Could $\mathcal{N} = 2$ Super-Conformal Algebra Be Relevant For TGD?

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Abstract

TGD has as is symmetries super-conformal symmetry (SCS), which is a huge extension of the ordinary SCS. For instance, the infinite-dimensional symplectic group plays the role of finite-dimension Lie-group as Kac-Moody group and the conformal weights for the generators of algebra corresponds to the zeros of fermionic zeta and their number of generators is therefore infinite.

The relationship of TGD SCS to super-conformal field theories (SCFTs) known as minimal models has remained without definite answer. The most general super-conformal algebra (SCA) assignable to string world sheets by strong form of holography has $\mathcal{N}$ equal to the number of spin states of leptonic and quark type fundamental spinors but the space-time SUSY is badly broken for it. Covariant constancy of the generating spinor modes is replaced with holomorphy - kind of "half covariant constancy". Right-handed neutrino and antineutrino are excellent candidates for generating $\mathcal{N}=2$ SCS with a minimal breaking of the corresponding space-time SUSY.

$\mathcal{N}=2$ SCS has also some inherent problems. The critical space-time dimension is $D=4$ but the existence of complex structure seems to require the space-time has metric signature different from Minkowskian: here TGD suggests a solution. $\mathcal{N}=2$ SCFTs are claimed also to reduce to topological QFTs under some conditions: this need not be a problem since TGD can be characterized as almost topological QFT. What looks like a further problem is that p-adic mass calculations require half-integer valued negative conformal weight for the ground state (and vanishing weight for massless states). One can however shift the scaling generator L_0 to get rid of problem: the shift has physical interpretation in TGD framework and must be half integer valued which poses the constraint $h=K/2$, $K=0,1,2..$ on the representations of SCA.

$\mathcal{N}=2$ SCA allows a spectral flow taking Ramond representations to Neveu-Scwartz variant of algebra. The physical interpretation is as super-symmetry mapping fermionic states to bosonic states. The representations of $\mathcal{N}=2$ SCA allowing degenerate states with positive central charge c and non-vanishing ground state conformal weight h give rise to minimal models allowing ADE classification, construction of partition functions, and even of n-point functions. This could make S-matrix of TGD exactly solvable in the fermionic sector. The ADE hierarchy suggests a direct interpretation in terms of orbifold hierarchy assignable to the hierarchy of Planck constants associated with the super-symplectic algebra: primary fields would correspond to orbifolds identified as coset spaces of ADE groups. Also an interpretation in terms of inclusions of hyper-finite factors is highly suggestive.

1 Introduction

The concrete realization of the super-conformal symmetry (SCS) in TGD framework has remained poorly understood. In particular, the question how SCS relates to super-conformal field theories (SCFTs) has remained an open question. The most general super-conformal algebra assignable to string world sheets by strong form of holography has $\mathcal{N}$ equal to the number of 4+4 =8 spin states of leptonic and quark type fundamental spinors but the space-time SUSY is badly broken for it. Covariant constancy of the generating spinor modes is replaced with holomorphy - kind of "half covariant constancy". I have considered earlier a proposal that $\mathcal{N}=4$ SCA could be realized in TGD framework but given up this idea. Right-handed neutrino and antineutrino are excellent candidates for generating $\mathcal{N}=2$ SCS with a minimal breaking of the corresponding space-time SUSY. Covariant constant neutrino is an excellent candidate for the generator of $\mathcal{N}=2$ SCS. The possibility of this SCS in TGD framework will be considered in the sequel.

1.1 Questions about SCS in TGD framework

This work was inspired by questions not related to $\mathcal{N}=2$ SCS, and it is good to consider first these questions.

1.1.1 Could the super-conformal generators have conformal weights given by poles of fermionic zeta?

The conjecture [K5] is that the conformal weights for the generators super-symplectic representation correspond to the negatives of $h=-ks_k$ of the poles s_k fermionic partition function $\zeta_F(ks)=\zeta(ks)/\zeta(2ks)$ defining fermionic partition function. Here k is constant, whose value must be fixed from the condition that the spectrum is physical. $\zeta(ks)$ defines bosonic partition function for particles whos energies are given by $log(p)$, p prime. These partition functions require complex temperature but is completely sensible in Zero Energy Ontology (ZEO), where thermodynamics is replaced with its complex square root.

For non-trivial zeros $2ks=1/2+iy$ of $\zeta(2ks)$ s would correspond pole $s=(1/2+iy)/2k$ of $z_F(ks)$. The corresponding conformal weights would be $h=(-1/2-iy)/2k$. For trivial zeros $2ks=-2n$, $n=1,2,..$ $s=-n/k$ would correspond to conformal weights $h=n/k>0$. Conformal confinement is assumed meaning that the sum of imaginary parts of of generators creating the state vanishes.

What can one say about the value of k? The pole of $\zeta(ks)$ at $s = 1/k$ would correspond to pole and conformal weight $h = -1/k$. For $k = 1$ the trivial conformal weights would be positive integers $h = 1, 2, ...$: this certainly makes sense. This gives for the real part for non-trivial conformal weights $h = -1/4$. By conformal confinement both pole and its conjugate belong to the state so that this contribution to conformal weight is negative half integers: this is consistent with the facts about super-conformal representations. For the ground state of super-conformal representation the conformal weight for conformally confined state would be $h = -K/2$. In p-adic mass calculations one would have $K = 6$ [K2].

The negative ground state conformal weights of particles look strange but p-adic mass calculations require that the ground state conformal weights of particles are negative: $h = -3$ is required.

1.1.2 What could be the origin of negative ground state conformal weights?

Super-symplectic conformal symmetries are realized at light-cone boundary and various Hamiltonians defined analogs of Kac-Moody generators are proportional functions $f(r_M)H_{J,m}H_A$, where $H_{J,m}$ correspond to spherical harmonics at the 2-sphere $R_M = constant$ and H_A is color partial wave in CP_2, $f(r_M)$ is a partial wave in radial light-like coordinate which is eigenstate of scaling operator $L_0 = r_M d/dR_M$ and has the form $(r_M/r_0)^{-h}$, where h is conformal weight which must be of form $h = -1/2 + iy$.

To get plane wave normalization for the amplitudes

$$(\frac{r_M}{r_0})^{-h} = (\frac{r_M}{r_0})^{-1/2} exp(iyx) \ , \ x = log(\frac{r_M}{r_0}) \ ,$$

one must assume $h = -1/2+iy$. Together with the invariant integration measure dr_M this gives for the inner product of two conformal plane waves $exp(iy_i x)$, $x = log(r_M/r_0)$ the desired expression $\int exp[iy_1 - y_2)x]dx = \delta(y_1 - y_2)$, where $dx = dr_M/r_M$ is scaling invariance integration measure. This is just the usual inner product of plane waves labelled by momenta y_i.

If r_M/r_0 can be identified as a coordinate along fermionic string (this need not be always the case) one can interpret it as real or imaginary part of a hypercomplex coordinate at string world sheet and continue these wave functions to the entire string world sheets. This would be very elegant realization of conformal invariance.

1.1.3 How to relate degenerate representations with $h > 0$ to the massless states constructed from tachyonic ground states with negative conformal weight?

This realization would however suggest that there must be also an interpretation in which ground states with negative conformal weight $h_{vac} = -k/2$ are replaced with ground states having vanishing conformal weights $h_{vac} = 0$ as in minimal SCAs and what is regarded as massless states have conformal weights $h = -h_{vac} > 0$ of the lowest physical state in minimal SCAs.

