# Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: Part III

M. Pitkänen,

February 2, 2024

Email: matpitka6@gmail.com.
http://tgdtheory.com/public_html/.
Postal address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland. ORCID: 0000-0002-8051-4364.

## Contents

**Abstract**

Cognitive representations are the basic topic of the third chapter related to $M^8-H$ duality. Cognitive representations are identified as sets of points in extension of rationals for algebraic varieties with "active" points containing fermion. The representations are discussed at both $M^8$- and $H$ level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces.

The notion is applied in various cases and the connection with $M^8-H$ duality is rather loose.

1. Extensions of rationals are essentially coders of information. There the possible analogy of extensions of rationals with genes deserves discussion. Extensions, which are not extensions of extensions would be analogous to genes. The notion of conserved gene as number theoretical analogy for Galois extensions as the Galois group of extension which is normal subgroup of Galois extension.
2. The possible physical meaning of the notion of perfectoid introduced by Peter Scholze is discussed in the framework of p-adic physics. Extensions of p-adic numbers involving roots of the prime defining the extension are involved and are not considered previously in TGD framework. There there possible physical meaning deserves discussion.
3. The construction of cognitive representation reduces to a well-known mathematical problem of finding the points of space-time surface with embedding space coordinates in given extension of rationals. The work of Kim and Coates represents new ideas in this respect and there is a natural connection with TGD.
4. One expects that large cognitive representations are winners in the number theoretical fight for survival. Strong form of holography suggests that it is enough to consider cognitive representations restricted to string world sheets and partonic 2-surfaces. If the 2-surface possesses large group of symmetries acting in extension of rationals, one can have large cognitive representations as orbit of point in extension. Examples of highly symmetric 2-D surfaces are geodesic spheres assignable to partonic 2-surfaces and cosmic strings and elliptic curves assignable with string world sheets and cosmic strings.
5. Rationals and their extensions give rise to a unique discretizations of space-time surface (for instance) - cognitive representation - having interpretation in terms of finite measurement resolution. There are however many open questions. Should one allow only octonionic polynomials defined as algebraic continuations of real polynomials or should one allow also analytic functions and regard polynomials as approximations. Zeta functions are especially interesting analytic functions and Defekind zetas characterize extensions of rationals and one can pose physically motivated questions about them.

# 1 Introduction

In the third chapter about $M^8-H$ duality the question whether the space-time surfaces in $M^8$ allow a global slicing by string world sheets $X^2$ defined by an integrable distribution of local tangent spaces $M^2(x) \subset M^4$ and their orthogonal duals or whether there is only a discrete set of surfaces $X^2$ is discussed. Discrete set is obtained by requiring that space-time surface or its normal space contains string world sheet as a complex (commutative) sub-manifold. By the strong form of holography (SH) this is enough to deduce the image of $X^4 \subset M^8$ in $H$ from the boundary data consisting of the $H$-images of $X^2$ and metrically 2-D light-like partonic orbits $X^3_L$ of topological dimension $D=3$.

Also the relation of $M^8-H$ duality to p-adic length scale hypothesis and dark matter hierarchy are discussed and it is shown that the notion of p-adic length scale emerging from p-adic mass calculations emerges also geometrically.

The fermionic aspects of $M^8-H$ duality are discussed: the basic purely number theoretic elements are the octonionic realization of $M^8$ spinors and the replacement of Dirac equation as a partial differential equation with an algebraic equation for octonionic spinors. Dirac equation for octonionic spinors is analogous to the algebraic momentum space variant of the ordinary Dirac equation. This provides also considerable understanding about the bosonic aspects of $M^8-H$ duality. In particular, the pre-images of $X^3_L \subset X^4 \subset H$ in $M^8$ correspond to mass shells for massless octonionic spinor modes realized as light-like 3-surfaces in $M^8$. One can say that $M^8$

picture realizes the momentum space dual of the modified Dirac equation in $X^4 \subset H$. Twistor Grassmannian picture supports the view that spinor modes also in $H$ are localized to $X_L^3 \subset X^4$, and obey the modified Dirac equation associated with Chern-Simons term.

Cognitive representations is the third basic topic of the chapter. Cognitive representations are identified as sets of points in an extension of rationals for algebraic varieties with "active" points containing fermion. The representations are discussed at both $M^8$- and $H$ level. General conjectures from algebraic geometry support the vision that these sets are concentrated at lower-dimensional algebraic varieties such as string world sheets and partonic 2-surfaces and their 3-D orbits identifiable also as singularities of these surfaces. For the earlier work related to adelic TGD and cognitive representations see [L15, L5, L7].

The notion is applied in various cases and the connection with $M^8 - H$ duality is rather loose.

1. Extensions of rationals are essentially coders of information. There the possible analogy of extensions of rationals with genes deserves discussion. Extensions, which are not extensions of extensions would be analogous to genes. The notion of conserved gene as number theoretical analogy for Galois extensions as the Galois group of extension which is normal subgroup of Galois extension.

2. The work of Peter Scholze [A7] based on the notion of perfectoid has raised a lot of interest in the community of algebraic geometers. One application of the notion relates to the attempt to generalize algebraic geometry by replacing polynomials with analytic functions satisfying suitable restrictions. Also in TGD this kind of generalization might be needed at the level of $M^4 \times CP_2$ whereas at the level of $M^8$ algebraic geometry might be enough. The notion of perfectoid as an extension of p-adic numbers $Q_p$ allowing all $p$:th roots of p-adic prime $p$ is central and provides a powerful technical tool when combined with its dual, which is function field with characteristic $p$.

   Could perfectoids have a role in TGD? The infinite-dimensionality of perfectoid is in conflict with the vision about finiteness of cognition. For other p-adic number fields $Q_q$, $q \neq p$ the extension containing $p$:th roots of $p$ would be however finite-dimensional even in the case of perfectoid. Furthermore, one has an entire hierarchy of almost-perfectoids allowing powers of $p^m$:th roots of p-adic numbers. The larger the value of $m$, the larger the number of points in the extension of rationals used, and the larger the number of points in cognitive representations consisting of points with coordinates in the extension of rationals. The emergence of almost-perfectoids could be seen in the adelic physics framework as an outcome of evolution forcing the emergence of increasingly complex extensions of rationals [L8].

3. The construction of cognitive representation represents a well-known mathematical problem of finding the points of space-time surface with embedding space coordinates in given extension of rationals. Number theorist Minhyong Kim [A3, A5] has speculated about very interesting general connection between number theory and physics. The reading of a popular article about Kim's work revealed that number theoretic vision about physics provided by TGD has led to a very similar ideas and suggests a concrete realization of Kim's ideas [L32]. In the following I briefly summarize what I call identification problem. The identification of points of algebraic surface with coordinates, which are rational or in extension of rationals, is in question. In TGD framework the embedding space coordinates for points of space-time surface belonging to the extension of rationals defining the adelic physics in question are common to reals and all extensions of p-adics induced by the extension. These points define what I call cognitive representation, whose construction means solving of the identification problem.

   Cognitive representation defines discretized coordinates for a point of "world of classical worlds" (WCW) taking the role of the space of spaces in Kim's approach. The symmetries of this space are proposed by Kim to help to solve the identification problem. The maximal isometries of WCW necessary for the existence of its Kähler geometry provide symmetries identifiable as symplectic symmetries. The discrete subgroup respecting extension of rationals acts as symmetries of cognitive representations of space-time surfaces in WCW, and one can identify symplectic invariants characterizing the space-time surfaces at the orbits of the symplectic group.

4. One expects that large cognitive representations are winners in the number theoretical fight for survival. Strong form of holography suggests that it is enough to consider cognitive representations restricted to string world sheets and partonic 2-surfaces. If the 2-surface possesses large group of symmetries acting in extension of rationals, one can have large cognitive representations as orbit of point in extension. Examples of highly symmetric 2-D surfaces are geodesic spheres assignable to partonic 2-surfaces and cosmic strings and elliptic curves assignable with string world sheets and cosmic strings [L41].

5. Rationals and their extensions give rise to a unique discretizations of space-time surface (for instance) - cognitive representation - having interpretation in terms of finite measurement resolution. There are howevever many open questions. Should one allow only octonionic polynomials defined as algebraic continuations of real polynomials or should one allow also analytic functions and regard polynomials as approximations. Zeta functions are especially interesting analytic functions and Dekekind zetas characterize extensions of rationals and one can pose physically motivated questions about them [L25].

# 2 About $M^8 - H$-duality, p-adic length scale hypothesis and dark matter hierarchy

$M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy as phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ are basic assumptions of TGD, which all reduce to number theoretic vision. In the sequel $M^8 - H$ duality, p-adic length scale hypothesis and dark matter hierarchy are discussed from number theoretic perspective.

Several new results emerge. Strong form of holography (SH) allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to $H$: SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

$M^8$ duality allows to relate p-adic length scales $L_p$ to differences for the roots of the polynomial defining the extension defining "special moments in the life of self" assignable causal diamond (CD) central in zero energy ontology (ZEO). Hence p-adic length scale hypothesis emerges both from p-adic mass calculations and $M^8 - H$ duality. It is proposed that the size scale of CD correspond to the largest dark scale $nL_p$ for the extension and that the sub-extensions of extensions could define hierarchy of sub-CDs. Skyrmions are an important notion if nuclear and hadron physics, $M^8 - H$ dyality suggests an interpretation of skyrmion number as winding number as that for a map defined by complex polynomial.

## 2.1 Some background

A summary of the basic notions and ideas involved is in order.

### 2.1.1 p-Adic length scale hypothesis

In p-adic mass calculations [K8] real mass squared is obtained by so called canonical identification from p-adic valued mass squared identified as analog of thermodynamical mass squared using p-adic generelization of thermodynamics assuming super-conformal invariance and Kac-Moody algebras assignable to isometries ad holonomies of $H = M^4 \times CP_2$. This implies that the mass squared is essentially the expectation value of sum of scaling generators associated with various tensor factors of the representations for the direct sum of super-conformal algebras and if the number of factors is 5 one obtains rather predictive scenario since the p-adic temperature $T_p$ must be inverse integer in order that the analogs of Boltzmann factors identified essentially as $p^{L_0/T_p}$.

The p-adic mass squared is of form $Xp + O(p^2)$ and mapped to $X/p + O(1/p^2)$. For the p-adic primes assignable to elementary particles ($M_{127} = 2^{127} - 1$ for electron) the higher order corrections are in general extremely small unless the coefficient of second order contribution is larger integer of order $p$ so that calculations are practically exact.

Elementary particles seem to correspond to p-adic primes near powers $2^k$. Corresponding p-adic length - and time scales would come as half-octaves of basic scale if all integers $k$ are allowed. For odd values of $k$ one would have octaves as analog for period doubling. In chaotic systems also the generalization of period doubling in which prime $p = 2$ is replaced by some other small prime appear and there is indeed evidence for powers of $p = 3$ (period tripling as approach to chaos). Many elementary particles and also hadron physics and electroweak physics seem to correspond to Mersenne primes and Gaussian Mersennes which are maximally near to powers of 2.

For given prime $p$ also higher powers of $p$ define p-adic length scales: for instance, for electron the secondary p-adic time scale is .1 seconds characterizing fundamental bio-rhythm. Quite generally, elementary particles would be accompanied by macroscopic length and time scales perhaps assignable to their magnetic bodies or causal diamonds (CDs) accompanying them.

This inspired p-adic length scale hypothesis stating the size scales of space-time surface correspond to primes near half-octaves of 2. The predictions of p-adic are exponentially sensitive to the value of $k$ and their success gives strong support for p-adic length scale hypothesis. This hypothesis applied not only to elementary particle physics but also to biology and even astrophysics and cosmology. TGD Universe could be p-adic fractal.

### 2.1.2 Dark matter as phases of ordinary matter with $h_{eff} = nh_0$

The identification of dark matter as phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ is second key hypothesis of TGD. To be precise, these phases behave like dark matter and galactic dark matter could correspond to dark energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes.

There are good arguments in favor of the identification $h = 6h_0$ [L3, L20]. "Effective" means that the actual value of Planck constant is $h_0$ but in many-sheeted space-time $n$ counts the number of symmetry related space-time sheets defining space-time surface as a covering. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is $nh_0$.

### 2.1.3 $M^8 - H$ duality

$M^8 - H$ duality ($H = M^4 \times CP_2$) [L29] has taken a central role in TGD framework. $M^8 - H$ duality allows to identify space-time regions as "roots" of octonionic polynomials $P$ in complexified $M^8$ - $M^8_c$ - or as minimal surfaces in $H = M^4 \times CP_2$ having 2-D singularities.

**Remark**:$O_c$,$H_c$,$C_c$,$R_c$ will be used in the sequel for complexifications of octonions, quaternions, etc.. number fields using commuting imaginary unit $i$ appearing naturally via the roots of real polynomials.

The precise form of $M^8 - H$ duality has however remained unclear. Two assumptions are involved.

1. Associativity stating that the tangent or normal space of at the point of the space-time space-time surface $M^8$ is associative - that is quaternionic. There are good reasons to believe that this is true for the polynomial ansatz everywhere but there is no rigorous proof.

2. The tangent space of the point of space-time surface at points mappable from $M^8$ to $H$ must contain fixed $M^2 \subset M^4 \subset M^8$ or an integrable distribution of $M^2(x)$ so that the 2-surface of $M^4$ determined by it belongs to space-time surface.

The strongest, global form of $M^8 - H$ duality states that $M^2(x)$ is contained to tangent spaces of $X^4$ at all points $x$. Strong form of holography (SH) states allows also the option for which this holds true only for 2-D surfaces - string world sheets and partonic 2-surfaces - therefore mappable to $H$ and that SH allows to determined $X^4 \subset H$ from this data. In the following a realization of this weaker form of $M^8 - H$ duality is found. Note however that one cannot exclude the possibility that also associativity is true only at these surfaces for the polynomial ansatz.

### 2.1.4 Number theoretic origin of p-adic primes and dark matter

There are several questions to be answered. How to fuse real number based physics with various p-adic physics? How p-adic length scale hypothesis and dark matter hypothesis emerge from TGD?

The properties of p-adic number fields and the strange failure of complete non-determinism for p-adic differential equations led to the proposal that p-adic physics might serve as a correlate for cognition, imagination, and intention. This led to a development of number theoretic vision which I call adelic physics. A given adele corresponds to a fusion of reals and extensions of various p-adic number fields induced by a given extension of rationals.

The notion of space-time generalizes to a book like structure having real space-time surfaces and their p-adic counterparts as pages. The common points of pages defining is back correspond to points with coordinates in the extension of rationals considered. This discretization of space-time surface is in general finite and unique and is identified as what I call cognitive representation. The Galois group of extension becomes symmetry group in cognitive degrees of freedom. The ramified primes of extension are exceptionally interesting and are identified as preferred p-adic primes for the extension considered.

The basic challenge is to identify dark scale. There are some reasons to expect correlation between p-adic and dark scales which would mean that the dark scale would depend on ramified primes, which characterize roots of the polynomial defining the extensions and are thus not defined completely by extension alone. Same extension can be defined by many polynomials. The naïve guess is that the scale is proportional to the dimension $n$ of extension serving as a measure for algebraic complexity (there are also other measures). p-Adic length scales $L_p$ would be proportional $nL_p$, $p$ ramified prime of extension? The motivation would be that quantum scales are typically proportional to Planck constant. It turns out that the identification of CD scale as dark scale is rather natural.

## 2.2 New results about $M^8 - H$ duality

In the sequel some new results about $M^8 - H$ duality are deduced. Strong form of holography (SH) allows to weaken the assumptions making possible $M^8 - H$ duality. It would be enough to map only certain complex 2-D sub-manifolds of quaternionic space-time surface in $M^8$ to $H$: SH would allow to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds would be determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and they form a discrete set.

### 2.2.1 Strong form of holography (SH)

Ordinary 3-D holography is forced by general coordinate invariance (GCI) and loosely states that the data at 3-D surfaces allows to determined space-time surface $X^4 \subset H$. In ZEO 3-surfaces correspond to pairs of 3-surfaces with members at the opposite light-like boundaries of causal diamond (CD) and are analogous to initial and final states of deterministic time evolution as Bohr orbit.

This poses additional strong conditions on the space-time surface.

1. The conjecture is that these conditions state the vanishing of super-symplectic Noether charges for a sub-algebra of super-symplectic algebra $SC_n$ with radial conformal weights coming as $n$-multiples of those for the entire algebra $SC$ and its commutator $[SC_n, SC]$ with the entire algebra: these conditions generalize super conformal conditions and one obtains a hierarchy of realizations.

   This hierarchy of minimal surfaces would naturally corresponds to the hierarchy of extensions of rationals with $n$ identifiable as dimension of the extension giving rise to effective Planck constant. At the level of Hilbert spaces the inclusion hierarchies for extensions could also correspond to the inclusion hierarchies of hyper-finite factors of type $I_1$ [K16] so that $M^8 - H$ duality would imply beautiful connections between key ideas of TGD.

2. Second conjecture is that the preferred extremals (PEs) are extremals of both the volume term and Kähler action term of the action resulting by dimensional reduction making possible the induction of twistor structure from the product of twistor spaces of $M^4$ and $CP_2$ to 6-D $S^2$ bundle over $X^4$ defining the analog of twistor space. These twistor spaces must have Kähler structure since action for 6-D surfaces is Kähler action - it exists only in these two cases [A2] so that TGD is unique.

Strong form of holography (SH) is a strengthening of 3-D holography. Strong form of GCI requires that one can use either the data associated either with

- light-like 3-surfaces defining partonic orbits as surfaces at which signature of the induced metric changes from Euclidian to Minkowskian or
- the space-like 3-surfaces at the ends of CD to determine space-time surface as PE (in case that it exists).

This suggests that the data at the intersections of these 2-surfaces defined by partonic 2-surfaces might be enough for holography. A slightly weaker form of SH is that also string world sheets intersecting partonic orbits along their 1-D boundaries is needed and this form seems more realistic.

SH allows to weaken strong form of $M^8 - H$ duality mapping space-time surfaces $X^4 \subset M^8$ to $H = M^4 \times CP_2$ that it allows to map only certain complex 2-D sub-manifolds of quaternionic space-time surface to $H$: SH allows to determine $X^4 \subset H$ from this 2-D data. Complex sub-manifolds are determined by conditions completely analogous to those determined space-time surface as quaternionic sub-manifold and only discrete set of them is obtained.

### 2.2.2 Space-time as algebraic surface in $M_c^8$ regarded complexified octonions

The octonionic polynomial giving rise to space-time surface as its "root" is obtained from ordinary real polynomial $P$ with rational coefficients by algebraic continuation. The conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees associativity and one can formulate this as rather convincing argument [?] Space-time surface $X_c^4$ is identified as a 4-D root for a $H_c$-valued "imaginary" or "real" part of $O_c$ valued polynomial obtained as an $O_c$ continuation of a real polynomial $P$ with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from $M_c^8$ to $M^8$. One could drop the subscripts "$_c$" but in the sequel they will be kept.

$M_c^4$ appears as a special solution for any polynomial $P$. $M_c^4$ seems to be like a universal reference solution with which to compare other solutions.

One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have $M^4$ projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of $P$. For monic polynomials these time values are algebraic integers and Galois group permutes them.

One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [?, ?] suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where $i$ commutes with the octonionic units and defines complexifiation of octonions. $i$ appears also in the roots defining complex extensions of rationals.

### 2.2.3 How do the solutions assignable to the opposite boundaries of CD relate to each other?

CD has two boundaries. The polynomials associated with them could be different in the general formulation discussed in [L44, L45] but they could be also same. How are the solutions associated with opposite boundaries of CD glued together in a continuous manner?

1. The polynomials assignable to the opposite boundaries of CD are allowed to be polynomials of $o$ *resp.* $(o - T)$: here $T$ is the distance between the tips of CD.
2. CD brings in mind the realization of conformal invariance at sphere: the two hemispheres correspond to powers of $z$ and $1/z$: the condition $z = \overline{1/z}$ at unit circle is essential and there is no real conjugation. How the sphere is replaced with 8-D CD which is also complexified. The absence of conjugation looks natural also now: could CD contain a 3-surface analogous to the unit circle of sphere at which the analog of $z = \overline{1/z}$ holds true? If so, one has

$P(o,z) = P(1/o,z)$ and the solutions representing roots fo $P(o,z)$ and $P(1/o,z)$ can be glued together.

Note that $1/o$ can be expressed as $\overline{o}/o\overline{o}$ when the Minkowskian norm squared $\overline{o}o$ is non-vanishing and one has polynomial equation also now. This condition is true outside the boundary of 8-D light-cone, in particular near the upper boundary of CD.

The counter part for the length squared of octonion in Minkowskian signature is light-one proper time coordinate $a^2 = t^2 - r^2$ for $M^8_+$. Replacing $o$ which scaled dimensionless variable $o_1 = o/(T/2)$ the gluing take place along $a = T/2$ hyperboloid.

One has algebraic holomorphy with respect to $o$ but also anti-holomorphy is possible. What could these two options correspond to? Could the space-time surfaces assignable to self and its time-reversal relate by octonionic conjugation $o \rightarrow \overline{o}$ relating two Fock vacuums annihilated by fermionic annihilation *resp.* creation operators?

In [L44, L45] the possibility that the sequence of SSFRs or BSFRs could involve iteration of the polynomial defining space-time surface - actually different polynomials were allowed for two boundaries. There are 3 options: each SSFR would involve the replacement $Q = P \circ .. \circ P \rightarrow P \circ Q$, the replacement occurs only when new "special moments in the life of self" defined by the roots of $P$ as $t = r_n$ balls of cd, or the replacement can occur in BSFR when the metabolic resources do not allow to continue the iteration (the increase of $h_{eff}$ during iteration increases the needed metabolic feed).

The iteration is compatible with the proposed picture. The assumption $P(0) = 0$ implies that iterates of $P$ contain also the roots of $P$ as roots - they are like conserved genes. Also the 8-D light-cone boundary remains invariant under iteration. Even more general function decompositions $P \rightarrow Q \rightarrow P$ are consistent with the proposed picture.

### 2.2.4 Brane-like solutions

One obtains also 6-D brane-like solutions to the equations.

1. In general the zero loci for imaginary or real part are 4-D but the 7-D light-cone $\delta M^8_+$ of $M^8$ with tip at the origin of coordinates is an exception [L10, L11, L12]. At $\delta M^8_+$ the octonionic coordinate $o$ is light-like and one can write $o = re$, where 8-D time coordinate and radial coordinate are related by $t = r$ and one has $e = (1+e_r)/\sqrt{2}$ such that one as $e^2 = e$.

   Polynomial $P(o)$ can be written at $\delta M^8_+$ as $P(o) = P(r)e$ and its roots correspond to 6-spheres $S^6$ represented as surfaces $t_M = t = r_N$, $r_M = \sqrt{r_N^2 - r_E^2} \leq r_N$, $r_E \leq r_N$, where the value of Minkowski time $t = r = r_N$ is a root of $P(r)$ and $r_M$ denotes radial Minkowski coordinate. The points with distance $r_M$ from origin of $t = r_N$ ball of $M^4$ has as fiber 3-sphere with radius $r = \sqrt{r_N^2 - r_E^2}$. At the boundary of $S^3$ contracts to a point.

2. These 6-spheres are analogous to 6-D branes in that the 4-D solutions would intersect them in the generic case along 2-D surfaces $X^2$. The boundaries $r_M = r_N$ of balls belong to the boundary of $M^4$ light-cone. In this case the intersection would be that of 4-D and 3-D surface, and empty in the generic case (it is however quite not clear whether topological notion of "genericity" applies to octonionic polynomials with very special symmetry properties).

3. The 6-spheres $t_M = r_N$ would be very special. At these 6-spheres the 4-D space-time surfaces $X^4$ as usual roots of $P(o)$ could meet. Brane picture suggests that the 4-D solutions connect the 6-D branes with different values of $r_n$.

   The basic assumption has been that particle vertices are 2-D partonic 2-surfaces and light-like 3-D surfaces - partonic orbits identified as boundaries between Minkowskian and Euclidian regions of space-time surface in the induced metric (at least at $H$ level) - meet along their 2-D ends $X^2$ at these partonic 2-surfaces. This would generalize the vertices of ordinary Feynman diagrams. Obviously this would make the definition of the generalized vertices mathematically elegant and simple.

   Note that this does not require that space-time surfaces $X^4$ meet along 3-D surfaces at $S^6$. The interpretation of the times $t_n$ as moments of phase transition like phenomena is suggestive. ZEO based theory of consciousness suggests interpretation as moments for state function reductions analogous to weak measurements ad giving rise to the flow of experienced time.

4. One could perhaps interpret the free selection of 2-D partonic surfaces at the 6-D roots as initial data fixing the 4-D roots of polynomials. This would give precise content to strong form of holography (SH), which is one of the central ideas of TGD and strengthens the 3-D holography coded by ZEO alone in the sense that pairs of 3-surfaces at boundaries of CD define unique preferred extremals. The reduction to 2-D holography would be due to preferred extremal property realizing the huge symplectic symmetries and making $M^8 - H$ duality possible as also classical twistor lift.

   I have also considered the possibility that 2-D string world sheets in $M^8$ could correspond to intersections $X^4 \cap S^6$? This is not possible since time coordinate $t_M$ constant at the roots and varies at string world sheets.

   Note that the compexification of $M^8$ (or equivalently octonionic $E^8$) allows to consider also different variants for the signature of the 6-D roots and hyperbolic spaces would appear for $(\epsilon_1, \epsilon_i, .., \epsilon_8)$, $epsilon_i = \pm 1$ signatures. Their physical interpretation - if any - remains open at this moment.

5. The universal 6-D brane-like solutions $S^6_c$ have also lower-D counterparts. The condition determining $X^2$ states that the $C_c$-valued "real" or "imaginary" for the non-vanishing $Q_c$-valued "real" or "imaginary" for $P$ vanishes. This condition allows universal brane-like solution as a restriction of $O_c$ to $M^4_c$ (that is $CD_c$) and corresponds to the complexified time=constant hyperplanes defined by the roots $t = r_n$ of $P$ defining "special moments in the life of self" assignable to CD. The condition for reality in $R_c$ sense in turn gives roots of $t = r_n$ a hyper-surfaces in $M^2_c$.

### 2.2.5 Explicit realization of $M^8 - H$ duality

$M^8 - H$ duality allows to map space-time surfaces in $M^8$ to $H$ so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in $M^8$ and as minimal surfaces with 2-D singularities in $H$ satisfying an infinite number of additional conditions stating vanishing of Noether charges for super-symplectic algebra actings as isometries for the "world of classical worlds" (WCW). Twistor lift allows variants of this duality. $M^8_H$ duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of extensions of rationals defining an evolutionary hierarchy. This forms the basis for the number theoretical vision about TGD.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions.

1. Associativity condition for tangent-/normal space is the first essential condition for the existence of $M^8 - H$ duality and means that tangent - or normal space is quaternionic.

2. The tangent space of space-time surface and thus space-time surface itself must contain a preferred $M^2_c \subset M^4_c$ or more generally, an integrable distribution of tangent spaces $M^2_c(x)$ and similar distribution of their complements $E^2c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in $R_c$ sense. $E^2_c(x)$ would correspond to partonic 2-surface.

One can imagine two realizations for this condition.

**Option I**: Global option states that the distributions $M^2_c(x)$ and $E^2_c(x)$ define slicing of $X^4_c$.

**Option II**: Only discrete set of 2-surfaces satisfying the conditions exist, they are mapped to $H$, and strong form of holography (SH) applied in $H$ allows to deduce space-time surfaces in $H$. This would be the minimal option.

That the selection between these options is not trivial is suggested by following.

1. For massless extremals (MEs, topological light rays) parameterized by light-like vector vector $k$ defining $M^2 \subset M^2 \times E^2 \subset M^4$ at each point and by space-like polarization vector $\epsilon$ depending on single transversal coordinate of $E^2$ [K1].

2. $CP_2$ coordinates have an arbitrary dependence on both $u = k \cdot m$ and $w = \epsilon \cdot m$ and can be also multivalued functions of $u$ and $w$. Single light-like vector $k$ is enough to identify $M^2$. $CP_2$ type extremals having metric and Kähler form of $CP_2$ have light-like geodesic as $M^4$ projection defining $M^2$ and its complement $E^2$ in the normal space.

3. String like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ are minimal surfaces and $X^2$ defines the distribution of $M^2(x) \subset M^4$. $Y^2$ ddefines the complement of this distribution.

