$ and using $(z_1^i, z_2^k)$ as coordinates in quaternionic space, the tangent vectors in quaternionic spaces can be written in terms of partial derivatives $\partial z_1^{1)}/\partial z_2^{k)}$ as

$$t_k^i \ = \ (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta_k^i) \ , \tag{4.5}$$

Here the first part corresponds to $RE(t^i)$ as quaternion and second part to $IM(t^i)$ as quaternion.

The condition that the vectors are parallel implies

$$\frac{\partial z_1^{1)}}{\partial z_2^{k)}} = 0 \ . \tag{4.6}$$

At the commutative 2-surface $X^2$ $z_1^{1)}$ is constant and $z_1^{2)}$ is a function of $z_2^{1)}$ and $z_2^{2)}$. One would have a graph of a function $z_1^{2)} = f_2(z_2^{k)})$ at $X^2$ but not elsewhere. One could regard $z_1^{1)}$ as an extremum of a function $z_1^{1)} = f_1(z_2^{k)})$.

How to interpret this result?

1. In the generic case this condition eliminates 1 dimension so that that 2-D surface would reduce to a 1-D curve.

2. If one poses constraints on the coefficients of $P(q)$ analogous to the conditions forcing the potential function for say cusp catastrophe to have degenerate extrema at the boundaries of the catastrophe one can get 2-D solution. For these values of parameters the conditions would be equivalent with $RE(P)=0$ or $IM(P)=0$ conditions.

   The vanishing of the gradient of $z_1^1$ would indeed correspond in the case of cups catastrophe to the condition for the co-incidence of two roots of the behavior variable $x$ as extremum of potential function $V(x,a,b)$ fixing the control variable $a$ as function of $b$.

   This would pose constraints on the coefficients of $P$ not all polynomials would be allowed. This kind of conditions would realize the idea of quantum criticality of TGD at the level of quaternion polynomials. This option is attractive if realizable also at the level of octonion polynomials. This turns out to be the case.

3. One would thus have two kinds of commutative/co-commutative surfaces. The generic 1-D surfaces and 2-D ones which are commutative/commutative and critical and assignable to string world sheets and partonic 2-surfaces. 1-D surfaces would correspond to fermion lines at the orbits of partonic 2-surfaces appearing in the twistor amplitudes in the interaction regions defined by CDS. 2-D surfaces would correspond to the orbits of fermionic strings connecting point-like fermions at their ends and serving as correlates for bound state entanglement for external fermions arriving into CD. This picture would allow also to understand why just some string world sheets and partonic 2-surfaces are selected.

The simplest manner to kill the proposal is to look for $P=q^2$ and $RE(P(q^2))=0$ surface. In this case this condition is indeed satisfied. One has

$$
\begin{array}{ll}
RE(P) = X^{1)} + X^{2)}I_1 \ , & \\
X^{1)} = (z_1^{1)})^2 - (z_1^{2)})^2 + (z_2^{1)})^2 - (z_2^{2)})^2 \ , & X^{2)} = 2z_1^{1)}z_1^{2)}I_1 \ , \\
IM(P) = Y^{1)} + Y^{2)}I_1 \ , & \\
Y^{1)} = (z_2^{1)} + \overline{z_2^{1)}})z_1^{1)} \ , & Y^{2)} = (z_2^{2)} + \overline{z_2^{2)}})z_1^{2)}
\end{array}
\quad (4.7)
$$

$X^{2)}=0$ gives $z_1^{1)}z_1^{2)}=0$ so that one has either $z_1^{1)}=0$ or $z_1^{2)}=0$. $X^{1)}=0$ gives for $z_1^{1)}=0$ $z_1^{2)}=\pm\sqrt{(z_2^{1)})^2+(z_2^{2)})^2}$.

The partial derivative $\partial z_1^{1)}/\partial z_2^{k)}$ is from implicit function theorem - following from the vanishing of the differential $d(RE(P))$ along the surface - proportional $\partial X^{1)}/\partial z_2^{k)}$, but vanishes as required.

Clearly, the quaternionic variant of the proposal survives in the simplest case its simplest test. 2-D character of the surface would be due to the criticality of $q^2$ making it possible to satisfy the conditions without the reduction of dimension.

### 4.2.5 Explicit form of associativity/quaternionicity conditions

Consider now the explicit conditions for associativity in the octonionic case.

1. One should calculate the octonionic tangent (normal) vectors $t^i$ for $X=0$ in associative ($Y=0$ in co-associative case) and show that there associators $Ass(t^i,t^j,t^k)$ vanish for all possible or all possible combinations $i,j,k$. In other words, one that one has

$$
Ass(t^i,t^j,t^k) = 0 \ , \quad i,j,k \in \{1,..,4\} \ , \quad Ass(a,b,c) \equiv (ab)c - a(bc) \ . \quad (4.8)
$$

One can cast the condition to simpler from by expressing $t^i$ as octonionic vectors $t^i_k E^k$:

$$Ass(E^a,E^b,E^b) = \equiv f^{abcd}E_d \ , \quad a,b,c,d \in \{1,..,7\} \ ,$$
$$f^{abcd} = \epsilon^{abe}\epsilon_e^{\ cd} - \epsilon^{aed}\epsilon^{bc}_{\ \ e} = 2\epsilon^{abe}\epsilon_e^{\ cd} \ . \quad (4.9)$$

The permutation symbols for a given triplet $i,j,k$ are structures constants for quaternionic inner product and completely antisymmetric (see `http://tinyurl.com/p42tqsq`).. $\epsilon_{ijk} = 1$ is true for the seven triplets $123, 145, 176, 246, 257, 347, 365$ defining quaternionic sub-spaces with 1-D intersections. The anti-associativity condition $(E_iE_j)E_k = -(E_iE_j)E_k$ holds true so that one has obtains the simpler expression for $f^{ijks}$ having values $\pm 2$.

Using this representation $Ass(t^i,t^j,t^k)$ reduces to 7 conditions for each triplet:

$$t^i_r t^j_s t^k_t f^{rstu} = 0 \ , \quad i,j,k \in \{1,..,4\} \ , \quad r,s,t,u \in \{1,..,7\} \ . \quad (4.10)$$

2. If the vanishing condition $X = 0$ or $Y = 0$ is crucial for associativity then every polynomial is its own case to be studied separately and a general principle behind associativity should be identified: the proposal is as a non-linear generalization of Cauchy-Riemann conditions. As the following little calculation shows, the vanishing condition indeed appears as one calculates partial derivatives $\partial z_1^{k)}/\partial z_2^{l)}$ in the expression for the tangent vectors of the surface deduced from the vanishing gradient of $X$ or $Y$.

3. I have proposed the octonionic polynomial ansatz already earlier but failed to prove that it gives associative tangent or normal spaces. Besides the intuitive geometric argument I failed to notice that the complex 8-D tangent vectors in coordinates $z_1^{k)}$ or $z_2^{k)}$ for complexified space-time surface and coordinates $(z_1^{k)}, z_2^{k)})$ for $o$ have components

$$\frac{\partial o^i}{\partial z^1_k} \leftrightarrow (\delta^i_k, \frac{\partial z_2^{i)}}{\partial z_1^{k)}})$$

or

$$(\frac{\partial o^i}{\partial z^2_k}) \leftrightarrow (\frac{\partial z_1^{i)}}{\partial z_2^{k)}}, \delta^i_k) \ . \quad (4.11)$$

These vectors correspond to complexified octonions $O_i$ given by

$$\delta^i_k E^k + \frac{\partial z_2^{i)}}{\partial z_1^{k)}} E^k E_4 \ , \quad (4.12)$$

where the unit octonions are given by $(E_0, E_1, E_2, E_3) = (1, I_1, I_2, I_3)$ and $(E_5, E_5, E_7, E_8) = (1, I_1, I_2, I_3)E_4$. The vanishing of the associators stating that one has

4. One can calculate the partial derivatives $\frac{\partial z_i^k}{\partial z_j^k}$ explicitly without solving the equations or the complex valued quaternionic components of $RE(P) \equiv X = 0$ or $IM(P) \equiv Y = 0$ (note that $X$ and $Y$ have for complex components labelled by $X^i$ and $Y^i$ respectively.

$$Y^i(z_1^{k)},z_2^{l)}) = c \in R \ , \quad i=1,...,4 \ , \quad \text{associativity} \ ,$$

or

$$X^i(z_1^{k)},z_2^{l)}) = c \in R \ , \quad i=1,...,4 \ , \quad \text{co-associativity} \ . \tag{4.13}$$

explicitly and check whether associativity holds true. The derivatives can be deduced from the constancy of $Y$ or $X$.

5. For instance, if one has $z_2^{k)}$ as function of $z_1^{k)}$, one obtains in the associative case

$$\begin{aligned} & RE(Y)^i{}_k + IM(Y)^i{}_k \frac{\partial z_2^{r)}}{\partial z_1^{k)}} = 0 \\ & RE(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_1^{k)}} \ , \qquad IM(Y)^i{}_k \equiv \frac{\partial Y^i}{\partial z_2^{k)}} \ . \end{aligned} \tag{4.14}$$

In co-associative case one must consider normal vectors expressible in terms of $Y^i$ so that $X$ is replaced with $Y$ in these equations.

This allows to solve the partial derivatives needed in associator conditions

$$\frac{\partial z_2^{i)}}{\partial z_1^{k)}} = \left[Im(Y)^{-1}\right]^i{}_r Re(Y)^r{}_k \ . \tag{4.15}$$

6. The vanishing conditions for the associators are however multilinear and one can multiply each factor by the matrix $IM(P)$ without affecting the condition so that $IM(P)^{-1}$ disappears and one obtains the conditions for vectors

$$\begin{aligned} & T^i_r T^j_s T^k_t f^{rstu} = 0 \ , \qquad i,j,k \in \{1,..,4\} \ , \quad r,s,t,u \in \{1,..,7\} \ , \\ & T^i = IM(Y)^i{}_k E^k - RE(Y)^i{}_k E^k E_4 \ . \end{aligned} \tag{4.16}$$

This form of conditions is computationally much more convenient.

How to solve these equations?

1. The antisymmetry of $f^{rstu}$ with respect to the first two indices $r,s$ leads one to ask whether one could have

$$T^i_r T^j_s T^k_t = 0 \tag{4.17}$$

for the 7 quaternionic triplets. This is guaranteed if one has either $RE(Y)^i{}_k = \partial Y^i/\partial z_1^k = 0$ (coquaternionic part of $T^i$) or $IM(Y)^i{}_k = \partial Y^i/\partial z_2^k = 0$ (co-quaternionic part of $T^i$) for *one* member in each triplet.

The study of the structure constants listed above shows that indices 1,2,3 are contained in all 7 triplets. Same holds for the indices appearing in any quaternionic triplet. Hence it is enough to require that three gradients $RE(Y)^ik = 0$ or $IM(Y)^i{}_k = 0$ $k \in \{1,2,3\}$ vanish.

This condition is obviously too strong since already single vanishing condition reduces the dimension of space-time variety to 3 in the generic case and it becomes trivially associative.

Octonionic automorphism group $G_2$ gives additional basis with their own quaternion triplets and the general condition would be that 3 partial derivatives vanish for a triplet obtained from the basic triplet $\{1, 2, 3\}$ by $G_2$ transformation. It is not quite clear to me whether the $G_2$ transformation can depend on position on space-time surface.

2. As noticed, the vanishing of all triplets is an un-necessarily strong condition. Already the vanishing of single gradient $RE(Y)^i{}_k$ or $IM(Y)^i{}_k$ reduces the dimension of the surface from 4 to 3 in the generic case. If one accepts that the dimension of associative surface is lower than 4 then single criticality condition is enough to obtain 3-D surface.

   In the generic case associativity holds true only at the 4-D tangent spaces of 3-surfaces at the ends of CD (at light-like partonic orbits it holds true trivially in 4-D) and that the twistor lift of Kähler action determines the space-time surfaces in their interior.

   In this case one can map only the boundaries of space-time surface by $M^8 - H$ duality to $H$. The criticality at these 3-surfaces dictates the boundary conditions and provides a solution to infinite number of conditions stating the vanishing of SSA Noether charges at space-like boundaries. These space-time regions would correspond to the regions of space-time surfaces inside CDs identifiable as interaction regions, where Kähler action and volume term couple and dynamics depends on coupling constants.

   The mappability of $M^8$ dynamics to $H$ dynamics in all space-time regions does not look feasible: the dynamics of octonionic polynomials involves no coupling constants whereas twistor lift of Kähler action involves couplings parameters. The dynamics would be non-associative in the geometric sense in the interior of CDs. Notice that also conformal field theories involve slight breaking of associativity and that octonions break associativity only slightly ($a(bc) = -(ab)c$ for octonionic imaginary units). I have discussed the breaking of associativity from TGD viewpoint in [K24] .

3. Twistor lift of Kähler action allows also space-time regions, which are minimal surfaces [L2] and for which the coupling between Kähler action and volume term vanishes. Preferred extremal property reduces to the existence of Hamilton-Jacobi structure as image of the quaternionic structure at the level of $M^8$. The dynamics is universal as also critical dynamics and independent of coupling constants so that $M^8 - H$ duality makes sense for it. External particles arriving into CD via its boundaries would correspond to critical 4-surfaces: I have discussed their interpretation from the perspective of physics and biology in [L3].

4. One should be able to produce associativity without the reduction of dimension. One can indeed hope of obtaining 4-D associative surfaces by posing conditions on the coefficients of the polynomial $P$ by requiring that one $RE(Y)^i_k$ or $IM(Y)^i_k$, $i = i_1$ -call it just $X_1$ - should vanish so that $Y^i$ would be critical as function of $z_1^k$ or $z_2^k$.

   At $X_1 = 0$ would have degenerate zero at the 4-surface. The decomposition of $X_1$ to a product of monomial factors with root in extension of rationals would have one or more factors appearing at least twice. The associative 4-surfaces would be ramified. Also the physically interesting p-adic primes are conjectured to be ramified in the sense that their decomposition to primes of extension of rationals contains powers of primes of extension. The ramification of the monomial factors is nothing but ramification for polynomials primes in field of rationals in terms of polynomial primes in its extension.

   This could lead to vanishing of say one triplet while keeping $D = 4$. This need not however give rise to associativity in which case also second $RE(Y)^i_i$ or $IM(Y)^i_k$, $i = i_2$, call it $X_2$, should vanish. The maximal number of $X_i$ would be $n_{max} = 3$. The natural condition consistent with quantum criticality of TGD Universe would be that the variety is associative but maximally quantum critical and has therefore dimension $D = 3$ or $D = 4$. Stronger condition allows only $D = 4$.

   These $n \leq 3$ additional conditions make the space-time surface analogous to a catastrophe with $n \leq 3$ behavior variables in Thom's classification of 7 elementary catastrophes with less than 11 control variables [A2]. Thom's theory does not apply now since it has only one potential function $V(x)$ (now $n \leq 3$ corresponding to the critical coordinates $Y^i$!) as a function of behaviour variables and control variables). Also the number of non-vanishing coefficients in the polynomial having values in an extension of rationals and acting as control variables is unlimited. In quaternionic case the number of potential functions is indeed 1 but the number of control variables unlimited.

5. One should be able to understand the $D = 3$ associative objects - say light-like 3-surfaces or 3-surfaces at the boundaries of CD - as 3-surfaces along which 4-D associative (co-associative) and non-associative (non-co-associative) surfaces are glued together.

   Consider a product $P$ of polynomials allowing 3-D surfaces as necessarily associative zero loci to which a small interaction polynomial vanishing at the boundaries of CD (proportional to $o^n$, $n > 1$) is added. Could $P$ allow 4-D surface as a zero locus of real or imaginary part and containing the light-like 3-surfaces thanks to the presence of additional parameters coming from the interaction polynomial. Can one say that this small interaction polynomial would generate 4-D space-time in some sense? 4-D associative space-time regions would naturally emerge from the increasing algebraic complexity both via the increase of the degree of the polynomial and the increase of the dimension of the extension of rationals making it easier to satisfy the criticality conditions!

   There are two regions to be considered: the interior and exterior of CD. Could associativity/co-associativity be possible outside CD but not inside CD so that one would indeed have free external particles entering to the non-associative interaction region. Why associativity conditions would be more difficult to satisfy inside CD? Certainly the space-likeness of $M^4$ points with respect to the preferred origin of $M^8$ in this region should be crucial since Minkowski norm appears in the expressions of $RE(P)$ and $IM(P)$.

Do the calculations for the associative case generalize to the co-associative case?

1. Suppose that one has possibly associative surface having $RE(P) = 0$. One would have $IM(P) = 0$ for dual space-time surface defining locally normal space of $RE(P) = 0$ surface. This would transform the co-associativity conditions to associativity conditions and the preceding arguments should go through essentially as such.

   Associative and co-associative surfaces would meet at singularity $RE(P) = IM(P) = 0$, which need not be point in Minkowskian signature (see $P = o^2$ example in the Appendix) and can be even 4-D! This raises the possibility that the associative and co-associative surfaces defined by $RE(P) = 0$ and $IM(P) = 0$ meet along 3-D light-like orbits partonic surfaces or 3-D ends of space-time surfaces at the ends of CD.

2. If $D = 3$ for associative surfaces are allowed besides $D = 4$ as boundaries of 4-surfaces, one can ask why not allow $D = 5$ for co-associative surfaces. It seems that they do not have a reasonable interpretation as a surface at which co-associative and non-co-associative 4-D space-time regions would meet. Or could they in some sense be geometric "co-boundaries" of 4-surfaces like branes in M-theory serve as co-boundaries of strings? Could this mean that 4-D space-time-surface is boundary of 5-D co-associative surface defining a TGD variant of brane with strings world sheets replaced with 4-D space-time surfaces?

   What came as a surprise that $P = o^2$ allows 5-D and 6-D surfaces as zero loci of $RE(P)$ or $IM(P)$ as shown in Appendix. The vanishing of the entire $o^2$ gives 4-D interior or exterior of CD forced also by associativity/co-associativity and thus maximal quantum criticality. This is very probably due to the special properties of $o^2$ as polynomial: in the generic case the zero loci should be 4-D.

This discussion can be repeated for complex/co-complex surfaces inside space-time surfaces associated with fermionic dynamics.

1. Associativity condition does not force string world sheets and partonic 2-surfaces but they could naturally correspond to commutative or co-commutative varieties inside associative/co-associative varieties.

In the generic case commutativity/co-commutativity allows only 1-D curves - naturally light-like fermionic world lines at the boundaries of partonic orbits and representing interacting point-like fermions inside CDs and used in the construction of twistor amplitudes [K26, K32]. There is coupling between Kähler part and volume parts of modified Dirac action inside CDs so that coupling constants are visible in the spinor dynamics and in dynamics of string world sheet.

2. At criticality one obtains 2-D commutative/co-commutative surfaces necessarily associated with external particles quantum critical in 4-D sense and allowing quaternionic structure. String world sheets would serve as correlates for bound state entanglement between fermions at their ends. Criticality condition would select string world sheets and partonic 2-surfacs from the slicing of space-time surface provided by quaternionic structure (having Hamilton-Jacobi structure as $H$-counterpart).

If associativity holds true and fixed $M_c^2$ is contained in the tangent space of space-time surface, one can map the $M^4$ projection of the space-time surface to a surface in $M^4 \times CP_2$ so that the quaternionic tangent space at given point is mapped to $CP_2$ point. One obtains 4-D surface in $H = M^4 \times CP_2$.

1. The condition that fixed $M_c^2$ belongs to the tangent space of $X_c^4$ is true in the sense that the coordinates $z_2^{k)}$ do not depend on $z_1^{1)}$ and $z_1^{2)}$ defining the coordinates of $M_c^2$. It is not clear whether this condition can be satisfied in the general case: octonionic polynomials are expected to imply this dependence un-avoidably.

   The more general condition allows $M_c^2$ to depend on position but assumes that $M_c^2$:s associated with different points integrate to a family 2-D surfaces defining a family of complexified string world sheets. In the similar manner the orthogonal complements $E_c^2$ of $M_c^2$ would integrate to a family of partonic 2-surfaces. At each point these two tangent spaces and their real projections would define a decomposition analogous to that define by light-like momentum vector and polarization vector orthogonal to it. This decomposition would define decomposition of quaternionic sub-spaces to complexified complex subspace and its co-complex normal space. The decomposition would correspond to Hamilton-Jacobi structure proposed to be central aspect of extremals [K22].

2. What is nice that this decomposition of $M_c^4$ ($M^4$) would (and of course should!) follow automatically from the octonionic decomposition. This decomposition is lower-dimensional analog to that of the complexified octonionic space induced by level sets of real octonionic polymials but at lower level and extremely natural due to the inclusion hierarchy of classical number fields. Also $M_c^2$ could have similar decomposition orthogonal complex curves by the value sets of polynomials. The hierarchy of grids means the realization of the coordinate grid consisting of quaternionic, complex, and real curves for complexified coordinates $o^k$ and their quaternionic and complex variants and is accompanied by corresponding real grids obtained by projecting to $M^4$ and mapping to $CP_2$.

   Thus these decompositions would be obtained from the octonionic polynomial decomposing it to real quaternionic and imaginary quaternionic parts first to get a grid of space-time surfaces as constant value sets of either real or imaginary part, doing the same for the non-constant quaternionic part of the octonionic polynomial to get similar grid of complexified 2-surfaces, and repeating this for the complexified complex octonionic part.

Unfortunately, I do not have computer power to check the associativity directly by symbolic calculation. I hope that the reader could perform the numerical calculations in non-trivial cases to this!

#### 4.2.6 General view about solutions to $RE(P) = 0$ and $IM(P) = 0$ conditions

The first challenge is to understand at general level the nature of $RE(P) = 0$ and $IM(P) = 0$ conditions. Appendix shows explicitly for $P(o) = o^2$ that Minkowski signature gives rise to unexpected phenomena. In the following these phenomena are shown to be completely general but not quite what one obtains for $P(o) = o^2$ having double root at origin.

1. Consider first the octonionic polynomials $P(o)$ satisfying $P(0) = 0$ restricted to the light-like boundary $\delta M^8_+$ assignable to 8-D CD, where the octonionic norm of $o$ vanishes.

   (a) $P(o)$ reduces along each light-ray of $\delta M^8_+$ to the same real valued polynomial $P(t)$ of a real variable $t$ apart from a multiplicative unit $E = (1+in)/2$ satisfying $E^2 = E$. Here $n$ is purely octonion-imaginary unit vector defining the direction of the light-ray.

   $IM(P) = 0$ corresponds to quaterniocity. If the $E^4$ ($M^8 = M^4 \times E^4$) projection is vanishing, there is no additional condition. 4-D light-cones $M^4_\pm$ are obtained as solutions of $IM(P) = 0$. Note that $M^4_\pm$ can correspond to any quaternionic subspace.

   If the light-like ray has a non-vanishing projection to $E^4$, one must have $P(t) = 0$. The solutions form a collection of 6-spheres labelled by the roots $t_n$ of $P(t) = 0$. 6-spheres are not associative.

   (b) $RE(PE) = 0$ corresponding to co-quaternionicity leads to $P(t) = 0$ always and gives a collection of 6-spheres.

2. Suppose now that $P(t)$ is shifted to $P_1(t) = P(t) - c$, $c$ a real number. Also now $M^4_\pm$ is obtained as solutions to $IM(P) = 0$. For $RE(P) = 0$ one obtains two conditions $P(t) = 0$ and $P(t-c) = 0$. The common roots define a subset of 6-spheres which for special values of $c$ is not empty.

The above discussion was limited to $\delta M^8_+$ and light-likeness of its points played a central role. What about the interior of 8-D CD?

1. The natural expectation is that in the interior of CD one obtains a 4-D variety $X^4$. For $IM(P) = 0$ the outcome would be union of $X^4$ with $M^4_+$ and the set of 6-spheres for $IM(P) = 0$. 4-D variety would intersect $M^4_+$ in a discrete set of points and the 6-spheres along 2-D varieties $X^2$. The higher the degree of $P$, the larger the number of 6-spheres and these 2-varieties.

2. For $RE(P) = 0$ $X^4$ would intersect the union of 6-spheres along 2-D varieties. What comes in mind that these 2-varieties correspond in $H$ to partonic 2-surfaces defining light-like 3-surfaces at which the induced metric is degenerate.

3. One can consider also the situation in the complement of 8-D CD which corresponds to the complement of 4-D CD. One expects that $RE(P) = 0$ condition is replaced with $IM(P) = 0$ condition in the complement and $RE(P) = IM(P) = 0$ holds true at the boundary of 4-D CD.

6-spheres and 4-D empty light-cones are special solutions of the conditions and clearly analogs of branes. Should one make the (reluctant-to-me) conclusion that they might be relevant for TGD at the level of $M^8$.

1. Could $M^4_+$ (or CDs as 4-D objects) and 6-spheres integrate the space-time varieties inside different 4-D CDs to single connected structure with space-time varieties glued to the 6-spheres along 2-surfaces $X^2$ perhaps identifiable as pre-images of partonic 2-surfaces and maybe string world sheets? Could the interactions between space-time varieties $X^4_i$ assignable with different CDs be describable by regarding 6-spheres as bridges between $X^4_i$ having only a discrete set of common points. Could one say that $X^2_i$ interact via the 6-sphere somehow. Note however that 6-spheres are not dynamical.

2. One can also have Poincare transforms of 8-D CDs. Could the description of their interactions involve 4-D intersections of corresponding 6-spheres?

3. 6-spheres in $IM(P) = 0$ case do not have image under $M^8 - H$ correspondence. This does not seem to be possible for $RE(P) = 0$ either: it is not possible to map the 2-D normal space to a unique $CP_2$ point since there is 2-D continuum of quaternionic sub-spaces containing it.

## 4.3 $M^8 - H$ duality: objections and challenges

In the following I try to recall all objections against the reduction of classical physics to octonionic algebraic geometry and against the notion of $M^8 - H$ duality and also invent some new counter arguments and challenges.

### 4.3.1 Can on really assume distribution of $M^2(x)$?

Hamilton-Jacobi structure means that $M^2(x)$ depends on position and $M^2(x)$ should define an integrable distribution integrating to a 2-D surface. For cosmic string extremals this surface would be minimal surface so that the term "string world sheet" is appropriate. There are objections.

1. It seems that the coefficients of octonionic polynomials cannot contain information about string world sheet, and the only possible choice seems to be that string world sheets and partonic 2-surfaces parallel to it assigned with integrable distribution of orthogonal complements $E^2(x)$ should be coded by quaternionic moduli. It should be possible to define quaternionic coordinates $q_i$ decomposing to pairs of complex coordinates to each choice of $M^2(x) \times E^2(x)$ decomposition of given $M^4_0$. Octonionic coordinates would be given by $o = q_1 + q_2 I_4$ where $q_i$ are associated with the same quaternionic moduli. The choice of Hamilton-Jacobi structure would not be ad hoc procedure anymore but part of the definition of solutions of field equations at the level of $M^8$.

2. It would be very nice if the quaternionic structure could be induced from a fixed structure defined for $M^8_c$ once the choice of curvilinear $M^4$ coordinates is made. Since Hamiltoni-Jacobi structure [K22] involves a choice of generalized Kähler form for $M^4$ and since quaternionic structure means that there is full $S^2$ of Kähler structures determined by quaternionic imaginary units (ordinary Kähler form for sub-space $E^8 \subset M^8_c$) the natural proposal is that Hamilton-Jacobi structures is determined by a particular local choice of the Kähler form for $M^4$ involving fixing of quaternionic imaginary unit fixing $M^2(x) \subset M^4_0$ identifiable as point of $S^2$. This might relate closely also to the fixing of twistor structure, which indeed involves also self-dual Kähler form and a similar choice.

3. One can argue that it is not completely clear whether massless extremals (MEs) [K22] allow a general Hamilton-Jacobi structure. It is certainly true that if the light-like direction and orthogonal polarization direction are constant, MEs exist. It is clear that if the form of field equations is preserved and thus reduces to contractions of various tensors with second fundamental form one obtains only contractions of light-like vector with itself or polarization vector and these contractions vanish. For spatially varying directions one could argue that light-like direction codes for a direction of light-like momentum and that problems with local conservation laws expressed by field equations might emerge.

### 4.3.2 Can one assign to the tangent plane of $X^4 \subset M^8$ a unique $CP_2$ point when $M^2$ depends on position

One should show that the choice $s(x) \in CP_2$ for a given distribution of $M^2(x) \subset M^4(x)$ is unique in order to realize the $M^8 - H$ correspondence as a map $M^8 \to H$. It would be even better if one had an analytic formula for $s(x)$ using tangent space-data for $X^4 \subset H$.

1. If $M^2(x) = M^2_0$ holds true but the tangent space $M^4(x)$ depends on position, the assignment of $CP_2$ point $s(x)$ to the tangent space of $X^4 \subset M^8$ is trivial. When $M^4(x)$ is not constant, the situation is not so easy.

2. The space $M^2(x) \subset M^4(x)$ satisfies also the constraint $M^2(x) \subset M^4_0$ since quaternionic moduli are fixed. To avoid confusion notice that $M^4(x)$ denotes tangent space of $X^4$ and is different from $M^4_0$ fixing the quaternionic moduli.

3. $M^2(x)$ determines the local complex subspace and its completion to quaternionic tangent space $M^4(x)$ determines a point $s(x)$ of $CP_2$. The idea is that $M^2_0$ defines a standard reference and that one should be able to map $M^2(x)$ to $M^2_0$ by $G_2$ automorphism mapping also the $s(x)$ to a unique point $s_0(x) \in CP_2$ defining the $CP_2$ point assignable to the point of $X^4 \subset M^8$.

4. One can assign to the point $x$ quaternionic unit vector $n(x)$ determining $M^2(x)$ as the direction of the preferred imaginary unit. The $G_2$ transformation must rotate $n(x)$ to $n_0$ defining $M_0^2$ and acts on $s$. $G_2$ transformation is not unique since $u_1 g u_2$ has the same effect for $u_i \subset U(2)$ leaving invariant the point of $CP_2$ for initial and final situation. Hence the equivalence classes of transformations should correspond to a point of 6-dimensional double coset space $U(2)\backslash G_2/U(2)$. Intuitively it seems obvious that the $s_0(x)$ is unique but proof is required.

### 4.3.3 What about the inverse of $M^8 - H$ duality?

$M^8 - H$ duality should have inverse in the critical regions of $X^4 \subset M^8$, where associativity conditions are satisfied. How could one construct the inverse of $M^8 - H$ duality in these regions? One should map space-time points $(m, s) \in M^4 \times CP_2$ to points $(m, e) = (m, f(m, s)) \in M^8$. $M_0^4 \supset M_0^2$ parameterized by $CP_2$ point can be chosen arbitrarily and one can require that it corresponds to some space-time point $(m_0, s_0) \in H$. $CP_2$ point $s(x)$ characterizes the quaternionic tangent space containing $M^2(x)$ and can choose $M_0^2$ to be $M^2(x_0)$ for conveniently chosen $x_0$. Coordinates $x$ can be used also for $X^4 \subset M^8$.

One obtains set of points $(m, e) = (m(x), f(m(x), s(x)) \in M^8$ and a distribution of 4-D spaces of labelled by $s(x)$. This requires that the 4-D tangent space spanned by the gradients of $m(x)$ and $f(m(x), s(x))$ and characterized by $s_1 \subset CP_2$ for given $M^2(x)$ by using the above procedure mapping the situation to that for $M_0^2$ is same as the tangent space determined by $s(x)$: $s(x) = s_1(x)$. Also the associativity conditions should hold true. One should have a formula for $s_1$ as function of tangent vectors of space-time surface in $M^8$. The ansatz based on algebraic geometry in $M_c^8$ should be equivalent with this ansatz. The problem is that the ansatz leads to algebraic functions which cannot be found explicitly. It might be that in practice the correspondence is easy only in the direction $M^8 \to H$.

### 4.3.4 What one can say about twistor lift of $M^8 - H$ duality?

One can argue that the twistor spaces $CP_1$ associated with $M^4$ and $E^4$ are in no way visible in the dynamics of octonion polynomials and in $M^8 - H$ duality. Hence one could argue that they are not needed for any reasonable purpose. I cannot decide whether this is indeed the case. There I will consider the existence of twistor lift of the $M^8$ and also the twistor lift $M^8 - H$ duality in the space-time regions, where the tangent spaces satisfy the conditions for the existence of the duality as a map $(m, e) \in M^8 \to (m, s) \in M^4 \times CP_2$ must be considered. This involves some non-trivial delicacies.

1. The twistor bundles of $M_c^4$ and $E_c^4$ would be simply $M_c^4 \times CP_1$ and $E_c^4 \times CP_1$. $CP_1 = S^2$ parameterizes direction vectors in 3-D Euclidian space having interpretation as unit quaternions so that this interpretation might make sense. The definition of twistor structure means a selection of a preferred quaternion unit and its representation as Kähler form so that these twistor bundles would have thus Kähler structure. Twistor lift replaces complex quaternionic surfaces with their twistor spaces with induced twistor structure.

2. In $M^8$ the radii of the spheres $CP_1$ associated with $M^4$ and $E^4$ would be most naturally identical whereas in $M^4 \times CP_2$ they can be different since $CP_2$ is moduli space. Is the value of the $CP_2$ radius visible at all in the classical dynamics in the critical associative/co-associative space-time regions, where one has minimal surfaces. Criticality would suggest that besides coupling constants also parameters with dimension of length should disappear from the field equations. At least for the known extremals such as massless extremals, $CP_2$ type extremals, and cosmic strings $CP_2$ radius plays no role in the equations. $CP_2$ radius comes however into play only in interaction regions defined by CDs since $M^8 - H$ duality works only at the 3-D ends of space-time surface and at the partonic orbits. Therefore the different radii for the $CP_1$ associated with $CP_2$ and $E^4$ cause no obvious problems.

Consider now the idea about twistor space as real part of octonionic twistor space regarded as quaternion-complex space.

1. One can regard $CP_1 = S^2$ as the space of unit quaternions and it is natural to replace it with the 6-sphere $S^6$ of octonionic imaginary units at the level of complexified octonions. The sphere of complexified (by $i$) unit octonions is non-compact space since the norm is complex valued and this generalization looks neither attractive nor necessary since the projection to real numbers would eliminate the complex part.

   The equations determining the twistor bundle of space-time surface can be indeed formulated as vanishing of the quaternionic imaginary part of $S^6$ coordinates, and one obtains a reduction to quaternionic sphere $S^2$ at space-time level.

   If $S^2$ is identified as sub-manifold $S^2 \subset S^6$, it can be chosen in very many manners (this is of course not necessary). The choices are parameterized by $SO(7)/SO(3) \times SO(4)$ having dimension $D = 12$. This choice has no physical content visible at the level of $H$. Note that the Kähler structure determining Hamilton-Jaboci structure is fixed by the choice of preferred direction $(M^2(x))$. If all these moduli are allowed, it seems that one has something resembling multiverse, the description at the level of $M^8$ is deeper one and one must ask whether the space-time surfaces with different twistorial, octonionic, and quaternionic moduli can interact.

2. The resulting octonionic analog of twistor space should be mapped by $M^8 - H$ corresponds to twistor space of space-time surface $T(M^4) \times T(CP_2)$. The radii of twistor spheres of $T(M^4)$ and $T(CP_2)$ are different and this should be also understood. It would seem that the radius of $T(M^4)$ at $H = M^4 \times CP_2$ side should correspond to that of $T(M^4)$ at $M^8$ side and thus to that of $S^6$ as its geodesic sphere: Planck length is the natural proposal inspired by the physical interpretation of the twistor lift. The radius of $T(CP_2)$ twistor sphere should correspond to that of $CP_2$ and is about $2^{12}$ Planck lengths.

   Therefore the scale of $CP_2$ would emerge as a scale of moduli space and does not seem to be present at the level of $M^8$ as a separate scale. $M^8$ level would correspond to what might be called Planckian realm analogous to that associated with strings before dynamical compactification which is now replaced with number theoretic compactification. The key question is what determines the ratio of the radii of $CP_2$ scale to Planck for which favored value is $2^{12}$ [K25]. Could quantum criticality determine this ratio?

# 5 Some challenges of octonionic algebraic geometry

Space-time surfaces in $H = M^4 \times CP_2$ identified as preferred extremals of twistor lift of Kähler action leads to rather detailed view about space-time surfaces as counterparts of particles. Does this picture follow from $X^4 \subset M^8$ picture and does this description bring in something genuinely new?

## 5.1 Could free many-particle states as zero loci for real or imaginary parts for products of octonionic polynomials

In algebraic geometry zeros for the products of polynomials give rise to disjoint varieties, which are disjoint unions of surfaces assignable to the individual surfaces and possibly having lower-dimensional intersections. For instance, for complex curves these intersections consist of points. For complex surfaces they are complex curves.

In the case of octonionic polynomial $P = RE(P) + IM(P)I_4$ ($Re$ and $Im$ are defined in quaternionic sense) one considers zeros of quaternionic polynomial $RE(P)$ or $IM(P)$.

1. Product polynomial $P = P_1P_2$ decomposes to

$$P = RE(P_1)RE(P_2) - IM(P_1)IM(P_2) + (RE(P_1)IM(P_1) + IM(P_1)RE(P_2)I_4 \ .$$

   One can require vanishing of $RE(P)$ or $IM(P)$.

   (a) $IM(P)$ vanishes for

$$(RE(P_1) = 0, RE(P_2) = 0)$$

or

$$I(m(P_1) = 0, IM(P_2) = 0) \ .$$

(b) $RE(P)$ vanishes for

$$(RE(P_1) = 0, IM(P_2) = 0)$$

or

$$IM(P_1) = 0, RE(P_2) = 0) \ .$$

One could reduce the condition $RE(P) = 0$ to $IM(P) = 0$ by replacing $P = P_1 + P_2 I_4$ with $P_2 - P_1 I_4$. If this condition is satisfied for the factors, it is satisfied also for the product. The set of surfaces is a commutative and associative algebra for the condition $IM(P) = 0$. Note that the quaternionic moduli must be same for the members of product. If one has quantum superposition of quaternionic moduli, the many-particle state involves a superposition of products with same moduli.

As found, the condition $IM(P) = 0$ can transform to $RE(P) = 0$ at singularities having $RE(P) = 0, IM(P) = 0$.

2. The commutativity of the product means that the products are analogous to many-boson states. $P^n$ would define an algebraic analog of Bose-Einstein condensate. Does this surface correspond to a state consisting of $n$ identical particles or is this artefact of representation? As a limiting case of product of different polynomials it might have interpretation as genuine $n$-boson states.

3. The product of two polynomials defines a union of disjoint surfaces having discrete intersection in Euclidian signature. In Minkowskian signature the vanishing of $q\overline{q}$ (conjugation does not affect the sign of $i$ and changes only the sign of $I_k$!) can give rise to 3-D light-cone. The non-commutativity of quaternions indeed can give rise to combinations of type $q\overline{q}$ in $RE(P)$ and $IM(P)$.

What about interactions?

1. Could one introduce interaction by simply adding a polynomial $P_{int}$ to the product? This polynomial should be small outside interaction region. CD would would define naturally interaction regions and the interaction terms should vanish at the boundaries of CD. This might be possible in Minkowskian signature, where $f(q^2)$ multiplying the interaction term might vanish at the boundary of CD: in Euclidian sector $q\overline{q} = 0$ would imply $q = 0$ but in Minkowskian sector it would give light-cone as solution. One should arrange $IM(P_{int})$ to be proportional to $q\overline{q}$ vanishing at the boundary of CD. Minkowskian signature could be crucial for the possibility to "turning interactions on".

2. If the imaginary part of the interaction term is proportional $f_1(q^2)f_2((q - T)^2)$ ($T$ is real and corresponds to the temporal distance between the tips of CD) with $f_i(0) = 0$, one could obtain asymptotic states reducing to disjoint unions of zero loci of $P^i$ at the boundaries of CD. If the order of of the perturbation terms is higher than the total order of polynomials $P^i$, one would obtain new roots and particle emission. Non-perturbative situation would correspond to a dramatic modification of the space-time surface as a zero locus of $IM(P)$. This picture would be $M^8$ counterpart for the reduction of preferred extremals to minimal surfaces analogous to geodesic lines near the boundaries of CD: preferred extremals reduce to extremals of both Kähler action and volume term in these regions [L2].

The singularities of scattering amplitudes at algebraic varieties of Grassmann manifolds are central in the twistor Grassmann program [B1, B4, B3]. Since twistor lift of TGD seems to be the correct manner to formulate classical TGD in $H$, one can wonder about the connection between space-time surfaces in $M_c^8$ and scattering amplitudes. Witten's formulation of twistor amplitudes in terms of algebraic curves in $CP_3$ suggests a formulation of scattering amplitudes in terms of the 4-D algebraic varieties in $M_c^8$ as of course, also TGD itself [K26, K32]! Could the huge multi-local Yangian symmetries of twistor Grassmann amplitudes reduce to octonion analyticity.

## 5.2 Two alternative interpretations for the restriction to $M^4$ subspace of $M_c^8$

One must complexify $M^8$ so that one has complexified octonions $M_c^8$. This means the addition of imaginary unit $i$ commuting with octonionic imaginary units. The vanishing of real or imaginary part of octonionic polynomial in quaternionic sense ($o = q_1 + Jq_2$) defines the space-time surface. Octonionic polynomial itself is obtained from a real polynomial by algebraic continuation so that in information theoretic sense space-time is 1-D. The roots of this real polynomial fix the polynomial and therefore also space-time surface uniquely. 1-D line degenerates to a discrete set of points of an extension in information theoretic sense. In p-adic case one can allow p-adic pseudo constants and this gives a model for imagination.

The octonionic roots $x + iy$ of the real polynomial need not however be real. There are two options.

1. The original proposal in [L11, L13] was that the *projection* from $M_c^8$ to real $M^4$ (for which $M^1$ coordinate is real and $E^3$ coordinates are imaginary with respect to $i$!) defines the real space-time surface mappable by $M^8 - H$ duality to $CP_2$.

2. An alternative option is that only the roots of the 4 vanishing polynomials as coordinates of $M_c^4$ belong to $M^4$ so that $m^0$ would be real root and $m^k$, $k = 1, ..., 3$ imaginary with respect to $i \to -i$. $M_c^8$ coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality.* This could make sense.

   What is remarkable that this could allow to identify CDs in very elegant manner: outside CD these 4 conditions would not hold true. This option looks more attractive than the first one. Why these conditions can be true just inside CD, should be understood.

Consider now this in detail.

1. One can think of starting from one of the 4 vanishing conditions for the components of octonionic polynomial guaranteeing associativity. Assuming real roots and continuing one by one through all 4 conditions to obtain 4-D Minkowskian real regions. The time coordinate of $M^4$ coordinates is real and others purely imaginary with respect to $i \to -i$. If this region does not connect 3-D surface at the boundaries of real CD, one must make a new trial.

   Cusp catastrophe determined as the zero locus of third order polynomial provides an example. There are regions with single real root, regions with two real roots (complex roots become real and identical) defining V-shaped boundary of cusp and regions with 3 real roots (the interior of the cusp).

2. The restriction of the octonionic polynomial to time axis $m^0$ identifiable as octonionic real axes is a real polynomial with algebraic coefficients. In this case the root and its conjugate with respect to $i$ would define the same surface. One could say that the Galois group of the real polynomial characterizes the space-time surface although at points other than those at real axis (time axis) the Galois group can be different.

   One could consider the local Galois group of the fourth quaternionic valued polynomial, say the part of quaternionic polynomial corresponding to real unit 1 when other components are required to vanish and give rise to coordinates in $M^8 \subset M_c^8$ - Minkowskian reality. The extension and its Galois group would depend on the point of space-time surface.

An interesting question is how strong conditions Minkowskian reality poses on the extension. Minkowskian reality seems to imply that $E^3$ roots are purely real so that for an octonionic polynomial obtained as a continuation of a *real* polynomial one expects that both root and complex conjugate should be allow and that Galois group should contain $Z_2$ reflection $i \to -i$. Space-time surface would be at least 2-sheeted. Also the model for elementary particles forces this conclusion on physical grounds. Real as opposite to imagined would mean Minkowskian reality in mathematical sense. In the case of polynomials this description would make sense in p-adic case by allowing the coefficients of the polynomial be pseudo constants.

3. What data one could use to fix the space-time surface? Can one start directly from the real polynomial and regard its coefficients as WCW coordinates? This would be easy and elegant. Space-time surface could be determined as Minkowskian real roots of the octonionic polynomial. The condition that the space-time surface has ends at boundaries of given CD and the roots are not Minkowskian real outside it would pose conditions on the polynomial. If the coefficients of the polynomial are p-adic pseudo constants, this condition might be easy to satisfy.