One could indeed start directly from the scaling invariant measure dr_M/r_M rather than allowing it to emerge from dr_M. This would require in the case of p-adic mass calculations that has representations satisfying Virasoro conditions for weight $h = -h_{vac} > 0$. p-Adic mass squared would be now shifted downwards and proportional to $L_0 + h_{vac}$. There seems to be no fundamental reason preventing this interpretation. One can also modify scaling generator L_0 by an additive constant term and this does not affect the value of c. This operation corresponds to replacing basis $\{z^n\}$ with basis $\{z^{n+1/2}\}$.

What makes this interpretation worth of discussing is that the entire machinery of conformal field theories with non-vanishing central charge and non-vanishing but positive ground state conformal weight becomes accessible allowing to determine not only the spectrum for these theories but also to determine the partition functions and even to construct n-point functions in turn serving as basic building bricks of S-matrix elements [K6].

ADE classification of these CFTs in turn suggests at connection with the inclusions of hyperfinite factors and hierarchy of Planck constants. The fractal hierarchy of broken conformal symmetries with sub-algebra defining gauge algebra isomorphic to entire algebra would give rise to dynamic symmetries and inclusions for HFFs suggest that ADE groups define Kac-Moody type symmetry algebras for the non-gauge part of the symmetry algebra.

1.2 Questions about $\mathcal{N} = 2$ SCS

$\mathcal{N} = 2$ SCFTs has some inherent problems. For instance, it has been claimed that they reduce to topological QFTs. Whether $\mathcal{N} = 2$ can be applied in TGD framework is questionable: they have critical space-time dimension $D = 4$ but since the required metric signature of space-time is wrong.

1.2.1 Inherent problems of $\mathcal{N} = 2$ SCS

$\mathcal{N} = 2$ SCS has some severe inherent problems.

1. $\mathcal{N} = 2$ SCS has critical space-time dimension $D = 4$, which is extremely nice. On the other, $\mathcal{N} = 2$ requires that space-time should have complex structure and thus metric signature (4,0), (0,4) or (2,2) rather than Minkowski signature. Similar problem is encountered in twistorialization and TGD proposal is Hamilton-Jacobi structure (se the appendix of [K4]), which is hybrid of hypercomplex structure and Kähler structure. There is also an old proposal by Pope et al [B7]that one can obtain by a procedure analogous to dimensional reduction $\mathcal{N} = 2$ SCS from a 6-D theory with signature (3,3). The lifting of Kähler action to twistor space level allows the twistor space of M^4 to have this signature and the degrees of freedom of the sphere S^2 are indeed frozen.

2. There is also an argument by Eguchi that $\mathcal{N} = 2$ SCFTs reduce under some conditions to mere topological QFTs [B5]. Thi looks bad but there is a more refined argument that $\mathcal{N} = 2$ SCFT transforms to a topological CFT only by a suitable twist [B4, B6]. This is a highly attractive feature since TGD can be indeed regarded as almost topological QF. For instance, Kähler action in Minkowskian regions could reduce to Chern-Simons term for a very general solution ansatz. Only the volume term having interpretation in terms of cosmological constant [K6] (extremely small in recent cosmology) would not allow this kind of reduction. The topological description of particle reactions based on generalized Feynman diagrams identifiable in terms of space-time regions with Euclidian signature of the induced metric would allow to build n-point functions in the fermionic sector as those of a free field theory. Topological QFT in bosonic degrees of freedom would correspond naturally to the braiding of fermion lines.

1.2.2 Can one really apply $\mathcal{N} = 2$ SCFTs to TGD?

TGD version of SCA is gigantic as compared to the ordinary SCA. This SCA involves super-symplectic algebra associated with metrically 2-dimensional light-cone boundary (light-like boundaries of causal diamonds) and the corresponding extended conformal algebra (light-like boundary is metrically sphere S^2). Both these algebras have conformal structure with respect to the light-like radial coordinate r_M and conformal algebra also with respect to the complex coordinate of S^2. Symplectic algebra replaces finite-dimensional Lie algebra as the analog of Kac-Moody algebra. Also light-like orbits of partonic 2-surfaces possess this SCA but now Kac-Moody algebra is defined by isometries of imbedding space. String world sheets possess an ordinary SCA assignable to isometries of the imbedding space. An attractive interpretation is that r_M at light-cone boundary corresponds to a coordinate along fermionic string extendable to a hypercomplex coordinate at string world sheet.

$\mathcal{N} = 8$ SCS seems to be the most natural candidate for SCS behind TGD: all fermion spin states would correspond to generators of this symmetry. Since the modes generating the symmetry are however only half-covariantly constant (holomorphic) this SUSY is badly broken at space-time level and the minimal breaking occurs for $\mathcal{N} = 2$ SCS generated by right-handed neutrino and antineutrino.

The key motivation for the application of minimal $\mathcal{N} = 2$ SCFTs to TGD is that SCAs for them have a non-vanishing central charge c and vacuum weight $h \geq 0$ and the degenerate character of ground state allows to deduce differential equations for n-point functions so that these theories are exactly solvable. It would be extremely nice is scattering amplitudes were basically determined by n-point functions for minimal SCFTs.

A further motivation comes from the following insight. ADE classification of $\mathcal{N} = 2$ SCFTs is extremely powerful result and there is connection with the hierarchy of inclusions of hyperfinite factors of type II_1, which is central for quantum TGD. The hierarchy of Planck constants assignable to the hierarchy of isomorphic sub-algebras of the super-symplectic and related algebras suggest interpretation in terms of ADE hierarchy a rather detailed view about a hierarchy of conformal field theories and even the identification of primary fields in terms of critical deformations.

The application $\mathcal{N} = 2$ SCFTs in TGD framework can be however challenged. The problem caused by the negative value of vacuum conformal weight has been already discussed but there are also other problems.

1. One can argue that covariantly constant right-handed neutrino - call it ν_R - defines a pure gauge super-symmetry and it has taken along time to decide whether this is the case or not. Taking at face value the lacking evidence for space-time SUSY from LHC would be easy but too light-hearted manner to get rid of the problem.

   Could it be that at space-time level covariantly constant right-handed neutrino (ν_R) and its antiparticle ($\overline{\nu_R}$) generates pure gauge symmetry so that the resulting sfermions correspond to zero norm states? The oscillator operators for ν_R at imbedding space level have commutator proportional to $p^k\gamma_k$ vanishing at the limit of vanishing massless four-momentum. This would imply that they generate sfermions as zero norm states. This argument is however formulated at the level of imbedding space: induced spinor modes reside at string world sheets and covariant constancy is replaced by holomorphy!

   At the level of induced spinor modes located at string world sheets the situation is indeed different. The anti-commutators are not proportional to $p^k\gamma_k$ but in Zero Energy Ontology (ZEO) can be taken to be proportional to $n^k\gamma_k$ where n_k is light-like vector dual to the light-like radial vector of the point of the light-like boundary of causal diamond CD (part of light-one boundary) considered. Therefore also constant ν_R and $\overline{\nu_R}$ are allowed as non-zero norm states and the 3 sfermions are physical particles. Both ZEO and strong form of holography (SH) would play crucial role in making the SCS dynamical symmetry.