   **Option I** is realized in all 3 cases. It is not clear whether $M^2$ can depend on position in the first 2 cases and also $CP_2$ point in the third case. It could be that only a discrete set of these string world sheets assignable to wormhole contacts representing massless particles is possible (**Option II**).

How these conditions would be realized?

1. The basic observation is that $X^2c$ can be fixed by posing to the non-vanishing $H_c$-valued part of octonionic polynomial $P$ condition that the $C_c$ valued "real" or "imaginary" part in $C_c$ sense for $P$ vanishes. $M_c^2$ would be the simplest solution but also more general complex sub-manifolds $X_c^2 \subset M_c^4$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

   These surfaces would be like the families of curves in complex plane defined by $u = 0$ an $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. As found, there are also classes special global solutions for which the choice of $M_c^2$ is global and does not depend on space-time point. The interpretation would be in terms of modes of classical massless fields characterized by polarization and momentum. If the identification of $M_c^2$ is correct, these surfaces are however unstable against perturbations generating discrete string world sheets and orbits of partonic 2-surfaces having interpretation space-time counterparts of quanta. That fields are detected via their quanta was the revolutionary observation that led to quantum theory. Could quantum measurement induce the instability decomposing the field to quanta at the level of space-time topology?

3. One can generalize this condition so that it selects 1-D surface in $X_c^2$. By assuming that $R_c$-valued "real" or "imaginary" part of quaternionic part of $P$ at this 2-surface vanishes. one obtains preferred $M_c^1$ or $E_c^1$ containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as complexified string. Together these kind 1-D surfaces in $R_c$ sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \to C_c \to H_c \to O_c$ realized as surfaces.

   This option could be made possible by SH. SH states that preferred extremals are determined by data at 2-D surfaces of $X^4$. Even if the conditions defining $X_c^2$ have only a discrete set of solutions, SH at the level of $H$ could allow to deduce the preferred extremals from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. Associativity and existence of $M^2(x)$ would be required only at the 2-D surfaces.

4. I have proposed that physical string world sheets and partonic 2-surfaces appear as singularities and correspond to 2-D folds of space-time surfaces at which the dimension of the quaternionic tangent space degenerates from 4 to 2 [L28] [K1]. This interpretation is consistent with a book like structure with 2-pages. Also 1-D real and imaginary manifolds could be interpreted as folds or equivalently books with 2 pages.

   For the singular surfaces the dimension quaternionic tangent or normal space would reduce from 4 to 2 and it is not possible to assign $CP_2$ point to the tangent space. This does not of course preclude the singular surfaces and they could be analogous to poles of analytic function. Light-like orbits of partonic 2-surfaces would in turn correspond to cuts.

5. What could the normal space singularity mean at the level of $H$? Second fundamental form defining vector basis in normal space is expected to vanish. This would be the case for minimal surfaces.

(a) String world sheets with Minkowskian signature (in $M^4$ actually) are expected to be minimal surfaces. In this case $T$ matters and string world sheets could be mapped to $H$ by $M^8 - H$ duality and SH would work for them.

(b) The light-like orbits of partonic 2-surfaces with Euclidian signature in $H$ would serve as analogs of cuts. $N$ is expected to matter and partonic 2-surfaces should be minimal surfaces. Their branching of partonic 2-surfaces is thus possible and would make possible (note the analogy with the branching of soap films) for them to appear as 2-D vertices in $H$.

The problem is to identify the pre-images of partonic 2-surfaces in $M^8$. The light-likeness of the orbits of partonic 2-surfaces (induced 4-metric changes its signature and degenerates to 3-D) should be important. Could light-likeness in this sense define the pre-images partonic orbits in $M^8$?

**Remark**: It must be emphasized that SH makes possible $M^8-H$ correspondence assuming that also associativity conditions hold true only at partonic 2-surfaces and string world sheets. Thus one could give up the conjecture that the polynomial ansatz implies that tangent or normal spaces are associative. Proving that this is the case for the tangent/normal spaces of these 2-surfaces should be easier.

#### 2.2.6 Does $M^8 - H$ duality relate hadron physics at high and low energies?

During the writing of this article I realized that $M^8 - H$ duality has very nice interpretation in terms of symmetries. For $H = M^4 \times CP_2$ the isometries correspond to Poincare symmetries and color $SU(3)$ plus electroweak symmetries as holonomies of $CP_2$. For octonionic $M^8$ the subgroup $SU(3) \subset G_2$ is the sub-group of octonionic automorphisms leaving fixed octonionic imaginary unit invariant - this is essential for $M^8 - H$ duality. $SU(3)$ is also subgroup of $SO(6) \equiv SU(4)$ acting as rotation on $M^8 = M^2 \times E^6$. The subgroup of the holonomy group of $SO(4)$ for $E^4$ factor of $M^8 = M^4 \times E^4$ is $SU(2) \times U(1)$ and corresponds to electroweak symmetries. One can say that at the level of $M^8$ one has symmetry breaking from $SO(6)$ to $SU(3)$ and from $SO(4) = SU(2) \times SO(3)$ to $U(2)$.

This interpretation gives a justification for the earlier proposal that the descriptions provided by the old-fashioned low energy hadron physics assuming $SU(2)_L \times SU(2)_R$ and acting acting as covering group for isometries $SO(4)$ of $E^4$ and by high energy hadron physics relying on color group $SU(3)$ are dual to each other.

#### 2.2.7 Skyrmions and $M^8 - H$ duality

I received a link (`https://tinyurl.com/ycathr3u`) to an article telling about research (`https://tinyurl.com/yddwhr2o`) carried out for skyrmions, which are very general condensed matter quasiparticles. They were found to replicate like DNA and cells. I realized that I have not clarified myself the possibility of skyrmions on TGD world and decided to clarify my thoughts.

1. *What skyrmions are?*

Consider first what skyrmions are.

1. Skyrmions are topological entities. One has some order parameter having values in some compact space S. This parameter is defined in say 3-ball such that the parameter is constant at the boundary meaning that one has effectively 3-sphere. If the 3rd homotopy group of S characterizing topology equivalence classes of maps from 3-sphere to S is non-trivial, you get soliton-llike entities, stable field configurations not deformable to trivial ones (constant value). Skyrmions can be assigned to space S which is coset space $SU(2)_L \times SU(2)_R/SU(2)_V$, essentially $S^3$ and are labelled by conserved integer-valued topological quantum number.

2. One can imagine variants of this. For instance, one can replace 3-ball with disk. $SO(3) = S^3$ with 2-sphere $S^2$. The example considered in the article corresponds to discretized situation in which one has magnetic dipoles/spins at points of say discretized disk such that spins have same direction about boundary circle. The distribution of directions of spin can give rise to skyrmion-like entity. Second option is distribution of molecules which do not have symmetry

axis so that as rigid bodies the space of their orientations is discretized version of $SO(3)$. The field would be the orientation of a molecule of lattice and one has also now discrete analogs of skyrmions.

3. More generally, skyrmions emerge naturally in old-fashioned hadron physics, where $SU(2)_L \times SU(2)_R/SU(2)_V$ involves left-handed, right-handed and vectorial subgroups of $SO(4) = SU(2)_L \times SU(2)_R$. The realization would be in terms of 4-component field $(\pi, \sigma)$, where $\pi$ is charged pion with 3 components - axial vector - and $\sigma$ which is scalar. The additional constraint $\pi \cdot \pi + \sigma^2 = constant$ defines 3-sphere so that one has field with values in $S^3$. There are models assigning this kind of skyrmion with nucleon, atomic nuclei, and also in the bag model of hadrons bag can be thought of as a hole inside skyrmion. These models seem to have something to do with reality so that a natural question is whether skyrmions might appear in TGD.

*2. Skyrmion number as winding number*

In TGD framework one can regard space-time as 4-surface in either octonionic $M^8_c$, $c$ refers here to complexification by an imaginary unit $i$ commuting with octonions, or in $M^4 \times CP_2$. For the solution surfaces $M^8$ has natural decomposition $M^8 = M^2 \times E^6$ and $E^6$ has $SO(6)$ as isometry group containing subgroup $SU(3)$ having automorphisms of octonions as subgroup leaving $M^2$ invariant. $SO(6) = SU(4)$ contains $SU(3)$ as subgroup, which has interpretation as isometries of $CP_2$ and counterpart of color gauge group. This supports $M^8 - H$ duality, whose most recent form is discussed in [L42].

The map $S^3 \to S^3$ defining skyrmion could be taken as a phenomenological consequence of $M^8 - H$ duality implying the old-fashioned description of hadrons involving broken $SO(4)$ symmetry (PCAC) and unbroken symmetry for diagonal group $SO(3)_V$ (CCV). The analog of $(\pi, sigma)$ field could correspond to a B-E condensate of pions $(\pi, sigma)$.

The obvious question is whether the map $S^3 \to S^3$ defining skyrmion could have a deeper interpretation in TGD framework. I failed to find any elegant formulation. One could however generalize and ask whether skyrmion like entities characterize by winding number are predicted by basic TGD.

1. In the models of nucleon and nuclei the interpretation of conserved topological skyrmion number is as baryon number. This number should correspond to the homotopy class of the map in question, essentially winding number. For polynomials of complex number degree corresponds to winding number. Could the degree $n = h_{eff}/h_0$ of polynomial $P$ having interpretation as effective Planck constant and measure of complexity - kind of number theoretic IQ - be identifiable as skyrmion number? Could it be interpreted as baryon number too?

2. For leptons regarded as local 3 anti-quark composites in TGD based view about SUSY [L34] the same interpretation would make sense. It seems however that the winding number must have both signs. Degree is $n$ is however non-negative.

   Here complexification of $M^8$ to $M^8_c$ is essential. One an allow both holomorphic and anti-holomorphic continuations of real polynomials $P$ (with rational coefficients) using complexification defined by commutative imaginary unit $i$ in $M^8_c$ so that one has polynomials $P(z)$ *resp.* $P(\overline{z})$ in turn algebraically continued to complexified octonionic polynomials $P(z, o)$ *resp.* $P(\overline{z}, o)$.

   Particles *resp.* antiparticles would correspond to the roots of octonionic polynomial $P(z, o)$ resp. $P(\overline{z}, o)$ meaning space-time geometrization of the particle-antiparticle dichotomy and would be conjugates of each other. This could give a nice physical interpretation to the somewhat mysterious complex roots of $P$.

*3. More detailed formulation*

To make this formulation more detailed on must ask how 4-D space-time surfaces correspond to 8-D "roots" for the "imaginary" ("real" ) part of complexified octonionic polynomial as surfaces in $M^8_c$.

1. Equations state the simultaneous vanishing of the 4 components of complexified quaternion valued polynomial having degree $n$ and with coefficients depending on the components of $O_c$, which are regarded as complex numbers $x + iy$, where $i$ commutes with octonionic units. The coefficients of polynomials depend on complex coordinates associated with non-vanishing "real" ("imaginary") part of the $O_c$ valued polynomial.

2. To get perspective, one can compare the situation with that in catastrophe theory in which one considers roots for the gradient of potential function of behavior variables $x^i$. Potential function is polynomial having control variables as parameters. Now behavior variable correspond "imaginary" ("real" ) part and control variables to "real" ("imaginary") of octonionic polynomial.

   For a polynomial with real coefficients the solution divides to regions in which some roots are real and some roots are complex. In the case of cusp catastrophe one has cusp region with 3-D region of the parameter defined by behavior variable $x$ and 2 control parameters with 3 real roots, the region in which one has one real root. The boundaries for the projection of 3-sheeted cusp to the plane defined by control variables correspond to degeneration of two complex roots to one real root.

   In the recent case it is not clear whether one cannot require the $M_c^8$ coordinates for space-time surface to be real but to be in $M^8 = M^1 + iE^7$ .

3. Allowing complex roots gives 8-D space-time surfaces. How to obtain real 4-D space-time surfaces?

   (a) One could project space-time surfaces to real $M^8$ to obtain 4-D real space-time surfaces. For $M^8$ this would mean projection to $M^1 + iE^7$ and in time direction the real part of root is accepted and is same for the root and its conjugate. For $E^7$ this would mean that imaginary part is accepted and means that conjugate roots correspond to different space-time surfaces and the notion of baryon number is realized at space-time level.

   (b) If one allows only real roots, the complex conjugation proposed to relate fermions and anti-fermions would be lost.

4. One can select for 4 complex $M_c^8$ coordinates $X^k$ of the surface and the remaining 4 coordinates $Y^k$ can be formally solved as roots of $n$:th degree polynomial with dynamical coefficients depending on $X^k$ and the remaining $Y^k$. This is expected to give rise to preferred extremals with varying dimension of $M^4$ and $CP_2$ projections.

5. It seems that all roots must be complex.

   (a) The holomorphy of the polynomials with respect to the complex $M_c^8$ coordinates implies that the coefficients are complex in the generic point $M_c^8$. If so, all 4 roots are in general complex but do not appear as conjugate pairs. The naïve guess is that the maximal number of solutions would be $n^4$ for a given choice of $M^8$ coordinates solved as roots. An open question is whether one can select subset of roots and what happens at $t = r_n$ surfaces: could different solutions be glued together at them.

   (b) Just for completeness one can consider also the case that the dynamical coefficients are real - this is true in the $E^8$ sector and whether it has physical meaning is not clear. In this case the roots come as real roots and pairs formed by complex root and its conjugate. The solution surface can be divided into regions depending on the character of 4 roots. The $n$ roots consist of complex root pairs and real roots. The members or complex root pairs are mapped to same point in $E^8$.

*4. Could skyrmions in TGD sense replicate?*

What about the observation that condensed matter skyrmions replicate? Could this have analog at fundamental level?

1. The assignment of conserved topological quantum number to the skyrmion is not consistent with replication unless the skyrmion numbers of outgoing states sum up to that of the initial state. If the system is open one can circumvent this objection. The replication would be like replication of DNA in which nucleotides of new DNA strands are brought to the system to form new strands.

2. It would be fascinating if all skyrmions would correspond to space-time surfaces at fundamental $M^8$ level. If so, skyrmion property also in magnetic sense could be induced by from a deeper geometric skyrmion property of the MB of the system. The openness of the system would be essential to guarantee conservation of baryon number. Here the fact that leptons and baryons have opposite baryon numbers helps in TGD framework. Note also ordinary DNA replication could correspond to replication of MB and thus of skyrmion sequences.

## 2.3 About p-adic length scale hypothesis and dark matter hierarchy

It is good to introduce first some background related to p-adic length scale hypothesis discussed in chapters of [K9] and dark matter hierarchy discussed in chapters [K6, K7], in particular in chatper [?].

### 2.3.1 General form of p-adic length scale hypothesis

The most general form of p-adic length scale hypothesis does not pose conditions on allowed p-adic primes and emerges from p-adic mass calculations [K3, K8, K10]. It has two forms corresponding to massive particles and massless particles.

1. For massive particles the preferred p-adic mass calculations based on p-adic thermodynamics predicts the p-adic mass squared $m^2$ to be proportional to $p$ or its power- the real counterpart of $m^2$ is proportional to $1/p$ or its power. In the simplest case one has

$$m^2 = \frac{X}{p}\frac{\hbar}{L_0} \quad ,$$

where $L_0$ is apart from numerical constant the length $R$ of $CP_2$ geodesic circle. $X$ is a numerical constant not far from unity. $X \geq 1$ is small integer in good approximation. For instance for electron one has $x = 5$.

By Uncertainty Principle the Compton length of particle is characterizing the size of 3-surfaces assignable to particle are proportional to $\sqrt{p}$:

$$L_c(m) = \frac{\hbar}{m} = \sqrt{\frac{1}{X}}L_p \quad , \quad L_p = \sqrt{p}L_0 = \quad .$$

Here $L_p$ is p-adic length scale and corresponds to minimal mass for given p-adic prime. p-Adic length scale would be would characterize the size of the 3-surface assignable to the particle and would correspond to Compton length.

2. For massless particles mass vanishes and the above picture is not possible unless there is very small mass coming from p-adic thermodynamics and determined by the size scale of CD - this is quite possible. The preferred time/spatial scales p-adic energy- equivalently 3-momentum are proportional to p-adic prime $p$ or its power. The real energy is proportional to $1/p$. At the embedding space level the size of scale causal diamond (CD) [L33] would be proportional to $p$: $L = T = pL_0$, $L_0 = T_0$ for $c = 1$. The interpretation in terms of Uncertainty Principle is possible.

There would be therefore two levels: space-time level and embedding space level . At the space-time level the primary p-adic length scale would be proportional to $\sqrt{p}$ whereas the p-adic length scale at embedding space-time would correspond to secondary p-adic length scale proportional to $p$. The secondary p-adic length scales would assign to elementary new physics in macroscopic scales. For electron the size scale of CD would be about .1 seconds, the time scale associated with the fundamental bio-rhythm of about 10 Hz.

3. A third piece in the picture is adelic physics [L15, L16] inspiring the hypothesis that effective Planck constant $h_{eff}$ given by $h_{eff}/h_0 = n$, $h = 6h_0$, labels the phases of ordinary matter identified as dark matter. $n$ would correspond to the dimension of extension of rationals.

   The connection between preferred primes and the value of $n = h_{eff}/h_0$ is interesting. One proposal is that preferred primes $p$ in p-adic length scale hypothesis determining the mass scale of particle correspond to so called ramified primes, which characterize the extensions. The p-adic variant of the polynomial defining space-time surfaces in $M^8$ picture would have vanishing discriminant in order $O(p)$. Since discriminant is proportional to the product of differences of different roots of the polynomial, two roots would be very near to each other p-adically. This would be mathematical correlate for criticality in p-adic sense.

   $M^8 - H$ duality [L29, L26] leads to the prediction that the roots $r_n$ of polynomial defining the space-time region in $M^8$ correspond to preferred time values $t = t_n =\propto r_n$- I have called $t = t_n$ "special moments in the life of self". Since the squares for the differences for the roots are proportional to ramified primes, these time differences would code for ramified primes assignable to the space-time surface. There would be several p-adic time scales involved and they would be coded by $t_{ij} = r_i - r_j$, whose moduli squared are divided by so called ramified primes defining excellent candidates for preferred p-adic primes. p-Adic physics would make itself visible at the level of space-time surface in terms of "special moments in the life of self".

4. p-Adic length scales emerge naturally from $M^8 - H$ duality [L29, L26]. Ramified primes would in $M^8$ picture appear as factors of time differences associated with "special moments in the life of self" associated with CD [L26]. One has $|t_i - t_j| \propto \sqrt{p_{ij}}$, $p_{ij}$ ramified prime. It is essential that square root of ramified prime appears here.

   This suggests strongly that p-adic length scale hypothesis is realized at the level of space-time surface and there are several p-adic length scales present coded to the time differences. Knowing of the polynomial would give information about p-adic physics involved. If dark scales correlate with p-adic length scales as proposed, the definition of dark scale should assume the dependence of ramified primes quite generally rather than as a result of number theoretic survival of fittest as one might also think.

   The factors $t_i - t_j$ are proportional - not only to the typically very large p-adic prime $p_{max}$ charactering the system - but also smaller primes or their powers. Could the scales in question be of form $l_p = \sqrt{X}\sqrt{p_{max}}L_0$ rather than p-adic length scales $L_{p_{ram}}$ defined by various ramified primes. Here $X$ would be integer consisting of small ramified primes.

   p-Adic mass calculations predict in an excellent approximation the mass of the particle is given by $m = (\sqrt{X}/\sqrt{p})m_0$, $X$ small integer and $m_0 = 1/L_0$. Compton length would be given by $L_c(p) = \sqrt{p}/\sqrt{X})L_0$. The identification $l_p = L_c(p)$ would be attractive but is not possible unless one has $X = 1$. In this case one would be considering p-adic length scale $L_p$. the interpretation in terms of multi-p-adicity seems to be the realistic option.

### 2.3.2 About more detailed form of p-adic length scale hypothesis

More specific form of p-adic length scale hypothesis poses conditions on physically preferred p-adic primes. There are several guesses for preferred primes. They could be primes near to integer powers $2^k$, where $k$ could be positive integer, which could satisfy additional conditions such as being odd, prime or be associated with Mersenne prime or Gaussian Mersenne. One can consider also powers of other small primes such as $p = 2, 3, 5$. p-Adic length scale hypothesis in is basic form would generalize the notion of period doubling. For odd values of $k$ one would indeed obtain period doubling, tripling, etc... suggesting strongly chaos theoretic origin.

*1. p-Adic length scale hypothesis in its basic form*

Consider first p-adic length scale hypothesis in its basic form.

1. In its basic form states that primes $p \simeq 2^k$ are preferred p-adic primes and correspond by p-adic mass calculations p-adic length scales $L_p \equiv L(k) \propto \sqrt{p} = 2^{k/2}$. Mersenne primes and primes associated with Gaussian Mersennes as especially favored primes and charged leptons ($k \in \{127, 113, 107\}$) and Higgs boson ($k = 89$) correspond to them. Also hadron

physics ($k = 107$) and nuclear physics ($k = 113$) correspond to these scales. One can assign also to hadron physics Mersenne prime and the conjecture is that Mersennes and Gaussian Mersennes define scaled variants of hadron physics and electroweak physics. In the length scale between cell membrane thickness fo 10 nm and nuclear size about 2.5 $\mu$m there are as many as 4 Gaussian Mersennes corresponding to $k \in \{151, 157, 163, 167\}$.

Mersenne primes correspond to prime values of $k$ and I have proposed that $k$ is prime for fundamental p-adic length scales quite generally. There are also however also other p-adic length scales - for instance, for quarks $k$ need not be prime - and it has remained unclear what criterion could select the preferred exponents $k$. One can consider also the option that odd values of $k$ defined fundamental p-adic length scales.

2. What makes p-adic length scale hypothesis powerful is that masses of say scaled up variant of hadron physics can be estimated by simple scaling arguments. It is convenient to use electron's p-adic length scale and calculate other p-adic length scales by scaling $L(k) = 2^{(k-127)/2}L(127)$.

Here one must make clear that there has been a confusion in the definitions, which was originally due to a calculational error.

1. I identified the p-adic length scale $L(151)$ mistakenly as $L(151) = 2^{(k-127)/2}L_e(127)$ by using instead of $L(127)$ electron Compton length $L_e \simeq L(127/\sqrt{5}$. The notation for these scales would be therefore $L_e(k)$ identified as $L_e(k) = 2^{(k-127)/2}L_e(127)$ and I have tried to use it systematically but failed to use the wrong notation in informal discussions.

2. This mistake might reflect highly non-trivial physics. It is scaled up variants of $L_e$ which seem to appear in physics. For instance, $L_e(151) \simeq 10$ nm corresponds to basic scale in living matter. Why the biological important scales should correspond to scaled up Compton lengths for electron? Could dark electrons with scaled up Compton scales equal to $L_e(k)$ be important in these scales? And what about the real p-adic length scales relate to these scales by a scaling factor $\sqrt{5} \simeq 2.23$?

*2. Possible modifications of the p-adic length scale hypothesis*

One can consider also possible modifications of the p-adic length scale hypothesis. In an attempt to understand the scales associated with INW structures in terms of p-adic length scale hypothesis it occurred to me that the scales which do not correspond to Mersenne primes or Gaussian Mersennes might be generated somehow from the these scales.

1. Geometric mean $L = \sqrt{L(k_1)L(k_2)}$ would length scale which would correspond to $L_p$ with $p \simeq 2^{(k_1+k_2)/2}$. This is of the required form only if $k = k_1 + k_2$ is even so that $k_1$ and $k_2$ are both even or odd. If one starts from Mersennes and Gaussian Mersennes the condition is satisfied. The value of $k = (k_1 + k_2)/2$ can be also even.

   **Remark**: The geometric mean $(127 + 107)/2 = 117$ of electronic and hadronic Mersennes corresponding to mass 16 MeV rather near to the mass of so called X boson [L4] (`https://tinyurl.com/ya3yuzeb`).

2. One can also consider the formula $L = (L(k_1)L(k_2)..L(k_n))^{1/n}$ but in this case the scale would correspond to prime $p \simeq 2^{k_1+...k_n)/n}$. Since $(k_1 + ..k_n)/n$ is integer only if $k_1 + ...k_n$ is proportional to $n$.

What about the allowed values of fundamental integers $k$? It seems that one must allow all odd integers.

1. If only prime values of $k$ are allowed, one can obtain obtain for twin prime pair $(k-1, k+1)$ even integer $k$ as geometric mean $\sqrt{k}$ if $k$ is square. If prime $k$ is not a member of this kind of pair, it is not possible to get integers $k-1$ and $k+1$. If only prime values of $k$ are fundamental, one could assign to $k = 89$ characterizing Higgs boson weak bosons $k = 90$ possibly characterizing weak bosons. Therefore it seems that one must allow all odd integers with the additional condition already explained.

2. Just for fun one can check whether $k = 161$ forced by the argument related to electroweak scale and $h_{eff}$ corresponds to a geometric mean of two Gaussian Mersennes. One has $k(k_1, k_2) = (k_2 + k_2)/2$ giving the list $k(151, 157) = 154$), $k(151, 163) = 157$ Gaussian Mersenne itself, $k(151, 167) = 159$, $k(157, 163) = 160$, $k(157, 167) = 162$, $k(163, 167) = 165$. Unfortunately, $k = 161$ does not belong to this set. If one allows all odd values of $k$ as fundamental, the problem disappears.

One can also consider refinements of p-adic length scale hypothesis in its basic form.

1. One can consider also a generalization of p-adic length scale hypothesis to allow length scales coming as powers of small primes. The small primes $p = 2, 3, 5$ assignable to Platonic solids would be especially interesting. $p = 2, 3, 5$ and also Fermat primes and Mersenne primes are maximally near to powers of two and their powers would define secondary and higher p-adic length scales. In this sense the extension would not actually bring anything new.

   There is evidence for the occurrence of long p-adic time scales coming as powers of 3 [?, ?] (`http://tinyurl.com/ycesc5mq`) and [K11] (`https://tinyurl.com/y8camqlt`. Furthermore, prime 5 and Golden Mean are related closely to DNA helical structure. Portion of DNA with L(151) contains 10 DNA codons and is the minimal length containing an integer number of codons.

2. The presence of length scales associated with 1 nm and 2 nm thick structures encourage to consider the possibility of p-adic primes near integers $2^k3^l5^m$ defining generators of multiplicative ideals of integers. They do not satisfy the maximal nearness criterion anymore but would be near to integers representable as products of powers of primes maximally near to powers of two.

What could be the interpretation of the integer $k$ appearing in $p \simeq 2^k$? Elementary particle quantum numbers would be associated with wormhole contacts with size scale of $CP_2$ whereas elementary particles correspond to p-adic size scale about Compton length. What could determine the size scale of wormhole contact? I have proposed that to p-adic length scale there is associated a scale characterizing wormhole contact and depending logarithmically on it and corresponds to $L_k = (1/2)log(p)L_0 = (k/2)log(2)L_0$. The generalization of this hypothesis to the case of $p \simeq 2^k3^l5^m...$ be straightforward and be $L_{k,l,m} = (1/2)(klog(2) + llog(3) + mlog(5) + ..)$.

### 2.3.3 Dark scales and scales of CDs and their relation to p-adic length scale hierarchy

There are two length scale hierarchies. p-Adic length scale hierarchy assignable to space-time surfaces and the dark hierarchy assignable to CDs. One should find an identification of dark scales and understand their relationship to p-adic length scales.

*1. Identification of dark scales*

The dimension $n$ of the extension provides the roughest measure for its complexity via the formula $h_{eff}/h_0 = n$. The basic - rather ad hoc - assumption has been that $n$ as dimension of extension defines not only $h_{eff}$ but also the size scale of CD via $L = nL_0$.

This assumption need not be true generally and already the attempt to understand gravitational constant [L43] as a prediction of TGD led to the proposal that gravitational Planck constant $h_{gr} = n_{gr}h_0 = GMm/v_0$ [?] could be coded by the data relating to a normal subgroup of Galois group appearing as a factor of $n$.

The most general option is that dark scale is coded by a data related to extension of its sub-extension and this data involves ramified primes. Ramified primes depend on the polynomial defining the extension and there is large number polynomials defining the same extension. Therefore ramified ramifies code information also about polynomial and dynamics of space-time surface.

First some observations.

1. For Galois extension the order $n$ has a natural decomposition to a product of orders $n_i$ of its normal subgroups serving also as dimensions of corresponding extensions: $n = \prod_i n_i$. This implies a decomposition of the group algebra of Galois group to a tensor product of state spaces with dimensions $n_i$ [L45].

