Could quantum randomness have something to do with classical chaos?

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Abstract

Tim Palmer has proposed that classical chaos and quantum randomness might be related. It came as a surprise to me that these to notions could a have deep relationship in TGD framework.

1. Strong form of Palmer's idea stating that quantum randomness reduces to classical chaos certainly fails but one can consider weaker forms of the idea. Even these variants fail in Copenhagen interpretation since strictly speaking there is no classical reality, only wave function coding for the knowledge about the system. Bohr orbits should be more than approximation and in TGD framework space-time surface as preferred extremal of action is analogous to Bohr orbit and classical physics defined by Bohr orbits is an exact part of quantum theory.

2. In the zero energy ontology (ZEO) of TGD the idea works in weaker form and has very strong implications for the more detailed understanding of ZEO and $M^8 - M^4 \times CP_2$ duality. Ordinary ("big") state functions (BSFRs) meaning the death of the system in a universal sense and re-incarnation with opposite arrow of time would involve quantum criticality accompanied by classical chaos assignable to the correspondence between geometric time and subjective time identified as sequence of "small" state function reductions (SSFRs) as analogs of weak measurements. The findings of Minev et al give strong support for this view and Libet's findings about active aspects of consciousness can be understood if the act of free will corresponds to BSFR.

M^8 picture identifies 4-D space-time surfaces X^4 as roots for "imaginary" or "real" part of octonionic polynomial $P_2 P_1$ obtained as a continuation of real polynomial $P_2(L - r) P_1(r)$, whose arguments have origin at the the tips of B and A and roots a the light-cone boundaries associated with tips. Causal diamond (CD) is identified intersection of future and past directed light-cones light-cones A and B. In the sequences of SSFRs $P_2(L - r)$ assigned to B varies and $P_1(r)$ assigned to A is unaffected. L defines the size of CD as distance $\tau = 2L$ between its tips.

Besides 4-D space-time surfaces there are also brane-like 6-surfaces corresponding to roots $r_{i,k}$ of $P_i(r)$ and defining "special moments in the life of self" having $t_i = r_{i,k}$ ball as M^4_+ projection. The number of roots and their density increases rapidly in the sequence of SSFRs. The condition that the largest root belongs to CD gives a lower bound to it size L as largest root. Note that L increases.

Concerning the approach to chaos, one can consider three options.

Option I: The sequence of steps consisting of unitary evolutions followed by SSFR corresponds to a functional factorization at the level of polynomials as sequence $P_2 = Q_1 \circ Q_2 \circ ... Q_n$. If the size of CD is assumed to increase, also the tip of active boundary of CD must shift so that the argument of P_2 $r - L$ is replaced in each iteration step to with updated argument with larger value of L.

Option II: A completely unexpected connection with the iteration of analytic functions and Julia sets, which are fractals assigned also with chaos interpreted as complexity emerges. In a reasonable approximation quantum time evolution by SSFRs could be induced by an iteration of a polynomial or even an analytic function: $P_2 = P_2 \to P_2^{\circ 2} \to ....$ For $P_2(0) = 0$ the roots of the iterate consists of inverse images of roots of P_2 by $P_2^{\circ - k}$ for $k = 0, ..., N - 1$.

Suppose that M^8 and X^4 are complexified and thus also $t = r$ and "real" X^4 is the projection of X_c^4 to real M^8. Complexify also the coefficients of polynomials P. If so, the Mandelbrot and Julia sets (http://tinyurl.com/cplj9pe and http://tinyurl.com/cvmr83g) characterizing fractals would have a physical interpretation in ZEO.

One approaches chaos in the sense that the $N - 1$:th inverse images of the roots of P_2 belonging to filled Julia set approach to points of Julia set of P_2 as the number N of iterations increases. Minimal L would increase with N if CD is assumed to contain all roots. The density of the roots in Julia set increases near L since the size of CD is bounded by the size Julia set. One could perhaps say that near the $t = L$ in the middle of CD the life of self when the size of CD has become almost stationary, is the most intensive.

Option III: A conservative option is to consider also real polynomials $P_2(r)$ with real argument r. Only non-negative real roots r_n are of interest whereas in the general case one considers all values of r. For a large N the new roots with possibly one exception would approach to the real Julia set obtained as a real projection of Julia set for complex iteration.

How the size L of CD is determined and when can BSFR occur?

Option I: If L is minimal and thus given by the largest (non-exceptional) root of iterate of P_2 in Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). L should smaller than the sizes of Julia sets of both A and B since the iteration gives no roots outside Julia sets.

Could BSFR become probable when L as the largest allowed root for iterate P_2 is larger than the size of Julia set of A? There would be no more new "special moments in the life of self" and this would make death (in universal sense) and re-incarnation with opposite arrow of time probable. The size of CD could decrease dramatically in the first iteration for P_1 if it is determined as the largest allowed root of P_1: the re-incarnated self would have childhood.

Option II: The size of CD could be determined in SSFR statistically as an allowed root of P_2. Since the density of roots increases, one would have a lot of choices and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time. BSFR would occur for the same reason as for the first option.

Contents

1 Introduction

There was an interesting guest post by Tim Palmer in the blog of Sabine Hosssenfelder (`http://tinyurl.com/yx7htn3u`).

1.1 Palmer's idea

Consider first what was said in the post "Undecidability, Uncomputability and the Unity of Physics. Part 1" by Tim Palmer.

1. I understood (perhaps mis-) that the idea is to reduce quantum randomness to classical chaos. If this is taken to mean that quantum theory reduces to chaos theory, I will not follow. The precise rules of quantum measurement having interpretation as measurements performed for the observables - typically generators of symmetries - are very restrictive and it is extremely difficult to image that classical physics could explain them. Quantum theory is much more than probability theory. Probabilities are essentially moduli squared for probability amplitudes and this gives rise to interference and entanglement. Therefore the idea of reducing state function reduction (SFR) and quantum randomness to classical chaos does not look promising. One could however consider the possibility classical chaos is in some sense as a correlate for quantum randomness or associated with state function reductions.

