Some considerations relating to the dynamics of quasicrystals

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Contents

1 The dynamics of quasicrystals, the dynamics of Kähler action, symbolic dynamics, and the dynamics of self-organization

The dynamics of quasicrystals looks to me very interesting because it shares several features of the dynamics of Kähler action defining the basic variational principle of classical TGD and defining the dynamics of space-time surfaces. In the following I will compare the basic features of the dynamics of quasicrystals to the dynamics of preferred extremals of Kähler action [K1].

Magnetic body carrying dark matter is the fundamental intentional agent in TGD inspired quantum biology and the cautious proposal is that magnetic flux sheets could define the grid of 3-planes (or more general 3-surfaces) defining quasi-periodic background fields favoring 4-D quasicrystals or more general structures in TGD Universe. Also 3-D quasicrystal like structures defined by grids of planes can be considered and 4-D quasicrystal structure could represent their time evolution.

Quite recently it has been reported that grids consisting of 2-D curved orthogonal surfaces characterize the architecture of neural wiring so that this hypothesis might make sense. This structure would be analogous to 2-D quasicrystal and its time evolution to 3-D quasicrystal.

1.1 The non-determinism for the dynamics of quasicrystals contra non-determinism of Kähler action

The dynamics of quasicrystals is non-deterministic in the sense that one cannot construct a unique quasicrystal by starting from a finite portion or even D-1-dimensional section of D-dimension quasicrystal thickened to a slice. Four-dimensional quasicrystals would therefore define a non-deterministic dynamics. This dynamics could serve as a geometric correlate for a full non-deterministic quantum dynamics involving also state function reductions. This requires that quantum classical correspondence is generalized so that also non-deterministic aspects of quantum dynamics are required to have geometric space-time correlates. The global empires of the 4-D quasicrystal could be interpreted as self-organization patterns whereas global empires would represent long range correlations.

This is very much analogous to 4-D spin glass degeneracy in TGD framework.

1. In TGD framework the preferred extremals of so called Kähler action define the dynamics of space-time surfaces. Kähler action [K4] is Maxwell action for the gauge field induced from the Kähler form of CP_2. Symplectic transformations of CP_2 act as abelian gauge transformations and therefore leave the induced Kähler form invariant. They do not however leave the induced metric invariant so that the action changes by a contribution assignable to classical gravitation. For vacuum extremals however the symplectic transformations act as symmetries.

2. This implies huge vacuum degeneracy. Every space-time surface for which CP_2 projection is Lagrangian manifold and thus having at most 2-D CP_2 projection has vanishing induced Kähler form and is therefore vacuum extremal: there is infinite number of 6-D vacuum sectors labelled by Lagrangian sub-manifolds of CP_2 transformed to each other by symplectic transformations. These vacuum extremals behave non-deterministically which means an analogy with quasicrystal dynamics and suggests that quasicrystals might define a simplified model for quantal self-organization.

3. Small deformations of these define non-vacuum extremals and It is very conceivable that part of the vacuum degeneracy remains and is manifested as multi-furcations. The number n of branches for a multi-furcation has interpretation in terms of effective Planck constant $\hbar_{eff} = n\hbar$ to which dark matter is assigned in TGD framework. This degeneracy is very much analogous to a 4-dimensional spin glass degeneracy meaning that space-time decomposes to deterministically behaving regions just like spin glass decomposes to magnetized regions with varying direction of magnetization.

4. The interpretation for the situation in TGD framework is in terms of quantum classical correspondence: not only quantum states correspond to space-time geometries as analogs of Bohr

orbits but also quantum jump sequences - which according to TGD inspired theory of consciousness define the contents of consciousness - have non-deterministic space-time geometries as geometric correlates. Space-time geometry and topology are like written text providing information about contents of consciousness.

5. Also p-adic topology as effective topology of space-time surfaces and natural topology for the landscape of extrema of Kähler function of WCW definings its Kähler geometry emerges naturally from this degeneracy. In physics obeying effective p-adic topology the counterpart would be short range chaos with long range correlations in the sense that one would have periodicity in the sense that physical states at time time at and $t + kp^n$, $k = 0, 1, ...p - 1$, $n \geq\geq 1$, would be very near to each other. The interpretation in terms of intentional action would be natural.

   One could also imagine of defining the analogs of empires as connected deterministic regions of space-time surface and the analogs of empires would be unions of disconnected components perhaps understandable in terms of p-adicity. Self-organization patterns would naturally correspond to these regions. Many-sheeted space-time would imply fractal hierarchy of self-organization patterns within self-organization patterns.

1.2 The dynamics of quasicrystals as a model for fundamental dynamics or high level symbolic dynamics?

Stephen Wolfram has suggested that cellular automatons could define the fundamental dynamics. It is not difficult to invent grave objections against this view. One of the objections is that this kind of dynamics is based on simple and rather ad hoc rules and applies to a society rather than to elementary particles. It is difficult to circumvent this counter argument.

One can however ask in what scale the symbolic dynamics does emerge? For few years ago my answer would have been "in biological length scales" (genetic code as symbolic dynamics). TGD Universe is however fractal, and this forces to ask whether this symbolic dynamics emerges already above CP_2 scale in some rudimentary form. In any case, even in this case the dynamics of self-organization would not be identifiable as the fundamental dynamics but as analogous to the rules of behavior in society.

The dynamics of quasicrystals brings indeed strongly in mind the dynamics of self-organization patterns prevailing at relatively high level of dynamical hierarchy. Symbolic dynamics prevailing at the level of biomolecules (genetic code) and at higher levels could be in question. This dynamics is dynamics for a society of conscious entities, which can decide whether to follow the rules or not. Rules as such do not matter too much: what is important that they make possible to predict the behavior of individuals and therefore make possible co-operation and formation of coherent and synchronous large scale structures making possible collective consciousness. In human society moral rules, laws, traffic rules, grammatical rules of language, etc... are examples about symbolic dynamics having very little to do with the laws of physics at fundamental level.

A natural question is whether the rules for building quasicrystals could provide a simplified model for this "social" dynamics - or perhaps even semi-realistic description - at the molecular level? Either quasicrystals or their building bricks - the arguments to be discussed later suggest that finite-sized quasicrystals - could be seen as a kind of society. The refusal to obey the rules guaranteeing the formation of larger quasicrystals would stop the quasicrystal growth and isolate the individual quasicrystal from the society. It could also lead to metabolic starvation: metabolic energy feed is indeed crucial element in living systems.

Quasicrystals could be seen as idealized structures having maximal complexity and therefore ability to represent information. Critical systems - also quantum critical ones - have a universal dynamics so that there is a large number of models making the same predictions for a given system. In practice this can be used to find the simplest possible model to simplify the mathematical description (say by finding the simplest conformal field theory to describe a 2-D critical system). From this point of view quasicrystals could be seen as an especially simple model possibly able to catch the universal properties of a real world system.

Does this self-organization dynamics then emerge only at and above bio-molecular scales or in all scales?

1. In TGD framework the classical dynamics at the fundamental level would be the geometrodynamics of space-time surfaces [K7]. Quantum Dynamics would be dictated by Dirac equation for WCW ("world of classical worlds") spinor fields and reduce to the modified Dirac equation for second quantized induced spinor fields at space-time surfaces.

   The fractality of the TGD Universe suggests that self-organization occurs in all length scales above CP_2 scale, which is about 10^4 times Planck scale. If so, structures analogous to finite pieces quasicrystals could appear in all scales down to CP_2 scale.

2. I have proposed a method for constructing preferred extremals of Kähler action [K8] and this recipe leads to an iteration procedure. Quite generally, iteration is known to lead to fractals as fixed sets of iteration. Therefore space-time surfaces could be seen as space-time correlates of self-organization patterns and fractals.

3. Fractality would mean that even inanimate matter should share some aspects assigned to living matter and that also systems like species and biosphere could behave like living organisms in some respects. Sheldrake has proposed is famous for his notion of memory at the level of entire species. He has also proposed that even inanimate systems could have "habits". For instance, minerals would have adopted the habit to crystallize to a particular crystal form. In this framework living matter would differ from mineral kingdom in that its habits would be much more flexible. I have discussed the implementation of Sheldrake's ideas in TGD framework [L1].

1.3 Could ordered water layers around biomolecules be modelled as quasicrystal like structure?

Water forms multilayered quasi-lattices (to be distinguished from quasicrystals!). These quasi-lattices around molecules are like ice coverings. These quasilattices have water molecule as a basic tedrahedral building blocks giving rise to icosahedral blocks (as suggested in discussions): the 4 electron pairs of water molecule are indeed located at the vertices of tedrahedron and for lattice like structures a regular tedrahedron is in question. Perhaps these quasi-lattices could be modelled as deformed quasicrystals.

This molecular ice would form a quasicrystal, which could somehow store information about environment via its structural degeneracy. If the information is conscious, it should be stored in the negentropic entanglement between the states of finite-sized quasi-lattices surrounding two separate molecules, and would have magnetic flux sheets connecting them as a space-time correlate. For 3-D quasicrystal like structure the lattice points would be in the intersections of 3 2-planes (or thin locally planar flux sheets) and define points of a lattice at which the analogs of coordinate planes meet.

Making these structures dynamical one would obtain 4-D quasicrystal like structures. In this case the intersections of 2 3-planes (or thin locally planar 3-D flux sheets) would give rise to 1-dimensional word lines of 3-D quasicrystal points whereas the intersections of 3 3-planes would correspond to points of 4-D lattice. What special could happens at these dynamically special points of space-time?

Zero energy ontology and TGD inspired theory of consciousness allows to consider a possible answer: the points of 4-D lattice correspond to CDs (causal diamonds) serving as space-time correlates for sub-selves identifiable as mental images. Quite generally quasi-periodically appearing mental images might be assigned to the points of quasi-lattice like structure.

Note that also cubic crystals can be constructed using grid consisting now of 3 orthogonal planes and the distances between grid planes serve as geometrical parametersm which magnetic body could vary. The constant deformation of the magnetic body would now however force rather large deformations of crystal structure probably impossible energetically. One can however ask whether phonons could be induced by the local deformations of the flux sheets of the grid inducing small oscillations of the lattice points. If the magnetic body indeed serves as the intentional agent using biological body, this connection might allow to understand the very special role of acoustic oscillations in hearing, speech, internal speech, and thought. For instance, could the reaction of magnetic flux sheets to sound give rise to hearing? And could the reaction of the quasilattice units to the oscillations of the flux sheets give rise to internal speech or induce even speech in sound organs? It has been argued that the structure of the intronic portion of DNA resembles that of language and this has led to proposal that acoustic waves propagating along DNA could code for language. If DNA is indeed accompanied by flux tubes and flux sheets, this idea would look rather natural in the recent context.

One of the basic findings of biology is that protein molecules are most of the time in a resting state in a folded configuration with globular form and surrounded by ordered water defining kind of ice covering. This state could represent conscious information realized as a negentropic entanglement between different molecules: kind of a molecular meditative state would be in question. In the presence of energy feed inducing "molecular summer" the molecular ice would melt, globular proteins would open and self-organize to form molecular aggregates as a reaction to the energy feed. After the energy feed stops, molecules would fold back to the globular form but the memory from the "molecular summer" would be stored to the negentropic entanglement between molecules.

The Indra's net formed by the magnetic flux tubes and sheets has become a standard part of TGD based view about living matter. The model for DNA as quantum computer [K3] involves flux sheets traversing through DNA strand and flux tubes connecting nucleotides to lipids of the cell membrane as well as flux sheets with the shape of cell membrane. This suggests that one actually has also in the case of DNA-cell membrane system three orthogonal grids of flux sheets at some scale and flux tubes condensed at the sheets of grids. These structure would organized the living matter to a well-organized geometric structure.

One of the first really crazy ideas related to the magnetic was the proposal that the magnetic bodies associated with living organisms could have shape reflecting the shape of the organism and its parts - even in the length scale of Earth [K5]. If one takes the flux sheets grids seriously, and replaces planes with closed surfaces obtained by scaling outer surfaces for parts of the organism, something like this is indeed expected.

2 What could be the variational principle behind self-organization?

Quasicrystals (say Penrose tilings) have a huge ground state degeneracy: given region of quasicrystal can be completed to infinite number of quasicrystals. For crystals the situation is different: local empire is the entire infinite crystal. Quasicrystals are clearly analogous to spin glass systems also possessing also large ground state degeneracy.

TGD Universe is a 4-D spin glass, and this degeneracy would imply non-determinism analogous to the non-determinism of quasi-crystal dynamics in 4-D 4-D Minkowski space) with local empires interpreted as self-organization patterns and global empires reflecting the long range correlations due to intentional action and obedience for social rules. In human society the ability to predict what person probably does next year in given day only by knowing his profession, would represent example about this kind of long range correlation caused basically by social forces.

2.1 Why Negentropy Maximization Principle should favor quasicrystals?

In TGD inspired theory of consciousness Negentropy Maximization Principle (NMP) [K6] is the basic variational principle. NMP states that the information contents of conscious experience is maximal. Therefore entanglement negentropy is expected to be the fundamental quantity.

1. Since conscious entities forming larger coherent structures (societies) are in question, it seems that one should characterize the quasi-lattice by a negentropy, which should be maximized (purely mathematically negentropy is very similar to entropy which is maximized for a closed system). This negentropy would *not* correspond to the negative of the ordinary thermodynamical entropy, which characterizes ensemble of particles rather than single coherent unit.