2. Could one actually identify several dark scales as the proposed identifications of gravitational, electromagnetic, etc variants of $h_{eff}$ suggest? The hierarchy of normal subgroups of Galois group of rationals corresponds to sub-groups with orders given by $N(i,1) = n_i n_{i-1}...n_{i-1}$ of $n$ define orders for the normal subgroups of Galois group. For extensions of $k-1$:th extension of rationals one has $N(i,k) = n_i n_{i-1}...n_{i-k}$. The most general option is that these normal subgroups provide only the data allowing to associate dark scales to each of them. The spectrum of $h_{eff}$ could correspond to the $\{N_{i,k}\}$ or at least the set $\{N_{i,1}\}$.

3. The extensions with prime dimension $n=p$ have no non-trivial normal subgroups and $n=p$ would hold for them. For these extensions the state space of group algebra is prime as Hilbert space and does not decompose to tensor product so that it would represent fundamental system. Could these extensions be of special interest physically? SSFRs would naturally involve state function reduction cascades proceeding downwards along hierarchy of normal subgroups and would represent cognitive measurements [L45].

The original guess was that dark scale $L_D = nL_p$, where $n$ is the order $n$ for the extensions and $p$ is a ramified prime for the extension. A generalized form would allow $L_D = N(i,1)L_{p_k}$ for the sub-extension such that $p_k$ is ramified prime for the sub-extension.

*2. Can one identify the size scale of CD as dark scale?*

It would be natural if the scale of CD would be determined by the extension of rationals. Or more generally, the scales of CD and hierarchy of sub-CDs associated with the extension would be determined by the inclusion hierarchy of extensions and thus correspond to the hierarchy of normal sub-groups of Galois group.

The simplest option would be $L_{CD} = L_D$ so that the size scales of sub-CD would correspond dark scales for sub-extension given by $L_{CD,i} = N(i,1)L_{p_k}$, $p_k$ ramified prime of sub-extension.

1. The differences $|r_i - r_j|$ would correspond to differences for Minkowski time of CD. CD need not contain all values of hyperplanes $t = r_i$ and the evolution by SSFR would gradually bring in day-light all roots $r_n$ of the polynomial $P$ defining space-time surface as "very special moments in the life of self". If the size scale of CD is so large that also the largest value of $|r_i|$ is inside the upper or lower half of CD, the size scale of CD would correspond roughly to the largest p-adic length scale.

   CD contains sub-CDs and these could correspond to normal subgroups of Galois extension as extension of extension of ....

2. One can ask what happens when all special moments $t = r_n$ have been experienced? Does BSFR meaning death of conscious entity take place or is there some other option? In [L44] I considered a proposal for how chaos could emerge via iterations of $P$ during the sequence of SSFRs.

   One could argue that when CD has reached by SSFRs following unitary evolutions a size for which all roots $r_n$ have become visible, the evolution could continues by the replacement of $P$ with $P \circ P$, and so on. This would give rise to iteration and space-time analog for the approach to chaos.

3. Eventually the evolution by SSFRs must stop. Biological arguments suggests that metabolic limitations cause the death of self since the metabolic energy feed is not enough to preserve the distribution of values of $h_{eff}$ (energies increase with $h_{eff} \propto Nn$, for $N$:th iteration and $h_{eff}$ is reduced spontaneously) [L46].

# 3 Fermionic variant of $M^8-H$ duality

The topics of this section is $M^8-H$ duality for fermions. Consider first the bosonic counterpart of $M^8-H$ duality.

1. The octonionic polynomial giving rise to space-time surface $X^4$ as its "root" is obtained from ordinary real polynomial $P$ with rational coefficients by algebraic continuation. The

conjecture is that the identification in terms of roots of polynomials of even real analytic functions guarantees associativity and one can formulate this as rather convincing argument [L10, L11, L12]. Space-time surface $X_c^4$ is identified as a 4-D root for a $H_c$-valued "imaginary" or "real" part of $O_c$ valued polynomial obtained as an $O_c$ continuation of a real polynomial $P$ with rational coefficients, which can be chosen to be integers. These options correspond to complexified-quaternionic tangent- or normal spaces. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The real 4-D space-time surface is projection of this surface from $M_c^8$ to $M^8$. One could drop the subscripts "$_c$" but in the sequel they will be kept.

$M_c^4$ appears as a special solution for any polynomial $P$. $M_c^4$ seems to be like a universal reference solution with which to compare other solutions.

One obtains also brane-like 6-surfaces as 6-spheres as universal solutions. They have $M^4$ projection, which is a piece of hyper-surface for which Minkowski time as time coordinate of CD corresponds to a root $t = r_n$ of $P$. For monic polynomials these time values are algebraic integers and Galois group permutes them.

2. One cannot exclude rational functions or even real analytic functions in the sense that Taylor coefficients are octonionically real (proportional to octonionic real unit). Number theoretical vision - adelic physics [L15], suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where $i$ commutes with the octonionic units and defines complexifiation of octonions. $i$ appears also in the roots defining complex extensions of rationals.

   The generalization of the relationship between reals, extensions of p-adic number fields, and algebraic numbers in their intersection is suggestive. The "world of classical worlds" (WCW) would contain the space-time surfaces defined by polynomials with general real coefficients. Real WCW would be continuous space in real topology. The surfaces defined by rational or perhaps even algebraic coefficients for given extension would represent the intersection of real WCW with the p-adic variants of WCW labelled by the extension.

3. $M^8 - H$ duality requires additional condition realized as condition that also space-time surface itself contains 2-surfaces having commutative (complex) tangent or normal space. These surfaces can be 2-D also in metric sense that is light-like 3-D surfaces. The number of these surfaces is finite in generic case and they do not define a slicing of $X^4$ as was the first expectation. Strong form of holography (SH) makes it possible to map these surfaces and their tangent/normal spaces to 2-D surfaces $M4 \times CP_2$ and to serve as boundary values for the partial differential equations for variational principle defined by twistor lift. Space-time surfaces in $H$ would be minimal surface apart from singularities.

Concerning $M^8 - H$ duality for fermions, there are strong guidelines: also fermionic dynamics should be algebraic and number theoretical.

1. Spinors should be octonionic. I have already earlier considered their possible physical interpretation. [L1].

2. Dirac equation as linear partial differential equation should be replaced with a linear algebraic equation for octonionic spinors which are complexified octonions. The momentum space variant of the ordinary Dirac equation is an algebraic equation and the proposal is obvious: $P\Psi = 0$, where $P$ is the octonionic continuation of the polynomial defining the space-time surface and multiplication is in octonionic sense. The conjugation in $O_c$ is induced by the conjugation of the commuting imaginary unit $i$. The square of the Dirac operator is real if the space-time surface corresponds to the projection $O_c \to M^8 \to M^4$ with real time coordinate and imaginary spatial coordinates so that the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for $O_c$ - a purely number theoretic notion.

   The masslessness condition restricts the solutions to light-like 3-surfaces $m_{kl}P^kP^l = 0$ in Minkowskian sector analogous to mass shells in momentum space - just as in the case of ordinary massless Dirac equation. $P(o)$ rather than octonionic coordinate $o$ would define momentum. These mass shells should be mapped to light-like partonic orbits in $H$.

3. This picture leads to the earlier phenomenological picture about induced spinors in $H$. Twistor Grassmann approach suggests the localization of the induced spinor fields at light-like partonic orbits in $H$. If the induced spinor field allows a continuation from 3-D partonic orbits to the interior of $X^4$, it would serve as a counterpart of virtual particle in accordance with quantum field theoretical picture.

## 3.1 $M^8 - H$ duality for space-time surfaces

It is good to explain $M^8 - H$ duality for space-time surfaces before discussing it in fermionic sector.

### 3.1.1 Space-time as 4-surface in $M_c^8 = O_c$

One can regard real space-time surface $X^4 \subset M^8$ as a $M^8-$-projection of $X_c^4 \subset M_c^8 = O_c$. $M_c^4$ is identified as complexified quaternions $H_c$ [L29, L42]. The dynamics is purely algebraic and therefore local an associativity is the basic dynamical principle.

1. The basic condition is associativity of $X^4 \subset M^8$ in the sense that either the tangent space or normal space is associative - that is quaternionic. This would be realized if $X_c^4$ as a root for the quaternion-valued "real" or "imaginary part" for the $O_c$ algebraic continuation of real analytic function $P(x)$ in octonionic sense. Number theoretical universality requires that the Taylor coefficients are rational numbers and that only polynomials are considered.

   The 4-surfaces with associative normal space could correspond to elementary particle like entities with Euclidian signature ($CP_2$ type extremals) and those with associative tangent space to their interaction regions with Minkowskian signature. These two kinds space-time surfaces could meet along these 6-branes suggesting that interaction vertices are located at these branes.

2. The conditions allow also exceptional solutions for any polynomial for which both "real" and "imaginary" parts of the octonionic polynomial vanish. Brane-like solutions correspond to 6-spheres $S^6$ having $t = r_n$ 3-ball $B^3$ of light-cone as $M^4$ projection: here $r_n$ is a root of the real polynomial with rational coefficients and can be also complex - one reason for complexification by commuting imaginary unit $i$. For scattering amplitudes the topological vertices as 2-surfaces would be located at the intersections of $X_c^4$ with 6-brane. Also Minkowski space $M^4$ is a universal solution appearing for any polynomial and would provide a universal reference space-time surface.

3. Polynomials with rational coefficients define EQs and these extensions form a hierarchy realized at the level of physics as evolutionary hierarchy. Given extension induces extensions of p-adic number fields and adeles and one obtains a hierarchy of adelic physics. The dimension $n$ of extension allows interpretation in terms of effective Planck constant $h_{eff} = n \times h_0$. The phases of ordinary matter with effective Planck constant $h_{eff} = nh_0$ behave like dark matter and galactic dark matter could correspond to classical energy in TGD sense assignable to cosmic strings thickened to magnetic flux tubes. It is not completely clear whether number galactic dark matter must have $h_{eff} > h$. Dark energy in would correspond to the volume part of the energy of the flux tubes.

   There are good arguments in favor of the identification $h = 6h_0$ [L19]. "Effective" means that the actual value of Planck constant is $h_0$ but in many-sheeted space-time $n$ counts the number of symmetry related space-time sheets defining $X^4$ as a covering space locally. Each sheet gives identical contribution to action and this implies that effective value of Planck constant is $nh_0$.

   The ramified primes of extension in turn are identified as preferrred p-adic primes. The moduli for the time differences $|t_r - t_s|$ have identification as p-adic time scales assignable to ramified primes [L42]. For ramified primes the p-adic variants of polynomials have degenerate zeros in $O(p) = 0$ approximation having interpretation in terms of quantum criticality central in TGD inspired biology.

4. During the preparation of this article I made a trivial but overall important observation. Standard Minkowski signature emerges as a prediction if conjugation in $O_c$ corresponds to

the conjugation with respect to commuting imaginary unit $i$ rather than octonionic imaginary units as though earlier. If the space-time surface corresponds to the projection $O_c \rightarrow M^8 \rightarrow M^4$ with real time coordinate and imaginary spatial coordinates the metric defined by the octonionic norm is real and has Minkowskian signature. Hence the notion of Minkowski metric reduces to octonionic norm for $O_c$ - a purely number theoretic notion.

### 3.1.2 Realization of $M^8 - H$ duality

$M^8 - H$ duality allows to $X^4 \subset M^8$ to $X^4 \subset H$ so that one has two equivalent descriptions for the space-time surfaces as algebraic surfaces in $M^8$ and as minimal surfaces with 2-D preferred 2-surfaces defining holography making possible $M^8-H$ duality and possibly appearing as singularities in $H$. The dynamics of minimal surfaces, which are also extremals of Kähler action, reduces for known extremals to purely algebraic conditions analogous to holomorphy conditions in string models and thus involving only gradients of coordinates. This condition should hold generally and should induce the required huge reduction of degrees of freedom proposed to be realized also in terms of the vanishing of super-symplectic Noether charges already mentioned [K12].

Twistor lift allows several variants of this basic duality [L40]. $M^8_H$ duality predicts that space-time surfaces form a hierarchy induced by the hierarchy of EQs defining an evolutionary hierarchy. This forms the basics for the number theoretical vision about TGD.

As already noticed, $X^4 \subset M^8$ would satisfy an infinite number of additional conditions stating vanishing of Noether charges for a sub-algebra $SSA_n \subset SSA$ of super-symplectic algebra $SSA$ actings as isometries of WCW.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions [L29].

1. Associativity condition for tangent-/normal spaces is the first essential condition for the existence of $M^8-H$ duality and means that tangent - or normal space is associative/quaternionic.
2. Each tangent space of $X^4$ at $x$ must contain a preferred $M^2_c(x) \subset M^4_c$ such that $M^2_c(x)$ define an integrable distribution and therefore complexified string world sheet in $M^4_c$. This gives similar distribution for their orthogonal complements $E^2c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in $R_c$ sense. $E^2_c(x)$ would correspond to partonic 2-surface. This condition generalizes for $X^4$ with quaternionic normal space. A possible interpretation is as a space-time correlate for the selection of quantization axes for energy (rest system) and spin.

One can imagine two realizations for the additional condition.

**Option I**: Global option states that the distributions $M^2_c(x)$ and $E^2_c(x)$ define a slicing of $X^4_c$.

**Option II**: Only a discrete set of 2-surfaces satisfying the conditions exist, they are mapped to $H$, and strong form of holography (SH) applied in $H$ allows to deduce $X^4 \subset H$. This would be the minimal option.

It seems that only **Option II** can be realized.

1. The basic observation is that $X^2_c$ can be fixed by posing to the non-vanishing $H_c$-valued part of octonionic polynomial $P$ condition that the $C_c$-valued "real" or "imaginary" part in $C_c$ sense for $P$ vanishes. $M^2_c$ would be the simplest solution but also more general complex sub-manifolds $X^2_c \subset M^4_c$ are possible. This condition allows only a discrete set of 2-surfaces as its solutions so that it works only for **Option II**.

   These surfaces would be like the families of curves in complex plane defined by $u = 0$ an $v = 0$ curves of analytic function $f(z) = u + iv$. One should have family of polynomials differing by a constant term, which should be real so that $v = 0$ surfaces would form a discrete set.

2. SH makes possible $M^8 - H$ duality assuming that associativity conditions hold true only at 2-surfaces including partonic 2-surfaces or string world sheets or perhaps both. Thus one can give up the conjecture that the polynomial ansatz implies the additional condition globally.

SH indeed states that PEs are determined by data at 2-D surfaces of $X^4$. Even if the conditions defining $X^2_c$ have only a discrete set of solutions, SH at the level of $H$ could allow to deduce the PEs from the data provided by the images of these 2-surfaces under $M^8 - H$ duality. The existence of $M^2(x)$ would be required only at the 2-D surfaces.

3. There is however a delicacy involved: $X^2$ might be 2-D only metrically but not topologically! The 3-D light-like surfaces $X^3_L$ indeed have metric dimension $D = 2$ since the induced 4-metric degenerates to 2-D metric at them. Therefore their pre-images in $M^8$ would be natural candidates for the singularities at which the dimension of the quaternionic tangent or normal space reduces to $D = 2$ [L28] [K1]. If this happens, SH would not be quite so strong as expected. The study of fermionic variant of $M^8 - H$-duality supports this conclusion.

One can generalize the condition selecting $X^2_c$ so that it selects 1-D surface inside $X^2_c$. By assuming that $R_c$-valued "real" or "imaginary" part of complex part of $P$ sense at this 2-surface vanishes. One obtains preferred $M^1_c$ or $E^1_c$ containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as a complexified string. Together these kind 1-D surfaces in $R_c$ sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_c \rightarrow C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

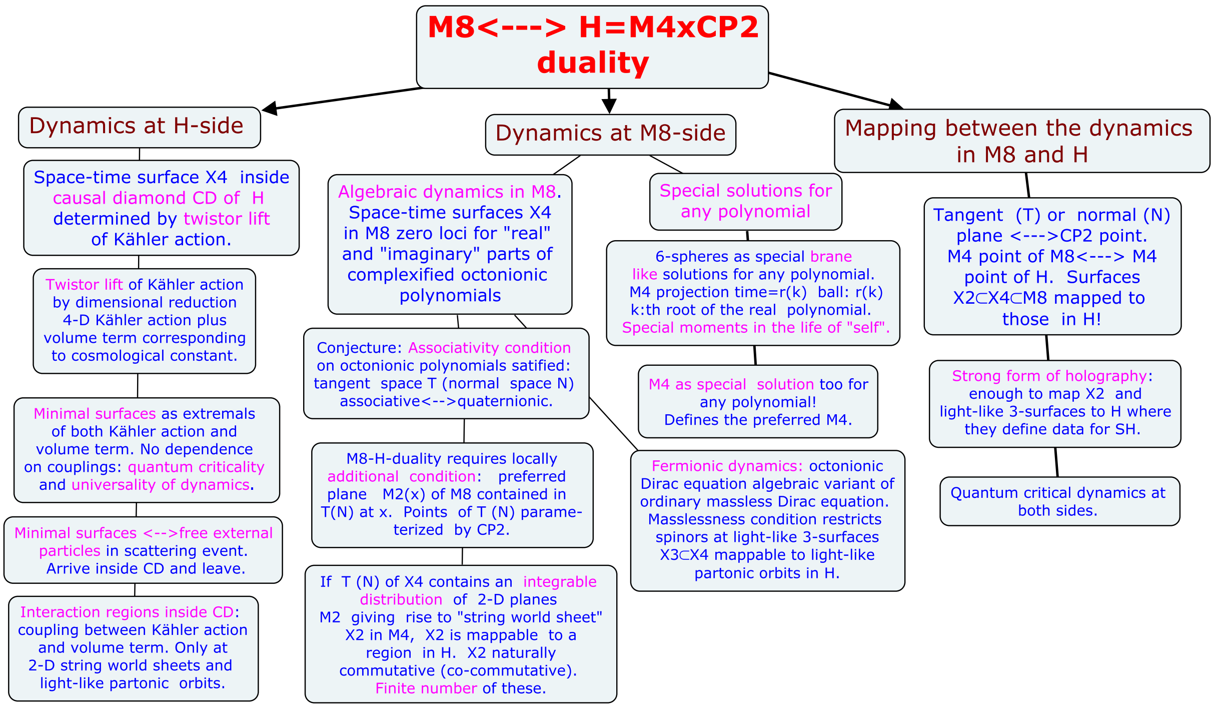


**Figure 1:** $M^8 - H$ duality.

## 3.2 What about $M^8 - H$ duality in the fermionic sector?

During the preparation of this article I become aware of the fact that the realization $M^8 - H$ duality in the fermionic sector has remained poorly understood. This led to a considerable integration of the ideas about $M^8 - H$ duality also in the bosonic sector and the existing phenomenological picture follows now from $M^8 - H$ duality. There are powerful mathematical guidelines available.

### 3.2.1 Octonionic spinors

By supersymmetry, octonionicity should have also fermionic counterpart.

1. The interpretation of $M^8_c$ as complexified octonions suggests that one should use complexified octonionic spinors in $M^8_c$. This is also suggested by SO(1,7) triality unique for dimension $d = 8$ and stating that the dimensions of vector representation, spinor representation and its conjugate are same and equal to $D = 8$. I have already earlier considered the possibility to interpret $M^8$ spinors as octonionic [L1]. Both octonionic gamma matrices and spinors have interpretation as octonions and gamma matrices satisfy the usual anti-commutation rules. The product for gamma matrices and gamma matrices and spinors is replaced with non-associative octonionic product.

2. Octonionic spinors allow only one $M^8$-chirality, which conforms with the assumption of TGD inspired SUSY that only quarks are fundamental fermions and leptons are their local composites [L34].

3. The decomposition of $X^2 \subset X^4 \subset M^8$ corresponding to $R \subset C \subset Q \subset O$ should have analog for the $O_c$ spinors as a tensor product decomposition. The special feature of dimension $D = 8$ is that the dimensions of spinor spaces associated with these factors are indeed 1, 2, 4, and 8 and correspond to dimensions for the surfaces!

   One can define for octonionic spinors associative/co-associative sub-spaces as quaternionic/co-quaternionic spinors by posing chirality conditions. For $X^4 \subset M^8_c$ one could define the analogs of projection operators $P_\pm = (1 \pm \gamma_5)/2$ as projection operators to either factor of the spinor space as tensor product of spinor space associated with the tangent and normal spaces of $X^4$: the analog of $\gamma_5$ would correspond to tangent or normal space depending on whether tangent or normal space is associative. For the spinors with definite chirality there would be no entanglement between the tensor factors. The condition would generalize the chirality condition for massless $M^4$ spinors to a condition holding for the local $M^4$ appearing as tangent/normal space of $X^4$.

4. The chirality condition makes sense also for $X^2 \subset X^4$ identified as complex/co-complex surface of $X^4$. Now $\gamma_5$ is replaced with $\gamma_3$ and states that the spinor has well-defined spin in the direction of axis defined by the decomposition of $X^2$ tangent space to $M^1 \times E^1$ with $M^1$ defining real octonion axis and selecting rest frame. Interpretation in terms of quantum measurement theory is suggestive.

What about tangent space quantum numbers in $M^8$ picture. In $H$-picture they correspond to spin and electroweak quantum numbers. In $M^8$ picture the geometric tangent space group for a rest system is product $SU(2) \times SU(2)$ with possible modifications due to octonionicity reducing tangent space group to those respecting octonionic automorphisms.

What about the sigma matrices for the octonionic gamma matrices? The surprise is that the commutators of $M^4$ sigma matries and those of E$^4$ sigma matrices close to the sama $SO(3)$ algebra allowing interpretation as representation for quaternionic automorphisms. Lorentz boosts are represented trivially, which conforms with the fact that octonion structure fixes unique rest system. Analogous result holds in $E^4$ degrees of freedom. Besides this one has unit matrix assignable to the generalize spinor structure of $CP_2$ so that also electroweak $U(1)$ factor is obtained.

One can understand this result by noticing that octonionic spinors correspond to 2 copies of a tensor products of the spinor doublets associated with spin and weak isospin. One has $2 \otimes 2 = 3 \oplus 1$ so that one must have $1 \oplus 3 \oplus 1 \oplus 3$. The octonionic spinors indeed decompose like $1+1+3+\overline{3}$ under $SU(3)$ representing automophisms of the octonions. $SO(3)$ could be interpreted as $SO(3) \subset SU(3)$. $SU(3)$ would be represented as tangent space rotations.

### 3.2.2 Dirac equation as partial differential equation must be replaced by an algebraic equation

Algebraization of dynamics should be also supersymmetric. The modified Dirac equation in $H$ is linear partial differential equation and should correspond to a linear algebraic equation in $M^8$.

1. The key observation is that for the ordinary Dirac equation the momentum space variant of Dirac equation for momentum eigenstates is algebraic! Could the interpretation for $M^8 - H$ duality as an analog of momentum-position duality of wave mechanics considered already earlier make sense! This could also have something to do with the dual descriptions of twistorial scattering amplitudes in terms of either twistor and momentum twistors. Already the earlier work excludes the interpretation of the octonionic coordinate $o$ as 8-momentum. Rather, $P(o)$ has this interpretation and $o$ corrresponds to embedding space coordinate.

2. The first guess for the counterpart of the modified Dirac equation at the level of $X^4 \subset M^8$ is $P\Psi = 0$, where $\Psi$ is octonionic spinor and the octonionic polynomial $P$ defining the space-time surface can be seen as a generalization of momentum space Dirac operator with octonion units representing gamma matrices. If associativity/co-associativity holds true, the equation becomes quaternionic/co-quaternionic and reduces to the 4-D analog of massless Dirac equation and of modified Dirac equation in $H$. Associativity hols true if also $\Psi$ satisfies associativity/co-associativity condition as proposed above.

3. What about the square of the Dirac operator? There are 3 conjugations involved: quaternionic conjugation assumed in the earlier work, conjugation with respect to $i$, and their combination. The analog of octonionic norm squared defined as the product $o_c o_c^*$ with conjugation with respect to $i$ only, gives Minkowskian metric $m_{kl}o^k\overline{o}^l$ as its real part. The imaginary part of the norm squared is vanishing for the projection $O_c \to M^8 \to M^4$ so that time coordinate is real and spatial coordinates imaginary. Therefore Dirac equation allows solutions only for the $M^4$ projection $X^4$ and $M^4$ ($M8$) signature of the metric can be said to be an outcome of quaternionicity (octonionicity) alone in accordance with the duality between metric and algebraic pictures.

   Both $P^\dagger P$ and $PP$ should annihilate $\Psi$. $P^\dagger P\Psi = 0$ gives $m_{kl}P^k\overline{P}^l = 0$ as the analog of vanishing mass squared in $M^4$ signature in both associative and co-associative cases. $PP\Psi = 0$ reduces to $P\Psi = 0$ by masslessness condition. One could perhaps interpret the projection $X_c^4 \to M^8 \to M^4$ in terms of Uncertainty Principle.

   There is a $U(1)$ symmetry involved: instead of the plane $M^8$ one can choose any plane obtained by a rotation $exp(i\phi)$ from it. Could it realize quark number conservation in $M^8$ picture?

   For $P = o$ having only $o = 0$ as root $Po = 0$ reduces to $o^\dagger o = 0$ and $o$ takes the role of momentum, which is however vanishing. 6-D brane like solutions $S^6$ having $t = r_n$ balls $B^3 \subset CD_4$ as $M^4$ projections one has $P = 0$ so that the Dirac equation trivializes and does not pose conditions on $\Psi$. $o$ would have interpretation as space-time coordinates and $P(o)$ as position dependent momentum components $P^k$.

   The variation of $P$ at mass shell of $M_c^8$ (to be precise) could be interpreted in terms of the width of the wave packet representing particle. Since the light-like curve at partonic 2-surface for fermion at $X_L^3$ is not a geodesic, mass squared in $M^4$ sense is not vanishing. Could one understand mass squared and the decay width of the particle geometrically? Note that mass squared is predicted also by p-adic thermodynamics [K8].

4. The masslessness condition restricts the spinors at 3-D light-cone boundary in $P(M^8)$. $M^8 - H$ duality [L29] suggests that this boundary is mapped to $X_L^3 \subset H$ defining the light-like orbit of the partonic 2-surface in $H$. The identification of the images of $P_kP^k = 0$ surfaces as $X_L^3$ gives a very powerful constraint on SH and $M^8 - H$ duality.

5. Also at 2-surfaces $X^2 \subset X^4$ an the variant Dirac equation would hold true and should commute with the corresponding chirality condition. Now $D^\dagger D\Psi = 0$ gives 2-D variant of masslessness condition with 2-momentum components represented by those of $P$. 2-D

masslessness locates the spinor to a 1-D curve $X^1_L$. Its $H$-image would naturally contain the boundary of the string word sheet at $X^3_L$ assumed to carry fermion quantum numbers and also the boundary of string world sheet at the light-like boundary of $CD_4$. The interior of string world sheet in $H$ would not carry induced spinor field.

6. The general solution for both 4-D and 2-D cases can be written as $\Psi = P\Psi_0$, $\Psi_0$ a constant spinor - this in a complete analogy with the solution of modified Dirac equation in $H$. $P$ depends on position: the WKB approximation using plane waves with position dependent momentum seems to be nearer to reality than one might expect.

### 3.2.3 The phenomenological picture at $H$-level follows from the $M^8$-picture

Remarkably, the partly phenomenological picture developed at the level of $H$ is reproduced at the level of $M^8$. Whether the induced spinor fields in the interior of $X^4$ are present or not, has been long standing question since they do not seem to have any role in the physical picture. The proposed picture answers this question.

Consider now the explicit realization of $M^8 - H$-duality for fermions.

1. SH and the expected analogy with the bosonic variant of $M^8 - H$ duality lead to the first guess. The spinor modes in $X^4 \subset M^8$ restricted to $X^2$ can be mapped by $M^8 - H$-duality to those at their images $X^2 \subset H$, and define boundary conditions allowing to deduce the solution of the modified Dirac equation at $X^4 \subset H$. $X^2$ would correspond to string world sheets having boundaries $X^1_L$ at $X^3_L$.

   The guess is not quite correct. Algebraic Dirac equation requires that the solutions are restricted to the 3-D and 1-D mass shells $P_kP^k = 0$ in $M^8$. This should remain true also in $H$ and $X^3_L$ and their 1-D intersections $X^1_L$ with string world sheets remain. Fermions would live at boundaries. This is just the picture proposed for the TGD counterparts of the twistor amplitudes and corresponds to that used in twistor Grassmann approach!