2. Second objection is that LHC has failed to detect sparticles. In TGD framework this objection cannot be taken seriously. The breaking of $\mathcal{N} = 2$ SUSY would be most naturally realized as different p-adic length scales for particle and sparticle. The mass formula would be the same apart from different p-adic mass scale. Sparticles could emerge at short p-adic length scale than those studied at LHC (labelled by Mersenne primes M_{89} and $M_{G,79} = (1+i)^79$).

   One the other hand, one could argue that since covariantly constant right-handed neutrino has no electroweak-, color- nor gravitational interactions, its addition to the state should not change its mass. Again the point is however that one considers only neutrinos at string world sheet so that covariant constancy is replaced with holomorphy and all modes of right-handed neutrino are involved. Kähler Dirac equation brings in mixing of left and right-handed neutrinos serving as signature for massivation in turn leading to SUSY breaking. One can of course ask whether the p-adic mass scales could be identical after all. Could the sparticles be dark having non-standard value of Planck constant $h_{eff} = n \times h$ and be created only at quantum criticality [K3].

This is a brief overall view about the most obvious problems and proposed solution of them in TGD framework and in the following I will discuss the details. I am of course not a SCFT professional. I however dare to trust my physical intuition since experience has taught to me that it is better to concentrate on physics rather than get drowned in poorly understood mathematical technicalities.

2 Some CFT backround

The construction of CFTs involves as the first step construction of irreducible unitary representations of conformal algebras. They are completely known for the central charge $0 \leq c \leq 1$. One can also construct modular invariant partition functions for tensor products possibly serving as partition functions of CFTs. Already Belavin, Polyakov and Zhamolodzhikov [B1] discovered in their pioneering paper so called minimal models with the defining property that the state space realizes only finite number of irreducible representations.

2.1 Modular invariant partition functions

The classification of modular functions leads to the ADE scheme [B3] (http://tinyurl.com/h9val5g). The physical picture is that the primary fields of minimal CFT correspond to deformations of a critical system in some configuration space. One can construct all minimal orbifold CFTs in orbifolds $G\backslash C^2$ of C^2 in which the discrete subgroup G of $SU(2)$ acts linearly [B8]. This is a minimal realization. ADE scheme enters via the ADE classification for the discrete subgroups of $SU(2)$ (see http://tinyurl.com/jyjplzc).

ADE classification gives an amazingly detailed view about the spectrum of minimal models and also about their partition functions [B3] (see http://tinyurl.com/zlhk3wu). More general rational CFTs can possess infinite families of Virasoro representations, which an be however organized to representations of W-algebra. So called WZW models provide an important example constructible for any semi-simple Lie algebra.

The decomposition of RCFT Hilbert space to sum over tensor products of spaces carrying irreducible unitary representation conformal algebra and is conjugate can be written as

$$H = \oplus_{j\bar{j}} N_{j\bar{j}} H_j \otimes H_{\bar{j}} \quad . \tag{2.1}$$

There are consistency conditions on the coefficients due to the conditions that the CFT must exist on any Riemann surface. Verlinde algebra (see http://tinyurl.com/y8p9muj6) expresses the fusion rules. The associative Verlinde algebra is finite-dimensional and has as its elements primary fields and its structure constants code for the fusion rules. Especially interesting primary fields are those which are simple in the sense that the product of two primary fields contains only one prime field.

It is good to understand how one ends up with the expression of partition function in conformal field theories.

1. Start from the fact that conformal invariance fixes the complex function by data at 1-dimensional curve and one can speak about analog of time evolution in direction orthogonal to this curve. Introduce Hamiltonian for the Euclidian "time" evolution in finite "time" interval defining an annulus at 2-D surface with boundaries identified as initial and final times. Assume periodic boundary conditions in Euclidian "time" direction so that the annulus effectively closes to a torus. The outcome is a conformal field theory at torus although one starts from conformal invariance at sphere or even Riemann surface with higher genus.

2. Torus has several conformally inequivalent variants since it can be obtained from complex plane by identifying the points differing by a translation generated by real unit 1 and complex number τ. The possible values of τ defines the moduli space for conformal equivalence classes of torus since the angle angle between the sides of this elementary cell and the ratio of the lengths of homologically non-trivial geodesics of torus are conformal invariants. Modular invariance however implies that the values of τ differing by PSL(2,Z) transformation are equivalent.

3. What happens if one applies this procedure at higher genus surface? If the annulus is around the handle of this kind of surface, one might have a problem since it is not clear whether periodic boundary conditions can be identified in terms of a compactification to torus - this kind of annulus cannot be physically compactified to a torus. One can also consider a Hamiltonian evolution associated with any curve characterized by homology class telling how many times the curve winds around various handles. Can one just use the parameter τ or should one take into account the homology class of the annulus.

   One can challenge the idea about Hamiltonian time evolution as a formal trick and consider the possibility that partition functions is defined for the entire 2-surface in moduli space. In this kind of situation it would be trivial for sphere.

4. One can write explicitly the expression for the Euclidian "time" evolution operator between the ends of annulus as an exponential:

$$exp(-H_{cycl}L) = exp\left[2\pi i\tau(L_0 - \frac{c}{24}) - 2\pi i\overline{tau}(\overline{L}_0 - \frac{c}{24}\right] \quad . \tag{2.2}$$

   Partition function is defined as the trace

$$Z(\tau) = Tr\left[exp(-H_{cycl}L)\right] \quad . \tag{2.3}$$

$$\chi_j(q) = Tr\left[exp\left[2\pi i\tau(L_0 - \tfrac{c}{24})\right]\right] = q^{h_j - \frac{c}{24}}\sum m_n q^n \quad , \quad q = exp(i2\pi\tau) \quad , \quad \overline{q} = exp(-i2\pi\overline{\tau}) \tag{2.4}$$

5. The decomposition of Hilbert space translates to a decomposition of the partition function as

$$Z(\tau) = \sum_{j\overline{j}} N_{j\overline{j}}\chi_j(q) \times \chi_{\overline{j}}(\overline{q}) \quad . \tag{2.5}$$

   Here one can wonder whether one could give up the interpretation in terms of Hamiltonian time evolution and consider just partition function in the moduli space of torus (or higher genus surface).

Modular invariance poses strong conditions of the expression of partition function of system as sum over products $chi_j\overline{\chi_{\overline{j}}}$ of characters assignable to irreducible unitary representations of Virasoro algebra. In the case of torus moduli correspond to complex plane whose points differing by a transformations by the discrete group SL(2,Z) are identified. The resulting moduli space has topology of torus. The generators of modular transformations are unit shift T: $\tau \to \tau + 1$ and inversion S: $\tau \to -1/\tau$ and it is enough to demand that the partition function is invariant under these transformations. The action of these transformations on characters induce an unitary automorphisms of the matrix $N_{j\overline{j}}$ and the condition is that the actions of S and T are trivial

$$TNT^{\dagger} = SNS^{\dagger} = N \quad . \tag{2.6}$$

It is interesting to relate this picture to TGD framework where one has string world sheets and partonic 2-surfaces.

1. The annulus picture applies to string world sheets. At the ends space-time surface at boundaries of CD one has fermionic strings connecting wormhole throat to another one along the first space-time sheet and returning back along second space-time sheet and forming thus a closed string, whose time evolution defines string space-time sheet as a cylindrical object. The strings at the ends of CD can get knotted and braided. They can also reconnect - the interpretation is in terms of standard stringy vertices. In fact this gives rise to 2-braiding possible because space-time dimension is 4.