2. In TGD Universe this negentropy would naturally be the number theoretic negentropy characterizing negentropic entanglement identified as a measure for conscious information. This information measure is assigned with the magnetic flux tubes connecting biomolecules and other units of living organism and even living organisms to larger coherent structures. In the case of quasicrystals flux tubes or flux sheets give rise to the long range constraints binding the units of quasi-crystal to each other.

3. The maximization of negentropy characterizing information content of conscious experience should be equivalent with the maximization of complexity as the number of almost degenerate ground states of quasicrystal. It is intuitively clear why quasicrystals would be favored over

crystals. But how quasicrystals could maximize entanglement negentropy? Why the entanglement negentropy would be large for quasicrystals? Why the large number of quasicrystals configurations would favor large entangelment negentropy.

If the entanglement is between two different quasicrystals, it means formation of quantum superposition of pairs of quasicrystal configurations and the higher the quasicrystal degeneracy, the larger the maximal entanglement negentropy. This conforms with the fact that quasicrystals are necessary of finite size. Most naturally the negentropic entanglement would be between the degenerate ground states of two finite-sized quasicrystals.

4. If degrees of freedom associated with the space-time geometry are entangled, the quantum dynamics at the level of "world of classical worlds" would be involved and by definition would not be describable by QFT in a fixed background space-time. One could speak about genuine quantum gravity: The Orch-OR proposal of Penrose and Hameroff is also a conjecture of similar character. One can also consider entanglement between states of some particle and quasicrystal but the negentropy content would be now much smaller due to the small number of particle states.

2.2 Maximal capacity to represent information with minimal metabolic energy costs as a basic variational principle?

The interpretation as a symbolic dynamics assignable to conscious entities would suggest that the maximization of the capacity to represent information (perhaps with minimal metabolic costs) could be the variational principle behind this dynamics. The number of different quasicrystals formed using the given rules should be maximal. This would give rise to very large number of states with nearly same energy allowing to represent the states of the external world (primitive sensory system). The larger the size of quasicrystal, the larger the number of degenerate configurations. Here of course physical constraints would pose an upper limit of the size.

But can one really assume rigid rules of construction giving rise to only quasicrystals? If the basic dynamical units are conscious entities they refuse to obey strict rules although they can decide to do so under "social pressures" (absence of metabolic energy feed can transform a sinner a saint!). Should these rules be an outcome of the variational principle alone? Or are they forced by some minimization principle - say minimization of metabolic energy feed - in presence of quasi-periodic background field configuration regarded as an external field favoring quasicrystals?

It seems that all configurations of the basic units must be accepted a priori: in principle even random spatial configurations of the basic units. For random configurations complexity would be maximal but co-operation minimal, long range correlations would be absent, and the ability to represent information would be minimal. For crystals long range correlations and co-operation would be maximal but crystal would have minimal capacity to represent and mimic. The natural manner to achieve long range correlations is to assume slowly varying quasi-periodic fields configurations representing the "social forces". In TGD framework these fields would naturally correspond to magnetic flux quanta serving as basic building bricks of magnetic bodies controlling biological body.

Note that by previous argument, the capacity to store conscious information is equivalent with ability to generate negentropic entanglement.

2.3 A possible realization for 4-D dynamics favoring quasicrystal like structures

Can one imagine a physical realization of 4-D quasicrystal dynamics in TGD framework? The basic problem is to understand how the rules for the formation of quasicrystals are forced. Certainly the hyper-plane grids associated with the basic polytope defining the quasicrystal force the long range correlations. But how to realize these grids physically?

1. In TGD Universe magnetic body acts as an intentional agent using biological body as a motor organ and sensory receptor. This suggests that the plane grids parallel to the faces of - say - icosahedron in the case of 3-D quasicrystal could in TGD Universe be realized as thin (and thus effectively 2-D) magnetic flux sheets forming the magnetic body around which the ordinary matter would self-organize to form a quasicrystal as a configuration sustainable by using

minimum metabolic energy feed. These grids would be part of the magnetic body responsible for the "social forces".

Rather remarkably, quite recent findings strongly suggest that brain involves an orthogonal grid of curved planes. Maybe this grid correspond to a quasi-lattice associated with a cubic basic unit serving as a basic information processing unit. Exact cubic crystal does not guarantee the needed ground state degeneracy and the deviation from it could be crucial in guaranteing large degeneracy of the basic structures.

2. Maybe the basic variational principle could be minimization of the metabolic energy feed in presence of fixed grid structure formed by flux sheets representing the slow dynamics to which the molecular dynamics would rapidly adapt. The intersections of the grid hyper-planes are good candidates for the equilibrium points and going outside them would require metabolic energy. The minimum of the magnetic energy $\mu \cdot B$ of magnetic dipole is reduced in the intersections of flux sheets if the effects of the magnetic fields sum up at the intersection. For the crossing of n orthogonal sheets there is an enhancement by $\sqrt{n}$ factor. The motor activities of the magnetic body itself would deform the quasicrystals: the flux sheets could be deformed and the distances between the flux sheets could also vary. This would lead to new quasicrystal configurations with high negentropic content.

   From the point of view of individual quasicrystal regarded as conscious entity fight for survival would be fighting for metabolic resources and fusion with a bigger quasicrystal could be one manner to guarantee the availability of metabolic energy.

3. Phason dynamics seems to allow both short range description in terms of permutations of basic units and long range hydrodynamical description. Two dynamics seem to be present: slow *resp.* fast dynamics in short *resp.* long scales. Maybe these paradoxical properties of phasons could be understood in this framework if the microscopic fast dynamics forced by the slow long length scale dynamics of flux sheets.

4. Also other than quasicrystal configurations would be possible but would require higher metabolic energy feed to preserve entanglement negentropy (amount of conscious information). In 4-D case one would have similar grids of thin and effectively 3-D magnetic flux sheets associated with the 3-D faces (maybe icosahedrons) of 4-D building brick of quasicrystals. Magnetic flux sheets would carry dark matter and give rise to negentropic entanglement between the units of the quasicrystal.

5. The negentropic entanglement between two different quasi-crystal like structures means quantum superposition of different space-time surfaces since the grids formed by the flux sheets would have different geometric parameters such as the distance between the flux sheets of the grid. Hence genuine quantum gravitational effects would be in question having no description in QFT framework and requiring description at the level of WCW.

2.4 Summary

The essential element of picture would be spin glass degeneracy giving a large number of ground states making possible highly negentropic entanglement betweemn separate spin glasses. Quasicrystals are not the only manner to satisfy this condition and 4-D quasi-lattices for which grid could contain also more general 3-surfaces than hyperplanes, can be considered. Grids of thin 3-surfaces would represent the rules forcing the quasi-lattice like configurations through localization to the wordlines defined by intersections of two 3-surfaces. TGD inspired quantum model for biology suggests concrete models for the grids of flux sheets involving also flux tubes topologically condensed on them as a manner to generate negentropic entanglement. Also fractal structures consisting of flux quanta inside flux quanta are highly suggestive.

The basic variational principle of quasicrystal dynamics (and its generalization to quasi-lattices) could be minimization of metabolic energy feed in presence of fixed configuration of the magnetic body obeying a relatively slow dynamics. The time scale of EEG is in the range .01-.1 seconds gives a first guess for the time scale of the dynamics of the magnetic body in scale of Earth. This time scale is to be compared to the time scale of 10^{-10} seconds of conformational dynamics bio-molecules.

Quasi-crystallization - or more generally, formation quasi-latticices - would be due to the existence of grids of thin 3-sheets parallel to the basic units of the 3-faces of 4-D basic unit of quasicrystal.

To show that this picture makes or does not make sense, one should be able to estimate reliably the metabolic energy feed needed to preserve a given negentropic entanglement entropy for a given configuration of the basic units (say clusters of water molecules) and to show that it is minimized for quasicrystal configurations in presence of the grid structure formed by flux sheets. This is probably relatively easy since the first guess for the equilibrium configurations corresponds to the highly symmetric crossing lines of for two 3-planes. One might also try to demonstrate the presence of negentropic entanglement between molecules, which are in resting state. This would be a direct demonstration for the notion of WCW and for non-trivial quantum gravity effects in living matter.

3 Could quasi-lattices and quasi-crystals emerge from the notion of p-adic manifold?

This section is inspired by the considerations of the new chapter "What p-adic icosahedron could mean? And what about p-adic manifold?" [K9]. The original purpose was to understand what the notion of p-adic icosahedron could mean but soon it turned out that the key challenge is to understand what p-adic manifold means. Also in TGD framework this is one of the basic challenges posed by the condition of number theoretical universality and the idea about algebraic continuation of physics between different number fields.

The basic problem is that p-adic topology is totally disconnected meaning that p-adic balls are either disjoint or nested so that the usual construction of manifold structure fails. The basic criticism against the notion of p-adic icosahedron, and more generally, the notion of p-adic manifold, is the technical complexity of the existing constructions by mathematicians.

TGD however suggests much simpler construction. The construction relies on a simple modification of the notion of manifold inspired by the interpretation of p-adic preferred extremals defining counterparts of real preferred extremals as cognitive representations of the latter. This requires a mapping from p-adic preferred extremals to real ones and vice versa. In manifold theory chart maps are the analogs of these maps and the only difference is that they are between different number fields.

What I have christened as canonical identification $I_{k,l}^Q$ mapping rationals $p^{rk}m/n$ with $|m|_p > p^{-k}$, $|n|_p > p^{-k}$, as $I_{k,l}^Q(p^{rk}(m/n)) = p^{-rk}I_{k,l}(m)/I_{k,l}(n)$, where $I_{k,l}(m = \sum m_n p^{nk}) = sum_{n<l} m_n p^{-nk}$ defines canonical identification for p-adic numbers m, n satisfying the above conditions in their pinary expansion with two cutoffs k and l. $I_{k,l}^Q$ is ill defined for irrational p-adic numbers since for them the representation as rational is not unique. A generalization to algebraic extensions is straightforward.

$I_{k,l}^Q$ is a compromise between the direct identification along common rationals favored by algebra and symmetries but being totally discontinuous without the cutoff $n < l$. This cutoff breaks symmetries slightly but guarantees continuity in finite measurement resolution defined by the pinary cutoff l. Symmetry breaking can be made arbitrarily small and has interpretation in terms of finite measurement resolution. Due to the pinary cutoff the chart map applied to various p-adic coordinates takes discrete set of rationals to discrete set of rationals and preferred extremal property can be used to make a completion to a real space-time surface. Uniqueness is achieved only in finite measurement resolution and is indeed just what is needed. Also general coordinate invariance is broken in finite measurement resolution. In TGD framework it is however possible to find preferred coordinates in order to minimize this symmetry breaking.

3.1 TGD based view about p-adic manifolds

The construction of p-adic manifold topology somehow overcoming the difficulty posed by the fact that p-adic balls are either disjoint or nested is necessary. It should also allow a close relationship between p-adic and real preferred extremals. It will be found that TGD leads naturally to a proposal of p-adic manifold topology [K9] based on canonical identification used to map the predictions of p-adic mass calculations to real numbers. This map would define coordinate charts for p-adic space-time surfaces - not as p-adic chart leafs as in the standard approach - but as real chart leafs. The real topology induced from real map leafs to the p-adic realm would be path-connected as required.

In TGD framework one must also require finite measurement resolution meaning that the canonical identification is characterized by pinary cutoff takes a discrete subset of rational points of p-adic preferred extremal to its real counterpart: for a subset of this subset rationals are mapped to themselves. One can complete this point set to a real preferred extremal in finite measurement resolution. This construction allows also to define p-adic integrals and differential forms in terms of their real counterparts by algebraic continuation. Therefore geometric notions like distance and volume make sense and there is a very close correspondence between real space-time geometries and their p-adic counterpart in the situations when they exist.

3.2 Can one consider a p-adic generalization of Penrose tiling and quasicrystals?

The mathematically rigorous generalization of Penrose Tilings and quasicrystals to p-adic context might be possible but is bound to be rather technical. The p-adic icosahedron as it is defined in the article does not seem very promising notion. The point is that it is defined in terms of fixed point set for subgroups of icosahedral group acting on Riemann sphere: the action in Euclidian 3-space is now more natural and certainly makes sense and actually simplifies the situation since Q_p^3 sd analog of E^3 is simplest possible 3-D p-adic manifold. It does not however allow Bruhat-Tits tree since the points of Q_p^n are not in 1-1 correspondence with the lattices of Q_p^n. The possibility to construct Bruhat-Tits tree is a special feature of projective spaces.

TGD based view about p-adic E^3 and S^2 as its sub-manifold allows to define also the counterpart of Penrose tiling and QCs in an elegant manner with a close relationship between real and p-adic variants of QC.

1. If one considers lattices in n-dimensional p-adic space Q_p^n replacing E^n, a more natural definition would be in terms of this space than in terms of sphere. For the counterpart of E^3 one can define the action of the subgroup A_5 of rotation group $SO(3)$ by introducing an algebraic extension of the p-adic numbers containing $cos(2\pi/5)$, $sin(2\pi/5)$ and $cos(2\pi/3)$, $sin(2\pi/3)$ and their products. What is interesting is that algebraic extension is forced automatically in p-adic context! In cut and project method [A2] the QC structure requires also this since the imbedded space has an algebraic dimension over integers equal to the dimension of the imbedding space over reals.

   Could it be that p-adic variants of QCs might provide number theoretic insights about QCs? Subspace would define algebraic extension of p-adic numbers and this extension would be such that it allows the representation of the isometry group of the Platonic solid possibly assignable to the QC.