The situation depends also on the coordinates used. For linear coordinates such as Minkowski coordinates Minkowskian reality looks natural. One can however consider also angle like coordinates representable only in terms of complex phases p-adically and coming as roots of unity and requiring complex extension: at H-side they are very natural. For instance, for $CP_2$ all coordinates would be naturally represented in this manner. For future light-cone one would have hyperbolic angle and 2 ordinary angles plus light-cone proper time which would be real and positive coordinate.

This picture conforms with the proposed picture. The point is that the time coordinate $m^k$ can be real in the sense that they are linear combinations of complex roots, say powers for the roots of unity. $E^4_c \subset M^8_c$ could be complex and contain also complex roots since $M^8 - H$ duality does not depend on whether tangent space is complex or not. Therefore would could have complex extensions.

## 5.3 Questions related to ZEO and CDs

Octonionic polynomials provide a promising approach to the understanding of ZEO and CDs. Light-like boundary of CD as also light-cone emerge naturally as zeros of octonionic polynomials. This does not yet give CDs and ZEO: one should have intersection of future and past directed light-cones. The intuitive picture is that one has a hierarchy of CDs and that also the space-time surfaces inside different CDs an interact.

### 5.3.1 Some general observations about CDs

It is good to list some basic features of CDS, which appear as both 4-D and 8-D variants.

1. There are both 4-D and 8-D CDs defined as intersections of future and past directed light-cones with tips at say origin 0 at real point $T$ at quaternionic or octonionic time axis. CDs can be contained inside each other. CDs form a fractal hierarchy with CDs within CDs: one can add smaller CDs with given CD in all possible manners and repeat the process for the sub-CDs. One can also allow overlapping CDs and one can ask whether CDs define the analog of covering of $O$ so that one would have something analogous to a manifold.

2. The boundaries of two CDs (both 4-D and 8-D) can intersect along light-like ray. For 4-D CD the image of this ray in $H$ is light-like ray in $M^4$ at boundary of CD. For 8-D CD the image is in general curved line and the question is whether the light-like curves representing fermion orbits at the orbits of partonic 2-surfaces could be images of these lines.

3. The 3-surfaces at the boundaries of the two 4-D CDs are expected to have a discrete intersection since 4 + 4 conditions must be satisfied (say $RE(P_i^k)) = 0$ for $i = 1, 2$, $k = 1, 4$. Along line octonionic coordinate reduces effectively to real coordinate since one has $E^2 = E$ for $E = (1 + in)/2$, $n$ octonionic unit. The origins of CDs are shifted by a light-like vector $kE$ so that the light-like coordinates differ by a shift: $t_2 = t_1 - k$. Therefore one has common zero for real polynomials $RE(P_1^k(t))$ and $RE(P_2^k(t - k))$.

Are these intersection points somehow special physically? Could they correspond to the ends of fermionic lines? Could it happen that the intersection is 1-D in some special cases? The example of $o^2$ suggest that this might be the case. Does 1-D intersection of 3-surfaces at boundaries of 8-D CDs make possible interaction between space-time surfaces assignable to separate CDs as suggested by the proposed TGD based twistorial construction of scattering amplitudes?

4. Both tips of CD define naturally an origin of quaternionic coordinates for $D = 4$ and the origin of octonionic coordinates for $D = 8$. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be along the real line (time axis) connecting the tips of CD. Only the translations in this specified direction are symmetries preserving the commutativity and associativity of the polynomial algebra.

5. One expects that also Lorentz boosts of 4-D CDs are relevant. Lorentz boosts leave second boundary of CD invariant and Lorentz transforms the other one. Same applies to 8-D CDs. Lorentz boosts define non-equivalent octonionic and quaternionic structures and it seems that one assume moduli spaces of them.

One can of course ask whether the still somewhat ad hoc notion of CD general enough. Should one generalize it to the analog of the polygonal diagram with light-like geodesic lines as its edges appearing in the twistor Grassmannian approach to scattering diagrams? Octonionic approach gives naturally the light-like boundaries assignable to CDs but leaves open the question whether more complex structures with light-like boundaries are possible. How do the space-time surfaces associated with different quaternionic structures of $M^8$ and with different positions of tips of CD interact?

### 5.3.2 The emergence of causal diamonds (CDs)

CDs are a key notion of zero energy ontology (ZEO). They should emerge from the number-theoretic dynamics somehow. How? In the following this question is approached from two different directions.

1. One can ask whether the emergence of CDs could be understood in terms of singularities of octonion polynomials located at the light-like boundaries of CDs. In Minkowskian case the complex norm $q\overline{q}_i$ is present in $P$. Could this allow to blow up the singular point to a 3-D boundary of light-cone and allow to understand the emergence of causal diamonds (CDs) crucial in ZEO. This question will be considered below.

2. These arguments were developed before the realization that the Minkowskian reality condition discussed in the previous section is natural for the space-time surfaces as roots of the 4 polynomials defining real or imaginary part of octonionic polynomial in quaternionic sense and giving $M^4$ point as a solution. Minkowskian reality can hold only in some regions of $M^4$ and an attractive conjecture is that it fails outside CD. CD would be a prediction of number theoretical dynamics and have counterpart also at the level of $H$.

Consider now the second approach in more detail. The study of the special properties for zero loci of general polynomial $P(o)$ at light-rays of $O$ indeed demonstrated that both 8-D land 4-D light-cones and their complements emerge naturally, and that the $M^4$ projections of these light-cones and even of their boundaries are 4-D future - or past directed light-cones. What one should understand is how CDs as their intersections, and therefore ZEO, emerge.

1. One manner to obtain CDs naturally is that the polynomials are sums $P(t) = \sum_k P_k(o)$ of products of form $P_k(o) = P_{1,k}(o)P_{2,k}(o-T)$, where $T$ is real octonion defining the time coordinate. Single product of this kind gives two disjoint 4-varieties inside future and past directed light-cones $M^4_+(0)$ and $M^4_-(T)$ for either $RE(P) = 0$ (or $IM(P) = 0$) condition. The complements of these cones correspond to $IM(P) = 0$ (or $RE(P) = 0$) condition.

2. If one has nontrivial sum over the products, one obtains a connected 4-variety due the interaction terms. One has also as special solutions $M^4_\pm$ and the 6-spheres associated with

the zeros $P(t)$ or equivalently $P_1(t_1) \equiv P(t)$, $t_1 = T - t$ vanishing at the upper tip of CD. The causal diamond $M^4_+(0) \cap M^4_-(T)$ belongs to the intersection.

**Remark**: Also the union $M^4_-(0) \cup M^4_+(T)$ past and future directed light-cones belongs to the intersection but the latter is not considered in the proposed physical interpretation.

3. The time values defined by the roots $t_n$ of $P(t)$ define a sequence of 6-spheres intersecting 4-D CD along 3-balls at times $t_n$. These time slices of CD must be physically somehow special. Space-time variety intersects 6-spheres along 2-varieties $X^2_n$ at times $t_n$. The varieties $X^2_n$ are perhaps identifiable as 2-D interaction vertices, pre-images of corresponding vertices in $H$ at which the light-like orbits of partonic 2-surfaces arriving from the opposite boundaries of CD meet.

   The expectation is that in $H$ one as generalized Feynman diagram with interaction vertices at times $t_n$. The higher the evolutionary level in algebraic sense is, the higher the degree of the polynomial $P(t)$, the number of $t_n$, and more complex the algebraic numbers $t_n$. $P(t)$ would be coded by the values of interaction times $t_n$. If their number is measurable, it would provide important information about the extension of rationals defining the evolutionary level. One can also hope of measuring $t_n$ with some accuracy! Octonionic dynamics would solve the roots of a polynomial! This would give a direct connection with adelic physics [L15] [L16].

   **Remark**: Could corresponding construction for higher algebras obtained by Cayley-Dickson construction solve the "roots" of polynomials with larger number of variables? Or could Cartesian product of octonionic spaces perhaps needed to describe interactions of CDs with arbitrary positions of tips lead to this?

4. Above I have considered only the interiors of light-cones. Also their complements are possible. The natural possibility is that varieties with $RE(P) = 0$ and $IM(P) = 0$ are glued at the boundary of CD, where $RE(P) = IM(P) = 0$ is satisfied. The complement should contain the external (free) particles, and the natural expectation is that in this region the associativity/co-associativity conditions can be satisfied.

5. The 4-varieties representing external particles would be glued at boundaries of CD to the interacting non-associative solution in the complement of CD. The interaction terms should be non-vanishing only inside CD so that in the exterior one would have just product $P(o) = P_{1,k_0}(o)P_{2,k_0}(o-T)$ giving rise to a disjoint union of associative varieties representing external particles. In the interior one could have interaction terms proportional to say $t^2(T-t)^2$ vanishing at the boundaries of CD in accordance with the idea that the interactions are switched one slowly. These terms would spoil the associativity.

**Remark**: One can also consider sums of the products $\prod_k P_k(o-T_k)$ of $n$ polynomials and this gives a sequence CDs intersecting at their tips. It seems that something else is required to make the picture physical.

## 5.4 About singularities of octonionic algebraic varieties

In Minkowskian signature the notion of singularity for octonionic polynomials involves new aspects as the study of $o^2$ singular at origin shows (see Appendix). The region in which $RE(o^2) = 0, IM(o^2) = 0$ holds true is 4-D rather than a discrete set of points as one would naively expect.

1. At singularity the local dimension of the algebraic variety is reduced. For instance, double cone of 3-space has origin as singular point where it becomes 0-dimensional. A more general example is local pinch in which cylinder becomes infinitely thin at some point. This kind of pinching could occur for fibrations as the fiber contracts to a lower-dimensional space along a sub-variety of the base space.

   A very simple analogy for this kind of singularity is the singularity of $P(x,y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$ defining the factorization of $P(x,y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

The signature of the singularity of algebraic variety determined by the conditions $P^i(z^j) = 0$ is the reduction of the maximal rank $r$ for the matrix formed by the partial derivatives $P^i_j \equiv \partial IM(P)^i/\partial z^j$ ("RE" could replace "IM"). Rank corresponds to the largest dimension of the minor of $P^i_j$ with non-vanishing determinant. Determinant vanishes when two rows of the minor are proportional to each other meaning that two tangent vectors become linearly dependent. When the rank is reduced by $\Delta r$, one has $r = r_{max} - \Delta r$ and the local dimension is locally reduced by $\Delta r$. One has hierarchy of singularities within singularities.

The conditions that all independent minors of the $P^i_j$ have reduced rank gives additional constraints and define a sub-variety of the algebraic variety. Note that the dimension of the singularity corresponds to $d_s = \Delta r$ in the sense that the dimension of tangent space at singularity is effectively $d_s$.

2. In the recent case there are 4 polynomials and 4 complex variables so that $IM(P)^i_j$ is $4 \times 4$-matrix. Its rank $r$ can have values in $r = 1, 2, 3, 2, 4$. One can use Thom's catastrophe theory as a guideline. Catastrophe decomposes to pieces of various dimensions characterized by the reduction of the rank of the matrix defined by the second derivatives $V_{ij} = \partial_i\partial_j V$ of the potential function defining the catastrophe. For instance, for cusp catastrophe with $V(x, a, b) = x^4 + ax^2 + bx$ one has V-shaped region in $(a, b)$ plane with maximal reduction of rank to $r = 0$ ($\partial_x^2 V = 0$) at the tip $(a, b) = 0$ at reduction to $r = 1$ at the sides of $V$, where two roots of $\partial_x V = 4x^3 + 2ax + b = 0$ co-incide requiring that the discriminant of this equation vanishes.

3. In the recent case $IM(P)$ takes the role of complex quaternion valued potential function and the 4 coordinates $z_1^{k)}$ that of behavior variable $x$ for cusp and $z_2^{k)}$ that of control parameters $(a, b)$. The reduction of the rank of $n \times n$ matrix by $\Delta r$ means that there are $r$ linearly independent rows in the matrix. These give $\Delta r$ additional conditions besides $IM(P) = 0$ so that the sub-variety along which the singularity takes places as dimension $r$. One can say that the $r$-dimensional tangent spaces integrate to the singular variety of dimension $r$.

   The analogy with branes would be realized as a hierarchical structure of singularities of the spacetime surfaces. This hierarchy of singularities would realize space-time correlates for quantum criticality, which is basic principle of quantum TGD. For instance, the reduction by 3-units would correspond to strings - say at the ends of CD and along the partonic orbits (fermion lines), and maximal reduction might correspond to discrete points - say the ends of fermion lines at partonic 2-surfaces. Also isolated intersection points can be regarded as singularities and are stably present but it does not make sense to add fermions to these points so that cognitive representations are not possible.

4. Note that also the associativity - and commutativity conditions already discuss involved the gradients of $IM(P)^i$ and $RE(P)^i$, which would suggests that these regions can be interpreted as singularities for which the dimension is not lowered by on unit since the vanishing conditions hold true identically by criticality.

There are two cases to be considered. The usual Euclidian case in which pinch reducing the dimension and the Minkowskian case in which metric dimension is reduced locally.

Consider first the Euclidian case.

1. In Euclidian case it is difficult to tell whether all values of $\Delta r$ are possible since octonion analyticity poses strong conditions on the singularities. The pinch could correspond to the singularity of the covering associated with the space-time surface defined by Galois group for the covering associated with $h_{eff}/h = n$ identifiable as the dimension of the extension [L9]. Therefore there would be very close connection between the extensions of rationals defining the Galois group and the extension of polynomial ring of 8 complex variables $z_i^k$, $i = 1, 2$, $k = 1, .., 4$ by algebraic functions. At the pinch, which would be algebraic point, the Galois group would have subgroup leaving the coordinates of the point invariant and some sheets of the covering defining roots would co-incide.

2. A very simple analogy for this kind of singularity is the singularity of $P(x, y) = y^2 - x = 0$ at origin: now the sheets $y = \pm\sqrt{x}$ co-incide at origin. The algebraic functions $y \mp \sqrt{x}$

defining the factorization of $P(x,y)$ co-incide at origin. Quite generally, two or more factors in the factorization of polynomial using algebraic functions co-incide at the singularity. This is completely analogous to the degeneracy or roots of polynomials of single variable.

3. Quaternion structure predicts the slicing of $M^4$ by string world sheets inducing that of space-time surfaces. One must ask whether singular space-time sheets emerge already for the slicing of $M^4$ by string world sheets. String world sheets could be considered as candidates for $\Delta r = 2$ singularities of this kind. The physical intuition strongly suggests that there indeed physically preferred string world sheets and identification as $\Delta r = 2$ singularities of Euclidian type is attractive. Partonic 2-surfaces are also candidates in this respect. Could some sheets of the $h_{eff}/h = n$ covering co-incide at string world sheets?

Consider next the Minkowskian case. At the level of $H$ the rank of the induced metric is reduced. This reduction need not be same as that for the matrix $P^i_j$ and it is of course not obvious that the partonic orbit allows description as a singularity of algebraic variety.

1. Could the matrix $P^i_j$ take a role analogous to the dual of induced metric and one might hope that the change of the sign for $P^i_j$ for a fixed polynomial at singular surface could be analogous to the change of the sign of $\sqrt{g_4}$ so that the idea about algebraization of this singularity at level of $M^8$ might make sense. The information about metric could come from the fact that $IM(P)$ depends on complex valued quaternion norm reducing to Minkowskian metric in Minkowskian sub-space.

2. The condition for the reduction of rank from its maximal value of $r = 4$ to $r = 3$ occurs if one has $det(P) = 0$, which defines co-dimension 1 surface as a sub-variety of space-time surface. The interpretation as co-incidence of two roots should make sense if $IM(P) = 0$. Root pairs would now correspond now to the points at different sides of the singular 3-surface.

   Minkowskian singularity cannot be identified as the 3-D space-like boundary of many-sheeted space-time surface located at the boundary of CD (induced metric is space-like).

   Could this sub-variety be identified as partonic orbit, the common boundary of the Euclidian and Minkowskian regions? This would require that associative region transforms to co-associative one here. $IM(P) = 0$ condition can transform to $RE(P) = 0$ condition if one has $P = 0$ at this surface. Minkowskian variant of point singularity ($P^i_j$ vanishes) would explode it to a light-like partonic orbit.

   What does this imply about the rank of singularity? The condition $IM(P) = RE(P) = 0$ does not reduce the rank if $P$ is linear polynomial and one could consider a hierarchy of reductions of rank. Since $q\overline{q}$ vanishes in Minkowskian sub-space at light-cone boundary rather than at point $q = 0$ only, there are reasons to expect that it appears in $P$ and reduces the rank by $\Delta r = 4$ (see Appendix for the discussion of $o^2$ case). The rank of the induced 4-metric is however reduced only by $\Delta r = 1$ at partonic orbit. If the complexified complex norm $z\overline{z}$, $z = z_1 + z_2 I_2$ can take the role of $q\overline{q}$, one has $\Delta r = 2$.

3. The reduction of rank to $r = 2$ would give rise to 2-surfaces, which are at the boundaries of 3-D singularities. If partonic orbits correspond to $\Delta r = 1$ singularities one could identify them as partonic 2-surfaces at the ends partonic orbits.

   Could the singularity at partonic 2-surface correspond to the reduction of the rank of the induced metric by 2 units? This is impossible in strict sense since there is only one light-like direction in signature $(1,-1,-1,-1)$. Partonic 2-surface singularity would however correspond to a corner for both Euclidian and Minkowskian regions at which the metrically 2-D but topologically 3-D partonic orbit meets the the space-like 3-surface along the light-like boundary of CD. Also the radial direction for space-like 3-surface could become light-like at partonic 2-surface if the $CP_2$ coordinates have vanishing gradient with respect to the light-like radial coordinate $r_M$ at the partonic 2-surface. In this sense the rank could be reduced by 2 units. The situation is analogous to that for fold singularity $y^2 - x = 0$.

   String world sheets cannot be subsets of $r = 3$ singularities, which suggests different interpretation for partonic 2-surfaces and string world sheets.

What could this different interpretation be?

1. Perhaps the most convincing interpretation of string world sheets/partonic 2-surfaces has been already discussed (this interpretation would generalize to associative space-time surfaces). They could be commutative/co-commutative (here permutation might be allowed!) sub-manifolds of associative regions of the space-time surface allowing quaternionic tangent spaces so that the notions of commutative and co-commutative make sense. The criticality conditions are satisfied without the reduction of dimension from $d = 2$ to $d = 1$. In non-associative regions string world sheets would reduce to 1-D curves. This would happen at the boundaries of partonic orbits and 3-surfaces at the ends of space-time surface and only the ends of strings at partonic orbits carrying fermion number would be needed to determine twistorial scattering amplitudes [K26, K32].

2. I have also considered an interpretation in terms of singularities of space-time surfaces represented as a sections of their own twistor bundle. Self-intersections of the space-time surface would correspond to 2-D surfaces in this case [L9] and perhaps identifiable as string world sheets. The interpretation mentioned above would be in terms of Euclidian singularities. If this is true, the question is only about whether these two interpretations are consistent with each other.

If I were forced to draw conclusion on basis of these notices, it would be that only $r = 4$ Minkowskian singularities could be interesting and at them $RE(P) = 0$ regions could be transformed to $IM(P) = 0$ regions. Furthermore, the reduction of rank for the induced metric cannot be equal to the reduction of the rank for $P^i_j$.

## 5.5 The decomposition of space-time surface to Euclidian and Minkowskian regions in octonionic description

The unavoidable outcome of $H$ picture is the decomposition of space-time surface to regions with Minkowskian or Euclidian signature of the induced metric. These regions are bounded by 3-D regions at which the signature of the induced metric is $(0, -1, -1, -1)$ due to the vanishing of the determinant of the induced metric. The boundary is naturally the light-like orbit of partonic 2-surface although one can consider also the possibility that these regions have boundaries intersecting along light-like curves defining boundaries of string world sheets. A more detailed view inspired by the study of extremals is following.

1. Let us assume that the above picture about decomposition of space-time surfaces in $H$ to two kinds regions takes place. The regions where the dynamicis universal minimal surface dynamics have associative pre-image in $M^8$. The regions where Kähler action and volume term couple the associative pre-image in $M^8$ exists only at the 3-D boundary regions and $M^8$ dynamics determines the boundary conditions for $H$ dynamics, which by holography is enough.

2. In the space-time regions having associative pre-image in $M^8$ one has a fibration of $X^4$ with with partonic surface as a local base and string world sheet as local fiber. In the interior of space-time region there are no singularities but at the boundary 2-D string world sheets becomes metrically 1-D as 1-D string boundary reduces metrically to 0-D structure analogous to a point. This reduction of dimension would be metric, but not topological.

   The singularity for plane curve $P(x, y) = y^2 - x^3 = 0$ at origin illustrates the difference between Minkowskian and Euclidian singularity. One has $(\partial_x P, \partial_y P) = (-3x^2, 2y)$ vanishing at origin so that $\Delta r = 1$ singularity is in question and the dimension of singular manifold is indeed $r = 0$. From $y = \pm x^{3/2}$, $x \geq 0$. The induced metric $g_{xx} = 1 + (dy/dx)^2 = 1 + (9/4)x$, $x \geq 0$ is however non-singular at origin.

3. If the Euclidian region with pre-image corresponds to a deformation of wormhole contact, the identification as image of a co-associative space-time region in $M^8$ is natural so that normal space is associative and contains also the preferred $M^2(x)$. In Minkowskian regions the identification as image of associative space-time region in $M^8$ is natural.

What can one say about the relationship of the $M^8$ counterparts of neighboring Minkowskian and Euclidian regions?

1. Do these regions intersect along light-like 3-surfaces, 1-D light-like curve (orbit of fermion) or is the intersection disrete set of points possibly assignable to the partonic 2-surface at the boundaries of CD? The $M^4$ projections of the inverse image of the light-like partonic orbit should co-incide but $E^4$ projections need not do so. They could be however mappable to the same partonic two surface in $M^8 - H$ correspondence or the images could have at least have light-like curve as common.

2. Is seems impossible for the space-time surfaces determined as zeros of octonionic polynomials to have boundaries. Rather, it seems that the boundary must be between Minkowskian and Euclidian regions of the space-time surface determined by the same octonionic polynomial. At the boundary also associate region would transform to co-associative region suggesting that $IM(P) = RE(P) = 0$ holds allowing to change the condition from $IM(P) = 0$ to $RE(P) = 0$.

Consider now in more detail whether this view can be realized.

1. In $H = M^4 \times CP_2$ the boundary between the Minkowskian and Euclidian space-time regions - light-like partonic 3-surface - is a singularity possible only in Minkowskian signature. Space-time surface $X^4$ at the boundary is effectively 3-D since one has $\sqrt{g_4} = 0$ meaning that tangent space is effectively 3-D. The 3-D boundary itself is metrically 2-D and this gives rise to the extended conformal invariance defining crucial distinction between TGD and super string models.

2. The singularities of $P(o)$ for $o$ identified as linear coordinate of $M^8_c$ were already considered. The singularities correspond to the boundaries of light-cone and the emergence of CDs can be understood. Could also the light-like orbits of partonic 2-surfaces be understood in the same manner? Does the pre-image of this singularity in $M^8$ emerge as a singularity of an algebraic variety determined by the vanishing of $IM(P)$ for the octonionic polynomial?

   What is common is that the rank of the induced metric by one unit also now. Now one has however also $det(g_4) = 0$. The singularities correspond to curved light-like 3-surfaces inside space-time surfaces rather than light-like surfaces in $M^8$: induced metric matters rather than $M^4$ metric.

3. Could also these regions correspond to singularities of octonionic polynomials at which $P(o) = 0$ is satisfied and associative region transforms to a co-associative region? For $M^2(x) = M_0^2$ this is impossible. Partonic 2-surfaces are planes $E^2$ now. One should have closed partonic 2-surfaces.

   Could the allowance of quaternionic structures with slicing of $X^4$ by string world sheets and partonic 2-surfaces help? If one has slicing of string world sheets by dual light-like curves corresponding to light-like coordinates $u$ and $v$, this slicing gives also rise to a slicing of light-like 3-surfaces and dual light-like coordinate. The pair $(u, v)$ in fact defines the analog of $z$ and $\overline{z}$ in hypercomplex case. Could the singularity of $P(o)$ using the quaternionic coordinates defined by $(u, v)$ and coordinates of partonic 2-surface allow to identify light-like partonic orbits with $det(g_4) = 0$ as a generalization of light-cone boundaries in $M^4$?

   The decomposition $M_0^4 = M_x^2 \times E^2(x)$ associated with quaternionic structure is independent of $E^4$. In the other hand, tangent space of space-time surface at point decomposes $M^2(x) \times E_T^2(x)$, where $E_T^2(x)$ is in general different from $E^2(x)$. Is this enough to obtain partonic 2-surfaces as singularities with $RE(P) = IM(P) = 0$?

The question whether the boundaries between Minkowskian and Euclidian can correspond to singular regions at which $P(o)$ vanishes and the surface $RE(P) = 0$ transforms to $IM(P) = 0$ surface remains open. What remains poorly understood is the role of the induced metric. My hope is that with a further work the picture could be made more detailed.

## 5.6 About rational points of space-time surface

What one can say about rational points of space-time surface?

1. An important special case corresponds to a generalization of so called rational surfaces for which a parametric representation in terms of 4 complex coordinates $t^k$ exists such that $o_1^k$ are *rational* functions of $t^k$. The singularities for 2-complex dimensional surfaces in $C^3$ or equivalently $CP_3$ are classified by Du Val [A5, A8] (see `http://tinyurl.com/ydz93hle`).

2. In [L9] [L5] I considered possible singularities of the twistor bundle. These would correspond typically 2-D self-intersections of the imbedding of space-time surfaces as 4-D base space of 6-D twistor bundle with sphere as a fiber. They could relate to string world sheets and partonic 2-surfaces and - as already found - are different from singularities at the level of $M_c^8$. The singularities of string world sheets and partonic 2-surfaces as hyper-complex and co-complex surfaces consist of points and could relate to the singularities at octonionic level.

As already mentioned, Bombieri-Lang conjecture (see `http://tinyurl.com/y887yn5b`) states that, for any variety $X$ of general type over a number field $k$, the set of $k$-rational points of $X$ is not Zariski dense (see `http://tinyurl.com/jm9fh74`) in $X$. Even more, the $k$-rational points are contained in a *finite* union of lower-dimensional sub-varieties of X.

This conjecture is highly interesting from TGD point of view if one believes in $M^8-H$ duality. Space-time surfaces $X^4 \subset M_c^8$ can be seen as $M^8 = M^4 \times E^4$ projections of zero loci for real or imaginary parts of octonionic polynomials in $o$. In complex sense they reduce to $M^4 \times E^4$ projections of algebraic co-dimension 4 surfaces in $C^8$. If Bombieri-Lang conjectures makes sense in this context, it would state that for a space-time surface $X^4 \subset M^8$ of general type the rational points are contained in a *finite* union of lower-dimensional sub-varieties. Also the conjecture of Vojta (see `http://tinyurl.com/y9sttuu4`) stating that varieties of general type cannot be potentially dense is known to be true for curves and support this general vision.

Could the finite union of sub-varieties correspond to string world sheets, partonic 2-surfaces, and their light-like orbits define singularities? But why just singular sub-varieties would be cognitively simple and have small Kodaira dimension $d_K$ allowing large number of rational points? In the case of partonic orbits one might understand this as a reduction of metric dimension. The orbit is effectively 2-dimensional partonic surface metrically and for the genera $g = 0, 1$ rational points are dense. For string world sheets with handle number smaller than 2 the situation is same.

The proposed realizations of associativity and commutativity provide additional support for this picture. Criticality guaranteeing associativity/commutativity would select preferred space-time surfaces as also string world sheets and partonic 2-surfaces.

Concluding, the general wisdom of algebraic geometry conforms with SH and with the vision about the localization of cognitive representations at 2-surfaces. There are of many possible options for detailed interpretation and certainly the above sketch cannot be correct at the level of details.

## 5.7 About $h_{eff}/h = n$ as the number of sheets of Galois covering

The following considerations were motivated by the observation of a very stupid mistake that I have made repeatedly in some articles about TGD. Planck constant $h_{eff}/h = n$ corresponds naturally to the number of sheets of the covering space defined by the space-time surface.

I have however claimed that one has $n = ord(G)$, where $ord(G)$ is the order of the Galois group $G$ associated with the extension of rationals assignable to the sector of "world of classical worlds" (WCW) and the dynamics of the space-time surface (what this means will be considered below).

This claim of course cannot be true since the generic point of extension $G$ has some subgroup $H$ leaving it invariant and one has $n = ord(G)/ord(H)$ dividing $ord(G)$. Equality holds true only for Abelian extensions with cyclic $G$. For singular points isotropy group is $H_1 \sup H$ so that $ord(H_1)/ord(H)$ sheets of the covering touch each other. I do not know how I have ended up to a conclusion, which is so obviously wrong, and how I have managed for so long to not notice my blunder.

This observation forced me to consider more precisely what the idea about Galois group acting as a number theoretic symmetry group really means at space-time level and it turned out that

$M^8 - H$ correspondence [L11] (see http://tinyurl.com/yd43o2n2) gives a precise meaning for this idea.

Consider first the action of Galois group (see http://tinyurl.com/y8grabt2 and http://tinyurl.com/ydze5psx).

1. The action of Galois group leaves invariant the number theoretic norm characterizing the extension. The generic orbit of Galois group can be regarded as a discrete coset space $G/H$, $H \subset G$. The action of Galois group is transitive for irreducible polynomials so that any two points at the orbit are $G$-related. For the singular points the isotropy group is larger than for generic points and the orbit is $G/H_1$, $H_1 \sup H$ so that the number of points of the orbit divides $n$. Since rationals remain invariant under $G$, the orbit of any rational point contains only single point. The orbit of a point in the complement of rationals under $G$ is analogous to an orbit of a point of sphere under discrete subgroup of $SO(3)$.

   $n = ord(G)/ord(H)$ divides the order $ord(G)$ of Galois group $G$. The largest possible Galois group for $n$-D algebraic extension is permutation group $S_n$. A theorem of Frobenius states that this can be achieved for $n = p$, $p$ prime if there is only single pair of complex roots (see http://tinyurl.com/y8grabt2). Prime-dimensional extensions with $h_{eff}/h = p$ would have maximal number theoretical symmetries and could be very special physically: p-adic physics again!

2. The action of $G$ on a point of space-time surface with imbedding space coordinates in $n$-D extension of rationals gives rise to an orbit containing $n$ points except when the isotropy group leaving the point is larger than for a generic point. One therefore obtains singular covering with the sheets of the covering touching each other at singular points. Rational points are maximally singular points at which all sheets of the covering touch each other.

3. At QFT limit of TGD the $n$ dynamically identical sheets of covering are effectively replaced with single one and this effectively replaces $h$ with $h_{eff} = n \times h$ in the exponent of action (Planck constant is still the familiar $h$ at the fundamental level). $n$ is naturally the dimension of the extension and thus satisfies $n \leq ord(G)$. $n = ord(G)$ is satisfied only if $G$ is cyclic group.

The challenge is to define what space-time surface as Galois covering does really mean!

1. The surface considered can be partonic 2-surface, string world sheet, space-like 3-surface at the boundary of CD, light-like orbit of partonic 2-surface, or space-time surface. What one actually has is only the data given by these discrete points having imbedding space coordinates in a given extension of rationals. One considers an extension of rationals determined by irreducible polynomial $P$ but in p-adic context also roots of $P$ determine finite-D extensions since $e^p$ is ordinary p-adic number.

2. Somehow this data should give rise to possibly unique continuous surface. At the level of $H = M^4 \times CP_2$ this is impossible unless the dynamics satisfies besides the action principle also a huge number of additional conditions reducing the initial value data ans/or boundary data to a condition that the surface contains a discrete set of algebraic points.

   This condition is horribly strong, much more stringent than holography and even strong holography (SH) implied by the general coordinate invariance (GCI) in TGD framework. However, preferred extremal property at level of $M^4 \times CP_2$ following basically from GCI in TGD context might be equivalent with the reduction of boundary data to discrete data if $M^8 - H$ correspondence [L11] (see http://tinyurl.com/yd43o2n2) is accepted. These data would be analogous to discrete data characterizing computer program so that an analog of computationalism would emerge [L7] (see http://tinyurl.com/y75246rk).

One can argue that somehow the action of discrete Galois group must have a lift to a continuous flow.

1. The linear superposition of the extension in the field of rationals does not extend uniquely to a linear superposition in the field reals since the expression of real number as sum of units of extension with real coefficients is highly non-unique. Therefore the naive extension of the extension of Galois group to all points of space-time surface fails.

2. The old idea already due to Riemann is that Galois group is represented as the first homotopy group of the space. The space with homotopy group $\pi_1$ has coverings for which points remain invariant under subgroup $H$ of the homotopy group. For the universal covering the number of sheets equals to the order of $\pi_1$. For the other coverings there is subgroup $H \subset \pi_1$ leaving the points invariant. For instance, for homotopy group $\pi_1(S^1) = Z$ the subgroup is $nZ$ and one has $Z/nZ = Z_p$ as the group of $n$-sheeted covering. For physical reasons its seems reasonable to restrict to finite-D Galois extensions and thus to finite homotopy groups.

   $\pi_1 - G$ correspondence would allow to lift the action of Galois group to a flow determined only up to homotopy so that this condition is far from being sufficient.

3. A stronger condition would be that $\pi_1$ and therefore also $G$ can be realized as a discrete subgroup of the isometry group of $H = M^4 \times CP_2$ or of $M^8$ ($M^8 - H$ correspondence) and can be lifted to continuous flow. Also this condition looks too weak to realize the required miracle. This lift is however strongly suggested by Langlands correspondence [K8, K27] (see `http://tinyurl.com/y9x5vkeo`).

The physically natural condition is that the preferred extremal property fixes the surface or at least space-time surface from a very small amount of data. The discrete set of algebraic points in given extension should serve as an analog of boundary data or initial value data.

1. $M^8 - H$ correspondence [L11] (see `http://tinyurl.com/yd43o2n2`) could indeed realize this idea. At the level of $M^8$ space-time surfaces would be algebraic varieties whereas at the level of $H$ they would be preferred extremals of an action principle which is sum of Kähler action and minimal surface term.

   They would thus satisfy partial differential equations implied by the variational principle and infinite number of gauge conditions stating that classical Noether charges vanish for a subgroup of symplectic group of $\delta M^4_\pm \times CP_2$. For twistor lift the condition that the induced twistor structure for the 6-D surface represented as a surface in the 12-D Cartesian product of twistor spaces of $M^4$ and $CP_2$ reduces to twistor space of the space-time surface and is thus $S^2$ bundle over 4-D space-time surface.

   The direct map $M^8 \rightarrow H$ is possible in the associative space-time regions of $X^4 \subset M^8$ with quaternionic tangent or normal space. These regions correspond to external particles arriving into causal diamond (CD). As surfaces in $H$ they are minimal surfaces and also extremals of Kähler action and do not depend at all on coupling parameters (universality of quantum criticality realized as associativity). In non-associative regions identified as interaction regions inside CDs the dynamics depends on coupling parameters and the direct map $M^8 \rightarrow CP_2$ is not possible but preferred extremal property would fix the image in the interior of CD from the boundary data at the boundaries of CD.

2. At the level of $M^8$ the situation is very simple since space-time surfaces would correspond to zero loci for $RE(P)$ or $IM(P)$ ($RE$ and $IM$ are defined in quaternionic sense) of an octonionic polynomial $P$ obtained from a real polynomial with coefficients having values in the field of rationals or in an extension of rationals. The extension of rationals would correspond to the extension defined by the roots of the polynomial $P$.

   If the coefficients are not rational but belong to an extension of rationals with Galois group $G_0$, the Galois group of the extension defined by the polynomial has $G_0$ as normal subgroup and one can argue that the relative Galois group $G_{rel} = G/G_0$ takes the role of Galois group.

   It seems that $M^8 - H$ correspondence could allow to realize the lift of discrete data to obtain continuous space-time surfaces. The data fixing the real polynomial $P$ and therefore also its octonionic variant are indeed discrete and correspond essentially to the roots of $P$.

3. One of the elegant features of this picture is that the at the level of $M^8$ there are highly unique linear coordinates of $M^8$ consistent with the octonionic structure so that the notion of a $M^8$ point belonging to extension of rationals does not lead to conflict with GCI. Linear coordinate changes of $M^8$ coordinates not respecting the property of being a number in extension of rationals would define moduli space so that GCI would be achieved.

Does this option imply the lift of $G$ to $\pi_1$ or to even a discrete subgroup of isometries is not clear. Galois group should have a representation as a discrete subgroup of isometry group in order to realize the latter condition and Langlands correspondence supports this as already noticed. Note that only a rather restricted set of Galois groups can be lifted to subgroups of $SU(2)$ appearing in McKay correspondence and hierarchy of inclusions of hyper-finite factors of type $II_1$ labelled by these subgroups forming so called ADE hierarchy in 1-1 correspondence with ADE type Lie groups [K16, K6] (see `http://tinyurl.com/ybavqvvr`). One must notice that there are additional complexities due to the possibility of quaternionic structure which bring in the Galois group SO(3) of quaternions.

**Remark**: After writing this article a considerable progress in understanding of $h_{eff}/h = n$ as number of sheets of Galois covering emerged. By $M^8$-duality space-time surface can be seen as zero locus for real or imaginary part (regarding octonions as sums of quaternionic real and imaginary parts) allows a nice understanding of space-time surface as an $h_{eff}/h = n$-fold Galois covering. $M^8$ is complexified by adding an imaginary unit $i$ commuting with octonionic imaginary units. Also space-time surface is complexified to 8-D surface in complexified $M^8$. One can say that ordinary space-time surface is the "real part" of this complexified space-time surface just like $x$ is the real part of a complex number $x + iy$. Space-time surface can be also seen as a root of $n$:th order polynomial with $n$ complex branches and the projections of complex roots to "real part" of $M^8$ define space-time surface as an $n$-fold covering space in which Galois group acts.

## 5.8 Connection with infinite primes

The idea about space-time surfaces as zero loci of polynomials emerged for the first time as I tried to understand the physical interpretation of infinite primes [K13], which were motivated by TGD inspired theory of consciousness. Infinite primes form an infinite hierarchy. At the lowest level the basic entity is the product $X = \prod_p p$ of all finite primes. The physical interpretation could be as an analog of fermionic sea with fermion states labelled by finite primes $p$.

1. The simplest infinite primes are of form $P = X \pm 1$ as is easy to see. One can construct more complex infinite primes as infinite integers of form $nX/r + mr$. Here $r$ is square free integer, $n$ is integer having no common factors with $r$, and $m$ can have only factors possessed also by $r$.

   The interpretation is that $r$ defines fermionic state obtained by kicking from Dirac sea the fermions labelled by the prime factors of $r$. The integers $n$ and $m$ define bosonic excitations in which $k$:th power of $p$ corresponds to $k$ bosons in state labelled by $p$. One can also construct more complex infinite primes as polynomials of $X$ and having no rational factors. In fact, $X$ becomes coordinate variable in the correspondence with polynomials.

2. This process can be repeated at the next level. Now one introduces product $Y = \prod_P P$ of all primes at the previous level and repeats the same construction. These infinite correspond to polynomials of $Y$ with coefficients given by rational functions of $X$. Primality means irreducibility in the field of rational functions so that solving $Y$ in terms of $X$ would give algebraic function.

3. At the lowest level are ordinary primes. At the next level the infinite primes are indeed infinite in real sense but have p-adic norms equal to unity. They can be mapped to polynomials $P(x_1)$ with rational coefficients and the simplest polynomials are monomials with rational root. Higher polynomials are irreducible polynomials with algebraic roots. At the third level of hierarchy one has polynomials $P(x_2|x_1)$ of two variables. They are polynomials of $x_1$ with coefficients with are rational functions of $x_1$. This hierarchy can be continued.

   One can define also infinite integers as products of infinite primes at various levels of hierarchy and even infinite rationals.

4. This hierarchy can be interpreted in terms of a repeated quantization of an arithmetic supersymmetric quantum field theory with elementary particles labelled by primes at given level of hierarchy. Physical picture suggests that the hierarchy of second quantizations is realized also in Nature and corresponds to the hierarchy of space-time sheets.

5. One could consider a mapping $P(x_n|x_{n-1}|..|x_1)$ by a diagonal projection $x_i = x$ to polynomials of single variable $x$. One could replace $x$ with complexified octonic coordinate $o_c$. Could this correspondence give rise to octonionic polynomials and could the connection with second quantization give classical space-time correlates of real quantum states assignable to infinite primes and integers? Even quantum states defining counterparts of infinite rationals could be considered. One could require that the real norm of these infinite rationals equals to one. They would define infinite number of real units with arbitrarily complex number theoretical anatomy. The extension of real numbers by these units would mean huge extension of the notion of real number and one could say that each real point corresponds to platonic defined by these units closed under multiplication.

   In ZEO zero energy states formed by pairs of positive and negative energy could correspond to these states physically. The condition that the ratio is unit would have also a physical interpretation in terms of particle content.

6. As already noticed, the notions of analyticity, quaternionicity, and octonionicity could be seen as a manifestation of polynomials in algebras defined by adding repeatedly a new non-commuting imaginary unit to already existing algebra. The dimension of the algebra is doubled in each step so that dimension comes as a power of 2. The algebra of polynomials with real coefficients is commutative and associative. This encourages the crazy idea that the spaces are indeed realized and the generalization of $M^8-H$ duality holds true at each level. At level $k$ the counterpart for $CP_2$ (for $k=3$) would be as moduli space for sub-spaces of dimension $2^{k-1}$ for which tangent space reduces to the algebra at level $k-1$. For $k=2$ $CP_1$ is the moduli space and could correspond to twistor sphere. Essentially Grassmannian $Gl(2^k, 2^{k-1})$ would be in question. This brings in mind twistor Grassmann approach involving hierarchy of Grassmannians too, which however allows all dimensions. What is interesting that the spinor bundle for space of even dimension $d$ has fiber with dimension $2^{d/2}$.

   The number of arguments for the hierarchy of polynomials assignable to the hierarchy of infinite primes increases by one at each step. Hence these two hierarchies are different.

The vanishing of the octonionic polynomials indeed allow a decomposition to products of prime polynomials with roots which in general are algebraic numbers and an exciting possibility is that the prime polynomials have interpretation as counterparts of elementary particles in very general sense.

Infinite primes can be mapped to polynomials and the most natural counterpart for the infinite rational would be as a complexified octonionic rational function $P_1(t)/P_2(t-T)$, where $T$ is real octonion, with coefficients in extension of rationals. This would naturally give the geometry CD. The assignment of opposite boundaries of CD to $P_1(t)$ and $P_2(t-T)$ is suggestive and identification of zero loci of $IM(P_1)$ and $IM(P_2)$ as incoming and outgoing particles would be natural. The zero and $\infty$ loci for $RE(P_1/P_2)$ would define interaction between these space-time varieties and should give rise to wormhole contacts connecting them. Note that the linearity of $IM(o_1o_2)$ in $IM(o_i)$ and non-linearity of $RE(o_1o_2)$ in $RE(o_i)$ would be a key element behind this identification. This idea will be discussed in more detail in the section "Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view".

# 6 Super variant of octonionic algebraic geometry and space-time surfaces as correlates for fermionic states

Could the octonionic level provide an elegant description of fermions in terms of super variant of octonionic algebraic geometry? Could one even construct scattering amplitudes at the level of $M^8$ using the variant of the twistor approach discussed in [K26, K32]?

The idea about super-geometry is of course very different from the idea that fermionic statistics is realized in terms of the spinor structure of "world of classical worlds" (WCW) but $M^8-H$ duality could however map these ideas and also number theoretic and geometric vision to each other. The angel of geometry and the devil of algebra could be dual to each other.

In the following I start from the notion of emergence generalized to the vision that entire physics emerges from the notion of number. This naturally leads to an identification of super-variants of various number fields, in particular of complexified octonions. After that super variants of $RE(P) = 0$ and $IM(P) = 0$ conditions are discussed, and the surprising finding is that the conditions might allow only single fermion states localized at strings. This would allow only single particle in the super-multiplet and would mean breaking of SUSY. This picture would be consistent with the earlier $H$ picture about construction of scattering amplitudes [K26, K32]. Finally the problems related to the detailed physical interpretation are discussed.