   One can also consider loops as handles attached to these annuli: since the induced metric is allowed to have Euclidian signature, they are in principle possible but involve always Euclidian regions around points, where the time direction of closed homologically trivial time loop defined by the time coordinate of Minkowski space changes. Preferred extremal property might forbid loop corrections in Minkowskian space-time regions but allow them inside Euclidian regions representing lines of scattering diagrams.

2. The moduli space for the conformal equivalence classes of partonic 2-surfaces is important in the TGD based model for family replication phenomenon [K1]. In TGD context one must construct modular invariant partition functions in these higher-dimensional moduli spaces - I call them elementary particle vacuum functionals. These partition functions do not allow interpretation in terms of Hamiltonian time evolution.

2.2 Degenerate conformal representations and minimal models

So called degenerate representations allow to construct minimal models with finite number of primary fields and derive also differential equations for their correlation functions. Degeneracy condition fixes the spectrum of so called minimal conformal field theories.

1. The conformal weight the ground state is fixed to $h \geq 0$. Virasoro conditions must be satisfied: it is enough that the generators L_1 and L_2 annihilate the ground state. The defining feature of degenerate representations is that they possess states with zero norm created by generates with negative conformal weights from the ground state.

2. Degenerate states are obtained as linear combinations of states constructible using products $\prod_k L_{-k}^{-n_k}$, $N = \sum_k n_k k$ of generators with total conformal weight $-N$ operating on ground state with weight h. Degeneracy means that some combination of the generators with total weight $-N$ annihilates the state. Besides this ordinary Virasoro conditions for generators with positive weight are satisfied. The existence of the degenerate state means that the metric of this sub-state space is degenerate so that its determinant - so called Kac determinant vanishes. This brings strongly in mind criticality: at criticality sub-representation is isolated from the larger representation and defines zero norm states. These would correspond to zero modes appearing at criticality and not contributing to the potential function.

3. Vanishing of Kac determinant gives a condition allowing to deduce a general formula for the allowed values of the central charge c defining the central extension of conformal algebra. One can factorize Kac determinant to a product form $\prod_n (h - h_n)$ and the eigenvalues h_n defined the ground state weights allowing the degeneracy. Unitarity gives a further condition on the representation and for $c < 1$ this dictates the spectrum of vacuum conformal weights completely.

One can deduce an explicit expression for the Kac determinant as function of c and h and this gives rise to the following fundamental formulas [B3] (see `http://tinyurl.com/h9val5g`) for the values of central charge c and ground state conformal weight h for which the determinant vanishes. For $c > 1$ the determinant does not vanish and is positive. For $c < 1$ situation is different.

$$\begin{array}{llll} c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \ , & p \ and \ q \ coprime \ , & p,q = 1,2,3,... \\ h = h_{r,s}(p,q) = \frac{[pr-qs]^2-(p-q)^2}{4pq} \ , & 1 \leq r \leq q-1, & 1 \leq s \leq p-1 \ . \end{array} \tag{2.7}$$

For these values of c and h the representation defined by dividing away zero norm states is irreducible and unitary. So called minimal models forming a special case of them and possessing finite number of primary fields correspond to these representations.

Why the degeneracy is so important? Suppose that primary conformal fields Φ_k have conformal weight h and satisfy the degeneracy condition. Then n-point functions satisfy also the appropriate form of the degeneracy condition being annihilated by the combination of Virasoro generators with total weight $-N$. This gives rise to n partial differential equations of order N for $\langle \Phi(z_1)...\Phi(z_n)\rangle$ allowing to solve the conformal field theory exactly. In TGD this generalizes would give a powerful tool to determine the correlation functions at string world sheets.

The standard example is provided by the $N = 2$ case. The operator $O = L_{-2} - \frac{3}{2h+1}L_{-1}^2$ generates from the ground state with conformal weight h zero norm state provided the condition $c = 2h(5-8h)/(2h+1)$ is satisfied. For $h = 1/2$ this gives $c = 1/2$. Primary fields of the CFT are annihilated by this operator as also n-point functions and this gives second order differential equations for the n-point functions.

If the proposed interpretation of negative conformal weights in TGD framework is correct then one can add the condition $h = K/2$ to the conditions fixing c and h. Although SCFT rather than CFT is expected to be interesting from TGD point of view, one can just for fun see the above conditions for c and h allow $h = K/2$. Direct calculation for $p = m, q = m+1$ shows that for $m = 4$ ($c = 1/2$), $x = 1$ and $x = 1/2$ are realized for ($r = 3, s = 1$) and ($r = 3, s = 2$) respectively. For $m = 5$ one obtains $x = 3$ corresponding to $r = 4$ and $s = 1$. For $m = 6$ one obtains $x = 5$. It is not clear $(p,q) = (m, m+1)$ allows to realize $h = K/2$ or even $h = 5/2$ and $h = 2$.

2.3 Minimal $\mathcal{N} = 2$ SCFTs

2.3.1 $\mathcal{N} = 2$ SCA

$\mathcal{N} = 2$ SCA is spanned by Virasoro generators L_n and their super counterparts G_r, where r is either integer (Ramond) or half-odd integer (Neveu-Schwartz) plus generators of conserved U(1) current J (se `http://tinyurl.com/yblzbovb`). Ramond and Neveu-Scwartz and these representations can be mapped to each other by spectral automorphism.

The commutation/anticommutation relations for $\mathcal{N} = 2$ algebra are given by

$$\begin{array}{l} [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad , \\ [L_m, J_n] = -nJ_{m+n} \quad , \\ [J_m, J_n] = \frac{c}{3}m\delta_{m+n,0} \quad , \\ \{G_r^+, G_s^-\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{c}{6}(r^2 - \frac{1}{4})\delta_{r+s,0} \quad , \\ \{G_r^+, G_s^+\} = 0 = \{G_r^-, G_s^-\} \quad , \\ [L_m, G_r^{\pm}] = (\frac{m}{2} - r)G_{r+m}^{\pm} \quad , \\ [J_m, G_r^{\pm}] = \pm G_{m+r}^{\pm} \quad . \end{array} \tag{2.8}$$

Also in the case of SCFTs one it is natural to search for sub-representations with ground state weight h and annihilated by some generator of conformal weight $-N$. In this case the operatiors would be monomials of Virasoro generators and their super counterparts and also now the vanishing of Kac-determinant [B2], whose expression was deduced by Boucher, Friedan and Kent, would allow to deduce information about allowed values of c and h. Also in this case the n-point functions $\langle\Phi(z_1)...\Phi(z_n)$ satisfy N:th order the differential equations implied by the condition that the generator in question annihilates the primary fields.

2.3.2 Spectral automorphism mapping Ramond and N-S representations to each other

Spectral automorphism maps both the algebra and its representations to new ones. The spectral automorphism mapping Ramond representation to N-S representation is given by

$$\begin{array}{l} \alpha(L_n) = L_n + \theta J_n + \frac{\theta^2}{6}\delta_{n,0} \quad , \\ \alpha(J_n) = J_n + \frac{\theta}{3}\delta_{n,0} \quad , \\ \alpha(G_r^{\pm}) = G_{r\pm\theta}^{\pm} \quad . \end{array} \tag{2.9}$$

The inverse of the automorphism is given by

$$\begin{array}{l} \alpha^{-1}(L_n) = L_n - \theta J_n + \frac{\theta^2}{6}\delta_{n,0} \quad , \\ \alpha^{-1}(J_n) = J_n - \frac{c}{3}\theta\delta_{n,0} \quad , \\ \alpha^{-1}(G_r^{\pm}) = G_{r\mp\theta}^{\pm} \quad . \end{array} \tag{2.10}$$

For $\theta = 1/2$ one obtains Ramond-NS spectral mapping.