2. One can also now define the icosahedron or any Platonic solid in terms of fixed points also now. Only discrete subgroups of the rotation group can be represented p-adically since algebraic extension is required. This brings in mind the notion of finite measurement resolution leading to a discretization of p-adically representable rotations and more general symmetries. For instance, without algebraic extension only rotations for which the rotation matrices are rational numbers are representable. It seems that finite subgroups of this kind are generated by rotations with rotation angle $\pi/2$ around various coordinate axes. Pythagorean triangles correspond to rationals values of cosine and sine and rotations for which rotation angle corresponds to Pythagorean angle define rational rotation matrices: these groups are discrete but contain infinite number of elements.

   Altogether this suggests a hierarchy of p-adic extensions leading to higher algebraic dimensions and larger discrete symmetries. This conforms with the general number theoretic vision about TGD.

3. Lattices in Q_p^n with integer coefficients make also sense and are characterized by n linearly independent (over p-adic integers) basic vectors $(a_1, ..., a_n)$. Most points of lattice would correspond to values of p-adic integers n_i in $\sum_i n_i a_i$ infinite as real numbers.

   Consider first a non-realistic option in which p-adic integers are mapped to p-adic integers as such. Note also that most of p-adic lattice points would map to real infinity. This kind of correspondence makes sense also for rationals but would give a totally discontinuous correpondence between reals and p-adics.

p-Adic manifold topology defined in terms of the canonical identification I_{kl} allows to interpret the p-adic lattice as a cognitive representation of the real one. The presence of pinary cutoffs k and l having interpretation in terms of finite cognitive resolution has two implications. Integers $n_i < p^k$ are mapped to themselves so that this portion of lattice is mapped to itself faitfully. The integers $k \leq n < l$ are not mapped to integers and the length of the image is bounded below. The real image of the p-adic lattice under I_{kl} is necessary compressed to a finite volume of E^3. This kind of compression and cutoff is natural for cognitive representations for which numerics with finite cutoff provides one particular analogy.

4. Could the notion of p-adic QC and Penrose tiling make sense if one considers p-adic counterparts of Euclidian space and a n-D cubic lattice with integer valued coefficients and spanned by unit vectors? Could the cut and project method generalize [A2]?

   This is not clear since projection would lead from a lattice in Q_p^n to a QC in lower-dimensional space which is associated with algebraic extension of Q_p but having algebraic dimension equal to n. If this space is K^m, K an algebraic extension of Q_p, one has $n = dim(K) \times m$. For prime values of n this would mean that $m = 1$ and one has n-D algebraic extension.

   Projection should be generalized to a map mapping points of n-D space to m-dimensional subspace K^m associated with algebraic extension of Q_p. Maybe it is better to formally extend Q_p^n to K^n and restrict the lattice to integer lattice in $Q_p^n \subset K^n$. In this manner the projection becomes well-defined as map from $Q_p^n \subset K^n$ to a subspace K^m of K^n. The basic condition could be that the points of the subspace K^m in K^n with algebraic dimension $n \times dim(K)$ define and m-dimensional subspace over K and n-dimensional subspace of Z_p.

   The "irrational angles" associated with the lower-dimensional subspace defining quasilattice defining algebraic extension of Q_p should be such that it allows the representation of the isometry group of the p-adic Platonic solid possibly assignable to the QC in question.

3.3 Cut and project construction of quasicrystals from TGD point of view

Cut and project [?] method is used to construct quasicrystals (QCs) in sub-spaces of a higher-dimensional linear space containing an ordinary space filling lattice, say cubic lattice. For instance, 2-D Penrose tiling is obtained as a projection of part of 5-D cubic lattice - known as Voronyi cell - around 2-D sub-space imbedded in five-dimensional space. The orientation of the 2-D sub-space must be chosen properly to get Penrose tiling. The nice feature of the construction is that it gives the entire 2-D QC. Using local matching rules the construction typically stops.

3.3.1 Sub-manifold gravity and generalization of cut and project method

The representation of space-time surfaces as sub-manifolds of 8-D $H = M^4 \times CP_2$ can be seen as a generalization of cut and project method.

1. The space-time surface is not anymore a linear 4-D sub-space as it would be in cut and project method but becomes curved and can have arbitrary topology. The imbedding space ceases to be linear $M^8 = M^4 \times E^4$ since E^4 is compactified to CP_2. Space-time surface is not a lattice but continuum.

2. The induction procedure geometrizing metric and gauge fields is nothing but projection for H metric and spinor connection at the continuum limit. Killing vectors for CP_2 isometries can be identified as classical gluon fields. The projections of the gamma matrices of H define induced gamma matrices at space-time surface. The spinors of H contain additional components allowing interpretation in terms of electroweak spin and hyper-charge.

3.3.2 Finite measurement resolution and construction of p-adic counterparts of preferred extremals forces "cut and project" via discretization

In finite measurement resolution realized as discretization by finite pinary cutoff one can expect to obtain the analog of cut and project since 8-D imbedding space is replaced with a lattice structure.

1. The p-adic/real manifold structure for space-time is induced from that for H so that the construction of p-adic manifold reduces to that for H.

2. The definition of the manifold structure for H in number theoretically universal manner requires for H discretization in terms of rational points in some finite region of M^4. Pinary cutoffs- two of them - imply that the manifold structures are parametrized by these cutoffs charactering measurement resolution. Second cutoff means that the lattice structure is piece of an infinite lattice. First cutoff means that only part of this piece is a direct imagine of real/p-adic lattice on p-adic/real side obtained by identifying common rationals (now integers) of real and p-adic number fields. The mapping of this kind lattice from real/p-adic side to p-adic/real side defines the discrete coordinate chart and the completion of this discrete structure to a preferred extremal gives a smooth space-time surface also in p-adic side if it is known on real side (and vice versa).

3. Cubic lattice structures with integer points are of course the simplest ones for the purposes of discretization and the most natural choice for M^4. For CP_2 the lattice is completely analogous to the finite lattices at sphere defined by orbits of discrete subgroups of rotation group and the analogs of Platonic solids emerge. Probably some mathematician has listed the Platonic solids in CP_2.

4. The important point is that this lattice like structure is defined at the level of the 8-D imbedding space rather than in space-time and the lattice structure at space-time level contains those points of the 8-D lattice like structure, which belong to the space-time surface. Finite measurement resolution suggests that all points of lattice, whose distance from space-time surface is below the measurement resolution for distance are projected to the space-time surface. Since space-time surface is curved, the lattice like structure at space-time level obtained by projection is more general than QC.

The lattice like structure results as a manifestation of finite measurement resolution both at real and p-adic sides and can be formally interpreted in terms of a generalization of cut and project but for a curved space-time surface rather than 4-D linear space, and for H rather than 8-D Minkowski space. It is of course far from clear whether one can obtain anything looking like say 3-D or 4-D version of Penrose tiling.

1. The size scale of CP_2 is so small (10^4 Planck lengths) that space-time surfaces with 4-D M^4 projection look like M^4 in an excellent first approximation and using M^4 coordinates the projected lattice looks like cubic lattice in M^4 except that the distances between points are not quite the M^4 distances but scaled by an amount determined by the difference between induced metric and M^4 metric. The effect is however very small if one believes on the general relativistic intuition.

   In TGD framework one however can have so called warped imbeddings of M^4 for which the component of the induced metric in some direction is scaled but curvature tensor and thus gravitational field vanishes. In time direction this scaling would imply anomalous time dilation in absence of gravitational fields. This would however cause only a the compression or expansion of M^4 lattice in some direction.

2. For Euclidian regions of space-time surface having interpretation as lines of generalized Feynman diagrams M^4 projection is 3-dimensional and at elementary particle level the scale associated with M^4 degrees of freedom is roughly the same as CP_2 scale. If CP_2 coordinates are used (very natural) one obtains deformation of a finite lattice-like structure in CP_2 analogous to a deformation of Platonic solid regarded as point set at sphere. Whether this lattice like structure could be seen as a subset of infinite lattice is not clear.

3. One can consider also string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ with 2-D M^4 projection and their deformations. In this case the projection of M^4 lattice to X^2 - having subset of two M^4 coordinates as coordinates - can differ considerably from a regular lattice since X^2 can be locally tilted with respect to M^4 lattice. This cannot however give rise to Penrose tiling requiring 5-D flat imbedding space. This argument applies also to 2-D string world sheets carrying spinor modes. In the idealized situation that string world sheet is plane in M^4 one might obtain an analog of Penrose tiling but with 4-D imbedding space.

The above quasi lattice like structures (QLs) are defined by a gravitational deformation of the cubic lattice of M^4. Is there any hope about the 4-D QLs in M^4 so that gravitation would give rise to the analogs of phason waves deforming them? Could cut and project method be generalized to give QL in M^4 as projection of 8-D cubic lattice in M^8?

3.3.3 $M^8 - H$ duality

Before considering an explicit proposal I try to describe what I call $M^8 - H$ duality ($H = M^4 \times CP_2$).

1. What I have christened $M^8 - H$ duality is a conjecture stating that TGD can be equivalently defined in M^8 or $M^4 \times CP_2$. This is the number theoretic counterpart of spontaneous compactification of string models but has nothing to do with dynamics: only two equivalent representations of dynamics would be in question.

2. Space-time surfaces (preferred extremals) in M^8 are postulated to be quaternionic sub-manifolds of M^8 possessing a fixed $M^2 \subset M^4 \subset M^8$ as sub-space of tangent space. "Quaternionic" means that the tangent space of M^4 is quaternionic and thus associative. Associativity conditions would thus determine classical dynamics. More generally, these subspaces $M^2 \subset M^8$ can form integrable distribution and they define tangent spaces of a 2-D sub-manifold of M^4. If this duality really holds true, space-time surfaces would define a lattice like structure projected from a cubic M^8 lattice. This of course does not guarantee anything: $M^8 - H$ duality itself suggests that these lattice like structures differ from regular M^4 crystals only by small gravitational effects.

3. The crucial point is that quaternionic sub-spaces are parametrized by CP_2. Quaternionic 4-surfaces of $M^8 = M^4 \times CP_2$ containing the fixed $M^2 \subset M^8$ can be mapped to those of $M^4 \times CP_2$ by defining M^4 coordinates as projections to preferred $M^4 \subset M^8$ and CP_2 coordinates as those specifying the tangent space of 4-surface at given point.

4. A second crucial point is that the preferred subspace $M^4 \subset M^8$ can be chosen in very many manners. This imbedding is a complete analog of the imbedding of lower-D subspace to higher-D one in cut and project method. M^4 can be identified as any 4-D subspace imbedded in M^4 and the group $SO(1,7)$ of 8-D Lorentz transformations defines different imbeddings of M^4 to M^8. The moduli space of different imbeddings of M^4 is the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and has dimension $D = 28 - 6 - 6 = 16$.

   When one fixes two coordinate axes as the real and one imaginary direction (physical interpretation is as an identification of rest system and spin quantization axes), one obtains $SO(1,7)/SO(2) \times SO(4)$ with higher dimension $D = 28 - 1 - 6 = 21$. When one requires also quaternionic structure one obtains the space $SO(1,7)/SU(1) \times SU(2)$ with dimension $D = 28 - 4 = 24$. Amusingly, this happens to be the number of physical degrees of freedom in bosonic string model.

3.3.4 How to obtain quasilattices and quasi-crystals in M^4?

Can one obtain quasi-lattice like structures (QLs) at space-time level in this framework? Consider first the space-time QLs possibly associated with the standard cubic lattice L^4_{st} of M^4 resulting as projections of the cubic lattice structure L^8_{st} of M^8.

1. Suppose that one fixes a cubic crystal lattice in M^8, call it L^8_{st}. Standard M^4 cubic lattice L^4_{st} is obtained as a projection to some M^4 sub-space of M^8 by simply putting 4 Euclidian coordinates for lattice points o constant. These sub-spaces are analogous to 2-D coordinate planes of E^3 in fixed Cartesian coordinates. There are $7!/3!4! = 35$ choices of this kind.

   One can consider also E_8 lattice is an interesting identification for the lattice of M^8 since E_8 is self-dual and defines the root lattice of the exceptional group E_8. E_8 is union of Z^8 and $(Z + 1/2)^8$ with the condition that the sum of all coordinates is an even integer. Therefore all lattice coordinates are either integers or half-integers. E_8 is a sub-lattice of 8-D cubic lattice with 8 generating vectors $e_i/2$, with e_i unit vector. Integral octonions are obtained from E_8 by scaling with factor 2. For this option one can imbed L^4_{st} as a sub-lattice to Z^8 or $(Z + 1/2)^8$.

2. Although $SO(1,3)$ leaves the imbedded 4-plane M^4 invariant, it transforms the 4-D crystal lattice non-trivially so that all 4-D Lorentz transforms are obtained and define different discretizations of M^4. These are however cubic lattices in the Lorentz transformed M^4 coordinates so that this brings nothing new. The QLs at space-time surface should be obtained as gravitational deformations of cubic lattice in M^4.

3. L^4_{st} indeed defines 4-D lattice at space-time surface apart from small gravitational effects in Minkowskian space-time regions. Elementary particles are identified in TGD a Euclidian space-time regions - deformed CP_2 type vacuum extremals. Also black-hole interiors are replaced with Euclidian regions: black-hole is like a line of a generalized Feynman diagram, elementary particle in some sense in the size scale of the black-hole. More generally, all physical objects, even in everyday scales, could possess a space-time sheet with Euclidian metric signature characterizing their size (AdS^5/CFT correspondence could inspire this idea). At these Euclidian space-time sheets gravitational fields are strong since even the signature of the induced metric is changed at their light-like boundary. Could it be that in this kind of situation lattice like structures, even QCs, could be formed purely gravitationally? Probably not: an interpretation as lattice vibrations for these deformations would be more natural.