## 6.1 About emergence

The notion of emergence is fashionable in the recent day physics, in particular, he belief is that 3-space emerges in some manner. In the sequel I consider briefly the standard view about emergence idea from TGD point of view, then suggest that the emergence in the deepest sense requires emergence of physics from the notion of number and that complexified octonions [L11] [L12, L13, L4, L10] are the most plausible candidate in this respect. After that I will show that number theory generalizes to super-number theory: super-number fields make sense and one can define the notion of super-prime. Every new step of progress creates worry about consistency with the earlier work, now the work done during last months with physics as octonionic algebraic geometry and also this aspect is discussed.

1. The notion of holography is behind the emergence of 3-space and implies that the notion of 2-space is taken as input. This could be justified by conformal invariance.

2. The key idea is that 3-space emerges somehow from entanglement. There is something that must entangle and this something must be labelled by points of space: one must introduce a discretised space. Then one must do some handwaving to make it 3-D - perhaps by arguing that holography based on 2-D holograms is unique by conformal invariance. The next hand-wave would replace this as a 3-D continuous space at infrared limit.

3. How to get space-time and how to get general coordinate invariance? How to get the symmetries of standard model and special relativity? Somehow all this must be smuggled into the theory when the audience is cheated to direct its attention elsewhere. This Münchausen trick requires a professional magician!

4. One attempt could take as starting point what I call strong form of holography (SH) in which 2-D data determine 4-D physics. Just like 2-D real analytic function determines analytic function of two complex variables in spacetime of 2 complex dimensions by analytic continuation (this hints strongly to quaternions). This is possible if conformal invariance is generalized to that for light-like 3-surfaces such as light-cone boundary. But the emergence magician should do the same without these.

   In TGD one could make this even simpler. Octonionic polynomials and rational functions are obtained from real polynomials of real variable by octonion-analytic continuation. And since polynomials and rational functions $P_1/P_2$ are in question their values at finite number of discrete points determined them if the orders of $P_1$ and $P_2$ are known!

   If one accepts adelic hierarchy based on extensions of rationals the coefficients of polynomials are in extensions of rationals and the situation simplifies further. The criticality conditions guaranteeing associativity for external particles is one more simplification: everything b becomes discrete. The physics at fundamental level could be incredibly simple: discrete number of points determines space-time surfaces as zero loci for $RE(P)$ or $IM(P)$ (octonions are decomposed to two quaternions gives $RE(o)$ and $IM(o)$).

   How this is mapped to physics leading to standard model emerging from the formulation in $M \times CP_2$ This map exists - I call it $M^8 - H$ duality - and takes space-time varieties in Minkowskian sector of complexified octonions to a space-time surface in $M^4 \times CP_2$ coding for standard model quantum numbers and classical fields.

How to get all this without bringing in octonionic imbedding space: this is the challenge for the emergence-magician! I am afraid this this trick is impossible. I will however propose a deeper for what emergence is. It would not be emergence of space-time and all physics from entanglement but from the notion of number, which is at the base of all mathematics. This view led to a discovery of the notion of super-number field, a completely new mathematical concept, which should show how deep the idea is.

## 6.2 Does physics emerge from the notion of number field?

Concerning emergence one can start from a totally different point of view. Even if one gets rid of space as something fundamental from Hilbert sapce and entanglement, one has not reached the most fundamental level. Structures like Hilbert space, manifold, etc. are not fundamental mathematical structures: they require the notion of number field. Number field is the fundamental notion.

Could entire physics emerge from the notion of number field alone: space-time, fermions, standard model interactions, gravitation? There are good hopes about this in TGD framework if one accepts $M^8-H$ duality and physics as octonionic algebraic geometry! One could however argue that fermions do not follow from the notion of number field alone. The real surprise was that formalizing this more precisely led to a realization that the very notion of number field generalizes to what one could call super-number field!

### 6.2.1 Emergence of physics from complexified octonionic algebraic geometry

Consider first the situation for number fields postponing the addition of attribute "super" later.

1. Number field endowed with basic arithmetic operations $+$, $-$, $\cdot$, $/$ is the basic notion for anyone wanting to make theoretical physics. There is a rich repertoire of number fields. Finite fields, rationals and their extensions, real numbers, complex numbers, quaternions, and octonions. There also p-adic numbers and their extensions induced by extensions of rationals and fusing into adele forming basic structure of adelic physics. Even the complex, quaternionic, and octonionic rationals and their extensions make sense. p-Adic variants of say octonions must be however restricted to have coefficients belonging to an extension of rationals unless one is willing to give up field property (the p-adic analog of norm squared can vanish in higher p-adic dimensions so that inverse need not exist).

   There are also function fields consisting of functions with local arithmetic operations. Analytic functions of complex variable provides the basic example. If function vanishes at some point its inverse element diverges at the same point. Function fields are derived objects rather than fundamental.

2. Octonions are the largest classical number field and are therefore the natural choice if one wants to reduce physics to the notion of number. Since one wants also algebraic extensions of rationals, it is natural to introduce the notion of complexified octonion by introducing an additional imaginary unit - call it $i$, commuting with the 7 octonionic imaginary units $I_k$. One obtains complexified octonions.

   That this is not a global number field anymore turns out to be a blessing physically. Complexified octonion $z_kE^k$ has $z_k = z_k + iy_k$. The complex valued norm of octonion is given by $z_0^2 + ...z_7^2$ (there is no conjugation involved. The norm vanishes at the complex surface $z_0^2 + ...z_7^2 = 0$ defining a 7-D surface in 7-D $O_c$ (the dimension is defined in complex sense). At this surface - complexified light-cone boundary - number field theory property fails but is preserved elsewhere since one can construct the inverse of octonion.

   At the real section $M^8$ (8-D Minkowski space with one real (imaginary) coordinate and 7 imaginary (real) coordinates the vanishing takes place also. This surface corresponds to the 7-D light-cone boundary of 8-D Minkowskian light-cone. This suggests that light-like propagation is basically due to the complexification of octonions implying local failure of the number field property. Same happens also in other real sections with $0 < n < 8$ real coordinates and $0 < m = 8 - n < 8$ imaginary coordinates and one obtains variant of light-cone with different signatures. Euclidian signature corresponding to $m = 0$ or $m = 8$ is an exception: light-cone boundary reduces to single point in this case and one has genuine number field - no propagation is possible in Euclidian signature.

Similar argument applies in the case of complexified quaternions $Q_c$ and complexified complex numbers $z_1 + z_2 I \in C_c$, where $I$ is octonionic imaginary unit. For $Q_c$ one obtains ordinary 3-D light-cone boundary in real section and 1-D light-cone boundary in the case of $C_c$. It seems that physics demands complexification! The restriction to real sector follows from the requirement that norm squared reduces to a real number. All real sectors are possible and I have already considered the question whether this should be taken as a prediction of TGD and whether it is testable.

### 6.2.2 Super-octonionic algebraic geometry

There is also a natural generalization of octonionic TGD to super-octonionic TGD based on octonionic triality. $SO(1,7)$ allows besides 8-D vector representations also spinor representations $8_c$ and $\overline{8}_c$. This suggests that super variant of number field of octonions might make sense. One would have $o = o_8 + o_{c,8} + \overline{0}_{c,8}$.

1. Should one combine $o_8$, $o_{c,8}$ and $\overline{o}_{c,8}$ to a coordinate triplet $(o_8, o_{c,8}, \overline{o}_{c,8})$ as done in supersymmetric theories to construct super-fields? The introduction of super-fields as primary dynamical variables is a good idea now since the very idea is to reduce physics to algebraic geometry at the level of $M^8$. Polynomials of super-octonions defining space-time varieties as zero loci for their real or imaginary part in quaternionic sense could however take the role of super fields. Space-time surface would correspond to zero loci for $RE(P)$ or $IM(P)$.

2. The idea about super-octonions should be consistent with the idea that we live in a complexified number field. How to define the notion of super-octonion? The tensor product $8 \otimes 8_c$ contains $8_c$ and $8 \otimes 8_{\overline{c}}$ contains $8_{\overline{c}}$ and one can use Glebsch-Gordan coefficients to contract $o$ and $\theta_c$ and $o$ and $\overline{\theta}_{c,n}$. The tensor product of $8_c$ and $8_{\overline{c}}$ defined using structure constants defining octonion product gives 8. Therefore one must have

$$o_s = o + \Psi_c \times \theta_{\overline{c}} + \Psi_{\overline{c}} \times \theta_c \ , \tag{6.1}$$

   where the products are octonion products. Super parts of super-coordinates would not be just Grassmann numbers but octonionic products of Grassmann numbers with octonionic spinors in $8_c$ and $\overline{8}_c$. This would bring in the octonionic analogs of spinor fields into the octonionic geometry.

   This seems to be consistent with super field theories since octonionic polynomials and even rational functions would give the analogs of super-fields. What TGD would provide would be an algebraic geometrization of super-fields.

3. What is the meaning of the conditions $RE(P) = 0$ and $IM(P) = 0$ for super-octonions? Does this condition hold true for all $d_G = 2^{16}$ super components of $P(o_s)$ or is it enough to pose the condition only for the octonionic part of $P(o)$? In the latter case $\Psi_c$ and $\Psi_{\overline{c}}$ would be free and this does not seem sensical and does not conform with octonionic super-symmetry. Therefore the first option will be studied in the sequel.

If super-octonions for a super variant of number field so that also inverse of super-octonion is well-defined, then even rational functions of complexified super-octonions makes sense and poles have interpretation in terms of 8-D light-fronts (partonic orbits at level of $H$). The notion must make sense also for other classical number fields, finite fields, rationals and their extensions, and p-adic numbers and their extensions. Does this structure form a generalization of number field to a super counter part of number field? The easiest manner to kill the idea is to check what happens in the case of reals.

1. The super-real would be of form $s = x + y\theta$, $\theta^2 = 0$. Sum and product are obviously well-defined. The inverse is also well-defined and given by $1/s = (x - y\theta))/x^2$. Note that for complex number $x + iy$ the inverse would be $\overline{z}/z\overline{z} = (x - yi)/(x^2 + y^2)$. The formula for super-inverse follows from the same formula as the inverse of complex number by defining conjugate of super-real $s$ as $\overline{s} = x - y\theta$ and the norm squared of $s$ as $|s|^2 = s\overline{s} = x^2$.

One can identify super-integers as $N = m + n\theta$. One can also identify super-real units as number of unit norm. Any number $1_n = 1+n\theta$ has unit norm and the norms form an Abelian group under multiplication: $1_m 1_n = 1_{m+n}$. Similar non-uniqueness of units occurs also for algebraic extensions of rationals.

2. Could one have super variant of number theory? Can one identify super-primes? Super-norm satisfies the usual defining property $|xy| = |x||y|$. Super-prime is defined only apart from the multiplicative factor $1_m$ giving not contribution to the norm. This is not a problem but a more rigorous formulation leads to the replacement of primes with prime ideals labelled by primes already in the ordinary number theory.

   If the norm of super-prime is ordinary prime it cannot decompose to a product of super-primes. Not all super-primes having given ordinary prime as norm are however independent. If super-primes $p + n\theta$ and $p + m\theta$ differ by a multiplication with unit $1_r = 1 + r\theta$, one has $n - m = pr$. Hence there are only $p$ super-primes with norm $p$ and they can be taken $p_s = p + k\theta$, $k \in \{0, p-1\}$. A structure analogous to a cyclic group $Z_p$ emerges.

   Note that also $\theta$ is somewhat analogous to prime although its norm is vanishing.

3. Just for fun, one an ask what is the super counterpart of Riemann Zeta. Riemann zeta can be regarded as an analog of thermodynamical partition function reducing to a product for partition functions for bosonic systems labelled by primes $p$. The contribution from prime $p$ is factor $1/(1-p^{-s})$. $p^{-s}$ is analogous to Boltzmann weight $N(E)exp(-E/T)$, where $N(E)$ is number of states with energy $E$. The degeneracy of states labelled by prime $p$ is for ordinary primes $N(p) = 1$. For super-primes the degeneracy is $N(p) = p$ and the weight becomes $1/(1-N(p)p^{-s}) = 1/(1-p^{-s+1})$. Super Riemann zeta is therefore $zeta(s-1)$ having critical line at $s = 3/2$ rather than at $s = 1/2$ and trivial zeros at real points $s = -1, -3, -5$, rather than at $s = -2, -4, -6, ...$

There are good reasons to expect that the above arguments work also for algebraic extensions of super-rationals and in fact for all number fields, even for super-variants of complex numbers, quaternions and octonions. This because the conditions for invertibility reduce to that for real numbers. One would have a generalization of number theory to super-number theory! Net search gives no references to anything like this. Perhaps the generalization has not been noticed because the physical motivation has been lacking. $M^8-H$ duality would imply that entire physics, including fermion statistics, standard model interactions and gravitation reduces to the notion of number in accordance with number theoretical view about emergence.

### 6.2.3 Is it possible to satisfy super-variants of $IM(P) = 0$ and $RE(P) = 0$ conditions?

Instead of super-fields one would have a super variant of octonionic algebraic geometry.

1. Super variants of the polynomials and even rational functions make sense and reduce to a sum of octonionic polynomials $P_{kl}\theta_1^k\theta_2^l$, where the integers $k$ and $l$ would be tentatively identified as fermion numbers and $\theta_k$ is a shorthand for a monomial of $k$ different thetas. The coefficients in $P_{kl} = P_{kl,n}o^n$ would be given by $P_{kl,n} = P_{n+k+l}B(n+k+l, k+l)$, where $B(r,s) = r!/(r-s)!s!$ is binomial coefficient. The space-time surfaces associated with $P_{kl}$ would be different and they need not be simultaneously critical, which could give rise to a breaking of supersymmetry.

   One would clearly have an upper bound for $k$ and $l$ for given CD. Therefore these many-fermion states must correspond to fundamental particles rather than many-fermion Fock states. One would obtain bosons with non-vanishing fermion numbers if the proposed identification is correct. Octonionic algebraic geometry for single CD would describe only fundamental particles or states with bounded fermion numbers. Fundamental particles would be indeed fundamental also geometrically.

2. One can also now define space-time varieties as zero loci via the conditions $RE(P_s)(o_s) = 0$ or $IM(P_s)(o_s) = 0$. One obtains a collection of 4-surfaces as zero loci of $P_{kl}$. One would have a correlation with between fermion content and algebraic geometry of the space-time surface unlike in the ordinary super-space approach, where the notion of the geometry remains rather

formal and there is no natural coupling between fermionic content and classical geometry. At the level of $H$ this comes from quantum classical correspondence (QCC) stating that the classical Noether charges are equal to eigenvalues of fermionic Noether charges.

In the definition of the first variant of super-octonions I followed the standard idea about what super-coordinates assuming that the super-part of super-octonion is just an anti-commuting Grassmann number without any structure: I just replaced $o$ with $o+\theta_k E^k + \overline{\theta}_k E^k$ regarding $\theta_k$ as anticommiting coordinates. Now $\theta_k$ receives octonionic coefficient: $\theta_k \rightarrow o_k\theta_k$. $\theta_k$ is now analogous to unit vector.

For the super-number field inspired formulation the situation is different since one assigns independent octonionic coordinates to anticommuting degrees of freedom. One has linear space with partially anti-commutative basis. $O_c$ is effectively replaced with $O_c^3$ so that one has 8+8+8=24-dimensional Cartesian product (it is amusing that the magic dimension 24 for physical polarizations of bosonic string models emerges).

What is the number of equations in the new picture? For $N$ super-coordinates one has $2^N$ separate monomials analogous to many-fermion states. Now one has $N = 8+8 = 16$ and this gives $2^{16}$ monomials! In the general case $RE = 0$ or $IM = 0$ gives 4 equations for each of the $d_G = 2^{16}$ monomials: the number of equations $RE = 0$ or $IM = 0$ is $4 \times 2^{16}$ and exceeds the number $d_O = 24$ of octonion valued coordinates. In the original interpretation these equations were regarded as independent and gave different space-time variety for each many-fermion state.

In the new framework these equations cannot be treated independently. One has 24 octonionic coordinates and $2^{16}$ equations. In the generic case there are no solutions. This is actually what one hopes since otherwise one would have a state involving superposition of many-fermion states with several fermion numbers.

The freedom to pose constraints on the coefficients of Grassmann parameters however allows to reduce degrees of freedom. All coefficients must be however expressible as products of $3\times 8 = 24$ components of super-octonion.

1. One can have solutions for which both $8_c$ part and $\overline{8}_c$ parts vanish. This gives the familiar 4 equations for 8 variables and 4-surfaces.

2. Consider first options, which fail. If $8_c$- or $\overline{8}_c$ part vanishes one has $d_G = 2^8$ and $4\times d_G = 4\times 64$ equations for $dO{=}8+8 = 16$ variables having no solutions in the generic case. The restriction of $8_c$ to its 4-D quaternionic sub-space would give $d_O = 4$ and $4d_G = 4\times 2^4 = 64$ conditions and 16 variables. The reduction to complex sub-space $z_1 + z_2 I$ of super-octonions would give $d_O = 2^2$ and $4\times 2^2 = 16$ conditions for $8+2 = 10$ variables.

3. The restriction to 1-D sub-space of super-octonions would give $4\times 2^1 = 8$ conditions and $8+1 = 9$ variables. Could the solution be interpreted as 1-D fermionic string assignable to the space-like boundary of space-time surface at the boundary of CD? Skeptic inside me asks whether this could mean the analog of $\mathcal{N} = 1$ SUSY, which is not consistent with $H$ picture.

   Second possibility is restriction to light-like subspace for which powers of light-like octonion reduce effectively to powers of real coordinate. Fermions would be along light-lines in $M^8$ and along light-like curves in $H$. The powers of super-octonion have super-part, which belongs to the 1-D super-space in question: only single fermion state is present besides scalar state.

4. There are probably other solutions to the conditions but the presence of fermions certainly forces a localization of fermionic states to lower-dimensional varieties. This is what happens also in $H$ picture. During years the localization of fermion to string worlds sheets and their boundaries has popped up again and again from various arguments. Could one hope that super-number theory provides the eventual argument.

   But how could one understand string world sheets in this framework? If they do not carry fermions at H-level, do they appear naturally as 2-D structures in the ordinary sense?

To sum up, although many details must be checked and up-dated, super-number theory provides and extremely attractive approach promising ultimate emergence as a reduction of physics to the notion of number. When physical theory leads to a discovery of new mathematics, one must take it seriously.

## 6.3 About physical interpretation

Super-octonionic algebraic geometry should be consistent with the $H$ picture in which baryon and lepton numbers as well as other standard model quantum numbers can be understood. There are still many details, which are not properly understood.

### 6.3.1 The interpretation of theta parameters

The interpretation of theta parameters is not completely straightforward.

1. The first interpretation is that $\theta_c$ and $\theta_{\overline{c}}$ correspond to objects with opposite fermion numbers. If this is not the case, one could perhaps define the conjugate of super-coordinate as octonionic conjugate $\overline{o}_s = \overline{o} + \overline{\theta}_1 + \overline{\theta}_2$. This looks ugly but cannot be excluded.

   There is also the question about spinor property. Octonionic spinors are 2-spinors with octonion valued components. Could one say that the coefficients of octonion units have been replaced with Grassmann numbers and the entire 2-component spinor is represented as a pair of $\theta_c$ and $\theta_{\overline{c}}$? The two components of spinor in massless theories indeed correspond to massless particle and its antiparticle.

2. One should obtain particles and antiparticles naturally as also separately conserved baryon and lepton numbers (I have also considered the identification of hadrons in terms of anyonic bound states of leptons with fractional charges).

   Quarks and leptons have different coupling to the induced Kähler form at the level of $H$. It seems impossible to understand this at the level of $M^8$, where the dynamics is purely algebraic and contains no gauge couplings.

   The difference between quarks and leptons is that they allow color partial waves with triality $t = \pm 1$ and triality $t = 0$. Color partial waves correspond to wave functions in the moduli space $CP_2$ for $M^4_0 \supset M^2_0$. Could the distinction between quarks and leptons emerge at the level of this moduli space rather than at the fundamental octonionic level? There would be no need for gauge couplings to distinguish between quarks and leptons at the level of $M^8$. All couplings would follow from the criticality conditions guaranteeing 4-D associativity for external particles (on mass shell states would be critical).

   If so, one would have only the super octonions and $\theta_c$ and $\theta_{\overline{c}}$ would correspond to fermions and antifermions with no differentiation to quarks or leptons. Fermion number conservation would be coded by the Grassmann algebra. Quantum classical correspondence (QCC) however suggests that it should be possible to distinguish between quarks and leptons already at $M^8$ level. Is it really enough that the distinction comes at the level of moduli space for CDs?

One can imagine also other options but they have their problems. Therefore this option will be considered in the sequel.

### 6.3.2 Questions about quantum numbers

The first questions relate to fermionic statistics.

1. Do super-octonions really realize fermionic statistics and how? The polynomials of super-octonions can have only finite degree in $\theta$ and $\theta_c$. One an say that only finite number of fermions are possible at given space-time point. As found, the conditions $IM(P) = 0$ and $RE(P) = 0$ might allow only single fermion strings as solutions perhaps assignable to partonic 2-surfaces.

   Can one allow for given CD arbitrary number of this kind of points as the idea that identical fermions can reside at different points suggests? Or is the number of fermions finite for given CD or correspond to the highest degree monomial of $\theta$ and $\theta_c$ in $P$?

   Finite fermion number of CD looks somewhat disappointing at first. The states with high fermion numbers would be described in terms of Cartesian products just like in condensed matter physics. Note however that space-time varieties with different octonionic time axes must be in any case described in this manner. It seems possible to describe the interactions using super-space delta functions stating that the interaction occur only in the intersection points of the space-time surfaces. The delta function would have also super-part as in SUSYs.

2. As found, the theta degree effectively reduces to $d = 1$ for the pointlike solutions, which by above argument are the only possible solutions besides purely bosonic solutions. Only single fermion would be allowed at given point. I have already earlier considered the question whether the partonic 2-surfaces can carry also many-fermion states or not [K26, K32], and adopted the working hypothesis that fermion numbers are not larger than 1 for given wormhole throat, possibly for purely dynamical reasons. This picture however looks too limited. The many fermion states might not however propagate as ordinary particles (the proposal has been that their propagator pole corresponds to higher power of $p^2$).

The $M^8$ description of particle quantum numbers should be consistent with $H$ description.

1. Can octonionic super geometry code for the quantum numbers of the particle states? It seems that super-octonionic polynomials multiplied by octonionic multi-spinors inside single CD can code only for the electroweak quantum numbers of fundamental particles besides their fermion and anti-fermion numbers. What about color?

   As already suggested, color corresponds to partial waves in $CP_2$ serving as moduli space for $M_0^4 \supset M_0^2$. Also four-momentum and angular momentum are naturally assigned with the translational degrees for the tip of CD assignable with the fundamental particle.

2. Quarks and leptons have different trialities at $H$ level. How can one understand this at $M^8$ level. Could the color triality of fermion be determined by the color representation assignable to the color decomposition of octonion as $8 = 1 + 1 + 3 + \overline{3}$. This decomposition occurs for all 3 terms in the super-octonion. Could the octet in question correspond to the term $D(8 \otimes 8_c; 8_c)_k^{mn} o_{c,m} \theta_{c,n} E^k$ and analogous $\theta_{\overline{c}}$ term in super octonion. Only this kind of term survives from the entire super-octonion polynomial at fermionic string for the solutions found.

3. There is however a problem: $8 = 1 + 1 + 3 + \overline{3}$ decomposition is not consistent with the idea that $\theta_c$ and $\theta_{\overline{c}}$ have definite fermion numbers. Quarks appear only as 3, not $\overline{3}$. Why $\overline{3}$ from $\theta$ term and 3 from $\theta_{\overline{c}}$ term should drop out as allowed single fermion state?

There are also other questions.

1. What about twistors in this framework? $M^4 \times CP_1$ as twistor space with $CP_1$ coding for the choice of $M_0^2 \subset M_0^4$ allows projection to the usual twistor space $CP_3$. Twistor wave functions describing spin elegantly would correspond to wave functions in the twistor space and one expects that the notion of super-twistor is well-defined also now. The 6-D twistor space $SU(3)/U(2) \times U(1)$ of $CP_2$ would code besides the choice of $M_0^4 \supset M_0^2$ also quantization axis for color hypercharge and isospin.

2. The intersection of space-time surfaces with $S^6$ giving analogs of partonic 2-surfaces might make possible for two sparticle lines to fuse to form a third one at these surfaces. This would define sparticle 3-vertex in very much the same manner as in twistor Grassmann approach to $\mathcal{N} = 4$ SUSY.

   $H$-picture however supports the alternative option that sparticles just scatter but there is no contact interaction defining analog of 3-vertex. If the lines can carry only single fermion, the $H$ picture about twistor diagrams [K26, K32] would be realized also at the level of $M^8$! This means breaking of SUSY since only single fermion states from the octonionic SUSY multiplet are realized. This would provide and easy - perhaps too easy - explanation for the failure to find SUSY at LHC.

3. What about the sphere $S^6$ serving as the moduli space for the choices of $M_+^8$? Should one have wave functions in $S^6$ or can one restrict the consideration to single $M_+^8$? As found, one obtains $S^6$ also as the zero locus of $Im(P) = 0$ for some radii identifiable as values $t_n$ of time coordinates given as roots of $P(t)$: as matter of fact, $S^6(t_n)$ is a solution of both $RE(P) = 0$ and $IM(P) = 0$. Can one identify the intersections $X^4 \cap S^6$ are 2-D as partonic 2-surfaces serving as topological vertices?

# 7 Could scattering amplitudes be computed in the octonionic framework?

Octonionic algebraic geometry might provide incredibly simple framework for constructing scattering amplitudes since now variational principle is involved and WCW reduces to a discrete set of points in extension of rationals.

## 7.1 Could scattering amplitudes be computed at the level of $M^8$?

It would be extremely nice if the scattering amplitudes could be computed at the octonionic level by using a generalization of twistor approach in ZEO finding a nice justification at the level of $M^8$. Something rather similar to $\mathcal{N}=4$ twistor Grassmann approach suggests itself.

1. In ZEO picture one would consider the situation in which the passive boundary of CD and members of state pairs at it appearing in zero energy state remain fixed during the sequence of state function reductions inducing stepwise drift of the active boundary of CD and change of states at it by unitary U-matrix at each step following by a localization in the moduli space for the positions of the active boundary.

2. At the active boundary one would obtain quantum superposition of states corresponding to different octonionic geometries for the outgoing particles. Instead of functional integral one would have sum over discrete points of WCW. WCW coordinates would be the coefficients of polynomial $P$ in the extension of rationals. This would give undefined result without additional constraints since rationals are a dense set of reals.

   Criticality however serves as a constraint on the coefficients of the polynomials and is expected to realize finite measurement resolution, and hopefully give a well defined finite result in the summation. Criticality for the outgoing states would realize purely number theoretically the cutoff due to finite measurement resolution and would be absolutely essential for the finiteness and well-definedness of the theory.

## 7.2 Interaction vertices for space-time surfaces with the same CD

Consider interaction vertices for space-time surfaces associated with given CD. At the level of $H$ the fundamental interactions vertices are partonic 2-surfaces at which 3 light-like partonic orbits meet. The incoming light-like sparticle lines scatter at this surface and they are not assumed to meet at single vertex. This assumption is motivated because it allows to avoid infinities but one must be ready to challenge it. It is essential that wormhole throats appear in pairs assignable to wormhole contacts and also contacts form pairs by the conservation of Kähler magnetic flux.

What could be the counterpart of this picture at level of $M^8$?

1. The simplest interaction could be associated with the common stable intersection points of the space-time regions. By dimensional consideration these intersections are stable and form a discrete set. This would however allow only 2-vertices involved in processes like mixing of states. In the generic case the intersection would consist of discrete points.

2. A stronger condition would be that these points belong to the extension of rationals defining adeles or is extension defined by the polynomial $P$. This would conform with the idea that scattering amplitudes involve only data associated with the points in the extension. The interaction points could be ramified points at which the action of a subgroup $H$ of Galois group $G$ would leave sheets of the Galois covering invariant so that some number of sheets would touch each other. I have discussed this proposal in [L9]. These points could be seen as analogs of interaction points in QFT description in terms of $n$-point functions and the sum over polynomials would give rise to the analog over integral over different $n$-point configurations.

3. A possible interpretation is that if the subgroup $H \subset G$ has $k$-elements the vertex represents meeting of $k$ sparticle lines and thus $k$-vertex would be in question. This picture is not what the $H$ view about twistor diagrams [K32] suggests: in these diagrams sparticle lines at the

light-like orbits of partonic 2-surfaces do not meet at single point but only scatter at partonic 2-surface, where three light-like orbits of partonic 2-surfaces meet.

4. An alternative interpretation is that $k$-vertex describes the decay of particle to $k$ fractional particles at partonic 2-surfaces and has nothing do with the usual interaction vertex.

This proposal need not describe usual particle scattering. Could the intersection of space-time varieties defined as zero loci for $RE(P_i)$ and $IM(P_i)$ with the special solutions $S^6(t_n)$ and $CD = M^4_+ \cap M^4_-$ define the loci of interaction? It is difficult to believe that these special solutions could be only a beauty spot of the theory. $X^2 = X^4 \cap S^6(t_n)$ is 2-D and $X^0 = X^4 \cap CD$ consists of discrete points.

Consider now the possible role of the singular $(RE(P) = IM(P) = 0)$ maximally critical surface $S^6(t_n)$ in the scattering.

1. As already found, the 6-D spheres $S^6$ with radii $t_n$ given by the zeros of $P(t)$ are universal and have interpretation as $t = t_n$ snapshots of 7-D spherical light front projection to $t = t_n$ 3-balls as cross sections of 4-D CD. Could the 2-D intersection $X^2 = X^4 \cap S^6(t_n)$ play a fundamental role in the description of interaction vertices?

2. Suppose that 3-vertices realize the dynamical realization of octonionic SUSY predicting large number of sparticles. Could one understand in this framework the 3-vertex for the orbits $X^3_i$ of partonic 2-surfaces meeting each other along their 2-D end defining partonic 2-surface and undersand how 3 fermions lines meet at single point in this picture?

3. Assume that 3 partonic orbits $X^3_i$, $i = 1, 2, 3$ meet at $X^2 = X^4 \cap S^6(t_n)$. That this occurs could be part of boundary conditions, which should follow from interaction consistency. If fermions just *go through* the $X^2_i$ in time direction they cannot meet at single point in the generic case. If the sparticle lines however can *move along* $X^2$ - maybe due the fact that an intersection $X^2 = X^4 \cap S^6(t_n)$ is in question - they intersect in the generic case and fuse to a third fermion line. Note that this portion of fermion line would be space-like whereas outside $X^2$ the line would be light-like. This can be used as an objection against the idea.

The picture allowing 3-vertices would be different from $H$ picture in which fermion lines only scatter and only 2+2 fermion vertex assignable to topological 3-vertex is fundamental.

1. One would have 2 wormhole contacts carrying fermion and third one carrying fermion anti-fermion pair at its opposite throats and analogous to boson. Of course, one can reproduce the earlier picture by giving up the condition about supersymmetric fermionic 3-vertex. On the other hand, the idea that interactions occur only at discrete points in extension of rationals is extremely attractive.

2. The surprising outcome from the construction of solutions of super-variants of $RE(P) = 0$ and $IM(P) = 0$ conditions was that if the superpart of super-octonion is non-vanishing, the variety can be only 1-D string like entity carrying one-fermion state. This does allow strings with higher fermion number so that the 3-vertex would not be possible! This suggests that fermionic lines appear as sub-varieties of space-time variety.

   If so the original picture [K32] applying at the level of $H$ applies also at the level of $M^8$. SUSY is broken dynamically allowing only single fermion states localized at strings and scattering of these occurs by classical interactions at the partonic 2-surfaces defining the topological vertices.

3. The only manner to have a point/line containing sparticle with higher fermion number would be as a singularity along which several branches of super-variety degenerate to single point/line: each variety would carry one fermion line. Unbroken octonionic SUSY would characterize singularities of the space-time varieties, which would be unstable so that SUSY would break. Singularities are indeed critical and thus unstable and also tend to possess enhanced symmetries.

What could be the interpretation of $X^0 = X^4 \cap CD$? For instance, could it be that these points code for 4-momenta classically so that quantum classical correspondence (QCC) would be realized also at the level of $M^8$ although there are no Noether charges now. But what about angular momenta? Could twistorialization realized in terms of the quaternionic structure of $M_0^4$ help here. What is the role of the intersections of 6-D twistor bundle of $X^4$ with 6-D twistor bundle of $M_0^4$ consisting of discrete points?

The interaction vertex would involve delta function telling that the interacting space-time varieties or their regions touch at same point of $M^8$. Delta function in theta parameter degrees of freedom and Grassmann integral over them would be also involved and guarantee fermion number conservation. Vertex factor should be determined by arguments used in Grassmannian twistor approach. I have developed a proposal in [K32] but this proposal allows only fermion number $\pm 1$ at fermion lines. Now all members of the multiplet would be allowed.

## 7.3 How could the space-time varieties associated with different CDs interact?

The interaction of space-time surfaces inside given CD is well-defined in the octonionic algebraic geometry. The situation is not so clear for different CDs for which the choice of the origin of octonionic coordinates is in general different and polynomial bases for different CDs do not commute nor associate.

The intuitive expectation is that 4-D/8-D CDs can be located everywhere in $M^4/M^8$. The polynomials with different origins neither commute nor are associative. Their sum is a polynomial whose coefficients are not real. How could one avoid losing the extremely beautiful associative and commutative algebra of polynomials?

1. Should one assume that the physics observable by single conscious observer corresponds to single CD defining the perceptive field of this observer [L17].

2. Or should one give up associativity and allow products (but not sums since one should give up the assumption that the coefficients of polynomials are real) of polynomials associated with different CDs as an analog for the formation of free many-particle states.

Consider first what happens for the single particle solutions defined as solutions of either $RE(P_i) = 0$ or $IM(P_i) = 0$.

1. The polynomials associated with different 8-D CDs do not commute nor associate. Should one allow their products so that one would still *effectively* have a Cartesian product of commutative and associative algebras? This would realize non-commutative and non-associative physics emerging in conformal field theories also at the level of space-time geometry.

2. If the CDs differ by a *real* (time) translation $o_2 = o_1 + T$ one still obtains $IM(P_1) = 0$ and $IM(P_2) = 0$ as solutions to $IM(P_1P_2) = 0$. This applies also to states with more particles. The identification would be in terms of external particles. For $RE(P_1P_2) = 0$ this is not the case. If the interior of CD corresponds to $RE(P_1P_2) = 0$, the dynamics in the interior is not only non-trivial but also non-commutative and non-associative. Non-trivial interaction would be obtained even without interaction terms in the polynomial vanishing at the boundaries of CD!

   Could one consider allowing only CDs with tips at the same real axis but having all sizes scales? This hierarchy of CD would characterize a particular hierarchy of conscious observers - selves having sub-selves (sub-CDs) [L17]. The allowance of only these CD would be analogous to a fixing of quantization axes.

3. What happens if one allows CDs differing by arbitrary octonion translation? Consider external particles. For $P_1$ and $P_2$ $RE$ and $IM$ are defined for different decompositions $o_i = RE(o_i) + n_i Im(o_i)$, where $n_i$, $i = 1, 2$ is a unit octonion.

   What decomposition should one use for $P_1P_2$? The decomposition for $P_1$ or $P_2$ or some other decomposition? One can express $P_2(o_2)$ using $o_1$ as coordinate but the coefficients multiplying powers of $o_1$ from *right* would not be real numbers anymore implying $IM(P_2)_1 \neq IM(P_2)_2$.

$IM(P_2)_1 = 0$ makes sense but the presence of particle 1 would have affected particle 2 or vice versa.

Could one argue that the coordinate systems satisfying the condition that some external particles described by $P_i$ have real coefficients and perhaps serving in the role of observers are preferred? Or could one imagine that $o_{12}$ is a kind of center of mass coordinate? In this case the 4-varieties associated with both particles would be affected. What is clear that the choice of the octonionic coordinate origin would affect the space-time varieties of external particles even if they could remain associative/critical.

4. Are there preferred coordinates in which criticality is preserved? For instance, can one achiever criticality for $P_2$ on coordinates of $o_1$ if $P_1$ is critical. Could one see this as a kind of number theoretic observer effect at the level of space-time geometry?

   **Remark:** $P_i(o)$ would reduce to a real polynomial at light-like rays with origin for $o_i$ irrespective of the octonionic coordinate used so that the spheres $S^6_i$ with origin at the origin of $o_i$ as solutions of $P_i(o) = 0$ would not be lost.

If one does not give up associativity and commutativity for polynomials, how can one describe the interactions between space-time surfaces inside different CDs at the level of $M^8$? The following proposal is the simplest one that one can imagine by assuming that interactions take place at discrete points of space-time surfaces with coordinates belonging an extension of rationals.

1. The most straightforward manner would be to introduce Cartesian powers of $O$ and CD:s inside these powers to describe the interaction between CDs with different origin. This would be analogous to what one does in condensed matter physics. What seems clear is that $M^8 - H$ correspondence should map all the factors of $(M^8)^n$ to the same $M^4 \times CP_2$ by a kind of diagonal projection.

   In topological 3-parton vertex $X^2$ three light-like partonic orbits along $X^4$ would meet. $X^2$ would be the contact of $X^4$ with $S^6$ associated with second 8-D CD. Together with SH this gives hopes about an elegant description of interactions in terms of connected space-time varieties.

2. The intersection $X^4_i \cap X^4_j$ consists of discrete set of points. This would suggest that the interaction means transfer of fermion between $X^4_1$ and $X^4_2$. The intersection of $X = S^6_1(t_m) \cap S^6_2(t_n)$ is 4-D and space-like. The intersection $X^4_i \cap X$ consists of discrete points could these discrete points allow to construct interaction vertices.

To make this more concrete, assume that the external particles outside the interaction CD (CD$_{int}$) defining the interaction region correspond to associative (or co-associative) space-time varieties with different CDs.

**Remark**: CDs are now 8-dimensional.

1. One can assign the external particles to the Cartesian factors of $(M^8)^n$ giving $(P_1, ..., P_n)$ just like one does in condensed matter physics for particles in 3-space $E^3$. Inside CD$_{int}$ the Cartesian factors would fuse to single factor and instead of Cartesian product one would have the octonionic product $P = \prod P_i$ plus the condition $RE(P) = 0$ (or $IM(P) = 0$: one should avoid too strong assumptions at this stage) would give to the space-time surface defining the interaction region.

2. $RE(P) = 0$ and $IM(P) = 0$ conditions make sense even, when the polynomials do not have origin at common real axis and give rise to 4 conditions for 8 polynomials of 8 complexified octonion components $P^i$. It is not possible to reduce the situation at the light-like boundaries of 8-D light-cone to a vanishing of polynomial $P(t)$ of real coordinate $t$ anymore, and one loses the the surfaces $S^6_i$ as special solutions and therefore also the partonic 2-surfaces $X^2_i = X^4 \cap S^6_i$. Should one assign all $X^2_i$ with the intersections of external particles with the two boundaries $\delta_\pm$ CD of CD defining the interaction region. They would intersect $\delta_\pm$CD at highly unique discrete points defining the sparticle interaction vertices. By 7-dimensionality of $\delta_\pm$CD the intersection points would be at the boundaries of 4-D CD and presumably at

light-like partonic orbits at which the induced metric is singular at $H$ side at least just as required by $H$ picture.

The most general external single-sparticle state would be defined by a product $P$ of mutually commuting and associating polynomials with tips of CD along common real axis and satisfying $IM(P_i) = 0$ or $RE(P_i) = 0$. This could give both free and bound states of constituents.

3. Different orders and associations for $P = \prod P_i$ give rise to different interaction regions. This requires a sum over the scattering amplitudes $\sum_p T(\prod_i P_{p(i)})$ associated with the permutations $p$: $(1, ..., n) \rightarrow (p(1), .., p(n))$ and $T = \sum_p U(p)T(P_{p(1)}...P_{p(n)})$ ($T(AB) + T(BA)$ in the simplest case) with suitable phase factors $U(p)$. Note that one does *not* have a sum over the polynomials $P_{p(1)}...P_{p(n)}$ but over the scattering amplitudes associated with them.

4. Depending on the monomial of theta parameters in super-octonion part of $P_i$, one has plus or minus signs under the exchange of $P_i$ and $P_j$. One can also have braid group as a lift of the permutation group. In this case given contribution to the scattering amplitude has a phase factor depending on the permutation (say $T = T(AB) + exp(i\theta)T(BA)$.

   One must also form the sum $T = \sum_{Ass} U(Ass)T(Ass(P))$ over all associations for a given permutation with phase factors $U(Ass)$. Here $T = T((AB)C) + UT(A(BC))$, $U$ phase factor, is the simplest case. One has "association statistics" as the analog of braid statistics. Permutations and associations have now a concrete geometric meaning at the level of space-time geometry - also at the level of $H$.

5. The geometric realization of permutations and associations could relate to the basic problem encountered in the twistorial construction of the scattering amplitudes. One has essentially sum over the cyclic permutations of the external particles but does not know how to construct the amplitudes for general permutations, which correspond to non-planar Feynman diagrams. The geometric realization of the permutations and associations would solve this problem in TGD framework.

## 7.4 Twistor Grassmannians and algebraic geometry

Twistor Grassmannians provide an application of algebraic geometry involving the above described notions [B2] (see `http://tinyurl.com/yd9tf2ya`). This approach allows extremely elegant expressions for planar amplitudes of $\mathcal{N} = 4$ SYM theory in terms of amplitudes formulated in Grassmannians $G(k, n)$.

It seems that this approach generalizes to TGD in such a manner that $CP_2$ degrees of freedom give rise to additional factors in the amplitudes having form very similar to the $M^4$ part of amplitudes and involving also $G(k, n)$ with ordinary twistor space $CP_3$ being replaced with the flag manifold $SU(3)/U(1) \times U(1)$: $k$ would now correspond to the number sparticles with negative weak isospin. Therefore the understanding of the algebraic geometry of twistor amplitudes could be helpful also in TGD framework.

### 7.4.1 Twistor Grassmannian approach very concisely

I try to compress my non-professional understanding of twistor Grassmann approach to some key points.

1. Twistor Grassmannian approach constructs the scattering amplitudes by fusing 3-vertices (+,-,-) (one positive helicity) and (-,+,+) (one negative helicity) to a more complex diagrams. All particles are on mass shell and massless but complex. If only real massless momenta are allowed the scattering amplitudes would allow only collinear gluons. Incoming particles have real momenta.

   **Remark**: Remarkably, $M^4 \times CP_2$ twistor lift of TGD predicts also complex Noether charges, in particular momenta, already at classical level. Quantal Noether charges should be hermitian operators with real eigenvalues, which suggests that total Noether charges are real. For conformal weights this condition corresponds to conformal confinement. Also $M^8 - H$ duality requires a complexification of octonions by adding commuting imaginary unit and allows to circumvent problems related to the Minkowski signature since the metric tensor can be regarded as Euclidian metric tensor defining complex value norm as bilinear $m^k m_{kl} m^l$ in complexified $M^8$ so that real metric is obtained only in sub-spaces with real or purely imaginary coordinates. The additional imaginary unit allows also to define what complex algebraic numbers mean.

The unique property of 3-vertex is that the twistorial formulation for the conservation of four-momentum implies that in the vertex one has either $\lambda_1 \propto \lambda_2 \propto \lambda_3$ or $\overline{\lambda}_1 \propto \overline{\lambda}_2 \propto \overline{\lambda}_3$. These cases correspond to the 2 3-vertices distinguished notationally by the color of the vertex taken to be white or black [B2].

**Remark**: One must allow octonionic super-space in $M^8$ formulation so that octonionic SUSY broken by $CP_2$ geometry reducing to the quaternionicity of 8-momenta in given scattering diagram is obtained.

2. The conservation condition for the total four-momentum is quadratic in twistor variables for incoming particles. One can linearize this condition by introducing auxiliary Grassmannian $G(k,n)$ over which the tree amplitude can be expressed as a residue integral. The number theoretical beauty of the multiple residue integral is that it can make sense also p-adically unlike ordinary integral.

   The outcome of residue integral is a sum of residues at discrete set of points. One can construct general planar diagrams containing loops from tree diagrams with loops by BCFW recursion. I have considered the possibility that BCFW recursion is trivial in TGD since coupling constants should be invariant under the addition of loops: the proposed scattering diagrammatics however assumed that scattering vertices reduce to scattering vertices for 2 fermions. The justification for renormalization group invariance would be number theoretical: there is no guarantee that infinite sum of diagrams gives simple function defined in all number fields with parameters in extension of rationals (say rational function).