2. The difficulty to combine general relativity (GRT) to quantum gravity was mentioned. The difficulty is basically due to the loss of Poincare symmetries in curved space-time. Already string models solve the problem by assuming that strings live in M^{10} or its spontaneous compactification. Strings are however 2-D, not 4-D, and this leads to a catastrophe. In TGD $H = M^4 \times CP_2$ allows to have Poincare invariance and conservation laws are not lost. In QFT picture this means that the existence of energy guarantees existence of Hamiltonian defining time evolution operator and S-matrix.

3. It was noticed that chaos in quantum theory cannot be assigned to Schrödinger equation. This is true and applies quite generally to unitary time evolution generated by unitary S-matrix acting linearly. It as also noticed that in statistical mechanism Liouville operator defines a linear equation for phase space probability distribution analogous to Schrödinger equation. Liouville equation allows the classical system to be non-linear and chaotic. Could Schrödinger equation in some sense replace Liouville equation in in quantum theory since phase space ceases to make sense by Uncertainty Principle.

   Could Schrödinger equation allow in some sense non-linear chaotic classical systems? In Copenhagen interpretation no classical system exists except at macroscopic limit as an approximation. One has only wave function coding for the knowledge about physical system changing in quantum measurement. There is no classical reality and there are no classical orbits of particle since one gives up the notion of Bohr orbit. Could Bohr orbit be more than approximation?

The author considers also the question about definition of chaos.

1. Chaos is difficult to define in GRT. The replacement time coordinate with its logarithm exponentially growing difference becomes linearly growing and one does not have chaos. By general coordinate invariance this definition of chaos does not therefore make sense.

2. Strange attractors are typical asymptotic situations in chaotic systems and can make sense also in general relativity (GRT). They represent lower dimensional manifolds to which the dynamics of the system is restricted asymptotically. It is not possible to predict to which strange attractor the chaotic dynamical system ends up. This definition of chaos makes sense also in GRT.

Remark: One must remember that the notion of chaos is often used in misleading sense. The increase of complexity looks like chaos for external observer but need not have anything to do with genuine chaos.

1.2 Could TGD allow realization of Palmer's idea in some form?

It came as a surprise to me that these to notions could a have deep relationship in TGD framework.

1. Strong form of Palmer's idea stating that quantum randomness reduces to classical chaos certainly fails but one can consider weaker forms of the idea. Even these variants fail in Copenhagen interpretation since strictly speaking there is no classical reality, only wave function coding for the knowledge about the system. Bohr orbits should be more than approximation and in TGD framework space-time surface as preferred extremal of action is analogous to Bohr orbit and classical physics defined by Bohr orbits is an exact part of quantum theory.

2. In the zero energy ontology (ZEO) of TGD the idea works in weaker form and has very strong implications for the more detailed understanding of ZEO and $M^8 - M^4 \times CP_2$ duality. Ordinary ("big") state functions (BSFRs) meaning the death of the system in a universal sense and re-incarnation with opposite arrow of time would involve quantum criticality accompanied by classical chaos assignable to the correspondence between geometric time and subjective time identified as sequence of "small" state function reductions (SSFRs) as analogs of weak measurements. The findings of Minev *et al* [L7] give strong support for this view [L7] and Libet's findings about active aspects of consciousness [J1] can be understood if the act of free will corresponds to BSFR.

M^8 picture identifies 4-D space-time surfaces X^4 as roots for "imaginary" or "real" part of octonionic polynomial $P_2 P_1$ obtained as a continuation of real polynomial $P_2(L - r)P_1(r)$, whose arguments have origin at the the tips of B and A and roots a the light-cone boundaries associated with tips. Causal diamond (CD) is identified intersection of future and past directed light-cones light-cones A and B. In the sequences of SSFRs $P_2(L - r)$ assigned to B varies and $P_1(r)$ assigned to A is unaffected. L defines the size of CD as distance $\tau = 2L$ between its tips.

Besides 4-S space-time surfaces there are also brane-like 6-surfaces corresponding to roots $r_{i,k}$ of $P_i(r)$ and defining "special moments in the life of self" having $t_i = r_{i,k}$ ball as M_+^4 projection. The number of roots and their density increases rapidly in the sequence of SSFRs. The condition that the largest root belongs to CD gives a lower bound to it size L as largest root. Note that L increases.

Concerning the approach to chaos, one can consider three options.

Option I: The sequence of steps consisting of unitary evolutions followed by SSFR corresponds to a functional factorization at the level of polynomials as sequence $P_2 = Q_1 \circ Q_2 \circ ... Q_n$. The size L of CD increases if it corresponds to the largest root, also the tip of active boundary of CD must shift so that the argument of P_2 $L - r$ is replaced in each iteration step to with updated argument with larger value of L identifiable as the largest root of P_2.

Option II: A completely unexpected connection with the iteration of analytic functions and Julia sets, which are fractals assigned also with chaos interpreted as complexity emerges. In a reasonable approximation quantum time evolution by SSFRs could be induced by an iteration of a polynomial or even an analytic function: $P_2 = P_2 \to P_2^{\circ 2} \to ....$ For $P_2(0) = 0$ the roots of the iterate consists of inverse images of roots of P_2 by $P_2^{\circ -k}$ for $k = 0, ..., N - 1$.

Suppose that M^8 and X^4 are complexified and thus also $t = r$ and "real" X^4 is the projection of X_c^4 to real M^8. Complexify also the coefficients of polynomials P. If so, the Mandelbrot and Julia sets (http://tinyurl.com/cplj9pe and http://tinyurl.com/cvmr83g) characterizing fractals would have a physical interpretation in ZEO.

Chaos is approached in the sense that the inverse images of the roots of P_2 assumed to belong to filled Julia set approaching to points of Julia set of P_2 as the number N of iterations increases in statistical sense. The size L as largest root of $P_2^{\circ N}$ would increase with N if CD is assumed to contain all roots. The density of the roots in Julia set increases near L since the size of CD is bounded by the size Julia set. One could perhaps say that near the $t = L$ in the middle of CD the life of self when the size of CD has become almost stationary, is the most intensive.

Option III: A conservative option is to consider only real polynomials $P_2(r)$ with real argument r. Only non-negative real roots r_n are of interest whereas in the general case one considers all values of r. For a large N the inverse iterates of the roots of P_2 would approach to the real Julia set obtained as a real projection of Julia set for complex iteration.