It seems that QLs are needed *already at the level of* M^4. M^8-H duality indeed provides a natural manner to obtain them.

1. The point is that the projections of L^8_{str} to sub-spaces M^4 defined as the $SO(1,7)$ Lorentz transforms of L^4_{st} define generalized QLs parametrized by 16-D moduli space $SO(1,7)/SO(1,3)\times SO(4)$. These QLs include also QCs. Presumably QC is a QL possessing a non-trivial point group just like Penrose tiling has the isometry group of dodecagon as point group and 3-D analog of Penrose tiling has the isometries of icosahedron as point group.

   This would allow to conclude that the discretization at the level of M^8 required by the definition of p-adic variants of preferred extremals as cognitive representations of their real counterparts would make possible 4-D QCs. M^8 formulation of TGD would explain naturally the QL lattices as discretizations forced by finite measurement resolution and cognitive resolution.

   A strong number theoretical constraint on these discretizations come from the condition that the 4-D lattice like structure corresponds to an algebraic extension of rationals. Even more, if this algebraic extension is 8-D (perhaps un-necessarily strong condition), there are extremely strong constraints on the 22-parameters of the imbedding. Note that in p-adic context the algebraic extension dictates the maximal isometry group identified as subgroup of $SO(1,7)$ assignable to the imbedding as the discussion of p-adic icosahedron demonstrates.

2. What about the physical interpretation of these QLs/QCs? As such QLs define only natural discretizations rather than physical lattices. It is of course quite possible to have also physical QLs/QCs such that the points - rather time like edge paths - of the discretization contain real particles. What about a "particle" localized to a point of 4-D lattice? In positive energy ontology there is no obvious answer to the question. In zero energy ontology the lattice point could correspond to a small causal diamond containing a zero energy state. In QFT context one would speak of quantum fluctuation. In p-adic context it would correspond to "though bubble" lasting for a finite time.

3. It is also possible to identify physical particles as edge paths of the 4-D QC, and one can consider time= constant snapshots as candidates for 3-D QCs. It is quite conceivable that the non-trivial point group of QCs favors them as physical QLs.

3.3.5 Expanding hyperbolic tesselations and quasi-tesselations obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

M^8-$M^4 \times CP_2$ duality and the discretization required by the notion of p-adic manifold relates in an interesting manner to expanding hyperbolic tesselations and quasi tesselations in $H^7 \subset M^8$, and possible expanding quasi-tesselations in obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

1. *Euclidian lattices* E_8, E_7, E_6

I have already considered E_8 lattice in M^8. The background space has however Minkowskian rather than Euclidian metric natural for the carrier space of the E_8 lattice. If one assigns some discrete subgroup of isometries to it, it is naturally subgroup of $SO(8)$ rather than $SO(1,7)$. Both these groups have $SO(7)$ as a subgroup meaning that preferred time direction is chosen as that associated with the real unit and considers a lattice formed from imaginary octonions.

E_8 lattice scaled up by a factor 2 to integer lattice allows octonionic integer multiplication besides sums of points so that the automorphism group of octions: discreted subgroups of $G_2 \subset SO(7)$ would be the natural candidates for point groups crystals or lattice like structures.

If one assumes also fixed spatial direction identified as a preferred imaginary unit, G_2 reduces to $SU(3) \subset SO(6) = SU(4)$ identifiable physically as color group in TGD framework. From this one ends up with the idea about $M^8 - M^4 \times CP_2$ duality. Different imbeddings of $M^4 \subset M^8$ are quaternionic sub-spaces containing fixed M^2 are labelled by points of CP_2.

All this suggests that E_7 lattice in time=constant section of even E^6 lattice is a more natural object lattice to consider. Kind of symmetry breaking scenario $E_8 \to E_7 \to E_6 \to G_2 \to SU(3)$ is suggestive. This Euclidian lattice would be completely analous to a slicing of 4-D space-time by 3-D lattices labelled by the value of time coordinate and is of course just what physical considerations suggest.

2. Hyperbolic tesselations

Besides crystals defined by a cubic lattice or associated with E_6 or E_7, one obtains an infinite number of hyperbolic tesselations in the case of M^8. These are much more natural in Minkowskian signature and could be also cosmologically very interesting. Quite generally, one can say that hyperbolic space is ideal for space-filling packings defined by hyperbolic manifolds H^n/Γ: they are completely analogous to space-filling packings of E^3 defined by discrete subgroups of translation group producing packings of E^3 by rhombohedra. One only replaces discrete translations with discrete Lorentz transformations. This is what makes these highly interesting from the point of view of quantum gravity.

1. In M^{n+1} one has tesselations of n-dimensional hyperboloid H^n defined by $t^2 - x1^2 - ... - x_n^2 = a^2 > 0$, where a defines Lorentz invariant which for $n = 4$ has interpretation as cosmic time in TGD framework. Any discrete subgroup Γ of the Lorentz group $SO(1,n)$ of M^{n+1} with suitable additional conditions (finite number of generators at least) allows a tesselation of H^n by basic unit H^n/Γ. These tesselations come as 1-parameter families labelled by the cosmic time parameter a. These 3-D tesselations participate cosmic expansion. Of course, also ordinary crystals are crystals only in spatial directions. One can of course discretize the values of a or some function of a in integer multiples of basic unit and assign to each copy of H^n/Γ a "center point" to obtain discretization of M^{n+1} needed for p-adicization.

2. For $n = 3$ one has M^4 and H^3, and this is very relevant in TGD cosmology. The parameter a defines a Lorentz invariant cosmic time for the imbeddings of Robertson-Walker cosmologies to $M^4 \times CP_2$. The tesselations realized as physical lattices would have natural interpretation as expanding 3-D lattice like structures in cosmic scales. What is new is that discrete translations are replaced by discrete Lorentz boosts, which correspond to discrete velocities and observationally to discrete red shifts for distant objecst. Interestingly, it has been found that red shift is quantized along straight lines [?]: "God's fingers" is the term used. I proposed for roughly two decades ago an explanation based on closed orbits of photons around cosmic strings [K2]. but explanation in terms of tesselations would also give rise to periodicity. A fascinating possibility is that these tesselation have defined macroscopically quantum coherent structures during the very early cosmology the the size scale of H^3/Γ was very small. One can also ask whether the macroscopic quantum coherence could still be there.

   Hyperbolic manifold property has purely local signatures such as angle surplus: the very fact that there are infinite number of hyperbolic tesselations is in conflict with the the fact that we have Euclidian 3-geometry in every day length scales. In fact, for critical cosmologies, which allow a one-parameter family of imbeddings to $M^4 \times CP_2$ (parameter characterizes the duration of the cosmology) one obtains flat 3-space in cosmological scales. Also overcritical cosmologies for which $a = constant$ section is 3-sphere are possible but only with a finite duration. Many-sheeted space-time picture also leads to the view that astrophysical objects co-move but do not co-expand so that the geometry of time=constant snapshot is Euclidian in a good approximation.

3. Does the notion of hyperbolic quasi-tesselation make sense?

Can one construct something deserving to be called quasi tesselations (QTs)? For QCs translational invariance is broken but in some sense very weakly: given lattice point has still an infinite number of translated copies. In the recent case translations are replaced by Lorentz transformations and discrete Lorentz invariance should be broken in similar weak manner.

If cut and project generalizes, QTs would be obtained using suitably chosen non-standard imbedding $M^4 \subset M^8$. Depending on what one wants to assume, M^4 is now image of M^4_{st} by an element of $SO(1,7)$, $SO(7)$, $SO(6)$ or G_2. The projection - call it P - must take place to M^4 sliced by scaled copies of H^3 from M^8_{st} sliced by scaled copies of H^7/Γ tesselation. The natural option is that P is directly from H^7 to $H^3 \subset H^7$ and is defined by a projecting along geodesic lines orthogonal to H^3. One can choose always the coordinates of M^4 and M^8 in such a manner that the coordinates of points of M^4 are $(t,x,y,z,0,0,0,0)$ with $t^2 - r^2 = a_4^2$ whereas for a general point of H^7 the coordinates are $(t,x,y,z,x_4,...x_7)$ with $t^2 - r^2 - r_4^2 = a_8^2$ for $H^3 \subset H^7$. The projection is in this case simply $(t,x,y,z,x_4,...,x_7) \rightarrow (t,x,y,z,0,...,0)$. The projection is non-empty only if one has $a_4^2 - a_8^2 \geq 0$ and the 3-sphere S^3 with radius $r_4 = \sqrt{a_4^2 - a_8^2}$ is projected to single point. The images of points from different copies of H^7/Γ are identical if S^3 intersects both copies. For r_4 much larger than the size of the projection $P(H^7/\Gamma)$ of single copy overlaps certainly occurs. This brings strongly in mind the overlaps of the dodecagons of Penrose tiling and icosahedrons of 3-D icosahedral QC. The point group of tesselation would be Γ.

4. Does one obtain ordinary H^3 tesselations as limits of quasi tesselations?

Could one construct expanding 3-D hyperbolic tesselations H_3/Γ_3 from expanding 7-D hyperbolic tesselations having H^7/Γ_7 as a basic building brick? This seems indeed to be the outcome at at the limit $r_4 \rightarrow 0$. The only projected points are the points of H^3 itself in this case. The counterpart of the group $\Gamma_7 \subset SO(1,7)$ is the group obtained as the intersection $\Gamma_3 = \Gamma_7 \cap SO(1,3)$: this tells that the allowed discrete symmetries do not lead out from H^3. This seems to mean that the 3-D hyperbolic manifold is H^3/Γ_3, and one obtains a space-filling 3-tesselation in complete analogy for what one obtains by projecting cubic lattice of E^7 to E^3 imbedded in standard manner. Note that $\Gamma_3 = \Gamma_7 \cap SO(1,3)$, where $SO(1,3) \subset SO(1,7)$, depends on imbedding so that one obtains an infinite family of tesselations also from different imbeddings parametrized by the coset space $SO(1,7)/SO(1,3)$. Note that if Γ_3 contains only unit element $H^3 \subset H^7/\Gamma_7$ holds true and tesselation trivializes.

3.4 Do Penrose tilings correspond to edge paths of Bruhat-Tits tree for projective sphere $P^1(Q_p)$?

Perhaps it deservers to be mentioned that there is an amusing co-incidence with Penrose tilings (see the book "In search of the Rieman zeros" [A3] by Lapidus, page 200) and between the representation of 2-adic numbers. This representation is in terms of a a tree containing only 3-vertices. Incoming edge represents n:th binary digit in the expansions $x = \sum x_n 2^n$, $x_n = 0,1$ and the two outgoing edges corresponds to the two values of the $n+1$:th binary digit. Each 2-adic number corresponds to a one particular edge path in this semi-infinite tree. This structure is very much analogous to Bruhat-Tits tree for p-adic projective line $P^1(Q_p)$ [A1] discussed in [K9].

A given Penrose tiling corresponds to semi-infinite bit string having only non-negative pinary digits and could be seen as a 2-adic integer. Two bit sequences describe same tiling if they differ from each other for a finite number bits only. Could the ends for the analog of Bruhat-Tits tree for p-adic integers (half-infinite paths beginning from some bit) be in one-one correspondence with Penrose tilings! Could one really describe 2-D Penrose tilings 2-adically? What about more general Penrose tilings and QCs? Maybe this conjecture is trivially true since Lapidus, who mentions this description of Penrose tilings, has written his book about p-adic strings [A3].

Unfortunately, I do not understand the arguments leading to the representation of Penrose tilings using bit sequences and whether this co-incidence has some deeper meaning.

4 About the notion of twisted quasicrystal

In the following I try to imagine some contents for the notion of twisted QC having in mind tetrahedrons glued together along their faces by twisting the glued tetrahedron with respect to the already existing one by angle γ.

1. What twist brings in my mind are nontrivial bundles in gauge theory. The simplest example is Möbius strip which can be seen as a bundle with circle as a base space and unit interval as fiber. It is obtained by gluing the ends of strip with a twist of π. In the recent case one could speak of twisted lattice or QC in which unit cells are glued to together with a twist.

2. The twist combined with the rotation induced by reflection in the common face of neigboring tetrahedra defines a rotation of tetrahedron analogous to a non-abelian gauge transformation assignable to gauge group SO(3). This leads to a gauge theory analogy and allows to see the conditions for closed twist path as stating that the holonomy of the gauge connection corresponds to the group of orientation preserving isometries of tetrahedron.

3. One possible analogy for the twist is in terms of a magnetic flux trough the face of tetrahedron. The flux would be associated with a flux tube carrying a helical magnetic field for which the twist per tetrahedron would be the same as the twist angle for tetrahedron (called face rotation angle in the summary article by Fang). For the most promising model of twisted QC this analogy leads to magnetostatic equations. This analogy does not take into account the non-Abelian character of the twist.

4.1 Does tetrahedral twisted QC make sense?

Consider the 3-D case with twisted tetrahedrons in mind.

1. Let us first introduce some terminology. Tetrahedron has four wall-neighbors which by definition share a common face with it. The remaining closest neighbors share only one vertex. Twist path runs by definition from tetrahedron connects only wall-neighbors and the sign of twist is same along the entire path. By choosing the orientation of the path properly the twist is always right-handed.