Central extension term contains par linear in m. This is changed as one finds by calculating the commutators of the transformed Virasoro generators and expressing it in in terms of transformed generators. This does not affect the value of c. No change occurs for $k = 2$ minimal representations with $Q = k/2(k+2) - 1/4 = 0$. Also the term linear in m remains unaffected if the $\theta = 1/2$ flow is modified to

$$\alpha(L_n) \quad = \quad L_n + \frac{1}{2}J_n + (\frac{1}{24} - \frac{Q_{N-S}}{2})\delta_{n,0} \quad . \tag{2.11}$$

Also the ground state is changed in the spectral flow and Q_{N-S} labels the ground state charge for the resulting N-S representation. For minimal SCAs the flow must label $(h, Q)_R$ to Ramond state to $(h, Q)_{N-S}$.

If the linear term of central extension is unaffected in the flow, the values of h and Q change as follows:

$$\begin{aligned} h_R \rightarrow h_{new,R} + \tfrac{c}{24} = h_{N-S} \ \ , \\ Q \rightarrow Q_{new,R} + \tfrac{c}{6} = Q_{N-S} \ \ . \end{aligned} \tag{2.12}$$

The simplest guess is that the change leaves (a, b) unchanged and just drops the $1/8$ term from h and Q. This condition determines the values of $h_{new,R}$ and $Q_{new,R}$ for minimal representations to

$$\begin{aligned} h_{new,R} = \tfrac{1}{8} - \tfrac{c}{24} = \tfrac{1}{8} - \tfrac{k}{8(k+2)} \ \ , \\ Q_{new,R} = \tfrac{1}{4} - \tfrac{1}{2k(k+2)} \ \ . \end{aligned} \tag{2.13}$$

2.3.3 Degenerate representations

The classification of unitary minimal super-conformal field theories is surprisingly well-understood [B9] (see `http://tinyurl.com/yctvyk2o`). ADE patterns are involved also in the classification of minimal SCFTs. The good news is that $\mathcal{N} = 2$ superstrings have critical dimension $D = 4$. The bad news is that the signature of the space-time metric is either (0,4), (2,2) or (4,0) rather than Minkowkian (1,3). This problem will be considered later in more detail.

I am not specialist and can only list the results. It is to be emphasized that not only the spectrum of basic parameters but also the partition functions are known, and correlation functions can be constructed.

1. The values of the central charge are given by

$$c = \frac{3k}{k+2} \ \ , \ \ k = 0, 1, 2... \tag{2.14}$$

   Central charge has values $c = 0, 1, 3/2, 9/5, ...$ and approaches $c = 3$ for large values of k.

2. The vacuum conformal weights and U(1) charges depend on two integer valued parameters a, b besides k

$$\begin{aligned} h_{ab} &= \frac{a(a+2) - b^2}{4(k+2)} + \frac{(a+b)_2^2}{8} \ \ , \\ Q_{ab} &= \frac{b}{2(k+2)} - \frac{(a+b)_2^2}{4} \ \ . \end{aligned} \tag{2.15}$$

   Here the conditions

$$a = 0, ..., k \ \ , \ \ |b - (a+b)_2| \leq a \ \ , \ \ (a+b)_2 \equiv a + b \mod 2 \tag{2.16}$$

   are satisfied. For Ramond type representations $(a + b)_2 = 1$ ($a + b$ is odd) is satisfied and for N-S type representations $(a + b)_2 = 0$ ($a + b$ is even) is satisfied. Note that $(h, Q) = (0, 0)$ is possible only for $(a, b) = 0$ in the case of $N - S$ representation. For Ramond representation this would give $(h, Q) = (1/8, -1/4)$.

3 Could $\mathcal{N} = 2$ super-conformal algebra be relevant for TGD?

Despite various objections already discussed in the introduction there are good reasons to pose the question of the title.

3.1 How does the ADE picture about SCFTs and criticality emerge in TGD?

The crucial question in TGD framework is how the ADE picture relates to criticality and SCFTs in 2 dimensions. That the SCFT would be defined in 2 dimensions follows from SH.

1. The connection of ADE with inclusions of hyperfinite factors and with the hierarchy of Planck constants defining a hierarchy of dark matters are basic conjectures of TGD.

2. Finite number of degrees of freedom is left when a H_+ sub-algebra of super-symplectic or some other conformal algebra isomorphic to the entire algebra G_+ and the commutator $[H_+, G_+]$ ("+" refers to non-negative conformal weights) annihilate the states. The conjecture is that this gives rise to a finite-dimensional ADE type algebra defining Kac-Moody algebra or gauge algebra whose constant generators however act non-trivially. Denote the resulting finite-D ADE group by A_+. The Kac-Moody algebra might act on fermionic strings whereas the super-symplectic algebra would at at the boundary of CD.

3. At criticality a phase transition changing the value of Planck constant and thus H_+ and A_+ take place. These phase transitions would have a natural description in ZEO: the group ADE group A_+ would be smaller or larger an the other end of space-time surface at the opposite boundary of CD.

4. If the groups $A_{+,i}$ and $A_{+,f}$ satisfy $A_{+,i} \subset A_{+,f}$, new degrees of freedom appear. They correspond to the coset space $A_{+,f}/A_{+,i}$. Coset spaces typically form orbifolds: in fact the term orbi-fold comes from the identification of orbifold as the space of orbits, now those of $A_{+,i}$ in $A_{+,f}$. One would have orbifolds of ADE groups belonging associated with the hierarchy of inclusions labelled perhaps by Planck constants.

5. The orbifolds $O = A_{+,f}/A_{+,i}$ are however orbifolds of ADE groups, which are in 1-1 correspondence with the finite ADE subgroups G of SU(2). Does this mean that the orbifold $O = A_{+,f}/A_{+,i}$ is somehow determined by orbifold $G\backslash SU(2)$? As far as orbifold property is considered, $A_{+,i}$ would be effectively finite-D $G \subset SU(2)$. Mathematician could probably answer this question immediately.

   This kind of reduction of relevant degrees of freedom takes place in catastrophe theory, where only very few degrees of freedom determine the type of catastrophe: also in this case criticality is involved and catastrophes correspond to a hierarchy of criticalities.

6. The hierarchy of Planck constants corresponds to a hierarchy of coverings of space-time surface determined by strong form of holography by those for string world sheets. Could the discrete ADE groups G act in both the fibers and bases of these coverings?

Orbifoldings correspond to pairs of ADE groups appearing in the tensor product of representations. The first guess is that this is due to pairing of Ramond and N-S representations but ADE pairs appear also for conformal minimal models without super-symmetry. Second guess is that the tensor product pairing in TGD framework reflects the fact that one has always a pair of wormhole throats associated with the wormhole contact.

Concluding, it would be very natural to identify the orbifold degrees of in $O = A_{+,f}/A_{+,i}$ primary fields of minimal SCFT. This makes sense if the orbifolding reduces effectively to that for SU(2) by finite discrete subgroup.

3.2 Degrees of freedom and dynamics

$N = 2$SCA or should be generated by the addition of right-handed neutrino or antineutrino to one-fermion state. The interpretation as a pure gauge symmetry seems plausible. Instead of trying to make ad hoc guesses by searching the enormous highly technical literature on the subject, it is better to try to build the physical picture first and hope that professionals could get motivated to perform detailed constructions.