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Could BSFR become probable when L as the largest allowed root for $P_2^{\circ N}$ is larger than the size of Julia set of A? There would be no more new "special moments in the life of self" and this would make death and re-incarnation with opposite arrow of time probable. The size of CD could decrease dramatically in the first iteration for P_1 if it is determined as the largest allowed root of P_1: the re-incarnated self would have childhood.

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The fact that fractals quite generally assignable to iteration (`http://tinyurl.com/ctmcdx5`) appear everywhere gives direct support for the ZEO based view about consciousness and self-organization and would give a completely new meaning for "self" in "self-organization" [L11]. Fractals, quantum measurement theory, theory of self-organization, and theory of consciousness would be closely related.

2 Could classical chaos and state function reduction relate to each other in TGD Universe?

In the sequel the idea about connection between chaos in some sense and state function reductions as they are understood in ZEO is discussed.

2.1 Classical physics is an exact part of quantum physics in TGD

Concerning the relation between classical and quantum the situation changes in TGD framework. Classical physics becomes an exact part of quantum theory. In zero energy ontology (ZEO) quantum states are superpositions of space-time surfaces preferred extremals of basic variational principle connecting 3-surfaces at opposite boundaries of causal diamond (CD). This solves the well-known basic problem of quantum measurement theory. Unitary time evolution operator or its generalization are totally different things from classical time evolution defined by highly non-linear field equations. There is nothing preventing quantum counterpart of chaos - it need not be classical chaos at space-time level but could correspond to some other form of chaos. Ordinary state function reduction in ZEO involves naturally quantum criticality involving long range quantum fluctuations assignable to chaotic systems so that the correlation between classical chaos defined in proper manner and state function reduction might make sense.

2.2 TGD space-time and $M^8 - H$ duality

$M^8 - H$ duality combined with zero energy ontology (ZEO) is central for the TGD inspired proposal for the connection between chaos and quantum.

2.2.1   Basic vision

Consider first what TGD space-time is.

1. In TGD framework space-times can be regarded 4-surfaces in $H = M^4 \times CP_2$ or in complexi-fiation of octonionic M^8. Linear Minkowski coordinates or Robertson-Walker coordinates for light-cone (used in TGD based cosmology) provide highly unique coordinate choice and this problem disappears. Exponential divergence in M^4 coordinates could be used as a symptom for a chaotic behavior.

2. The solutions of field equations are preferred extremals satisfying extremely powerful addi-tional conditions giving rise to a huge generalization of the ordinary 2-D conformal symmetry to 4-D context. In fact, twistor twist of TGD predicts that one has minimal surfaces, which are also extremals of 4-D Kähler action apart from 2-D singularities identifiable as string world sheets and partonic 2-surfaces having a number theoretical interpretation. The huge symmetries act as maximal isometry group of "world of classical worlds" (WCW) consisting of preferred extremals connecting pair of 3-surfaces, whose members are located at boundaries of causal diamond (CD). These symmetries strongly suggest that TGD represents completely integrable system and thus non-chaotic and diametrical opposite of a chaotic system. There-fore the chaos - if present - would be something different.

$M^8 - H$ duality suggests an analogous picture at the level of M^8. $M^8 - H$ duality in itse most restrictive form states that space-time surfaces are characterized by "roots" of rational polynomials extended to complexified octonionic ones by replacing the real coordinate by octonionic coordinate o [L2, L3, L4].

1. One can define the imaginary and real parts $IM(P)$ and $RE(P)$ of $P(o)$ in octonionic sense by using the decomposition of octonions $o = q_1 + I_4 q_2$ to two quaternions so that $IM(P)$ and $RE(P)$ are quaternion valued. For 4-D space-time surfaces one has either $IM(P) = 0$ or $RE(P) = 0$ in the generic case. The curve defined by the vanishing of imaginary or real part of complex function serves as the analog.

2. If the condition $P(0) = 0$ is satisfied, the boundary of δM_+^8 of M^8 light-cone is special. By the light-likeness of δM_+^8 points the polynomial $P(o)$ at δM_+^8 reduces to ordinary real polynomial $P(r)$ of the radial M^4 coordinate r identifiable as linear M^4 time coordinate t: $r = t$.

   Octonionic roots $P(o) = 0$ at M^8 light-cone reduce to roots $t = r_n$ of the real polynomial $P(r)$ and give rise to 6-D exceptional solutions with $IM(P) = RE(P) = 0$ vanish. The solutions are located to δM_+^8 and have topology of 6-sphere S^6 having 3-balls B^3 with $t = r_n$ as of M_+^4 projections. The "fiber" at point of B^3 with radial M^4 coordinate $r_M \leq r_n$ is 3-sphere $S^3 \subset E^4 \subset M^8 = M^4 \times E^4$ contracting to point at the δM_+^4.

   These 6-D objects are analogous to 5-branes in string theory and define "special moments in the life of self". At these surfaces the 4-D "roots" for $IM(P)$ or $RE(P)$ intersect and intersection is 2-D partonic surface having interpretation as a generalization of vertex for particles generalized to 3-D surfaces (instead of strings). In string theory string world sheets have boundaries at branes. Strings are replaced with space-time surfaces and branes with "special moments in the life of self".

   Quite generally, one can consider gluing 4-D "roots" for different polynomials P_1 and P_2 at surface $t = r_n$ when r_n is common root. For instance, P and its iterates $P^{\circ N}$ having r_n and the lower inverse iterates as common roots can be glued in this manner.

3. It is possible complexify M^8 and thus also r. Complexification is natural since the roots of P are in general complex. Also 4- space-time surface is complexified to 8-D surface and real space-time surface can be identified as its real projection.

To sum up, space-time surfaces would be coded a polynomial with rational or at most algebraic coefficients. Essentially the discrete data provided by the roots r_n of P would dictate the space-time surface so that one would have extremely powerful form of holography.

One can consider generalizations of the simplest picture.

1. One can also consider a generalization of polynomials to general analytic functions F of octo-nions obtained as octonionic continuation of a real function with rational Taylor coefficients: the identification of space-time surfaces as "roots" of $IM(F)$ or $RE(F)$ makes sense.