3. The general form of the Grassmannian integrand in $G(k,n)$ can be deduced and follows from Yangian invariance meaning that one has conformal symmetries and their duals which expand to full infinite-dimensional Yangian symmetry. The denominator of the integrand of planar tree diagram is the product of determinants of $k \times k$ minors for the $k \times n$ matrix providing representation of a point of $G(k,n)$ unique apart from $SL(k,k)$ transformations. Only minors consisting of $k$ consecutive columns are assumed in the product. The residue integral is determined by the poles of the denominator. There are also dynamical singularities allowing the amplitude to be non-vanishing only for some special configurations of the external momenta.

4. On mass-shell diagrams obtained by fusing 3-vertices are highly redundant. One can describe the general diagram by using a disk such that its boundary contains the external particles with positive or negative helicity. The diagram has certain number $n_F$ of faces. There are moves, which do not affect the amplitude and it is possible to reduce the number of faces to minimal one: this gives what is called reduced diagram. Reduced diagrams with $n_F$ faces define a unique $n_F - 1$-dimensional sub-manifold of $G(k,n)$ over which the residue integral can be defined. Since the dimension of $G(k,n)$ is finite, also $n_F$ is finite so that the number of diagrams is finite.

5. On mass shell diagrams can be labelled by the permutations of the external lines. This gives a connection with 1+1-dimensional QFTs and with braid group. In 1+1-D integral QFTs however scattering matrix induces only particle exchanges.

   The permutation has simple geometric description: one starts from the boundary point of the diagram and moves always from left or right depending on the color of the point from which one started. One arrives some other point at the boundary and the final points are different for different starting points so that the process assigns a unique perturbation for a given diagram. Diagrams which are obtained by moves from each other define the same permutation. BFCW bridge which is a manner to obtain new Yangian invariant corresponds to a permutation of consecutive external particles in the diagram.

6. The poles of the denominator determine the value of the multiple residue integrals. If one allowed all minors, one would have extremely complex structure of singularities. The allowance only cyclically taken minors simplifies the situation dramatically. Singularities correspond to $n$ subgroups of more than 2 collinear k-vectors implying vanishing of some of the minors.

7. Algebraic geometry comes in rescue in the understanding of singularities. Since residue integral is in question, the choice is rather free and only the homology equivalence class of the cell decomposition matters. The poles for a hierarchy with poles inside poles since given singularity contains sub-singularities. This hierarchy gives rise to a what is known as cell composition - stratification - of Grassmannian consisting of varieties with various dimensions. These sub-varieties define representatives for the homology group of Grassmannian. Schubert cells already mentioned define this kind of stratification.

   **Remark**: The stratification has very strong analogy of the decomposition of catastrophe in Thom's catastrophe theory to pieces of various dimensions. The smaller the dimension, the higher the criticality involved. A connection with quantum criticality of TGD is therefore highly suggestive.

   Cyclicity implies a reduction of the stratification to that for positive Grassmannians for which the points are representable as $k \times n$ matrices with non-negative $k \times k$ determinants. This simplifies the situation even further.

   Yangian symmetries have a geometric interpretation as symmetries of the stratification: level 1 Yangian symmetries are diffeomorphisms preserving the cell decomposition.

### 7.4.2 Problems of twistor approach

Twistor approach is extremely beautiful and elegant but has some problems.

1. The notion of twistor structure is problematic in curved space-times. In TGD framework the twistor structures of $M^4$ and $CP_2$ ($E^4$) induce twistor structure of space-time surface and the problem disappears just like the problems related to classical conservation laws are circumvented. Complexification of octonions allows to solve the problems related to the metric signature in twistorialization.

2. The description of massive particles is a problem. In TGD framework $M^8$ approach allows to replace massive particles with particles with octonionic momenta light-like in 8-D sense belonging to quaternionic subspace for a given diagram. The situation reduces to that for ordinary twistors in this quaternionic sub-space but since quaternionic sub-space can vary, additional degrees of freedom bringing in $CP_2$ emerge and manifest themselves as transversal 8-D mass giving real mass in 4-D sense.

3. Non-planar diagrams are also a problem. In TGD framework a natural guess is that they correspond to various permutations of free particle octonionic polynomials. Their product defines interaction region in the interior of CD to which free particles satisfying associativity conditions (quantum criticality) arrive. If the origins of polynomials are not along same time axis, the polynomials do not commute nor associate. One must sum over their permutations and for each permutation over its associations.

## 7.5 About the concrete construction of twistor amplitudes

At $H$-side the ground states of super-conformal representations are given by the anti-symmetrized products of the modes of $H$-spinor fields labelled by four-momentum, color quantum numbers, and electroweak (ew) quantum numbers. At partonic 2-surface one has finite number of many fermion states. Single fermion states are assigned with $H$-spinor basis and the fermion states form a representation of a finite-D Clifford algebra.

$M^8$ picture should reproduce the physical equivalent of $H$ picture: in particular, one should understand four-momentum, color quantum numbers, ew quantum numbers, and $B$ and $L$. $M^8-H$ correspondence requires that the super-twistorial description of scattering amplitudes in $M^8$ is equivalent with that in $H$.

The $M^8$ picture is roughly following.

1. The ground states of super-conformal representations expressible in terms of spinor modes of $H$ correspond at level of $M^8$ wave functions in super variant of the product $T(M^4) \times T(CP_2)$ of twistor spaces of $M^4$ and $CP_2$. This twistor space emerges naturally in $M^8 - H$ correspondence from the quaternionicity condition for 8-momenta.

2. Bosonic $M^8$ degrees of freedom translate to wave functions in the product $T(M^4) \times T(CP_2)$ labelled by four-momentum and color. Super parts of the $M^4$ and $CP_2$ twistors code for spin and ew degrees of freedom and fermion numbers. Only a finite number of spin-ew spin states is possible for a given fundamental particle since one has finite-D Grassmann algebra.

3. Contrary to the earlier expectations [K32], the view about scattering diagrams is very similar to that in $\mathcal{N} = 4$ SUSY. The analog of 3-gluon vertex is fundamental and emerges naturally from number theoretic vision in which scattering diagrams defines a cognitive representation and vertices of the diagram correspond to fusion of sparticle lines.

### 7.5.1 Identification of $H$ quantum numbers in terms of $M^8$ quantum numbers

The first challenge is to understand how $M^8 - H$ correspondence maps $M^8$ quantum numbers to $H$ quantum numbers. At the level of $M^8$ one does not have action principle and conservation laws must follow from the properties of wave functions in various moduli spaces assignable to 4-D and 8-D CDs that is quaternion and octonion structures. The symmetries of the moduli spaces would dictate the properties of wave functions.

There are three types of symmetries and quantum numbers.

*1. WCW quantum numbers*

At level of $H$ the quantum numbers in WCW"vibrational"degrees of freedom are associated with the representations of super-symplectic group acting as isometries of WCW. Super-symplectic generators correspond to Hamiltonians labelled by color and angular momentum quantum numbers for $SU(3) \times SO(3)$. In $M^4_\pm$ there are also super-symplectic conformal weights assignable to the radial light-coordinate in $\delta M^4_\pm$. These conformal weights could be complex and might relate closely to the zeros of Riemann zeta [K23]. Physical states should however have integer valued conformal weights (conformal confinement).

At the level of $M^8$ WCW "vibrational"degrees of freedom are discrete and correspond to the degree of the octonionic polynomial $P$ and its coefficients in the extension of rationals considered. WCW integration reduces to a discrete sum, which should be well-defined by the criticality conditions on the coefficients of the polynomials. $M^8 - H$ correspondence guarantees that 4-varieties in $M^8$ are mappable to space-time surfaces in $H$. Therefore also quantum numbers should be mappable to each other.

There are also spinorial degrees of freedom associated with WCW spinors with spin-like quantum numbers assignable to fermionic oscillator operators labelled by spin, ew quantum numbers, fermion numbers, and by super-symplectic conformal weights.

*2. Quantum numbers assignable to isometries of $H$.*

These quantum numbers are special assignable to the ground states of the representations of Kac-Moody algebras associated with light-like partonic orbits.

1. The isometry group of $H$ consists of Poincare group and color group for $CP_2$. $M^8$ isometries correspond to $8-D$ Poincare group. Only $G_2$ respects given octonion structure and 8-D Lorentz transformations transform to each other different octonion structures. Quantum numbers consist of 8-momentum and analogs of spin and ew spin. $M^8 - H$ correspondence is non-trivial since one must map light-like quaternionic 8-momenta to 4-momenta and color quantum numbers.

2. There are quantum numbers assignable to cm spinor degrees of freedom. They correspond for both $M^8$ and $H$ to 8-D spinors and give rise to spin and ew quantum numbers. For these quantum numbers $M^8 - H$ correspondence is trivial. At the level of $H$ baryon and lepton numbers are assignable to the conserved chiralities of $H$-spinors.

Quantum classical correspondence (QCC) is a key piece of TGD.

1. At the level of $H$ QCC states that the eigenvalues of the fermionic Noether charges are equal to the classical bosonic Noether charges in Cartan algebra implies that fermionic quantum number as also ew quantum numbers and spin have correlates at the level of space-time geometry.

2. A the level of $M^8$ QCC is very concrete. Both bosonic and superpart of octonions have the decomposition $1+\overline{1}+3+\overline{3}$ under color rotations. Each monomial of theta parameters characterizes one particular many-fermion state containing leptons/antileptons and quarks/antiquarks. Leptons/antileptons are assignable to complexified octonionic units $(1 \pm iI_1)/\sqrt{2}$ defining preferred octonion plane $M_2$ and quarks/antiquarks are assignable to triplet and antitriplet, which also involve complexified octonion units. One obtains breaking of SUSY in the sense that space-time varieties assignable to different theta monomials are different (one can argue that the sum $8_s + \overline{8}_s$ can be regarded as real).

   Purely leptonic and antileptonic varieties correspond to 1 and $\overline{1}$ and quark and antiquark varieties to 3 and $\overline{3}$ and the monomial transforms as a tensor product of thetas. The monomial has well defined quark and lepton numbers and the interpretation is that it characterizes fundamental sparticle. At the level of $H$ this kind of correspondence follows form QCC.

3. Also super-momentum leads to a characterization of spin and fermion numbers of the state since delta function expressing conservation of super-momentum codes the supersymmetry for scattering amplitudes and gives rise to vertices conserving fermion numbers. Does this mean QCC in the sense that the super parts of super-momentum and super twistor should be associated with space-time varieties with same fermion and spin content?

*How the light-like quaternionic 8-momenta are mapped to $H$ quantum numbers?*

The key challenge is to understand how the light-like quaternionic 8-momenta are mapped to massive $M^4$ momenta and color quantum numbers.

1. One has wave function in the space of $CP_2$ quaternionic four-momenta. $M_0^4$ momentum can be identified as $M_0^2$ projection and in general massive unless $M_0^2$ and $M_0^4$ are chosen so that the light-like $M^8$ momentum belongs to $M_0^2$. The situation is analogous to that in the partonic description of hadron scattering.

   The space of quaternionic sub-spaces $M_0^4 \supset M_0^2$ with this property is parameterized by $CP_2$, and one obtains color partial waves. The inclusion of the choice of quantization axis extends this space to $T(CP_2) = SU(3)/U(1)\times U(1)$. Without quaternionicity/associativity condition the space of momenta would correspond to $M^8$.

   The wave functions in the moduli space for the position of the tip of CD and for the choice $M_0^2 \supset M_0^4$ specifying $M_0^4$ twistor structure and choice of quantization axis of spin correspond to wave functions in the twistor space $CP_3$ of $M_\pm^4$ coding for momentum and spin.

   **Remark**: The inclusion of $M^4$ spin quantization axis characterized by the choice of $M_0^2$ extends $M_0^4$ to geometric twistor space $T(M^4) = M_0^4 \times S^2 \supset M_0^2$ having bundle projection to $CP_3$. Twistorialization means essentially the inclusion of the choice of various quantization axis as degrees of freedom. This space is for symmetry group $G$ the space $G/H$, where $H$ is the Cartan sub-group of $G$. This description might make sense also at the level of super-symplectic and super-Kac-Moody symmetries.

2. Ordinary octonionic degrees of freedom for super-octonions in $M^8$ must be mapped to $M^4 \times CP_2$ cm degrees of freedom. Super octonionic parts should correspond to fermionic and spin and electroweak degrees of freedom. The space of super-twistorial states should same as the space of the super-symplectic grounds states describable in terms $H$-spinor modes.

3. One has wave function in the moduli space of CDs. The states in $M^8$ are labelled by quaternionic super-momenta. Bosonic part must correspond to four-momentum and color and super-part to spin and ew quantum numbers of $CP_2$. This part of the moduli space wave function is characterized by the spin and ew spin quantum numbers of the fundamental particle. Wave functions in the super counterpart of $T(M^4) \times T(CP_2)$ allow to characterize

these degrees of freedom without the introduction of spinors and should correspond to the ground states of super-conformal representations in $H$.

It seems that $H$-description is an abstract description at the level moduli spaces and $M^8$ description for single space-time variety represents reduction to the primary level, where number theory dictates the dynamics.

### 7.5.2 Octonionic twistors and super-twistors

How to define octonionic twistors? Or is it enough to identify quaternionic/associative twistors as sub-spaces of octonionic twistors?

*1. Ordinary twistors and super-twistors*

Consider first how ordinary twistors and their super counterparts could be defined, and how they could allow an elegant description of spin and ew quantum numbers as quantum numbers analogous to angular momenta.

1. Ordinary twistors are defined as pairs of 2-spinors giving rise to a representation of four-momentum. The spinors are complex spinors transforming as a doublet representation of SL(2,C) and its conjugate.

   The 2-spinors are related by incidence relation, a linear condition in which $M^4$ coordinates represented as $2 \times 2$ matrix appears linearly [K32]. The expression of four-momentum is bilinear in the spinors and invariant under complex scalings of the 2-spinors compensating each other so that instead of 8-D space one has actually 6-D space, which reduces to $CP_3$ to which the geometric twistor space $M^4 \times S^2$ has a projection.

2. For light-like four-momenta $p$ the determinant of the matrix having the two 2-spinors as rows and representing $p$ as a point of $M^4$ vanishes. Wave functions in $CP_3$ allow to describe spin in terms of bosonic wave function. What is so beautiful is that this puts particles with different spin in a democratic position.

   Super-twistors allow to integrate the states constructible as many-fermion states of $\mathcal{N}$ elementary fermions in the same representations involving several spins. The many-fermion states - sparticles - are in 1-1-correspondence with Grassmann algebra basis.

3. The description of massless particles in terms of $M^4$ (super-)twistors is elegant but one encounters problems in the case of massive particles [K28, K26, K32].

*2. Octonionic twistors at the level of $M^8$?*

How to define octonionic twistors at the level of $M^8$?

1. At the level of $M^8$ one has light-like 8-momenta. The $M^4$ momentum identified as $M_0^4$ projection can there be massive. This solves the basic problem of the standard twistor approach.

2. The additional assumption is that the 8-momenta in given vertex of scattering diagram belong to the same quaternionic sub-space $M_0^4 \subset M^8$ satisfying $M_0^4 \supset M_0^2$. This effectively transforms momentum space $M^4 \times E^4$ to $M^4 \times CP_2$. A stronger condition is that all momenta in a given diagram belong to the same sub-space $M_0^4 \supset M_0^2$.

   **Remark**: Quaternionicity implies that the 8-momentum is time-like or light-like if one requires that quaternionicity for an arbitrary choice of the octonionic structure (the action of 8-D Poincare group gives rise transforms octonionic structures to each other).

3. Complex 2-spinors are replaced with complexified octonionic spinors which must be consistent quaternionicity condition for 8-momenta. A good guess is that the spinors belong to a quaternionic sub-space of octonions too. This is expected to transform them effectively to quaternionic spinors. Without effective quaternionicity the number of 2-spinor components would be 8 rather than 4 times larger than for ordinary 2-spinors.

   **Remark:** One has complexified octonions ($i$ commutes with the octonionic imaginary units $E_k$).

4. Octonionic/quaternionic twistors should be pairs of octonionic/quaternionic 2-spinors determined only modulo octonionic/quaternionic scaling. If quaternionicity holds true, the number of 2-spinor components is 4 times larger than usually. Does this mean that one has basically quaternionic twistors plus moduli space $CP_2$ for $M_0^4 \supset M_0^2$. One should be able to express octonionic twistors as bi-linears formed from 2 octonionic/quaternionic 2-spinors. Octonionic option should give the octonionic counterpart $OP_3$ of Grassmannian $CP_3$, which does not however exist.

   **Remark**: Octonions allow only projective plane $OP_2$ as the octonionic counterpart of $CP_2$ (see `http://tinyurl.com/ybwaeu2s`) but do not allow higher-D projective spaces nor Grassmannians (see `http://tinyurl.com/ybm8ubef`, whereas reals, complex numbers, and quaternions do so. The non-existence of Grassmannians for rings obtained by Cayley-Dickson construction could mean that $M^8 - H$ correspondence and TGD do not generalize beyond octonions.

   Does the restriction to quaternionic 8-momenta the Grassmannians to be quaternionic (sub-spaces of octonions). This would give quaternionic counterpart $HP_3$ of $CP_3$. Quaternions indeed allow projective spaces and Grassmannians and (see `http://tinyurl.com/y9htjstc` and `http://tinyurl.com/y87gpq8l`).

   **Remark**: One can wonder whether non-commutativity forces to distinguish between left- and right Grassmannians (points as lines $\{c(q_1,..,q_n)|c \in H\}$ or as lines as lines $\{(q_1,..,q_n)c|c \in H\}$.

5. Concerning the generalization to octonionic case, it is crucial to realize that the $2\times 2$-matrix representing four-momentum as a pair 2-spinor can be regarded as an element in the sub-space of complexified quaternions. The representation of four-momentum would be as sum of $p_8 = p_1^k\sigma_k + I_4 p_2^k\sigma_k$, where $I_4$ octonionic imaginary unit orthogonal to $\sigma_k$ representing quaternionic units.

   No! The twistorial representation of the 4-momentum is already quaternionic! Choosing the decomposition of $M^8$ to quaternionic sub-space and its complement suitably, one has $IM(p_8) = 0$ for quaternionic 8-momenta and one obtains standard representation of 4-momentum in this sub-space! The only new element is that one has now moduli specifying the quaternionic sub-space. If the sub-space contains a fixed $M_0^2$ one obtains just $CP_2$ and ordinary twistor codes for the choices of $M_0^2$. If the choice of color quantization axes matters as it indeed does, one has twistor space $SU(3)/U(1)\times U(1)$ instead of $CP_2$. This would suggest that ordinary representation of scattering amplitudes reduces apart from the presence of $CP_2$ twistor to the usual representation.

   One can hope for a reduction to ordinary twistors and projective spaces, moduli space $CP_2$ for quaternion structures, and moduli space for the choices of real axis of octonion structures. One can even consider the possibility [K32] of using standard $M_0^2$ with the property that $M^8$ momentum reduces to $M_0^2$ momentum and coding the information about real $M_0^2$ to moduli. This could reduce the twistor space to $RP(3)$ associated with $M_0^2$ is considered and solve the problems related to the signature of $M^4$. Note however that the complexification of octonions in any case allow to regard the metric as Euclidian albeit complexified so that these problems should disappear.

*3. Octonionic super-twistors at the level of $M^8$?*

Should one generalize the notion of super-twistor to octonionic context or can one do by using only the moduli space and the fact that octonionic geometry codes for various components of octonion as analog of super-field? It seems that super-twistors are needed.

1. It seems that super-twistors are needed. Octonionic super-momentum would appear in the super variant of momentum conserving delta function resulting in the integration over translational moduli. In twistor Grassmann approach this delta function is super-twistorialized and this leads to the amazingly simple expressions for the scattering amplitudes.

2. At the level of $M^8$ one should generalize ordinary momentum to super-momentum and perform super-twistorialization. Different monomials of theta parameters emerging from super

part of momentum conserving delta function (for $\mathcal{N}=1$ one has $\delta(\theta-\theta_0)=exp(i\theta-\theta_0)/i)$ correspond to different spin states of the super multiplet and anti-commutativity guarantees correct statistics. At the level of $H$ the finite-D Clifford algebra of 8-spinors at fixed point of $H$ gives states obtained as monomials or polynomials for the components of super-momentum in $M^8$.

3. Octonionic super-momentum satisfying quaternionicity condition can be defined as a combination of ordinary octonionic 8-momentum and super-parts transforming like $8_s$ and $\overline{8}_s$. One can express the octonionic super-momentum as a bilinear of the super-spinors defining quaternionic super-twistor. Quaternionicity is assumed at least for the octonionic super-momenta in the same vertex. Hence the $M^4$ part of the super-twistorialization reduces to that in SUSYs and one obtains standard formulas. The new elements is the super-twistorialization of $T(CP_2)$.

   **Remark**: Octonionic SUSY involving $8+8_s+\overline{8}_s$ would be an analog of $\mathcal{N}=8$ SUSY associated with maximal supergravity (see `http://tinyurl.com/nv3aajy`) and in $M^4$ degrees of freedom twistorialization should be straightforward.

   The octonionic super-momentum belongs to a quaternionic sub-space labelled by $CP_2$ point and corresponds to a particular sub-space $M^2_0$ in which it is light-like (has no other octonionic components). $M^2_0$ is characterized by point of $S^2$ point of twistor space $M^4\times S^2$ having bundle projection to $CP_3$.

4. That the twistor space $T(CP_2)=SU(3)/U(1)\times U(1)$ coding for the color quantization axes rather than only $CP_2$ emerges must relate to the presence of electroweak quantum numbers related to the super part of octonionic momentum. Why the rotations of $SU(2)\times U(1)\subset SU(3)$ have indeed interpretation also as tangent space-rotations interpreted as electroweak rotations. The transformations having an effect on the choice of quantization axies are parameterized by $S^2$ relating naturally to the choice of $SO(4)$ quantization axis in $E^4$ and coded by the geometric twistor space $T(E^4)=E^4\times S^2$.

5. Since the super-structure is very closely related to the construction of the exterior algebra in the tangent space, super-twistorialization of $T(CP_2)$ should be possible. Octonionic triality could be also in a key role and octonionic structure in the tangent space of $SU(3)$ is highly suggestive. $SU(3)$ triality could relate to the octonionic triality.

   $SU(3)/U(1)\times U(1)$ is analogous with the ordinary twistor space $CP_3$ obtained from $C^4$ as a projective space. Now however $U(1)\times U(1)$ instead of group of complex scalings would define the equivalence classes. Generalization of projective space would be in question. The super-part of twistor would be obtained as $U(1)\times U(1)$ equivalence class and gauge choice should be possible to get manifestly 6-D representation. One can ask whether the $CP_2$ counterparts of higher- D Grassmannians appear at the level of generalized twistor diagrams: could the spaces $SU(n)/G$, $H$ Cartan group correspond to these spaces?

*4. How the wave functions in super-counterpart of $T(CP_2)$ correspond to quantum states in $CP_2$ degrees of freedom?*

In $CP_2$ spinor partial waves have vanishing triality $t=0$ for leptonic chirality and $t=\pm 1$ for quarks and antiquarks. One can say that the triality $t\neq 1$ states are possible thanks to the anomalous hypercharge equal to fractional electromagnetic charge $Y_A=Q_{em}$ of quarks: this gives also correlation between color quantum numbers and electroweak quantum numbers which is wrong for spinor partial waves. The super-symplectic and super Kac-Moody algebras however bring in vibrationals degrees of freedom and one obtains correct quantum number assignments [K9].

This mechanism should have a counterpart at the level of the super variant of the twistor space $T(CP_2)=SU(3)/U(1)\times U(1)$. The group algebra of $SU(3)$ gives the scalar wave functions for all irreps of $SU(3)$ as matrix elements. Allowing only matrix elements that are left- or right invariant under $U(1)\times U(1)$ one obtains all irreps realized in $T(CP_2)$ as scalar wave functions. These representations have $t=0$. The situation would be analogous for scalar functions in $CP_2$. One must however obtain also electroweak quantum numbers and $t\neq 0$ colored states. Here the octonionic algebraic geometry and superpart of the $T(CP_2)$ should come in rescue. The electroweak degrees of freedom in $CP_2$ should correspond to the super-parts of twistors.

The $SU(3)$ triplets assignable to the triplets 3 and $\overline{3}$ of space-time surfaces would make possible also the $t = \pm 1$ states. Color would be associated with the octonionic geometry. The simplest possibility would be that one has just tensor products of the triplets with $SU(3)/U(1) \times U(1)$ partial waves. In the case of $CP_2$ there is however a correlation between color partial waves and electroweak quantum numbers and the same is expected also now between super-part of the twistor and geometric color wave function: minimum correlation is via $Y_A = Q_{em}$. The minimal option is that the number theoretic color for the octonionic variety modifies the transformation properties of $T(CP_2)$ wave function only by a phase factor due to $Y_A = Q_{em}$ as in the case of $CP_2$.

The most elegant outcome would be that super-twistorial state basis in $T(M^4) times T(CP_2)$ is equivalent with the state basis defined by super-symplectic and super Kac-Moody representations in $H$.

### 7.5.3 About the analogs of twistor diagrams

There seems to be a strong analogy with the construction of twistor amplitudes in $\mathcal{N} = 4$ SUSY [B1, B4, B3] and one can hope of obtaining a purely geometric analog of SUSY with dynamics of fields replaced by the dynamics of algebraic super-octonionic surfaces.

1. Number theoretical vision leads to the proposal that the scattering amplitudes involve only data at discrete points of the space-time variety belonging to extension of rationals defining cognitive representation. The identification of these points has been already considered in the case of partonic orbits entering to the partonic 2-vertex and for the regions of space-time surfaces intersecting at discrete set of points. Scattering diagrams should therefore correspond to polygons with vertices of polygons defining cognitive representation and lines assignable to the external fundamental particles with given quark and lepton numbers having correlates at the level of space-time geometry. This occurs also in twistor Grassmannian approach [B1, B4, B3].

   Since polynomials determine space-time surfaces, this data is enough to determine the space-time variety completely. Indeed, the zeros of $P(t)$ determining the space-time variety give also rise to a set of spheres $S^6(t_n)$ and partonic 2-surfaces $X^2(t_n) = X^4 \cap S^6(t_n)$, where $t_n$ is root of $P(t)$. The discretization need not mean a loss of information. The scattering amplitudes would be expressible as an analog of $n$-point function with points having coordinates in the extension of rationals.

2. (Super) octonion as"field"in $X^4$ is dynamically analogous to (super) gauge potentials and super-octonion to its super variant. (Super) gauge potentials are replaced with $M^8$ (super-) octonion coordinate and gauge interactions are geometrized. Here I encounter a problem with terminology. Neither sparticle nor sboson sounds good. Hence I will talk about sparticles.

3. The amplitude for a given space-time variety contains no information $M^8$-momentum. $M^8$-momentum emerges as a label for a wave function in the moduli space of 4-D and 8-D CDs involving both translational and orientational degrees of freedom. For fixed time axis the orientational degrees of freedom reduce to rotational degrees of freedom identifiable in terms of the twistor sphere $S^2$. The delta functions expressing conservation of 8-D quaternionic super-momentum in $M^8$ coming from the integration over the moduli space of 8-D translations.

   As found, quaternionicity of 8-momenta implies that standard $M^4$ twistor description of momenta applies but one obtains $CP_2$ twistors as additional contribution. This is of course what one would intuitively expect.

   8-D momentum conservation in turn translates to the conservation of momentum and color quantum numbers in the manner described. The amplitudes in momentum and color degrees of freedom reduce to kinematics as in SUSYs. It is however not clear whether one should also perform number theoretical discretization of various moduli spaces.

   In any case, it seems that all the details of the scattering amplitudes related to moduli spaces reduce to symmetries and the core of calculations reduces to the construction of space-time varieties as zero loci of octonionic polynomials and identification of the points of the 4-varieties in extension of rationals. Classical theory would indeed be an exact part of the quantum theory.

4. Quaternionic 8-D light-likeness reduces the situation to the level of ordinary complex and thus even positive (real) Grassmannians. This is crucial from the p-adic point of view. $CP_2$ twistors characterizes the moduli related to the choice of quaternionic sub-space, where 8-momentum reduces to ordinary 4-momentum. $M^4$ parts of the scattering amplitudes in twistor Grassmann approach should be essentially the same as in $\mathcal{N}=4$ SUSY apart from the replacement of super degrees of freedom with super-octonionic ones. The challenge is to generalize the formalism so that it applies also to $CP_2$ twistors. The challenge would be to generalize the formalism so that it applies also to $CP_2$ twistors. The $M^4$ and $CP_2$ degrees of freedom are expected to factorize in twistorial amplitudes. A good guess is that the scattering amplitudes are obtained as residue integrals in the analogs of Grassmannians associated with $T(CP_2)$. Could one have Grassmannians also now?

   Consider the formula of tree amplitude for $n$ gluons with $k$ negative helicities conjectured Arkani-Hamed et al in the twistor Grassmannian approach [B3]. The amplitude follows from the twistorial representation for momentum conservation and is equal to an $k \times n$-fold multiple residue integral over the complex variables $C_{\alpha a}$ defining coordinates for Grassmannian $Gl(n,k)$ and reduces to a sum over residues. The integrand is the inverse for the product of all $k \times k$ minors of the matrix $C_{\alpha a}$ in cyclic order and the resides corresponds to zeros for one or more minors. This part does not depend on twistor variables. The dependence on $n$ twistor variables comes from the product $\prod_{\alpha=1}^{k} \delta(C_{\alpha a} W^a)$ of $k$ delta functions related to momentum conservation. $W^a$ denotes super-twistors in the 8-D representation, which is linear. One has projective invariance and therefore a reduction to $T(M^4) = CP_3 = SU(4)/SU(3) \times U(1)$.

   Could this formula generalize almost as such to $T(CP_2)$ and come from the conservation of $E^4$ momentum? One has $n$ sparticles to which super-twistors in $T(CP_2)$ are assigned. The first guess is that the sign of helicity are replaced by the sign of electroweak isospin - essentially $E^4$ spin at the level of $M^8$. For electromagnetic charge identified as the analog of helicity one would have problems in the case of neutrinos. $T(M^4) = CP_3 = SU(4)/SU(3) \times U(1)$ is replaced with $T(CP_2) = SU(3)/U(1) \times U(1)$. $T(CP_2)$ does not have a representation as a projective space but there is a close analogy since the group of complex scalings is replaced with $U(1) \times U(1)$. The (apparent) linearity is lost but one represent the points of $T(CP_2)$ as exponentials of $su(3)$ Lie-algebra elements with vanishing $u(1) \times u(1)$ part. The resulting 3 complex coordinates are analogous to two complex $CP_2$ coordinates. The basic difference between $M^4$ and $CP_2$ degrees of freedom would come from the exponential representation of twistors.

5. By Yangian invariance one should obtain very similar formulas for the amplitudes except that one has instead of $\mathcal{N}=4$ SUSY $\mathcal{N}=8$ octonionic SUSY analogous to $\mathcal{N}=8$ SUGRA.

### 7.5.4 Trying to understand the fundamental 3-vertex

Due to its unique twistorial properties as far as realization of four-momentum conservation is considered 3-vertex is fundamental in the construction of scattering diagrams in twistor Grassmannian approach to $\mathcal{N}=4$ SYM [B2] (see `http://tinyurl.com/yd9tf2ya`). Twistor Grassmann approach suggests that 3-vertex with complexified light-like 8-momenta represents the basic building brick representing from which more complex diagrams can be constructed using the BCFW recursion formula [B2]. In TGD 3-vertex generalized to 8-D light-like quaternionic momenta should be highly analogous to the 4-D 3-vertex and in a well-defined sense reduce to it if all momenta of the diagram belong to the same quaternionic sub-space $M^4_0$. It is however not completely clear how 3-vertex emerges in TGD framework.

1. A possible identification of the 3-vertex at the level of $M^8$ would be as a vertex at which 3 sparticle lines with light-like complexified quaternionic 8-momenta meet. This vertex would be associated with the partonic vertex $X^2(t_n) = X^4 \cap S^6(t_n)$. Incoming sparticle lines at the light-like partonic orbits identified as boundaries of string world sheets (for entangled states at least) would be light-like.

   Does the fusion of two sparticle lines to third one require that either or both fusing lines become space-like - say pieces of geodesic line inside the Euclidian space-time region- bounded by the partonic orbit? The identification of the lines of twistor diagrams as carriers of light-like complexified quaternionic momenta in 8-D sense does not encourage this interpretation (also classical momenta are complex). Should one pose the fusion of the light-like lines as a boundary condition? Or should one give up the idea that sparticle lines make sense inside interaction region?

2. As found, one can challenge the assumption about the existence of string world sheets as commutative regions in the non-associative interaction region. Could one have just fermion lines as light-like curves at partonic orbits inside CD? Or cannot one have even them?

   Even if the polynomial $\prod_i P_i$ defining the interaction region is product of polynomials with origins of octonionic coordinates not along the same real line, the 7-D light-cones of $M^8$ associated with the particles still make sense in the sense that $P_i(o_i) = 0$ reduces at it to $P_i(t_i) = 0$, $t_i$ real number, giving spheres $S^6(t_i(n))$ and partonic 2-surfaces and vertices $X_2(t_i(n))$. The light-like curves as geodesics the boundary of 7-D light-cones mapped to light-like curves along partonic orbits in $H$ would not be lost inside interaction regions.

3. At the level of $H$ this relates to a long standing interpretational problem related to the notion of induced spinor fields. SH suggests strongly the localization of the induced spinor fields at string world sheets and even at sparton lines in absence of entanglement. Super-conformal symmetry however requires that induced spinor fields are 4-D and thus seems to favor de-localization. The information theoretic interpretation is that the induced spinor fields at string world sheets or even at sparton lines contain all information needed to construct the scattering amplitudes. One can also say that string world sheets and sparton lines correspond to a description in terms of an effective action.

#### 7.5.5 Could the $M^8$ view about twistorial scattering amplitudes be consistent with the earlier $H$ picture?

The proposed $M^8$ picture involving super coordinates of $M^8$ and super-twistors does not conform with the earlier proposal for the construction of scattering amplitudes at the level of $H$ [K32]. In $H$ picture the introduction of super-space does not look natural, and one can say that fundamental fermions are the only fundamental particles [K26, K32]. The $H$ view about super-symmetry is as broken supersymmetry in which many fermion states at partonic 2-surfaces give rise to supermultiplets such that fermions are at different points. Fermion 4-vertex would be the fundamental vertex and involve classical scattering without fusion of fermion lines. Only a redistribution of fermion and anti-fermion lines among the orbits of partonic 2-surfaces would take place in scattering and one would have kind of OZI rule.

Could this $H$ view conform with the recent $M^8$ view much closer to the SUSY picture. The intuitive idea without a rigorous justification has been that the fermion lines at partonic 2-surfaces correspond to singularities of many-sheeted space-time surface at which some sheets co-incide. $M^8$ sparticle consists effectively of $n$ fermions at the same point in $M^8$. Could it be mapped by $M^8-H$ duality to $n$ fermions at distinct locations of partonic 2-surface in $H$?

$M^8-H$ correspondence maps the points of $M^4 \subset M^4 \times E^4$ to points of $M^4 \subset M^4 \times CP_2$. The tangent plane of space-time surface containing a preferred $M^2$ is mapped to a point of $CP_2$. If the effective $n$-fermion state $M^8$ is at point at which $n$ sheets of space-time surface co-incide and if the tangent spaces of different sheets are not identical, which is quite possible and even plausible, the point is indeed mapped to $n$ points of $H$ with same $M^4$ coordinates but different $CP_2$ coordinates and sparticle would be mapped to a genuine many-fermion state. But what happens to scalar sparticle. Should one regard it as a pure gauge degree of freedom in accordance with the chiral symmetry at the level of $M^8$ and $H$?

# 8 From amplituhedron to associahedron

Lubos has a nice blog posting (see `http://tinyurl.com/y7ywhxew`) explaining the proposal represented in the newest article by Nima Arkani-Hamed, Yuntao Bai, Song He, Gongwang Yan [**?**]see `http://tinyurl.com/ya8zstll`). Amplituhedron is generalized to a purely combinatorial notion of associahedron and shown to make sense also in string theory context (particular bracketing).

The hope is that the generalization of amplituhedron to associahedron allows to compute also the contributions of non-planar diagrams to the scattering amplitudes - at least in $\mathcal{N} = 4$ SYM. Also the proposal is made that color corresponds to something less trivial than Chan-Paton factors.

The remaining problem is that 4-D conformal invariance requires massless particles and TGD allows to overcome this problem by using a generalization of the notion of twistor: masslessness is realized in 8-D sense and particles massless in 8-D sense can be massive in 4-D sense.

In TGD non-associativity at the level of arguments of scattering amplitude corresponds to that for octonions: one can assign to space-time surfaces octonionic polynomials and induce arithmetic operations for space-time surface from those for polymials (or even rational or analytic functions). I have already earlier [L11] demonstrated that associahedron and construction of scattering amplitudes by summing over different permutations and associations of external particles (space-time surfaces). Therefore the notion of associahedron makes sense also in TGD framework and summation reduces to "integration" over the faces of associahedron. TGD thus provides a concrete interpretation for the associations and permutations at the level of space-time geometry.

In TGD framework the description of color and four-momentum is unified at the level and the notion of twistor generalizes: one has twistors in 8-D space-time instead of twistors in 4-D space-time so Chan-Paton factors are replaced with something non-trivial.

## 8.1 Associahedrons and scattering amplitudes

The following describes briefly the basic idea between associahedrons.

### 8.1.1 Permutations and associations

One starts from a non-commutative and non-associative algebra with product (in TGD framework this algebra is formed by octonionic polynomials with real coefficients defining space-time surfaces as the zero loci of their real or imaginary parts in quaternionic sense. One can indeed multiply space-time surface by multiplying corresponding polynomials! Also sum is possible. If one allows rational functions also division becomes possible.

All permutations of the product of n elements are in principle different. This is due to non-commutativity. All associations for a given ordering obtained by scattering bracket pairs in the product are also different in general. In the simplest case one has either $a(bc)$ or $(ab)c$ and these 2 give different outcomes. These primitive associations are building bricks of general associations: for instance, abc does not have well-defined meaning in non-associative case.

If the product contains $n$ factors, one can proceed recursively to build all associations allowed by it. Decompose the n factors to groups of $m$ and $n-m$ factors. Continue by decomposing these two groups to two groups and repeat until you have have groups consisting of 1 or two elements. You get a large number of associations and you can write a computer code computing recursively the number $N(n)$ of associations for $n$ letters.

Two examples help to understand. For $n = 3$ letters one obviously has $N(n = 3) = 2$. For $n = 4$ one has $N(4) = 5$: decompose first $abcd$ to $(abc)d$, $a(bcd)$ and $(ab)(cd)$ and then the two 3 letter groups to two groups: this gives $N(4) = 2 + 2 + 1 = 5$associations and associahedron in 3-D space has therefore 5 faces.

### 8.1.2 Geometric representation of association as face of associahedron

Associations of $n$ letters can be represented geometrically as so called Stasheff polytope (see `http://tinyurl.com/q9ga785`). The idea is that each association of $n$ letters corresponds to a face of polytope in $n-2$-dimensional space with faces represented by the associations.

Associahedron is constructed by using the condition that adjacent faces (now 2-D polygons) intersecting along common face (now 1-D edges). The number of edges of the face codes for the structure particular association. Neighboring faces are obtained by doing minimal change which means replacement of some $(ab)c$ with $a(bc)$ appearing in the association as a building bricks or vice versa. This means that the changes are carried out at the root level.

### 8.1.3 How does this relate to particle physics?

In scattering amplitude letters correspond to external particles. Scattering amplitude must be invariant under permutations and associations of the external particles. In particular, this means that one sums over all associations by assigning an amplitude to each association. Geometrically this means that one "integrates" over the boundary of associahedron by assigning to each face an amplitude. This leads to the notion of associahedron generalizing that of amplituhedron.

Personally I find it difficult to believe that the mere combinatorial structure leading to associahedron would fix the theory completely. It is however clear that it poses very strong conditions on the structure of scattering amplitudes. Especially so if the scattering amplitudes are defined in terms of "volumes" of the polyhedrons involved so that the scattering amplitude has singularities at the faces of associahedron.

An important constraint on the scattering amplitudes is the realization of the Yangian generalization of conformal symmetries of Minkowski space. The representation of the scattering amplitudes utilizing moduli spaces (projective spaces of various dimensions) and associahedron indeed allows Yangian symmetries as diffeomorphisms of associahedron respecting the positivity constraint. The hope is that the generalization of amplituhedron to associahedron allows to generalize the construction of scattering amplitudes to include also the contribution of non-planar diagrams of at $\mathcal{N} = 4$ SYM in QFT framework.

## 8.2 Associations and permutations in TGD framework

Also in the number theoretical vision about quantum TGD one encounters associativity constraings leading to the notion of associahedron. This is closely related to the generalization of twistor approach to TGD forcing to introduce 8-D analogs of twistors [L11] (see `http://tinyurl.com/yd43o2n2`).

### 8.2.1 Non-associativity is induced by octonic non-associativity

As found in [L11], non-associativity at the level of space-time geometry and at the level of scattering amplitudes is induced from octonionic non-associativity in $M^8$.

1. By $M^8-H$ duality ($H = M^4 \times CP_2$) the scattering are assignable to complexified 4-surfaces in complexified $M^8$. Complexified $M^8$ is obtained by adding imaginary unit $i$ commutating with octonionic units $I_k$, $k = 1,,..,7$. Real space-time surfaces are obtained as restrictions to a Minkowskian subspace complexified $M^8$ in which the complexified metric reduces to real valued 8-D Minkowski metric. This allows to define notions like Kähler structure in Minkowskian signature and the notion of Wick rotations ceases to be ad hoc concept. Without complexification one does not obtain algebraic geometry allowing to reduces the dynamics defined by partial differential equations for preferred extremals in $H$ to purely algebraic conditions in $M^8$. This means huge simplications but the simplicity is lost at the QFT-GRT limit when many-sheeted space-time is replaced with slightly curved piece of $M^4$.

2. The real 4-surface is determined by a vanishing condition for the real or imaginary part of octonionic polynomial with $RE(P)$ and $IM(P)$ defined by the composition of octonion to two quaternions: $o = RE(o) + I_4 IM(o)$, where $I_4$ is octonionic unit orthogonal to a quaternionic sub-space and $RE(o)$ and $IM(o)$ are quaternions. The coefficients of the polynomials are assumed to be real. The products of octonionic polynomials are also octonionic polynomials (this holds for also for general power series with real coefficients (no dependence on $I_k$. The product is not however neither commutative nor associative without additional conditions. Permutations and their associations define different space-time surfaces. The exchange of particles changes space-time surface. Even associations do it. Both non-commutativity and non-associativity have a geometric meaning at the level of space-time geometry!

3. For space-time surfaces representing external particles associativity is assumed to hold true: this in fact guarantees $M^8-H$ correspondence for them! For interaction regions associativity does not hold true but the field equations and preferred extremal property allow to construct the counterpart of space-time surface in $H$ from the boundary data at the boundaries of $CD$ fixing the ends of space-time surface.

Associativity poses quantization conditions on the coefficients of the polynomial determining it. The conditions are interpreted in terms of quantum criticality. In the interaction region identified naturally as causal diamond (CD), associativity does not hold true. For instance, if external particles as space-time surfaces correspond to vanishing of $RE(P_i)$ for polynomials representing particles labelled by $i$, the interaction region (CD) could correspond to the vanishing of $IM(P_i)$ and associativity would fail. At the level of $H$ associativity and criticality corresponds to minimal surface property so that quantum criticality corresponds to universal free particle dynamics having no dependence on coupling constants.

4. Scattering amplitudes must be commutative and associative with respect to their arguments which are now external particles represented by polynomials $P_i$ This requires that scattering amplitude is sum over amplitudes assignable to 4-surfaces obtained by allowing all permutations and all associations of a given permutation. Associations can be described combinatorially by the associahedron!

   **Remark:**. In quantum theory associative statistics allowing associations to be represented by phase factors can be considered (this would be associative analog of Fermi statistics). Even a generalization of braid statistics can be considered.

Yangian variants of various symmetries are a central piece also in TGD although supersymmetries are realized in different manner and generalized to super-conformal symmetries: these include generalization of super-conformal symmetries by replacing 2-D surfaces with light-like 3-surfaces, supersymplectic symmetries and dynamical Kac-Moody symmetries serving as remnants of these symmetries after supersymplectic gauge conditions characterizing preferred extremals are applied, and Kac-Moody symmetries associated with the isometries of $H$ . The representation of Yangian symmetries as diffeomorphisms of the associahedron respecting positivity constraint encourages to think that associahedron is a useful auxiliary tool also in TGD.