   For 2-D case constant octonionic spinors $\Psi_0$ and gamma matrix algebra are equivalent with the ordinary Weyl spinors and gamma matrix algebra and can be mapped as such to $H$. This gives one additional reason for why SH must be involved.

2. At the level of $H$ the first guess is that the modified Dirac equation $D\Psi = 0$ is true for $D$ based on the modified gamma matrices associated with both volume action and Kähler action. This would select preferred solutions of modified Dirac equation and conform with the vanishing of super-symplectic Noether charges for $SSA_n$ for the spinor modes. The guess is not quite correct. The restriction of the induced spinors to $X^3_L$ requires that Chern-Simons action at $X^3_L$ defines the modified Drac action.

3. The question has been whether the 2-D modified Dirac action emerges as a singular part of 4-D modified Dirac action assignable to singular 2-surface or can one assign an independent 2-D Dirac action assignable to 2-surfaces selected by some other criterion. For singular surfaces $M^8 - H$ duality fails since tangent space would reduce to 2-D space so that only their images can appear in SH at the level of $H$.

   This supports the view that singular surfaces are actually 3-D mass shells $M^8$ mapped to $X^3_L$ for which 4-D tangent space is 2-D by the vanishing of $\sqrt{g_4}$ and light-likeness. String world sheets would correspond to non-singular $X^2 \subset M^8$ mapped to $H$ and defining data for SH and their boundaries $X^1_L \subset X^3_L$ and $X^1_L \subset CD_4$ would define fermionic variant of SH.

What about the modified Dirac operator $D$ in $H$?

1. For $X^3_L$ modified Dirac equation $D\Psi = 0$ based on 4-D action $S$ containing volume and Kähler term is problematic since the induced metric fails to have inverse at $X^3_L$. The only possible action is Chern-Simons action $S_{CS}$ used in topological quantum field theories and now defined as sum of C-S terms for Kähler actions in $M^4$ and $CP_2$ degrees of freedom. The presence of $M^4$ part of Kähler form of $M^8$ is forced by the twistor lift, and would give rise to small CP breaking effects explaining matter antimatter asymmetry [L34]. $S_{C-S}$ could emerge as a limit of 4-D action.

The modified Dirac operator $D_{C-S}$ uses modified gamma matrices identified as contractions $\Gamma^{\alpha}_{CS} = T^{\alpha k}\gamma_k$, where $T^{\alpha k} = \partial L_{CS}/\partial(\partial_\alpha h^k)$ are canonical momentum currents for $S_{C-S}$ defined by a standard formula.

2. $CP_2$ part would give conserved Noether currents for color in and $M^4$ part Poincare quantum numbers: the apparently small CP breaking term would give masses for quarks and leptons! The bosonic Noether current $J_{B,A}$ for Killing vector $j^k_A$ would be proportional to $J^{\alpha}_{B,A} = T^{\alpha}_k j_A k$ and given by $J_{B,A} = \epsilon^{\alpha\beta\gamma}[J_{\beta\gamma}A_k + A_\beta J_{\gamma k}]j^k_A$.

   Fermionic Noether current would be $J_{F,A} = \overline{\Psi}J^{\alpha}\Psi$ 3-D Riemann spaces allow coordinates in which the metric tensor is a direct sum of 1-D and 2-D contributions and are analogous to expectation values of bosonic Noether currents. One can also identify also finite number of Noether super currents by replacing $\overline{\Psi}$ or $\Psi$ by its modes.

3. In the case of $X^3_L$ the 1-D part light-like part would vanish. If also induced Kähler form is non-vanishing only in 2-D degrees of freedom, the Noether charge densities $J^t$ reduce to $J^t = JA_k j^k_A$, $J = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}$ defining magnetic flux. Modified Dirac operator would reduce to $D = JA_k\gamma^k D_t$ and 3-D solutions would be covariantly constant spinors along the light-like geodesics parameterized by the points 2-D cross section. One could say that the number of solutions is finite and corresponds to covariantly constant modes continued from $X^1_L$ to $X^3_L$. This picture is just what twistor Grassmannian approach led to [L24].

### 3.2.4 A comment inspired by the ZEO based quantum measurement theory

I cannot resist the temptation to make a comment relating to quantum measurement theory inspired by zero energy ontology (ZEO) extending to a theory of consciousness [L33, L45, L46].

I have proposed [L42, L44] that the time evolution by "big" state function reductions (BSFRs) could be induced by iteration of real polynomial $P$ - at least in some special cases. The foots of the real polynomial $P$ would define a fractal at the limit of larger number of iterations. The roots of $n$-fold iterate $\circ^n P$ would contain the inverse images under $\circ^{-n+1}P$ of roots of $P$ and for $P(0) = 0$ the inverse image $\circ^n P$ would consist of inverse images under $\circ^{-k}P$, $k = 0, ...., n-1$, of roots of $P$.

Also the mass shells for $\circ^n P$ would be unions of inverses images under $\circ^{-k}P$, $k = 0, , ...., n-1$, of roots of $P$. This gives rather concrete view about evolution of $M^4$ projections of the partonic orbits. A rough approximate expression for the largest root of real $P$ approximated as $P(x) \simeq a_n x^n + an - 1ix^{n-1}$ for large $x$ is $x_{max} \sim a_n/a_{n-1}$. For $\circ^n P$ one obtains the same estimate. This suggests that the size scales of the partonic orbits are same for the iterates. The mass shells would not differ dramatically: could they have an interpretation in terms of mass splitting?

The evolution by iteration would add new partonic orbits and preserve the existing ones: this brings in mind conservation of genes in biological evolution. This is true also for a more general evolution allowing general functional decomposition $Q \to Q \circ P$ to occur in BSFR.

### 3.2.5 What next in TGD?

The construction of scattering amplitudes has been the dream impossible that has driven me for decades. Maybe the understanding of fermionic $M^8 - H$ duality provides the needed additional conceptual tools. The key observation is utterly trivial but far reaching: there are 3 possible conjugations for octonions corresponding to the conjugation of commutative imaginary unit or of octonionic imaginary units or both of them. 1st norm gives a real valued norm squared in Minkowski signature natural at $M^8$ level! Second one gives a complex valued norm squared in Euclidian signature. 1st and 2nd norms are equivalent for octonions light-like with respect to the first norm. The 3rd conjugation gives a real-valued Euclidian norm natural at the level of Hilbert space.

1. $M^8$ picture looks simple. Space-time surfaces in $M^8$ can be constructed from real polynomials with real (rational) coefficients, actually knowledge of their roots is enough. Discrete data - roots of the polynomial!- determine space-time surface as associative or co-associative region! Besides this one must pose additional condition selecting 2-D string world sheets and 3-D light-like surfaces as orbits of partonic 2-surfaces. These would define strong form of holography (SH) allowing to map space-time surfaces in $M^8$ to $M^4 \times CP_2$.

2. Could SH generalize to the level of scattering amplitudes expressible in terms of n-point functions of CFT?! Could the n points correspond to the roots of the polynomial defining space-time region!

   Algebraic continuation to quaternion valued scattering amplitudes analogous to that giving space-time sheets from the data coded SH should be the key idea. Their moduli squared are real - this led to the emergence of Minkowski metric for complexified octonions/quaternions) would give the real scattering rates: this is enough! This would mean a number theoretic generalization of quantum theory.

3. One can start from complex numbers and string world sheets/partonic 2-surfaces. Conformal field theories (CFTs) in 2-D play fundamental role in the construction of scattering string theories and in modelling 2-D statistical systems. In TGD 2-D surfaces (2-D at least metrically) code for information about space-time surface by strong holography (SH) .

   Are CFTs at partonic 2-surfaces and string world sheets the basic building bricks? Could 2-D conformal invariance dictate the data needed to construct the scattering amplitudes for given space-time region defined by causal diamond (CD) taking the role of sphere $S^2$ in CFTs. Could the generalization for metrically 2-D light-like 3-surfaces be needed at the level of "world of classical worlds" (WCW) when states are superpositions of space-time surfaces, preferred extremals?

The challenge is to develop a concrete number theoretic hierarchy for scattering amplitudes: $R \rightarrow C \rightarrow H \rightarrow O$ - actually their complexifications.

1. In the case of fermions one can start from 1-D data at light-like boundaries LB of string world sheets at light-like orbits of partonic 2-surfaces. Fermionic propagators assignable to LB would be coded by 2-D Minkowskian QFT in manner analogous to that in twistor Grassmann approach. n-point vertices would be expressible in terms of Euclidian n-point functions for partonic 2-surfaces: the latter element would be new as compared to QFTs since point-like vertex is replaced with partonic 2-surface.

2. The fusion (product?) of these Minkowskian and Euclidian CFT entities corresponding to different realization of complex numbers as sub-field of quaternions would give rise to 4-D quaternionic valued scattering amplitudes for given space-time sheet. Most importantly: there moduli squared are real for both norms.

   It is not quite clear whether one must use the 1st Minkowskian norm requiring "time-like" scattering amplitudes to achieve non-negative probabilities or use the 3rd norm to get the ordinary positive-definite Hilbert space norm. A generalization of quantum theory (CFT) from complex numbers to quaternions (quaternionic "CFT") would be in question.

3. What about several space-time sheets? Could one allow fusion of different quaternionic scattering amplitudes corresponding to different quaternionic sub-spaces of complexified octonions to get octonion-valued non-associative scattering amplitudes. Again scattering rates would be real. This would be a further generalization of quantum theory.

There is also the challenge to relate $M^8$- and $H$-pictures at the level of WCW. The formulation of physics in terms of WCW geometry [K12, L39] leads to the hypothesis that WCW Kähler geometry is determined by Kähler function identified as the 4-D action resulting by dimensional reduction of 6-D surfaces in the product of twistor spaces of $M^4$ and $CP_2$ to twistor bundles having $S^2$ as fiber and space-time surface $X^4 \subset H$ as base. The 6-D Kähler action reduces to the sum of 4-D Kähler action and volume term having interpretation in terms of cosmological constant.

The question is whether the Kähler function - an essentially geometric notion - can have a counterpart at the level of $M^8$.

1. SH suggests that the Kähler function identified in the proposed manner can be expressed by using 2-D data or at least metrically 2-D data (light-like partonic orbits and light-like boundaries of CD). Note that each WCW would correspond to a particular CD.

2. Since 2-D conformal symmetry is involved, one expects also modular invariance meaning that WCW Kähler function is modular invariant, so that they have the same value for $X^4 \subset H$ for which partonic 2-surfaces have induced metric in the same conformal equivalence class.

3. Also the analogs of Kac-Moody type symmetries would be realized as symmetries of Kähler function. The algebra of super-symplectic symmetries of the light-cone boundary can be regarded as an analog of Kac-Moody algebra. Light-cone boundary has topology $S^2 \times R_+$ where $R_+$ corresponds to radial light-like ray parameterized by radial light-like coordinate $r$. Super symplectic transformations of $S^2 \times CP_2$ depend on the light-like radial coordinate $r$, which is analogous to the complex coordinate $z$ for he Kac-Moody algebras.

   The infinitesimal super-symplectic transformations form algebra SSA with generators proportional to powers $r^n$ . The Kac-Moody invariance for physical states generalizes to a hierarchy of similar invariances. There is infinite fractal hierarchy of sub-algebras $SSA_n \subset SSA$ with conformal weights coming as $n$-multiples of those for SSA. For physical states $SSA_n$ and $[SSA_n, SSA]$ would act as gauge symmetries. They would leave invariant also Kähler function in the sector $WCW_n$ defined by $n$. This would define a hierarchy of sub- WCWs of the WCW assignable to given CD.

   The sector $\mathrm{WCW}_n$ could correspond to extensions of rationals with dimension $n$, and one would have inclusion hierarchies consisting of sequences of $n_i$ with $n_i$ dividing $n_{i+1}$. These inclusion hierarchies would naturally correspond to those for hyper-finite factors of type $\mathrm{II}_1$ [K16].

# 4 Cognitive representations and algebraic geometry

The general vision about cognition is realized in terms of adelic physics as physics of sensory experience and cognition [L15, L14]. Rational points and their generalization as ratios of algebraic integers for geometric objects would define cognitive representations as points common for real and various p-adic variants of the space-time surface. The finite-dimensionality for induced p-adic extensions allows also extensions of rationals involving root of $e$ and its powers. This picture applies both at space-time level, embedding space level, and at the level of space-time surfaces but basically reduces to embedding space level. Hence counting of the (generalized) rational points for geometric objects would be determination of the cognitive representability.

## 4.1 Cognitive representations as sets of generalized rational points

The set of rational points depends on the coordinates chosen and one can argue that one must allow different cognitive representations and classify them according to their effectiveness.

How uniquely the $M^8_c$ coordinates can be chosen?

(a) Polynomial property allows only linear transformations of the complex octonionic coordinates with coefficients which belong to the extension of rationals used. This poses extremely strong restrictions on the allowed representations once the quaternionic moduli defining a foliation of $M^4_0$ is chosen. One has therefore moduli space of quaternionic structures. One must also fix the time axis in $M^4$ assignable to real octonions.

(b) One can also define several inequivalent octonionic structures and associate a moduli space to these. The moduli space for octonionic structures would correspond to the space of $M^4_0 \subset M^8$s as quaternionic planes containing fixed $M^2_0$. One can allow even allow Lorentz transforms mixing real and imaginary octonionic coordinates. It seems that these moduli are not relevant at the level of $H$.

What could the precise definition of rationality?

(a) The coordinates of point are rational in the sense defined by the extension of rationals used. Suppose that one considers parametric representations of surfaces as maps from space-time surface to embedding space. Suppose that one uses as space-time coordinates subset of preferred coordinates for embedding space. These coordinate changes cannot be global and one space-time surface decomposes to regions in which different coordinates apply.

(b) The coordinate transformations between over-lapping regions are birational in the sense that both the map and its inverse are in terms of rational functions. This makes the notion of rationality global.

(c) When cognitively easy rational parametric representations are possible? For algebraic curves with $g \geq 2$ in $CP_2$ represented as zeros of polynomials this cannot the case since the number of rational points is finite for instance for $g \geq 2$ surfaces. There is simple explanation for this. Solving second complex coordinate in terms of the other one gives it as an algebraic function for $g \geq 2$: this must be the reason for the loss of dense set of rational points. For elliptic surfaces $y^2 - x^3 - ax - b = 0$ $y^2$ is however polynomial of $x$ and one can find rational parametric representation by taking $y^2$ as coordinate [L7]. For $g = 0$ one has linear equations and one obtains dense set of rational points. For conic sections one can also have dense set of rational points but not always. Generalizing from this it would seem that the failure to have rational parametric representation is the basic reason for the loss of dense set of rational points.

This picture does not work for general surfaces but generalizes for algebraic varieties defined by several polynomial equations. The co-dimension $d_c = 1$ case is however unique and the most studied one since for several polynomial equations one encounters technical difficulties when the intersection of the surfaces defined by the $d_c$ polynomials need not be complete for $d_c > 1$. In the recent situation one has $d_c = 4$ but octonion analyticity could be powerful enough symmetry to solve the problem of non-complete intersections by eliminating them or providing a physical interpretation for them.

## 4.2 Cognitive representations assuming $M^8 - H$ duality

Many questions should be answered.

(a) Can one generalize the results applying to algebraic varieties? Could the general vision about rational and potentially dense set of rational points generalize?. At $M^8$ side the description of space-time surfaces as algebraic varieties indeed conforms with this picture. Could one understand SH from the fact that real analyticity octonionic polynomials are determined by ordinary polynomial real coordinate completely? In information theoretic sense sense SH reduces to 1-D holography and the polynomial property makes the situation effectively discrete since finite number of points of real axis allows to determine the octonionic polynomial completely! It is a pity that one cannot measure octonionic polynomial directly!

(b) Also the notion of Zariski dimension should make sense in TGD at $M^8$ side. Preferred extremals define the notion of closed set for given CD at $M^8$ side? It would indeed seem that one define Zariski topology at the level of $M^8_c$. Zariski topology would require 4-surfaces, string world sheets, or partonic 2-surfaces and even 1-D curves. This picture conforms with the recent view about TGD and resembles the M-theory picture, where one has branes. SH suggests that the analog of Zariski dimension of space-time surface reduces to that for strings world sheets and partonic 2-surfaces and that even these are analogous to 1-D curves by complex analyticity. Integrability of TGD and preferred extremal property would indeed suggest simplicity.

$M^8 - H$ hypothesis suggests that these conjectures make sense also at $H$ side. String world sheets, partonic 2-surface, space-like 3-surfaces at the ends of space-time surface at boundaries of CD, and light-like 3-surfaces correspond to closed sets also at the level of WCW in the topology most natural for WCW.

(c) Also the problems related to Minkowskian signature could be solved. String world sheets are problematic because of the Minkowskian signature. They however have the topology of disk plus handles suggesting immediately a vision about cognitive representations in terms of rational points. One can can complexify string world sheets and it seems possible to apply the results of algebraic geometry holding true in Euclidian signature. This would be analogous to the Wick rotation used in QFTs and also in twistor Grassmann approach.

(d) What about algebraic geometrization of the twistor lift? How complex are twistor spaces of $M^4$, $CP_2$ and space-time surface? How can one generalize twistor lift to the level of $M^8$. $S^2$ bundle structure and the fact that $S^2$ allows a dense set of rational suggests that the complexity of twistor space is that of space-time surface itself so that the situation actually reduces to the level of space-time surfaces.

Suppose one accepts $M^8 - H$ duality requiring that the tangent space of space-time surface at given point $x$ contains $M^2(x)$ such that $M^2(x)$ define an integrable distribution giving rise to string world sheets and their orthogonal complements give rise to partonic 2-surfaces. This would give rise to a foliation of the space-time surface by string world sheets and partonic 2-surface conjecture on basis of the properties of extremals of Kähler action. As found these foliations could correspond to quaternion structures that is allowed choices of quaterionic coordinates.

Should one define cognitive representations at the level of $M^8$ or at the level of $M^4 \times CP_2$? Or both? For $M^8$ option the condition that space-time point belongs to an extension of rationals applies at the level of $M^8$ coordinates. For $M^4 \times CP_2$ option cognitive representations are at the level of $M^4$ and $CP_2$ parameterizing the points of $M^4$ and their tangent spaces. The formal study of partial differential equations alone does not help much in counting the number of rational points. One can define cognitive representation in very many ways, and some cognitive representation could be preferred only because they are more efficient than others. Hence both cognitive representations seems to be acceptable.

Some cognitive representations are more efficient than others. General coordinate invariance (GCI) at the level of cognition is broken. The precise determination of cognitive efficiency is a challenge in itself. For instance, the use of coordinates for which coordinate lines are orbits of subgroups of the symmetry group should be highly efficient. Only coordinate transformations mediated by bi-rational maps can take polynomial representations to polynomial representations. It might well be that only a rational (in generalized sense) sub-group $G_2$ of octonionic automorphisms is allowed. For rational surfaces allowing parametric representation in terms of polynomial functions the rational points form a dense set.

The cognitive resolution for a dense set of rational points is unrealistically high since cognitive representation would contain infinite number of points. Hence one must tighten the notion of cognitive representation. The rational points must contain a fermion. Fermions are indeed are identified as correlates for Boolean cognition [K2]. This would suggests a view in which cognitive representations are realized at the light-like orbits of partonic 2-surfaces at which Minkowskian associative and Euclidian co-associative space-time surfaces meet. The general wisdom is that rational points are localized to lower-dimensional sub-varieties (Bombieri-Lang conjecture): this conforms with the view that fermion lines reside at the orbits of partonic 2-surfaces.

## 4.3 Are the known extremals in $H$ easily cognitively representable?

Suppose that one takes TGD inspired adelic view about cognition seriously. If cognitive representations correspond to rational points for an extension of rationals, then the surfaces allowing large number of this kind of points are easily representable cognitively by adding fermions to these points. One could even speculate that mathematical cognition invents those geometric objects, which are easily cognitively representable and thus have a large number of rational points.

### 4.3.1 Could the known extremals of twistor lift be cognitively easy?

Also TGD is outcome of mathematical cognition. Could the known extremals of the twistor lift of Kähler action be cognitively easy? This is suggested by the fact that even such a pariah class theoretician as I am have managed to discover then! Positive answer could be seen as support for the proposed description of cognition!

(a) If one believes in $M^8-H$ duality and the proposed identification of associative and co-associative space-time surfaces in terms of algebraic surfaces in octonionic space $M^8_c$, the generalization of the results of algebraic geometry should give overall view about the cognitive representations at the level of $M^8$. In particular, surfaces allowing rational parametric representation (polynomials would have rational coefficients) would allow dense set or rational points since the images of rational points are rational. Rationals are understood here as ratios of algebraic integers in extension of rationals.

(b) Also for $H$ the existence of parameter representation using preferred $H$-coordinates and rational functions with rational coefficients implies that rational points are dense. If $M^8-H$ correspondence maps the parametric representations in terms of rational functions to similar representations, dense set of rational points is preserved in the correspondence. There is however no obvious reason why $M^8-H$ duality should have this nice property.

One can even play with the idea that the surfaces, which are cognitively difficult at the $M^8$ side, might be cognitively easy at $H$-side or vice versa. Of course, if the explicit representation as algebraic functions makes sense at $M^8$ side, this side looks cognitively ridiculously easy as compared to $H$ side. The preferred extremal property and SH can however change the situation.

(c) At $M^8$ side and for a given point of $M^4$ there are several points of $E^4$ (or vice versa) if the degree of the polynomial is larger than $n=1$ so that for the image of the surface $H$ there are several $CP_2$ points for a given point of $M^4$ (or vice versa) depending on the choice of coordinates. This is what the notion of the many-sheeted space-time predicts.

(d) The equations for the surface at $H$ side are obtained by a composite map assigning first to the coordinates of $X^4 \subset M^8$ point of $M^4 \times E^4$, and then assigning to the points of $X^4 \subset M^8$ $CP_2$ coordinates of the tangent space of the point. At this step the slightly non-local tangent space information is fed in and the surfaces in $M^4 \times CP_2$ cannot be given by zeros of polynomials. The indeed satisfy instead of algebraic equations partial differential equations given by the Kähler action for the twistor lift TGD. Algebraic equations instead of partial differential equations suggests that the $M^8$ representation is much simpler than $H$-representation. On the other hand, reduction to algebraic equations at $M^8$ side could have interpretation in terms of the conjectured complete integrability of TGD [K1, K14].

### 4.3.2 Testing the idea about self-reference

In any case, it is possible to test the idea about self-reference by looking whether the known extremals in $H$ are cognitively easy and even have a dense set of rational points in natural coordinates. Here I will consider the situation at the level of $M^4\times CP_2$. It was already found that the known extremals can have inverse images in $M^8$.

(a) Canonically imbedded $M^4$ with linear coordinates and constant $CP_2$ coordinates rational is the simple example about preferred extremal and it seems that TGD based cosmology at microscopic relies on these extremals. In this case it is obvious that one has a dense set of rational points at both sides. Could this somehow relate to the fact that physics as physics $M^4$ was discovered before general relativity?

Canonically imbedded $M^4$ corresponds to a first order octonionic polynomial for which imaginary part is put to constant so that tangent space is same everywhere and corresponds to a constant $CP_2$ coordinate.

(b) $CP_2$ type extremals have 4-D $CP_2$ projection and light-like geodesic line of $M^4$ as $M^4$ projection. One can choose the time parameter as a function of $CP_2$ coordinates in infinitely many ways. Clearly the rational points are dense in any $CP_2$ coordinates.

(c) Massless extremals (MEs) are given as zeros of arbitrary functions of $CP_2$ coordinates and 2 $M^4$ coordinates representing local light-like direction and polarization direction orthogonal to it. In the simplest situation these directions are constant. In the general case light-like direction would define tangent space of string world sheet giving rise also to a distribution of ortogonal polarization planes. This is consistent with the general properties of the $M^8$ representation and corresponds to the decomposition of quaternionic tangent plane to complex plane and its complement. One can ask whether one should allow only polynomials with rational coefficients as octonionic polynomials.

(d) String like objects $X^2 \times Y^2$ with $X^2 \subset M^4$ a minimal surface and $Y^2$ complex or Lagrangian surface of $CP_2$ are also basic extremals and their deformations in $M^4$ directions are expected to give rise to magnetic flux tubes.

If $Y^2$ is complex surface with genus $g = 0$ rational points are dense. Also for $g = 1$ one obtains a dense set of rational points in some extension of rationals. For elliptic curves one has lattice of rational points. What happens for Lagrangian surfaces $Y^2$? In this case one does not have complex curves but real co-dimension 2 surfaces. There is no obvious objection why these surfaces would not be possible.

(e) What about string world sheets? If the string world is static $M^2 \subset M^4$ one has a dense set of rational points. One however expects something more complex. If the string world sheet is rational map $M^2$ to its orthogonal complement $E^2$ one has rational surface. For rotating strings this does not make sense except for certain period of time. If the choice of the quaternion structure corresponds to a choice of minimal surface in $M^4$ as integrable distribution for $M^2(x)$, the coordinates associated with the Hamilton-Jacobi structure could make the situation simple.

If one restricts the consideration the intersections of partonic 2-surfaces and string world sheets at two boundaries of CD the situation simplifies and the question is only about the rationality of the $M^4$ coordinates at rational points of $Y^2 \subset CP_2$. This would simplify the situation enormously and might even allow to use existing knowledge.

(f) The slicing of of space-time surfaces by string world sheets and partonic 2-surfaces required by Hamilton-Jacobi structure could be seen as a fibering analogous to that possessed by elliptic surfaces. This suggest that $M^8$ counterparts of spacetime surfaces are not of general type in Kodaira classification and that the number of rational points can be large. If the existence of Hamilton-Jacobi structure does not allow handles, this factor would be cognitively simple. This would however suggests that fermion number is not localized at the ends of strings only - as assumed in the construction of scattering amplitudes inspired by twistor Grassmann approach [K5] - but also to the interior of the light-like curves inside string world sheets.

## 4.4 Twistor lift and cognitive representations

What about twistor lift of TGD replacing space-time surfaces with their twistor spaces. Consider first $M^8$ side.

(a) At $M^8$ side $S^2$ seems to introduce nothing new. One might expect that the situation does not change at $H$-side since space-time surfaces are obtained essentially by dimensional reduction and the possible problem relates to the choice of base space as section of is twistor bundle and the embedding of space-time as base space could have singularities at the boundary of Euclidian and Minkowskian space-time regions as discussed in [L7].

At the side of $M^8$ the proposed induction of twistor structure is just a projection of the twistor sphere $S^6$ to its geodesic sphere and one has 4-D moduli space for geodesic spheres $S^2 \subset S^6$. If one interprets the choice of $S^2 \subset S^6$ as as a section in the moduli space, the moduli of $S^2$ can depend on the point of space-time surface. Note that there are is also a position dependent choice of preferred point of $S^2$ representing Kähler form, and this choice is good candidate for giving rise to Hamilton-Jacobi structures with position dependent $M^2$.

(b) The notion of Kodaira dimension is defined also for co-dimension 4 algebraic varieties in $M^8_c$. The cognitively easiest spacetime surfaces would allow rational parametric representation with complex coordinates serving as parameters. If this is not possible, one has algebraic functions, which makes the situation much more complex so that the number of rational points would be small.

(c) For some complex enough extensions of rationals the set of rational points can be dense. $g \geq 2$ genera are basic example and one expects also in more general case that polynomials involving powers larger than $n = 4$ make the situation problematic. The condition that real or imaginary part of real analytic octonionic polynomial is in question is a strong symmetry expected to faciliate cognitive representability.

(d) The general intuitive wisdom from algebraic geometry is that the rational points are dense only in lower-dimensional sub-varieties (Bombieri-Lang and Vojta conjectures mentioned in the first section). The general vision inspired by SH and the proposal for the construction of twistor amplitudes indeed is that the algebraic points (rational in generalized sense) defining cognitive representations are associated with the intersections of string world sheets and partonic 2-surfaces to which fermions are assigned. This would suggest that partonic 2-surfaces and string world sheets contain the cognitive representation, which under additional conditions can contain very many points.

(e) An interesting question concerns the $M^8$ counterparts of partonic 2-surfaces as space-time regions with Minkowskian and Euclidian signature. The partonic orbits representing the boundaries between these regions should be mapped to each other by $M^8 - H$ duality. This conforms with the fact that induced metric must have degenerate signature $(0, -1, -1, -1)$ at partonic orbits. Can one assume that the topologies of partonic 2-surfaces at two sides are identical? Consider partonic 2-surface of genus $g$ in $M^4 \times CP_2$ - say at the boundary of CD. It should be inverse image of a 2-surface in $M^4 \times E^4$ such that the tangent space of this surface labelled by $CP_2$ coordinates is mapped to a 2-surface in $M^4 \times CP_2$. If the inverse of $M^8 - H$ correspondence is continuous one expects that $g$ is preserved.