2. What is intriguing that for space-time surfaces for which $IM(F_1) = 0$ and $IM(F_2) = 0$, one has $IM(F_1F_2) = RE(F_1)IM(F_2) + IM(F_1)RE(F_2) = 0$. One can multiply space-time surfaces by multiplying the polynomials. Multiplication is possible also when one has $RE(F_1) = 0$ and $IM(F_2) = 0$ or $RE(F_2) = 0$ or $IM(F_1) = 0$ since one has $RE(F_1F_2) = RE(F_1)RE(F_2) - IM(F_1)IM(F_2) = 0$.

   For $IM(F) = 0$ type space-time surfaces one can even define polynomials analytic functions of the space-time surface with rational Taylor coefficients. One could speak of functions having space-time surface as argument, space-time surface itself would behave like number.

3. One can also form functional composites $P \circ Q$ (also for analytic functions with complex coefficients). Since $P \circ Q$ at $IM(Q) = 0$ surface is quaternionic, its image by P is quate-rionic and satisfies $IM(P \circ Q) = 0$ so that one obtains a new solution. One can iterate space-time surfaces defined by $Im(P) = 0$ condition by iterating these polynomials to give $P, P^{circ2}, ..., P^{\circ N}...$ From $IM(P) = 0$ solutions one obtains a solutions with $RE(Q) = 0$ by multiplying the M^8 coordinates with I_4 appearing in $o = q_1 + I_4 q_2$.

   The $Im(P) = 0$ solutions can be iterated to give $P \to P \circ P \to ..$, which suggests that the sequence of SSFRs could at least approximately correspond to the dynamics of iterations and generalizations of Mandelbrot and Julia sets and other complex fractals and also their space-time counterparts. Chaos (or rather, complexity theory) including also these fractals could be naturally part of TGD!

2.2.2   Building many-particle states at the level of M^8

The polynomials defining surfaces in M^8 are defined in preferred M^8 coordinates with preferred selection of M^8 time axis M^1 as real octonionic axis and one octonionic imaginary axes charac-terizing subspace $M^2 \subset M^8$. $M^4 \subset M^8$ is quaternionic subspace containing M^2. Different choices of $M^4 \sup M^2$ are labelled by points of CP_2 and $M^8 - H$ duality maps these choices to points of CP_2.

The origin of $M8$ coordinates coordinates must be at M^1 so that the 8-D Poincare symmetry reduces to time translations and rotations of around spatial coordinate axis M^2 respecting the rationality of polynomial coefficients or in more general case the extension of rationals associated with the coefficients. This corresponds to a selection of quantization axis for energy and angular momentum and could have a deeper meaning in quantum measurement theory.

The Lorentz transformations of M^4 change the direction of time axis and also M^2 in the general case and generate new octonionic structure and quaternonic structure. One should understand how space-time regions as roots of octonionic polynomials with different rest frames relate to each other.

The intuitive picture is that each particle as a region determined by octonionic polynomial corresponds to its own CD and rest frame determined by its 4-momentum in fixed coordinate frame for M^4. Also quantization axis of spin fixed. One can assign CD for to interacting many particle system with common rest frame. One can speak of external (incoming and outgoing) free particles with their own CDs characterizing their rest systems. The challenges is to related the polynomials P_n associated with the external particles to the polynomial characterizing the interacting system.

1. Assume that the polynomial defining the CD is product P_1P_2 of polynomials P_1 and P_2 assignable to its active and passive boundaries with origins of octonionic coordinates at the tips $t = 0$ and $t = \tau$ of CD. If the space-time surface reduces to the root of P_1 at passive boundary and root of $IM(P_2)$ at active boundary, one could say that the 3-surfaces at these boundaries correspond to P_1 and P_2 asymptotically. If these conditions are true everywhere, one has two un-correlated space-time surfaces, which does not make sense. $IM(P_1)RE(P_2) + RE(P_1)IM(P_2) = 0$ indeed allows more general solutions than $IM(P_1) = 0$ and $IM(P_2) = 0$ everywhere. The fact that the boundaries correspond to special 6-D brane like solutions in M^8 suggests that it is possible to pose the boundary condition $IM(P_1) = 0$ *resp.* $IM(P_2) = 0$ at the boundaries.

2. The formation of products is possible also at the boundaries so that one can assume that P_i at the boundary of many-particle CD is with product $P_i = \prod_k P_{ik}$. The boundary conditions would read read $P_{ik} = 0$ at active *resp.* passive boundary of many-particle CD respectively. The interpretation would be that P_{ik} corresponds to an external particle which is in interacting state at active boundary. In the interior of many-particle CD only the condition $Im(P_1 P_2) = 0$ would hold true so that interactions of particles would have algebraic description.

3. One should also understand how the external particles characterized by CDs with different rest frame are glued to the boundary of many-particle CD. Assume that M^4 is same for all these particles so that CP_2 coordinates are same. The boundaries of 4-D CDs are 3-D light-cones with different origins so that their M^4 intersection is 2-D defining a 2-D surface at the boundary of CD. The interpretation in terms of partonic 2-surface suggests itself. The partonic 2-surfaces of free particle and its interacting variant would be same at the intersection.

   The gluing should correspond to a root $t = r_n$ of polynomial defining a "special moment in the life of self". The roots of P_1 and its Lorentz boots as values of coordinates at light-radial geodesic are related by Lorentz boost and are not same in general. One could require that the root r_n and its Lorentz boost belong to the 2-D interaction of two light-cones and thus define two points of partonic 2-surface. These points would not be identical and the interaction would be non-local in the scale of partonic 2-surface. It seems that the condition that root r_n and Lorentz boost $L(rm)$ co-incide would pose too strong constraints on external momenta.

2.3 In what sense chaos/complexity could emerge in TGD Universe?

Consider now in what sense chaos (or complexity, one must be precise here) could emerge in TGD framework?