Consider first the degrees of freedom involved.

1. In bosonic sector one has at the fundamental level deformations of string world sheets (possibly of partonic 2-surfaces too). There are also deformations of string world sheets in CP_2 degrees of freedom: the latter could be assigned with electroweak gauge bosons and SU(3) Killing vectors related to color gauge potentials defining representation spaces for Kac-Moody algebras involved. $\mathcal{N} = 2$ SCA should determine correlation functions for these. At higher abstraction level the dynamical variables would correspond to representations of ADE groups assignable to inclusions of HFFs and primary fields would correspond to orbifolds of groups assignable to the hierarchy of Planck constants.

2. In M^4 degrees of freedom there are 2 degrees of freedom orthogonal to string world sheets which correspond to complex coordinate. They would give rise to 2 additional tensor factors to the super Virasoro algebra, which should have 5 tensor factors if p-adic mass calculations are taken at face value. $N = 2$SCA should have this number of tensor factors.

3. There are also fermionic degrees of freedom associated with the induced spinors at string world sheets and they would contribute to SCA too.

What one can say about the dynamics?

1. The dynamics at the level of physical particles would be essentially due to the non-trivial topological vertex in which 3 light-like 3-surfaces join along their ends. This dynamics would have huge symmetry generalizing the duality symmetry of hadronic string models: scattering diagram would be analogous to a computation with vertices having identification as algebraic operations and all computations connecting given sets of objects in initial and final state would be equivalent. This symmetry would allow to move the ends of internal lines so that loops could be transformed to tadpoles and snipped away giving a braided tree diagram as minimal scattering diagram. Something analogous to this happens for twistor Grassmann diagrams.

2. To the lines meeting at vertices defined by partonic 2-surfaces one can assign the fundamental four-fermion vertex [K6] defining second dynamics. This vertex does not however correspond to ordinary fermion vertex involving quartic term in fermion fields but corresponds to redistribution of fermion lines between the 3-legs. Therefore fermion dynamics would be free and this would allow to avoid divergences. The tensor net construction [K6] suggests for a very elegant description of these computations in terms of so called perfect tensors defining the nodes of the net and defining isometries between any leg and its complement with each leg involving unitary braiding operation.

3. The third dynamics would be at the level of Kähler action defined by the functional integral for the exponent of Kähler action. Quantum criticality motivates the proposal is that it is RG invariant in the sense that loop corrections vanish since Kähler coupling strength is analogous to critical temperature and is piecewise constant so that coupling constant evolution is discrete and the values fo α_K are labelled by a subset of p-adic primes.

3.3 Covariantly constant right-handed neutrinos as generators of super-conformal symmetries

As explained in the introduction, holomorphic right-handed neutrinos could generate the super-conformal symmetries with minimal breaking. Also other fermionic spin states (at imbedding base level) would generate super-conformal symmetries but they would be badly broken.

1. At imbedding space level massless modes of right-handed neutrino are covariantly constant in CP_2 and do not mix with left handed neutrinos. On the other hand, *induced* (as opposed to imbedding space -) right-handed neutrino spinors, which are not constant, mix with the left handed neutrino spinor modes and they are physical degrees of freedom. This follows from the mixing of the M^4 and CP_2 contributions to modified gamma matrices determined by the Kähler action and are essentially contractions of canonical momentum currents with imbedding space gamma matrices.

2. Induced spinor modes at string world sheets must carry vanishing weak W and possibly also Z fields to guarantee that em charge is well-defined. SH implies that the data at string world sheets are enough to construct the quantum theory. The assumption about localization is thus natural but not actually necessary, and it is not even clear whether Kähler-Dirac equation is really consistent with the localization at string world sheets although the special properties of Kähler Dirac gamma matrices (in particular, the degenerate character of the effective space-time metric defined by their anti-commutators) suggests this.

3. One must not forget that the conformal structure of solutions is extremely powerful and makes the situation almost independent of the Dirac action used. Dirac equation reduces essentially to holomorphy and to the condition that other half of the modified gamma matrices annihilate the spinor mode. One can therefore ask whether string world sheets could be minimal surfaces and whether Dirac equation in the induced metric could be satisfied at string world sheets. The trace of the second fundamental form giving rise to a term mixing M^4 chiralities vanishes in this case but there is still the mixing of gamma matrices inducing mixing of M^4 chiralities serving as a signal for massivation in M^4 sense.

4. The interpretation of $\mathcal{N} = 2$ supersymmetry possibly generated by right-handed neutrino has remained unresolved. As explained in the introduction, this problem disappears in ZEO since the boundary of CD allows anti-commutators of holomorphic ν_R oscillator operators to be non-vanishing also for constant mode and one obtains constant modes with non-vanishing norm to which space-time $\mathcal{N} = 2$ SUSY can be assigned.

5. A further complication is brought by the recent progress in twistorialization of Kähler action [K6]. It adds to the Kähler action extremely small volume term, and this term could spoil the idea about localization of the modes at string world sheets. Again the conformal structure of the solutions would save the situation if one does not require localization to string world sheets. The picture would be in accordance with SH.

3.4 Is $\mathcal{N} = 2$ SCS possible?

Could one assign $\mathcal{N} = 2$SCA with these degrees of freedom?

1. $\mathcal{N} = 2$SCA can be associated with any Super-Kac Moody algebra defined by simple Lie group by coset construction (see `http://tinyurl.com/yd2zqjvz`), in particular for $CP_2 = SU(3)/SU(2) \times U(1)$. The Kac-Moody algebra defined by the product of color group and electroweak group is not simple, but the fact that electroweak group holonomy group of CP_2 strongly suggests that $\mathcal{N} = 2$ SCA is possible. This would take care of color and electroweak degrees of freedom.

2. There are also 2 degrees of freedom corresponding to M^4 deformations of string world sheet orthogonal to the sheet. Free field construction would assign $\mathcal{N} = 2$ to the degrees of freedom orthogonal to the string world sheet but the central charge is $c = 3 > 3k/(k+2)$ for the unitary $\mathcal{N} = 2$ SCFTs. Personally I do not see any reason why one could not have tensor product of several $\mathcal{N} = 2$SCAs with different central charges.

There are some objections against the idea of understanding the correlation functions of this dynamics in terms of $\mathcal{N} = 2$ SCA.

1. $\mathcal{N} = 2$ SCA is claimed to require (2,2) signature for the metric of the target space in stringy realization: in Minkowskian resp. Euclidian space-time regions the induced metric has signature (1,-1,-1,-1) resp.(-1,-1,-1,-1). To my best understanding the target space is associated with one particular realization so that this objection need not be crucial. Note that also in twistor Grassmann approach (2,2) signature plays also special role making things well-defined whereas in other signature one must apply Wick-rotation.

2. There is also an argument that $\mathcal{N} = 2$ SCFTs reduce to topological QFTs. TGD is indeed almost topological QFT and inside the string world sheets one expects the S-matrix to reduce to braiding S-matrix. The non-triviality of the scattering amplitudes would come from topology: one could assign the points of n-points functions to the ends of different legs of the diagrams.