Consider next the $H$-side. There is a conjecture that for Cartesian product the Kodaira dimension is sum $d_K = \sum_i d_{K,i}$ of the Kodaira dimensions for factors. Suppose that $CP_1$ fiber as surface in the 12-D twistor bundle $T(M^4) \times T(CP_2)$ has Kodaira dimension $d_K(CP_1) = -\infty$ (it is expected to be rational surface) then the fact that the bundle decomposes to Cartesian product locally and rational points are pairs of rational points in the factors, is indeed consistent with the proposal. $S^2$ would give dense set of rational points in $S^2$ and the bundle would have infinite number of rational points.

In TGD context, it is however space-time surface which matters. Space-time surface as section of the bundle would not however have a dense set of points in the general case and the relevant Kodaira dimension be $d_K = d_K(X^4)$. One can of course ask whether the space-time surface as an algebraic section (not many of them) of the twistor bundle could chosen to be cognitively simple.

## 4.5 What does cognitive representability really mean?

The following considerations reflect the ideas inspired by Face Book debate with Santeri Satama (SS) relating to the notion of number and the notion of cognitive representation.

SS wants to accept only those numbers that are constructible, and SS mentioned the notion of demonstrability due to Gödel. According to my impression demonstrability means that number can be constructed by a finite algorithm or at least that the information needed to construct the number can be constructed by a finite algorithm although the construction itself would not be possible as digit sequence in finite time. If the constructibility condition is taken to extreme, one is left only with rationals.

As a physicists, I cannot consider starting to do physics armed only with rationals: for instance, continuous symmetries and the notion of Riemann manifold would be lost. My basic view is that we should identify the limitations of cognitive representability as limitations for what can exist. I talked about cognitive representability of numbers central in the adelic physics approach to TGD. Not all real numbers are cognitively representable and need not be so.

Numbers in the extensions of rationals would be cognitively representable as points with coordinates in an extension of rationals. The coordinates themselves are highly unique in the octonionic approach to TGD and different coordinates choices for complexified octonionic $M^8$ are related by transformations changing the moduli of the octonion structure. Hence one avoids problems with general coordinate invariance). Not only algebraic extensions of rationals are allowed. Neper number $e$ is an exceptional transcendental in that $e^p$ is p-adic number and finite-D extensions of p-adic numbers by powers for root of $e$ are possible.

My own basic interest is to find a deeper intuitive justification for why algebraic numbers shoud be cognitively representable. The naïve view about cognitive representability is that the number can be produced in a finite number of steps using an algorithm. This would leave only rationals under consideration and would mean intellectual time travel to ancient Greece.

Situation changes if one requires that only the information about the construction of number can be produced in a finite number of steps using an algorithm. This would replace construction with the recipe for construction and lead to a higher abstraction level. The concrete construction itself need not be possible in a finite time as bit sequence but could be possible physically ($\sqrt{2}$ as a diagonal of unit square, one can of course wonder where to buy ideal unit squares). Both number theory and geometry would be needed.

Stern-Brocot tree associated with partial fractions indeed allows to identify rationals as finite paths connecting the root of S-B tree to the rational in question. Algebraic numbers can be identified as infinite periodic paths so that finite amount of information specifies the path. Transcendental numbers would correspond to infinite non-periodic paths. A very close analogy with chaos theory suggests itself.

### 4.5.1 Demonstrability viz. cognitive representability

SS talked about demonstrable numbers. According to Gödel demonstrable number would be representable by a formula $G$, which is provable in some axiom system. I understand this that $G$ would give a recipe for constructing that number. In computer programs this can even mean infinite loop, which is easy to write but impossible to realize in practice. Here comes the possibility that demonstrability does not mean constructibility in finite number of steps but only a finite recipe for this.

The requirement that all numbers are demonstrable looks strange to me. I would talk about cognitive representability and reals and p-adic number fields emerge unavoidably as prerequisites for this notion: cognitive representation must be about something in order to be a representation.

About precise construction of reals or something bigger - such as surreals - containing them, there are many views and I am not mathematician enough to take strong stance here. Note however that if one accepts surreals as being demonstrable (I do not really understand what this could mean) one also accept reals as such. These delicacies are not very interesting for the formulation of physics as it is now.

The algorithm defining $G$ defines a proof. But what does proof mean? Proof in mathematical sense would reduce in TGD framework be a purely cognitive act and assignable to the p-adic sectors of adele. Mathematicians however tend to forget that for physicist the demonstration is also experimental. Physicist does not believe unless he sees: sensory perception is needed. Experimental proofs are what physicists want. The existence of $\sqrt{2}$ as a diagonal of unit square is experimentally demonstrable in the sense of being cognitively representable but not deducible from the axioms for rational numbers. As a physicist I cannot but accept both sensory and cognitive aspects of existence.

Instead of demonstrable numbers I prefer to talk about cognitively representable numbers.

(a) All numbers are cognizable (p-adic) or sensorily perceivable (real). These must form continua if one wants to avoid problems in the construction of physical theories, where continuous symmetries are in a key role.

Some numbers but not all are also *cognitively representable* that is being in the intersection reals and p-adics - that is in extension of rationals if one allows extensions of p-adics induced by extensions of rationals. This generalizes to intersection of space-time surfaces with real/p-adic coordinates, which are highly unique linear coordinates at octonionic level so that objections relating to a loss of general coordinate invariance are circumvented. General coordinate transformations reduce to automorphisms of octonions.

The relationship to the axiom of choice is interesting. Should axiom of choice be restricted to the points of complexified octonions with coordinates in extensions of rationals? Only points in the extensions could be selected and this selection process would be physical in the sense that fermions providing realization of quantum Boolean algebra would reside at these points [K2]. In preferred octonionic coordinates the $M^8$ coordinates of these points would be in given extension of rationals. At the limit of algebraic numbers these points would form a dense set of reals.

**Remark:** The spinor structure of "world of classical worlds" (WCW) gives rise to WCW spinors as fermionic Fock states at given 3-surface. In ZEO many-fermion states have interpretation in terms of superpositions of pairs of Boolean statements $A \rightarrow B$ with $A$ and $B$ represented as many-fermion states at the ends of space-time surface located at the opposite light-like boundaries of causal diamond (CD). One could say that quantum Boolean logic emerges as square root of Kähler geometry of WCW.

At partonic 2-surfaces these special points correspond to points at which fermions can be localized so that the representation is physical. Universe itself would come in rescue to make representability possible. One would not anymore try to construct mathematics and physics as distinct independent disciplines.

Even observer as conscious entity is necessarily brought into both mathematics and physics. TGD Universe as a spinor field in WCW is re-created state function reduction by reduction and evolves: evolution for given CD corresponds to the increase of the size of extension of rationals in statistical sense. Hence also mathematics with fixed axioms is replaced with a q dynamical structure adding to itself new axioms discovery by discovery [L16, L15].

(b) Rationals as cognitively representable numbers conforms with naïve intuition. One can however criticize the assumption that also algebraic numbers are such. Consider $\sqrt{2}$: one can simply define it as length of diagonal of unit square and this gives a meter stick of length $\sqrt{2}$: one can represent any algebraic number of form $m + n\sqrt{2}$ by using meter stricks with length of 1 and $\sqrt{2}$. Cognitive representation is also sensory representation and would bring in additional manner to represent numbers.

Note that algebraic numbers in $n$-dimensional extension are points of $n$-dimensional space and their cognitive representations as points on real axis obtained by using the meter sticks assignable to the algebraic numbers defining base vectors. This should generalize to the roots arbitrary polynomials with rational or even algebraic coefficients. Essentially projection form $n$-D extension to 1-D real line is in question. This kind of projection might be important in number theoretical dynamics. For instance, quasi-periodic quasi-crystals are obtained from higher-D periodic crystals as projections.

$n$-D algebraic extensions of p-adics induced by those of rationals might also related to our ability to imagine higher-dimensional spaces.

(c) In TGD Universe cognitive representability would emerge from fundamental physics. Extensions of rationals define a hierarchy of adeles and octonionic surfaces are defined as zero loci for real or imaginary parts (in quaternionic sense) of polynomials of real argument with coefficients in extension continued to octonionic polynomials [L9]. The zeros of real polynomial have a direct physical interpretation and would represent algebraic numbers physically. They would give the temporal positions of partonic 2-surfaces representing particles at light-like boundary of CD.

(d) Note that all calculations with algebraic numbers can be done without using approximations for the genuinely algebraic numbers defining the basis for the extension. This actually simplifies enormously the calculation and one avoids accumulating errors. Only at the end one represents the algebraic units concretely and is forced to use rational approximation unless one uses above kind of cognitive representation.

For these reasons I do not feel any need to get rid of algebraics or even transcendentals. Sensory aspects of experience require reals and cognitive aspects of experience require p-adic numbers fields and one ends up with adelic physics. Cognitive representations are in the intersection of reality and various p-adicities, something expressible as formulas and concrete physical realizations or at least finite recipes for them.

### 4.5.2 What the cognitive representability of algebraic numbers could mean?

Algebraic numbers should be in some sense simple in order to be cognitively representable.

(a) For rationals representation as partial fractions produces the rational number by using a finite number of steps. One starts from the top of Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq`) and moves to right or left at each step and ends up to the rational number appearing only once in S-B tree.

(b) Algebraic numbers cannot be produced in a finite number of steps. During the discussion I however realized that one can produce the information needed to construct the algebraic number in a finite number of steps. One steps to a new level of abstraction by replacing the object with the information allowing to construct the object using infinite number of steps but repeating the same sub-algorithm with finite number of steps: infinite loop would be in question.

Similar abstraction takes place as one makes a step from the level of space-time surface to the level of WCW. Space-time surface with a continuum of points is represented by a finite number of WCW coordinates, in the octonionic representation of space-time surface by the coefficients of polynomial of finite degree belonging to an extension of rationals [L9]. Criticality conditions pose additional conditions on the coefficients. Finite number of algebraic points at space-time surface determines the entire space-time surface under these conditions! Simple names for complex things replacing the complex things is the essence of cognition!

(c) The interpretation for expansions of numbers in given base suggests an analog with complexity theory and symbolic dynamics associated with division. For cognitively representable numbers the information about this dynamics should be coded by an algorithm with finite steps. Periodic orbit or fixed point orbit would be the dynamical analog for simplicity. Non-periodic orbit would correspond to complexity and possibly also chaos.

These ideas led to two approaches in attempt to understand the cognitive representability of algebraic numbers.

*1. Generalized rationals in extensions of rationals as periodic orbits for the dynamics of division*

The first approach allows to represent ratios of algebraic integers for given extension using periodic expansion in the base so that a finite amount of information is needed to code the number if one accepts the numbers defining the basis of the algebraic extension as given.

(a) Rationals allow periodic expansion with respect to any base. For p-adic numbers the base is naturally prime. Therefore the information about rational is finite. One can see the expansion as a periodic orbit in dynamics determining the expansion by division $m/n$ in given base. Periodicity follows from the fact that the output of the division algorithm for a given digit has only a finite number of outcomes so that the process begins to repeat itself sooner or later.

(b) This generalizes to generalized rationals in given extension of rationals defined as ratios of algebraic integers. One can reduce the division to the construction of the expansion of ordinary rational identified as number theoretic norm $|N|$ of the denominator in the extension of rationals considered.

The norm $|N|$ of $N$ is the determinant $|N| = det(N)$ for the linear map of extension induced by multiplication with $N$. $det(N)$ is ordinary (possibly p-adic) integer. This is achieved by multiplying $1/N$ by $n-1$ conjugates of $N$ both in numerator and denominator so that one obtains product of $n-1$ conjugates in the numerator and $det(N)$ in the denominator. The computation of $1/N$ as series in the base used reduces to that in the case of rationals.

(c) One has now periodic orbits in $n$-dimensional space defined by algebraic extensions which for ordinary rationals reduced to periodic orbits in 1-D space. This supports the interpretation of numbers as orbits of number theoretic dynamics determining the next digit of the generalized rational for given base. This picture also suggests that transcendentals correspond to non-periodic orbits. Some transcendentals could still allow a finite algorithm: in this case the dynamics would be still deterministic. Some transcendentals would be chaotic.

(d) Given expansion of algebraic number is same for all extensions of rationals containing the extension in question and the ultimate extension corresponds to algebraic numbers.

The problem of this approach is that the algebraic numbers defining the extension do not have representation and must be accepted as irreducibles.

*2. Algebraic numbers as infinite periodic orbits in the dynamics of partial fractions*

Second approach is based on partial fractions and Stern-Brocot tree (see `http://tinyurl.com/yb6ldekq`, see also `http://tinyurl.com/yc6hhboo`) and indeed allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. I had managed to not become aware of this possibility and am grateful for SS for mentioning the representation of algebraics in terms of S-B tree.

(a) The definition S-B tree is simple: if $m/n$ and $m'/n'$ are any neighboring rationals at given level in the tree, one adds $(m+m')/(n+n')$ between them and obtains in this manner the next level in the tree. By starting from $(0/1)$ and $(1/0)$ as representations of zero and $\infty$ one obtains $(0/1)(1/1)(1/0)$ as the next level. One can continue in this manner ad infinitum. The nodes of S-B tree represent rational points and it can be shown that given rational appears only once in the tree.

Given rational can be represented as a finite path beginning from $1/1$ at the top of tree consisting of left moves $L$ and right moves $R$ and ending to the rational which appears only once in S-B tree. Rational can be thus constructured by a sequences $R^{a_0}L^{a_1}L^{a_2}....$ characterized by the sequence $a_0; a_1, a_2....$ For instance, $4/11 = 0 + 1/(2+x)$ , $x = 1/(1+1/3)$ corresponds to $R^0L^2R^1L^{3-1}$ labelled by $0; 2, 1, 3$.

(b) Algebraic numbers correspond to infinite but periodic paths in S-B tree in the sense that some sequence of $L$:s and $R$:s characterized by sequences of non-negative integers starts to repeat itself. Periodicity means that the information needed to construct the number is finite.

The actual construction as a digit sequence representing algebraic number requires infinite amount of time. In TGD framework octonionic physics would come in rescue and construct algebraic numbers as roots of polynomials having concrete interpretations as coordinate values assignable to fermions at partonic 2-surfaces.

(c) Transcendentals would correspond to non-periodic infinite sequences of $L$:s and $R$:s. This does not exclude the possibility that these sequences are expressible in terms of some rule involving finite number of steps so that the amount of information would be also now finite. Information about number would be replaced by information about rule.

This picture conforms with the ideas about transition to chaos. Rationals have finite paths. A possible dynamical analog is particle coming at rest due to the dissipation. Algebraic numbers would correspond to periodic orbits possible in presence of dissipation if there is external feed of energy. They would correspond to dynamical self-organization patterns.

**Remark**: If one interprets the situation in terms of conservative dynamics, rationals would correspond to potential minima and algebraic numbers closed orbits around them.

The assignment of period doubling and p-pling to this dynamics as the dimension of extension increases is an attractive idea. One would expect that the complexity of periodic orbits increases as the degree of the defining irreducible polynomial increases. Algebraic numbers as maximal extension of rationals possibly also containing extension containing all rational roots of $e$ and transcendentals would correspond to chaos.

Transcendentals would correspond to non-periodic orbits. These orbits need not be always chaotic in the sense of being non-predictable. For instance, Neper number $e$ can be said to be p-adically algebraic number ($e^p$ is p-adic integer albeit infinite as real integer). Does the sequence of $L$:s and $R$:s allow a formula for the powers of $L$ and $R$ in this case?

(d) TGD should be an integrable theory. This suggests that scattering amplitudes involve only cognitive representations as number theoretic vision indeed strongly suggests [L9]. Cognitively representable numbers would correspond to the integrable sub-dynamics [L18]. Also in chaotic systems both periodic and chaotic orbits are present. Complexity theory for characterization of real numbers exists. The basic idea is that complexity is measured by the length of the shortest program needed to code the bit sequences coding for the number.

### 4.5.3 Surreals and ZEO

The following comment is not directly related to cognitive representability but since it emerged during discussion, I will include it. SS favors surreals (see `http://tinyurl.com/86jatas`) as ultimate number field containing reals as sub-field. I must admit that my knowledge and understanding of surreals is rather fragmentary.

I am agnostic in these issues and see no conflict between TGD view about numbers and surreals. Personally I however like very much infinite primes, integers, and rationals over surreals since they allow infinite numbers to have number theoretical anatomy [K13]. A further reason is that the construction of infinite primes resembles structurally repeated second quantization of the arithmetic number field theory and could have direct space-time correlate at the level of many-sheeted space-time. One ends up also to a generalization of real number. Infinity can be seen as something related to real norm: everything is finite with respect to various p-adic norms.

Infinite rationals with unit real norm and various p-adic norms bring in infinitely complex number theoretic anatomy, which could be even able to represent even the huge WCW and the space of WCW spinor fields. One could speak of number theoretical holography or algebraic Brahman=Atman principle. One would have just complexified octonions with infinitely richly structure points.

Surreals are represented in terms of pairs of sets. One starts the recursive construction from empty set identified as 0. The definition says that the pairs (.|.) of sets defining surreals $x$ and $y$ satisfy $x \leq y$ if the left hand part of $x$ as set is to left from the pair defining $y$ and the right hand part of $y$ is to the right from the pair defining $x$. This does not imply that one has always $x < y$, $y < x$ or $x = y$ as for reals.

What is interesting that the pair of sets defining surreal $x$ is analogous to a pair of states at boundaries of CD defining zero energy state. Is there a connection with zero energy ontology (ZEO)? One could perhaps say at the level of CD - forgetting everything related to zero energy states - following. The number represented by $CD_1$ - say represented as the distance between its tip - is smaller than than the number represented by $CD_2$, if $CD_1$ is inside $CD_2$. This conforms with the left and righ rule if left and right correspond to the opposite boundaries of CD. A more detailed definition would presumably say that $CD_1$ can be moved so that it is inside $CD_2$.

What makes this also interesting is that CD is the geometric correlate for self, conscious entity, also mathematical mental image about number.

# 5 Galois groups and genes

In an article discussing a TGD inspired model for possible variations of $G_{eff}$ [L21], I ended up with an old idea that subgroups of Galois group could be analogous to conserved genes in that they could be conserved in number theoretic evolution. In small variations such as above variation Galois subgroups as genes would change only a little bit. For instance, the dimension of Galois subgroup would change.

The analogy between subgoups of Galois groups and genes goes also in other direction. I have proposed long time ago that genes (or maybe even DNA codons) could be labelled by $h_{eff}/h = n$ . This would mean that genes (or even codons) are labelled by a Galois group of Galois extension (see `http://tinyurl.com/zu5ey96`) of rationals with dimension $n$ defining the number of sheets of space-time surface as covering space. This could give a concrete dynamical and geometric meaning for the notion of gene and it might be possible some day to understand why given gene correlates with particular function. This is of course one of the big problems of biology.

## 5.1 Could DNA sequence define an inclusion hierarchy of Galois extensions?

One should have some kind of procedure giving rise to hierarchies of Galois groups assignable to genes. One would also like to assign to letter, codon and gene and extension of rationals and its Galois group. The natural starting point would be a sequence of so called intermediate Galois extensions $E^H$ leading from rationals or some extension $K$ of rationals to the final extension $E$. Galois extension has the property that if a polynomial with coefficients in $K$ has single root in $E$, also other roots are in $E$ meaning that the polynomial with coefficients $K$ factorizes into a product of linear polynomials. For Galois extensions the defining polynomials are irreducible so that they do not reduce to a product of polynomials.

Any sub-group $H \subset Gal(E/K))$ leaves the intermediate extension $E^H$ invariant in element-wise manner as a sub-field of $E$ (see `http://tinyurl.com/y958drcy`). Any subgroup $H \subset Gal(E/K))$ defines an intermediate extension $E^H$ and subgroup $H_1 \subset H_2 \subset ...$ define a hierarchy of extensions $E^{H_1} > E^{H_2} > E^{H_3}...$ with decreasing dimension. The subgroups $H$ are normal - in other words $Gal(E)$ leaves them invariant and $Gal(E)/H$ is group. The order $|H|$ is the dimension of $E$ as an extension of $E^H$. This is a highly non-trivial piece of information. The dimension of $E$ factorizes to a product $\prod_i |H_i|$ of dimensions for a sequence of groups $H_i$.

Could a sequence of DNA letters/codons somehow define a sequence of extensions? Could one assign to a given letter/codon a definite group $H_i$ so that a sequence of letters/codons would correspond a product of some kind for these groups or should one be satisfied only with the assignment of a standard kind of extension to a letter/codon?

Irreducible polynomials define Galois extensions and one should understand what happens to an irreducible polynomial of an extension $E^H$ in a further extension to $E$. The degree of $E^H$ increases by a factor, which is dimension of $E/E^H$ and also the dimension of $H$. Is there a standard manner to construct irreducible extensions of this kind?

(a) What comes into mathematically uneducated mind of physicist is the functional decomposition $P^{m+n}(x) = P^m(P^n(x))$ of polynomials assignable to sub-units (letters/codons/genes) with coefficients in $K$ for a algebraic counterpart for the product of sub-units. $P^m(P^n(x))$ would be a polynomial of degree $n+m$ in $K$ and polynomial of degree $m$ in $E^H$ and one could assign to a given gene a fixed polynomial obtained as an iterated function composition. Intuitively it seems clear that in the generic case $P^m(P^n(x))$ does not decompose to a product of lower order polynomials. One could use also polynomials assignable to codons or letters as basic units. Also polynomials of genes could be fused in the same manner.

(b) If this indeed gives a Galois extension, the dimension $m$ of the intermediate extension should be same as the order of its Galois group. Composition would be non-commutative but associative as the physical picture demands. The longer the gene, the higher the algebraic complexity would be. Could functional decomposition define the rule for who extensions and Galois groups correspond to genes? Very naïvely, functional decomposition in mathematical sense would correspond to composition of functions in biological sense.

(c) This picture would conform with $M^8 - M^4 \times CP_2$ correspondence [L9] in which the construction of space-time surface at level of $M^8$ reduces to the construction of zero loci of polynomials of octonions, with rational coefficients. DNA letters, codons, and genes would correspond to polynomials of this kind.

## 5.2 Could one say anything about the Galois groups of DNA letters?

A fascinating possibility is that this picture could allow to say something non-trivial about the Galois groups of DNA letters.

(a) Since $n = h_{eff}/h$ serves as a kind of quantum IQ, and since molecular structures consisting of large number of particles are very complex, one could argue that $n$ for DNA or its dark variant realized as dark proton sequences can be rather large and depend on the evolutionary level of organism and even the type of cell (neuron viz. soma cell). On the other, hand one could argue that in some sense DNA, which is often thought as information processor, could be analogous to an integrable quantum field theory and be solvable in some sense. Notice also that one can start from a background defined by given extension $K$ of rationals and consider polynomials with coefficients in $K$. Under some conditions situation could be like that for rationals.

(b) The simplest guess would be that the 4 DNA letters correspond to 4 non-trivial finite groups with smaller possible orders: the cyclic groups $Z_2, Z_3$ with orders 2 and 3 plus 2 finite groups of order 4 (see the table of finite groups in `http://tinyurl.com/j8d5uyh`). The groups of order 4 are cyclic group $Z_4 = Z_2 \times Z_2$ and Klein group $Z_2 \oplus Z_2$ acting as a symmetry group of rectangle that is not square - its elements have square equal to unit element. All these 4 groups are Abelian. Polynomial equations of degree not larger than 4 can be solved exactly in the sense that one can write their roots in terms of radicals.

(c) Could there exist some kind of connection between the number 4 of DNA letters and 4 polynomials of degree less than 5 for whose roots one an write closed expressions in terms of radicals as Galois found? Could it be that the polynomials obtained by a a repeated functional composition of the polynomials of DNA letters have also this solvability property?

This could be the case! Galois theory states that the roots of polynomial are solvable by radicals if and only if the Galois group is solvable meaning that it can be constructed from abelian groups using Abelian extensions (see `https://cutt.ly/4RuXmGo`).

Solvability translates to a statement that the group allows so called sub-normal series $1 < G_0 < G_1 ... < G_k$ such that $G_{j-1}$ is normal subgroup of $G_j$ and $G_j/G_{j-1}$ is an abelian group. An equivalent condition is that the derived series $G \triangleright G^{(1)} \triangleright G^{(2)} \triangleright ...$ in which $j+1$:th group is commutator group of $G_j$ ends to trivial group. If one constructs the iterated polynomials by using only the 4 polynomials with Abelian Galois groups, the intuition of physicist suggests that the solvability condition is guaranteed! Wikipedia article also informs that for finite groups solvable group is a group whose composition series has only factors which are cyclic groups of prime order.

Abelian groups are trivially solvable, nilpotent groups are solvable, p-groups (having order, which is power prime) are solvable and all finite p-groups are nilpotent. Every group with order less than 60 elements is solvable. Fourth order polynomials can have at most $S_4$ with 24 elements as Galois groups and are thus solvable. Fifth order polynomials can have the smallest non-solvable group, which is alternating group $A_5$ with 60 elements as Galois group and in this case are not solvable. $S_n$ is not solvable for $n > 4$ and by the finding that $S_n$ as Galois group is favored by its special properties (see `https://arxiv.org/pdf/1511.06446.pdf`).

$A_5$ acts as the group icosahedral orientation preserving isometries (rotations). Icosahedron and tetrahedron glued to it along one triangular face play a key role in TGD inspired model of bio-harmony and of genetic code [L2, L22]. The gluing of tetrahedron increases the number of codons from 60 to 64. The gluing of tetrahedron to icosahedron also reduces the order of isometry group to the rotations leaving the common face fixed and makes it solvable: could this explain why the ugly looking gluing of tetrahedron to icosahedron is needed? Could the smallest solvable groups and smallest non-solvable group be crucial for understanding the number theory of the genetic code.

An interesting question inspired by $M^8-H$-duality [L9] is whether the solvability could be posed on octonionic polynomials as a condition guaranteeing that TGD is integrable theory in number theoretical sense or perhaps following from the conditions posed on the octonionic polynomials. Space-time surfaces in $M^8$ would correspond to zero loci of real/imaginary parts (in quaternionic sense) for octonionic polynomials obtained from rational polynomials by analytic continuation. Could solvability relate to the condition guaranteeing $M^8$ duality boiling down to the condition that the tangent spaces of space-time surface are labelled by points of $CP_2$. This requires that tangent or normal space is associative (quaternionic) and that it contains fixed complex sub-space of octonions or perhaps more generally, there exists an integrable distribution of complex subspaces of octonions defining an analog of string world sheet.

What could the interpretation for the events in which the dimension of the extension of rationals increases? Galois extension is extensions of an extension with relative Galois group $Gal(rel) = Gal(new)/Gal(old)$. Here $Gal(old)$ is a normal subgroup of $Gal(new)$. A highly attractive possibility is that evolutionary sequences quite generally (not only in biology) correspond to this kind of sequences of Galois extensions. The relative Galois groups in the sequence would be analogous to conserved genes, and genes could indeed correspond to Galois groups [K4] [L9]. To my best understanding this corresponds to a situation in which the new polynomial $P_{m+n}$ defining the new extension is a polynomial $P_m$ having as argument the old polynomial $P_n(x)$: $P_{m+n}(x) = P_m(P_n(x))$.

What about the interpretation at the level of conscious experience? A possible interpretation is that the quantum jump leading to an extension of an extension corresponds to an emergence of a reflective level of consciousness giving rise to a conscious experience about experience. The abstraction level of the system becomes higher as is natural since number theoretic evolution as an increase of algebraic complexity is in question.

This picture could have a counterpart also in terms of the hierarchy of inclusions of hyperfinite factors of type $II_1$ (HFFs). The included factor $M$ and including factor $N$ would correspond to extensions of rationals labelled by Galois groups $Gal(M)$ and $Gal(N)$ having $Gal(M) \subset Gal(M)$ as normal subgroup so that the factor group $Gal(N)/Gal(M)$ would be the relative Galois group for the larger extension as extension of the smaller extension. I have indeed proposed [L23] that the inclusions for which included and including factor consist of operators which are invariant under discrete subgroup of $SU(2)$ generalizes so that all Galois groups are possible. One would have Galois confinement analogous to color confinement: the operators generating physical states could have Galois quantum numbers but the physical states would be Galois singlets.