1. Chaos (or complexity) could be an approximate property emerging in number theoretical discretization for cognitive representations labelled by extensions of rationals as the dimension of extension and therefore algebraic complexity increases ad the number of points in cognitive representation as points of M^8 with coordinates in the extension of rationals increases. The minimal number of points corresponds to the degree of the polynomial determining the extension. At the limit of maximal complexity the extension would consists of algebraic numbers and the cognitive representation would be dense subset of space-time surface. It is not clear whether the roots r_n are also dense along time axis.

2. Also transcendental extensions of rationals can be considered. Typically they are infinite-D in both real and p-adic sectors. Exponential function is however number-theoretically completely unique. Neper number e and its roots define infinite-D extensions of rationals but - rather remarkably - finite-dimensional extensions of p-adic numbers since e^p is ordinary p-adic number. Extension of rationals would become infinite-D but the induced extensions of rationals would remain finite-D in accordance with the idea that cognition is always finite-D.

   Could one allow e and its roots and thus exponential functions besides polynomials? Could exponential divergence be the hallmark of chaos or perhaps the first step in the transition to transcendental chaos (or rather, complexity)? Could chaos (complexity) in real sense be possible for extensions of rationals generated by a root of e? One can however argue that the finite dimension of induced p-adic extensions means that cognitive chaos is not yet present.

   For general transcendentals the dimensions of p-adic extensions are infinite and one would have also cognitive chaos (infinite complexity). Could the transition to chaos means the emergence of analytic functions with rational coefficients having also roots, which are transcendentals. Chaos would mean that one can only approximated f analytic function as a polynomial giving approximation for the roots. By $M^8 - H$ duality these roots would correspond to values of M^4 time inside light-cone, preferred moments of time [L9]. These would become transcendental and in general p-adic extension would become infinite-D.

3. An interesting analogy with real numbers emerges. Real numbers have expansion in powers of any integer, in particular any prime p. The sequence defined by the coefficients of the expansion are analogous to an orbit of a discrete dynamical system. For transcendentals the expansion is unpredictable and analogous to a chaotic orbit.

   For rationals this expansion is periodic so that one has analog of a periodic orbit. This applies also to expansion of rationals formed from the integers in finite-D extensions of rationals. One must of course accept that the algebraic numbers defining the roots do not allow periodic expansion but one can do all calculations in extension and perform approximation only at end of computation. Therefore the extensions of rationals represent also islands of order in the ocean of trancendental chaos. Could one see he gradual increase of the dimension of extension of rationals as a transition to chaos: of course, chaos would be wrong term since increase in algebraic complexity, which corresponds to evolution in TGD Universe is in question. Cognition becomes more and more refined.

4. As found, space-time surfaces behave like numbers and one can have functions having space-time surface as argument. Could the picture about emergence of chaos for reals be translated to the level of space-time surfaces identified as "roots" of octonion analytic function in M^8? The polynomial space-time surfaces would represent islands of order in chaos defined by general analytic functions with rational Taylor coefficients.

2.3.1 Can one imagine a connection between quantum randomness and chaos?

To my view, the reduction of quantum randomness to classical chaos is definitely excluded. Quantum classical correspondence allows however to consider a looser connection between quantum randomness and chaos.

1. The following considerations lead to a formulation of a more precise view about the sequence of steps consisting of a unitary evolution followed by SSFR as a a model of self. $M^8 - H$ duality involving representation of space-time surface in terms of a polymial with rational coefficients leads to an approximate model of the quantal time evolution by SSFRs as quantum counterpart for an iteration of a polynomial map, and gives a direct connection with chaos as algebraic complexity in the sense of generalization of Mandelbrot and Julia sets (`http://tinyurl.com/cplj9pe` and `http://tinyurl.com/cvmr83g`).

   The identification of time evolution as iteration $P \to P^{\circ 2} \to ...$ is very probably only an approximation. More general picture would assume that the corresponds to a functional factorization of P as $P = P_1 \circ P_2 \circ ... \circ P_n$. Even this assumption can be only approximate.

2. The fixed points of iteration would correspond to asymptotics for the evolution of space-time surface defined by iteration and approach of CD to a fixed point CD. This conforms with the idea that fixed points of iteration as representations of fractals, criticality and chaos. Chaos understood as genuine chaos could correspond to a fluctuation of the arrow of time in the sequence of SSFRs as a fixed point of iteration is reached.

It must be of course made clear that the view about $M^8 - H$ duality already considered and the view about the emergence of fractals to be discussed are only one of the many options that one can imagine and involve many poorly understood aspecs. Only time will tell whether the proposals work and how they must be improved.

2.3.2 Chaos and time

TGD Universe has gigantic symmeries [K1, K2] and looks like a completely integrable system and the idea about genuine chaos at space-time level does not look attractive. $M^8 - H$ duality suggests that chaos - actually complexity - in the sense of Mandelbrot fractals looks more promising idea. ZEO int turn suggests that chaos could be associated with the relationship between geometric and subjective time in the sense that the orderings of the two times would not be strictly identical.

1. Often the chaos is taken to mean increase of complexity (Mandelbrot and Julia sets), which actually means a diametric opposite of chaos. In TGD framework a more promising connection is between finite measurement resolution and complexity as that for extension of rationals. For trivial extensions of rationals the points of cognitive representation have rational M^8 (and becase also H-) coordinates. All other points fail to have a cognitive representation. For extensions of rationals the number of points in cognitive representations increases: the increase of cognitive complexity has actually nothing to do with emergence of a genuine chaos. Here one must be however very cautious and one must consider ZEO view about state function reduction in detail to see what happens.

2. $M^8 - H$ duality allows to consider a concrete example. The roots r_n of real rational polynomials P or even analytic functions correspond "special moments in the life of self". Could the increase of complexity be understood in terms of what happens for the roots. The number of these moments equals to the degree n of P and cognitive representation more and more complex since the dimension of extension equals to n: this could occur in BSFRs at least. The clock defined by the moments roots $t = r_k$ could become more precise. It will be found that in presence of quantum criticality the emerging complexity could also correspond to a genuine chaos.

3. One can define clock time as a temporal distance τ between tips of CD after "small" state function reduction (SSFR), which corresponds to weak measurement in standard picture. Passive boundary and the states at the passive boundary of CD remain unchanged (generalized Zeno effect) and the states at active boundary is change. Also the distance between tips of CD changes but increases in statistical sense.