### 8.2.2 Is color something more than Chan-Paton factors?

Nima et al talk also about color structure of the scattering amplitudes usually regarded as trivial. It is claimed that this is actually not the case and that there is non-trivial dynamics involved. This is indeed the case in TGD framework. Also color quantum numbers are twistorialized in terms of the twistor space of $CP_2$, and one performs a twistorialization at the level of $M^8$ and $M^4 \times CP_2$. At the level of $M^8$ momenta and color quantum numbers correspond to associative 8-momenta. Massless particles are now massless in 8-D sense but can be massive in 4-D sense. This solves one of the basic difficulty of the ordinary twistor approach. A further bonus is that the choice of the imbedding space H becomes unique: only the twistor spaces of $S^4$ (and generalized twistor space of $M^4$ and $CP_2$ have Kähler structure playing a crucial role in the twistorialization of TGD. To sum up, all roads lead to Rome. Everyone is well-come to Rome!

## 8.3 Questions inspired by quantum associations

Associations have (or seem to have) different meaning depending on whether one is talking about cognition or mathematics. In mathematics the associations correspond to different bracketings of mathematical expressions involving symbols denoting mathematical objects and operations between them. The meaning of the expression - in the case that it has meaning - depends on the bracketing of the expression. For instance, one has $a(b+c) \neq (ab)+c$ , that is $ab+ac \neq ab+c$). Note that one can change the order of bracket and operation but not that of bracket and object.

For ordinary product and sum of real numbers one has associativity: $a(bc) = (ab)c$ and $a+(b+c) = (a+b)+c$. Most algebraic operations such as group product are associative. Associativity of product holds true for reals, complex numbers, and quaternions but not for octonions and this would be fundamental in both classical and quantum TGD.

The building of different associations means different groupings of $n$ objects. This can be done recursively. Divide first the objects to two groups, divide these tow groups to two groups each, and continue until you jave division of 3 objects to two groups - that is $abc$ divided into $(ab)c$ or $a(bc)$. Numbers 3 and 2 are clearly the magic numbers.

This inspire several speculative quetions related to the twistorial construction of scattering amplitudes as associative singlets, the general structure of quantum entanglement, quantum measurement cascade as formation of association, the associative structure of many-sheeted space-time as a kind of linguistic structure, spin glass as a strongly associative system, and even the tendency of social structures to form associations leading from a fully democratic paradise to cliques of cliques of ... .

1. In standard twistor approach 3-gluon amplitude is the fundamental building brick of twistor amplitudes constructed from on-shell-amplitudes with complex momenta recursively. Also in TGD proposal this holds true. This would naturally follow from the fact that associations can be reduced recursively to those of 3 objects. 2- and 3-vertex would correspond to a fundamental associations. The association defined 2-particle pairing (both associated particles having either positive or negative helicities for twistor amplitudes) and 3-vertex would have universal structure although the states would be in general decompose to associations.

2. Consider first the space-time picture about scattering [L11]. CD defines interaction region for scattering amplitudes. External particles entering or leaving CD correspond to associative space-time surfaces in the sense that the tangent space or normal space for these space-time surfaces is associative. This gives rise to $M^8 - H$ correspondence.

   These surfaces correspond to zero loci for the imaginary parts (in quaternionic sense) for octonionic polynomial with coefficients, which are real in octonionic sense. The product of $\prod_i P_i)$ of polynomials with same octonion structure satisfying $IM(P_i) = 0$ has also vanishing imaginary part and space-time surface corresponds to a disjoint union of surfaces associated with factors so that these states can be said to be non-interacting.

   Neither the choice of quaternion structure nor the choice of the direction of time axis assignable to the octonionic real unit need be same for external particles: if it is the particles correspond to same external particle. This requires that one treats the space of external particles (4-surfaces) as a Cartesian product of of single particle 4-surfaces as in ordinary scattering theory.

   Space-time surfaces inside CD are non-associative in the sense that the neither normal nor tangent space is associative: $M^8 - M^4 \times CP_2$ correspondence fails and space-time surfaces inside CD must be constructed by applying boundary conditions defining preferred extremals. Now the real part of $RE(\prod_i P_i)$ in quaternionic sense vanishes: there is genuine interaction even when the incoming particles correspond to the same octonion structure since one does not have union of surfaces with vanishing $RE(P_i)$. This follows from s rather trivial observation holding true already for complex numbers: imaginary part of $zw$ vanishes if it vanishes for $z$ and $w$ but this does not hold true for the real part. If octonionic structures are different, the interaction is present irrespective of whether one assumes $RE(\prod_i P_i) = 0$ or $IM(\prod_i P_i) = 0$. $RE(\prod_i P_i) = 0$ is favoured since for $IM(\prod_i P_i) = 0$ one would obtain solutions for which $IM(P_i) = 0$ would vanish for the $i$:th particle: the scattering dynamics would select $i$:th particle as non-interacting one.

3. The proposal is that the entire scattering amplitude defined by the zero energy state - is associative, perhaps in the projective sense meaning that the amplitudes related to different associations relate by a phase factor (recall that complexified octonions are considered), which could be even octonionic. This would be achieved by summing over all possible associations.

4. Quantum classical correspondence (QCC) suggests that in ZEO the zero energy states - that is scattering amplitudes determined by the classically non-associative dynamics inside CD - form a representation for the non-associative product of space-time surfaces defined by the condition $RE(\prod_i P_i) = 0$. Could the scattering amplitude be constructed from products of octonion valued single particle amplitudes. This kind of condition would pose strong constraints on the theory. Could the scattering amplitudes associated with different associations be octonionic - may be differing by octonion-valued phase factors - and could only their sum be real in octonionic sense (recall that complexified octonions involving imaginary unit $i$ commuting with the octonionic imaginary units are considered)?

One can look the situation also from the point of view of positive and negative energy states defining zero energy states as they pairs.

1. The formation of association as subset is like formation of bound state of bound states of ... . Could each external line of zero energy state have the structure of association? Could also the internal entanglement associated with a given external line be characterized in terms of association.

   Could the so called monogamy theorem stating that only two-particle entanglement can be maximal correspond to the decomposing of $n = 3$ association to one- and two-particle associations? If quantum entanglement is behind associations in cognitive sense, the cognitive meaning of association could reduce to its mathematical meaning.

   An interesting question relates to the notion of identical particle: are the many-particle states of identical particles invariant under associations or do they transform by phase factor under association. Does a generalization of braid statistics make sense?

2. In ZEO based quantum measurement theory the cascade of quantum measurements proceeds from long to short scales and at each step decomposes a given system to two subsystems. The cascade stops when the reduction of entanglement is impossible: this is the case if the entanglement probabilities belong to an extension of extension of rationals characterizing the extension in question. This cascade is nothing but a formation of an association! Since only the state at the second boundary of CD changes, the natural interpretation is that state function reduction mean a selection of association in 3-D sense.

3. The division of $n$ objects to groups has also social meaning: all social groups tend to divide into cliques spoiling the dream about full democracy. Only a group with 2 members - Romeo and Julia or Adam and Eve - can be a full democracy in practice. Already in a group of 3 members 2 members tend to form a clique leaving the third member outside. Jules and Catherine, Jim and Catherine, or maybe Jules and Jim! Only a paradise allows a full democracy in which non-associativity holds true. In ZEO it would be realized only at the quantum critical external lines of scattering diagram and quantum criticality means instability. Quantum superposition of all associations could realize this democracy in 4-D sense.

A further perspective is provided by many-sheeted space-time providing classical correlate for quantum dynamics.

1. Many-sheeted space-time means that physical states have a hierarchical structure - just like associations do. Could the formation of association (AB) correspond basically to a formation of flux tube bond between A and B to give AB and serve as space-time correlate for (negentropic) entanglement. Could ((AB)C) would correspond to (AB) and (C) "topologically condensed" to a larger surface. If so, the hierarchical structure of many-sheeted space-time would represent associations and also the basic structures of language.

2. Spin glass (see `http://tinyurl.com/y9yyq8ga`) is a system characterized by so called frustrations. Spin glass as a thermodynamical system has a very large number of minima of free energy and one has fractal energy landscape with valleys inside valleys. Typically there is a competition between different pairings (associations) of the basic building bricks of the system.

   Could spin glass be describable in terms of associations? The modelling of spin glass leads to the introduction of ultrametric topology characterizing the natural distance function for the free energy landscape. Interestingly, p-adic topologies are ultrametric. In TGD framework I have considered the possibility that space-time is like 4-D spin glass: this idea was originally inspired by the huge vacuum degeneracy of Kähler action. The twistor lift of TGD breaks this degeneracy but 4-D spin glass idea could still be relevant.

# 9 Gromov-Witten invariants, Riemann-Roch theorem, and Atyiah-Singer index theorem from TGD point of view

Gromov-Witten (G-W) invariants, Riemann-Roch theorem (RR), and Atyiah-Singer index theorem (AS) are applied in advanced algebraic geometry, and it is interesting to see whether they could have counterparts in TGD framework. The basic difference between TGD and conventional algebraic geometry is due to the adelic hierarchy demanding that the coefficients of polynomials involved are in given extension of rationals. Continuous moduli spaces are replaced with discrete ones by number theoretical quantization due to the criticality guaranteeing associativity of tangent or normal space. $M^8 - H$ duality brings in powerful consistency conditions: counting of allowed combinations of coefficients of polynomials on $M^8$ side and counting of dimensions on $H$ side using AS should give same results. $M^8 - H$ duality might be in fact analogous to the mirror symmetry of M-theory.

## 9.1 About the analogs of Gromow-Witten invariants and branes in TGD

Gromow-Witten invariants, whose definition was discussed in [L12], play a central role in superstring theories and M-theory and are closely related to branes. For instance, partition functions can be expressed in terms of these invariants giving additional invariants of symplectic and algebraic geometries. Hence it is interesting to look whether they could be important also in TGD framework.

1. As such the definition of G-W invariants discussed in [L12] do not make sense in TGD framework. For instance, space-time surface is not a closed symplectic manifold whereas $M^8$ and $H$ are analogs of symplectic spaces. Minkowskian regions of space-time surface have Hamilton-Jacobi structure at the level of both $M^8$ and $H$ and this might replace the symplectic structure. Space-time surfaces are not closed manifolds.

   Physical intuition however suggests that the generalization exists. The fact that Minkowskian metric and Euclidian metric for complexified octonions are obtained in various sectors for which complex valued length squared is real suggests that signature is not a problem. Kähler form for complexified $z$ gives as special case analog of Kähler form for $E^4$ and $M^4$.

2. The quantum intersection defines a description of interactions in terms of string world sheets. If I have understood G-W invariant correctly, one could have for $D > 4$-dimensional symplectic spaces besides partonic $2k-2$-D surfaces also surfaces with smaller but even dimension identifiable as branes of various dimensions. Branes would correspond to a generalization of relative cohomology. In TGD framework one has $2k = 4$ and the partonic 2-surfaces have dimension 2 so that classical intersections consisting of discrete points are possible and stable for string world sheets and partonic 2-surfaces. This is a unique feature of 4-D space-time.

   One might think a generalization of G-W invariant allowing to see string world sheets as connecting the spaced-like 3-surfaces at the boundaries of CDs and light-like orbits of partonic 2-surfaces. The intersection is not discrete now and marked points would naturally correspond to the ends points of strings at partonic 2-surfaces associated with the boundaries of CD and with the vertices of topological scattering diagrams.

3. The idea about 2-D string world sheet as interaction region could generalize in TGD to space-time surface inside CD defining 4-D interaction region. In [L13] one indeed ends up with amazingly similar description of interactions for $n$ external particles entering CD and represented as zero loci for quaternion valued “real” part $RE(P)$ or “imaginary” part $IM(P)$ for the complexified octonionic polynomial.

   Associativity forces quantum criticality posing conditions on the coefficients of the polynomials. Polynomials with the origin of octonion coordinate along the same real axis commute and associate. Since the origins are different for external particles in the general case, the polynomials representing particles neither commute nor associate inside the interaction region defined by CD but one can also now define zero loci for both $RE(\prod P_i)$ and $IM(\prod P_i)$ giving $P_i = 0$ for some $i$. Now different permutations and different associations give rise to different interaction regions and amplitude must be sum over all these.

3-vertices would correspond to conditions $P_i = 0$ for 3 indices $i$ simultaneously. The strongest condition is that 3 partonic 2-surfaces $X_i^2$ co-incide: this condition does not satisfy classical dimension rule and should be posed as essentially 4-D boundary condition. Two partonic 2-surfaces $X_i^2(t_i(n))$ intersect at discrete set of points: could one assume that the sparticle lines intersect and there fusion is forced by boundary condition? Or could one imagine that partonic 2-surfaces turns back in time and second partonic 2-surface intersects it at the turning point?

4. In 4-D context string world sheets are associated with magnetic flux tubes connecting partonic orbits and together with strings serve as correlates for negentropic entanglement assignable to the p-adic sectors of the adele considered, to attention in consciousness theory, and to remote mental interactions in general and occurring routinely between magnetic body and biological body also in ordinary biology. This raises the question whether "quantum touch" generalizes from 2-D string world sheets to 4-D space-time surface (magnetic flux tubes) connecting 3-surfaces at the orbits and partonic orbits.

5. The above formulation applies to closed symplectic manifolds $X$. One can however generalize the formulation to algebraic geometry. Now the algebraic curve $X^2$ is characterized by genus $g$ and order of polynomial $n$ defining it. This formulation looks very natural in $M^8$ picture.

An interesting question is whether the notion of brane makes sense in TGD framework.

1. In TGD branes inside space-time variety are replaced by partonic 2-surfaces and possibly by their light-like orbits at which the induced metric changes signature. These surfaces are metrically 2-D. String world sheets inside space-time surfaces have discrete intersection with the partonic 2-surfaces. The intersection of strings as space-like *resp.* light-like boundaries of string world sheet with partonic orbit sheet *resp.* space-like 3-D ends of space-time surface at boundaries of CD is also discrete classically.

2. An interesting question concerns the role of 6-spheres $S^6(t_n)$ appearing as special solutions to the octonionic zero locus conditions solving both $RE(P_n) = 0$ and $IM(P_n) = 0$ requiring $P_n(o) = 0$. This can be true at 7-D light cone $o = et$, $e$ light-like vector and $t$ a real parameter. The roots $t_n$ of $P(t) = 0$ give 6-spheres $S^6(t_n)$ with radius $t_n$ as solutions to the singularity condition. As found, one can assign to each factor $P_i$ in the product of polynomials defining many-particle state in interaction region its own partonic 2-surfaces $X^2(t_n)$ related to the solution of $P_i(t) = 0$

   Could one interpret 6-spheres as brane like objects, which can be connected by 2-D "free" string world sheets as 2-varieties in $M^8$ and having discrete intersection with them implied by the classical dimension condition for the intersection. Free string world sheets would be something new and could be seen as trivially associative surfaces whereas 6-spheres would represent trivially co-associative surfaces in $M^8$.

   The 2-D intersections of $S^6(t_n)$ with space-time surfaces define partonic 2-surfaces $X^2$ appearing at then ends of space-time and as vertices of topological diagrams. Light-like sparticle lines along parton orbits would fuse at the partonic 2-surfaces and give rise to the analog of 3-vertex in $\mathcal{N} = 4$ SUSY.

Some further TGD inspired remarks are in order.

1. Virasoro conjecture generalizing Witten conjecture involves half Virasoro algebra. Super-Virasoro algebra algebra and its super-symplectic counterpart (SSA) play a key role in the formulation of TGD at level of $H$. Also these algebras are half algebras. The analogs of super-conformal conformal gauge conditions state that sub-algebra of SSA with conformal weights coming as $n$-ples of those for entire algebra and its commutator with entire SSA give rise to vanishing Noether charges and annihilate physical states.

   These conditions are conjecture to fix the preferred extremals and serve as boundary conditions allowing the formulation of $M^8 - H$ correspondence inside space-time regions (interaction regions), where the associativity conditions fail to be true and direct $M^8 - H$

correspondence does not make sense. Non-trivial solutions to these conditions are possible only if one assumes half super-conformal and half super-symplectic algebras. Otherwise the generators of the entire SSA annihilate the physical states and all SSA Noether charges vanish. The invariance of partition function for string world sheets in this sense could be interpreted in terms of emergent dynamical symmetries.

2. Just for fun one can consider the conjecture that the reduction of quantum intersections to classical intersections mediated by string world sheets implies that the numbers of string world sheets as given by the analog of G-W invariants are integers.

## 9.2 Does Riemann-Roch theorem have applications to TGD?

Riemann-Roch theorem (RR) (see `http://tinyurl.com/mdmbcx6`) is a central piece of algebraic geometry. Atyiah-Singer index theorem is one of its generalizations relating the solution spectrum of partial differential equations and topological data. For instance, characteristic classes classifying bundles associated with Yang-Mills theories (see `http://tinyurl.com/y9xvkhyy`) have applications in gauge theories and string models.

The advent of octonionic approach to the dynamics of space-time surfaces inspired by $M^8-H$ duality [L11] [L12, L13] gives hopes that dynamics at the level of complexified octonionic $M^8$ could reduce to algebraic equations plus criticality conditions guaranteeing associativity for space-time surfaces representing external particles, in interaction region commutativity and associativity would be broken. The complexification of octonionic $M^8$ replacing norm in flat space metric with its complexification would unify various signatures for flat space metric and allow to overcome the problems due to Minkowskian signature. Wick rotation would not be a mere calculational trick.

For these reasons time might be ripe for applications of possibly existing generalization of RR to TGD framework. In the following I summarize my admittedly unprofessional understanding of RR discussing the generalization of RR for complex algebraic surfaces having real dimension 4: this is obviously interesting from TGD point of view.

I will also consider the possible interpretation of RR in TGD framework. One interesting idea is possible identification of light-like 3-surfaces and curves (string boundaries) as generalized poles and zeros with topological (but not metric) dimension one unit higher than in Euclidian signature.

### 9.2.1 Could a generalization of Riemann-Roch theorem be useful in TGD framework?

The generalization of RR for algebraic varieties, in particular for complex surfaces (real dimension equal to 4) exists. In $M^8$ picture the complexified metric Minkowskian signature need not cause any problems since the situation can be reduced to Euclidian sector. Clearly, this picture would provide a realization of Wick rotation as more than a trick to calculate scattering amplitudes.

Consider first the motivations for the desire of having analog of Riemann-Roch theorem (RR) at the level of space-time surfaces in $M^8$.

1. It would be very nice if partonic 2-surfaces would have interpretation as analogs of zeros or poles of a meromorphic function. RR applies to the divisors characterizing meromorphic functions and 2-forms, and one could hope of obtaining information about the dimensions of these function spaces giving rise to octonionic space-time varieties. Note however that the reduction to real polynomials or even rational functions might be already enough to give the needed information. Rational functions are required by the simplest generalization whereas the earlier approach assumed only polynomials. This generalization does not however change the construction of space-time varieties as zero loci of polynomials in an essential manner as will be found.

2. One would like to count the degeneracies for the intersections of 2-surfaces of space-time surface and here RR might help since its generalization to complex surfaces involves intersection form as was found in the brief summary of RR for complex surfaces with real dimension 4 (see Eq. 3.5).

   In particular, one would like to know about the intersections of partonic 2-surfaces and string world sheets defining the points at which fermions reside. The intersection form reduces the problem via Poincare duality to 2-cohomology of space-time surfaces. More generally, it is known that the intersection form for 2-surfaces tells a lot about the topology of 4-D manifolds (see `http://tinyurl.com/y8tmqtef`). This conforms with SH. Gromow-Witten invariants [L4] (see `http://tinyurl.com/ybobccub`) are more advanced rational valued invariants but might reduce to integer valued in variants in TGD framework [L13].

There are also other challenges to which RR might relate.

1. One would like to know whether the intersection points for string world sheets and partonic 2-surfaces can belong in an extension of rationals used for adele. If the points belong to cognitive representations and subgroup of Galois group acts trivially then the number of points is reduces as the points at its orbit fuse together. The sheets of the Galois covering would intersect at point. The images of the fused points in $H$ could be disjoint points since tangent spaces need not be parallel.

2. One would also like to have idea about what makes partonic 2-surfaces and string world sheets so special. In 2-D space-time one would have points instead of 2-surfaces. The obvious idea is that at the level of $M^8$ these 2-surfaces are in some sense analogous to poles and zeros of meromorphic functions. At the level of $H$ the non-local character of $M^8-H$ would imply that preferred extremals are solutions of an action principle giving partial differential equations.

### 9.2.2 What could be the analogs of zeros and poles of meromorphic function?

The basic challenge is to define what notions like pole, zero, meromorphic function, and divisor could mean in TGD context. The most natural approach based on a simple observation that rational functions need not define map of space-time surface to itself. Even though rational function can have pole inside CD, the point $\infty$ need not belong to the space-time variety defined the rational functions. Hence one can try the modification of the original hypothesis by replacing the octonionic polynomials with rational functions. One cannot exclude the possibility that although the interior of CD contains only finite points, the external particles outside CD could extend to infinity.

1. For octonionic analytic polynomials the notion of zero is well-defined. The notion of pole is well-defined only if one allows rational functions $R = P_1(o)/P_2(o)$ so that poles would correspond to zeros for the denominator of rational function. 0 and $\infty$ are both unaffected by multiplication and $\infty$ also by addition so that they are algebraically special. There are several variants of this picture. The most general option is that for a given variety zeros of both $P_i$ are allowed.

2. The zeros of $IM(P_1) = 0$ and $IM(P_2) = 0$ would give solutions as unions of surfaces associated with $P_i$. This is because $IM(o_1o_2) = IM(o_1)RE(o_2) + IM(o_2)RE(o_1)$. There is no need to emphasize how important this property of $IM$ for product is. One might say that one has two surfaces which behave like free non-interacting particles.

3. These surfaces should however interact somehow. The intuitive expectation is that the two solutions are glued by wormhole contacts connecting partonic 2-surfaces corresponding to $IM(P_1) = 0$ and $IM(P_2) = 0 = \infty$. For $RE(P_i) = 0$ and $RE(P_i) = \infty$ the solutions do not reduce to separate solutions $RE(P_1) = 0$ and $RE(P_2) = 0$. The reason is that the real part of $o_1o_2$ satisfies $Re(o_1o_2) = Re(o_1)Re(o_2) - Im(o_1)Im(o_2)$. There is a genuine interaction, which should generate the wormhole contact. Only at points for which $P_1 = 0$ and $P_2 = 0$ holds true, $RE(P_1) = 0$ and $RE(P_2) = 0$ are satisfied simultaneously. This happens in the discrete intersection of partonic 2-surfaces.

4. Elementary particles correspond even for $h_{eff} = h$ to two-sheeted structures with partonic surfaces defining wormhole throats. The model for elementary particles requires that particles are minimally 2-sheeted structures since otherwise the conservation of monopole Kähler magnetic flux cannot be satisfied: the flux is transferred between space-time sheets through wormhole contacts with Euclidian signature of induced metric and one obtains closed flux loop. Euclidian wormhole contact would connect the two Minkowskian sheets. Could the Minkowskian sheets corresponds to zeros $IM(P_i)$ for $P_1$ and $P_2$ and could wormhole contacts emerge as zeros of $RE(P_1/P_2)$?

One can however wonder whether this picture could allow more detailed specification. The simplest possibility would be following. The basic condition is that CD emerges automatically from this picture.

1. The simplest possibility is that one has $P_1(o)$ and $P_2(T-o)$ with the origin of octions at the "lower" tip of CD. One would have $P_1(0)=0$ and $P_2(0)=0$. $P_1(o)$ would give rise to the "lower" boundary of CD and $P_2(T-o)$ to the "upper" boundary of CD.

   ZEO combined with the ideas inspired by infinite rationals as counterparts of space-time surfaces connecting 3-surfaces at opposite boundaries of CD [K13] would suggest that the opposite boundaries of CD could correspond zeros and poles respectively and the ratio $P_1(o)/P_2(T-o)$ and to zeros of $P_1$ resp. $P_2$ assignable to different boundaries of CD. Both light-like parton orbits and string world sheets would interpolate between the two boundaries of CD at which partonic 2-surface would correspond to zeros and poles.

   The notion divisor would be a straightforward generalization of this notion in the case of complex plane. What would matter would be the rational function $P_1(t)/P_2(T-t)$ extended from the real (time) axis of octonions to the entire space of complexified octonions. Positive degree of divisor would multiply $P_1(t)$ with $(t-t_1)^m$ inducing a new zero at or increasing the order of existing zero at $t_1$. Negative orders $n$ would multiply the denominator by $(t-t_1)^n$.

2. One can also consider the possibility that both boundaries of CD emerge for both $P_1$ and $P_2$ and without assigning either boundary of CD with $P_i$. In this case $P_i$ would be sum over terms $P_{ik} = P_{ia_k}(o)P_{ib_k}(T-o)$ of this kind of products satisfying $P_{ia_k}(0)=0$ and $P_{ib_k}(0)=0$.

One can imagine also an alternative approach in which 0 and $\infty$ correspond to opposite tips of CD and have geometric meaning. Now zeros and poles would correspond to 2-surfaces, which need not be partonic. Note that in the case of Riemann surfaces $\infty$ can represent any point. This approach does not however look attractive.

### 9.2.3 Could one generalize RR to octonionic algebraic varieties?

RR is associated with complex structure, which in TGD framework seems to make sense independent of signature thanks to complexification of octonions. Divisors are the key notion and characterize what might be called local winding numbers. De-Rham cohomology is replaced with much richer Dolbeault cohomology (see `http://tinyurl.com/y7cvs5sx`) since the notion of continuity is replaced with that of meromorphy. Symplectic approach about which G-W invariants for symplectic manifolds provide an example define a different approach and now one has ordinary cohomology.

An interesting question is whether $M^8-H$-duality corresponds to the mirror symmetry of string models (see `http://tinyurl.com/yc2m2e5m`) relating complex structures and symplectic structures. If this were the case, $M^8$ would correspond to complex structure and $H$ to symplectic structure.

RR for curves gives information about dimensions for the spaces of meromorphic functions having poles with order not higher than specified by divisor. This kind of interpretation would be very attractive now since the poles and zeros represented as partonic 2-surfaces would have direct physical interpretation in terms of external particles and interaction vertices. RR for curves involves poles with orders not higher than specified by the divisor and gives a formula for the dimension of the space of meromorphic functions fora given divisor. As a special case give the dimension $l(nD)$ for a given divisor.

Could something similar be true in TGD framework?

1. Arithmetic genus makes sense for polynomials $P(t)$ since $t$ can be naturally complexified giving a complex curve with well-defined arithmetic genus. What could correspond to the intersection form for 2-surfaces representing $D$ and $K-D$? The most straightforward possibility is that partonic 2-surfaces correspond to poles and zeros.

   Divisor $-D$ would correspond to the inverse of $P_2/P_1$ representing it. $D-K$ would also a well-defined meaning provided the canonical divisor associated with holomorphic 2-form has well-defined meaning in the Dolbeault cohomology of the space-time surface with complex structure. RR would give direct information about the space of space-time varieties defined by $RE(P) = 0$ or $IM(P) = 0$ condition.

One could hope of obtaining information about intersection form for string world sheets and partonic 2-surfaces. Whether the divisor $D - K$ has anything to do string world sheets, is of course far from clear.

2. Complexification means that field property fails in the sense that complexified Euclidian norm vanishes and the inverse of complexified octonion/quaternion/complex number is infinite formally. For Euclidian sector with real coordinates this does not happen but does take place when some coordinates are real and some imaginary so that signature is effectively Minkowskian signature.

   At 7-D light-cone of $M^8$ the condition $P(o) = 0$ reduces to a condition for real polynomial $P(t) = 0$ giving roots $t_n$. Partonic 2-varieties are intersections of 4-D space-time varieties with 6-spheres with radii $t_n$. There are good reasons to expect that the 3-D light-like orbits of partonic 3-surfaces are intersections of space-time variety with 7-D light-cone boundary and their $H$ counterparts are obtained as images under $M^8 - H$ duality.

   For light-like complefixied octonionic points the inverse of octonion does not exist since the complexified norm vanishes. Could the light-like 3-surfaces as partonic orbits correspond to images under $M^8 - H$ duality for zeros and/or poles as 3-D light-like surfaces? Could also the light-like boundaries of strings correspond to this kind of generalized poles or zeros? This could give a dynamical realization for the notions of zero and pole and increase the topological dimension of pole and zero for both 2-varieties and 4-varieties by one unit. The metric dimension would be unaffected and this implies huge extension of conformal symmetries central in TGD since the light-like coordinate appears as additional parameter in the infinitesimal generators of symmetries.

Could one formulate the counterpart of RR at the level of $H$? The interpretation of $M^8 - H$ duality as analog of mirror symmetry (see `http://tinyurl.com/yc2m2e5m`) suggests this. In this case the first guess for the identification of the counterpart of canonical divisor could be as Kähler form of $CP_2$. This description would provide symplectic dual for the description based on divisors at the level of $M^8$. G-W invariants and their possible generalization are natural candidates in this respect.

## 9.3 Could the TGD variant of Atyiah-Singer index theorem be useful in TGD?

Atyiah-Singer index theorem (AS) is one of the generalizations of RR and has shown its power in gauge field theories and string models as a method to deduce the dimensions of various moduli spaces for the solutions of field equations. A natural question is whether AS could be useful in TGD and whether the predictions of AS at $H$ side could be consistent with $M^8 - H$ duality suggesting very simple counting for the numbers of solutions at $M^8$ side as coefficient combinations of polynomials in given extension of rationals satisfying criticality conditions. One can also ask whether the hierarchy of degrees $n$ for octonion polynomials could correspond to the fractal hierarchy of generalized conformal sub-algebras with conformal weights coming as $n$-multiples for those for the entire algebras.

Atyiah-Singer index theorem (AS) and other generalizations of RR involve extremely abstract concepts. The best manner to get some idea about AS is to learn the motivations for it. The article `http://tinyurl.com/yc49lljp` gives a very nice general view about the motivations of Atyiah-Singer index theorem and also avoids killing the reader with details.

Solving problems of algebraic geometry is very demanding. The spectrum of solutions can be discrete (say number of points of space-time surface having linear $M^8$ coordinates in an extension of rationals) or continuous such as the space of roots for $n$:th order polynomials with real coefficients.

An even more difficult challenge is solving of partial differential equations in some space, call it $X$, of say Yang-Mills gauge field coupled to matter fields. In this case the set of solutions is typically continuous moduli space.

One can however pose easier questions. What is the number of solutions in counting problem? What is the dimension of the moduli space of solutions? Atiyiah-Singer index theorem relates this number - analytic index - to topological index expressible in terms of topological invariants assignable to complexified tangent bundle of $X$ and to the bundle structure - call it field bundle - accompanying the fields for which field equations are formulated.

### 9.3.1 AS very briefly

Consider first the assumptions of AS.

1. The idea is to study perturbations of a given solution and linearize the equations in some manifold $X$ often assumed to be compact. This leads to a linear partial differential equations defined by linear operator $P$. One can deduce the dimension of the solution space of $P$. This number defines the dimension of the tangent space of solution space of full partial differential equations, call it moduli space.

2. The idea is to assign to the partial differential operator $P$ its symbol $\sigma(P)$ obtained by replacing derivatives with what might be called momentum components. The reversal of this operaion is familiar from elementary wave mechanics: $p_i \rightarrow id/dx^i$. This operation can be formulated in terms of co-tangent bundle. The resulting object is purely algebraic. If this matrix is reversible for all momentum values and points of $X$, one says that the operator is elliptic.

   Note that for field equations in Minkowski space $M^4$ the invertibility constraint is not satisfied and this produces problems. For instance, for massive $M^4$ d'Alembertian for scalar field the symbol is four-momentum squared, which vanishes, when on-mass shell condition is satisfied. Wick rotation is somewhat questionable manner to escape this problem. One replaces Minkowski space with its Euclidian counterpart or by 4-sphere. If all goes well the dimension of the solution space does not depend on the signature of the metric.

3. In the general case one studies linear equation of form $DP = f$, where $f$ is homogenuity term representing external perturbation. $f$ can also vanish. Quite generally, one can write the dimension of the solution space as

   $$Ind_{anal}(P) = dim(ker(P)) - dim(coker(P)) \ . \tag{9.1}$$

   $ker(P)$ denotes the solution space for $DP = 0$ without taking into account the possible restrictions coming from the fact that $f$ can involve part $f_0$ satisfying $Df_0 = 0$ (for instance, $f_0$ corresponds to resonance frequency of oscillator system) nor boundary conditions guaranteing hermiticity. Indeed, the hermitian conjugate $D^\dagger$ of $D$ is not automatically identical with $D$. $D^\dagger$ is defined in terms of the inner product for small perturbations as

   $$\langle D^\dagger P_1^* | DP_2 \rangle = \langle P_1 | DP_2 \rangle \ . \tag{9.2}$$

   The inner product involves integration over $X$ and partial integrations transfer the action of partial derivatives from $P_2$ to $P_1^*$. This however gives boundary terms given by surface integral and hermiticity requires that they vanish. This poses additional conditions on $P$ and contributes to $dim(coker(P))$.

The challenge is to calculate $Ind_{anal}(P)$ and here AS is of enormous help. AS relates analytical index $Ind_{anal}(P)$ for $P$ to topological index $Ind_{top}(\sigma(P))$ for its symbol $\sigma(P)$.

1. $Ind_{top}(\sigma(P))$ involves only data associated with the topology $X$ and with the bundles associated with field variables. In the case of Yang-Mills fields coupled to matter the bundle is the bundle associated with the matter fields with a connection determined by Yang-Mills gauge potentials. So called Todd class $Td(X)$ brings in information about the topology of complexified tangent bundle.

2. $Ind_{top}(\sigma(P))$ is not at all easy to define but is rather easily calculable as integrals of various invariants assignable to the bundle structure involved. Say instanton density for YM fields and various topological invariants expressing the topological invariants associated with the metric of the space. What is so nice and so non-trivial is that the dimension of the moduli space for non-linear partial differential equations is determined by topological invariants. Much of the dynamics reduces to topology.

The expression for $Ind_{top}(\sigma(P))$ involves besides $\sigma_P$ topological data related to the field bundle and to the complexified tangent bundle. The expression $Ind_{top}$ as a function of the symbol $\sigma(P)$ is given by

$$Ind_{top}(\sigma(P)) = (-1)^n \langle ch(\sigma(P)) \cdot Td(T_C(X), [X]) \rangle \ . \tag{9.3}$$

The expression involves various topological data.

1. Dimension of $X$.
2. The quantity $\langle x.y \rangle$ involving cup product $x.y$ of cohomology classes, which contains a contribution in the highest homology group $H^n(X)$ of $X$ corresponding to the dimension of $X$ and is contracted with this fundamental class $[X]$. $\langle x.y \rangle$ denotes matrix trace for the operator $ch(\sigma(P))$ formed as polynomial of $\sigma(P)$. $[X]$ denotes so called fundamental class fr $X$ belonging to $H^n$ and defines the orientation of $X$.
3. Chern character $ch_E(t)$ (see `http://tinyurl.com/ybavu66h`). I must admit that I ended up to a garden of branching paths while trying to understand the definition of $ch_E$ is. In any case, $ch_E(t)$ characterizes complex vector bundle $E$ expressible in terms of Chern classes (see `http://tinyurl.com/y8jlaznc`) of $E$. $E$ is the bundle assignable to field variables, say Yang Mills fields and various matter fields.

   Both direct sums and tensor products of fiber spaces of bundles are possible and the nice feature of Chern class is that it is additive under tensor product and multiplicative under direct sum. The fiber space of the entire bundle is now direct sum of the tangent space of $X$ and field space, which suggests that $Ind(top)$ is actually the analog of Chern character for the entire bundle.

   $t = \sigma P$ has interpretation as an argument appearing in the definition of Chern class generalized to Chern character. $t = \sigma(P)$ would naturally correspond to a matrix valued argument of the polynomial defining Chern class as cohomology element. $ch(\sigma(P))$ is a polynomial of the linear operator defined by symbol $\sigma(P)$. $ch_E$ for given complex vector bundle is a polynomial, whose coefficients are relatively easily calculable as topological invariants assignable to bundle $E$. $E$ must be the field bundle now.
4. Todd class $Td(T_C(X))$ for the complexified tangent bundle (see `http://tinyurl.com/yckv4w84`) appears also in the expression. Note that also now the complexification occurs. The cup product gives element in $H^n(X)$, which is contracted with fundamental class $[X]$ and integrated over $X$.

### 9.3.2 AS and TGD

The dynamics of TGD involves two levels: the level of complexified $M^8$ (or equivalently $E^8$) and the level of $H$ related to $M^8 - H$ correspondence.

1. At the level of $M^8$ one has algebraic equations rather than partial differential equations and the situation is extremely simple as compared to the situation for a general action principle. At the level of $H$ one has action principle and partial differential equations plus infinite number of gauge conditions selecting preferred extremals and making dynamics for partial differential equations dual to the dynamics determined by purely number theoretic conditions.

   The space-time varieties representing external particles outside CDs in $M^8$ satisfy associativity conditions for tangent space or normal space and reducing to criticality conditions for the real coefficients of the polynomials defining the space-time variety. In the interior of CDs associativity conditions are not satisfied but the boundary conditions fix the values of the coefficients to be those determined by criticality conditions guaranteing associativity outside the CD.

In the interiors space-time surfaces of CDs $M^8$-duality does not apply but associativity of tangent spaces or normal spaces at the boundary of CD fixes boundary values and minimal surface dynamics and strong form of holography (SH) fixes the space-time surfaces in the interior of CD.

2. For the $H$-images of space-time varieties in $H$ under $M^8-H$ duality the dynamics is universal coupling constant independent critical dynamics of minimal surfaces reducing to holomorphy in appropriate sense. For minimal surfaces the 4-D Kähler current density vanishes so that the solutions are 4-D analogs of geodesic lines outside CD. Inside CD interactions are coupled on and this current is non-vanishing. Infinite number of gauge conditions for various half conformal algebras in generalized sense code at $H$ side for the number theoretical critical conditions at $M^8$ side. The sub-algebra with conformal weights coming as $n$-ples of the entire algebra and its commutator with entire algebra gives rise to vanishing classical Noether charges. An attractive assumption is that the value of $n$ at $H$ side corresponds to the order $n$ of the polynomials at $M^8$ side.

3. The coefficients of polynomials $P(o)$ determining space-time varieties are real numbers (also complexified reals can be considered without losing associativity) restricted to be numbers in extension of rationals. This makes it possible to speak about p-adic variants of the space-time surfaces at the level of $M^8$ at least.

Could Atyiah-Singer theorem have relevance for TGD?

1. For real polynomials it is easy to calculate the dimension of the moduli space by counting the number of independent real (in octonionic sense) coefficients of the polynomials of real variable (one cannot exclude that the coefficients are in complex extension of rationals). Criticality conditions reduce this number and the condition that coefficients are in extension of rationals reduces it further. One has quite nice overall view about the number of solutions and one can see them as subset of continuous moduli space. If $M^8 - H$ duality really works then this gives also the number of preferred extremals at $H$ side.

2. This picture is not quite complete. It assumes fixing of 8-D CD in $M^8$ as well as fixing of the decomposition $M^2 \subset M^4 \subset M^4 \times E^4$. This brings in moduli space for different choices of octonion structures (8-D Lorentz group is involved). Also moduli spaces for partonic 2-surfaces are involved. Number theoretical universality seems to require that also these moduli spaces have only points with coordinates in extension of rationals involved.

3. In principle one can try to formulate the counterpart of AS at $H$ side for the linearization of minimal surface equations, which are nothing but the counterpart of massless field equations in a fixed background metric. Note that additional conditions come from the requirement that the term from Kähler action reduces to minimal surface term.

Discrete sets of solutions for the extensions of rationals should correspond to each other at the two sides. One can also ask whether the dimensions for the effective continuous moduli spaces labelled by $n$ characterizing the sub-algebras of various conformal algebras isomorphic to the entire algebra and those for the polynomials of order $n$ satisfying criticality conditions. One would have a number theoretic analog for a particle in box leading to the quantization of momenta.

All this is of course very speculative and motivated only by the general physical vision. If the speculations were true, they would mean huge amount of new mathematics.

# 10 Could the precursors of perfectoids emerge in TGD?

In algebraic-geometry community the work of Peter Scholze [A9] (see `http://tinyurl.com/y7h2sms7`) introducing the notion of perfectoid related to p-adic geometry has raised a lot of interest. There are two excellent popular articles about perfectoids: the first article in AMS (see `http://tinyurl.com/ydx38vk4`) and second one in Quanta Magazine (see `http://tinyurl.com/yc2mxxqh`). I had heard already earlier about the work of Scholze but was too lazy to even attempt to understand what is buried under the horrible technicalities of modern mathematical prose. Rachel Francon re-directed my attention to the work of Scholze (see `http://tinyurl.com/yb46oza6`). The work of Scholze is interesting also from TGD point of view since the construction of p-adic geometry is a highly non-trivial challenge in TGD.

1. One should define first the notion of continuous manifold but compact-open characteristic of p-adic topology makes the definition of open set essential for the definition of topology problematic. Even single point is open so that hopes about p-adic manifold seem to decay to dust. One should pose restrictions on the allowed open sets and p-adic balls with radii coming as powers of $p$ are the natural candidates. p-Adic balls are either disjoint or nested: note that also this is in conflict with intuitive picture about covering of manifold with open sets. All this strangeness originates in the special features of p-adic distance function known as ultra-metricity. Note however that for extensions of p-adic numbers one can say that the Cartesian products of p-adic 1-balls at different genuinely algebraic points of extension along particular axis of extension are disjoint.

2. At level of $M^8$ the p-adic variants of algebraic varieties defined as zero loci of polynomials do not seem to be a problem. Equations are algebraic conditions and do not involve derivatives like partial differential equations naturally encountered if Taylor series instead of polynomials are allowed. Analytic functions might be encountered at level of $H = M^4 \times CP_2$ and here p-adic geometry might well be needed.

   The idea is to define the generalization of p-adic algebraic geometry in terms of p-adic function fields using definitions very similar to those used in algebraic geometry. For instance, generalization of variety corresponds to zero locus for an ideal of p-adic valued function field. p-Adic ball of say unit radius is taken as the basic structure taking the role of open ball in the topology of ordinary manifolds. This kind of analytic geometry allowing all power series with suitable restrictions to function field rather than allowing only polynomials is something different from algebraic geometry making sense for p-adic numbers and even for finite fields.

3. One would like to generalize the notion of analytic geometry even to the case of number fields with characteristic $p$ ($p$-multiple of element vanishes), in particular for finite fields $F_p$ and for function fields $F_p[t]$. Here one encounters difficulties. For instance, the factorial $1/n!$ appearing as normalization factor of forms diverges if $p$ divides it. Also the failure of Frobenius homomorphism to be automorphism for $F_p[t]$ causes difficulties in the understanding of Galois groups.

The work of Scholze has led to a breakthrough in unifying the existing ideas in the new framework provided by the notion of perfectoid. The work is highly technical and involves infinite-D extension of ordinary p-adic numbers adding all powers of all roots $p^{1/p^m}$, $m = 1, 2....$ Formally, an extension by powers of $p^{1/p^\infty}$ is in question.

This looks strange at first but it guarantees that all p-adic numbers in the extension have $p$:th roots, one might say that one forms a $p$-fold covering/wrapping of extension somewhat analogous to complex numbers. This number field is called perfectoid since it is perfect meaning that Frobenius homomorphism $a \rightarrow a^p$ is automorphism by construction. $Frob$ is injection always and by requiring that $p$:th roots exist always, it becomes also a surjection.

This number field has same Galois groups for all of its extensions as the function field $G[t]$ associated with the union of function fields $G = F_p[t^{1/p^m}]$. Automorphism property of $Frob$ saves from the difficulties with the factorization of polynomials and p-adic arithmetics involving remainders is replaced with purely local modulo $p$ arithmetics.

## 10.1 About motivations of Scholze

Scholze has several motivations for this work. Since I am not a mathematician, I am unable to really understand all of this at deep level but feel that my duty as user of this mathematics is at least to try!