# 6 Could the precursors of perfectoids emerge in TGD?

In algebraic-geometry community the work of Peter Scholze [A7] (see http://tinyurl.com/y7h2sms7) introducing the notion of perfectoid related to p-adic geometry has raised a lot of interest. There are two excellent popular articles about perfectoids: the first article in AMS (see http://tinyurl.com/ydx38vk4) and second one in Quanta Magazine (see http://tinyurl.com/yc2mxxqh). I had heard already earlier about the work of Scholze but was too lazy to even attempt to understand what is buried under the horrible technicalities of modern mathematical prose. Rachel Francon re-directed my attention to the work of Scholze (see http://tinyurl.com/yb46oza6). The work of Scholze is interesting also from TGD point of view since the construction of p-adic geometry is a highly non-trivial challenge in TGD.

(a) One should define first the notion of continuous manifold but compact-open characteristic of p-adic topology makes the definition of open set essential for the definition of topology problematic. Even single point is open so that hopes about p-adic manifold seem to decay to dust. One should pose restrictions on the allowed open sets and p-adic balls with radii coming as powers of $p$ are the natural candidates. p-Adic balls are either disjoint or nested: note that also this is in conflict with intuitive picture about covering of manifold with open sets. All this strangeness originates in the special features of p-adic distance function known as ultra-metricity. Note however that for extensions of p-adic numbers one can say that the Cartesian products of p-adic 1-balls at different genuinely algebraic points of extension along particular axis of extension are disjoint.

(b) At level of $M^8$ the p-adic variants of algebraic varieties defined as zero loci of polynomials do not seem to be a problem. Equations are algebraic conditions and do not involve derivatives like partial differential equations naturally encountered if Taylor series instead of polynomials are allowed. Analytic functions might be encountered at level of $H = M^4 \times CP_2$ and here p-adic geometry might well be needed.

The idea is to define the generalization of p-adic algebraic geometry in terms of p-adic function fields using definitions very similar to those used in algebraic geometry. For instance, generalization of variety corresponds to zero locus for an ideal of p-adic valued function field. p-Adic ball of say unit radius is taken as the basic structure taking the role of open ball in the topology of ordinary manifolds. This kind of analytic geometry allowing all power series with suitable restrictions to function field rather than allowing only polynomials is something different from algebraic geometry making sense for p-adic numbers and even for finite fields.

(c) One would like to generalize the notion of analytic geometry even to the case of number fields with characteristic $p$ ($p$-multiple of element vanishes), in particular for finite fields $F_p$ and for function fields $F_p[t]$. Here one encounters difficulties. For instance, the factorial $1/n!$ appearing as normalization factor of forms diverges if $p$ divides it. Also the failure of Frobenius homomorphism to be automorphism for $F_p[t]$ causes difficulties in the understanding of Galois groups.

The work of Scholze has led to a breakthrough in unifying the existing ideas in the new framework provided by the notion of perfectoid. The work is highly technical and involves infinite-D extension of ordinary p-adic numbers adding all powers of all roots $p^{1/p^m}$, $m = 1, 2....$ Formally, an extension by powers of $p^{1/p^\infty}$ is in question.

This looks strange at first but it guarantees that all p-adic numbers in the extension have $p$:th roots, one might say that one forms a $p$-fold covering/wrapping of extension somewhat analogous to complex numbers. This number field is called perfectoid since it is perfect meaning that Frobenius homomorphism $a \to a^p$ is automorphism by construction. $Frob$ is injection always and by requiring that $p$:th roots exist always, it becomes also a surjection.

This number field has same Galois groups for all of its extensions as the function field $G[t]$ associated with the union of function fields $G = F_p[t^{1/p^m}]$. Automorphism property of $Frob$ saves from the difficulties with the factorization of polynomials and p-adic arithmetics involving remainders is replaced with purely local modulo $p$ arithmetics.

## 6.1 About motivations of Scholze

Scholze has several motivations for this work. Since I am not a mathematician, I am unable to really understand all of this at deep level but feel that my duty as user of this mathematics is at least to try!

(a) Diophantine equations is a study of polynomial equations in several variables, say $x^2 + 2xy + y = 0$. The solutions are required to be integer valued: in the example considered $x = y = 0$ and $x = -y = -1$ is such a solution. For integers the study of the solution is very difficult and one approach is to study these equations modulo $p$ that is reduced the equations to finite field $G_p$ for any $p$. The equations simplify enormously since ane has $a^p = a$ in $F_p$. This identity in fact defines so called Frobenius homomorphism acting as automorphism for finite fields. This holds true also for more complex fields with characteristic $p$ say the ring $F_p[t]$ of power series of $t$ with coefficients in $F_p$.

The powers of variables, say $x$, appearing in the equation is reduced to at most $x^{p-1}$. One can study the solutions also in p-adic number fields. The idea is to find first whether finite field solution, that is solution modulo $p$, does exist. If this is the case, one can calculate higher powers in $p$. If the series contains finite number of terms, one has solution also in the sense of ordinary integers.

(b) One of the related challenges is the generalization of the notion of variety to a geometry defined in arbitrary number field. One would like to have the notion of geometry also for finite fields, and for their generalizations such as $F_p[t]$ characterized by characteristic $p$ ($px = 0$ holds true for any element of the field). For fields of characteristic 1 - extensions of rationals, real, and p-adic number fields) $xp = 0$ not hold true for any $x \neq 0$. Any field containing rationals as sub-field, being thus local field, is said to have characteristic equal to 1. For local fields the challenge is relatively easy.

(c) The situation becomes more difficult if one wants a generalization of differential geometry. In differential geometry differential forms are in a key role. One wants to define the notion of differential form in fields of characteristic $p$ and construct a generalization of cohomology theory. This would generalize the notion of topology to p-adic context and even for finite fields of finite character. A lot of work has been indeed done and Grothendieck has been the leading pioneer.

The analogs of cohomology groups have values in the field of p-adic numbers instead of ordinary integers and provide representations for Galois groups for the extensions of rationals inducing extensions of p-adic numbers and finite fields.

In ordinary homology theory non-contractible sub-manifolds of various dimensions correspond to direct summands $Z$ (group of integers) for homology groups and by Poincare duality those for cohomology groups. For Galois groups $Z$ is replaced with $Z_N$. $N$ depends on extension to which Galois group is associated and if $N$ is divisible by $p$ one encounters technical problems.

There are many characteristic $p$- and p-adic cohomologies such as etale cohomology, chrystalline cohomology, algebraic de-Rham cohomology. Also Hodge theory for complex differential forms generalizes. These cohomologies should be related by homomorphism and category theoretic thinking the proof of the homomorphism requires the construction of appropriate functor between them.

The integrals of forms over sub-varieties define the elements of cohomology groups in ordinary cohomology and should have p-adic counterparts. Since p-adic numbers are not well-ordered, definite integral has no straightforward generalization to p-adic context. One might however be able to define integrals analogous to those associated with differential forms and depending only on the topology of sub-manifold over which they are taken. These integrals would be analogous to multiple residue integrals, which are the crux of the twistor approach to scattering amplitudes in super-symmetric gauge theories. One technical difficulty is that for a field of finite characteristic the derivative of $X^p$ is $pX^{p-1}$ and vanishes. This does not allow to define what integral $\int X^{p-1} dX$ could mean. Also $1/n!$ appears as natural normalization factor of forms but if $p$ divides it, it becomes infinite.

## 6.2 Attempt to understand the notion of perfectoid

Consider now the basic ideas behind the notion of perfectoid.

(a) For finite finite fields $F_p$ Frobenius homomorphism $a \to a^p$ is automorphism since one has $a^p = a$ in modulo $p$ arithmetics. A field with this property is called perfect and all local fields are perfect. Perfectness means that an algebraic number in any extension $L$ of perfect field $K$ is a root of a separable minimal polynomial. Separability means that the number of roots in the algebraic closure of $K$ of the polynomial is maximal and the roots are distinct.

(b) All fields containing rationals as sub-fields are perfect. For fields of characteristic $p$ $Frob$ need not be a surjection so that perfectness is lost. For instance, for $F_p[t]$ $Frob$ is trivially injection but surjective property is lost: $t^{1/p}$ is not integer power of $t$.

One can however extend the field to make it perfect. The trick is simple: add to $F_p[t]$ all fractional powers $t^{1/p^n}$ so that all $p$:th roots exist and $Frob$ becomes and automorphism. The automorphism property of $Frob$ allows to get rid of technical problems related to a factorization of polynomials. The resulting extension is infinite-dimensional but satisfies the perfectness property allowing to understand Galois groups, which play key role in various cohomology theories in characteristic $p$.

(c) Let $K = Q_p[p^{1/p^\infty}]$ denote the infinite-dimensional extension of p-adic number field $Q_p$ by adding all powers of $p^m$:th roots for all all $m = 1, 2, ....$ This is not the most general option: $K$ could be also only a ring. The outcome is perfect field although it does not of course have Frobenius automorphism since characteristic equals to 1.

One can divide $K$ by $p$ to get $K/p$ as the analog of finite field $F_p$ as its infinite-dimensional extension. $K/p$ allows all $p$:th roots by construction and $Frob$ is automorphism so that $K/p$ is perfect by construction.

The structure obtained in this manner is closely related to a perfect field with characteristic $p$ having same Galois groups for all its extensions. This object is computationally much more attractive and allows to prove theorems in p-adic geometry. This motivates the term perfectoid.

(d) One can assign to $K$ another object, which is also perfectoid but has characteristic $p$. The correspondence is as follows.

   i. Let $F_p$ be finite field. $F_p$ is perfect since it allows trivially all $p$:th roots by $a^p = a$. The ring $F_p[t]$ is however not prefect since $t^{1/p^m}$ is not integer power of $t$. One must modify $F_p[t]$ to obtain a perfect field. Let $G_m = F_p[t^{1/p^m}]$ be the ring of formal series in powers of $t^{1/p^m}$ defining also function field. These serious are called t-adic and one can define t-adic norm.

   ii. Define t-adic function field $K_b$ called the **tilt** of $K$ as

$$K_b = \cup_{m=1,...}(K/p)[t^{1/pm}][t] \ .$$

   One has all possible power series with coefficients in $K/p$ involving all roots $t^{1/p^m}$, $m = 1, 2, ...$, besides powers of positive integer powers of $t$. This function field has characteristic $p$ and all roots exist by construction and $Frob$ is automorphism. $K_b/t$ is perfect meaning that the minimal polynomials for the for given analog of algebraic number in any of its extensions allows separable polynomial with maximal number of roots in its closure.

This sounds rather complicated! In any case, $K_b/t$ has same number theoretical structure as $Q_p[p^{1/p^\infty}]/p$ meaning that Galois groups for all of its extensions are canonically isomorphic to those for extensions of $K$. Arithmetics modulo $p$ is much simpler than p-adic arithmetic since products are purely local and there is no need to take care about remainders in arithmetic operations, this object is much easier to handle.

Note that also p-adic number fields fields $Q_p$ as also $F_p = Q_p/p$ are perfect but the analog of $K_b = F_b[t]$ fails to be perfect.

### 6.3 Second attempt to understand the notions of perfectoid and its tilt

This subsection is written roughly year after the first version of the text. I hope that it reflects a genuine increase in my understanding.

(a) Scholze introduces first the notion of perfectoid. This requires some background notions. The characteristic $p$ for field is defined as the integer $p$ (prime) for which $px = 0$ for all elements $x$. Frobenius homomorphism (Frob familiarly) is defined as $Frob : x \to x^p$. For a field of characteristic $p$ $Frob$ is an algebra homomorphism mapping product to product and sum to sum: this is very nice and relatively easy to show even by a layman like me.

(b) Perfectoid is a field having either characteristic $p = 0$ (reals, p-adics for instance) or for which $Frob$ is a surjection meaning that $Frob$ maps at least one number to a given number $x$.

(c) For finite fields $Frob$ is identity: $x^p = x$ as proved already by Fermat. For reals and p-adic number fields with characteristic p=0 it maps all elements to unit element and is not a surjection. Field is perfect if it has either $p = 0$ (reals, p-adics) or if Frobenius is surjection. Finite fields are obviously perfectoids too.

Scholze introduces besides perfectoids $K$ also what he calls tilt $K_b$ of the perfectoid. $K_b$ is infinite-D extension of p-adic numbers by iterated $p$:th roots p-adic numbers: the units of the extension correspond to the roots $p^{1/p^k}$. They are something between p-adic number fields and reals and leads to theorems giving totally new insights to arithmetic geometry. Unfortunately, my technical skills in mathematics are hopelessly limited to say anything about these theorems.

(a) As we learned during the first student year of mathematics, real numbers can be defined as Cauchy sequences of rationals converging to a real number, which can be also algebraic number or transcendental. The elements in the tilt $K_b$ would be this kind of sequences.

(b) Scholze starts from (say) p-adic numbers and considers infinite sequence of iterates of $1/p$:th roots. At given step $x \to x^{1/p}$. This gives the sequence $(x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, ...)$ identified as an element of the tilt $K_b$. At the limit one obtains $1/p^\infty$ root of $x$.

   **Remark:** For finite fields each step is trivial ($x^p = x$) so that nothing interesting results: one has $(x, x, x, x, ...)$

   i. For p-adic number fields the situation is non-trivial. $x^{1/p}$ exists as p-adic number for all p-adic numbers with unit norm having $x = x_0 + x_1p + ....$ In the lowest order $x \simeq x_0$ the root is just $x$ since $x$ is effectively an element of finite field in this approximation. One can develop the $x^{1/p}$ to a power series in $p$ and continue the iteration. The sequence obtained defines an element of tilt $K_b$ of field K, now p-adic numbers.

   ii. If the p-adic number $x$ has norm $p^n$, $n \neq 0$ and is therefore not p-adic unit, the root operation makes sense only if one performs an extension of p-adic numbers containing all the roots $p^{1/p^k}$ . These roots define one particular kind of extension of p-adic numbers and the extension is infinite-dimensional since all roots are needed. One can approximate $K_b$ by taking only finite number iterated roots.

(c) The tilt is said to be fractal: this is easy to understand from the presence of the iterated $p$:th root. Each step in the sequence is like zooming. One might say that p-adic scale becomes $p$:th root of itself. In TGD the p-adic length scale $L_p$ is proportional to $p^{1/2}$: does the scaling mean that the p-adic length scale would defined hierarchy of scales proportional to $p^{1/2kp}$: root of itself and approach the $CP_2$ scale since the root of $p$ approaches unity. Tilts as extensions by iterated roots would improve the length scale resolution.

One day later after writing this I got the feeling that I might have vaguely understood one more important thing about the tilt of p-adic number field: changing of the characteristic

0 of p-adic number field to characteristics $p > 0$ of the corresponding finite field for its tilt. What could this mean?

(a) Characteristic $p$ ($p$ is the prime labelling p-adic number field) means $px = 0$. This property makes the mathematics of finite fields extremely simple: in the summation one need not take care of the residue as in the case of reals and p-adics. The tilt of the p-adic number field would have the same property! In the infinite sequence of the p-adic numbers coming as iterated $p$:th roots of the starting point p-adic number one can sum each p-adic number separately. This is really cute if true!

(b) It seems that one can formulate the arithmetics problem in the tilt where it becomes in principle as simple as in finite field with only p elements! Does the existence of solution in this case imply its existence in the case of p-adic numbers? But doesn't the situation remain the same concerning the existence of the solution in the case of rational numbers? The infinite series defining p-adic number must correspond a sequence in which binary digits repeat with some period to give a rational number: rational solution is like a periodic solution of a dynamical system whereas non-rational solution is like chaotic orbit having no periodicity? In the tilt one can also have solutions in which some iterated root of $p$ appears: these cannot belong to rationals but to their extension by an iterated root of $p$.

The results of Scholze could be highly relevant for the number theoretic view about TGD in which octonionic generalization of arithmetic geometry plays a key role since the points of space-time surface with coordinates in extension of rationals defining adele and also what I call cognitive representations determining the entire space-time surface if $M^8 - H$ duality holds true (space-time surfaces would be analogous to roots of polynomials). Unfortunately, my technical skills in mathematics needed are hopelessly limited.

TGD inspires the question is whether this kind of extensions could be interesting physically. At the limit of infinite dimension one would get an ideal situation not realizable physically if one believes that finite-dimensionality is basic property of extensions of p-adic numbers appearing in number theoretical quantum physics (they would related to cognitive representations in TGD). Adelic physics [L15] involves all finite-D extensions of rationals and the extensions of p-adic number fields induced by them and thus also cutoffs of extensions of type $K_b$- which I have called precursors of $K_b$.

### 6.3.1 How this relates to Witt vectors?

Witt vectors provide an alternative representation of p-adic arithmetics of p-adic integers in which the sum and product are reduced to purely local digit-wise operations for each power of p for the components of Witt vector so that one need not worry about carry pinary digit.

(a) The idea is to consider the sequence consisting pinary cutoffs to p-adic number $xmodp^n$ and identify p-adic integer as this kind of sequence as $n$ approaches infinity. This is natural approach when one identifies finite measurement resolution or cognitive resolution as a cutoff in some power of $p^n$. One simply forms the numbers $X_n = x\ mod\ p^{n+1}$: for numbers $1, ..., p-1$ they are called Teichmueller representatives and only they are needed to construct the sequences for general $x$. One codes this sequence of pinary cutoffs to Witt vector.

(b) The non-trivial observation made by studying sums of p-adic numbers is that the sequence $X_0, X_1, X_2, ...$ of approximations define a sequence of components of Witt vector as $W_0 = X_0$, $W_1 = X_0^p + pX_1$, $W_2 = X_0^(p^2) + pX_1^p + p^2X_2$, ... or more formally $W_n = Sum_{i<n} p^i Z X_i^[ p^(n-i)]$.

(c) The non-trivial point is that Witt vectors form a commutative ring with local digit-wise multiplication and sum modulo $p$: there no carry digits. Effectively one obtains infinite Cartesian power of finite field $F_p$. This means a great simplification in arithmetics. One can do the arithmetics using Witt vectors and deduce the sum and product from their product.

(d) Witt vectors are universal. In particular, they generalize to any extension of p-adic numbers. Could Witt vectors bring in something new from physics point of view? Could they allow a formulation for the hierarchy of pinary cutoffs giving some new insights? For instance, neuro-computationalist might ask whether brain could perform p-adic arithmetics using a linear array of modules (neurons or neuron groups) labelled by $n = 1, 2, ...$ calculates sum or product for component $W_n$ of Witt vector? No transfer of carry bits between modules would be needed. There is of course the problem of transforming p-adic integers to Witt vectors and back - it is not easy to imagine a natural realization for a module performing this transformation. Is there any practical formulation for say p-adic differential calculus in terms of Witt vectors?

I would seem that Witt vectors might relate in an interesting manner to the notion of perfectoid. The basic result proved by Petter Scholtze is that the completion $\cup_n Q_p(p^{1/p^n})$ of p-adic numbers by adding $p^n$:th roots and the completion of Laurent series $F_p((t))$ to $\cup_n F_p((t^{1/p^n}))$ have isomorphic absolute Galois groups and in this sense are one and same thing. On the other hand, p-adic integers can be mapped to a subring of $F_p(t)$ consisting of Taylor series with elements allowing interpretation as Witt vectors.

## 6.4 TGD view about p-adic geometries

As already mentioned, it is possible to define p-adic counterparts of $n$-forms and also various p-adic cohomologies with coefficient field taken as p-adic numbers and these constructions presumably make sense in TGD framework too. The so called rigid analytic geometry is the standard proposal for what p-adic geometry might be.

The very close correspondence between real space-time surfaces and their p-adic variants plays realized in terms of cognitive representations [L17, L16, L9] plays a key role in TGD framework and distinguishes it from approaches trying to formulate p-adic geometry as a notion independent of real geometry.

Ordinary approaches to p-adic geometry concentrate the attention to single p-adic prime. In the adelic approach of TGD one considers both reals and all p-adic number fields simultaneously.

Also in TGD framework Galois groups take key role in this framework and effectively replace homotopy groups and act on points of cognitive representations consisting of points with coordinates in extension of rationals shared by real and p-adic space-time surfaces. One could say that homotopy groups at level of sensory experience are replaced by Galois at the level of cognition. It also seems that there is very close connection between Galois groups and various symmetry groups. Galois groups would provide representations for discrete subgroups of symmetry groups.

In TGD framework there is strong motivation for formulating the analog of Riemannian geometry of $H = M^4 \times CP_2$ for p-adic variants of $H$. This would mean p-adic variant of Kähler geometry. The same challenge is encountered even at the level of "World of Classical Worlds" (WCW) having Kähler geometry with maximal isometries. p-Adic Riemann geometry and $n$-forms make sense locally as tensors but integrals defining distances do not make sense p-adically and it seems that the dream about global geometry in p-adic context is not realizable. This makes sense: p-adic physics is a correlate for cognition and one cannot put thoughts in weigh or measure their length.

### 6.4.1 Formulation of adelic geometry in terms of cognitive representations

Consider now the key ideas of adelic geometry and of cognitive representations.

(a) The king idea is that p-adic geometries in TGD framework consists of p-adic balls of possibly varying radii $p^n$ assignable to points of space-time surface for which the preferred embedding space coordinates are in the extension of rationals. At level of $M^8$

octonion property fixes preferred coordinates highly uniquely. At level of $H$ preferred coordinates come from symmetries.

These points define a cognitive representation and inside p-adic points the solution of field equations is p-adic variant of real solution in some sense. At $M^8$ level the field equations would be algebraic equations and real-p-adic correspondence would be very straightforward. Cognitive representations would make sense at both $M^8$ level and $H$ level.

**Remark:** In ordinary homology theory the decomposition of real manifold to simplexes reduces topology to homology theory. One forgets completely the interiors of simplices. Could the cognitive representations with points labelling the p-adic balls could be seen as analogous to decompositions to simplices. If so, homology would emerge as something number theoretically universal. The larger the extension of rationals, the more precise the resolution of homology would be. Therefore p-adic homology and cohomology as its Poincare dual would reduce to their real counterparts in the cognitive resolution used.

(b) $M^8 - H$ correspondence would play a key role in mapping the associative regions of space-time varieties in $M^8$ to those in $H$. There are two kinds of regions. Associative regions in which polynomials defining the surfaces satisfy criticality conditions and non-associative regions. Associative regions represent external particles arriving in CDs and non-associative regions interaction regions within CDs.

(c) In associative regions one has minimal surface dynamics (geodesic motion) at level of $H$ and coupling parameters disappear from the field equations in accordance with quantum criticality. The challenge is to prove that $M^8 - H$ correspondence is consistent with the minimal surface dynamics n $H$. The dynamics in these regions is determined in $M^8$ as zero loci of polynomials satisfying quantum criticality conditions guaranteeing associativity and is deterministic also in p-adic sectors since derivatives are not involved and pseudo constants depending on finite number of pinary digits and having vanishing derivative do not appear. $M^8 - H$ correspondence guarantees determinism in p-adic sectors also in $H$.

(d) In non-associative regions $M^8 - H$ correspondence does not make sense since the tangent space of space-time variety cannot be labelled by $CP_2$ point and the real and p-adic $H$ counterparts of these regions would be constructed from boundary data and using field equations of a variational principle (sum of the volume term and Kähler action term), which in non-associative regions gives a dynamics completely analogous to that of charged particle in induced Kähler field. Now however the field characterizes extended particle itself.

Boundary data would correspond to partonic 2-surfaces and string world sheets and possibly also the 3-surfaces at the ends of space-time surface at boundaries of CD and the light-like orbits of partonic 2-surfaces. At these surfaces the 4-D (!) tangent/normal space of space-time surface would be associative and could be mapped by $M^8 - H$ correspondence from $M^8$ to $H$ and give rise to boundary conditions.

Due to the existence of p-adic pseudo-constants the p-adic dynamics determined by the action principle in non-associative regions inside CD would not be deterministic in p-adic sectors. The interpretation would be in terms of freedom of imagination. It could even happen that boundary values are consistent with the existence of space-time surface in p-adic sense but not with the existence of real space-time surfaces. Not all that can be imagined is realizable.

At the level of $M^8$ this vision seems to have no obvious problems. Inside each ball the same algebraic equations stating vanishing of $IM(P)$ (imaginary part of $P$ in quaternionic sense) hold true. At the level of $H$ one has second order partial differential equations, which also make sense also p-adically. Besides this one has infinite number of boundary conditions stating the vanishing of Noether charges assignable to sub-algebra super-symplectic algebra and its commutator with the entire algebra at the 3-surfaces at the boundaries of CD. Are these two descriptions really equivalent?

During writing I discovered an argument, which skeptic might see as an objection against $M^8 - H$ correspondence.

(a) $M^8$ correspondence maps the space-time varieties in $M^8$ in non-local manner to those in $H = M^4 \times CP_2$. $CP_2$ coordinates characterize the tangent space of space-time variety in $M^8$ and this might produce technical problems. One can map the real variety to $H$ and find the points of the image variety satisfying the condition and demand that they define the "spine" of the p-adic surface in p-adic $H$.

(b) The points in extensions of rationals in $H$ need not be images of those in $M^8$ but should this be the case? Is this really possible? $M^4$ point in $M^4 \times E^4$ would be mapped to $M^4 \subset M^4 \times CP_2$: this is trivial. 4-D associative tangent/normal space at $m$ containing preferred $M^2$ would be characterized by $CP_2$ coordinates: this is the essence of $M^8 - H$ correspondence. How could one guarantee that the $CP_2$ coordinates characterizing the tangent space are really in the extension of rationals considered? If not, then the points of cognitive representation in $H$ are not images of points of cognitive representation in $M^8$. Does this matter?

### 6.4.2 Are almost-perfectoids evolutionary winners in TGD Universe?

One could take perfectoids and perfectoid spaces as a mere technical tool of highly refiner mathematical cognition. Since cognition is basic aspect of TGD Universe, one could also ask perfectoids or more realistically, almost-perfectoids, could be an outcome of cognitive evolution in TGD Universe?

(a) p-Adic algebraic varieties are defined as zero loci of polynomials. In the octonionic $M^8$ approach identifying space-time varieties as zero loci for RE or IM of octonionic polynomial (RE and IM in quaternionic sense) this allows to define p-adic variants of space-time surfaces as varieties obeying same polynomial equations as their real counterparts provided the coefficients of octonion polynomials obtainable from real polynomials by analytic continuation are in an extension of rationals inducing also extension of p-adic numbers.

The points with coordinates in the extension of rationals common to real and p-adic variants of $M^8$ identified as cognitive representations are in key role. One can see p-adic space-time surfaces as collections of "monads" labelled by these points at which Cartesian product of 1-D p-adic balls in each coordinate degree. The radius of the p-adic ball can vary. Inside each ball the same polynomial equations are satisfied so that the monads indeed reflect other monads.

Kind of algebraic hologram would be in question consisting of the monads. The points in extension allow to define ordinary real distance between monads. Only finite number of monads would be involved since the number of points in extension tends to be finite. As the extension increases, this number increases. Cognitive representations become more complex: evolution as increase of algebraic complexity takes place.

(b) Finite-dimensionality for the allowed extensions of p-adic number fields is motivated by the idea about finiteness of cognition. Perfectoids are however infinite-dimensional. Number theoretical universality demands that on only extensions of p-adics induced by those of rationals are allowed and defined extension of the entire adele. Extensions should be therefore be induced by the same extension of rationals for all p-adic number fields.

Perfectoids correspond to an extension of $Q_p$ apparently depending on $p$. This dependence is in conflict with number theoretical universality if real. This extension could be induced by corresponding extension of rationals for all p-adic number fields. For p-adic numbers $Q_q$ $q \neq p$ all equation $a^{p^n} = x$ reduces to $a^n = x \ mod \ p$ and this in term to $a^m = x \ mod \ p$, $m = n \ mod \ p$. Finite-dimensional extension is needed to have all roots of required kind! This extension is therefore finite-D for all $q \neq p$ and infinite-D for $p$.

(c) What about infinite-dimensionality of the extension. The real world is rarely perfect and our thoughts about it even less so, and one could argue that we should be happy with almost-perfectoids! "Almost" would mean extension induced by powers of $p^{1/p^m}$ for large enough $m$, which is however not infinite. A finite-dimensional extension approaching perfectoid asymptotically is quite possible!

(d) One could see the almost perfectoid as an outcome of evolution and perfectoid as the asymptotic states. High dimension of extension means that p-adic numbers and extension of rationals have large number of common numbers so that also cognitive representations contain a large number of common points. Maybe the p-adic number fields, which are evolutionary winners, have managed to evolve to especially high-dimensional almost-perfectoids! Note however that also the roots of $e$ can be considered as extensions of rationals since corresponding p-adic extensions are finite-dimensional. Similar evolution can be considered also now.