The minimal models seem however to have the same symmetries as TGD and could therefore give some idea about what might be expected. $h = K/2$ condition for the representations of degenerate representations of $\mathcal{N}=2$ SCA follows if h corresponds to the actual conformal weight of a massless state shifted to zero by redefinition of the scaling generator L_0 by shift $L_0 \to L_0 - h$. In the alternative picture this shift would map vacuum state with vanishing conformal weight to that with negative conformal weight $-h$. If $-h$ is sum over conformal weights $-1/2$ for the "wave functions" at light-cone boundary are proportional to $r_M^{-1/2}$ factor then it must be negative half integer and one has $h = K/2$.

This picture conforms also with the hypothesis that the poles of fermionic zeta determine the conformal weights for the generators of super-conformal symmetry with physical states assumed to satisfy conformal confinement implying that the imaginary parts of generators of SCA remain hidden. Note that the number of generators for the SCAs would be infinite unlike for ordinary SCAs: this would be also due to the fact that symplectic group is infinite-dimensional. Conformal confinement allows how the reduction of the conformal algebra at string world sheets to the ordinary super-conformal algebra. Also thermalization would occur only for this algebra.

For these reasons it is interesting to look what one obtains now by applying $h = K/2$ condition

1. $\mathcal{N}=2$ super-conformal symmetry algebra (see `http://tinyurl.com/yd2zqjvz`) involving besides Virasoro generators also generators for U(1) current and their super-counterparts is a reasonable candidate in TGD framework where classical Kähler current is conserved. The addition of right-handed neutrino or its antiparticle is an excellent candidate for generating exact $\mathcal{N}=2$ space-time supersymmetry as super-gauge symmetry as already explained. The conservation of quark and lepton numbers however allows to consider badly broken conformal SUSY algebra with larger value of $\mathcal{N}$.

2. The infinite-D symplectic algebra replaces the Kac-Moody algebra at light-cone boundary. At the light-like orbits of partons one obtains the counterpart of Kac-Moody algebra associated with the isometries of H and holonomies of CP_2. One might hope that p-adic thermodynamics involving only super-Virasoro generators is not affected at all by these complications. The states of additional algebras would only define the ground states of the Kac-Moody typ Super-Virasoro representations assignable to string world sheets (no thermalization in super-symplectic nor Kac-Moody degrees of freedom would occur), and the quantum numbers in question would correspond to quantum numbers of massless particles with massive excitations having mass scale defined by CP_2 mass scale.

3.5 How to circumvent the signature objection against $\mathcal{N}=2$ SCFT?

As already noticed $\mathcal{N}=2$ SCA is claimed to require (2,2) signature for the metric of the target space in the stringy realization. The problem is that $\mathcal{N}=2$ super-conformal symmetry requires space-time to have complex structure. Could one circumvent this objection?

The first attempt is based on the observation that the notion of Kähler structure generalizes in TGD framework to what I have called Hamilton-Jacobi structure. This means that the complex structure is hybrid of hypercomplex structure in longitudinal tangent space M^2 and of ordinary complex structure in transversal space E^2. The signature poses also problem in the definition of twistor structure and is circumvented using this construction.

The second attempt is based on the twistor lift of Kähler action.

1. Pope et al [B7] (see `http://tinyurl.com/jnon4fh`) propose that one might start from 6-D theory space-time signature (1,1,1-1,1,-1) with $\mathcal{N}=2$ supersymmetry and perform kind of dimensional reduction freezing 2 time coordinates of a 6-D space to obtain $\mathcal{N}=2$ superstrings in the resulting effectively 4-dimensional space-time with signature (1,-1,-1,-1).

2. The twistor lift of TGD replaces space-time surface with its 6-D twistor space. One can choose the metric signature of the sphere S^2 having radius of order Planck constant defining the fiber of twistor space $M^4 \times S^2$ to be (1,1) or (-1,-1). For (1,1) one obtains signature (1,1,1,-1,-1,-1). Dimensional reduction is involved and the analog for the freezing of S^2 time dimensions takes place. This suggests that one could have $\mathcal{N}=2$ symmetry at the level of twistor spaces of space-time surfaces.

3. These two approaches seem to be very closely related in TGD framework.

Third trial would be based on the idea that the signature of the effective metric defined by the anticommutators of the modified gamma matrices appearing in modified Dirac action takes care of the problem by giving signature (1,1,-1,-1) for the effective metric. The following argument does not support this option.

1. In Kähler-Dirac action the modified gamma matrices define effective space-time metric $G^{\alpha\beta}$ via their anticommutators. The physical role of $G^{\alpha\beta}$ has remained obscure. One has $G^{\alpha\beta} = T^{\alpha}_k T^{beta}_l h^{kl}$, where T^{α}_k is the canonical momentum current.

2. There are two contributions to T^{α}_k corresponding to Kähler action and extremely small volume term suggested by the twistor lift of Kähler action having interpretation in terms of cosmological constant. Let us write Kähler action density as $L_K = kJ^{\mu\nu}J_{mu\nu}\sqrt{g}/2$ and volume action density as $L_{vol} = K\sqrt{g}$. One can write T^{α}_k as

$$
\begin{aligned}
T^{\alpha}_k &= [T^{\alpha\beta}[g]g_{k\beta} + T^{\alpha\beta}[J]J_{k,\beta} \quad , \\
g_{k\beta} &= h_{kl}\partial_\beta h^l \quad , \quad J_{k,\beta} = J_{kl}\partial_\beta h^l \quad ,
\end{aligned}
\tag{3.1}
$$

The tensors appearing in this formula can be expressed in a concise notation as

$$
\begin{aligned}
T[g] &= T[K,g] + T[vol,g] \quad , \\
T[K,g] &= \frac{\partial L_K}{\partial g} \equiv k[J \circ J - \frac{1}{4}Tr(J \circ J)\sqrt{g} \equiv T_{K,1} + T_{K,2} \quad , \\
T[vol,g] &= \frac{\partial L_{vol}}{\partial g} = \frac{K}{2}g \quad , \\
T[J] &= \frac{\partial L_K}{\partial J} = kJ\sqrt{g} \quad ,
\end{aligned}
\tag{3.2}
$$

$\circ$ denotes product of tensors defined by contraction. $T^{\alpha\beta}[g]$ is energy momentum tensor and $T^{\alpha\beta}[J] = kJ^{\alpha\beta}$ is its analog coming from variations with respect to induced Kähler form. The following formulas will be used.

$$
g_{k\mu}g^k_{\ \nu} = g_{\mu\nu} \quad , \quad g_{k\mu}J^k_{\ \nu} = J_{\mu\nu} \quad , \quad J_{k\mu}J^k_{\ \nu} = -s_{\mu\nu}
\tag{3.3}
$$

Here s refers to CP_2 metric. G can be written in compact notation as

$$
\begin{aligned}
G &= G[g,g] + G[J,J] + 2G[g,J] \quad , \\
G[g,g] &= T \circ T \quad , \\
G[J,J] &= -T[J] \circ s \circ T[J] = -k^2 J \circ s \circ J \times det(g) \quad , \\
G[g,J] &= T \circ J \circ T[J] = kT \circ J \circ J \times \sqrt{g} = T \circ T_{K,1} \quad .
\end{aligned}
\tag{3.4}
$$

The expression for Gboils down to

$$\begin{aligned} G &= 4T_{K,1}\circ T_{K,1}+4T_{K,1}\circ T_{K,2}+T_{K,2}\circ T_{K,2}\\ &- kJ\circ s\circ J+KT_{K,1}+\frac{kK}{2}T_{1K}\\ &+ \frac{K^2}{4}g \ . \end{aligned} \tag{3.5}$$

The terms are quartic, quadratic, and zeroth order in J. One should disentangle these terms and be able to see whether the signature of G could be (1,1,-1,-1) in the vicinity of string world sheets. I have not been able to identify any obvious mechanism.