2. Numerical work has led to a conjecture that the twist angle has the value

$$\gamma = arccos((3\phi - 1)/4) \ . \tag{4.1}$$

3. Twisted tetrahedron construction is not unique. The sign of the twist angle can be $\pm\gamma$ and these choices are inequivalent and give rise to periodic helical structures with periods 3 and 5 so that one has chirality breaking which must relate to the construction involving reflection of the unit vectors connecting the centers of the wall-neighbor tetrahedra (see below). For a given tetrahedron one has several options depending on the signs of the twist angle at the four walls. This will be discussed in detail below.

4. The article by group summarizes the recent situation in attempt to find tetrahedral QC as a packing with high packing fraction. Several closed twist paths are claimed and referred to as $3G, 4G, 5G, 20G$. Here n in "nG" refers to the number of tetrahedrons involved. Their twist anagles (face rotation angle in the article) are listed. I do not know whether the explicit expressions for these angles have been derived analytically or numerically. For rather small values of n considered, the identifications look rather convincing.

5. Twisted QC studied in numerical experimentation has problems. Gaps can be formed and the tetrahedral twist paths can collide. The generic (and catastrophic) situation is that the colliding tetrahedra do not meet face-by-face. The desired situation is that if the collision happens the two tetrahedra relate just like tetrahedron and its twisted wall-neighbor - that is meet face-to-face without any transversal shift. If the twist angle γ is irrational multiple of 2π, closed twist paths are definitely out of question since for closed paths the rotation around a closed twist path yields the same tetrahedron in rotated configuration. If one has $\gamma = (m/n)(2\pi)/3$, one can have closed twist paths containing kn tetrahedrons.

The big dream would be a tetrahedral QC with packing fraction equal to one. Already Aristotle claimed that space could be packed completely by tetrahedra. The article in New York Times states that Aristotle was wrong: it would be nice to see the argument destroying Artistotle's dream. I do not know whether there exist rigorous upper limits for the packing fraction destroying this dream. If Aristotle was wrong, gaps are unavoidable even if one allows twisted QC.

4.2 Conditions for a closed twist path

The task is to formulate the conditions for having a closed twist path precisely. One can start from torus knot analogy as a simpler situation. Torus knot is obtained by rotating clock (disk with pointer) around a circle. The orbit of the end of pointer defines the knot and it is closed if the travel last multiple of 12 hours in other words twis is multiple of 2π.

What one obtains if one replaces disk (clock table) with a tetrahedron - a clock with four faces? Also now one has center of mass degrees of freedom and orientational (rigid body -) degrees of freedom characterized by an element of rotation group SO(3) parametrized in terms of Euler angles. One must have a closed path in the ideal situation in both cm and rigid body degrees of freedom. One can of course consider also the possibility that the path is not quite closed in center of mass degrees of freedoms so that one has "offset". In rigid body degrees of freedom one must get orientation preserving isometry of tetrahedron leaving it invariant: this group contains 12 elements and is generated by rotations leaving one vertex invariant and can be also interpreted group A_4 as even permutations of four objects.

4.2.1 Exponential representation of rotation

One can use for a rotation by an angle γ around axes parallel to unit vector e (say twist) exponential representation

$$T = exp(\gamma e \cdot T) = cos(\gamma) \times Id_{3\times 3} + cos(\gamma) e \cdot T \ . \tag{4.2}$$

Id is unit matrix and γ is the twist angle and $T = (T_x, T_y, T_z)$ is matrix valued vector defined by infinitesimal generators of rotations around various coordinate axis. The expressions for T_i are given by

$$\begin{aligned} T_x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \\ T_y &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ T_z &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.3}$$

4.2.2 Closedness of twist path in cm degrees of freedom

In cm degrees of freedom one has the discrete path consisting of segments of constant length between cm positions of subsequent tetrahedra and by the irrationality of tetrahedral angle defined as angle between the faces of tetrahedron (dihedral angle is given by $\Phi_d = arccos(1/3)$). This path cannot be closed without twists.

1. The condition for closedness states that the segment path closes says that the sum of the unit vectors E_i associated with the segments vanishes. Without twist this is not possible. With twists one obtains the condition

$$\sum E_i = 0 \ . \tag{4.4}$$

E_i connects the centers of $i-1$:th tetrahedron and twisted i:th tetrahedron. e_i denote the corresponding unit vectors before the twist.

2. One must find the explicit relationship between the sets of $\{e_{k)}(i)\}$ and $\{e_{k)}(i+1)\}$ of unit vectors for the three choices of the wall-neighbor at the first step. Since these vectors point from center of mass to the center of a wall they do not depend on the cm position of the tetrahedron so that the relationship is same at each step apart from the choice of the wall-neighbor. A mere reflection in the common face relates the unit vectors

$$e_{k)i} \rightarrow P(e_{k)i}) = e_{k)i+1} = e_{k)i} - 2e_{k)i} \cdot e_i e_i \ . \tag{4.5}$$

The application of the reflection P twice gives identity transformation. The occurrence of a reflection in the construction should relate closely to the breaking of chirality in the sense that different signs of γ are found to give helical structures with period 3 and 5 respectively. Reflection could be accompanied by a reflection permutating two faces of the $i+1$:th tetrahedron: in this manner the net transformaton would be rotation.

3. Two subsequent tetrahedra along twist path have before the twist three common vertices and edges and 1 new vertex with distance unit vectors for two wall-neighbor tetrahedra $i-1$ and i are related by the twiest

$$\begin{aligned} E_i &= T_i e_i = T_i P_i e_{i-1} \ , \\ T_i &= exp(\gamma e_i \cdot T) = cos(\gamma) \times Id_{3\times 3} + sin(\gamma) e_i \cdot T \ . \end{aligned} \tag{4.6}$$

Here $T_i = exp(e_i \cdot T) = cos(\gamma) Id_{3\times 3} + e_i \cdot T$ denotes the twist around axis e_i by angle γ performed at i:th step rotating the 3 faces of the tetrahedron emerging at i:th step γ is referred to as "face rotation angle" in the article. At each step there are three alternative choices corresponding to the 3 faces of i:th tetrahedron rotated by T_i.

4. The standard basis for $e_{k)}(0)$ assignable to the first tetrahedron is obtained by using the standard representation $\epsilon_{1/2} = (\pm 1, 0, -1/2)$ and $\epsilon_{3/4} = (0, \pm 1, 1/2)$ for the vertices of tetrahedron of edge length 2. The unit vectors $e^{k)}$ orthogonal to the faces of the standard tetrahedron are given by

$$e_{k)}(0) = \sqrt{\frac{3}{2}} \sum_{k \neq i} \epsilon_k \ . \tag{4.7}$$

One can construct the sequence of E_i by repeating the basic step and study under what conditions the process yields closed orbits or finite number plane classes. Certainly the twist angle must relate to the dihedral angle which is the angle between two subsequent segments.

5. The condition above allows an explicit numerical representation by using standard representation of twist matrices around the normal of common face and reflection of unit vectors e_i with respect to the common face at a given step.

The idea about finite number of plane classes as a signature of QC property is very beautiful and should be testable by looking with twist angle lead to finite number of unit vectors e_i. A good guess is that the unit vectors in this case define an orbit of discrete subgroup of SO(3) and therefore can be assigned to the vertex or - more naturally- face of some regular polyhedron - that is Platonic solid. This hypothesis would fix the number of plane classes to be the number of faces of tetrahdron, cube, octahedron, dodecahedron, and icosahedron. The connection with the Platonic solids is very attractive since it would also give a connection with the grid construction of 3-D Penrose tilings.

Also infinite discrete sub-groups of SO(3) can be considered if one allows infinite number of plane classes as number of faces of the corresponding polyhedron like structure. It is not clear to me whether the maximization of packing fraction necessarily correlates with the number of plane classes. It would be the underlying discrete symmetry, which would give rise to long range order and also to a maximization of packing fraction.

4.2.3 Closedness of the path in rigid body degrees of freedom

The path should close in orientational degrees of freedom or more generally, represents isometry of tetrahedron (12 element group of even permutations of four elements, call it G) leaving it invariant. By choosing the twist angle suitable this might be achieved.

What distinguishes the situation from simple torus knot analogy is that the twists nowe around several different axis rather than single one: one axis defined by each face. The total rotation of tetrahedron around a closed path due to these twists combined with the reflections with respect to common face must be trivial. Twists are however non-commutative and the angles do not add like for ordinary clock table in the above example. Rather, the tetrahedron making an imaginary journey through the twist path suffest $SO(3)$ rotation which must reduce to an element of G:

$$T = \prod_i T_i P_i = g \in G \ . \tag{4.8}$$

The axes around which the twist takes place is defined by unit vector e_i connecting cm:s of two subsquent tetrahedra. The explicit expression of twist angle in terms of e_i has been already given using the exponential formula and once the twist path is fixed, T can be calculated. Also the action of the reflection P_i has been also calculated.

The conditions for the closedness of the twist path in cm and rigid body degrees of freedom should be satisfied for closed twists paths, and it should be possible to study numerically these conditions for various numbers of tetrahedra for the path and find the allowed values of twist angle. As one goes around any closed twist-path, the net twist is multiple of $2\pi/3$ around some of the four axis. If closed twist-paths exist at all, this condition might kill the hypothesis.

4.3 The analogy with non-abelian gauge theory with gauge group $SO(3)$

One might hope that the non-Abelian character of the twists could be taken into account by applying the generalization of magnetic flux as a non-integrable phase factor along twist path determined as product of twists along it $T = \prod_i T_i$. This would lead to an analogy with lattice gauge theory with gauge group $SO(3)$. The problem is that one has $T = \prod_i T_i P_i$. One should extend the gauge group to include also reflections so that not all gauge transformation could not be generated continuously from unity. An alternative possibility is that P_i is accompanied by a mapping to each other two walls of the $i+1$:th tetrahedron. This changes the ordering of the unit vectrors from center to the face centers but leaves the tetrahedron invariant. This would transform P_i to a rotation: $P_i \rightarrow R_i$ so that gauge group analogy would hold true.

In gauge theory analogy one has the identification

$$T_i = Pexp(i \int_{i-1 \rightarrow i} A_\mu dx^\mu$$

holding true with integration defined as an ordered integral along the arc segment connecting the centers of subsequent tetrahedra at twist path. For Abelian gauge group ($T_i = exp(i \int_{i-1\rightarrow i} A_\mu dx^\mu)$ the condition $T = 1$ for a closed curve would reduce to a quantization of magnetic flux through the 2-surface enclosed by the closed curve $\oint A_\mu dx^\mu = \int B \cdot da = n2\pi$. This is obtained by using the exponential representation for the elements T_i. If magnetic field vanishes, the condition $T = 1$ is identically satisfied.

For a closed twist path one must have $T \in G$, where G is the isometry group of the tetrahedron containing orientation preserving isometries. In lattice gauge theories one replaces gauge potentials with the factors T_i assigned with the edges of lattice. In the recent case QC gauge theory with lattice replaced with twisted QC seems to be the appropriate analogy. An important distinction is that gauge rotations have in the recent case direct geometric meaning. I do not know whether gauge theories in quasilattices have been studied.

If T around closed path reduces to a unit element, one would say that the gauge potential has trivial holonomy group and gauge field is trivial. Now G acts as a discrete holonomy group which says that the orbit of tetrahedron around twist path brings it back as a isometry rotated version. Periodicity of T along the twist paths (say 3- or 5-periodicity) would give hopes about finite number of plane classes. When gauge field vanishes but connection has non-trivial holonomy, one has topological QFT so that one would have analogy of topological QFT. Againts this observation it is interesting that the connection between QCs and topological insulators for which two-element group Z_2 plays the role of holonomy group has been recently emphasized (see this).

In gauge theory gauge potentials couple to matter fields, say spinor fields. In particular, the analog of covariantly constant field along twist path satisfying $D_\mu \Psi = 0$ would be obtained by applying non-integrable phase factor $exp(i\int_{i-1\rightarrow i} A_\mu dx^\mu)$ to the value of Ψ at the center of $i-1$:th tetrahedron to obtain the value of Ψ at the center of $+1$:th tetrahedron. The analogy is obvious: replace Ψ with a field having the possible orientations of tetrahedron as values and define the action of gauge transfrmation as T_iR_i or T_iP_i. One could also say that tetrahedron suffers a parallel translation along twist path such that the net translation is isometry.

4.4 Some attempts to construct twisted QCs

In the following some options for the rules governing what happens at given tetrahedron. The attempts rely on the notion of twist flux which is still an Abelian notion and originally inspired by the erratic treatment of twists as elements of SO(2) rather than SO(3). The emerging idea is that the twist paths can be regarded as a kind of flux tubes with helical winding of the magnetic field characterized by twist angle. The basic idea is that the flux tubes do not close: if this occurs, the conditions described above apply. This notion is not consistent with the numerical construction proceeding layer by layer from the central tetrahedron.

The basic notion is that of twist flux. The sign of twist flux is fixed by the sign of the twist angle $\pm\gamma$ and its magnitude can be taken to be γ. Depending on sign of γ at given face of tetrahedron one can say that there is twist flux $\pm gamma$ though that face. This analogy does not take into account the non-abelian character of the twists.

4.4.1 Spherically symmetric option

The first trial might be a failure but led to potentially useful analogies with hydrodynamic flow and electromagnetic fields and also to a realization that the rules for each tetrahedron must be essentially the same.

1. One starts from a fixed tetrahedron and glues wall-neighbors with the same twist angle (twist to say right at each face). In hydrodynamic analogy this tetrahedron serves as a source of fluid feeding same amount of fluid through all four faces. For the wall-neighbors the rules cannot same as for this "seed" tetrahedron. Each wall-neighbor has one incoming flux and can have 1, 2, or 3 outgoing ones. 3 outgoing fluxes are selected in the hope that one would obtain approximately radial twist paths. The rules are not democratic and this is alarming.