To get some perspective mote that for large primes such as $M_{127} = 2^{127} - 1$ characterizing electron the lowest almost perfectoid would give powers of $M_{127}^{1/M_{127}} = (2^{127} - 1)^{1/(2^{127}-1)} \sim 1 + log(2)2^{-120}$! The lattice of points in extension is extremely dense near real unit. The density of of points in cognitive representations near this point would be huge. Note that the length scales comes as negative powers of two, which brings in mind p-adic length scale hypothesis [K9].

Although the octonionic formulation in terms of polynomials (or rational functions identifying space-time varieties as zeros or poles of $RE(P)$ or $IM(P)$ is attractive in its simplicity, one can also consider the possibility of allowing analytic functions of octonion coordinate obtained from real analytic functions. These define complex analytic functions with commutative imaginary unit used to complexify octonions. Could meromorphic functions real analytic at real axis having only zeros and poles be allowed? The condition that all p-adic variants of these functions exist simultaneously is non-trivial. Coefficients must be in the extension of rationals considered and convergence poses restrictions. For instance, $e^x$ converges only for $|x|_p < 1$. These functions might appear at the level of $H$.

# 7 Secret Link Uncovered Between Pure Math and Physics

I learned about a possible existence of a very interesting link between pure mathematics and physics (see `http://tinyurl.com/y86bckmo`). The article told about ideas of number theorist Minhyong Kim working at the University of Oxford. As I read the popular article, I realized it is something very familiar to me but from totally different view point.

Number theoretician encounters the problem of finding rational points of an algebraic curve defined as real or complex variant in which case the curve is 2-D surface and 1-D in complex sense. The curve is defined as root of polynomials polynomials or several of them. The polynomial have typically rational coefficients but also coefficients in extension of rationals are possible.

For instance, Fermat's theorem is about whether $x^n + y^n = 1$, $n = 1, 2, 3, ...$ has rational solutions for $n \geq 1$. For $n = 1$, and $n = 2$ it has, and these solutions can be found. It is now known that for $n > 2$ no solutions do exist. Quite generally, it is known that the number is finite rather than infinite in the generic case.

A more general problem is that of finding points in some algebraic extension of rationals. Also the coefficients of polynomials can be numbers in the extension of rationals. A less demanding problem is mere counting of rational points or points in the extension of rationals and a lot of progress has been achieved in this problem. One can also dream of classifying the surfaces by the character of the set of the points in extension.

I have consider the identification problem earlier in [L9] and I glue here a piece of text summarizing some basic results. The generic properties of sets of rational points for algebraic curves are rather well understood. Mordelli conjecture proved by Falting as a theorem (see `http://tinyurl.com/y9oq37ce`) states that a curve over $Q$ with genus $g = (d-1)(d-2)/2 > 1$ (degree $d > 3$) has only finitely many rational points.

(a) Sphere $CP_1$ in $CP_2$ has rational points as a dense set. Quite generally rational surfaces, which by definition allow parametric representation using polynomials with rational coefficients (encountered in context of Du Val singularities characterized by the extended

Dynkin diagrams for finite subgroups of $SU(2)$) allow dense set of rational points [A4, A6]).

$g = 0$ does not yet guarantee that there is dense set of rational points. It is possible to have complex conics (quadratic surface) in $CP_2$ with no rational points. Note however that this depends on the choice of the coordinates: if origin belongs to the surface, there is at least one rational point

(b) Elliptic curve $y^2 - x^3 - ax - b$ in $CP_2$ (see http://tinyurl.com/lovksny) has genus $g = 1$ and has a union of lattices of rational points and of finite cyclic groups of them since it has origin as a rational point. This lattice of points are generated by translations. Note that elliptic curve has no singularities that is self intersections or cusps (for $a = 0, b = 0$ origin is a singularity).

$g = 1$ does not guarantee that there is infinite number of rational points. Fermat's last theorem and $CP_2$ as example. $x^d + y^d = z^d$ is projectively invariant statement and therefore defines a curve with genus $g = (d-1)(d-2)/2$ in $CP_2$ (one has $g = 0, 0, 2, 3, 6, 10, ...$). For $d > 2$, in particular $d = 3$, there are no rational points.

(c) $g \geq 2$ curves do not allow a dense set of rational points nor even potentially dense set of rational points.

In my article [L9] providing TGD perspective about the role of algebraic geometry in physics, one can find basic results related to the identification problem including web links and references to literature.

## 7.1 Connection with TGD and physics of cognition

The identification problem is extremely difficult even for mathematicians - to say nothing about humble physicist like me with hopelessly limited mathematical skills. It is however just this problem which I encounter in TGD inspired vision about adelic physics [L16, L15, L9]. Recall that in TGD space-times are 4-surfaces in $H = M^4 \times CP_2$, preferred extremals of the variational principle defining the theory [K12, L24].

(a) In this approach p-adic physics for various primes $p$ provide the correlates for cognition: there are several motivations for this vision. Ordinary physics describing sensory experience and the new p-adic physics describing cognition for various primes $p$ are fused to what I called adelic physics. The adelic physics is characterized by extension of rationals inducing extensions of various p-adic number fields. The dimension $n$ of extension characterizes kind of intelligence quotient and evolutionary level since algebraic complexity is the larger, the larger the value of $n$ is. The connection with quantum physics comes from the conjecture that $n$ is essentially effective Planck constant $h_{eff}/h_0 = n$ characterizing a hierarchy of dark matters. The larger the value of $n$ the longer the scale of quantum coherence and the higher the evolutionary level, the more refined the cognition.

(b) An essential notion is that of cognitive representation [K11] [L15, L9]. It has several realizations. One of them is the representation as a set of points common to reals and extensions of various p-adic number fields induced by the extension of rationals. These space-time points have points in the extension of rationals considered defining the adele. The coordinates are the embedding space coordinates of a point of the space-time surface. The symmetries of embedding space provide highly unique embedding space coordinates.

(c) The gigantic challenge is to find these points common to real number field and extensions of various p-adic number fields appearing in the adele.

(d) If this were not enough, one must solve an even tougher problem. In TGD the notion of "world of classical worlds" (WCW) is also a central notion [K12]. It consists of space-time surfaces in embedding space $H = M^4 \times CP_2$, which are so called preferred extremals of the action principle of theory. Quantum physics would reduce to geometrization of WCW and construction of classical spinor fields in WCW and representing basically many-fermion states: only the quantum jump would be genuinely quantal in quantum theory.

There are good reasons to expect that space-time surfaces are minimal surfaces with 2-D singularities, which are string world sheets - also minimal surfaces [L24, L28]. This gives nice geometrization of gauge theories since minimal surfaces equations are geometric counterparts for massless field equations.

One must find the algebraic points, the cognitive representation, for all these preferred extremals representing points of WCW (one must have preferred coordinates for H - the symmetries of embedding space crucial for TGD and making it unique, provide the preferred coordinates)!

(e) What is so beautiful is that in given cognitive resolution defined by the extension of rationals inducing the discretization of space-time surface, the cognitive representation defines the coordinates of the space-time surfaces as a point of WCW. In finite cognitive and measurement resolution this huge infinite-dimensional space WCW discretizes and the situation can be handled using finite mathematics.

## 7.2 Connection with Kim's work

So: what is then the connection with the work and ideas of Kim. There has been a lot of progress in understanding the problem: here I an only refer to the popular article.

(a) One step of progress has been the realization that if one uses the fact that the solutions are common to both reals and various p-adic number fields helps a lot. The reason is that for rational points the rationality implies that the solution of equation representable as infinite power series of $p$ contains only finite number powers of $p$. If one manages to prove the this happens for even single prime, a rational solution has been found.

The use of reals and all p-adic numbers fields is nothing but adelic physics. Real surfaces and all its p-adic variants form pages of a book like structure with infinite number of pages. The rational points or points in extension of rationals are the cognitive representation and are points common to all pages in the back of the book.

This generalizes also to algebraic extensions of rationals. Solving the number theoretic problem is in TGD framework nothing but finding the points of the cognitive representation. The surprise for me was that this viewpoint helps in the problem rather than making it more complex.

There are however problematic situations in some cases the hypothesis about finite set of algebraic points need not make sense. A good example is Fermat for $x + y = 1$. All rational points and also algebraic points are solutions. For $x^2 + y^2 = 1$ the set of Pythagorean triangles characterizing the solutions is infinite. How to cope with these situations in which one has accidental symmetries as one might say?

(b) Kim argues that one can make even further progress by considering the situation from even wider perspective by making the problem even bigger. Introduce what the popular article (see `http://tinyurl.com/y86bckmo`) calls the space of spaces. The space of string world sheets is what string models suggests. WCW is what TGD suggests. One can get a wider perspective of the problem of finding algebraic points of a surface by considering the problem in the space of surfaces and at this level it might be possible to gain much more understanding. The notion of WCW would not mean horrible complication of a horribly complex problem but possible manner to understand the problem!

The popular article mentioned in the beginning mentions so called Selmer varieties as a possible candidate for the space of spaces. From the Wikipedia article (see `http://tinyurl.com/y27so3f2`) telling about Kim one can find a link to an article [A3] related to Selmer varieties. This article goes over my physicist's head but might give for a more mathematically oriented reader some grasp about what is involved. One can find also a list of publications of Kim (see `http://people.maths.ox.ac.uk/kimm/`.

Kim also suggests that the spaces of gauge field configurations could provide the spaces of spaces. The list contains an article [A5] with title *Arithmetic Gauge Theory: A Brief Introduction* (see http://tinyurl.com/y66mphkh) , which might help physicist to understand the ideas. An arithmetic variant of gauge theory could provide the needed space of spaces.

## 7.3 Can one make Kim's idea about the role of symmetries more concrete in TGD framework?

The crux of the Kim's idea is that somehow symmetries of space of spaces could come in rescue in the attempts to understand the rational points of surface. The notion of WCW suggest in TGD framework rather concrete realization of this idea that I have discussed from the point of view of construction of quantum states.

(a) A little bit more of zero energy ontology (ZEO) is needed to follow the argument. In ZEO causal diamonds (CDs) are central. CDs are defined as intersections of future and past directed light-cones with points replaced with $CP_2$ and forming a scale hierarchy are central. Space-time surfaces are preferred extremals with ends at the opposite boundaries of CD indeed looking like diamond. Symplectic group for the boundaries of causal diamond (CD) is the group of isometries of WCW [K12] [L24]. Maximal isometry group is required to guarantee that the WCW Kähler geometry has Riemann connection - this was discovered for loop spaces by Dan Freed [A1]. Its Lie algebra has structure of Kac- Moody algebra with respect to the light-like radial coordinate of the light-like boundary of CD, which is piece of light-cone boundary. This infinite-D group plays central role in quantum TGD: it acts as maximal group of WCW isometries and zero energy states are invariant under its action at opposite boundaries.

(b) As one replaces space-time surface with a cognitive representation associated with an extension of rationals, WCW isometries are replaced with their infinite discrete subgroup acting in the number field define by the extension of rationals defining the adele. These discrete isometries do not leave the cognitive representation invariant but replace with it new one having the same number of points and one obtains entire orbit of cognitive representations. This is what the emergence of symmetries in wider conceptual framework would mean.

(c) One can in fact construct invariants of the symplectic group. Symplectic transformations leave invariant the Kähler magnetic fluxes associated with geodesic polygons with edges identified as geodesic lines of $H$. There are also higher-D symplectic invariants. The simplest polygons are geodesic triangles. The symplectic fluxes associated with the geodesic triangles define symplectic invariants characterizing the cognitive representation. For the twistor lift one must allow also $M^4$ to have analog of Kähler form and it would be responsible for CP violation and matter antimatter asymmetry [L6]. Also this defines symplectic invariants so that one obtains them for both $M^4$ and $CP_2$ projections and can characterize the cognitive representations in terms of these invariants. Note that the existence of twistor lift fixes the choice of $H$ uniquely since $M^4$ and $CP_2$ are the only 4-D spaces allowing twistor space with Kähler structure [A2] necessary for defining the twistor lift of Kähler action.

    More complex cognitive representations in an extension containing the given extension are obtained by adding points with coordinates in the larger extension and this gives rise to new geodesic triangles and new invariants. A natural restriction could be that the polynomial defining the extension characterizing the preferred extremal via $M^8-H$ duality defines the maximal extension involved.

(d) Also in this framework one can have accidental symmetries. For instance, $M^4$ with $CP_2$ coordinates taken to be constant is a minimal surface, and all rational and algebraic points for given extension belong to the cognitive representation so that they are infinite. Could this has something to do with the fact that we understand $M^4$ so well and have even identified space-time with Minkowski space! Linear structure would be cognitively easy for the same reason and this could explain why we must linearize.

$CP_2$ type extremals with light-like $M^4$ geodesic as $M^4$ projection is second example of accidental symmetries. The number of rational or algebraic points with rational $M^4$ coordinates at light-like curve is infinite - the situation is very similar to $x + y = 1$ for Fermat. Simplest cosmic strings are geodesic sub-manifolds, that is products of plane $M^2 \subset M^4$ and $CP_2$ geodesic sphere. Also they have exceptional symmetries.

What is interesting from the point of view of proposed model of cognition is that these cognitively easy objects play a central role in TGD: their deformations represent more complex dynamical situations. For instance, replacing planar string with string world sheet replaces cognitive representation with a discrete or perhaps even finite one in $M^4$ degrees of freedom.

(e) A further TGD based simplification would be $M^8 - H$ ($H = M^4 \times CP_2$) duality in which space-time surfaces at the level of $M^8$ are algebraic surfaces, which are mapped to surfaces in H identified as preferred extremals of action principle by the $M^8 - H$ duality [L9]. Algebraic surfaces satisfying algebraic equations are very simple as compared to preferred extremals satisfying partial differential equations but "preferred" is what makes possible the duality. This huge simplification of the solution space of field equations guarantees holography necessitated by general coordinate invariance implying that space-time surfaces are analogous to Bohr orbits. It would also guarantee the huge symmetries of WCW making it possible to have Kähler geometry.

This suggests in TGD framework that one finds the cognitive representation at the level of $M^8$ using methods of algebraic geometry and maps the points to H by using the $M^8 - H$ duality. TGD and octonionic variant of algebraic geometry would meet each other.

It must be made clear that now solutions are not points but 4-D surfaces and this probably means also that points in extension of rationals are replaced with surfaces with embedding space coordinates defining function in extensions of rational functions rather than rationals. This would bring in algebraic functions. This might provide also a simplification by providing a more general perspective. Also octonionic analyticity is extremely powerful constraint that might help.

# 8 Cognitive representations for partonic 2-surfaces, string world sheets, and string like objects

Cognitive representations are identified as points of space-time surface $X^4 \subset M^4 \times CP_2$ having embedding space coordinates in the extension of of rationals defined by the polynomial defined by the $M^8$ pre-image of $X^4$ under $M^8 - H$ correspondence [L10, L11, L36, L29, L27, L25]. Cognitive representations have become key piece in the formulation of scattering amplitudes [L31] . One might argue that number theoretic evolution as increase of the dimension of the extension of rationals favors space-time surfaces with especially large cognitive representations since the larger the number of points in the representation is, the more faithful the representation is.

One can pose several questions if one accepts the idea that space-time surfaces with large cognitive representations are survivors.

(a) Preferred p-adic primes are proposed to correspond to the ramified primes of the extension [L38]. The proposal is that the p-adic counterparts of space-time surfaces are identifiable as imaginations whereas real space-time surfaces correspond to realities. p-Adic space-time surfaces would have the embedding space points in extension of rationals as common with real surfaces and large number of these points would make the representation realistic. Note that the number of points in extension does not depend on p-adic prime.

Could some extensions have an especially high number of points in the cognitive representation so that the corresponding ramified primes could be seen as survivors in number theoretical fight for survival, so to say? Galois group of the extension acts on cognitive representation. Galois extension of an extension has the Galois group of the original extension as normal subgroup so that ormal Galois group is analogous to a conserved gene.

(b) Also the type of extremal matters. For instance, for instance canonically imbedded $M^4$ and $CP_2$ contain all points of extension. These surfaces correspond to the vanishing of real or imaginary part (in quaternionic sense) for a linear octonionic polynomial $P(o) = o$! As a matter of fact, this is true for all known preferred extremals under rather mild additional conditions. Boundary conditions posed at both ends of CD in ZEO exclude these surfaces and the actual space-time surfaces are expected to be their deformations.

(c) Could the surfaces for which the number of points in cognitive representation is high, be the ones most easily discovered by mathematical mind? The experience with TGD supports positive answer: in TGD the known extremals [K1] are examples of such mathematical objects! If so, one should try to identify mathematical objects with high symmetries and look whether they allow TGD realization.

(d) One must also specify more precisely what cognitive representation means. Strong form of holography (SH) states that the information gives at 2-D surfaces - string world sheets and partonic 2-surfaces - is enough to determine the space-time surfaces. This suggests that it is enough to consider cognitive representation restricted to these 2-surfaces. What kind of 2-surfaces are the cognitively fittest one? It would not be surprising if surfaces with large symmetries acting in extension were favored and elliptic curves with discrete 2-D translation group indeed turn out to be assigable string world sheets as singularities and string like objects. In the case of partonic 2-surfaces geodesic sphere of $CP_2$ is similar object.

All known extremals, in particular preferred extremals, are good candidates in this respect because of their high symmetries. By strong form of holography (SH) partonic 2-surfaces and string world sheets are expected to give rise to cognitive representations. Also cosmic strings are expected to carry them. Under what conditions these representations are large?

## 8.1 Partonic 2-surfaces as seats of cognitive representations

One can start from SH and look the situation more concretely. The situation for partonic 2-surfaces has been considered already earlier [L37, L26] but deserves a separate discussion.

(a) Octonionic polynomials allow special solutions for which the entire polynomial vanishes. This happens at 6-sphere $S^6$ at the boundary of 8-D light-cone. $S^6$ is analogous to brane and has radius $R = r_n$, which is a root of the real polynomial with rational coefficients algebraically continued to the octonionic polynomial.

$S^6$ has the ball $B^3$ of radius $r_n$ of the light-cone $M^4_+$ with time coordinate $t = r_n$ as analog of base space and sphere $S^3$ of $E^4$ with radius $R = \sqrt{r_n^2 - r^2}$, $r$ the radial coordinate of $B^3$ as an analog of fiber. The analog of the fiber contracts to a point at the boundary of the light-cone. The points with $B^3$ projection and $E^4$ coordinates in extension of rationals belong to the cognitive representation. The condition that $R^2 = x_i x^i = r_n^2 - r^2$ is square of a number of extension is rather mild and allows infinite number of solutions.

(b) The 4-D space-time surfaces $X^4$ are obtained as generic solutions of $Im(P(o)) = 0$ or $Re(P(o)) = 0$. Their intersection with $S^6$ - partonic 2-surface $X^2$ - is 2-D. The assumption is that the incoming and outgoing 4-D space-time surfaces representing orbits of particles in topological sense are glued together at $X^2$ and possibly also in their interiors. $X^2$ serves as an analog of vertex for 3-D particles. This gives rise to topological analogs of Feynman diagrams.

In the generic case the number of points in cognitive representation restricted to $X^2$ is finite unless the partonic 2-surface $X^2$ is special - say correspond to a geodesic spere of $S^6$.

(c) The discrete isometries and conformal symmetries of the cognitive representation restricted to $X^2$ possibly represented as elements of Galois group might play a role. For $X^2 = S^2$ the finite discrete subgroups of $SO(3)$ giving rise to finite tessellations and appearing in ADE correspondence might be relevant. For genera $g = 01, 2$ conformal symmetry $Z_2$ is always possible but for higher genera only in the case of hyper-elliptic surfaces- this used to explain why only $g = 0, 1, 2$ correspond to observed particles [K3] whereas higher genera could be regarded as many-particle states of handles having continuous mass spectrum. Torus is an exceptional case and one can ask whether discrete subgroup of its isometries could be realized.

(d) In TGD inspired theory of consciousness [L17, L26] the moments $t = r_n$ corresponds to "very special moments in the life of self". They would be also cognitively very special - kind of eureka moments with a very large number of points in cognitive representation. The question is whether these surfaces might be relevant for understanding the nature of mathematical consciousness and how the mathematical notions emerge at space-time level.

## 8.2 Ellipticity

Surfaces with discrete translational symmetries is a natural candidate for a surface with very large cognitive representation. Are their analogs possible? The notions of elliptic function, curve, and surface suggest themselves as a starting point.

(a) Elliptic functions (`http://tinyurl.com/gpugcnh`) have 2-D discrete group of translations as symmetries and are therefore doubly periodic and thus identifiable as functions on torus.

Weierstrass elliptic functions $\mathcal{P}(z; \omega_1, \omega_2)$ (`http://tinyurl.com/ycu8oa4r`) are defined on torus and labelled by the conformal equivalence class $\lambda = \omega_1/\omega_2$ of torus identified as the ratio $\lambda = \omega_1/\omega_2$ of the complex numbers $\omega_i$ defining the periodicities of the lattice involved. Functions $\mathcal{P}(z; \omega_1, \omega_2)$ are of special interest as far as elliptic curves are considered and defines an embedding of elliptic curve to $CP_2$ as will be found.

If the periods are in extension of rationals then values in the extension appear infinitely many times. Elliptic functions are not polynomials. Although the polynomials giving rise to octonionic polynomials could be replaced by analytic functions it seems that elliptic functions are not the case of primary interest. Note however that the roots $r_n$ could be also complex and could correspond to values of elliptic function forming a lattice.

(b) Elliptic curves (`http://tinyurl.com/lovksny`) are defined by the polynomial equation

$$y^2 = P(x) = x^3 + ax + b \ . \tag{8.1}$$

An algebraic curve of genus 1 allowing 2-D discrete translations as symmetries is in question. If a point of elliptic curve has coordinates in extension of rationals then 2-D discrete translation acting in extension give rise to infinite number of points in the cognitive representation. Clearly, the 2-D vectors spanning the lattice defining the group must be in extension of rationals.

One can indeed define commutative sum $P + Q$ for the points of the elliptic curve. The detailed definition of the group law and its geometric illustration can be found in Wikipedia article (`http://tinyurl.com/lovksny`).

(a) Consider real case for simplicity so that elliptic curve is planar curve. $y^2 = P(x) = x^3 + ax + b$ must be non-negative to guarantee that $y$ is real. $P(x) \geq 0$ defines a curve in upper $(x, y)$ plane extending from some negative value $x_{min}$ corresponding to $y^2 = P(x_{min}) = 0$ to the right. Given value of $y$ can correspond to 3 real roots or 1 real root of $P_y(x) = y^2 - P(x)$. At the two extrema of $P_y(x)$ 2 real roots co-incide.

The graph of $y = \pm\sqrt{P(x)}$ is reflection symmetric having two branches beginning from $(x_{min}, y = 0)$.

(b) The negative $-P$ is obtained by reflection with respect to x-axis taking $y_P$ to $-y_P$. Neutral element $O$ is identified as point a infinity (assuming compactification of the plane to a sphere) which goes to itself under reflection $y \rightarrow -y$.

(c) One assigns to the points $P$ and $Q$ of the elliptic curve a line $y = sx + d$ containing them so that one has $s = (y_p - y_Q)/(x_P - x_Q)$. In the generic case the line intersects the elliptic curve also at third point $R$ since $P_{y=sx+d}(x)$ is third order polynomial having three roots $(x_P, x_Q, x_R)$. It can happen that 2 roots are complex and one has 1 real root. At criticalityfor the transiton from 3 to 1 real roots one has $x_Q = x_R$.

Geometrically one can distinguish between 4 cases.

- The roots $P, Q, R$ of $P_{y=sx+d}(x)$ are different and finite: one defines the sum as $P + Q = -R$.
- $P \neq Q$ and $Q = R$ (roots $Q$ and $R$ are degenerate): $P + Q + Q = O$ giving $R = -P/2$.
- $P$ and $Q$ are at a line parallel to y-axis and one has $R = O$: $P + Q + O = O$ and $P = -Q$.
- $P$ is double root of $P_{y=sx+d}(x)$ with tangent parallel to $y$-axis at the point $(x_{min}, y = 0)$ at which the elliptic curve begins so that one has $R = O$: $P + P + O = O$ gives $P = -P$. This corresponds to torsion.

(d) Elliptic surfaces (see `http://tinyurl.com/yc33a6dg`) define a generalization of elliptic curves and are defined for 4-D complex manifolds. Fiber is required to be smooth and has genus 1.

## 8.3 String world sheets and elliptic curves

In twistor lift of TGD space-time surfaces identifiable as minimal surfaces with singularities, which are string world sheets and partonic 2-surfaces. Preferred extremal property means that space-time surfaces are extremals of both Kähler action and volume action except at singularities.

Are string world sheets with very large number of points in cognitive representation possible? One has right to expect that string world sheets allow special kind of symmetries allowing large, even infinite number of points at the limit of large sheet and related by symmetries acting in the extension of rationals. If one of the points is in the extension, also other symmetry related points are in the extension. For a non-compact group, say translation one would have infinite number of points in the representation but the finite size of CD would pose a limitation to the number of points.

String world sheets are good candidates for the realization of elliptic curves.

(a) The general conjecture is that preferred extremals allow what I call Hamilton-Jacobi structure for $M^4$ [K12]. The distribution of tangent spaces having decomposition $M^4(x) = M^2(x) \times E^2(x)$ would be integrable giving rise to a family of string world sheets $Y^2$ and partonic 2-surfaces $X^2$ more general than those defined above. $X^2$ and $Y^2$ are orthogonal to each other at each point of $X^4$. One can introduce local light-cone coordinates $(u, v)$ for $Y^2$ and local $E^2$ complex coordinate $w$ for $X^2$.

(b) Space-time surface itself would be a deformation of $M^4$ with Hamilton-Jacobi structure in $CP_2$ direction. $w$ coordinate as function $w(z)$ of $CP_2$ complex coordinate $z$ or vice versa would define the string world sheet. This would be a transversal deformation of the basic string world sheet $Y^2$: stringy dynamics is indeed transversal.

(c) The idea about maximal cognitive representation suggests that $w \leftrightarrow z$ correspondence defines elliptic curve. One would have $y^2 = P(x) = x^3 + ax + b$ with either $(y = w, x = z)$ or $(y = z, x = w)$. A natural conjecture is that for the space-time surface corresponding to a given extension $K$ of rationals the coefficients $a$ an $b$ belong to $K$ so that the algebraic complexity of string world sheet would increase in number theoretic evolution [L35]. The orbit of a algebraic point at string world sheet would be lattice made finite by the size of CD. Elliptic curves would define very special deformed string world sheets in space-time.

(d) It is interesting to consider the pre-image of given point $y$ ($y = w$ or $y = z$) covering point $x$. One has $y = \pm\sqrt{u}$, $u = P(x)$ corresponding to group element and its negative: there are two points of covering given value of $u$. $u = P(x)$ covers 3 values of $x$. The values of $x$ would belong to 6-fold covering of rationals. The number theoretic interpretation for the effective Planck constant $h_{eff} = nh_0$ states that $n$ is the number of sheets for space-time surface as covering.

There is evidence that $h_{eff} = h$ corresponds to $n = 6$ [L3]. Could 6-fold covering of rationals be fundamental since it gives very large cognitive representation at the level of string world sheets?

For extensions $K$ of rationals the $x$ coordinates for the points of cognitive representation would belong to 6-D extension of $K$.

(e) Ellipticity condition would apply on the string world sheets themselves. In the number theoretic vision string world sheets would correspond at $M^8$ level to singularities at which the quaternionic tangent space degenerates to 2-D complex space. Are these conditions consistent with each other? It would seem that the two conditions would select cognitively very special string world sheets and partonic 2-surfaces defining by strong form of holography (SH) space-time surface as a hologram in SH. Consciousness theorist interested in mathematical cognition might ask whether the notion of elliptic surfaces have been discovered just because it is cognitively very special. In the case of partonic 2-surfaces geodesic sphere of $CP_2$ is similar object.

## 8.4 String like objects and elliptic curves

String like objects - cosmic strings - and their deformations, are fundamental entities in TGD based cosmology and astrophysics and also in TGD inspired quantum biology. One can assign elliptic curves also to string like objects.

(a) Quite generally, the products $X^2 \times Y^2 \subset M^4$ of string world sheets $X^2$ and complex surfaces $Y^2$ of $CP_2$ define extremals that I have called cosmic strings [K1].