   The statistical nature of the change implies that the ordering for subjective time as sequence of SSFRs is not quite the same as that for τ (one could of course assume that only increase of the CD size is possible in BSFR but this would be an ad hoc assumption). This corresponds to a kind of quantum randomness due to the state function reductions. If the number of roots is large and the average time chronon is small, the changes of time order could occur often. Could this have interpretation as a genuine chaos in short time scales due to SSFRs? This need not correspond to a genuine chaos at the level of space-surfaces as preferred extremals. Chaos as algebraic complexity could however increases and would be consistent with complete integrability: this happends in n increases in BSFRs.

2.3.3 Chaos in death according to ZEO

The assignment of a genuine chaos to death looks natural from what we know about biological death. Could this assignment make sense in ZEO where BSFR corresponds in a well-defined sense to death?

1. Recall that BSFR corresponds to ordinary state function reduction in which the arrow of geometric time identifiable as distance between the tips of CD changes: self dies and re-incarnates with an opposite arrow of time. The active boundary of CD becomes passive. The passive boundary becomes active and the size of CD starts to statistically increase in opposite time direction in SSFRs. The former passive boundary CD can remain at the critical moment but could also shift towards the former active boundary - the re-incarnated self would have small CD and could have "childhood."

   The continual increase of CD looks strange. Also our mental images would increase in size and unless one makes special assumptions (say that the average change of the size of CD is proportional to its size (scaling)) one ends up with difficulties. Time evolution as stepwise scaling would be indeed natural.

2. Under what conditions does BSFR - death and reincarnation - occur? A quantum criticality implying instability against BSFR should be involved. The size scales of CD as temporal distances τ between its tips would have critical values τ_{cr} at which death of self in this universal sense could take place. τ_{cr} could be integer multiple of CP_2 length scale with allowed integers being primes of preferred primes allowed by p-adic length scale hypothesis. Criticality

indeed involves long range fluctuations assigned with chaotic behavior: the simplest example is the transition to chaos in convection as energy feed to the system increases.

3. A concrete model for SSFRs [L12] suggests that one can assign to CD temperature T satisfying $T \propto 1/\tau$ so that the evolution of self would correspond to T as analog of cosmic temperature. Death could correspond to a critical temperature T_{cr} (τ_{cr}) and would be unavoidable. The quantum criticality assignable to death could correspond to the emergence of a genuine temporal chaos. The time order would become more and more ill-defined, and time τ would go forth and back so that eventually one would $\tau = \tau_{crit}$ as size of CD and death would occur. This however requires that the number of roots r_n increases so that also their density increases. This requires that the degree of the polynomial P defining the extension increases. Can this be consistent with the assumption that passive boundary does not change?

   Remark: Why I take this seriously is that I have had near death experience being in clinically unconscious but actually conscious state and I experienced quite literally the flow of time forth and back and was fighting to preserve the usual arrow of time.

4. This picture applies to all BSFRs and SSFRs and therefore to ordinary state functions reductions in all scales: the findings of Minev *et al* [L7] can be understood if the arrow of time indeed changes [L7]. There would be a connection between state function reductions and chaos understood as genuine chaos. The idea that this chaos corresponds to a strange attractor at space-time level is not plausible. Rather it could be analogous to chaos in the sense of an attractor of iteration of complex function by functional decomposition. Fixed point is also a fractal and corresponds to criticality.

2.3.4   What gives rise to the lethal quantum criticality, BSFR, and death?

What could give rise to quantum criticality leading to death and reincarnation of self as BSFR?

1. If P remains the same during SSFRs, one could think that once the CD size is so large that all "special moments in the life of self" have been experienced as time values $\tau = r_n$, the system is ready to die. But how could this give rise to quantum criticality?

2. Assume that CD is defined as the intersection of future and past light-cones and the polynomial P corresponds to a product $P_1(r)P_2(L - r)$ of polynomials associated with these two light-cones such that P_i vanishes at the tip of its light-cone corresponding to $r = 0$ *resp.* $L - r = 0$. P_1 associated with the passive boundary of CD would not change in SSFRs but P_2 associated with the active boundary would change. Most importantly its degree would increase and the number of roots and their density would increase too. Eventually the density of active roots would become so high that death as BSFR is bound to occur as event $\tau = \tau_{cr}$.

   Remark: One can consider two options: real M^8 and real r or complexified M^8 and complex r.

3. As already noticed, if the space-time surface reduces to the root of P_1 at passive boundary and root of P_2 at active boundary, one could say that the 3-surfaces at these boundaries correspond to P_1 and P_2 asymptotically. The fact that the boundaries correspond to special 6-D brane like solutions in M^8 suggests that the boundary conditions are possible.

4. The statistically increasing extension of rationals would correspond to "personal" evolution for the changing part of self during life cycle. Note that $n = h_{eff}/h_0$ corresponds to the scale of quantum coherence thus increasing. This extension would define the evolutionary level of the unchanging part ("soul") during the next re-incarnation.

2.3.5   Could polynomial iteration approximate quantum time evolution by SSFRs in statistical sense?

I have considered rather concrete models for the counterpart of S-matrix for given space-time surface [L5, L6, L13] but the deeper understanding of the sequence of SSFRs is still lacking although quite concrete proposals already exists.

Number theoretical vision suggests that also the time evolution by SSFRs should reduce to number theory being induced by some natural number theoretical dynamics.

1. The most general option is that in each SSFR a superposition over extensions defined by various polynomials with varying rational coefficients is generated. The idea about the correspondence of the sequence of SSFRs with a functional decomposition of polynomials is however attractive.

2. The sequence of unitary evolutions brings strongly in mind the iteration $U \to U^2 \to U^3$.... One can however consider also the possibly $U \to U_1 U \to U_2 U_1 U$.... The obvious guess for the iteration of U is that it is induced by a functional iteration of polynomial P_2 assigned to the active boundary of CD $P_2 \to P_2 \circ P_2 \to$ .... The more general option would not be iteration anymore but a composition of form $P_2 \to P_3 \circ P_2 \to$ .....