2. Situation could be seen as a discrete analog for a radial electric field derivable from potential depending on radial coordinate. A twist in a given direction would represent a local value of electric field in that direction. The potential function represents total twist for any curve from origin to the given radial distance r and is therefore linear in the radial coordinate. This means that the charge density as divergence of the constant radial electric field behaves like 1/r. If the analogy with potential function really holds true, one can consider any point as the initial point of construction. One could also model twists as electric fluxes orthogonal to the wall in question.

3. The optimistic expectation would be that all twist paths are more or less radial and non-closed. Unfortunately, the twist paths cannot even approximately radial as becomes clear by using some visual imagination. The graphical representation of numerical simulations by Julio demonstrated the same. The point is that each tetrahedron has one input flux and yields 3 output fluxes in

widely differing angles. Eventually this leads to a turning of twist paths backwards. Closed twist paths tend to form and collisions are unavoidable, and it would be a miracle if they would happen in face to face and obeying the twist rule. Physically the analog system is hydrodynamic flow with each tetrahedron generating new liquid or a tetrahedral lattice like structure with two units of charge at every lattice cell. Both are physically unrealistic.

4. The 3G,4G,5G, and 20G structures are examples about closed structures obtained by using what seems to be this kind of constructing starting from center and proceeding layer by layer radially outwards.

The lesson could be that the rules must be as democratic as possible. Either they are same for all tetrahedra or they can differ only for tetrahedron and its wall-neighbors by a kind of reflection. This leaves only two options to consider.

4.4.2 Tetrahedra as maximal sources or sinks of twist flux

For the first option tetrahedron serves as a source of 4 twist fluxes and its wall-neighbors as sinks of 4 twist fluxes. Hence the rules for wall-neighbors are mirror images of each other. The electrostatic analogy is a lattice of (twisted) tetrahedra for which tetrahedron and its wall-neighbor have always opposite charges of four units. For paths consisting of wall neighbors the total twist vanishes if the number of tetrahedra is even. It is still possible that one can have undesired collisions and gaps.

This option could be tested numerically. One could consider tetrahedral lattice and look what closed paths formed by wall-neighbors look like and compare them to twist paths for the proposed twist angle γ. If the modifications are not too large, one might hope of obtaining a twisted variant of the tetrahedral crystal. This is not expected to produce genuine twisted QC.

4.4.3 The net twist fluxes entering and leaving given tetrahedron are same

Second option assumes that the incoming net flux to a tetrahedron is the same as the outgoing net flux so that tetrahedra do not serve as sources of flux. There are two units of incoming flux and two units of outgoing flux. These rules are completely democratic. In this case twist paths would be analogous to magnetic flux tubes (no magnetic charges). Incompressible hydrodynamic flow is second analogy. At each tetrahedron two flux tubes meet so that two magnetic field patterns/hydrodynamic flows would define the analogy. These analogy systems are physical since each tetrahedron is neutral/there are now sources of fluid.

1. Suppose that one starts from some tetrahedron by selecting the wall pairs 12 and 34 to be connected by a flux tube. In other words, the first (second) flux comes through wall 1 (3) and leaves through wall 2 (4). For the wall-neighbor the outgoing flux through wall is incoming and can go out through any other wall so that there are three options to choose. If one continues without considering possible constraints from other twist paths one has 3^n options for a path containing n tetrahedrons. These correspond to different flux paths. A combinatorial explosion takes place and it is not possible to test numerically whether one can avoid undesired collisions and whether the analogs of DNA strands can form a QC like structure.

   The non-uniqueness of the rules is both blessing and curse. It allows to avoid undesired collisions, formation of closed paths, and gaps on one hand but also leads to a combinatorial explosion so that numerical work can only demonstrate that the rules fail to produce twisted QCs. The real believer would hope that the combinatorial explosion could give rise to infinite number of twisted QCs.

2. The conditions that closed twist paths are allowed has been already considered and one obtains conditions for the closing of twist path in cm and rotational degrees of freedom. One can also consider the possibility that there are no closed twist paths. This might help to avoid collisions. This gives a strong additional constraint since magnetic flux lines/tubes - now analogs of twist paths - are not denied by Maxwell's equations without additional conditions.

   (a) The magnetic field serving as analogy should have vanishing divergence (no sources). This condition ($\nabla \cdot B = 0$) is satisfied for $B = \nabla \times A$, A vector potential.

(b) If the flow is also irrotational ($\nabla \times B = 0$), there are no closed flux lines and the rotation $\oint B \cdot dl$ around closed curves vanishes (this means vanishing of net twist angle in the analogy). $\nabla \times B = 0$ gives $\nabla \times (\nabla \times A) = \nabla^2 A - \nabla(\nabla \cdot A) = 0$. For the magnetic flux tubes the twist can also have a concrete meaning: the magnetic field along flux tube would be helical with a twist ϕ per tetrahedron: this however makes magnetic field rotational in the scale defined by the thickness of the flux tube so that the condition $\nabla \times B = 0$ seems to be too strong.

In Maxwell's theory one can also pose the gauge condition $\nabla \cdot A = 0$ giving Laplace equation $\nabla^2 A = 0$ for A. Apart from gauge condition, one obtains magneto-static Maxwell's equations from the analog.

(c) As noticed, the condition $\nabla \times B = 0$ is un-necessary strong and excludes helical flows. For instance, one can have infinitely long straight flux tubes carrying helical magnetic field for which $\nabla \times B$ is non-vanishing and parallel to the plane orthogonal to the axis of the flux tube. The irrotational character of the magnetic field in length scales longer than flux tube thickness is certainly enough. This kind of non-closed flux tubes can be also curved so that even this condition is too strong. Hydrodynamic flow is a good example: the flow inside pipelines can have eddies: all that matters is that pipelines themselves are not closed.

(d) Note that if twist angle is rational multiple of 2π, only $B = \nabla \times A$ can be assumed. One can ask whether the analogy in 4-D case is defined by full Maxwell's equations for radiation fields. If so, one might have some hopes about connection with physics.

3. One must not forget that there would be two magnetic fields corresponding to the two flux tubes through each tetrahedron. In Maxwell's theory one would have a superposition of these fields and this can mean the breakdown of the analogy. In TGD framework superposition of fields is replaced with the superposition of their effects on test particle since the field patterns typically correspond to different space-time shees. One can consider an explicit realization of the twisted QC in terms of flux tubes identified as tubular space-time sheets assignable separately to the two field patterns. In the earlier speculative article I indeed suggested that QCs or more general quasi-lattice like structures formed by the flux tubes might be important in making living matter to behave coherently: the magnetic body formed by the web of flux tubes would control ordinary living matter. For flux tube network the weakening of the condition $\nabla \times B = 0$ is very natural so that topological field quantization seems to provide a more natural analogy system.

4. Also a helical incompressible hydrodynamic flow along a pipeline network without closed pipelines serves as an analogy for the situation. The usefulness of both analogies is that a pattern of (say) magnetic field or velocity field satisfying the conditions could serve as as a heuristic template for constructing QCs possibly satisfying the consistency conditions and allow to overcome the combinatorial explosion. If the analogy is taken very seriously, magnetic fields (imcompressible hydrodynamic flows) without closed flux loops and having quasi-periodic structure could be in one-one correspondence with twisted QCs.

5. A further analogy for the twist path is provided by a DNA strand. For DNA strand the twist between subsequent nucleotides is in a good approximation $\phi = 2\pi/20$ corresponds to 18 degrees ($\pi/10$) and is not far from the claimed twist angle 15.522 degrees (it would be interesting to find the precise value of twist angle for DNA!). Twist paths would correspond to a network of infinitely long DNA strands and each tetrahedron - analog of DNA nucleotide - is shared by two DNA strands.

   What is amusing that TGD inspired model for living matter indeed assigns flux tubes to DNA strands and also transversal flux tubes connecting DNA nucleotides to cell membrane. The structure in question could of course be more general than twisted QC.

To summarize, at this moment it seems that magnetic fields without gauge condition could serve as a useful analogy system as also incompressible hydrodynamic flow. Whether one can allow closed flux lines remains an open question. The idea about finite number of plane classes is very attractive and it could be that the solutions of the condition are in 1-1 correspondence with Platonic solids. The twist might be realized as a twist of a helical magnetic field per tetrahedron. If this kind of field configurations with quasiperiodic structure can be shown to exist, there are hopes about coping with the combinatorial explosion and even about twisted QCs.

If ideal twisted tetrahedral QC exists, it has packing fraction equal to 1. I do not know whether there exists mathematical results giving some upper limit of the packing fraction of tetrahedral arrangements.

4.5 Could tetrahedral QC inherit its long range order from icosahedral QC?

The proposed twist angle involves Golden Mean. Also icosahedral QC involves Golden Mean via the isometry group of icosahedron, which is $A_5 \times Z_2$. This suggests that icosahedral QC might allow a variant in which tetrahedron and some other object appear as basic building bricks just as 2-D Penrose tiling has two basic building bricks.

One could indeed try to build QC with gaps from tetrahedra using icosahedral QC and putting inside each supercell a tetrahedron and maximizing its size by a suitable rotation to avoid intersections with neighboring tetrahedra. "Suitable rotations" could correspond to the twists and QC long range order would be inherited from icosahedral QC. This would give the analog of Penrose tiling with regular tetrahedron and gaps defining two (or possibly more) basic building bricks.

4.5.1 Icosahedral Penrose tiling

One can try to guess what icosahedral Penrose tiling looks like by starting from 2-D Penrose tiling first. Penrose tiling consists of 2 kinds of of rhombis (thick colored blue and thing with green color in the picture). The tiling has as super-cells decagons rather than pentagons (this conforms with the fact that phase factors $exp(in2pi/5)$ define 10-D real representation of Z_5 as required by group theoretical approach that I discussed earlier). They either meet edge-to-edge (green-blue or blue-blue) or penetrate inside each other somewhat having a common green.

The naive generalization to 3-D would replace decagons with icosahedrons. Icosahedrons could meet face-to-face (thick-thin or thick-thick) or share a thin rhombohedron. The guess is too simple. In analogy with Penrose tiling using two different rhombi as tiles icosahedral QC uses thick and thin rhombohedra. Rhombohedron is defined as a generalization of cube obtained by replacing its faces with rhombi. There are three different super-cells which can be partially overlapping in the sense that they share one or more rhombohedra. The supercells correspond to icosahedron, its dual dodecahedron, icosadodecahedron. Each of these has as a group of isometries $A_5 \times Z_2$ containing the group A_4 of tetrahedral rotations as a subgroup: this gives hopes about imbedding tetrahedra inside the supercells in a natural manner and obtaining tetrahedral QC: visual inspection shows that these hopes are realized.

4.5.2 An attempt to construct tetrahedral QC from icosahedral QC

Consider a construction of a candidate for a tetrahedral QC (my very first QC!) by imbedding tetrahedra inside supercells. The first guess that tetrahedra are inside super-cells cannot be satisfied for all supercells which can also intersect. Therefore tetrahedra can intersect. One could avoid them triviallly by scaling down the size of the tetrahedra. A proper choice of the orientation of tetrahedrons minimize the intersections. Also proper twist of the tetrahedra could help. It is assumed that the three supercells have same circumradius.

Note: This assumption must be checked!

1. How to imbed tetrahedra inside super-cells?

One should be able to imbed inside each supercell a regular tetrahedron with vertices at the surface of the sphere of same circumradius as for icosahedron and center coinciding with that of supercells. The group of orientable isometries of tetrahedron is A_4 and indeed subgroup of the icosahedral isometry group $A_5 \times Z_2$ so that there are hopes about this.

1. Consider first icosahedron. *If the tetrahedral vertices are a subset of icosahedral ones* then the intersections of tetrahedra belonging to the possible neighboring icosahedra meeting it face-to-face could be avoided. The numbers of vertices, edges and faces of icosahedron are $V = 12, E = 30$, and $F = 20$ so that Euler characteristic equals to $-V + E - F = 2$.

A direct visual inspection of icosahedral geometry shows that one can divide the 20 faces to 4 groups of 5 equilateral triangles in 1-1 correspondence with the 4 vertices of tetrahedron. The 30 edges of icosahedron in turn decompose to 6 groups consisting of 5 edges defining pentagon and forming a boundary for the 5-group of triangles. These pentagons can be assigned to the edges of tetrahedron identified as lines orthogonal to the pentagons going through their center and the vertex to which the pentagon is assigned with.

Hence it seems clear that one can assign to each icosahedron a tetrahedron sharing 4 vertices of the icosahedron. There are 5 inequivalent choices of this kind from the fact that single vertex identified as tetrahedron vertex fixes the rest so that 5=20/4 inequivalent selections of this vertex are possible.

2. For dodecahedron ones has $V = 20, E = 30$, and $F = 12$. Dodecahedron is obtained as a dual of icosahedron and the tetrahedron inside icosahedron is mapped to its dual which is also tetrahedron having $V = 4, E = 6, F = 4$. Now vertices of tetrahedron are in the center of group of 3 pentagons and there are 4 groups of this kind as one finds the Wikipedia illustration. In this case there are 3 equivalent choices of the tetrahedron. In both cases one can consider dual tetrahedra which leak partially out of the supercell in question.