1. Diophantine equations is a study of polynomial equations in several variables, say $x^2+2xy+y=0$. The solutions are required to be integer valued: in the example considered $x=y=0$ and $x=-y=-1$ is such a solution. For integers the study of the solution is very difficult and one approach is to study these equations modulo $p$ that is reduced the equations to finite field $G_p$ for any $p$. The equations simplify enormously since ane has $a^p = a$ in $F_p$. This identity in fact defines so called Frobenius homomorphism acting as automorphism for finite fields. This holds true also for more complex fields with characteristic $p$ say the ring $F_p[t]$ of power series of $t$ with coefficients in $F_p$.

   The powers of variables, say $x$, appearing in the equation is reduced to at most $x^{p-1}$. One can study the solutions also in p-adic number fields. The idea is to find first whether finite field solution, that is solution modulo $p$, does exist. If this is the case, one can calculate higher powers in $p$. If the series contains finite number of terms, one has solution also in the sense of ordinary integers.

2. One of the related challenges is the generalization of the notion of variety to a geometry defined in arbitrary number field. One would like to have the notion of geometry also for finite fields, and for their generalizations such as $F_p[t]$ characterized by characteristic $p$ ($px=0$ holds true for any element of the field). For fields of characteristic 1 - extensions of rationals, real, and p-adic number fields) $xp=0$ not hold true for any $x\neq 0$. Any field containing rationals as sub-field, being thus local field, is said to have characteristic equal to 1. For local fields the challenge is relatively easy.

3. The situation becomes more difficult if one wants a generalization of differential geometry. In differential geometry differential forms are in a key role. One wants to define the notion of differential form in fields of characteristic $p$ and construct a generalization of cohomology theory. This would generalize the notion of topology to p-adic context and even for finite fields of finite character. A lot of work has been indeed done and Grothendieck has been the leading pioneer.

   The analogs of cohomology groups have values in the field of p-adic numbers instead of ordinary integers and provide representations for Galois groups for the extensions of rationals inducing extensions of p-adic numbers and finite fields.

   In ordinary homology theory non-contractible sub-manifolds of various dimensions correspond to direct summands $Z$ (group of integers) for homology groups and by Poincare duality those for cohomology groups. For Galois groups $Z$ is replaced with $Z_N$. $N$ depends on extension to which Galois group is associated and if $N$ is divisible by $p$ one encounters technical problems.

   There are many characteristic $p$- and p-adic cohomologies such as etale cohomology, chrystalline cohomology, algebraic de-Rham cohomology. Also Hodge theory for complex differential forms generalizes. These cohomologies should be related by homomorphism and category theoretic thinking the proof of the homomorphism requires the construction of appropriate functor between them.

   The integrals of forms over sub-varieties define the elements of cohomology groups in ordinary cohomology and should have p-adic counterparts. Since p-adic numbers are not well-ordered, definite integral has no straightforward generalization to p-adic context. One might however be able to define integrals analogous to those associated with differential forms and depending only on the topology of sub-manifold over which they are taken. These integrals would be analogous to multiple residue integrals, which are the crux of the twistor approach to scattering amplitudes in super-symmetric gauge theories. One technical difficulty is that for a field of finite characteristic the derivative of $X^p$ is $pX^{p-1}$ and vanishes. This does not allow to define what integral $\int X^{p-1}dX$ could mean. Also $1/n!$ appears as natural normalization factor of forms but if $p$ divides it, it becomes infinite.

## 10.2 Attempt to understand the notion of perfectoid

Consider now the basic ideas behind the notion of perfectoid.

1. For finite finite fields $F_p$ Frobenius homomorphism $a \rightarrow a^p$ is automorphism since one has $a^p = a$ in modulo $p$ arithmetics. A field with this property is called perfect and all local fields are perfect. Perfectness means that an algebraic number in any extension $L$ of perfect field $K$ is a root of a separable minimal polynomial. Separability means that the number of roots in the algebraic closure of $K$ of the polynomial is maximal and the roots are distinct.

2. All fields containing rationals as sub-fields are perfect. For fields of characteristic $p$ $Frob$ need not be a surjection so that perfectness is lost. For instance, for $F_p[t]$ $Frob$ is trivially injection but surjective property is lost: $t^{1/p}$ is not integer power of $t$.

   One can however extend the field to make it perfect. The trick is simple: add to $F_p[t]$ all fractional powers $t^{1/p^n}$ so that all $p$:th roots exist and $Frob$ becomes and automorphism. The automorphism property of $Frob$ allows to get rid of technical problems related to a factorization of polynomials. The resulting extension is infinite-dimensional but satisfies the perfectness property allowing to understand Galois groups, which play key role in various cohomology theories in characteristic $p$.

3. Let $K = Q_p[p^{1/p^\infty}]$ denote the infinite-dimensional extension of p-adic number field $Q_p$ by adding all powers of $p^m$:th roots for all all $m = 1, 2, ....$ This is not the most general option: $K$ could be also only a ring. The outcome is perfect field although it does not of course have Frobenius automorphism since characteristic equals to 1.

   One can divide $K$ by $p$ to get $K/p$ as the analog of finite field $F_p$ as its infinite-dimensional extension. $K/p$ allows all $p$:th roots by construction and $Frob$ is automorphism so that $K/p$ is perfect by construction.

   The structure obtained in this manner is closely related to a perfect field with characteristic $p$ having same Galois groups for all its extensions. This object is computationally much more attractive and allows to prove theorems in p-adic geometry. This motivates the term perfectoid.

4. One can assign to $K$ another object, which is also perfectoid but has characteristic $p$. The correspondence is as follows.

   (a) Let $F_p$ be finite field. $F_p$ is perfect since it allows trivially all $p$:th roots by $a^p = a$. The ring $F_p[t]$ is however not prefect since $t^{1/p^m}$ is not integer power of $t$. One must modify $F_p[t]$ to obtain a perfect field. Let $G_m = F_p[t^{1/p^m}]$ be the ring of formal series in powers of $t^{1/p^m}$ defining also function field. These serious are called t-adic and one can define t-adic norm.

   (b) Define t-adic function field $K_b$ called the **tilt** of $K$ as

   $$K_b = \cup_{m=1,...}(K/p)[t^{1/pm}][t] \ .$$

   One has all possible power series with coefficients in $K/p$ involving all roots $t^{1/p^m}$, $m = 1, 2, ...$, besides powers of positive integer powers of $t$. This function field has characteristic $p$ and all roots exist by construction and $Frob$ is automorphism. $K_b/t$ is perfect meaning that the minimal polynomials for the for given analog of algebraic number in any of its extensions allows separable polynomial with maximal number of roots in its closure.

   This sounds rather complicated! In any case, $K_b/t$ has same number theoretical structure as $Q_p[p^{1/p^\infty}]/p$ meaning that Galois groups for all of its extensions are canonically isomorphic to those for extensions of $K$. Arithmetics modulo $p$ is much simpler than p-adic arithmetic since products are purely local and there is no need to take care about remainders in arithmetic operations, this object is much easier to handle.

   Note that also p-adic number fields fields $Q_p$ as also $F_p = Q_p/p$ are perfect but the analog of $K_b = F_b[t]$ fails to be perfect.

## 10.3 Second attempt to understand the notions of perfectoid and its tilt

This subsection is written roughly year after the first version of the text. I hope that it reflects a genuine increase in my understanding.

1. Scholze introduces first the notion of perfectoid. This requires some background notions. The characteristic $p$ for field is defined as the integer $p$ (prime) for which $px = 0$ for all elements $x$. Frobenius homomorphism (Frob familiarly) is defined as $Frob : x \to x^p$. For a field of characteristic $p$ $Frob$ is an algebra homomorphism mapping product to product and sum to sum: this is very nice and relatively easy to show even by a layman like me.

2. Perfectoid is a field having either characteristic $p = 0$ (reals, p-adics for instance) or for which $Frob$ is a surjection meaning that $Frob$ maps at least one number to a given number $x$.

3. For finite fields $Frob$ is identity: $x^p = x$ as proved already by Fermat. For reals and p-adic number fields with characteristic p=0 it maps all elements to unit element and is not a surjection. Field is perfect if it has either $p = 0$ (reals, p-adics) or if Frobenius is surjection. Finite fields are obviously perfectoids too.

Scholze introduces besides perfectoids $K$ also what he calls tilt $K_b$ of the perfectoid. $K_b$ is infinite-D extension of p-adic numbers by iterated $p$:th roots p-adic numbers: the units of the extension correspond to the roots $p^{1/p^k}$. They are something between p-adic number fields and reals and leads to theorems giving totally new insights to arithmetic geometry. Unfortunately, my technical skills in mathematics are hopelessly limited to say anything about these theorems.

1. As we learned during the first student year of mathematics, real numbers can be defined as Cauchy sequences of rationals converging to a real number, which can be also algebraic number or transcendental. The elements in the tilt $K_b$ would be this kind of sequences.

2. Scholze starts from (say) p-adic numbers and considers infinite sequence of iterates of $1/p$:th roots. At given step $x \to x^{1/p}$. This gives the sequence $(x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, ...)$ identified as an element of the tilt $K_b$. At the limit one obtains $1/p^\infty$ root of $x$.

   **Remark:** For finite fields each step is trivial ($x^p = x$) so that nothing interesting results: one has $(x, x, x, x, ...)$

   (a) For p-adic number fields the situation is non-trivial. $x^{1/p}$ exists as p-adic number for all p-adic numbers with unit norm having $x = x_0 + x_1 p + ....$ In the lowest order $x \simeq x_0$ the root is just $x$ since $x$ is effectively an element of finite field in this approximation. One can develop the $x^{1/p}$ to a power series in $p$ and continue the iteration. The sequence obtained defines an element of tilt $K_b$ of field K, now p-adic numbers.

   (b) If the p-adic number $x$ has norm $p^n$, $n \neq 0$ and is therefore not p-adic unit, the root operation makes sense only if one performs an extension of p-adic numbers containing all the roots $p^{1/p^k}$ . These roots define one particular kind of extension of p-adic numbers and the extension is infinite-dimensional since all roots are needed. One can approximate $K_b$ by taking only finite number iterated roots.

3. The tilt is said to be fractal: this is easy to understand from the presence of the iterated $p$:th root. Each step in the sequence is like zooming. One might say that p-adic scale becomes $p$:th root of itself. In TGD the p-adic length scale $L_p$ is proportional to $p^{1/2}$: does the scaling mean that the p-adic length scale would defined hierarchy of scales proportional to $p^{1/2kp}$: root of itself and approach the $CP_2$ scale since the root of $p$ approaches unity. Tilts as extensions by iterated roots would improve the length scale resolution.

One day later after writing this I got the feeling that I might have vaguely understood one more important thing about the tilt of p-adic number field: changing of the characteristic 0 of p-adic number field to characteristics $p > 0$ of the corresponding finite field for its tilt. What could this mean?

1. Characteristic $p$ ($p$ is the prime labelling p-adic number field) means $px = 0$. This property makes the mathematics of finite fields extremely simple: in the summation one need not take care of the residue as in the case of reals and p-adics. The tilt of the p-adic number field would have the same property! In the infinite sequence of the p-adic numbers coming as iterated $p$:th roots of the starting point p-adic number one can sum each p-adic number separately. This is really cute if true!

2. It seems that one can formulate the arithmetics problem in the tilt where it becomes in principle as simple as in finite field with only p elements! Does the existence of solution in this case imply its existence in the case of p-adic numbers? But doesn't the situation remain the same concerning the existence of the solution in the case of rational numbers? The infinite series defining p-adic number must correspond a sequence in which binary digits repeat with some period to give a rational number: rational solution is like a periodic solution of a dynamical system whereas non-rational solution is like chaotic orbit having no periodicity? In the tilt one can also have solutions in which some iterated root of $p$ appears: these cannot belong to rationals but to their extension by an iterated root of $p$.

The results of Scholze could be highly relevant for the number theoretic view about TGD in which octonionic generalization of arithmetic geometry plays a key role since the points of space-time surface with coordinates in extension of rationals defining adele and also what I call cognitive representations determining the entire space-time surface if $M^8 - H$ duality holds true (space-time surfaces would be analogous to roots of polynomials). Unfortunately, my technical skills in mathematics needed are hopelessly limited.

TGD inspires the question is whether this kind of extensions could be interesting physically. At the limit of infinite dimension one would get an ideal situation not realizable physically if one believes that finite-dimensionality is basic property of extensions of p-adic numbers appearing in number theoretical quantum physics (they would related to cognitive representations in TGD). Adelic physics [L15] involves all finite-D extensions of rationals and the extensions of p-adic number fields induced by them and thus also cutoffs of extensions of type $K_b$- which I have called precursors of $K_b$.

### 10.3.1 How this relates to Witt vectors?

Witt vectors provide an alternative representation of p-adic arithmetics of p-adic integers in which the sum and product are reduced to purely local digit-wise operations for each power of p for the components of Witt vector so that one need not worry about carry pinary digit.

1. The idea is to consider the sequence consisting pinary cutoffs to p-adic number $xmodp^n$ and identify p-adic integer as this kind of sequence as $n$ approaches infinity. This is natural approach when one identifies finite measurement resolution or cognitive resolution as a cutoff in some power of $p^n$. One simply forms the numbers $X_n = x \ mod \ p^{n+1}$: for numbers $1, ..., p-1$ they are called Teichmueller representatives and only they are needed to construct the sequences for general $x$. One codes this sequence of pinary cutoffs to Witt vector.

2. The non-trivial observation made by studying sums of p-adic numbers is that the sequence $X_0, X_1, X_2, ...$ of approximations define a sequence of components of Witt vector as $W_0 = X_0$, $W_1 = X_0^p + pX_1$, $W_2 = X_0^(p^2) + pX_1^p + p^2X_2$, ... or more formally $W_n = Sum_{i<n} p^i Z X_i^[ p^(n-i)]$.

3. The non-trivial point is that Witt vectors form a commutative ring with local digit-wise multiplication and sum modulo $p$: there no carry digits. Effectively one obtains infinite Cartesian power of finite field $F_p$. This means a great simplification in arithmetics. One can do the arithmetics using Witt vectors and deduce the sum and product from their product.

4. Witt vectors are universal. In particular, they generalize to any extension of p-adic numbers. Could Witt vectors bring in something new from physics point of view? Could they allow a formulation for the hierarchy of pinary cutoffs giving some new insights? For instance, neuro-computationalist might ask whether brain could perform p-adic arithmetics using a linear array of modules (neurons or neuron groups) labelled by $n = 1, 2, ...$ calculates sum or product for component $W_n$ of Witt vector? No transfer of carry bits between modules

would be needed. There is of course the problem of transforming p-adic integers to Witt vectors and back - it is not easy to imagine a natural realization for a module performing this transformation. Is there any practical formulation for say p-adic differential calculus in terms of Witt vectors?

I would seem that Witt vectors might relate in an interesting manner to the notion of perfectoid. The basic result proved by Petter Scholtze is that the completion $\cup_n Q_p(p^{1/p^n})$ of p-adic numbers by adding $p^n$:th roots and the completion of Laurent series $F_p((t))$ to $\cup_n F_p((t^{1/p^n}))$ have isomorphic absolute Galois groups and in this sense are one and same thing. On the other hand, p-adic integers can be mapped to a subring of $F_p(t)$ consisting of Taylor series with elements allowing interpretation as Witt vectors.

## 10.4 TGD view about p-adic geometries

As already mentioned, it is possible to define p-adic counterparts of $n$-forms and also various p-adic cohomologies with coefficient field taken as p-adic numbers and these constructions presumably make sense in TGD framework too. The so called rigid analytic geometry is the standard proposal for what p-adic geometry might be.

The very close correspondence between real space-time surfaces and their p-adic variants plays realized in terms of cognitive representations [L17] [L16, L11] plays a key role in TGD framework and distinguishes it from approaches trying to formulate p-adic geometry as a notion independent of real geometry.

Ordinary approaches to p-adic geometry concentrate the attention to single p-adic prime. In the adelic approach of TGD one considers both reals and all p-adic number fields simultaneously.

Also in TGD framework Galois groups take key role in this framework and effectively replace homotopy groups and act on points of cognitive representations consisting of points with coordinates in extension of rationals shared by real and p-adic space-time surfaces. One could say that homotopy groups at level of sensory experience are replaced by Galois at the level of cognition. It also seems that there is very close connection between Galois groups and various symmetry groups. Galois groups would provide representations for discrete subgroups of symmetry groups.

In TGD framework there is strong motivation for formulating the analog of Riemannian geometry of $H = M^4 \times CP_2$ for p-adic variants of $H$. This would mean p-adic variant of Kähler geometry. The same challenge is encountered even at the level of "World of Classical Worlds" (WCW) having Kähler geometry with maximal isometries. p-Adic Riemann geometry and $n$-forms make sense locally as tensors but integrals defining distances do not make sense p-adically and it seems that the dream about global geometry in p-adic context is not realizable. This makes sense: p-adic physics is a correlate for cognition and one cannot put thoughts in weigh or measure their length.

### 10.4.1 Formulation of adelic geometry in terms of cognitive representations

Consider now the key ideas of adelic geometry and of cognitive representations.

1. The king idea is that p-adic geometries in TGD framework consists of p-adic balls of possibly varying radii $p^n$ assignable to points of space-time surface for which the preferred imbedding space coordinates are in the extension of rationals. At level of $M^8$ octonion property fixes preferred coordinates highly uniquely. At level of $H$ preferred coordinates come from symmetries.

   These points define a cognitive representation and inside p-adic points the solution of field equations is p-adic variant of real solution in some sense. At $M^8$ level the field equations would be algebraic equations and real-p-adic correspondence would be very straightforward. Cognitive representations would make sense at both $M^8$ level and $H$ level.

   **Remark:** In ordinary homology theory the decomposition of real manifold to simplexes reduces topology to homology theory. One forgets completely the interiors of simplices. Could the cognitive representations with points labelling the p-adic balls could be seen as analogous to decompositions to simplices. If so, homology would emerge as something number theoretically universal. The larger the extension of rationals, the more precise the resolution of homology would be. Therefore p-adic homology and cohomology as its Poincare dual would reduce to their real counterparts in the cognitive resolution used.

2. $M^8 - H$ correspondence would play a key role in mapping the associative regions of space-time varieties in $M^8$ to those in $H$. There are two kinds of regions. Associative regions in which polynomials defining the surfaces satisfy criticality conditions and non-associative regions. Associative regions represent external particles arriving in CDs and non-associative regions interaction regions within CDs.

3. In associative regions one has minimal surface dynamics (geodesic motion) at level of $H$ and coupling parameters disappear from the field equations in accordance with quantum criticality. The challenge is to prove that $M^8 - H$ correspondence is consistent with the minimal surface dynamics n $H$. The dynamics in these regions is determined in $M^8$ as zero loci of polynomials satisfying quantum criticality conditions guaranteeing associativity and is deterministic also in p-adic sectors since derivatives are not involved and pseudo constants depending on finite number of pinary digits and having vanishing derivative do not appear. $M^8 - H$ correspondence guarantees determinism in p-adic sectors also in $H$.

4. In non-associative regions $M^8 - H$ correspondence does not make sense since the tangent space of space-time variety cannot be labelled by $CP_2$ point and the real and p-adic $H$ counterparts of these regions would be constructed from boundary data and using field equations of a variational principle (sum of the volume term and Kähler action term), which in non-associative regions gives a dynamics completely analogous to that of charged particle in induced Kähler field. Now however the field characterizes extended particle itself.

   Boundary data would correspond to partonic 2-surfaces and string world sheets and possibly also the 3-surfaces at the ends of space-time surface at boundaries of CD and the light-like orbits of partonic 2-surfaces. At these surfaces the 4-D (!) tangent/normal space of space-time surface would be associative and could be mapped by $M^8 - H$ correspondence from $M^8$ to $H$ and give rise to boundary conditions.

   Due to the existence of p-adic pseudo-constants the p-adic dynamics determined by the action principle in non-associative regions inside CD would not be deterministic in p-adic sectors. The interpretation would be in terms of freedom of imagination. It could even happen that boundary values are consistent with the existence of space-time surface in p-adic sense but not with the existence of real space-time surfaces. Not all that can be imagined is realizable.

At the level of $M^8$ this vision seems to have no obvious problems. Inside each ball the same algebraic equations stating vanishing of $IM(P)$ (imaginary part of $P$ in quaternionic sense) hold true. At the level of $H$ one has second order partial differential equations, which also make sense also p-adically. Besides this one has infinite number of boundary conditions stating the vanishing of Noether charges assignable to sub-algebra super-symplectic algebra and its commutator with the entire algebra at the 3-surfaces at the boundaries of CD. Are these two descriptions really equivalent?

During writing I discovered an argument, which skeptic might see as an objection against $M^8 - H$ correspondence.

1. $M^8$ correspondence maps the space-time varieties in $M^8$ in non-local manner to those in $H = M^4 \times CP_2$. $CP_2$ coordinates characterize the tangent space of space-time variety in $M^8$ and this might produce technical problems. One can map the real variety to $H$ and find the points of the image variety satisfying the condition and demand that they define the "spine" of the p-adic surface in p-adic $H$.

2. The points in extensions of rationals in $H$ need not be images of those in $M^8$ but should this be the case? Is this really possible? $M^4$ point in $M^4 \times E^4$ would be mapped to $M^4 \subset M^4 \times CP_2$: this is trivial. 4-D associative tangent/normal space at $m$ containing preferred $M^2$ would be characterized by $CP_2$ coordinates: this is the essence of $M^8 - H$ correspondence. How could one guarantee that the $CP_2$ coordinates characterizing the tangent space are really in the extension of rationals considered? If not, then the points of cognitive representation in $H$ are not images of points of cognitive representation in $M^8$. Does this matter?

### 10.4.2 Are almost-perfectoids evolutionary winners in TGD Universe?

One could take perfectoids and perfectoid spaces as a mere technical tool of highly refiner mathematical cognition. Since cognition is basic aspect of TGD Universe, one could also ask perfectoids or more realistically, almost-perfectoids, could be an outcome of cognitive evolution in TGD Universe?

1. p-Adic algebraic varieties are defined as zero loci of polynomials. In the octonionic $M^8$ approach identifying space-time varieties as zero loci for RE or IM of octonionic polynomial (RE and IM in quaternionic sense) this allows to define p-adic variants of space-time surfaces as varieties obeying same polynomial equations as their real counterparts provided the coefficients of octonion polynomials obtainable from real polynomials by analytic continuation are in an extension of rationals inducing also extension of p-adic numbers.

   The points with coordinates in the extension of rationals common to real and p-adic variants of $M^8$ identified as cognitive representations are in key role. One can see p-adic space-time surfaces as collections of "monads" labelled by these points at which Cartesian product of 1-D p-adic balls in each coordinate degree. The radius of the p-adic ball can vary. Inside each ball the same polynomial equations are satisfied so that the monads indeed reflect other monads.

   Kind of algebraic hologram would be in question consisting of the monads. The points in extension allow to define ordinary real distance between monads. Only finite number of monads would be involved since the number of points in extension tends to be finite. As the extension increases, this number increases. Cognitive representations become more complex: evolution as increase of algebraic complexity takes place.

2. Finite-dimensionality for the allowed extensions of p-adic number fields is motivated by the idea about finiteness of cognition. Perfectoids are however infinite-dimensional. Number theoretical universality demands that on only extensions of p-adics induced by those of rationals are allowed and defined extension of the entire adele. Extensions should be therefore be induced by the same extension of rationals for all p-adic number fields.

   Perfectoids correspond to an extension of $Q_p$ apparently depending on $p$. This dependence is in conflict with number theoretical universality if real. This extension could be induced by corresponding extension of rationals for all p-adic number fields. For p-adic numbers $Q_q$ $q \neq p$ all equation $a^{p^n} = x$ reduces to $a^n = x \ mod \ p$ and this in term to $a^m = x \ mod \ p$, $m = n \ mod \ p$. Finite-dimensional extension is needed to have all roots of required kind! This extension is therefore finite-D for all $q \neq p$ and infinite-D for $p$.

3. What about infinite-dimensionality of the extension. The real world is rarely perfect and our thoughts about it even less so, and one could argue that we should be happy with almost-perfectoids! "Almost" would mean extension induced by powers of $p^{1/p^m}$ for large enough $m$, which is however not infinite. A finite-dimensional extension approaching perfectoid asymptotically is quite possible!

4. One could see the almost perfectoid as an outcome of evolution and perfectoid as the asymptotic states. High dimension of extension means that p-adic numbers and extension of rationals have large number of common numbers so that also cognitive representations contain a large number of common points. Maybe the p-adic number fields, which are evolutionary winners, have managed to evolve to especially high-dimensional almost-perfectoids! Note however that also the roots of $e$ can be considered as extensions of rationals since corresponding p-adic extensions are finite-dimensional. Similar evolution can be considered also now.

   To get some perspective mote that for large primes such as $M_{127} = 2^{127} - 1$ characterizing electron the lowest almost perfectoid would give powers of $M_{127}^{1/M_{127}} = (2^{127} - 1)^{1/(2^{127}-1)} \sim 1 + log(2)2^{-120}$! The lattice of points in extension is extremely dense near real unit. The density of of points in cognitive representations near this point would be huge. Note that the length scales comes as negative powers of two, which brings in mind p-adic length scale hypothesis [K18].

Although the octonionic formulation in terms of polynomials (or rational functions identifying space-time varieties as zeros or poles of $RE(P)$ or $IM(P)$ is attractive in its simplicity, one can also consider the possibility of allowing analytic functions of octonion coordinate obtained from real analytic functions. These define complex analytic functions with commutative imaginary unit used to complexify octonions. Could meromorphic functions real analytic at real axis having only zeros and poles be allowed? The condition that all p-adic variants of these functions exist simultaneously is non-trivial. Coefficients must be in the extension of rationals considered and convergence poses restrictions. For instance, $e^x$ converges only for $|x|_p < 1$. These functions might appear at the level of $H$.

# 11 Cognitive representations and algebraic geometry

The general vision about cognition is realized in terms of adelic physics as physics of sensory experience and cognition [L15, L14]. Rational points and their generalization as ratios of algebraic integers for geometric objects would define cognitive representations as points common for real and various p-adic variants of the space-time surface. The finite-dimensionality for induced p-adic extensions allows also extensions of rationals involving root of $e$ and its powers. This picture applies both at space-time level, imbedding space level, and at the level of space-time surfaces but basically reduces to imbedding space level. Hence counting of the (generalized) rational points for geometric objects would be determination of the cognitive representability.

## 11.1 Cognitive representations as sets of generalized rational points

The set of rational points depends on the coordinates chosen and one can argue that one must allow different cognitive representations and classify them according to their effectiveness.

How uniquely the $M^8_c$ coordinates can be chosen?

1. Polynomial property allows only linear transformations of the complex octonionic coordinates with coefficients which belong to the extension of rationals used. This poses extremely strong restrictions on the allowed representations once the quaternionic moduli defining a foliation of $M^4_0$ is chosen. One has therefore moduli space of quaternionic structures. One must also fix the time axis in $M^4$ assignable to real octonions.

2. One can also define several inequivalent octonionic structures and associate a moduli space to these. The moduli space for octonionic structures would correspond to the space of $M^4_0 \subset M^8$s as quaternionic planes containing fixed $M^2_0$. One can allow even allow Lorentz transforms mixing real and imaginary octonionic coordinates. It seems that these moduli are not relevant at the level of $H$.

What could the precise definition of rationality?

1. The coordinates of point are rational in the sense defined by the extension of rationals used. Suppose that one considers parametric representations of surfaces as maps from space-time surface to imbedding space. Suppose that one uses as space-time coordinates subset of preferred coordinates for imbedding space. These coordinate changes cannot be global and one space-time surface decomposes to regions in which different coordinates apply.

2. The coordinate transformations between over-lapping regions are birational in the sense that both the map and its inverse are in terms of rational functions. This makes the notion of rationality global.

3. When cognitively easy rational parametric representations are possible? For algebraic curves with $g \geq 2$ in $CP_2$ represented as zeros of polynomials this cannot the case since the number of rational points is finite for instance for $g \geq 2$ surfaces. There is simple explanation for this. Solving second complex coordinate in terms of the other one gives it as an algebraic function for $g \geq 2$: this must be the reason for the loss of dense set of rational points. For elliptic surfaces $y^2 - x^3 - ax - b = 0$ $y^2$ is however polynomial of $x$ and one can find rational parametric representation by taking $y^2$ as coordinate [L9]. For $g = 0$ one has linear equations

and one obtains dense set of rational points. For conic sections one can also have dense set of rational points but not always. Generalizing from this it would seem that the failure to have rational parametric representation is the basic reason for the loss of dense set of rational points.

This picture does not work for general surfaces but generalizes for algebraic varieties defined by several polynomial equations. The co-dimension $d_c = 1$ case is however unique and the most studied one since for several polynomial equations one encounters technical difficulties when the intersection of the surfaces defined by the $d_c$ polynomials need not be complete for $d_c > 1$. In the recent situation one has $d_c = 4$ but octonion analyticity could be powerful enough symmetry to solve the problem of non-complete intersections by eliminating them or providing a physical interpretation for them.

## 11.2 Cognitive representations assuming $M^8 - H$ duality

Many questions should be answered.

1. Can one generalize the results applying to algebraic varieties? Could the general vision about rational and potentially dense set of rational points generalize?. At $M^8$ side the description of space-time surfaces as algebraic varieties indeed conforms with this picture. Could one understand SH from the fact that real analyticity octonionic polynomials are determined by ordinary polynomial real coordinate completely? In information theoretic sense sense SH reduces to 1-D holography and the polynomial property makes the situation effectively discrete since finite number of points of real axis allows to determine the octonionic polynomial completely! It is a pity that one cannot measure octonionic polynomial directly!

2. Also the notion of Zariski dimension should make sense in TGD at $M^8$ side. Preferred extremals define the notion of closed set for given CD at $M^8$ side? It would indeed seem that one define Zariski topology at the level of $M^8_c$. Zariski topology would require 4-surfaces, string world sheets, or partonic 2-surfaces and even 1-D curves. This picture conforms with the recent view about TGD and resembles the M-theory picture, where one has branes. SH suggests that the analog of Zariski dimension of space-time surface reduces to that for strings world sheets and partonic 2-surfaces and that even these are analogous to 1-D curves by complex analyticity. Integrability of TGD and preferred extremal property would indeed suggest simplicity.

   $M^8 - H$ hypothesis suggests that these conjectures make sense also at $H$ side. String world sheets, partonic 2-surface, space-like 3-surfaces at the ends of space-time surface at boundaries of CD, and light-like 3-surfaces correspond to closed sets also at the level of WCW in the topology most natural for WCW.

3. Also the problems related to Minkowskian signature could be solved. String world sheets are problematic because of the Minkowskian signature. They however have the topology of disk plus handles suggesting immediately a vision about cognitive representations in terms of rational points. One can can complexify string world sheets and it seems possible to apply the results of algebraic geometry holding true in Euclidian signature. This would be analogous to the Wick rotation used in QFTs and also in twistor Grassmann approach.

4. What about algebraic geometrization of the twistor lift? How complex are twistor spaces of $M^4$, $CP_2$ and space-time surface? How can one generalize twistor lift to the level of $M^8$. $S^2$ bundle structure and the fact that $S^2$ allows a dense set of rational suggests that the complexity of twistor space is that of space-time surface itself so that the situation actually reduces to the level of space-time surfaces.

Suppose one accepts $M^8 - H$ duality requiring that the tangent space of space-time surface at given point $x$ contains $M^2(x)$ such that $M^2(x)$ define an integrable distribution giving rise to string world sheets and their orthogonal complements give rise to partonic 2-surfaces. This would give rise to a foliation of the space-time surface by string world sheets and partonic 2-surface conjecture on basis of the properties of extremals of Kähler action. As found these foliations could correspond to quaternion structures that is allowed choices of quaterionic coordinates.

Should one define cognitive representations at the level of $M^8$ or at the level of $M^4 \times CP_2$? Or both? For $M^8$ option the condition that space-time point belongs to an extension of rationals applies at the level of $M^8$ coordinates. For $M^4 \times CP_2$ option cognitive representations are at the level of $M^4$ and $CP_2$ parameterizing the points of $M^4$ and their tangent spaces. The formal study of partial differential equations alone does not help much in counting the number of rational points. One can define cognitive representation in very many manners, and some cognitive representation could be preferred only because they are more efficient than others. Hence both cognitive representations seems to be acceptable.

Some cognitive representations are more efficient than others. General coordinate invariance (GCI) at the level of cognition is broken. The precise determination of cognitive efficiency is a challenge in itself. For instance, the use of coordinates for which coordinate lines are orbits of subgroups of the symmetry group should be highly efficient. Only coordinate transformations mediated by bi-rational maps can take polynomial representations to polynomial representations. It might well be that only a rational (in generalized sense) sub-group $G_2$ of octonionic automorphisms is allowed. For rational surfaces allowing parametric representation in terms of polynomial functions the rational points form a dense set.

The cognitive resolution for a dense set of rational points is unrealistically high since cognitive representation would contain infinite number of points. Hence one must tighten the notion of cognitive representation. The rational points must contain a fermion. Fermions are indeed are identified as correlates for Boolean cognition [K1]. This would suggests a view in which cognitive representations are realized at the light-like orbits of partonic 2-surfaces at which Minkowskian associative and Euclidian co-associative space-time surfaces meet. The general wisdom is that rational points are localized to lower-dimensional sub-varieties (Bombieri-Lang conjecture): this conforms with the view that fermion lines reside at the orbits of partonic 2-surfaces.

## 11.3 Are the known extremals in $H$ easily cognitively representable?

Suppose that one takes TGD inspired adelic view about cognition seriously. If cognitive representations correspond to rational points for an extension of rationals, then the surfaces allowing large number of this kind of points are easily representable cognitively by adding fermions to these points. One could even speculate that mathematical cognition invents those geometric objects, which are easily cognitively representable and thus have a large number of rational points.

### 11.3.1 Could the known extremals of twistor lift be cognitively easy?

Also TGD is outcome of mathematical cognition. Could the known extremals of the twistor lift of Kähler action be cognitively easy? This is suggested by the fact that even such a pariah class theoretician as I am have managed to discover then! Positive answer could be seen as support for the proposed description of cognition!

1. If one believes in $M^8 - H$ duality and the proposed identification of associative and co-associative space-time surfaces in terms of algebraic surfaces in octonionic space $M^8_c$, the generalization of the results of algebraic geometry should give overall view about the cognitive representations at the level of $M^8$. In particular, surfaces allowing rational parametric representation (polynomials would have rational coefficients) would allow dense set or rational points since the images of rational points are rational. Rationals are understood here as ratios of algebraic integers in extension of rationals.

2. Also for $H$ the existence of parameter representation using preferred $H$-coordinates and rational functions with rational coefficients implies that rational points are dense. If $M^8 - H$ correspondence maps the parametric representations in terms of rational functions to similar representations, dense set of rational points is preserved in the correspondence. There is however no obvious reason why $M^8 - H$ duality should have this nice property.

   One can even play with the idea that the surfaces, which are cognitively difficult at the $M^8$ side, might be cognitively easy at $H$-side or vice versa. Of course, if the explicit representation as algebraic functions makes sense at $M^8$ side, this side looks cognitively ridiculously easy as compared to $H$ side. The preferred extremal property and SH can however change the situation.

3. At $M^8$ side and for a given point of $M^4$ there are several points of $E^4$ (or vice versa) if the degree of the polynomial is larger than $n = 1$ so that for the image of the surface $H$ there are several $CP_2$ points for a given point of $M^4$ (or vice versa) depending on the choice of coordinates. This is what the notion of the many-sheeted space-time predicts.

4. The equations for the surface at $H$ side are obtained by a composite map assigning first to the coordinates of $X^4 \subset M^8$ point of $M^4 \times E^4$, and then assigning to the points of $X^4 \subset M^8$ $CP_2$ coordinates of the tangent space of the point. At this step the slightly non-local tangent space information is fed in and the surfaces in $M^4 \times CP_2$ cannot be given by zeros of polynomials. The indeed satisfy instead of algebraic equations partial differential equations given by the Kähler action for the twistor lift TGD. Algebraic equations instead of partial differential equations suggests that the $M^8$ representation is much simpler than $H$-representation. On the other hand, reduction to algebraic equations at $M^8$ side could have interpretation in terms of the conjectured complete integrability of TGD [K22, K28].

### 11.3.2 Testing the idea about self-reference

In any case, it is possible to test the idea about self-reference by looking whether the known extremals in $H$ are cognitively easy and even have a dense set of rational points in natural coordinates. Here I will consider the situation at the level of $M^4 \times CP_2$. It was already found that the known extremals can have inverse images in $M^8$.

1. Canonically imbedded $M^4$ with linear coordinates and constant $CP_2$ coordinates rational is the simple example about preferred extremal and it seems that TGD based cosmology at microscopic relies on these extremals. In this case it is obvious that one has a dense set of rational points at both sides. Could this somehow relate to the fact that physics as physics $M^4$ was discovered before general relativity?

   Canonically imbedded $M^4$ corresponds to a first order octonionic polynomial for which imaginary part is put to constant so that tangent space is same everywhere and corresponds to a constant $CP_2$ coordinate.

2. $CP_2$ type extremals have 4-D $CP_2$ projection and light-like geodesic line of $M^4$ as $M^4$ projection. One can choose the time parameter as a function of $CP_2$ coordinates in infinitely many manners. Clearly the rational points are dense in any $CP_2$ coordinates.

3. Massless extremals (MEs) are given as zeros of arbitrary functions of $CP_2$ coordinates and 2 $M^4$ coordinates representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant. In the general case light-like direction would define tangent space of string world sheet giving rise also to a distribution of ortogonal polarization planes. This is consistent with the general properties of the $M^8$ representation and corresponds to the decomposition of quaternionic tangent plane to complex plane and its complement. One can ask whether one should allow only polynomials with rational coefficients as octonionic polynomials.

4. String like objects $X^2 \times Y^2$ with $X^2 \subset M^4$ a minimal surface and $Y^2$ complex or Lagrangian surface of $CP_2$ are also basic extremals and their deformations in $M^4$ directions are expected to give rise to magnetic flux tubes.

   If $Y^2$ is complex surface with genus $g = 0$ rational points are dense. Also for $g = 1$ one obtains a dense set of rational points in some extension of rationals. For elliptic curves one has lattice of rational points. What happens for Lagrangian surfaces $Y^2$? In this case one does not have complex curves but real co-dimension 2 surfaces. There is no obvious objection why these surfaces would not be possible.

5. What about string world sheets? If the string world is static $M^2 \subset M^4$ one has a dense set of rational points. One however expects something more complex. If the string world sheet is rational map $M^2$ to its orthogonal complement $E^2$ one has rational surface. For rotating strings this does not make sense except for certain period of time. If the choice of the quaternion structure corresponds to a choice of minimal surface in $M^4$ as integrable

distribution for $M^2(x)$, the coordinates associated with the Hamilton-Jacobi structure could make the situation simple.

If one restricts the consideration the intersections of partonic 2-surfaces and string world sheets at two boundaries of CD the situation simplifies and the question is only about the rationality of the $M^4$ coordinates at rational points of $Y^2 \subset CP_2$. This would simplify the situation enormously and might even allow to use existing knowledge.

6. The slicing of of space-time surfaces by string world sheets and partonic 2-surfaces required by Hamilton-Jacobi structure could be seen as a fibering analogous to that possessed by elliptic surfaces. This suggest that $M^8$ counterparts of spacetime surfaces are not of general type in Kodaira classification and that the number of rational points can be large. If the existence of Hamilton-Jacobi structure does not allow handles, this factor would be cognitively simple. This would however suggests that fermion number is not localized at the ends of strings only - as assumed in the construction of scattering amplitudes inspired by twistor Grassmann approach [K26] - but also to the interior of the light-like curves inside string world sheets.

## 11.4 Twistor lift and cognitive representations

What about twistor lift of TGD replacing space-time surfaces with their twistor spaces. Consider first $M^8$ side.

1. At $M^8$ side $S^2$ seems to introduce nothing new. One might expect that the situation does not change at $H$-side since space-time surfaces are obtained essentially by dimensional reduction and the possible problem relates to the choice of base space as section of is twistor bundle and the imbedding of space-time as base space could have singularities at the boundary of Euclidian and Minkowskian space-time regions as discussed in [L9].

   At the side of $M^8$ the proposed induction of twistor structure is just a projection of the twistor sphere $S^6$ to its geodesic sphere and one has 4-D moduli space for geodesic spheres $S^2 \subset S^6$. If one interprets the choice of $S^2 \subset S^6$ as as a section in the moduli space, the moduli of $S^2$ can depend on the point of space-time surface. Note that there are is also a position dependent choice of preferred point of $S^2$ representing Kähler form, and this choice is good candidate for giving rise to Hamilton-Jacobi structures with position dependent $M^2$.

2. The notion of Kodaira dimension is defined also for co-dimension 4 algebraic varieties in $M^8_c$. The cognitively easiest spacetime surfaces would allow rational parametric representation with complex coordinates serving as parameters. If this is not possible, one has algebraic functions, which makes the situation much more complex so that the number of rational points would be small.

3. For some complex enough extensions of rationals the set of rational points can be dense. $g \geq 2$ genera are basic example and one expects also in more general case that polynomials involving powers larger than $n = 4$ make the situation problematic. The condition that real or imaginary part of real analytic octonionic polynomial is in question is a strong symmetry expected to faciliate cognitive representability.

4. The general intuitive wisdom from algebraic geometry is that the rational points are dense only in lower-dimensional sub-varieties (Bombieri-Lang and Vojta conjectures mentioned in the first section). The general vision inspired by SH and the proposal for the construction of twistor amplitudes indeed is that the algebraic points (rational in generalized sense) defining cognitive representations are associated with the intersections of string world sheets and partonic 2-surfaces to which fermions are assigned. This would suggest that partonic 2-surfaces and string world sheets contain the cognitive representation, which under additional conditions can contain very many points.

5. An interesting question concerns the $M^8$ counterparts of partonic 2-surfaces as space-time regions with Minkowskian and Euclidian signature. The partonic orbits representing the boundaries between these regions should be mapped to each other by $M^8 - H$ duality. This conforms with the fact that induced metric must have degenerate signature $(0, -1, -1, -1)$

at partonic orbits. Can one assume that the topologies of partonic 2-surfaces at two sides are identical? Consider partonic 2-surface of genus $g$ in $M^4 \times CP_2$ - say at the boundary of CD. It should be inverse image of a 2-surface in $M^4 \times E^4$ such that the tangent space of this surface labelled by $CP_2$ coordinates is mapped to a 2-surface in $M^4 \times CP_2$. If the inverse of $M^8 - H$ correspondence is continuous one expects that $g$ is preserved.

Consider next the $H$-side. There is a conjecture that for Cartesian product the Kodaira dimension is sum $d_K = \sum_i d_{K,i}$ of the Kodaira dimensions for factors. Suppose that $CP_1$ fiber as surface in the 12-D twistor bundle $T(M^4) \times T(CP_2)$ has Kodaira dimension $d_K(CP_1) = -\infty$ (it is expected to be rational surface) then the fact that the bundle decomposes to Cartesian product locally and rational points are pairs of rational points in the factors, is indeed consistent with the proposal. $S^2$ would give dense set of rational points in $S^2$ and the bundle would have infinite number of rational points.

In TGD context, it is however space-time surface which matters. Space-time surface as section of the bundle would not however have a dense set of points in the general case and the relevant Kodaira dimension be $d_K = d_K(X^4)$. One can of course ask whether the space-time surface as an algebraic section (not many of them) of the twistor bundle could chosen to be cognitively simple.

## 11.5 What does cognitive representability really mean?

The following considerations reflect the ideas inspired by Face Book debate with Santeri Satama (SS) relating to the notion of number and the notion of cognitive representation.

SS wants to accept only those numbers that are constructible, and SS mentioned the notion of demonstrability due to Gödel. According to my impression demonstrability means that number can be constructed by a finite algorithm or at least that the information needed to construct the number can be constructed by a finite algorithm although the construction itself would not be possible as digit sequence in finite time. If the constructibility condition is taken to extreme, one is left only with rationals.

As a physicists, I cannot consider starting to do physics armed only with rationals: for instance, continuous symmetries and the notion of Riemann manifold would be lost. My basic view is that we should identify the limitations of cognitive representability as limitations for what can exist. I talked about cognitive representability of numbers central in the adelic physics approach to TGD. Not all real numbers are cognitively representable and need not be so.

Numbers in the extensions of rationals would be cognitively representable as points with coordinates in an extension of rationals. The coordinates themselves are highly unique in the octonionic approach to TGD and different coordinates choices for complexified octonionic $M^8$ are related by transformations changing the moduli of the octonion structure. Hence one avoids problems with general coordinate invariance). Not only algebraic extensions of rationals are allowed. Neper number $e$ is an exceptional transcendental in that $e^p$ is p-adic number and finite-D extensions of p-adic numbers by powers for root of $e$ are possible.