   The boundary conditions at the boundary of CD and at gluing points - possibly $t = r_n$ surfaces to which 6-branes are assignable as special solutions and identified as "special moments in the life of self" could make the superpositions of functional composites more probable contributions in the superposition. The polynomial $P \circ Q$ has same roots as Q (for $P(0) = Q(0) = 0$) and this favors conservative state function reductions preserving the state already achieved.

   Iteration would be even more conservative option. If the solutions assignable to P and Q are to be glued together along brane with $t = r_n$ they must share r_n as root. This would favor iterations if one has superposition over different rational coefficient values for P and Q with fixed degree.

   Remark: Also critical points of Q as zeros of derivative are preserved in $Q \to P \to Q$ as one finds by applying chain rule. For iteration both the new critical points/roots of $P \circ P^{\circ k}$ are inverse images of critical points/roots of $P^{\circ k}$. Only roots are of significance in the picture considered.

3. Superpositions of different iterates generated in the unitary time evolution preceding SSFR are required by the model of temporal chaos. SSFR selects extension of rationals and thus fixed iteration. In statistical sense the degree of iteration is bound to increase so that in statistical sense quantum iteration reduces to classical one. At the limit of fixed point of iteration the number of critical points $t = p_n$ and roots $t = r_n$ of the iterate increases as also their density along time axis and temporal chaos emerges leading to fluctuation of CD size τ.

4. Iteration of the real polynomial P satisfying $P(0) = 0$ would mean that one would have a series extensions obtained as powers of generating extension: E, $E \circ E$, $E \circ E \circ E$,... conserving the roots of E provided the polynomials involved vanish at origin: $P(0) = 0$. The proposal has been that biological evolution corresponds to a more general series of extensions $E_1, E_2 \circ E_1, E_3 \circ E_2 \circ E_1, ...$ Also now Galois groups in the series of them would be conserved. I have proposed that Galois groups are analogs of conserved genes [L1, L4].

The proposed picture is only one possibility to interpret evolution of self as iteration leading to chaos in the proposed sense.

1. One could argue that the polynomial $P_{nk} = P_n \circ .... \circ P_n$ associated with the active boundary remains the same during SSFRs as long as possible. This because the increase of degree from nk to $n(k+1)$ in $P_{nk} \to P_{nk} \circ P_n$ increases h_{eff} by factor $(k+1)/k$ so that the metabolic feed needed to preserve the value of h_{eff} increases.

   Rather, when all roots of the polynomials P assignable to the active boundary of CD are revealed in the gradual increase of CD preserving P_{nk}, the transition $P_{nk} \to P_{nk} \circ P_n$ could occur provided the metabolic resources allow this. Otherwise BSFR occurs and self dies and re-incarnates. The idea that BSFR occurs when metabolic resources are not available is discussed in [L14].

2. Could $P_{nk} \to P_{nk} \circ P_n$ occur only in BSFRs so that the degree n of P would be preserved during single life cycle of self - that n can increase only in BSFRs was indeed the original guess.

2.4 Basic facts about iteration of real polynomials

The iteration of real polynomials and also more general functions can be understood graphically. Assign to a x point $y = f(x)$ of the graph and reflect through the line $y = x$ and project to the graph to obtain the image point $x_1 = f(x)$. Fixed points $x = f(x)$ correspond to the intersections of the line $y = x$ and graph $y = f(x)$. The magnitude $|df/dx|$ at the intersection point determines whether it is attractor ($|df/dx| < 1$ or repellor ($|df/dx| \geq 1$) in which case large jumps in the value of x can occur, as one can easily check. Quite generally iteration in the part of the graph below (above) $y = x$ decreases (increases) x. Real polynomial $c - x^2$ provides a simple example.

Feigenbaum discovered by iterating logistic map numerically (`http://tinyurl.com/u3zwmar`) that the approach to chaos - not only for logistic map but - for real functions $f(x)$ with one quadratic maximum and depending on a varied parameter a is universal. Period-doubling bifurcations occur at parameter values satisfying at the limit $n \to \infty$

$$\frac{a_{N-1} - a_{N-2}}{a_N - a_{N-1}} \to 4.669201609... \ .$$

Second universality relates to the widths of tines - distances between the branches of bifurcation - appearing in the sequence of bifurcations. The ratio between width of the tine to widths of its sub-tines approaches at the limit $N \to \infty$ to constant given by

$$\alpha = 2.502907875095892822283902873218... \ .$$

.

In TGD framework conservative option would correspond to real M^4 so that the coordinates t and r would be real and the polynomials P_1 an P_2 would have real coefficients. The time evolution by iterations of P_2 would reduce to an iteration of a real polynomial P_2.

The number of real roots is in general smaller than the degree n of the polynomial. Only non-negative roots can be considered since one as $r \geq 0$ and $r = 0$ is a root. This condition could generalize to complex polynomials of complexified r as a condition $Re(r_c) \geq 0$ guaranteeing that roots are in the upper half plane for the variable $z = ir_c$.

The real polynomial $P(x)$ of degree n one has either positive or negative values between neighboring roots and at least one extremum between them. The n roots of $P_n(x)$ gives rise to Nn roots in N:th iteration and only non-negative ones are allowed. Since the roots are below the axis $y = s$, the root is obtained from the inverse of the roots by reflecting with respect to $y = x$ and projecting to the graph. The inverse of this operation increases the root. One has special case of complex iteration.

2.5 What about TGD analogs of Mandelbrot -, Julia-, and Fatou sets?

What about the interpretation of Mandelbrot -, Julia-, and Fatou sets (`http://tinyurl.com/cplj9pe` and `http://tinyurl.com/cvmr83g`) in the proposed picture? Could the iteration of P_2 define analogs of Mandelbrot and Julia fractals? This would give the long-sought-for connection between quantum physics and Mandelbrot and Julias sets, which are simply too beautiful objects to lack a physical application. Period-doubling bifurcations (`http://tinyurl.com/t2swmdg`) are involved with the iteration of real functions and relate closely to the complex fractals when the polynomials considered have real coefficients.