3. Icosidodecahedron is a fusion of isosahedron and dodecahedron having $V = 20 + 12 = 32$, $E = 30 + 30 = 60$, and $F = 20 + 12$. Also the the isometry group of icosadodecahedron is icosahedral group, and one expects that the vertices of tetrahedron belong to an of the isometry group. It is not clear to me whether there is only single orbit or possible two corresponding to a union of the vertices of icosahedron and dodecahedron.

   Each 5-group of triangles of icosahedron is replaced with a group consisting of 5 triangles surrounding a pentagon. In a dual manner 3-group of pentagons surrounding single triangle replaces group of three pentagons for dodecahedron. One can identify two tetrahedra for which vertices are however not vertices of icosadodecahedron.

   (a) For the first tetrahedron the groups of three pentagons surrounding triangle define the 4 regions assignable to the vertices of tetrahedron. The vertex associated with the group of pentagons around triangle corresponds to the center point of the triangle. There are 5 choices of the tetrahedron as in the case of icosahedron.

   (b) For the second tetrahedron the groups of five triangles surrounding one pentagon define the 4 groups and the vertex corresponds to the center of the pentagon. The corners of both tetrahedra "leak out" from the icosadodecahedron. There are 3 choices of the tetrahedron as in the case of dodecahedron.

   One has 3+5=8 choices of the tetrahedra in this case.

2. How to avoid collisions of tetrahedra?

The possible problems in the construction of tetrahedral QC are caused by the overlap of the supercells and by the fact that for icosadodecahedron the corners of tetrahedron "leak out" and can intersect.

1. A proper choice of the "standard" orientations of the tetrahedra allows to minimize the intersections (5 for icosahedron, 3 or dodecahedron, and 5+3 for icosadodecahedron).

2. One can consider the possibility of making small twists and in this manner to achieve meeting face-by-face for neighboring tetrahedra. For icosahedron and dodecahedron the twists force three corners outside the supercell. The twist axes would go through one vertex of "standard" tetrahedron without twist (5 *resp.* 3 standard choices). After twist the three vertices of tetrahedron would penetrate to neighboring supercells but this need not be catastrophe since there is a freedom to choose each neighboring tetrahedron in several besides manners besides making a twist to eliminate intersections. Twists could make possible for tetrahedra to meet in face-to-face manner and in this manner to maximize the packing fraction. One cannot have a maximal packing fraction since the faces can meet face-to-face only partially so that there is an "offset".

3. The scaling down of the tetrahedron size is the trivial manner to achieve this but does not look attractive. It would trivially give a tetrahedral QC analogous to an elastic lattice for which basic units can be rotated. One could imagine even scaling up the sizes of the tetrahedra using proper twists and choices of the orientations.

Note: One should check whether the supercells have indeed the same size. For instance, if icosadodecahedron has larger size than dodecahedron and icosahedron one could use a tetrahedron with a smaller circumradius than icosadodecaheron to avoid the leakage.

To sum up, the key idea encouraging to take the construction at least half-seriously is group theoretic: icosahedral group has the group of tetrahedral rotations as subgroup and this allows to imbed tetrahedra to the three supercells in question. Twist is second key concept. Also scaling of tetrahedra might be involved.

3. An argument leading to the idea about tetrahedral QC

The article Dense Crystalline Dimer Packings of Regular Tetrahedra of Glotzner et al about tetrahedral packings shows the basic competing factors in packing and leads to the understanding why icosahedral QC could lead to the tetrahedral QC with large packing fraction in the manner described.

1. Suppose that one surrounds the Platonic solid with a sphere and that one has an optimal packing of spheres touching each other inducing polyhedral packing for polyhedra inside spheres. Tetrahedron inside a given sphere has very probably the lowest fractional volume amongst all Platonic solids so that the packing fraction is the worst one for the induced packing. Tetrahedron has simply too small number of vertices meaning a lot of empty volume inside sphere. At the limit of very large number of vertices for polyhedron approaches packing of spheres and I think this is well understood limit:$p = \pi/\sqrt{18}$ from the article.

2. To achieve a high packing fraction one must be increase the radius of spheres (and of imbedded polyhedra) so that they intersect. By rotating the polyhedra suitably one can avoid their intersections. Obviously parallel faces minimize the empty volume.

   Now the number of different face orientations (plane classes) is the variable which should be minimized to get the optimum packing fraction amongst all polyhedra. For cubes it is minimal (3) and cubes indeed allow ideal packing with packing fraction $p = 1$. Tetrahedra with 4 different face orientations should come next.

Sphere is however not the optimum choice for the packing from which to start the construction. Any polyhedron inside sphere is a better choice if one has one polyhedron inside polyhedron with same circumradius equal to that of sphere.

1. In the case of icosahedral QC the three supercells replace spheres and already intersect themselves. Could this represent the optimal situation concerning intersections? The task is to select among finite number of orientations of tetrahedra (3,5, 3+5) and by a suitable choice of twists find the optimum packing.

2. In principle the twist angle and twist axis are different for each supercell. Maximization very often has best solution which is highly symmetric. Hence maximum packing fraction suggests a completely symmetric solution in the sense that twist angle is same everywhere and around axis which in some sense is same for all supercells.

   Reflection with respect to a common face or face contained inside neighbor is the natural manner to define the standard orientation of the neighboring icosahedra, and the selection of preferred tetrahedron and twist axes could be "same" in this reference system.

4. Understanding icosahedral QC from group theoretic arguments and cut and project construction

The inspection of the illustration of icosahedral QC at the page 306 of the book "Crystallography of Quasicrystals" by Steurer and Deloudi (thanks to Julio) demonstrates that the circumradii of isosahedra (I) are smaller than those of dodecahedra (D) which cannot be too far from those of icosadodecahedra (ID). This is not catatrophe. One can scale up the radii of isosahedral tetrahedra and by suitable twists might get rid of intersections and obtain better packing ratio.

There are several questions to be answered.

1. Why all three orbits- dodecahedral, icosahedral, icosadodecahedral, of the icosahedral group appear?

2. Why the sizes of these orbits are what they are? Can one calculate their sizes.

3. Can one estimate packing fraction statistically for the ideal case defined by maximal tetrahedron radius presumably associated with the icosadodecahedron by the inspection of the figure. QC has fractal structure: could this mean that the ratio depends on the length scale used. What is its value at the limit of infinite QC?

The group theoretical approach to the construction of icosahedral QC (explained earlier) using cut and project method involving projection of a cubic lattice in 3-D complex spaces to 3-D real subspace suggests an answer to these questions.

1. One starts from the space of wave functions in the discrete space defined by the 12 faces of icosahedron. This has real dimension 12. The isometry group is $A_5 \times Z_2$ so that the representation decomposes to a direct sum of two real 6-D representations corresponding to the two Z_2 parities (Z_2 permutes the opposite faces of icosahedron). 6-D representations in turn can be regarded as complex 3-D representations and decompose to direct sums of two real 3-D representations. The 3-D subspace to which the projection takes places is identified as a subspace parallel to either of these real 3-D representations.

2. The orbits of the icosahedral group in the complex 3-D representation space project to the orbits of isosahedral group in real 3-D representation space. All three orbits are realized. This could be understood in the following manner.

   (a) Consider the Voronyi cell around the 3-D real subspace or 3 rather-space parallel to it. The real 3-space decomposes all 6-cubes along it into two parts. 6-cube has $2^6 = 64$ vertices. A reasonable assumption is that it gives rise to two or more orbits of the icosahedral group. D,I, and ID have 12, 20, and 12+20=32 vertices respectively. There are two options corresponding to 64=12+32+32 and 64=32+32. The latter option is more natural.

   (b) Consider the projection of the icosahedral group orbit with 32 vertice to real 3-space. We know that it can be only I, D, or ID. This allows 3 different orientations with respect to real 3-space or more generally, equivalence classes of orientations producing the same projection.
   In the case of ID one obtains pentagons surrounded by 5 triangles and triangles surrounded by 3 pentagons. In the case of D (I) the orientation is such that the inverse images of ID triangles (pentagons) are projected to point and the projection reduces to D (I).

   (c) Note that Voronyi cell implies projections from both sides of the real 3-plane which can intersect. This makes possible intersections of the supercells. The visual inspection of the figure indeed shows that the icosahedra seem to intersect dodecahedra so that they should correspond to projections from different sides of the 3-plane (do these sides correspond to complex conjugates in complex 3-D space?)

3. SOS theorem (!) can be used to estimate edge lengths. The claim is that all edge lengths are equal.The non-vanishing edge lengths for projections are identical and the sum over edge lengths squared is simply the number Na^2, where a is the non-vanishing edge length. For ID all edge projections have the same length. One can argue that there must be two orbits for faces since faces are either triangles or pentagons. The duality between vertices and faces would then suggest that there are two orbits for vertices. Assuming this and applying SOS to single orbit gives in all three cases that the ratio of the length squared of projection equals to the ratio of dimensions: $a^2 = 3/6 = 1/2$ giving $a = 1/\sqrt{2}$. All edges would have the same length.

4. Edge lengths allow to predict the circumradii of supercells for 6-D unit cube. One obtains for the three cases the following results:

$$\begin{aligned} r(I) &= \frac{\sqrt{10+2\sqrt{5}}}{4\sqrt{2}} = 0.62750 \ , \\ r(D) &= \frac{\sqrt{3}\Phi}{2^{3/2}} = 0.99084 \ , \\ r(ID) &= \frac{\Phi}{\sqrt{2}} \simeq 1.141 \ . \end{aligned} \tag{4.9}$$

These predictions seem to be consistent with the visual inspection of the figure below. IDs indeed look largest but are very near to Ds in size whereas Is are considerably smaller.

These data allow to deduce a statistical formula for the packing fraction assuming that the tetrahedron radius equals to a radius of I,D, or ID.

1. Consider first a situation that the circumradius $r(T)$ of tetrahedron equals to that of ID: $r(T) = r(ID)$. In this case the circumradii of icosahedral and dodecahedral tetrahedra must be scaled accordingly and these tetrahedra leak out. The intersections should be handled by twists. Another option would $r(T) = r(D)$. A suitable twist could allow to avoid intersections.

2. Suppose that one can estimate the packing fraction as the ratio

$$\begin{aligned} p &= \frac{V(T)}{V(average)} \ , \\ V(average) &= f(D)V(D) + f(I)V(I) + f(ID)V(ID) \ . \end{aligned} \tag{4.10}$$

Here $f(X) = N(X)/(N(D)+N(I)+N(ID)$ is the frequency and $N(X)$ the number of supercells of type $X = I, D, ID$, which can be estimated by using large enough sample of QC.

$$\begin{aligned} p &= \frac{V(T)}{V(ID)} \times \frac{1}{f(D)\frac{V(D)}{V(ID)} + f(I)\frac{V(I)}{V(ID)} + f(ID)} \ , \\ f(I) + f(D) + f(ID) &= 1 \ . \end{aligned} \tag{4.11}$$

3. If the circumradius of tetrahedron equals to that of ID one obtains using the formulas

$$\begin{array}{l} \frac{V(T)}{a^3(T)} = \frac{1}{6\sqrt{2}} \simeq .11785 \ , \\ r(T) = r(ID) = \Phi \times a(ID) \ , \\ a(T) = \sqrt{\frac{8}{3}}r(T) = \sqrt{\frac{8}{3}}\Phi \times a(ID) \end{array} \tag{4.12}$$

the result

$$\frac{V(T)}{a^3(ID)} = \frac{1}{6\sqrt{2}} \times (\sqrt{\frac{8}{3}}\Phi)^3 \simeq 2.1739 \ . \tag{4.13}$$

Here $r(T)$ denotes the circumradius of tetrahedron.

4. One must calculate the volume ratios appearing in the formula for the packing fraction. The volumes of the I,D, and ID are obtained by using the general formulas for their volumes (the formulas can be found from Wikipedia: see this, this, and this) and the relation $a_D = a(I) = a(ID)/\sqrt{2}$ as

$$\begin{array}{l} \frac{V(I)}{a(ID)^3} = 2^{-3/2} \times \frac{5}{12}(3+\sqrt{5}) \simeq .77135 \ , \\ \frac{V(D)}{a(ID)^3} = 2^{-3/2} \times \frac{1}{4}(15+7\sqrt{5}) \simeq 2.7093 \ , \\ \frac{V(ID)}{a(ID)^3} = 2^{-3/2} \times \frac{1}{6}(45+17\sqrt{5}) \simeq 4.89169 \ . \end{array}$$

The small value of $V(I)$ is consistent with the visual inspection of the figure. The scale of $V(ID)$ should be larger than $V(D)$ by factor 1.2177. This seems to be consistent with the figure. In any case, from this one obtains

$$\begin{array}{l} \frac{V(T)}{V(ID)} \simeq .44441 \ , \\ \frac{V(I)}{V(ID)} = 0.15769 \ , \\ \frac{V(D)}{V(ID)} = 0.55386 \ , \end{array} \tag{4.14}$$

5. Using the values of volume ratios one obtains an approximate estimate for the packing fraction in terms of frequencies of occurrence of I,D, and ID:

$$
\begin{aligned}
p &= \frac{V(T)}{V(ID)} \times \frac{1}{f(D)\frac{V(D)}{V(ID)} + f(I)\frac{V(I)}{V(ID)} + f(ID)} \\
&= \frac{.44441}{f(D)\times 0.55386 + f(I)\times 0.15769 + 1 - f(D) - f(I)} \quad . \qquad (4.15)
\end{aligned}
$$

The packing fraction is predicted to have the lower bound $p \geq 0.80239$ corresponding to $f(D) = 1$ if the circuradius of tetrahedron is that for ID. From the illustrations one has that $f(D) = f(I)$ holds in a reasonable approximation. The lower limit $p \leq 1$ would give $f(D) = f(I) \leq .82827$ which gives negative $f(ID)$. $f(D) > f(I)$ is therefore required.