My own basic interest is to find a deeper intuitive justification for why algebraic numbers shoud be cognitively representable. The naive view about cognitive representability is that the number can be produced in a finite number of steps using an algorithm. This would leave only rationals under consideration and would mean intellectual time travel to ancient Greece.

Situation changes if one requires that only the information about the construction of number can be produced in a finite number of steps using an algorithm. This would replace construction with the recipe for construction and lead to a higher abstraction level. The concrete construction itself need not be possible in a finite time as bit sequence but could be possible physically ($\sqrt{2}$ as a diagonal of unit square, one can of course wonder where to buy ideal unit squares). Both number theory and geometry would be needed.

Stern-Brocot tree associated with partial fractions indeed allows to identify rationals as finite paths connecting the root of S-B tree to the rational in question. Algebraic numbers can be identified as infinite periodic paths so that finite amount of information specifies the path. Transcendental numbers would correspond to infinite non-periodic paths. A very close analogy with chaos theory suggests itself.

### 11.5.1 Demonstrability viz. cognitive representability

SS talked about demonstrable numbers. According to Gödel demonstrable number would be representable by a formula $G$, which is provable in some axiom system. I understand this that $G$ would give a recipe for constructing that number. In computer programs this can even mean infinite loop, which is easy to write but impossible to realize in practice. Here comes the possibility that demonstrability does not mean constructibility in finite number of steps but only a finite recipe for this.

The requirement that all numbers are demonstrable looks strange to me. I would talk about cognitive representability and reals and p-adic number fields emerge unavoidably as prerequisites for this notion: cognitive representation must be about something in order to be a representation.

About precise construction of reals or something bigger - such as surreals - containing them, there are many views and I am not mathematician enough to take strong stance here. Note however that if one accepts surreals as being demonstrable (I do not really understand what this could mean) one also accept reals as such. These delicacies are not very interesting for the formulation of physics as it is now.

The algorithm defining $G$ defines a proof. But what does proof mean? Proof in mathematical sense would reduce in TGD framework be a purely cognitive act and assignable to the p-adic sectors of adele. Mathematicians however tend to forget that for physicist the demonstration is also experimental. Physicist does not believe unless he sees: sensory perception is needed. Experimental proofs are what physicists want. The existence of $\sqrt{2}$ as a diagonal of unit square is experimentally demonstrable in the sense of being cognitively representable but not deducible from the axioms for rational numbers. As a physicist I cannot but accept both sensory and cognitive aspects of existence.

Instead of demonstrable numbers I prefer to talk about cognitively representable numbers.

1. All numbers are cognizable (p-adic) or sensorily perceivable (real). These must form continua if one wants to avoid problems in the construction of physical theories, where continuous symmetries are in a key role.

   Some numbers but not all are also *cognitively representable* that is being in the intersection reals and p-adics - that is in extension of rationals if one allows extensions of p-adics induced by extensions of rationals. This generalizes to intersection of space-time surfaces with real/p-adic coordinates, which are highly unique linear coordinates at octonionic level so that objections relating to a loss of general coordinate invariance are circumvented. General coordinate transformations reduce to automorphisms of octonions.

   The relationship to the axiom of choice is interesting. Should axiom of choice be restricted to the points of complexified octonions with coordinates in extensions of rationals? Only points in the extensions could be selected and this selection process would be physical in the sense that fermions providing realization of quantum Boolean algebra would reside at these points [K1]. In preferred octonionic coordinates the $M^8$ coordinates of these points would be in given extension of rationals. At the limit of algebraic numbers these points would form a dense set of reals.

   **Remark:** The spinor structure of "world of classical worlds" (WCW) gives rise to WCW spinors as fermionic Fock states at given 3-surface. In ZEO many-fermion states have interpretation in terms of superpositions of pairs of Boolean statements $A \to B$ with $A$ and $B$ represented as many-fermion states at the ends of space-time surface located at the opposite light-like boundaries of causal diamond (CD). One could say that quantum Boolean logic emerges as square root of Kähler geometry of WCW.

   At partonic 2-surfaces these special points correspond to points at which fermions can be localized so that the representation is physical. Universe itself would come in rescue to make representability possible. One would not anymore try to construct mathematics and physics as distinct independent disciplines.

   Even observer as conscious entity is necessarily brought into both mathematics and physics. TGD Universe as a spinor field in WCW is re-created state function reduction by reduction and evolves: evolution for given CD corresponds to the increase of the size of extension of rationals in statistical sense. Hence also mathematics with fixed axioms is replaced with a q dynamical structure adding to itself new axioms discovery by discovery [L16, L15].

2. Rationals as cognitively representable numbers conforms with naive intuition. One can however criticize the assumption that also algebraic numbers are such. Consider $\sqrt{2}$: one can simply define it as length of diagonal of unit square and this gives a meter stick of length $\sqrt{2}$: one can represent any algebraic number of form $m + n\sqrt{2}$ by using meter stricks with length of 1 and $\sqrt{2}$. Cognitive representation is also sensory representation and would bring in additional manner to represent numbers.

   Note that algebraic numbers in $n$-dimensional extension are points of $n$-dimensional space and their cognitive representations as points on real axis obtained by using the meter sticks assignable to the algebraic numbers defining base vectors. This should generalize to the roots arbitrary polynomials with rational or even algebraic coefficients. Essentially projection form $n$-D extension to 1-D real line is in question. This kind of projection might be important in number theoretical dynamics. For instance, quasi-periodic quasi-crystals are obtained from higher-D periodic crystals as projections.

   $n$-D algebraic extensions of p-adics induced by those of rationals might also related to our ability to imagine higher-dimensional spaces.

3. In TGD Universe cognitive representability would emerge from fundamental physics. Extensions of rationals define a hierarchy of adeles and octonionic surfaces are defined as zero loci for real or imaginary parts (in quaternionic sense) of polynomials of real argument with coefficients in extension continued to octonionic polynomials [L11]. The zeros of real polynomial have a direct physical interpretation and would represent algebraic numbers physically. They would give the temporal positions of partonic 2-surfaces representing particles at light-like boundary of CD.

4. Note that all calculations with algebraic numbers can be done without using approximations for the genuinely algebraic numbers defining the basis for the extension. This actually simplifies enormously the calculation and one avoids accumulating errors. Only at the end one represents the algebraic units concretely and is forced to use rational approximation unless one uses above kind of cognitive representation.

For these reasons I do not feel any need to get rid of algebraics or even transcendentals. Sensory aspects of experience require reals and cognitive aspects of experience require p-adic numbers fields and one ends up with adelic physics. Cognitive representations are in the intersection of reality and various p-adicities, something expressible as formulas and concrete physical realizations or at least finite recipes for them.

### 11.5.2 What the cognitive representability of algebraic numbers could mean?

Algebraic numbers should be in some sense simple in order to be cognitively representable.

1. For rationals representation as partial fractions produces the rational number by using a finite number of steps. One starts from the top of Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq`) and moves to right or left at each step and ends up to the rational number appearing only once in S-B tree.

2. Algebraic numbers cannot be produced in a finite number of steps. During the discussion I however realized that one can produce the information needed to construct the algebraic number in a finite number of steps. One steps to a new level of abstraction by replacing the object with the information allowing to construct the object using infinite number of steps but repeating the same sub-algorithm with finite number of steps: infinite loop would be in question.

   Similar abstraction takes place as one makes a step from the level of space-time surface to the level of WCW. Space-time surface with a continuum of points is represented by a finite number of WCW coordinates, in the octonionic representation of space-time surface by the coefficients of polynomial of finite degree belonging to an extension of rationals [L11]. Criticality conditions pose additional conditions on the coefficients. Finite number of algebraic points at space-time surface determines the entire space-time surface under these conditions! Simple names for complex things replacing the complex things is the essence of cognition!

3. The interpretation for expansions of numbers in given base suggests an analog with complexity theory and symbolic dynamics associated with division. For cognitively representable numbers the information about this dynamics should be coded by an algorithm with finite steps. Periodic orbit or fixed point orbit would be the dynamical analog for simplicity. Non-periodic orbit would correspond to complexity and possibly also chaos.

These ideas led to two approaches in attempt to understand the cognitive representability of algebraic numbers.

*1. Generalized rationals in extensions of rationals as periodic orbits for the dynamics of division*

The first approach allows to represent ratios of algebraic integers for given extension using periodic expansion in the base so that a finite amount of information is needed to code the number if one accepts the numbers defining the basis of the algebraic extension as given.

1. Rationals allow periodic expansion with respect to any base. For p-adic numbers the base is naturally prime. Therefore the information about rational is finite. One can see the expansion as a periodic orbit in dynamics determining the expansion by division $m/n$ in given base. Periodicity follows from the fact that the output of the division algorithm for a given digit has only a finite number of outcomes so that the process begins to repeat itself sooner or later.

2. This generalizes to generalized rationals in given extension of rationals defined as ratios of algebraic integers. One can reduce the division to the construction of the expansion of ordinary rational identified as number theoretic norm $|N|$ of the denominator in the extension of rationals considered.

   The norm $|N|$ of $N$ is the determinant $|N| = det(N)$ for the linear map of extension induced by multiplication with $N$. $det(N)$ is ordinary (possibly p-adic) integer. This is achieved by multiplying $1/N$ by $n-1$ conjugates of $N$ both in numerator and denominator so that one obtains product of $n-1$ conjugates in the numerator and $det(N)$ in the denominator. The computation of $1/N$ as series in the base used reduces to that in the case of rationals.

3. One has now periodic orbits in $n$-dimensional space defined by algebraic extensions which for ordinary rationals reduced to periodic orbits in 1-D space. This supports the interpretation of numbers as orbits of number theoretic dynamics determining the next digit of the generalized rational for given base. This picture also suggests that transcendentals correspond to non-periodic orbits. Some transcendentals could still allow a finite algorithm: in this case the dynamics would be still deterministic. Some transcendentals would be chaotic.

4. Given expansion of algebraic number is same for all extensions of rationals containing the extension in question and the ultimate extension corresponds to algebraic numbers.

The problem of this approach is that the algebraic numbers defining the extension do not have representation and must be accepted as irreducibles.

*2. Algebraic numbers as infinite periodic orbits in the dynamics of partial fractions*

Second approach is based on partial fractions and Stern-Brocot tree (see `http://tinyurl.com/yb6ldekq`, see also `http://tinyurl.com/yc6hhboo`) and indeed allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. I had managed to not become aware of this possibility and am grateful for SS for mentioning the representation of algebraics in terms of S-B tree.

1. The definition S-B tree is simple: if $m/n$ and $m'/n'$ are any neighboring rationals at given level in the tree, one adds $(m+m')/(n+n')$ between them and obtains in this manner the

next level in the tree. By starting from (0/1) and (1/0) as representations of zero and $\infty$ one obtains (0/1)(1/1)(1/0) as the next level. One can continue in this manner ad infinitum. The nodes of S-B tree represent rational points and it can be shown that given rational appears only once in the tree.

Given rational can be represented as a finite path beginning from 1/1 at the top of tree consisting of left moves $L$ and right moves $R$ and ending to the rational which appears only once in S-B tree. Rational can be thus constructured by a sequences $R^{a_0}L^{a_1}L^{a_2}$.... characterized by the sequence $a_0; a_1, a_2$.... For instance, $4/11 = 0+1/(2+x)$ , $x = 1/(1+1/3)$ corresponds to $R^0L^2R^1L^{3-1}$ labelled by $0; 2, 1, 3$.

2. Algebraic numbers correspond to infinite but periodic paths in S-B tree in the sense that some sequence of $L$:s and $R$:s characterized by sequences of non-negative integers starts to repeat itself. Periodicity means that the information needed to construct the number is finite.

   The actual construction as a digit sequence representing algebraic number requires infinite amount of time. In TGD framework octonionic physics would come in rescue and construct algebraic numbers as roots of polynomials having concrete interpretations as coordinate values assignable to fermions at partonic 2-surfaces.

3. Transcendentals would correspond to non-periodic infinite sequences of $L$:s and $R$:s. This does not exclude the possibility that these sequences are expressible in terms of some rule involving finite number of steps so that the amount of information would be also now finite. Information about number would be replaced by information about rule.

   This picture conforms with the ideas about transition to chaos. Rationals have finite paths. A possible dynamical analog is particle coming at rest due to the dissipation. Algebraic numbers would correspond to periodic orbits possible in presence of dissipation if there is external feed of energy. They would correspond to dynamical self-organization patterns.

   **Remark**: If one interprets the situation in terms of conservative dynamics, rationals would correspond to potential minima and algebraic numbers closed orbits around them.

   The assignment of period doubling and p-pling to this dynamics as the dimension of extension increases is an attractive idea. One would expect that the complexity of periodic orbits increases as the degree of the defining irreducible polynomial increases. Algebraic numbers as maximal extension of rationals possibly also containing extension containing all rational roots of $e$ and transcendentals would correspond to chaos.

   Transcendentals would correspond to non-periodic orbits. These orbits need not be always chaotic in the sense of being non-predictable. For instance, Neper number $e$ can be said to be p-adically algebraic number ($e^p$ is p-adic integer albeit infinite as real integer). Does the sequence of $L$:s and $R$:s allow a formula for the powers of $L$ and $R$ in this case?

4. TGD should be an integrable theory. This suggests that scattering amplitudes involve only cognitive representations as number theoretic vision indeed strongly suggests [L11]. Cognitively representable numbers would correspond to the integrable sub-dynamics [L18]. Also in chaotic systems both periodic and chaotic orbits are present. Complexity theory for characterization of real numbers exists. The basic idea is that complexity is measured by the length of the shortest program needed to code the bit sequences coding for the number.

### 11.5.3 Surreals and ZEO

The following comment is not directly related to cognitive representability but since it emerged during discussion, I will include it. SS favors surreals (see `http://tinyurl.com/86jatas`) as ultimate number field containing reals as sub-field. I must admit that my knowledge and understanding of surreals is rather fragmentary.

I am agnostic in these issues and see no conflict between TGD view about numbers and surreals. Personally I however like very much infinite primes, integers, and rationals over surreals since they allow infinite numbers to have number theoretical anatomy [K13]. A further reason is that the construction of infinite primes resembles structurally repeated second quantization of the arithmetic number field theory and could have direct space-time correlate at the level of many-sheeted space-time. One ends up also to a generalization of real number. Infinity can be seen as something related to real norm: everything is finite with respect to various p-adic norms.

Infinite rationals with unit real norm and various p-adic norms bring in infinitely complex number theoretic anatomy, which could be even able to represent even the huge WCW and the space of WCW spinor fields. One could speak of number theoretical holography or algebraic Brahman=Atman principle. One would have just complexified octonions with infinitely richly structure points.

Surreals are represented in terms of pairs of sets. One starts the recursive construction from empty set identified as 0. The definition says that the pairs (.|.) of sets defining surreals $x$ and $y$ satisfy $x \leq y$ if the left hand part of $x$ as set is to left from the pair defining $y$ and the right hand part of $y$ is to the right from the pair defining $x$. This does not imply that one has always $x < y$, $y < x$ or $x = y$ as for reals.

What is interesting that the pair of sets defining surreal $x$ is analogous to a pair of states at boundaries of CD defining zero energy state. Is there a connection with zero energy ontology (ZEO)? One could perhaps say at the level of CD - forgetting everything related to zero energy states - following. The number represented by $CD_1$ - say represented as the distance between its tip - is smaller than than the number represented by $CD_2$, if $CD_1$ is inside $CD_2$. This conforms with the left and righ rule if left and right correspond to the opposite boundaries of CD. A more detailed definition would presumably say that $CD_1$ can be moved so that it is inside $CD_2$.

What makes this also interesting is that CD is the geometric correlate for self, conscious entity, also mathematical mental image about number.

# 12 Galois groups and genes

In an article discussing a TGD inspired model for possible variations of $G_{eff}$ [L19], I ended up with an old idea that subgroups of Galois group could be analogous to conserved genes in that they could be conserved in number theoretic evolution. In small variations such as above variation Galois subgroups as genes would change only a little bit. For instance, the dimension of Galois subgroup would change.

The analogy between subgoups of Galois groups and genes goes also in other direction. I have proposed long time ago that genes (or maybe even DNA codons) could be labelled by $h_{eff}/h = n$ . This would mean that genes (or even codons) are labelled by a Galois group of Galois extension (see `http://tinyurl.com/zu5ey96`) of rationals with dimension $n$ defining the number of sheets of space-time surface as covering space. This could give a concrete dynamical and geometric meaning for the notion of gene and it might be possible some day to understand why given gene correlates with particular function. This is of course one of the big problems of biology.

## 12.1 Could DNA sequence define an inclusion hierarchy of Galois extensions?

One should have some kind of procedure giving rise to hierarchies of Galois groups assignable to genes. One would also like to assign to letter, codon and gene and extension of rationals and its Galois group. The natural starting point would be a sequence of so called intermediate Galois extensions $E^H$ leading from rationals or some extension $K$ of rationals to the final extension $E$. Galois extension has the property that if a polynomial with coefficients in $K$ has single root in $E$, also other roots are in $E$ meaning that the polynomial with coefficients $K$ factorizes into a product of linear polynomials. For Galois extensions the defining polynomials are irreducible so that they do not reduce to a product of polynomials.

Any sub-group $H \subset Gal(E/K))$ leaves the intermediate extension $E^H$ invariant in element-wise manner as a sub-field of $E$ (see `http://tinyurl.com/y958drcy`). Any subgroup $H \subset Gal(E/K))$ defines an intermediate extension $E^H$ and subgroup $H_1 \subset H_2 \subset ...$ define a hierarchy of extensions $E^{H_1} > E^{H_2} > E^{H_3}...$ with decreasing dimension. The subgroups $H$ are normal - in other words $Gal(E)$ leaves them invariant and $Gal(E)/H$ is group. The order $|H|$ is the dimension of $E$ as an extension of $E^H$. This is a highly non-trivial piece of information. The dimension of $E$ factorizes to a product $\prod_i |H_i|$ of dimensions for a sequence of groups $H_i$.

Could a sequence of DNA letters/codons somehow define a sequence of extensions? Could one assign to a given letter/codon a definite group $H_i$ so that a sequence of letters/codons would correspond a product of some kind for these groups or should one be satisfied only with the assignment of a standard kind of extension to a letter/codon?

Irreducible polynomials define Galois extensions and one should understand what happens to an irreducible polynomial of an extension $E^H$ in a further extension to $E$. The degree of $E^H$ increases by a factor, which is dimension of $E/E^H$ and also the dimension of $H$. Is there a standard manner to construct irreducible extensions of this kind?

1. What comes into mathematically uneducated mind of physicist is the functional decomposition $P^{m+n}(x) = P^m(P^n(x))$ of polynomials assignable to sub-units (letters/codons/genes) with coefficients in $K$ for a algebraic counterpart for the product of sub-units. $P^m(P^n(x))$ would be a polynomial of degree $n+m$ in $K$ and polynomial of degree $m$ in $E^H$ and one could assign to a given gene a fixed polynomial obtained as an iterated function composition. Intuitively it seems clear that in the generic case $P^m(P^n(x))$ does not decompose to a product of lower order polynomials. One could use also polynomials assignable to codons or letters as basic units. Also polynomials of genes could be fused in the same manner.

2. If this indeed gives a Galois extension, the dimension $m$ of the intermediate extension should be same as the order of its Galois group. Composition would be non-commutative but associative as the physical picture demands. The longer the gene, the higher the algebraic complexity would be. Could functional decomposition define the rule for who extensions and Galois groups correspond to genes? Very naively, functional decomposition in mathematical sense would correspond to composition of functions in biological sense.

3. This picture would conform with $M^8 - M^4 \times CP_2$ correspondence [L11] in which the construction of space-time surface at level of $M^8$ reduces to the construction of zero loci of polynomials of octonions, with rational coefficients. DNA letters, codons, and genes would correspond to polynomials of this kind.

## 12.2 Could one say anything about the Galois groups of DNA letters?

A fascinating possibility is that this picture could allow to say something non-trivial about the Galois groups of DNA letters.

1. Since $n = h_{eff}/h$ serves as a kind of quantum IQ, and since molecular structures consisting of large number of particles are very complex, one could argue that $n$ for DNA or its dark variant realized as dark proton sequences can be rather large and depend on the evolutionary level of organism and even the type of cell (neuron viz. soma cell). On the other, hand one could argue that in some sense DNA, which is often thought as information processor, could be analogous to an integrable quantum field theory and be solvable in some sense. Notice also that one can start from a background defined by given extension $K$ of rationals and consider polynomials with coefficients in $K$. Under some conditions situation could be like that for rationals.

2. The simplest guess would be that the 4 DNA letters correspond to 4 non-trivial finite groups with smaller possible orders: the cyclic groups $Z_2, Z_3$ with orders 2 and 3 plus 2 finite groups of order 4 (see the table of finite groups in `http://tinyurl.com/j8d5uyh`). The groups of order 4 are cyclic group $Z_4 = Z_2 \times Z_2$ and Klein group $Z_2 \oplus Z_2$ acting as a symmetry group of rectangle that is not square - its elements have square equal to unit element. All these 4 groups are Abelian. Polynomial equations of degree not larger than 4 can be solved exactly in the sense that one can write their roots in terms of radicals.

3. Could there exist some kind of connection between the number 4 of DNA letters and 4 polynomials of degree less than 5 for whose roots one an write closed expressions in terms of radicals as Galois found? Could it be that the polynomials obtained by a a repeated functional composition of the polynomials of DNA letters have also this solvability property?

   This could be the case! Galois theory states that the roots of polynomial are solvable by radicals if and only if the Galois group is solvable meaning that it can be constructed from abelian groups using Abelian extensions (see https://en.wikipedia.org/wiki/Solvable_group).

Solvability translates to a statement that the group allows so called sub-normal series $1 < G_0 < G_1 ... < G_k$ such that $G_{j-1}$ is normal subgroup of $G_j$ and $G_j/G_{j-1}$ is an abelian group. An equivalent condition is that the derived series $G \triangleright G^{(1)} \triangleright G^{(2)} \triangleright ...$ in which $j+1$:th group is commutator group of $G_j$ ends to trivial group. If one constructs the iterated polynomials by using only the 4 polynomials with Abelian Galois groups, the intuition of physicist suggests that the solvability condition is guaranteed! Wikipedia article also informs that for finite groups solvable group is a group whose composition series has only factors which are cyclic groups of prime order.

Abelian groups are trivially solvable, nilpotent groups are solvable, p-groups (having order, which is power prime) are solvable and all finite p-groups are nilpotent. Every group with order less than 60 elements is solvable. Fourth order polynomials can have at most $S_4$ with 24 elements as Galois groups and are thus solvable. Fifth order polynomials can have the smallest non-solvable group, which is alternating group $A_5$ with 60 elements as Galois group and in this case are not solvable. $S_n$ is not solvable for $n > 4$ and by the finding that $S_n$ as Galois group is favored by its special properties (see https://arxiv.org/pdf/1511.06446.pdf).

$A_5$ acts as the group icosahedral orientation preserving isometries (rotations). Icosahedron and tetrahedron glued to it along one triangular face play a key role in TGD inspired model of bio-harmony and of genetic code [L1, L20]. The gluing of tetrahedron increases the number of codons from 60 to 64. The gluing of tetrahedron to icosahedron also reduces the order of isometry group to the rotations leaving the common face fixed and makes it solvable: could this explain why the ugly looking gluing of tetrahedron to icosahedron is needed? Could the smallest solvable groups and smallest non-solvable group be crucial for understanding the number theory of the genetic code.

An interesting question inspired by $M^8 - H$-duality [L11] is whether the solvability could be posed on octonionic polynomials as a condition guaranteeing that TGD is integrable theory in number theoretical sense or perhaps following from the conditions posed on the octonionic polynomials. Space-time surfaces in $M^8$ would correspond to zero loci of real/imaginary parts (in quaternionic sense) for octonionic polynomials obtained from rational polynomials by analytic continuation. Could solvability relate to the condition guaranteeing $M^8$ duality boiling down to the condition that the tangent spaces of space-time surface are labelled by points of $CP_2$. This requires that tangent or normal space is associative (quaternionic) and that it contains fixed complex sub-space of octonions or perhaps more generally, there exists an integrable distribution of complex subspaces of octonions defining an analog of string world sheet.

What could the interpretation for the events in which the dimension of the extension of rationals increases? Galois extension is extensions of an extension with relative Galois group $Gal(rel) = Gal(new)/Gal(old)$. Here $Gal(old)$ is a normal subgroup of $Gal(new)$. A highly attractive possibility is that evolutionary sequences quite generally (not only in biology) correspond to this kind of sequences of Galois extensions. The relative Galois groups in the sequence would be analogous to conserved genes, and genes could indeed correspond to Galois groups [K4] [L11]. To my best understanding this corresponds to a situation in which the new polynomial $P_{m+n}$ defining the new extension is a polynomial $P_m$ having as argument the old polynomial $P_n(x)$: $P_{m+n}(x) = P_m(P_n(x))$.

What about the interpretation at the level of conscious experience? A possible interpretation is that the quantum jump leading to an extension of an extension corresponds to an emergence of a reflective level of consciousness giving rise to a conscious experience about experience. The abstraction level of the system becomes higher as is natural since number theoretic evolution as an increase of algebraic complexity is in question.

This picture could have a counterpart also in terms of the hierarchy of inclusions of hyperfinite factors of type $II_1$ (HFFs). The included factor $M$ and including factor $N$ would correspond to extensions of rationals labelled by Galois groups $Gal(M)$ and $Gal(N)$ having $Gal(M) \subset Gal(M)$ as normal subgroup so that the factor group $Gal(N)/Gal(M)$ would be the relative Galois group for the larger extension as extension of the smaller extension. I have indeed proposed [L21] that the inclusions for which included and including factor consist of operators which are invariant under discrete subgroup of $SU(2)$ generalizes so that all Galois groups are possible. One would have Galois confinement analogous to color confinement: the operators generating physical states could have Galois quantum numbers but the physical states would be Galois singlets.

# 13 A possible connection with family replication phenomenon?

In TGD framework the genus $g$ of the partonic 2-surfaces is proposed to label fermion families [K2, K9, K10]. One can characterize by genus $g$ the topology of light-like partonic orbits and identify the three fermion generators as 2-surfaces with genus $g = 0, 1, 2$ with the special property that they are always hyper-elliptic. Quantum mechanically also topological mixing giving rise to CKM mixing is possible. The view is that given connected 3-surface can contain several light-like 3-surface with different genera. For instance, hadrons would be such surfaces.

There are however questions to be answered.

1. The genera $g = 0, 1, 2$ assigned with the free fermion families correspond to Riemann surfaces, which are always hyper-elliptic allowing therefore $Z_2$ as a global conformal symmetry. These complex curves correspond to degrees $n = 2, 3, 4$ for the corresponding polynomials. For $n \leq 4$ can write explicit solutions for the roots of the polynomials. Could there be a deep connection between particle physics and mathematical cognition?
2. The homology and genus for 2-surfaces of $CP_2$ correlate with each other [A10]: is this consistent with the proposed topologicization of color hypercharge implying color confinement?
3. $h_{eff}/h = n$ hypothesis means that dark variant of particle particle characterized by genus $g$ is $n$-fold covering of this surface. In the general case the genus of covering is different. Is this consistent with the genus-generation correspondence?
4. The degree of complex curve correlates with the genus of the curve. Is generation-genus correspondence consistent with the assumption that partonic 2-surfaces have algebraic curve as $CP_2$ projection (this need not be the case)?

## 13.1 How the homology charge and genus correlate?

Complex surfaces in $CP_2$ are highly interesting from TGD point of view.

1. The model for elementary particles assumes that the partonic 2-surfaces carrying fermion number are homologically non-trivial, in other words they carry Kähler magnetic monopole flux having values $q = \pm 1$ and $q = \pm 2$. The idea is that color hyper charge $Y = \{\pm 2/3, \pm 1/3\}$ is proportional to $n$ for quarks and color confinement topologizes to the vanishing of total homology charge [K10].
2. The explanation of the family replication phenomenon [K2] in terms of genus-generation correspondence states that the three quarks and lepton generations correspond to the three lowest genera $g = 0, 1, 2$ for partonic 2-surfaces. Only these genera are always hyper-elliptic allowing thus a global $Z_2$ conformal symmetry. The physical vision is that for higher genera the handles behave like free particles. Is this proposal consistent with the proposal for the topologization of color confinement?

There is a result [A10] (page 124) stating that if the homology charge $q$ is divisible by 2 then one must have $g \geq q^2/4 - 1$. If $q$ is divisible by $h$, which is odd power of prime, one has $g \geq (q^2/4 - 1) - (q^2/4h^2)$. For $q = 2$ the theorem allows $g \geq 0$ so that all genera with color hyper charge $Y = \pm 2/3$ are realized.

The theorem says however nothing about $q = 0, 1$. These charges can be assigned to the two different geodesic spheres of $CP_2$ with $g = 0$ remaining invariant under SO(3) and U(2) subgroups of SU(3) respectively. Is $g > 0$ possible for $q = 1$ as the universality of topological color confinement would require? For $q = 3$ one would have $g \geq 1$. For $q = 4$ $h = 2$ divides $q$ and one has $g \geq 2$. It would seem $g \geq 5$. The conditions become more restrictive for higher $q$, which suggests that for $q = 0, 1$ one has $g \geq 0$ so that the topologization of color hypercharge would make sense.

## 13.2 Euler characteristic and genus for the covering of partonic 2-surface

Hierarchy of Planck constants $h_{eff}/h = n$ means a hierarchy of space-time surfaces identifiable as $n$-fold coverings. The proposal is that the number of sheets in absence of singularities is maximal possible and equals to the dimension of the extension dividing the order of its Galois group.

The Euler characteristic of $n$-fold covering in absence of singular points is $\chi_n = n\chi$. If there are singular (ramified) points these give a correction term given by Riemann-Hurwitz formula (see http://tinyurl.com/y7n2acub.)

In absence of singularities one has from $\chi = -2(g-1)$ and $\chi_n = n\chi$

$$g_n = n(g-1)+1 \ . \tag{13.1}$$

For $n = 1$ this indeed gives $g_1 = g$ independent of $g$. One can also combine this with the formula $g = (d-1)(d-2)/2$ holding for non-singular algebraic curves of degree $d$.

Singularities are unavoidable at algebraic points of cognitive representations at which some subgroup of Galois group leaves the point invariant (say rational point in ordinary sense). One can consider the possibility that fermions are located at the singular points at which several sheets of covering touch each other. This would give a correction factor to the formula. If the projection map from the covering to based is of form $\Pi(z) = z^n$ at the singular point $P$, one says that singularity has ramimifaction index $e_P = n$ and the algebraic genus would increase to

$$g_n = n(g-1)+1+\frac{1}{2}\sum_P (e_P-1) \ . \tag{13.2}$$

Indeed, singularities mean that sheets touch each other at singular points and this increases connectivity.

Under what conditions the genus of dark partonic surface with $n > 1$ can be same as that of the ordinary partonic surface representing visible matter? For the genera $g = 0$ and $g = 1$ this is possible so that these genera would be in an exceptional role also from the point of view of dark matter.

1. For $g = 1$ one has $g_n = g = 1$ independent of $n$ in absence of singular point. Torus topology (assignable to muon and (c,s) quarks) is exceptional. In presence of singularities the genus would increase by the $\sum_P (e_P-1)/2$ independent of the value of $n$. The lattice of points for elliptic surfaces would suggest existence of infinite number of singular points if the abelian group operations preserve the singular character of the points so that the genus would become infinite.

2. For $g = 0$ one would have $g_n = -n+1$ in absence of singularities. Only $n = 1$ - ordinary matter - is possible without singularities. Dark matter is however possible if singularities are allowed. For sphere one would obtain $g_n = -n+1+\sum_P (e_P-1)/2 \geq 0$. The condition $n \leq \sum_P (e_P-1)/2+1$ must therefore hold true for $g \geq 0$.

   The condition $g_n = -n+1+\sum_P (e_P-1)/2 = g = 0$ gives $\sum_P (e_P-1) = 2(n-1)$. For spherical topology it is possible to have dense set of rational points so that it is possible create cognitive representations with arbitrary number of points which can be also singular. One might argue that this kind of situation corresponds to a non-perturbative phase.

3. For $g = 2$ one would have $g_n = n+1+\sum_P (e_P-1)/2$ and genus would grow with $n$ even in absence of singularities and would be very large for large values of $h_{eff}$. $g_n = 2$ is obtained with $n = 1$ (ordinary matter) and no singular points not even allowed for $n = 1$. $g_n = g = 2$ is not possible for $n > 1$.

   Note that dark $g \geq 2$ fermions cannot correspond to lower generation fermions with singular points of covering. More generally, one could say that $g \geq 2$ fermions can exists only with standard value of Planck constant unless they are singular coverings of $g < 2$ fermions.

What is clear that the model of dark matter predicts breaking of universality. This breaking is not seen in the standard model couplings but makes it visible in amore delicate manner and might allow to understand why the masses of fermions increase with generation index.

## 13.3 All genera are not representable as non-singular algebraic curves

Suppose for a moment that partonic 2-surfaces correspond to rational maps of algebraic curves in $CP_2$ to $M^4$ that is deformations of these curves in $M^4$ direction. This assumption is of course questionable but deserves to be sttudied.

The formula (for algebraic curve see http://tinyurl.com/nt6tkey)

$$g = \frac{(d-1)(d-2)}{2} + \frac{\sum \delta_s}{2} \ ,$$

where $\delta_s > 0$ characterizes the singularity, does not allow all genera for algebraic curves for $\sum \delta_s = 0$: one has $g = 0, 0, 1, 3, 6, 10, ..$ for $d = 1, 2, ....$

For instance, $g = 2$, which would correspond in TGD to third quark or lepton generation is not possible without singularities for $d = 3$ curve having $g = 1$ without singularities!

This raises questions. Could the third fermion generation actually correspond to $g = 3$? Or does it correspond to $g = 2$ 2-surface of $CP_2$, which is more general surface than algebraic curve meaning that it is not representable as complex surface? Or could third generation fermions correspond to $g = 0$ or $g = 1$ curves with singular point of covering by Galois group so that several sheets touch each other?

To sum up, if the results for algebraic varieties generalize to TGD framework, they suggest notable differences between different fermion families. Universality of standard model interactions says that the only differences between fermion families are due to the differ masses. It is not clear whether the different masses could be due to the differences at number theoretical level and dark matter sectors.

1. All genera can appear as as ordinary matter ($d = 1$). Dark variants of $g = 1$ states have $g_d = 1$ automatically in absence of singular points. Dark variants of $g = 0$ states must have singular point in order to give $g_n = 0$. Dark variants of $g = 2$ states with $g_d = 2$ are obtained from $g = 1$ states with singularities. The special role of the two lowest is analogous to their special role for algebraic curves.

2. If one assumes that partonic 2-surfacs correspond to algebraic curves, one obtains again that $g = 2$ surfaces must correspond to singular $g = 0$ and $g = 1$ which could be dark in TGD sense.

# 14 Secret Link Uncovered Between Pure Math and Physics

I learned about a possible existence of a very interesting link between pure mathematics and physics (see http://tinyurl.com/y86bckmo). The article told about ideas of number theorist Minhyong Kim working at the University of Oxford. As I read the popular article, I realized it is something very familiar to me but from totally different view point.

Number theoretician encounters the problem of finding rational points of an algebraic curve defined as real or complex variant in which case the curve is 2-D surface and 1-D in complex sense. The curve is defined as root of polynomials polynomials or several of them. The polynomial have typically rational coefficients but also coefficients in extension of rationals are possible.

For instance, Fermat's theorem is about whether $x^n + y^n = 1$, $n = 1, 2, 3, ...$ has rational solutions for $n \geq 1$. For $n = 1$, and $n = 2$ it has, and these solutions can be found. It is now known that for $n > 2$ no solutions do exist. Quite generally, it is known that the number is finite rather than infinite in the generic case.

A more general problem is that of finding points in some algebraic extension of rationals. Also the coefficients of polynomials can be numbers in the extension of rationals. A less demanding problem is mere counting of rational points or points in the extension of rationals and a lot of progress has been achieved in this problem. One can also dream of classifying the surfaces by the character of the set of the points in extension.

I have consider the identification problem earlier in [L11] and I glue here a piece of text summarizing some basic results. The generic properties of sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see http://tinyurl.com/y9oq37ce) states that a curve over $Q$ with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

1. Sphere $CP_1$ in $CP_2$ has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A5, A8]).

   $g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in $CP_2$ with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

2. Elliptic curve $y^2 - x^3 - ax - b$ in $CP_2$ (see `http://tinyurl.com/lovksny`) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

   $g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and $CP_2$ as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in $CP_2$ (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

3. $g \geq 2$ curves do not allow a dense set of rational points nor even potentially dense set of rational points.

In my article [L11] providing TGD perspective about the role of algebraic geometry in physics, one can find basic results related to the identification problem including web links and references to literature.

## 14.1 Connection with TGD and physics of cognition

The identification problem is extremely difficult even for mathematicians - to say nothing about humble physicist like me with hopelessly limited mathematical skills. It is however just this problem which I encounter in TGD inspired vision about adelic physics [L16, L15, L11]. Recall that in TGD space-times are 4-surfaces in $H = M^4 \times CP_2$, preferred extremals of the variational principle defining the theory [L32, L22].

1. In this approach p-adic physics for various primes $p$ provide the correlates for cognition: there are several motivations for this vision. Ordinary physics describing sensory experience and the new p-adic physics describing cognition for various primes $p$ are fused to what I called adelic physics. The adelic physics is characterized by extension of rationals inducing extensions of various p-adic number fields. The dimension $n$ of extension characterizes kind of intelligence quotient and evolutionary level since algebraic complexity is the larger, the larger the value of $n$ is. The connection with quantum physics comes from the conjecture that $n$ is essentially effective Planck constant $h_{eff}/h_0 = n$ characterizing a hierarchy of dark matters. The larger the value of $n$ the longer the scale of quantum coherence and the higher the evolutionary level, the more refined the cognition.

2. An essential notion is that of cognitive representation [K12] [L15, L11]. It has several realizations. One of them is the representation as a set of points common to reals and extensions of various p-adic number fields induced by the extension of rationals. These space-time points have points in the extension of rationals considered defining the adele. The coordinates are the imbedding space coordinates of a point of the space-time surface. The symmetries of imbedding space provide highly unique imbedding space coordinates.

3. The gigantic challenge is to find these points common to real number field and extensions of various p-adic number fields appearing in the adele.

4. If this were not enough, one must solve an even tougher problem. In TGD the notion of "world of classical worlds" (WCW) is also a central notion [L32]. It consists of space-time surfaces in imbedding space $H = M^4 \times CP_2$, which are so called preferred extremals of

the action principle of theory. Quantum physics would reduce to geometrization of WCW and construction of classical spinor fields in WCW and representing basically many-fermion states: only the quantum jump would be genuinely quantal in quantum theory.

There are good reasons to expect that space-time surfaces are minimal surfaces with 2-D singularities, which are string world sheets - also minimal surfaces [L22, L24]. This gives nice geometrization of gauge theories since minimal surfaces equations are geometric counterparts for massless field equations.

One must find the algebraic points, the cognitive representation, for all these preferred extremals representing points of WCW (one must have preferred coordinates for H - the symmetries of imbedding space crucial for TGD and making it unique, provide the preferred coordinates)!

5. What is so beautiful is that in given cognitive resolution defined by the extension of rationals inducing the discretization of space-time surface, the cognitive representation defines the coordinates of the space-time surfaces as a point of WCW. In finite cognitive and measurement resolution this huge infinite-dimensional space WCW discretizes and the situation can be handled using finite mathematics.

## 14.2 Connection with Kim's work

So: what is then the connection with the work and ideas of Kim. There has been a lot of progress in understanding the problem: here I an only refer to the popular article.

1. One step of progress has been the realization that if one uses the fact that the solutions are common to both reals and various p-adic number fields helps a lot. The reason is that for rational points the rationality implies that the solution of equation representable as infinite power series of $p$ contains only finite number powers of $p$. If one manages to prove the this happens for even single prime, a rational solution has been found.

   The use of reals and all p-adic numbers fields is nothing but adelic physics. Real surfaces and all its p-adic variants form pages of a book like structure with infinite number of pages. The rational points or points in extension of rationals are the cognitive representation and are points common to all pages in the back of the book.

   This generalizes also to algebraic extensions of rationals. Solving the number theoretic problem is in TGD framework nothing but finding the points of the cognitive representation. The surprise for me was that this viewpoint helps in the problem rather than making it more complex.

   There are however problematic situations in some cases the hypothesis about finite set of algebraic points need not make sense. A good example is Fermat for $x + y = 1$. All rational points and also algebraic points are solutions. For $x^2+y^2 = 1$ the set of Pythagorean triangles characterizing the solutions is infinite. How to cope with these situations in which one has accidental symmetries as one might say?

2. Kim argues that one can make even further progress by considering the situation from even wider perspective by making the problem even bigger. Introduce what the popular article (see `http://tinyurl.com/y86bckmo`) calls the space of spaces. The space of string world sheets is what string models suggests. WCW is what TGD suggests. One can get a wider perspective of the problem of finding algebraic points of a surface by considering the problem in the space of surfaces and at this level it might be possible to gain much more understanding. The notion of WCW would not mean horrible complication of a horribly complex problem but possible manner to understand the problem!

   The popular article mentioned in the beginning mentions so called Selmer varieties as a possible candidate for the space of spaces. From the Wikipedia article (see `http://tinyurl.com/y27so3f2`) telling about Kim one can find a link to an article [A4] related to Selmer varieties. This article goes over my physicist's head but might give for a more mathematically oriented reader some grasp about what is involved. One can find also a list of publications of Kim (see `http://people.maths.ox.ac.uk/kimm/`.

Kim also suggests that the spaces of gauge field configurations could provide the spaces of spaces. The list contains an article [A7] with title *Arithmetic Gauge Theory: A Brief Introduction* (see `http://tinyurl.com/y66mphkh`) , which might help physicist to understand the ideas. An arithmetic variant of gauge theory could provide the needed space of spaces.

## 14.3 Can one make Kim's idea about the role of symmetries more concrete in TGD framework?

The crux of the Kim's idea is that somehow symmetries of space of spaces could come in rescue in the attempts to understand the rational points of surface. The notion of WCW suggest in TGD framework rather concrete realization of this idea that I have discussed from the point of view of construction of quantum states.

1. A little bit more of zero energy ontology (ZEO) is needed to follow the argument. In ZEO causal diamonds (CDs) are central. CDs are defined as intersections of future and past directed light-cones with points replaced with $CP_2$ and forming a scale hierarchy are central. Space-time surfaces are preferred extremals with ends at the opposite boundaries of CD indeed looking like diamond. Symplectic group for the boundaries of causal diamond (CD) is the group of isometries of WCW [L32] [L22]. Maximal isometry group is required to guarantee that the WCW Kähler geometry has Riemann connection - this was discovered for loop spaces by Dan Freed [A1]. Its Lie algebra has structure of Kac- Moody algebra with respect to the light-like radial coordinate of the light-like boundary of CD, which is piece of light-cone boundary. This infinite-D group plays central role in quantum TGD: it acts as maximal group of WCW isometries and zero energy states are invariant under its action at opposite boundaries.

2. As one replaces space-time surface with a cognitive representation associated with an extension of rationals, WCW isometries are replaced with their infinite discrete subgroup acting in the number field define by the extension of rationals defining the adele. These discrete isometries do not leave the cognitive representation invariant but replace with it new one having the same number of points and one obtains entire orbit of cognitive representations. This is what the emergence of symmetries in wider conceptual framework would mean.

3. One can in fact construct invariants of the symplectic group. Symplectic transformations leave invariant the Kähler magnetic fluxes associated with geodesic polygons with edges identified as geodesic lines of $H$. There are also higher-D symplectic invariants. The simplest polygons are geodesic triangles. The symplectic fluxes associated with the geodesic triangles define symplectic invariants characterizing the cognitive representation. For the twistor lift one must allow also $M^4$ to have analog of Kähler form and it would be responsible for CP violation and matter antimatter asymmetry [L8]. Also this defines symplectic invariants so that one obtains them for both $M^4$ and $CP_2$ projections and can characterize the cognitive representations in terms of these invariants. Note that the existence of twistor lift fixes the choice of $H$ uniquely since $M^4$ and $CP_2$ are the only 4-D spaces allowing twistor space with Kähler structure [A3] necessary for defining the twistor lift of Kähler action.