3.6 The necessity of Kac-Moody algebra of $SU(2)\times U(1)$

An interesting observation [B8] (see `http://tinyurl.com/hdy66lt`) is that the central charge $c=3k/(k+2)$ emerges by Sugawara construction of the (Super-)Virasoro algebra for $SU(2)$ for (Super-)Kac-Moody algebra with central charge k.

1. In the general case one has following expressions for the central charge c and ground state weight h of the Super Virasoro algebra associated with Super-Kac-Moody algebra

$$\begin{aligned} c &= \frac{kdim(G)}{k+g} \ ,\\ h(\lambda) &= \frac{C(\lambda)}{2(k+g)} \ . \end{aligned} \tag{3.6}$$

   C is Casimir operator in representation λ of G and g is the dual Coxeter number (half of the value of Casimir in fundamental representation).

2. If one accepts these formulas for c and h, the $\mathcal{N}=2$ SUSY fixes Kac-Moody group to be $SU(2)$ or possibly electroweak $SU(2)\times U(1)$ as physical intuition suggests. The value $c=3k_1/(k_1+1)$ requires $k=2k_1$ and $h=K/2$ gives $C(\lambda)=j(j+1)=2K(k_1+1)$.

3. What is the interpretation of $SU(2)$? Electroweak $SU(2)$ operating in fermionic electro-weak spin degrees of freedom is a natural candidate and would require and also allow the inclusion of also U(1) factor naturally identifiable as the U(1) charge of the $\mathcal{N}=2$ SCFT. In fact, the detailed study of Ramond representations show that U(1) factor must contribute to the ground state conformal weight in order to satisfy $h=K/2$ condition.

3.7 $h=K/2$ condition for Ramond representations

The question is whether $h=K/2$ suggested by the conformal invariance for the radial coordinate at light-like boundary can be achieved for these representations. Consider first Ramond type representations.

1. The condition on the allowed values $h=K/2$ of the ground state conformal weight gives

$$\begin{aligned} &h_{ab}=\tfrac{a(a+2)-b^2}{4(k+2)}+\tfrac{1}{8}=\tfrac{K}{2} \ , \quad 0\leq a\leq k \ , \quad b\leq a+1 \ ,\\ &Q_{ab}=\tfrac{b}{2(k+2)}-\tfrac{1}{4} \ . \end{aligned} \tag{3.7}$$

   Also the value of U(1) charge is given.

2. A possible manner to get rid of the problematic 1/8 term is to assume

$$-\frac{b^2}{4(k+2)} + \frac{1}{8} = 0 \tag{3.8}$$

satisfied under the conditions

$$k = 2k_1 \ , \quad b^2 = k_1 + 1 \ . \tag{3.9}$$

This fixes the spectrum of k_1 to values $0, 3, 8, 15, 24, 35, ...$ and non-negative integer b satisfying $|b-1| < a$ determines the value of k_1.

3. As a consequence, one obtains the condition

$$\frac{a(a+2)}{4(k+2)} = \frac{K}{2} \ . \tag{3.10}$$

This condition can be satisfied if one has

$$a = k = K \ . \tag{3.11}$$

Second option $a = k + 2 = K - 2$ does not satisfy the condition $a \leq k$.

4. Altogether one obtains

$$\begin{aligned} & k = 2k_1 \ , \quad k_1 = b^2 - 1 \ , \quad a = k = K \leq k \ , \\ & c = \tfrac{3k_1}{k_1+1} \ , \quad Q = \tfrac{1}{4}(\tfrac{1}{b} - 1) \ . \end{aligned} \tag{3.12}$$

U(1) charge is quantized unless one as $b = 1$ giving $k_1 = 0$ so that one has also $k = 0$. One can ask whether the fractionization of U(1) charge could relate to the charge fractionization possibly related to the hierarchy of Planck constants and/or to the braid statistics. Should one require that physical states have integer charge? Could conformal confinement imply vanishing of ground state U(1) charge automatically? This is is true if complex conjugate conformal weights correspond to opposite U(1) charges.

It is interesting to see whether this picture is consistent with the predictions of the $SU(2) \times U(1)$ Kac-Moody algebra option.

1. Ramond option corresponds naturally to the half-odd integers spin for the Super-Kac-Moody associated with SU(2) as will be found. For physical reasons one can expect that also U(1) tensor factor is present and adds to the vacuum conformal weight. From the general expression of the conformal weight one expects that the term 1/8 is this contribution.

   This would suggests the condition in SU(2) degrees of freedom in terms of half odd integer spin $j = (2r+1)/2$

$$\frac{a(a+2) - b^2}{4(k+2)} = \frac{a(a+2)}{4(k+2)} - \frac{1}{8} = \frac{(2r+1)(2r+3)}{8(k+2)} = \frac{K}{2} - \frac{1}{8} \ . \tag{3.13}$$

This gives the conditions

$$2a(a+2) - k + 2 = (2r+1)(2r+3) \ , \quad \frac{(2r+1)(2r+3)}{k+2} = 4K - 1 \ . \tag{3.14}$$

This condition can be satisfied if $k+2$ divides the numerator - say $(2r+1)$ or $(2r+3)$. The conclusion is that the U(1) factor must be present, which in turn supports the interpretation in terms of gauge group of electroweak interactions and extended holonomy group of CP_2 needed to obtain respectable spinor structure.

3.8 $h = K/2$ condition for N-S type representations

One can look the situation also for the N-S type representations. In this case one expects that spin is even. It is rather clear that the interpretation is in terms of sfermions is not correct. Spin for N-S states is even, which encourages the interpretation as bosonic states involving fermion and antifermion at same or opposite throats of wormhole contact.

1. The values of ground state conformal weight and U(1) charge are assumed to be given by

$$\begin{aligned} h_{ab} &= \frac{a(a+2)-b^2}{4(k+2)} = \frac{K}{4} \ , \\ Q_{ab} &= \frac{b}{2(k+2)} \ . \end{aligned} \tag{3.15}$$

2. In the case of SU(2) Kac-Moody algebra one would have $h_{ab} = j(j+1)/2(k+2)$, which would give

$$a(a+2) - b^2 = 2j(j+1) \ , \quad \frac{j(j+1)}{k+2} = K \ . \tag{3.16}$$

   Two solutions of the latter equation are

   - $j = k+2$ giving $k = K-3$ and $j = K-1$
   - $j+1 = k+2$ given $k = K-1$ and $j = K$.

   The values of $j$are integers as expected.

3. The condition $a(a+2) - b^2 = j(j+1)$ gives a further number theoretic constraint. Special solutions are $a = j-1, b = 0$ and $a = j = b^2$.

To sum up, $\mathcal{N} = 2$ superconformal theories provide an attractive approach in attempts to gain a more detailed understanding of the super-conformal invariance at string world sheets. The fermionic n-point functions as restricted to string world sheets in turn could correspond to n-point functions for a CFT assignable to partonic 2-surfaces and one should understand the relationship between these two CFTs. More generally, strong form of holography allows to except CFT description for both the spin and orbital degrees of freedom of WCW and one should understand their relationship. It must be however emphasized that the actual SCA in TGD corresponds to the number $\mathcal{N} = \forall$ of spin states for H-spinors. The corresponding space-time SUSY is expected to be badly broken.

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