It could be that the assumption that the tetrahedral circumradius equals to that for ID is too optimistic and a more general formula allowing scaling of the maximally optistic tetrahedron size by factor λ gives $p \rightarrow \lambda^3 p$. If tetrahedron has same circuradius as D, one obtains $\lambda = (r(D)/r(ID)^3 = .16990$. which gives rather small packing fraction for all values of $f(X)$.

6. One must estimate numerically the frequencies f_I using a sufficiently large sample of QC. One cannot exclude the possibility that p is scale dependent. On basis of the picture below one cannot make reliable estimates: large enough 3-dimensional sample would be needed.

The conclusion is that it might be possible to obtain a rather high packing fraction if the tetrahedron has circumradiud of dodecahedron using twists to avoid the intersections.

5. 3-D Penrose tiling as a dual of icosahedral QC and the dual of tetrahedral twisted QC?

For a novice like me it is easy to confuse notions like 3-D Penrose tiling and icosahedral QC. There seems however to be a deep difference. The point is that the cubic crystal of higher-dimensional space has a dual obtained by replacing the vertices of the cubes with the centers of their faces. In 3-D case cube and octahedron relate by this duality. The projections of both crystals to the lower-dimensional space are expected to define also QCs and these QCs should be dual in a well-defined sense.

Icosahedral QC and 3-D Penrose tiling seem to be dual to each other. For the icosahedral QC the basic unit is formed by the $2^6 = 64$ vertices of 6-cube whereas for the 3-D Penrose tiling by the 12 vertices dual to the vertices of 6-cube. n-D cube has $2n$ faces and for $n = 6$ one indeed obtains 12 faces characterized by unit vectors from origin to the centers of faces.

As described in the article of Dietl and Eschenburg, the projections of these vectors 2 suitably chosen 3-D space define the 12 vertices of icosahedron and one obtains space-filling QC for which icosahedrons can intersect. The analogy with 2-D Penrose tiling is complete. For 2-D Penrose tiling the 10 unit vectors pointing to the centers of the 10 faces of 5-cube define are projected to a suitable plane and define 10 sides of the basic unit appearing in Penrose tiling: also now the basic units can intersect.

For 3-D Penrose tiling the twisted tetrahedral QC construction would proceed just like I suggested first using the assumption that icosahedral symmetry implies decomposition to possibly overlapping icosahedrons. Since tetrahedron is its own dual, the natural expectation is that the two constructions are related by a duality. The 3-D Penrose tiling however provides a considerably simpler approach.

Since neighboring icosahedrons can intersect, also the the tetrahedrons inside them can do so. One can choose the imbedding of the tetrahedron inside icosahedron to minimize intersections of the neighboring tetrahedra. There are 5 different imbeddings to consider. One can also perform a twist around one of the 4 tetrahedral axes to minimize the intersections. The optimum situation would be that neighboring tetrahedra have parallel faces: the entire faces cannot intersect so that there is necessarily an "offset".

It is not possible to estimate the packing fraction by considering only single icosahedron since the isosahedra can intersect. One must estimate the packing fraction statistically as the ratio $p = V/N(T)V(T)$, where V is large enough volume of 3-D Penrose tiling containing integer number of possibly intersecting icosahedra, $V(T)$ is the volume of tetrahedron having same circumradius as icosahedron, and $N(T)$ is the number of tetrahedra in volume V. The lower bound for packing fraction is just the ratio of volumes of tetrahedron and icosahedron having same circumradius R. The formulas of Wikipedia give for the volumes of tetrahedron and icosahedron expressed in terms of R read as

$$\begin{aligned} V(T) &= \frac{(\frac{3}{8})^{3/2}}{6\sqrt{2}} \times R^3 \\ V(I) &= \frac{5}{12}(3+\sqrt{5})(\frac{2^3}{\Phi 5^{1/2}})^{3/2}R^3 \ , \\ \Phi &= \frac{1}{2}(1+\sqrt{5}) \ . \end{aligned} \quad (4.16)$$

This gives the lower bound

$$p \ > \ \frac{V(T)}{V(I)} = \frac{(2^2\Phi)^{3/2}}{3^{3/2}5^{1/4}(3+5^{1/2})} = .20235 \ . \quad (4.17)$$

The lower bound is rather small (smaller than the earlier incorrect estimate) so that the intersections of isosahedra should eliminate a considerable portion of volume. For instance, for $p = .81$ the average volume if isosahedron not intersecting neighbors should be about 1/4 of its real volume would correspond a scaling down of effective icosahedron circumradius by a factor $\lambda = 1/4^{1/3} \simeq .62996$. One might hope that the rather large number of neighbors (isohedron has 20 faces) could induce a large reduction of volume not intersected by neighbors.

One cannot exclude the scaling up of the tetrahedral volume by a factor $\lambda^{3/2}$ to improve p. Skeptic can also ask whether the proposed 3-D Penrose tiling really gives maximal packing fraction: maybe the packing fraction is indeed extremum but minimum instead of maximum and maximum for the icosahedral QC?

There might be an interesting relationship to E_8 lattice known to define a 4-D QC having as 3-D cross section an icosahedral QC. The symmetry group of this QC is Weyl group of roots system of E_8 generated by reflections known as G_0. G_0 has as a subgroup the group G_1 acting as isometries of 4-D regular polytope known as 600-cell and defining the point group of 4-D QC. This 4-D QC has as cross-section 3-D QC with icosahedral symmetry. Could one consider also now a dual construction giving at the first step 4-D object with 16 3-facets and having as a cross section icosahedron having 20 2-facets.

6. Geometric realization for the sequence of Fibonacci numbers?

The construction recipe for icosadodecahedron can be thought of as scaling the sphere containing either icosahedron/dodecahedron upwards by keeping the sizes of triangles/pentagons constant (this brings in mind the jitterbug transformation). In both cases the outcome contains empty regions which are filled with pentagons/triangles with the same edge length to surround the unscaled triangle/pentagon so that a complete filling results. The radius of the expanded sphere is fixed from the condition that the edge lengths of pentagons and triangles are same. The ratio of the total area of 12 pentagons to that for 20 triangles is

$$r = 12 \times A_5/20 \times A_3 = \frac{\sqrt{3}}{5cos(\pi/5)} \simeq 0.42819 \ .$$

The ratio is smaller than one as it should be. From Golden Triangle one can deduce by elementary considerations the expression

$$\cos(\pi/5) = \frac{3+5\Phi}{2(2+3\Phi)} \ .$$

Dodecahedron, icosahedron, and icosadodecahedron = dodecahedron+ icosahedron correspond to Fibonacci numbers 3,5,8. The two interpretations for the decomposition of the icosadodecahedron correspond to the commutativity of sum operation: 3+5=5+3. 3+5 means that triangles are surrounded by three pentagons, and 5+3 that pentagons are surrounded by triangles. The process leading to icosadodecahedron is also analogous to growth process (Fibonacci numbers indeed define growth process!) or to cosmic expansion with astrophysical objects preserving their sizes and empty regions filled with new objects (new matter from vacuum energy.

These analogies make one wonder whether this process could be continued indefinitely to produce a geometric realization for the entire Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$ giving 8=3+5, 5+8=13, 8+13=21, etc... by similar rules expressing the commutativity of addition and associativity of addition giving 3+(3+5)=3+(5+3)= (3+5)+3=(5+3)+3=(3+3)+5=5+(3+3). For instance , 3+(3+5) would mean a decomposition in which triangles are surrounded by three groups (triangle+3 pentagons).

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REFERENCES

Mathematics

[A1] G. Cornelissen and F. Kato. The p-Adic Icosahedron. http://www.ams.org/notices/200507/fea-cornelissen.pdf, 2006.

[A2] N. G. de Brujin. Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II. *Nederl. Akad. Wetensch. Indag. Math.*, 43(1):39,52,53–66, 1981.

[A3] M. L. Lapidus. *In Search of the Riemann Zeros: Strings, Fractal Membrans and Noncommutative Spacetimes.* American Mathematical Society, 2008.

Books related to TGD

[K1] M. Pitkänen. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#class, 2006.

[K2] M. Pitkänen. Cosmic Strings. In *Physics in Many-Sheeted Space-Time.* Onlinebook. http://tgdtheory.com/public_html/tgdclass/tgdclass.html#cstrings, 2006.

[K3] M. Pitkänen. DNA as Topological Quantum Computer. In *Genes and Memes.* Onlinebook. http://tgdtheory.com/public_html/genememe/genememe.html#dnatqc, 2006.

[K4] M. Pitkänen. Identification of the Configuration Space Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#kahler, 2006.

[K5] M. Pitkänen. Magnetospheric Sensory Representations. In *Magnetospheric Consciousness.* Onlinebook. http://tgdtheory.com/public_html/magnconsc/magnconsc.html#srepres, 2006.

[K6] M. Pitkänen. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness.* Onlinebook. http://tgdtheory.com/public_html/tgdconsc/tgdconsc.html#nmpc, 2006.

[K7] M. Pitkänen. *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook.http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html, 2006.

[K8] M. Pitkänen. The Recent Vision About Preferred Extremals and Solutions of the Modified Dirac Equation. In *Quantum Physics as Infinite-Dimensional Geometry.* Onlinebook. http://tgdtheory.com/public_html/tgdgeom/tgdgeom.html#dirasvira, 2012.

[K9] M. Pitkänen. What p-Adic Icosahedron Could Mean? And What about p-Adic Manifold? In *TGD as a Generalized Number Theory.* Onlinebook. http://tgdtheory.com/public_html/tgdnumber/tgdnumber.html#picosahedron, 2013.

Articles about TGD

[L1] M. Pitkänen. Sheldrakes Morphic Fields and TGD View about Quantum Biology. http://tgdtheory.com/articles/sheldrake.pdf, 2011.

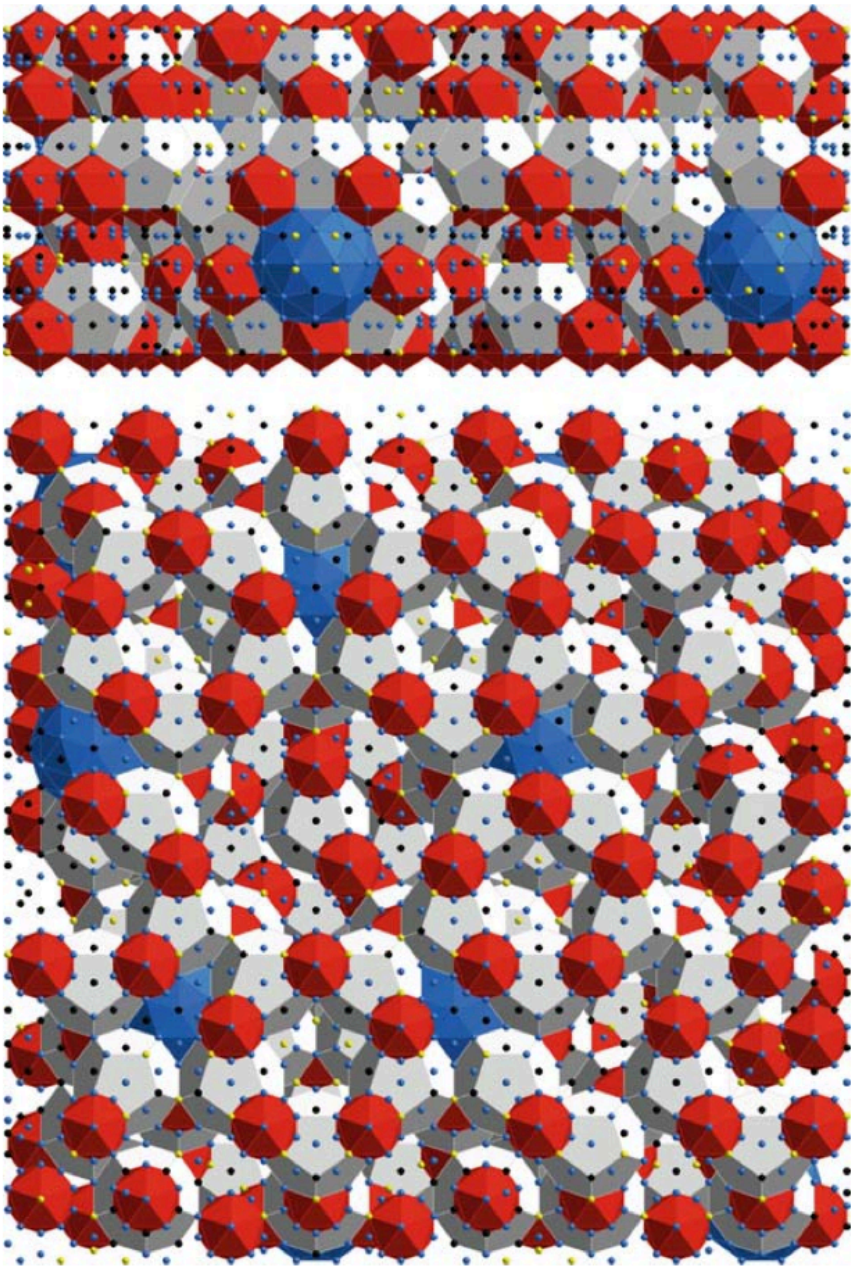

Figure 1: Figure illustrating the supercells of icosahedral QC (taken from page 306 of the book "Crystallography of Quasicrystals" by Steurer and Deloudi).