(b) Elliptic curves allow a standard embedding to $CP_2$ as complex surfaces constructible in terms of Weierstrass elliptic function $\mathcal{P}(z)$ (`http://tinyurl.com/ycu8oa4r`) satisfying the identity

$$[\mathcal{P}'(z)]^2 = [\mathcal{P}(z)]^3 - g_2\mathcal{P}(z) - g_3 \quad . \tag{8.2}$$

Here $g_2$ and $g_3$ are modular invariants. This identity is of the same form as the condition $y^2 = x^3 + ax + b$ with identifications $y = \mathcal{P}'(z), x = \mathcal{P}(z)$ and $(a = -g_2, b = -g_3)$. From the expression

$$y^2 = x(x-1)(x-\lambda) \tag{8.3}$$

in terms of the modular invariant $\lambda = \omega_1/\omega_2$ of torus one obtains

$$g_2 = \frac{4^{1/3}}{3}(\lambda^2 - \lambda + 1 \quad , \quad g_3 = \frac{1}{27}(\lambda+1)(2\lambda^2 - 5\lambda + 2) \quad . \tag{8.4}$$

Note that third root of $a$ appears in the formula. The so called modular discriminant

$$\Delta = g_2^3 - 27g_3^2 = \lambda^2(\lambda-1)^2 \quad . \tag{8.5}$$

vanishes for $\lambda = 0$ and $\lambda = 1$ for which the lattice degenerates.

(c) The embedding of the elliptic curve to $CP_2$ can be expressed in projective coordinates of $CP_2$ as

$$(z^1, z^2, z^3) = (\xi^1, \xi^2, 1) = (\frac{\mathcal{P}'(w)}{2}, \mathcal{P}(w), 1) \ . \tag{8.6}$$

# 9 Are fundamental entities discrete or continuous and what discretization at fundamental level could mean?

There was an interesting FB discussion about discrete and continuum. I decided to write down my thoughts and emphasize those points that I see as important.

## 9.1 Is discretization fundamental or not?

The conversation inspired the question whether discreteness is something fundamental or not. If it is assumed to be fundamental, one encounters problems. The discrete structures are not unique. One has deep problem with the known space-time symmetries. Symmetries are reduced to discrete subgroup or totally lost. A further problem is the fact that in order to do physics, one must bring in topology and length measurements.

In discrete situation topology, in particular space-time dimension, must be put in via homology effectively already meaning use of embedding to Euclidian space. Length measurement remains completely ad hoc. The construction of discrete metric is highly non-unique procedure and the discrete analog of of say Einstein's theory (Regge calculus) is rather clumsy. One feeds in information, which was not there by using hand weaving arguments like infrared limit. It is possible to approximate continuum by discretization but discrete to continuum won't go.

In hype physics these hand weaving arguments are general. For instance, the emergence of 3-space from discrete Hilbert space is one attempt to get continuum. One puts in what is factually a discretization of 3-space and then gets 3-space back at IR limit and shouts "Eureka!".

## 9.2 Can one make discretizations unique?

Then discussion went to numerics. Numerics is for mathematicians same as eating for poets. One cannot avoid it but luckily you can find people doing the necessary programming if you are a professor. Finite discretization is necessary in numerics and is highly unique.

I do not have anything personal against discretization as a numerical tool. Just the opposite, I see finite discretization as absolutely essential element of adelic physics as an attempt to describe also the correlates of cognition in terms of p-adic physics with p-adic space-time sheets as correlates of "thought bubbles" [L15, L16]. Cognition is discrete and finite and uses rational numbers: this is the basic clue.

(a) Cognitive representations are discretizations of (for instance) space-time surface. One can say that physics itself builds its cognitive representation in all scales using p-adic space-time sheets. They should be unique once measurement resolution is characterized if one is really talking about fundamental physics.

The idea abou tp-adic physics as physics of cognition indeed led to powerful calculational recipes. In p-adic thermodynamics the predictions come in power series of p-adic prime p and for the values of p assignable to elementary particles the two lowest terms give practically exact result [K8]. Corrections are of order $10^{-76}$ for electron characterized by Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38}$.

(b) Adelic physics [L15] provides the formulation of p-adic physics: it is assumed that cognition is universal. Adele is a book like structure having as pages reals and extensions of various p-adic number fields induced by given extension of rationals. Each extension of rationals defines its own extension of the rational adele by inducing extensions of p-adic number fields. Common points between pages consist of points in extension of rationals. The books associated with the adeles give rise to an infinite library.

At space-time level the points with coordinates in extension define what I call cognitive representation. In the generic case it is discrete and has finite number of points. The loss of general coordinate invariance is the obvious objection. In TGD however the symmetries of the embedding space fix the coordinates used highly uniquely. $M^8 - H$ duality ($H = M^4 \times CP_2$) and octonionic interpretation implies that $M^8$ octonionic linear coordinates are highly unique [L9, L29]. Note that $M^8$ must be complexified. Different coordinatizations correspond to different octonionic structures- to different moduli - related by Poincare transformations of $M^8$. Only rational time translations as transformations of octonionic real coordinate are allowed as coordinate changes respecting octonionic structure.

(c) Discretization by cognitive representation is unique for given extension of rationals defining the measurement resolution. At the limit of algebraic numbers algebraic points form a dense set of real space-time surface and p-adic space-time surfaces so that the measurement resolution is ideal. One avoids the usual infinities of quantum field theories induced by continuous delta functions, which for cognitive representations are replaced with Kronecker deltas. This seems to be the best that one can achieve with algebraic extensions of rationals. Also for transcendental extensions the situation is discrete.

This leads to a number theoretic vision about second quantization of induced spinor fields central for the construction of gamma matrices defining the spinor structure of "world of classical worlds" (WCW) providing the arena of quantum dynamics in TGD analogous to the super-space of Wheeler [K12]. One ends up to a construction allowing to understand TGD view about SUSY as necessary aspect of second quantization of fermions and leads to the conclusions that in the simplest scenario only quarks are elementary fermions and leptons can be seen as their local composites analogous to super partners.

(d) Given polynomial defining space-time surfaces in $M^8$ defines via its roots extension of rationals. The hierarchy of extensions defines an evolutionary hierarchy. The dimension n of extension defines kind of IQ measuring algebraic complexity and n corresponds also to effective Planck constant labelling phases of dark matter in TGD sense so that a direct connection with physics emerges.

Embedding space assigns to a discretization a natural metric. Distances between points of metric are geodesic distances computed at the level of embedding space.

(e) An unexpected finding was that the equations defining space-time surfaces as roots of real or imaginary parts of octonionic polynomials have also 6-D brane like entities with topology of $S^6$ as solutions [L26, L36]. These entities intersect space-time surfaces at 3-D sections for which linear $M^4$ time is constant. 4-D roots can be glued together along these branes. These solutions turn out to have an interpretation in TGD based theory of quantum measurement extending to a theory of consciousness. The interpretation as moments of "small" state function reductions as counterparts of so called weak measurements. They could correspond to special moments in the life of conscious entity.

## 9.3 Can discretization be performed without lattices?

For a systems obeying dynamics defined by partial differential equations, the introduction of lattices seems to be necessary aspect of discretization. The problem is that the replacement of derivatives with discrete approximations however means that there is no hope about exact results. In the general case the discretization for partial differential equations involving derivatives forces to introduce lattice like structures. This is not needed in TGD.

(a) At the level of $M^8$ ordinary polynomials give rise to octonionic polynomials and space-time surfaces are algebraic surfaces for which imaginary or real part of octonionic polynomial in quaternionic sense vanishes. The equations are purely algebraic involving no partial derivatives and there is no need for lattice discretization.

For surfaces defined by polynomials the roots of polynomial are enough to fix the polynomials and therefore also the space-time surface uniquely: discretization is not an approximation but gives an exact result! This could be called number theoretical holography and generalizes the ordinary holography. Space-time surfaces are coded by the roots of polynomials with rational coefficients.

(b) What about the field equations at the level of $H = M^4 \times CP_2$? $M^8 - H$ duality maps these surfaces to preferred extremals as 4-surfaces in $H$ analogous to Bohr orbits. Twistor lift of TGD predicts that they should be minimal surfaces with 2-D singularities being also extremals of 4-D Kähler action. The field equations would reduce locally to purely algebraic conditions. In properly chosen coordinates for $H$ they are expected to be determined in terms of polynomials coding for the same extension of rationals as their $M^8$ counterparts so that the degree should be same [L29]. This would allow to deduce the partial derivatives of embedding space for the image surfaces without lattice approximation.

(c) The simplest assumption is that the polynomials have rational coefficients. Number theoretic universality allows to consider also algebraic coefficients. In both cases also WCW is discretized and given point -space-time surface in QCD has coordinates given by the points of the number theoretically universal cognitive representation of the space-time surface. Even real coefficients are possible. This would allow to obtain WCW as a continuum central for the construction of WCW metric but is not consistent with number theoretical universality.

Can one have polynomial/functions with rational coefficients and discretization of WCW without lattice but without losing WCW metric? Maybe the same trick that works at space-time level works also in WCW!

i. The group WCW isometries is identified as symplectic transformations of $\delta M^4_\pm \times CP_2$ ($\delta M^4_\pm$ denotes light-cone boundary) containing the boundary of causal diamond CD. The Lie algebra $Sympl$ of this group is analogous half-Kac Moody algebra having symplectic transformations of $S^2 \times CP_2$ as counterpart of finite-D Lie group has fractal structure containing infinite number of sub-algebras $Sympl_n$ isomorphic to algebra itself: the conformal weights assignable to radial light-like coordinate are $n$-multiples of those for the entire algebra. Note that conformal weights of $Sympl$ are non-negative.

ii. One formulation for the preferred extremal property is in terms of infinite number of analogs of gauge conditions stating the vanishing of classical and also Noether charges for $Sympl_n$ and $[Sympl_n, Sympl]$. The conditions generalize to the super-counterpart of $Sympl$ and apply also to quantum states rather than only space-time surfaces. In fact, while writing this I realized that - contrary to the original claim - also the vanishing of the Noether charges of higher commutators is required so that effectively $Sympl_n$ would define normal subgroup of $Sympl$. These conditions does not follow automatically.

The Hamiltonians of $Sympl(S^2 \times CP_2)$ are also labelled by the representations of the product of the rotation group $SO(3) \subset SO(3,1)$ of $S^2$ and color group $SU(3)$ together forming the analog of the Lie group defining Kac-Moody group. This group does not have have the fractal hierarchy of subgroups. The strongest condition is that the algebra corresponding to Hamiltonian isometries does not annihilate the physical states.

The space of states satisfying the gauge conditions is finite-D and that WCW becomes effectively finite-dimensional. A coset space associated with $Sympl$ would be in question and it would have maximal symmetries as also WCW. The geometry of the reduced WCW, WCW$_{red}$ could be deduced from symmetry considerations alone.

iii. Number theoretic discretization would correspond to a selection of points of this subspace with the coordinates in the extension of rationals.The metric of WCW$_{red,n}$ at the points of discretization would be known and no lattice discretization would be needed. The gauge conditions are analogous to massless Dirac equation in WCW and could be solved in the points of discretization without introducing the lattice to approximate derivatives. As a matter fact, Dirac equation can be formulated solely in terms of the generators of *Sympl*.

iv. This effectively restricts WCW to $WCW_{red,n}$ in turn reduced to its discrete subset - since infinite number of WCW coordinates are fixed. If this sub-space can be regarded as realization of infinite number of algebraic conditions by polynomials with rational coefficients one can assign to it extension of rationals defining naturally the discretization of WCW$_{red,n}$. This extension is naturally the same as for space-time surfaces involved so that the degree of polynomials defining WCW$_{red,n}$ would be naturally $n$ and same as that for the polynomial defining the space-time surface. WCW$_{red,n}$ would decompose to union of spaces WCW$_{red,E_n}$ labelled by extensions $E_n$ of rationals with same dimension $n$.
There is analogy with gauge fixing. WCW$_{red,E_n}$ is a coset space of WCW defined by the gauge conditions. One can represent this coset space as a sub-manifold of WCW by taking one representative point from each coset. This choice is not unique but one can hope finding a gauge choice realized by an infinite number of polynomials of degree $n$ defining same extension of rationals as the polynomial defining the space-time surfaces in question.

v. WCW spinor fields would be always restricted to finite-D algebraic surface of WCW$_{red,E_n}$ expressible in terms of algebraic equations. Finite measurement resolution indeed strongly suggests that WCW spinor field mode is non-vanishing only in a region parameterized in WCW by finite number of parameters. There is also a second manner to see this. WCW$_{red,E_n}$ could be also seen as $n+4$-dimensional surface in $WCW$.

vi. One can make this more concrete. Cognitive representation by points of space-time surface with coordinates in the extension - possibly satisfying additional conditions such as belonging to the 2-D vertices at which space-time surfaces representing different roots meet - provides WCW coordinates of given space-time surface. Minimum number of points corresponds to the dimension of extension so that the selection of coordinate can be redundant. As the values of these coordinates vary, one obtains coordinatization for the sector of WCW$_{red,E_n}$. An interesting question is whether one could represent the distances of space-time surfaces in this space in terms of the data provided by the points of discretization.
An interesting question is whether one can represent the distances of space-time surfaces in this space in terms of the data provided by the points of cognitive representation. One can define distance between two disjoint surfaces as the minimum of distance between the points of 2-surfaces. Could something like this work now? The points would be restricted to the cognitive representations. Could one define the distance between two cognitive representations with same number N of points in the following manner.
Consider all bipartitions formed by the cognitive representations obtained by connecting their points together in 1-1 manner. There are N! bipartitions of this kind if the number of points is N. Calculate the sum of the squares of the embedding space distances between paired points. Find the bipartition for which this distance squared is minimum and define the distance between cognitive representations as this distance. This definition works also when the numbers of points are different.

vii. If there quantum states are the basic objects and there is nothing "physical" behind them one can ask how we can imagine mathematical structures which different from basic structure of TGD. Could quantum states of TGD Universe in some sense represent all mathematical objects which are internally consistent. One could indeed say that at the level of WCW all $n+4$-D manifolds can be represented concretely in terms of WCW spinor fields localized to $n$-D subspaces of WCW. WCW spinor fields can represent concept of 4-surface of $\mathrm{WCW}_{red,n}$ as a quantum superposition of its instance and define at the same time $n+4$-D surfaces [L39] [L28, L32, L31, L39].

## 9.4 Simple extensions of rationals as codons of space-time genetic code

A fascinating idea is that extensions of rationals define the analog of genetic code for space-time surfaces, which would therefore represent number theory and also finite groups.

i. The extensions of rationals define an infinite hierarchy: the proposal is that the dimension of extensions corresponds to the integer $n$ characterizing subalgebra $Sympl_n$. This would give direct correspondence between the inclusions of HFFs assigned to the hierarchy of algebras $Sympl_n$ and hierarchy of extensions of rationals with dimension $n$.
Galois group for a extension of extension contains Galois group of extension as normal subgroup and is therefore *not simple*. Extension hierarchies correspond to inclusion hierarchies for normal subgroups. Simple Galois groups are in very special position and associated with what one might call simple extensions serving as fundamental building bricks of inclusion hierarchies. They would be like elementary particles and define fundamental space-time regions. Their Galois groups would act as groups of physical symmetries.

ii. One can therefore talk about elementary space-time surfaces in $M^8$ and their compositions by function composition of octonionic polynomials. Simple groups would label elementary space-time regions. They have been classified: (see `http://tinyurl.com/y3xh4hrh`). The famous Monster groups are well-known examples about simple finite groups and would have also space-time counterparts. Also the finite subgroups of Lie groups are special and those of $SU(2)$ are associated with Platonic solids and seem to play key role in TGD inspired quantum biology. In particular, vertebrate genetic code can be assigned to icosahedral group.

iii. There is also an analogy with genes. Extensions with simple Galois groups could be seen as codons and sequences of extension obtained by functional composition as analogs of genes. I have even conjectured that the space-time surfaces associated with genes could quite concretely correspond to extensions of extensions of ...

## 9.5 Are octonionic polynomials enough or are also analytic functions needed?

I already touched the question whether also analytic functions with rational coefficients (number theoretical universality) might be needed.

i. The roots of analytic functions generate extension of rationals. If the roots involve transcendental numbers they define infinite extensions of rationals. Neper number $e$ is very special in this sense since $e^p$ is ordinary p-adic number for all primes $p$ so that the induced extension is finite-dimensional. One could thus allow it without losing number theoretical universality. The addition of $\pi$ gives infinite-D extension but one could do by adding only roots of unity to achieve finite-D extensions with finite accuracy of phase measurement. Phases would be number theoretically universal but not angles.

ii. One could of course consider only transcendental functions with rational roots. Trigonometric function $sin(x/2\pi)$ serves as a simple example. One can also argue that since physics involves in an essential manner trigonometric functions via Fourier analysis, the inclusion of analytic functions with algebraic roots must be allowed.

iii. What about analytic functions as limits of polynomials with rational coefficients such that the number of roots becomes infinite at the limit? Also their imaginary and real part can vanish in quaternionic sense and could define space-time surfaces

- analogs of transcendentals as space-time surfaces. It is not clear whether these could be allowed or not.

Could one have a universal polynomial like function giving algebraic numbers as the extension of rationals defined by its algebraic roots? Could Riemann zeta (see http://tinyurl.com/nfbkrsx) code algebraic numbers as an extension via its roots. I have conjectured that roots of Riemann zeta are algebraic numbers: could they span all algebraic numbers?

It is known that the real or imaginary part of Riemann zeta along $s = 1/2$ critical line can approximate any function to arbitrary accuracy: also this would fit with universality. Could one think that the space-time surface defined as root of octonionic continuation of zeta could be universal entity analogous to a fixed point of iteration in the construction of fractals? This does not look plausible.

(d) One can construct iterates of Riemann zeta having at least the same roots as zeta by the rule

$$\begin{aligned} &f_0(s) = \zeta(s) \ , \\ &f_n(s) = \zeta(f_{n-1}(s)) - \zeta(0), \ \ \zeta(0) = -1/2 \ . \end{aligned} \tag{9.1}$$

$\zeta$ is not a fixed point of this iteration as the fractal universality would suggest. The set of roots however is. Should one be happy with this.

(e) Riemann zeta has also counterpart in all extensions of rationals known as Dedekind zeta (see http://tinyurl.com/y5grktv) [L13, L38, L30]. Riemann zeta is therefore not unique. One can ask whether Dedekind zetas associated with simple Galois groups are special and whether Dedekind zetas associated with extensions of extensions of .... can be constructed by using the Dedekind zetas of simple extensions. How do the roots of Dedekind zeta depend on the associated extension of rationals? How the roots of Dedekind zeta for extension of extension defined by composite of two polynomials depend on extensions involved? Are the roots union for the roots associated with the composites?

(f) What about forming composites of Dedekind zetas? Categorical according to my primitive understanding raises the question whether a composition of extensions could correspond to a composition of functions. Could Dedekind zeta for a composite of extensions be obtained from a composite of Dedekind zetas for extensions? Requiring that roots of extension $E_1$ are preserved would give formula

$$\zeta_{D,E_1E_2} = \zeta_{D,E_1} \circ \zeta_{D,E_2} - \zeta_{D,E_1}(0) \ . \tag{9.2}$$

The zeta function would be obtained by an iteration of simple zeta functions labelled by simple extensions. The inverse image for the set of roots of $\zeta_{D,E_1}$ under $\zeta_{D,E_2}$ that is the set $\zeta^{-1}_{D,E_2}(roots(\zeta_{D,E_1})$ would define also roots of $\zeta_{D,E_1E_2}$. This looks rather sensible.

But what about iteration of Riemann zeta, which corresponds to trivial extension? Riemann $\zeta$ is not invariant under iteration although its roots are. Should one accept this and say that it is the set of roots which defines the invariant. Could one say that the iterates of various Dedekind zetas define entities which are somehow universal.

# REFERENCES

## Mathematics

[A1] Freed DS. The Geometry of Loop Groups, 1985.

[A2] N. Hitchin. Kählerian twistor spaces. *Proc London Math Soc*, 8(43):133–151, 1981.. Available at: https://tinyurl.com/pb8zpqo.

[A3] Coates J and Kim M. Selmer varieties for curves with CM Jacobians. *Kioto J Math.*, 50(4):827–852, 2010.Available at: https://people.maths.ox.ac.uk/kimm/papers/cm3.pdf.

[A4] McKay J. Cartan matrices, finite groups of quaternions, and kleinian singularities. *Proc AMS*, 1981. Available at: https://tinyurl.com/ydygjgge.

[A5] Kim M. Arithmetic Gauge Theory: A Brief Introduction. *Mod Phys Lett A*, 33(29), 2018. Available at: https://www.worldscientific.com/doi/10.1142/S0217732318300124.

[A6] Reid M. The du val singularities an, dn, e6, e7, e8. Available at:https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf.

[A7] Scholze P. Perfectoid spaces, 2011. Available at: https://arxiv.org/abs/1111.4914.

## Theoretical Physics

## Books related to TGD

[K1] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I. https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/prext.pdf, 2023.

[K2] Pitkänen M. Conscious Information and Intelligence. In *TGD Inspired Theory of Consciousness: Part II. https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/intsysc.pdf, 2023.

[K3] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K4] Pitkänen M. Could Genetic Code Be Understood Number Theoretically? In *Genes and Memes: Part I. https://tgdtheory.fi/tgdhtml/genememe1.html*. Available at: https://tgdtheory.fi/pdfpool/genenumber.pdf, 2023.

[K5] Pitkänen M. From Principles to Diagrams. In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/diagrams.pdf, 2023.

[K6] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck1.html., 2023.

[K7] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part II.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck2.html., 2023.

[K8] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K9] Pitkänen M. *p-Adic length Scale Hypothesis.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/padphys.html., 2023.

[K10] Pitkänen M. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mass3.pdf, 2023.

[K11] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II. https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K12] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K13] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K14] Pitkänen M. The classical part of the twistor story. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistorstory.pdf, 2023.

[K15] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

[K16] Pitkänen M. Was von Neumann Right After All? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/vNeumann.pdf, 2023.


## Articles about TGD


[L1] Pitkänen M. General ideas about octonions, quaternions, and twistors. Available at: https://tgdtheory.fi/public_html/articles/oqtwistor.pdf., 2014.

[L2] Pitkänen M. Geometric theory of harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[L3] Pitkänen M. Hydrinos again. Available at: https://tgdtheory.fi/public_html/articles/Millsagain.pdf., 2016.

[L4] Pitkänen M. X boson as evidence for nuclear string model. Available at: https://tgdtheory.fi/public_html/articles/Xboson.pdf., 2016.

[L5] Pitkänen M. Are higher structures needed in the categorification of TGD? Available at: https://tgdtheory.fi/public_html/articles/nstructures.pdf., 2017.

[L6] Pitkänen M. Breaking of $CP$, $P$, and $T$ in cosmological scales in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/rotuniverse.pdf., 2017.

[L7] Pitkänen M. Could McKay correspondence generalize in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2017.

[L8] Pitkänen M. Could the precursors of perfectoids emerge in TGD? Available at: https://tgdtheory.fi/public_html/articles/perfectoid.pdf., 2017.

[L9] Pitkänen M. Does $M^8-H$ duality reduce classical TGD to octonionic algebraic geometry? Available at: https://tgdtheory.fi/public_html/articles/ratpoints.pdf., 2017.

[L10] Pitkänen M. Does $M^8-H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: https://tgdtheory.fi/public_html/articles/ratpoints1.pdf., 2017.

[L11] Pitkänen M. Does $M^8-H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: https://tgdtheory.fi/public_html/articles/ratpoints2.pdf., 2017.

[L12] Pitkänen M. Does $M^8-H$ duality reduce classical TGD to octonionic algebraic geometry?: part III. Available at: https://tgdtheory.fi/public_html/articles/ratpoints3.pdf., 2017.

[L13] Pitkänen M. Generalization of Riemann Zeta to Dedekind Zeta & Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/dedekind.pdf., 2017.

[L14] Pitkänen M. p-Adicization and adelic physics. Available at: https://tgdtheory.fi/public_html/articles/adelicphysics.pdf., 2017.

[L15] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L16] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L17] Pitkänen M. Re-examination of the basic notions of TGD inspired theory of consciousness. Available at: https://tgdtheory.fi/public_html/articles/conscrit.pdf., 2017.

[L18] Pitkänen M. What could be the role of complexity theory in TGD? Available at: https://tgdtheory.fi/public_html/articles/numberchaos.pdf., 2017.

[L19] Pitkänen M. On Hydrinos Again. *Pre-Space-Time Journal*, 8(1), 2017. See also https://tgtheory.fi/public_html/articles/Millsagain.pdf.

[L20] Pitkänen M. Dark valence electrons and color vision. Available at: https://tgdtheory.fi/public_html/articles/colorvision.pdf., 2018.

[L21] Pitkänen M. Is the hierarchy of Planck constants behind the reported variation of Newton's constant? Available at: https://tgdtheory.fi/public_html/articles/variableG.pdf., 2018.

[L22] Pitkänen M. New results in the model of bio-harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonynew.pdf., 2018.

[L23] Pitkänen M. TGD view about coupling constant evolution. Available at: https://tgdtheory.fi/public_html/articles/ccevolution.pdf., 2018.

[L24] Pitkänen M. The Recent View about Twistorialization in TGD Framework. Available at: https://tgdtheory.fi/public_html/articles/smatrix.pdf., 2018.

[L25] Pitkänen M. Are fundamental entities discrete or continous and what discretization at fundamental level could mean? Available at: https://tgdtheory.fi/public_html/articles/zetaspecu.pdf., 2019.

[L26] Pitkänen M. $M^8-H$ duality and consciousness. Available at: https://tgdtheory.fi/public_html/articles/M8Hconsc.pdf., 2019.

[L27] Pitkänen M. $M^8-H$ duality and the two manners to describe particles. Available at: https://tgdtheory.fi/public_html/articles/susysupertwistor.pdf., 2019.

[L28] Pitkänen M. Minimal surfaces: comparison of the perspectives of mathematician and physicist. Available at: https://tgdtheory.fi/public_html/articles/minimalsurfaces.pdf., 2019.

[L29] Pitkänen M. New results related to $M^8-H$ duality. Available at: https://tgdtheory.fi/public_html/articles/M8Hduality.pdf., 2019.

[L30] Pitkänen M. Quantum self-organization by $h_{eff}$ changing phase transitions. Available at: https://tgdtheory.fi/public_html/articles/heffselforg.pdf., 2019.

[L31] Pitkänen M. Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals . Available at: https://tgdtheory.fi/public_html/articles/symplorbsm.pdf., 2019.

[L32] Pitkänen M. Secret Link Uncovered Between Pure Math and Physics. Available at: https://tgdtheory.fi/public_html/articles/KimTGD.pdf., 2019.

[L33] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L34] Pitkänen M. SUSY in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/susyTGD.pdf., 2019.

[L35] Pitkänen M. TGD inspired theory of consciousness and living systems. Available at: https://tgdtheory.fi/public_html/articles/badenbaden.pdf., 2019.

[L36] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, and Inclusions of Hyperfinite Factors. Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2019.

[L37] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors. Available at: https://tgdtheory.fi/public_html/articles/McKaygeneral.pdf., 2019.

[L38] Pitkänen M. Trying to understand why ramified primes are so special physically. Available at: https://tgdtheory.fi/public_html/articles/ramified.pdf., 2019.

[L39] Pitkänen M. Twistor lift of TGD and WCW geometry. Available at: https://tgdtheory.fi/public_html/articles/wcwnewest.pdf., 2019.

[L40] Pitkänen M. Twistors in TGD. Available at: https://tgdtheory.fi/public_html/articles/twistorTGD.pdf., 2019.

[L41] Pitkänen M. When do partonic 2-surfaces and string world sheets define large cognitive representations? Available at: https://tgdtheory.fi/public_html/articles/elliptic.pdf., 2019.

[L42] Pitkänen M. About $M^8 - H$-duality, p-adic length scale hypothesis and dark matter hierarchy. Available at: https://tgdtheory.fi/public_html/articles/paddarkscales.pdf., 2020.

[L43] Pitkänen M. Can TGD predict the value of Newton's constant? Available at: https://tgdtheory.fi/public_html/articles/Gagain.pdf., 2020.

[L44] Pitkänen M. Could quantum randomness have something to do with classical chaos? Available at: https://tgdtheory.fi/public_html/articles/chaostgd.pdf., 2020.

[L45] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: https://tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[L46] Pitkänen M. When does "big" state function reduction as universal death and reincarnation with reversed arrow of time take place? Available at: https://tgdtheory.fi/public_html/articles/whendeath.pdf., 2020.