   More complex cognitive representations in an extension containing the given extension are obtained by adding points with coordinates in the larger extension and this gives rise to new geodesic triangles and new invariants. A natural restriction could be that the polynomial defining the extension characterizing the preferred extremal via $M^8 - H$ duality defines the maximal extension involved.

4. Also in this framework one can have accidental symmetries. For instance, $M^4$ with $CP_2$ coordinates taken to be constant is a minimal surface, and all rational and algebraic points for given extension belong to the cognitive representation so that they are infinite. Could this has something to do with the fact that we understand $M^4$ so well and have even identified space-time with Minkowski space! Linear structure would be cognitively easy for the same reason and this could explain why we must linearize.

   $CP_2$ type extremals with light-like $M^4$ geodesic as $M^4$ projection is second example of accidental symmetries. The number of rational or algebraic points with rational $M^4$ coordinates at light-like curve is infinite - the situation is very similar to $x + y = 1$ for Fermat. Simplest cosmic strings are geodesic sub-manifolds, that is products of plane $M^2 \subset M^4$ and $CP_2$ geodesic sphere. Also they have exceptional symmetries.

What is interesting from the point of view of proposed model of cognition is that these cognitively easy objects play a central role in TGD: their deformations represent more complex dynamical situations. For instance, replacing planar string with string world sheet replaces cognitive representation with a discrete or perhaps even finite one in $M^4$ degrees of freedom.

5. A further TGD based simplification would be $M^8 - H$ ($H = M^4 \times CP_2$) duality in which space-time surfaces at the level of $M^8$ are algebraic surfaces, which are mapped to surfaces in H identified as preferred extremals of action principle by the $M^8 - H$ duality [L11]. Algebraic surfaces satisfying algebraic equations are very simple as compared to preferred extremals satisfying partial differential equations but “preferred” is what makes possible the duality. This huge simplification of the solution space of field equations guarantees holography necessitated by general coordinate invariance implying that space-time surfaces are analogous to Bohr orbits. It would also guarantee the huge symmetries of WCW making it possible to have Kähler geometry.

   This suggests in TGD framework that one finds the cognitive representation at the level of $M^8$ using methods of algebraic geometry and maps the points to H by using the $M^8 - H$ duality. TGD and octonionic variant of algebraic geometry would meet each other.

   It must be made clear that now solutions are not points but 4-D surfaces and this probably means also that points in extension of rationals are replaced with surfaces with imbedding space coordinates defining function in extensions of rational functions rather than rationals. This would bring in algebraic functions. This might provide also a simplification by providing a more general perspective. Also octonionic analyticity is extremely powerful constraint that might help.

# 15 Are fundamental entities discrete or continuous and what discretization at fundamental level could mean?

There was an interesting FB discussion about discrete and continuum. I decided to write down my thoughts and emphasize those points that I see as important.

## 15.1 Is discretization fundamental or not?

The conversation inspired the question whether discreteness is something fundamental or not. If it is assumed to be fundamental, one encounters problems. The discrete structures are not unique. One has deep problem with the known space-time symmetries. Symmetries are reduced to discrete subgroup or totally lost. A further problem is the fact that in order to do physics, one must bring in topology and length measurements.

In discrete situation topology, in particular space-time dimension, must be put in via homology effectively already meaning use of imbedding to Euclidian space. Length measurement remains completely ad hoc. The construction of discrete metric is highly non-unique procedure and the discrete analog of of say Einstein's theory (Regge calculus) is rather clumsy. One feeds in information, which was not there by using hand weaving arguments like infrared limit. It is possible to approximate continuum by discretization but discrete to continuum won't go.

In hype physics these hand weaving arguments are general. For instance, the emergence of 3-space from discrete Hilbert space is one attempt to get continuum. One puts in what is factually a discretization of 3-space and then gets 3-space back at IR limit and shouts "Eureka!".

## 15.2 Can one make discretizations unique?

Then discussion went to numerics. Numerics is for mathematicians same as eating for poets. One cannot avoid it but luckily you can find people doing the necessary programming if you are a professor. Finite discretization is necessary in numerics and is highly unique.

I do not have anything personal against discretization as a numerical tool. Just the opposite, I see finite discretization as absolutely essential element of adelic physics as an attempt to describe also the correlates of cognition in terms of p-adic physics with p-adic space-time sheets as correlates of "thought bubbles" [L15, L16]. Cognition is discrete and finite and uses rational numbers: this is the basic clue.

1. Cognitive representations are discretizations of (for instance) space-time surface. One can say that physics itself builds its cognitive representation in all scales using p-adic space-time sheets. They should be unique once measurement resolution is characterized if one is really talking about fundamental physics.

   The idea abou tp-adic physics as physics of cognition indeed led to powerful calculational recipes. In p-adic thermodynamics the predictions come in power series of p-adic prime p and for the values of p assignable to elementary particles the two lowest terms give practically exact result [K9]. Corrections are of order $10^{-76}$ for electron characterized by Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38}$.

2. Adelic physics [L15] provides the formulation of p-adic physics: it is assumed that cognition is universal. Adele is a book like structure having as pages reals and extensions of various p-adic number fields induced by given extension of rationals. Each extension of rationals defines its own extension of the rational adele by inducing extensions of p-adic number fields. Common points between pages consist of points in extension of rationals. The books associated with the adeles give rise to an infinite library.

   At space-time level the points with coordinates in extension define what I call cognitive representation. In the generic case it is discrete and has finite number of points. The loss of general coordinate invariance is the obvious objection. In TGD however the symmetries of the imbedding space fix the coordinates used highly uniquely. $M^8 - H$ duality ($H = M^4 \times CP_2$) and octonionic interpretation implies that $M^8$ octonionic linear coordinates are highly unique [L11, L25]. Note that $M^8$ must be complexified. Different coordinatizations correspond to different octonionic structures- to different moduli - related by Poincare transformations of $M^8$. Only rational time translations as transformations of octonionic real coordinate are allowed as coordinate changes respecting octonionic structure.

3. Discretization by cognitive representation is unique for given extension of rationals defining the measurement resolution. At the limit of algebraic numbers algebraic points form a dense set of real space-time surface and p-adic space-time surfaces so that the measurement resolution is ideal. One avoids the usual infinities of quantum field theories induced by continuous delta functions, which for cognitive representations are replaced with Kronecker deltas. This seems to be the best that one can achieve with algebraic extensions of rationals. Also for transcendental extensions the situation is discrete.

   This leads to a number theoretic vision about second quantization of induced spinor fields central for the construction of gamma matrices defining the spinor structure of "world of classical worlds" (WCW) providing the arena of quantum dynamics in TGD analogous to the super-space of Wheeler [L32]. One ends up to a construction allowing to understand TGD view about SUSY as necessary aspect of second quantization of fermions and leads to the conclusions that in the simplest scenario only quarks are elementary fermions and leptons can be seen as their local composites analogous to super partners.

4. Given polynomial defining space-time surfaces in $M^8$ defines via its roots extension of rationals. The hierarchy of extensions defines an evolutionary hierarchy. The dimension n of extension defines kind of IQ measuring algebraic complexity and n corresponds also to effective Planck constant labelling phases of dark matter in TGD sense so that a direct connection with physics emerges.

   Imbedding space assigns to a discretization a natural metric. Distances between points of metric are geodesic distances computed at the level of imbedding space.

5. An unexpected finding was that the equations defining space-time surfaces as roots of real or imaginary parts of octonionic polynomials have also 6-D brane like entities with topology

of $S^6$ as solutions [L23, L29]. These entities intersect space-time surfaces at 3-D sections for which linear $M^4$ time is constant. 4-D roots can be glued together along these branes. These solutions turn out to have an interpretation in TGD based theory of quantum measurement extending to a theory of consciousness. The interpretation as moments of "small" state function reductions as counterparts of so called weak measurements. They could correspond to special moments in the life of conscious entity.

## 15.3 Can discretization be performed without lattices?

For a systems obeying dynamics defined by partial differential equations, the introduction of lattices seems to be necessary aspect of discretization. The problem is that the replacement of derivatives with discrete approximations however means that there is no hope about exact results. In the general case the discretization for partial differential equations involving derivatives forces to introduce lattice like structures. This is not needed in TGD.

1. At the level of $M^8$ ordinary polynomials give rise to octonionic polynomials and space-time surfaces are algebraic surfaces for which imaginary or real part of octonionic polynomial in quaternionic sense vanishes. The equations are purely algebraic involving no partial derivatives and there is no need for lattice discretization.

   For surfaces defined by polynomials the roots of polynomial are enough to fix the polynomials and therefore also the space-time surface uniquely: discretization is not an approximation but gives an exact result! This could be called number theoretical holography and generalizes the ordinary holography. Space-time surfaces are coded by the roots of polynomials with rational coefficients.

2. What about the field equations at the level of $H = M^4 \times CP_2$? $M^8 - H$ duality maps these surfaces to preferred extremals as 4-surfaces in $H$ analogous to Bohr orbits. Twistor lift of TGD predicts that they should be minimal surfaces with 2-D singularities being also extremals of 4-D Kähler action. The field equations would reduce locally to purely algebraic conditions. In properly chosen coordinates for $H$ they are expected to be determined in terms of polynomials coding for the same extension of rationals as their $M^8$ counterparts so that the degree should be same [L25]. This would allow to deduce the partial derivatives of imbedding space for the image surfaces without lattice approximation.

3. The simplest assumption is that the polynomials have rational coefficients. Number theoretic universality allows to consider also algebraic coefficients. In both cases also WCW is discretized and given point -space-time surface in QCD has coordinates given by the points of the number theoretically universal cognitive representation of the space-time surface. Even real coefficients are possible. This would allow to obtain WCW as a continuum central for the construction of WCW metric but is not consistent with number theoretical universality.

   Can one have polynomial/functions with rational coefficients and discretization of WCW without lattice but without losing WCW metric? Maybe the same trick that works at space-time level works also in WCW!

   (a) The group WCW isometries is identified as symplectic transformations of $\delta M^4_\pm \times CP_2$ ($\delta M^4_\pm$ denotes light-cone boundary) containing the boundary of causal diamond CD. The Lie algebra *Sympl* of this group is analogous half-Kac Moody algebra having symplectic transformations of $S^2 \times CP_2$ as counterpart of finite-D Lie group has fractal structure containing infinite number of sub-algebras $Sympl_n$ isomorphic to algebra itself: the conformal weights assignable to radial light-like coordinate are $n$-multiples of those for the entire algebra. Note that conformal weights of *Sympl* are non-negative.

   (b) One formulation for the preferred extremal property is in terms of infinite number of analogs of gauge conditions stating the vanishing of classical and also Noether charges for $Sympl_n$ and $[Sympl_n, Sympl]$. The conditions generalize to the super-counterpart of *Sympl* and apply also to quantum states rather than only space-time surfaces. In fact, while writing this I realized that - contrary to the original claim - also the vanishing of the Noether charges of higher commutators is required so that effectively $Sympl_n$ would define normal subgroup of $Sympl$. These conditions does not follow automatically.

The Hamiltonians of $Sympl(S^2 \times CP_2)$ are also labelled by the representations of the product of the rotation group $SO(3) \subset SO(3,1)$ of $S^2$ and color group $SU(3)$ together forming the analog of the Lie group defining Kac-Moody group. This group does not have have the fractal hierarchy of subgroups. The strongest condition is that the algebra corresponding to Hamiltonian isometries does not annihilate the physical states.

The space of states satisfying the gauge conditions is finite-D and that WCW becomes effectively finite-dimensional. A coset space associated with $Sympl$ would be in question and it would have maximal symmetries as also WCW. The geometry of the reduced WCW, WCW$_{red}$ could be deduced from symmetry considerations alone.

(c) Number theoretic discretization would correspond to a selection of points of this sub-space with the coordinates in the extension of rationals.The metric of WCW$_{red,n}$ at the points of discretization would be known and no lattice discretization would be needed. The gauge conditions are analogous to massless Dirac equation in WCW and could be solved in the points of discretization without introducing the lattice to approximate derivatives. As a matter fact, Dirac equation can be formulated solely in terms of the generators of $Sympl$.

(d) This effectively restricts WCW to $WCW_{red,n}$ in turn reduced to its discrete subset - since infinite number of WCW coordinates are fixed. If this sub-space can be regarded as realization of infinite number of algebraic conditions by polynomials with rational coefficients one can assign to it extension of rationals defining naturally the discretization of WCW$_{red,n}$. This extension is naturally the same as for space-time surfaces involved so that the degree of polynomials defining WCW$_{red,n}$ would be naturally $n$ and same as that for the polynomial defining the space-time surface. WCW$_{red,n}$ would decompose to union of spaces WCW$_{red,E_n}$ labelled by extensions $E_n$ of rationals with same dimension $n$.

There is analogy with gauge fixing. WCW$_{red,E_n}$ is a coset space of WCW defined by the gauge conditions. One can represent this coset space as a sub-manifold of WCW by taking one representative point from each coset. This choice is not unique but one can hope finding a gauge choice realized by an infinite number of polynomials of degree $n$ defining same extension of rationals as the polynomial defining the space-time surfaces in question.

(e) WCW spinor fields would be always restricted to finite-D algebraic surface of WCW$_{red,E_n}$ expressible in terms of algebraic equations. Finite measurement resolution indeed strongly suggests that WCW spinor field mode is non-vanishing only in a region parameterized in WCW by finite number of parameters. There is also a second manner to see this. WCW$_{red,E_n}$ could be also seen as $n+4$-dimensional surface in $WCW$.

(f) One can make this more concrete. Cognitive representation by points of space-time surface with coordinates in the extension - possibly satisfying additional conditions such as belonging to the 2-D vertices at which space-time surfaces representing different roots meet - provides WCW coordinates of given space-time surface. Minimum number of points corresponds to the dimension of extension so that the selection of coordinate can be redundant. As the values of these coordinates vary, one obtains coordinatization for the sector of WCW$_{red,E_n}$. An interesting question is whether one could represent the distances of space-time surfaces in this space in terms of the data provided by the points of discretization.

An interesting question is whether one can represent the distances of space-time surfaces in this space in terms of the data provided by the points of cognitive representation. One can define distance between two disjoint surfaces as the minimum of distance between the points of 2-surfaces. Could something like this work now? The points would be restricted to the cognitive representations. Could one define the distance between two cognitive representations with same number N of points in the following manner.

Consider all bipartitions formed by the cognitive representations obtained by connecting their points together in 1-1 manner. There are N! bipartitions of this kind if the number of points is N. Calculate the sum of the squares of the imbedding space distances between paired points. Find the bipartition for which this distance squared is minimum and define the distance between cognitive representations as this distance. This definition works also when the the numbers of points are different.

(g) If there quantum states are the basic objects and there is nothing "physical" behind them one can ask how we can imagine mathematical structures which different from basic structure of TGD. Could quantum states of TGD Universe in some sense represent all mathematical objects which are internally consistent. One could indeed say that at the level of WCW all $n+4$-D manifolds can be represented concretely in terms of WCW spinor fields localized to $n$-D subspaces of WCW. WCW spinor fields can represent concept of 4-surface of WCW$_{red,n}$ as a quantum superposition of its instance and define at the same time $n+4$-D surfaces [L32] [L24, L28, L27, L31].

## 15.4 Simple extensions of rationals as codons of space-time genetic code

A fascinating idea is that extensions of rationals define the analog of genetic code for space-time surfaces, which would therefore represent number theory and also finite groups.

(a) The extensions of rationals define an infinite hierarchy: the proposal is that the dimension of extensions corresponds to the integer $n$ characterizing subalgebra $Sympl_n$. This would give direct correspondence between the inclusions of HFFs assigned to the hierarchy of algebras $Sympl_n$ and hierarchy of extensions of rationals with dimension $n$.

Galois group for a extension of extension contains Galois group of extension as normal subgroup and is therefore *not simple*. Extension hierarchies correspond to inclusion hierarchies for normal subgroups. Simple Galois groups are in very special position and associated with what one might call simple extensions serving as fundamental building bricks of inclusion hierarchies. They would be like elementary particles and define fundamental space-time regions. Their Galois groups would act as groups of physical symmetries.

(b) One can therefore talk about elementary space-time surfaces in $M^8$ and their compositions by function composition of octonionic polynomials. Simple groups would label elementary space-time regions. They have been classified: (see `http://tinyurl.com/y3xh4hrh`). The famous Monster groups are well-known examples about simple finite groups and would have also space-time counterparts. Also the finite subgroups of Lie groups are special and those of $SU(2)$ are associated with Platonic solids and seem to play key role in TGD inspired quantum biology. In particular, vertebrate genetic code can be assigned to icosahedral group.

(c) There is also an analogy with genes. Extensions with simple Galois groups could be seen as codons and sequences of extension obtained by functional composition as analogs of genes. I have even conjectured that the space-time surfaces associated with genes could quite concretely correspond to extensions of extensions of ...

## 15.5 Are octonionic polynomials enough or are also analytic functions needed?

I already touched the question whether also analytic functions with rational coefficients (number theoretical universality) might be needed.

(a) The roots of analytic functions generate extension of rationals. If the roots involve transcendental numbers they define infinite extensions of rationals. Neper number $e$ is very special in this sense since $e^p$ is ordinary p-adic number for all primes $p$ so that the induced extension is finite-dimensional. One could thus allow it without losing number theoretical universality. The addition of $\pi$ gives infinite-D extension but one could do by adding only roots of unity to achieve finite-D extensions with finite accuracy of phase measurement. Phases would be number theoretically universal but not angles.

(b) One could of course consider only transcendental functions with rational roots. Trigonometric function $sin(x/2\pi)$ serves as a simple example. One can also argue that since physics involves in an essential manner trigonometric functions via Fourier analysis, the inclusion of analytic functions with algebraic roots must be allowed.

(c) What about analytic functions as limits of polynomials with rational coefficients such that the number of roots becomes infinite at the limit? Also their imaginary and real part can vanish in quaternionic sense and could define space-time surfaces - analogs of transcendentals as space-time surfaces. It is not clear whether these could be allowed or not.

Could one have a universal polynomial like function giving algebraic numbers as the extension of rationals defined by its algebraic roots? Could Riemann zeta (see `http://tinyurl.com/nfbkrsx`) code algebraic numbers as an extension via its roots. I have conjectured that roots of Riemann zeta are algebraic numbers: could they span all algebraic numbers?

It is known that the real or imaginary part of Riemann zeta along $s = 1/2$ critical line can approximate any function to arbitrary accuracy: also this would fit with universality. Could one think that the space-time surface defined as root of octonionic continuation of zeta could be universal entity analogous to a fixed point of iteration in the construction of fractals? This does not look plausible.

4. One can construct iterates of Riemann zeta having at least the same roots as zeta by the rule

$$\begin{array}{l} f_0(s) = \zeta(s) \ , \\ f_n(s) = \zeta(f_{n-1}(s)) - \zeta(0), \ \ \zeta(0) = -1/2 \ . \end{array} \tag{15.1}$$

$\zeta$ is not a fixed point of this iteration as the fractal universality would suggest. The set of roots however is. Should one be happy with this.

5. Riemann zeta has also counterpart in all extensions of rationals known as Dedekind zeta (see `http://tinyurl.com/y5grktv`) [L33, L30, L26]. Riemann zeta is therefore not unique. One can ask whether Dedekind zetas associated with simple Galois groups are special and whether Dedekind zetas associated with extensions of extensions of .... can be constructed by using the Dedekind zetas of simple extensions. How do the roots of Dedekind zeta depend on the associated extension of rationals? How the roots of Dedekind zeta for extension of extension defined by composite of two polynomials depend on extensions involved? Are the roots union for the roots associated with the composites?

6. What about forming composites of Dedekind zetas? Categorical according to my primitive understanding raises the question whether a composition of extensions could correspond to a composition of functions. Could Dedekind zeta for a composite of extensions be obtained from a composite of Dedekind zetas for extensions? Requiring that roots of extension $E_1$ are preserved would give formula

$$\zeta_{D,E_1E_2} = \zeta_{D,E_1} \circ \zeta_{D,E_2} - \zeta_{D,E_1}(0) \ . \tag{15.2}$$

The zeta function would be obtained by an iteration of simple zeta functions labelled by simple extensions. The inverse image for the set of roots of $\zeta_{D,E_1}$ under $\zeta_{D,E_2}$ that is the set $\zeta_{D,E_2}^{-1}(roots(\zeta_{D,E_1})$ would define also roots of $\zeta_{D,E_1E_2}$. This looks rather sensible.

But what about iteration of Riemann zeta, which corresponds to trivial extension? Riemann $\zeta$ is not invariant under iteration although its roots are. Should one accept this and say that it is the set of roots which defines the invariant. Could one say that the iterates of various Dedekind zetas define entities which are somehow universal.

# 16 Summary and future prospects

In the following I give a brief summary about what has been done. I concentrate on $M^8 - H$ duality since the most significant results are achieved here.

It is fair to say that the new view answers the following a long list of open questions.

1. When $M^8 - H$ correspondence is true (to be honest, this question emerged during this work!)? What are the explicit formulas expressing associativity of the tangent space or normal space of the 4-surface?

   The key element is the formulation in terms of complexified $M^8$ - $M^8_c$ - identified in terms of octonions and restriction $M^8_c \rightarrow M^8$. One loses the number field property but for polynomials ring property is enough. The level surfaces for real and imaginary parts of octonionic polynomials with real coefficients define 4-D surfaces in the generic case.

   Associativity condition is an additional condition reducing the dimension of the space-time surface unless some components of $RE(P)$ or $IM(P)$ are critical meaning that also their gradients vanish. This conforms with the quantum criticality of TGD and provides a concrete first principle realization for it.

   An important property of $IM(P_1P_2)$ is its linearity with respect to $IM(P_i)$ implying that this condition gives the surfaces $IM(P_i) = 0$ as solutions. This generalizes by induction to $IM(P_1P_2...P_n)$. For $RE(P_1P_2) = 0$ linearity does not hold true and there is a genuine interaction. A physically attractive idea idea is that $RE(P_1P_2) = 0$ holds true inside CDs and for wormhole contacts between space-time sheets with Minkoskian signature. One can generalizes this also to $IM(P_1/P_2)$ and $RE(P_1/P_2)$ if rational functions are allowed. Note however that the origins of octonionic coordinates in $P_i$ must be on the octonionic real line.

2. How this picture corresponds to twistor lift? The twistor lift of Kähler action (dimensionally reduced Kähler action in twistor space of space-time surface) one obtains two kinds of space-time regions. The regions, which are minimal surfaces and obey dynamics having no dependence on coupling constants, correspond naturally to the critical regions in $M^8$ and $H$.

   There are also regions in which one does not have extremal property for both Kähler action and volume term and the dynamics depends on coupling constant at the level of $H$. These regions are associative only at their 3-D ends at boundaries of CD and at partonic orbits, and the associativity conditions at these 3-surfaces force the initial values to satisfy the conditions guaranteeing preferred extremal property. The non-associative space-time regions are assigned with the interiors of CDs. . The particle orbit like space-time surfaces entering to CD are critical and correspond to external particles.

   It has later turned out [L22] that it might be possible to take the associativity conditions to extreme in the sense that they would hold everywhere apart from a set of discrete points and space-time surface would be minimal surfaces at all points except this finite set of points. There would be transfer of conserved quantities assignable to the volume term and the 4-D Kähler action (coming as dimensionally reduced 6-D Kähler action for the twistor lift of TGD) only at these points and elementary fermions would be naturally assignable to these points.

3. The surprise was that $M^4 \subset M^8$ is naturally co-associative. If associativity holds true also at the level of $H$, $M^4 \subset H$ must be associative. This is possible if $M^8 - H$ duality maps tangent space in $M^8$ to normal space in $H$ and vice versa.

4. The connection to the realization of the preferred extremal property in terms of gauge conditions of subalgebra of SSA is highly suggestive. Octonionic polynomials critical at the boundaries of space-time surfaces would determine by $M^8 - H$ correspondence the solution to the gauge conditions and thus initial values and by holography the space-time surfaces in $H$.

5. A beautiful connection between algebraic geometry and particle physics emerges. Free many-particle states as disjoint critical 4-surfaces can be described by products of corresponding

polynomials satisfying criticality conditions. These particles enter into CD , and the non-associative and non-critical portions of the space-time surface inside CD describe the interactions. One can define the notion of interaction polynomial as a term added to the product of polynomials. It can vanish at the boundary of CD and forces the 4-surface to be connected inside CD. It also spoils associativity: interactions are switched on. For bound states the coefficients of interaction polynomial are such that one obtains a bound state as associative space-time surface.

6. This picture generalizes to the level of quaternions. One can speak about 2-surfaces of space-time surface with commutative or co-commutative tangent space. Also these 2-surfaces would be critical. In the generic case commutativity/co-commutativity allows only 1-D curves.

   At partonic orbits defining boundaries between Minkowskian and Euclidian space-time regions inside CD the string world sheets degenerate to the 1-D orbits of point like particles at their boundaries. This conforms with the twistorial description of scattering amplitudes in terms of point like fermions.

   For critical space-time surfaces representing incoming states string world sheets are possible as commutative/co-commutative surfaces (as also partonic 2-surfaces) and serve as correlates for (long range) entaglement) assignable also to macroscopically quantum coherent system ($h_{eff}/h = n$ hierarchy implied by adelic physics).

7. The octonionic polynomials with real coefficients form a commutative and associative algebra allowing besides algebraic operations function composition. Space-time surfaces therefore form an algebra and WCW has algebra structure. This could be true for the entire hierarchy of Cayley-Dickson algebras, and one would have a highly non-trivial generalization of the conformal invariance and Cauchy-Riemann conditions to their n-linear counterparts at the $n$:th level of hierarchy with $n = 1, 2, 3, ..$ for complex numbers, quaternions, octonions,... One can even wonder whether TGD generalizes to this entire hierarchy!

8. In the original version of this article I did not realize that there are two options for realizing the idea that the $M_c^4$ projection of space-time surface in $M_c^8$ must belong to $M^4$.

   (a) I proposed that the *projection* from $M_c^8$ to real $M^4$ (for which $M^1$ coordinate is real and $E^3$ coordinates are imaginary with respect to $i$!) defines the real space-time surface mappable by $M^8 - H$ duality to $CP_2$ [L11].

   (b) An alternative option, which I have not considered in the original versions of [L11, L13] is that only the roots of the 4 vanishing polynomials as coordinates of $M_c^4$ belong to $M^4$ so that $m^0$ would be real root and $m^k$, $k = 1, ..., 3$ imaginary with respect to $i \to -i$. $M_c^8$ coordinates would be invariant ("real") under combined conjugation $i \to -i, I_k \to -I_k$. In the following I will speak about this property as *Minkowskian reality*. This could make sense. Outside CD these conditions would not hold true. This option looks more attractive than the first one. Why these condition can be true just inside CD, should be understood.

9. The use of polynomials or rational functions could be also an approximation. Analytic functions of real variable extended to octonionic functions would define the most general space-time surfaces but the limitations of cognition would force to use polynomial approximation. The degree $n$ of the polynomial determining also $h_{eff} = nh_0$ would determine the quality of the approximation and at the same time the "IQ" of the system.

All big pieces of quantum TGD are now tightly interlinked.

1. The notion of causal diamond (CD) and therefore also ZEO can be now regarded as a consequence of the number theoretic vision and $M^8 - H$ correspondence, which is also understood physically.

2. The hierarchy of algebraic extensions of rationals defining evolutionary hierarchy corresponds to the hierarchy of octonionic polynomials.

3. Associative varieties for which the dynamics is critical are mapped to minimal surfaces with universal dynamics without any dependence on coupling constants as predicted by twistor lift of TGD. The 3-D associative boundaries of non-associative 4-varieties are mapped to initial values of space-time surfaces inside CDs for which there is coupling between Kähler action and volume term.

4. Free many particle states as algebraic 4-varieties correspond to product polynomials in the complement of CD and are associative. Inside CD the addition of interaction terms vanishing at its boundaries spoils associativity and makes these varieties connected.

5. The super variant of the octonionic algebraic geometry makes sense, and one obtains a beautiful correlation between the fermion content of the state and corresponding space-time variety. This suggests that twistorial construction indeed generalizes. Criticality for the external particles giving rise to additional constraints on the coefficients of polynomials could make possible to have well-define summation over corresponding varieties.

What mathematical challenges one must meet?

1. One should prove more rigorously that criticality is possible without the reduction of dimension of the space-time surface.

2. One must demonstrate that SSA conditions can be true for the images of the associative regions (with 3-D or 4-D). This would obviously pose strong conditions on the values of coupling constants at the level of $H$.

Concerning the description of interactions there are several challenges.

1. Do associative space-time regions have minimal surface extremals as images in $H$ and indeed obeying universal critical dynamics? As found, the study of the known extremals supports this view.

2. Could one construct the scattering amplitudes at the level of $M^8$? Here the possible problems are caused by the exponents of action (Kähler action and volume term) at $H$ side. Twistorial construction [K32] however leads to a proposal that the exponents actually cancel. This happens if the scattering amplitude can be thought as an analog of Gaussian path integral around single extremum of action and conforms with the integrability of the theory. In fact, nothing prevents from defining zero energy states in this manner! If this holds true then it might be possible to construct scattering amplitudes at the level of $M^8$.

3. What about coupling constants? Coupling constants make themselves visible at $H$ side both via the vanishing conditions for Noether charges in sub-algebra of SSA and via the values of the non-vanishing Noether charges. $M^8 - H$ correspondence determining the 3-D boundaries of interaction regions within CDs suggests that these couplings must emerge from the level $M^8$ via the criticality conditions posing conditions on the coefficients of the octonionic polynomials coding for interactions.

   Could all coupling constant emerge from the criticality conditions at the level of $M^8$? The ratio of $R^2/l_P^2$ of $CP_2$ scale and Planck length appears at $H$ level. Also this parameter should emerge from $M^8 - H$ correspondence and thus from criticality at $M^8$ level. Physics would reduce to a generalization of the catastrophe theory of Rene Thom!

4. The description of interactions at the space-time surface associated with single CD should be $M^8$ counterpart of the $H$ picture in which 3 light-like partonic orbits meet at common end topological vertex - defined by a partonic 2-surface and fermions scatter without touching. Now one has octonionic sparticle lines and interaction vertex becomes possible. This conforms with the idea that interactions take place at discrete points belonging to the extension of rationals. The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

The partonic 2-surfaces defining topological vertices would naturally correspond to the intersections $X^2 = X^4 \cap S^6(t_n)$, which satisfy $RE(P) = IM(P) = 0$ and are singular and doubly critical. If sparticle lines are allowed to move along this space-like 2-surface (the line becomes space-like) they can intersect and give rise to a fusion vertex producing the third fermionic line.

5. Real analyticity requires that the octonionic polynomials have real coefficients. This forces the origin of octonionic coordinates to be at real line (time axis) in the octonionic sense, and guarantees the associativity and commutativity of the polynomials. Arbitrary CDs cannot be located along this line. Can one assume that all CDs involved with *observable* processes satisfy this condition?

   If not, how do the 4-varieties associated with octonionic polynomials with different origins interact? How could one avoid losing the extremely beautiful associative and commutative algebra? It seems that one cannot form their products and sums and must form the Cartesian product of $M^8$:s with different tips for CDS and formulate the interaction in this framework. In the case of space-time surfaces associated with different CDs the discrete intersections of space-time surfaces would define the interaction vertices.

6. Super-octonionic geometry suggests that the twistorial construction of scattering amplitudes in $\mathcal{N} = 4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in an appropriate extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

   If scattering diagrams are associated with discrete cognitive representations, one obtains a generalization of super-twistor formalism involving polygons. Super-octonions as counterparts of super gauge potentials are well-defined if octonionic 8-momenta are quaternionic: indeed, Grassmannians have quaternionic counterparts but not octonionic ones. There are good hopes that the twistor Grassmann approach to $\mathcal{N} = 4$ SUSY generalizes. The core part in the calculation of the scattering diagram would reduce to the construction of octonionic 4-varieties and identifying the points belonging to the extension of rationals considered. The rest would be dictated by symmetries and integrations over various moduli spaces, which should be number theoretically universal so that residue calculus strongly suggests itself.

7. What is the connection with super conformal variant of Yangian symmetry, whose generalization in TGD framework is highly suggestive? Twistorial construction of scattering amplitudes at the level of $M^8$ looks highly promising idea and could also realize Yangian supersymmetry. The conjecture is that the twistorial amplitudes decompose to $M^4$ and $CP_2$ parts with similar structure with $E^4$ spin (electroweak isospin) replacing ordinary spin and that the integrands in Grassmannians emerging from the conservation of $M^4$ and $E^4$ 4-momenta are identical in the two cases and thus guarantee Yangian supersymmetry in both sectors. The only difference would be due to the product of delta functions associated with the "negative helicities" (weak isospins with negative sign) expressible as a delta function in the complement of $SU(3)$ Cartan algebra $U(1) \times U(1)$ by using exponential map.

It is appropriate to close with a question about fundamentals.

1. The basic structure at $M^8$ side consists of complexified octonions. The metric tensor for the complexified inner product for complexified octonions (no complex conjugation with respect to $i$ for the vectors in the inner product) can be taken to have any signature $(\epsilon_1, ..., \epsilon_8)$, $\epsilon_i = \pm 1$. By allowing some coordinates to be real and some coordinates imaginary one obtains effectively any signature from say purely Euclidian signature. What matters is that the restriction of complexified metric to the allowed sub-space is real. These sub-spaces are linear Lagrangian manifolds for Kähler form representing the commuting imaginary unit $i$. There is analogy with wave mechanics. Why $M^8$ -actually $M^4$ - should be so special real section? Why not some other signature?

2. The first observation is that the $CP_2$ point labelling tangent space is independent of the signature so that the problem reduces to the question why $M^4$ rather than some other signature $(\epsilon_1, .., \epsilon_4)$. The intersection of real subspaces with different signatures and same origin $(t, r) = 0$ is the common sub-space with the same signature. For instance, for $(1, -1, -1, -1)$ and $(-1, -1, -1, -1)$ this subspace is 3-D $t = 0$ plane sharing with CD the lower tips of CD. For $(-1, 1, 1, 1)$ and $(1, 1, 1, 1)$ the situation is same. For $(1, -1, -1, -1)$ and $(1, 1, -1, -1)$ $z = 0$ holds in the intersection having as common with the lower boundary of CD the boundary of 3-D light-cone. One obtains in a similar manner boundaries of 2-D and 1-D light-cones as intersections.

3. What about CDs in various signatures? For a fully Euclidian signature the counterparts for the interiors of CDs reduce to 4-D intervals $t \in [0, T]$ and their exteriors and thus the space-time varieties representing incoming particles reduce to pairs of points $(t, r) = (0, 0)$ and $(t, r) = (T, 0)$: it does not make sense to speak about external particles. For other signatures the external particles correspond to 4-D surfaces and dynamics makes sense. The CDs associated with the real sectors intersect at boundaries of lower dimensional CDs: these lower-dimensional boundaries are analogous to subspaces of Big Bang (BB) and Big Crunch (BC).

4. I have not found any good argument for selecting $M^4 = M^{1,3}$ as a unique signature. Should one allow also other real sections? Could the quantum numbers be transferred between sectors of different signature at BB and BC? The counterpart of Lorentz group acting as a symmetry group depends on signature and would change in the transfer. Conservation laws should be satisfied in this kind of process if it is possible. For instance, in the leakage from $M^4 = M^{1,3}$ to $Mi, j$, say $M^{2,2}$, the intersection would be $M^{1,2}$. Momentum components for which signature changes, should vanish if this is true. Angular momentum quantization axis normal to the plane is defined by two axis with the same signature. If the signatures of these axes are preserved, angular momentum projection in this direction should be conserved. The amplitude for the transfer would involve integral over either boundary component of the lower-dimensional CD.

   Could the leakage between signatures be detected as disappearance of matter for CDs in elementary particle scales or lab scales?

5. One can also raise a question about the role of WCW geometry as a continuous infinite-D geometry: could the discretization by cognitive representations making WCW effectively discrete mean its loss? It seems that this cannot be the case. At least in the real sector continuum must be present and the discretization reflects only the discreteness of cognitive representations. In principle continuous WCW could make sense also in p-adic sectors of the adele.

   The identification of space-time surfaces as zero loci of polynomials generalizes to rational functions and even transcendental functions although the existence of the p-adic counterparts of these functions requires additional conditions. Could one interpret the representation in terms of polynomials and possibly rational functions as an approximation? Could the hierarchy of approximations obtained in this manner give rise to a hierarchy of hyper-finite factors of type $II_1$ defining a hierarchy of measurement resolutions [K16]?

# 17 Appendix: $o^2$ as a simple test case

Octonionic polynomial $o^2$ serves as a simple testing case. $o^2$ is not irreducible so that its properties might not be generic and it might be better to study polynomial of form $P(o) = o + po^2$ instead.

Before continuing, some conventions are needed.

1. The convention is that in $M^8 = M^1 \times E^7$ $E^7$ corresponds to purely imaginary complexified octonions in both octonionic sense and in the sense that they are proportional to $i$. $M^1$ corresponds to octonions real in both senses. This corresponds to the signature $(1, -1, -1, -1, ...)$ for $M^8$ metric obtained as restriction of complexified metric. For $M^4 = M^1 \times E^3$ analogous conventions hold true.

2. Conjugation $o = o_0 + o_k I_k \rightarrow \overline{o} \equiv o_0 - I_k o_k$ does not change the sign of $i$. Quaternions can be decomposed to real and imaginary parts and some notation is needed. The notation $q = Re(q) + Im(q)$ seems to be the least clumsy one: here $Im(q)$ is 3-vector.

The explicit expression in terms of quaternionic decomposition $o = q_1 + q_2 I_4$ reads as

$$P(o) = o^2 = q_1^2 - q_2\overline{q}_2 + (q_1q_2 + q_2\overline{q}_1)I_4 \ . \tag{17.1}$$

$o$ corresponds to complexified octonion and there are two options concerning the interpretation of $M^4$ and $E^4$. $M^4$ could correspond to quaternionic or co-quaternionic sub-space. I have assumed the first interpretation hitherto but actually the identification is not obvious. This two cases are different and must be treated both.

With these notations quaternionic inner product reads as

$$\begin{aligned} q_1q_2 &= Re(q_1q_2) + Im(q_1q_2) \ , \\ Re(q_1q_2) &= Re(q_1)Re(q_2) - Im(q_1)\cdot Im(q_2) \ , \\ Im(q_1q_2) &= Re(q_1)Im(q_2) + Re(q_2)Im(q_1) + Im(q_1)\times Im(q_2) \ . \end{aligned} \tag{17.2}$$

Here $a\cdot b$ denotes the inner product of 3-vectors and $a\times b$ their cross product.

Note that one has real and imaginary parts of octonions as two quaternions and real and imaginary parts of quaternions. To avoid confusion, I will use $RE$ and $IM$ to denote the decomposition of octonions to quaterions and $Re$ and $Im$ for the decomposition of quaternions to real and imaginary parts.

One can express the $RE(o^2)$ as

$$\begin{aligned} RE(o^2) &\equiv X \equiv q_1^2 - q_2\overline{q}_2 \ , \\ Re(X) &= Re(q_1)^2 - Im(q_1)\cdot Im(q_2) - (Re(q_2)^2 + Im(q_2)\cdot Im(q_2)) \ , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) \ . \end{aligned} \tag{17.3}$$

For $IM(o^2)$ one has

$$\begin{aligned} IM(o^2) &\equiv Y = q_1q_2 + q_2\overline{q}_1 \\ Re(Y) &= 2Re(q_1)Re(q_2) \ , \\ Im(Y) &= Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1)\times Im(q_2) \ . \end{aligned} \tag{17.4}$$

The essential point is that only $RE(o^2)$ contains the complexified Euclidian norm $q_2\overline{q_2}$ which becomes Minkowskian of Euclidian norm depending on whether one identifies $M^4$ as associative or co-associative surface in $o_c^8$.

## 17.1 Option I: $M^4$ is quaternionic

Consider first the condition $RE(o^2) = 0$. The condition decomposes to two conditions stating the vanishing of quaternionic real and imaginary parts:

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1)\cdot Im(q_2) - (Re(q_2)^2 + Im(q_2)\cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{E^4}(q_2) = 0 \ , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ . \end{aligned} \tag{17.5}$$

$Im(X) = 0$ is satisfied for $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has two options. This gives 1-D line in time direction of 3-D hyperplane as a solution for $M^4$ factor.

$Re(X) = 0$ states $N_{M^4}(q_1) = N_{E^4}(q_2)$. $q_2$ coordinate itself is free. $N_{E^4}(q_2)$ is negative so that $q_1$ must be space-like with respect to the $N_{M^4}$ so that only the solution $Re(q_1) = 0$ is possible. Therefore one has $Re(q_1) = 0$ and $N_{M^4}(q_1) = N_{E^4}(q_2)$.

One can assign to each $E^4$ point a section of hyperboloid with $t = 0$ hyper-plane giving a sphere and the surface is 6-dimensional sphere bundle like variety! This is completely unexpected result and presumably is due to the additional accidental symmetries due to the octonionicity. Also the fact that $o^2$ is not irreducible polynomial is a probably reason since for $o$ the surface is 4-D. The addition of linear term is expected to remove the degeneracy.

Consider next the case $IM(o^2) = 0$. The conditions read now as

$$\begin{aligned} Re(Y) &= 2Re(q_1)Re(q_2) = 0 \ , \\ Im(Y) &= Re(q_1)Im(q_2) - Re(q_2)Im(q_1) + Im(q_1) \times Im(q_2) = 0 \ . \end{aligned} \tag{17.6}$$

Since cross product is orthogonal to the factors $Im(Y) = 0$ condition requires that $Im(q_1)$ and $Im(q_2)$ are parallel vectors: $Im(q_1) = \lambda Im(q_2)$ and one has the condition $Re(q_1) = \lambda Re(q_2)$ implying $q_1 = \Lambda q_2$. Therefore to each point of $E^4$ is associated a line of $M^4$. The surface is 5-dimensional.

It is interesting to look what the situation is if both conditions are true so that one would have a singularity. In this case $Re(q_1) = 0$ and $Re(q_1) = \lambda Re(q_2)$ imply $\lambda = 0$ so that $q_1 = 0$ is obtained and the solution reduces to 4-D $E^4$, which would be co-associative.

## 17.2 Option II: $M^4$ is co-quaternionic

This case is obtained by the inspection of the previous calculation by looking what changes the identification of $M^4$ as co-quaternionic factor means. Now $q_1$ is Euclidian and $q_2$ Minkowskian coordinate and $q_2\overline{q}_2$ gives Minkowskian rather than Euclidian norm.

Consider first $RE(o^2) = 0$ case.

$$\begin{aligned} Re(X) &= Re(q_1)^2 - Im(q_1) \cdot Im(q_2) - (Re(q_2)^2 + Im(q_2) \cdot Im(q_2)) \equiv N_{M^4}(q_1) - N_{M^4}(q_2) = 0 \ , \\ Im(X) &= Im(q_1^2) = 2Re(q_1)Im(q_1) = 0 \ . \end{aligned} \tag{17.7}$$

$N_{M^4}(q_1) - N_{M^4}(q_2) = 0$ condition holds true now besides the condition $Re(q_1) = 0$ or $Im(q_1) = 0$ so that one has also now two options.

1. For $Re(q_1) = 0$ $N_{M^4}(q_1)$ is non-positive and this must be the case for $N_{M^4}(q_2))$ so that the *exterior* of the light-cone is selected. In this case the points of $M^4$ with fixed $N_{M^4}$ give rise to a 2-D intersection with $Re(q_1) = 0$ hyper-plane that is sphere so that one has 6-D surface, kind of sphere bundle.

2. For $Im(q_1) = 0$ Minkowski norm is positive and so must be corresponding norm in $E^4$ so that in $E^4$ surface has future ligt-cone as projection. This surface is 4-D. The emergence of future light-cone might provide justification for the emergence of CDs and zero energy ontology.

For $IM(o^2)$ the discussion is same as in quaternionic case since norm does not appear in the equations.

At singularity both $RE(o^2)$ and $IM(o^2) = 0$ vanish. The condition $q_1 = \Lambda q_2$ reduces to $\Lambda = 0$ so that $q_1 = 0$ is only allowed. This leaves only light-cone boundary under consideration.

The appearance of surfaces with dimension higher than 4 raises the question whether something is wrong. One could of course argue that associativity allows also lower than 4-D surfaces as associative surfaces and higher than 4-D surfaces as co-associative surfaces. At $H$-level one can say that one has 4-D surfaces. A good guess is that this behavior disappears when the linear term is absent and origin ceases to be a singularity.

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