1. In the simplest situation both Mandelbrot and Julia sets are fractals associated with the iteration of complex polynomial $P_c(z) = z^2 + c$ where z and c are complex numbers (note that in TGD would have $c = 0$ in this case). One can consider also more general polynomials and even rational functions, in particular polynomial $f = P_2$ defined earlier, and replace $z = 0$ with any critical point satisfying $df/dz = 0$. Even meromorphic transcendental functions can be considered: what is required that the image contains the domain.

2. Mandelbrot set M is defined as the region of the plane spanned by the values of c for which the iteration starting from the critical point z_{cr} does not lead to infinity. Physically the restriction to Mandelbrot set looks natural.

3. For rational functions Julia set J_c (`http://tinyurl.com/cplj9pe` corresponds to a fixed value of c, and is defined as points z for which are unstable in the sense that for an arbitrary small perturbation of z iteration can lead to infinity. Inside J_c the iteration is repelling: $|f(w) - f(z)| > |w - z|$ for all w in neighbourhood of z within J_c. One can say that the behavior is chaotic within J_c and regular in its complement - Fatou set. Julia set can contain also cycles and iteration in J_c leads to these cycles. These cycles are analogs of the limit cycles appearing in the iteration of real-valued function discovered by Feigenbaum (`http://tinyurl.com/u3zwmar`).

   For polynomials Julia set can be identified as the boundary of the filled Julia set consisting of points for which iterates remain bounded. Also the inverse iterates in this set remain bounded. The filled J_c - denote it by $J_{c,in}$ - can be regarded as a set of points, which are inverse images of fixed points of the polynomial. All points except at most two points of J_c can be regarded as points in the limiting set for the union $\cup_n f^{-n}(z)$ of the inverse images for the points z in filled Julia set. Julia set and its complement Fatou set are invariant under both P and P^{-1} and therefore also under their functional powers. Julia set is the set of pre-images for practically any point of J_c: this can be used for computational purposes. If I have understood correctly there can be single exceptional point for which this is not the case. J_c can be regarded as a fractal curve. For parameter values inside M J_c is connected, which seems counter intuitive. For c outside the M, Julia set is a discrete Cantor space, Fatou dust.

   What is remarkable from TGD point of view is that the new roots obtained in N:th step of iteration are $N - 1$:th inverse images of the roots of P. Since polynomial iteration takes sufficiently distant points to ∞, its inverse does the opposite so that the roots of $P^{\circ N}$ are bounded: this strongly suggests that the roots of $P^{\circ N}$ are in J_c if those of P_2 are. One can say that the situation becomes chaotic at the large N limit since the number of roots increases without bound.

4. Fatou set F_c can be identified as the complement of Julia set. Fatou set fills the complex plane densely and has disjoint components, which contain at least one point with $df/dz = 0$ unless Fatou set contains $z = \infty$. Note however that critical point is ot fixed point as in gradient dynamics. This allows to deduce the number or at least upper bound for the number of components of Fatou set, which equals to the degree n of polynomial in the generic case. All components have entire J_c accumulation points. Since the points of J_c are infinitely near to more than 2 disjoint sets for Fatou set with more than 2 components, J_c cannot be a smooth curve in this case being thus fractal. However, the Julia set of $P = z^2 + c$ is also fractal although Fatou set has only two components corresponding to the critical point $z = 0$ and $z = \infty$.

A couple of examples are in order: for $P(z) = z^2$ Julia set is unit circle S^1 and Fatou set has interior and exterior of S^1 as its components. The cycles in Julia set correspond to roots of unity and the orbits of other points form dense sets of unit circle. For $P(z) = z^2 - 2$ Julia set is the interval $(-2, 2)$ having fixed points as its ends. Fatou set has only one component as the complement of Julia set. For $P(z) = z^2 + c$, c complex Julia set is in general fractal. Hence the roots of the polynomial need not belong to Julia set.

2.5.1   Emergence of Mandelbrot and Julia sets from ZEO assuming $M^8 - H$ duality

Consider now the application to TGD assuming $M^8 - H$ duality [L2, L3, L4, L10] .

1. In TGD framework complex numbers $x + iy$ emerge in the complexification of M^8 and i commutes with octonionic units. If space-time surfaces are identified as real projection of their complexified variants obtained as roots of polynomials one can consider also polynomials with complexified coefficients c. Note that c would be complex rational but one can also consider complex algebraic numbers. The most general situation corresponds to analytic functions with complex rational Taylor coefficients. Complex argument with complex coefficients is possible space-time surface is identified by projection the complex space-time surface to real part of complexified M^8 [L2, L3, L3].

2. The complexified light-like coordinate r at the active boundary CD defines the analog of z plane in which iterates of P_2 act. r corresponds directly to the complexified linear time coordinate t of M^8 (time-axis connects tips of CD) and the roots r_n of P_2 correspond to the "special moments in the life of self" as time values $t = r_n$. Assume that $P_2(0)$ vanishes so that r_n are also roots of iterates.

3. Julia set J_c bounds filled Julia set $J_{c,in}$ of the complexified r-plane, whose interior points remain inside $J_{c,in}$ in the iterations by fixed P_2. Julia set J_c is connected but the Fatou set as its complement has several components labelled by the $n-1$ points p_k satisfying $dP_2(z)/dz = 0$ and by $z = \infty$ so that Fatou set has n components. The inverse iterates of roots need not belong to Fatou sets not containing ∞ or to the filled Julia set.

4. There are several Mandelbrot sets and the extrema of P_2 satisfying $dP_2/dr = 0$ label them. The extrema of P_2 are also extrema of its iterates. There are $n-1$ extrema p_k. In the real case they can be classified as either attractors or repellors but in complex situation they correspond to saddle points. Denote by $M(p_n)$ the region of parameter space of polynomial coefficients c for which the iteration starting starting at $p(n)$ does not lead outside it.

   In the real case the iteration of P_2 leads to the attractors $t = p_k$. In complex case the situation is not so simple and the basic of attraction is replaced with the Fatou set $F_c(p_k)$.

   Since c parameterizes points in the space of polynomials characterizing space-time surfaces in TGD, Mandelbrot set can be defined as a sub-space of "world of classical worlds" (WCW). Inside $M(p_n)$ the iteration maps r_n to a point $M_{in}(r_n)$. Note that also new roots emerge in each iteration and the Mandelbrot set for the iterates contains more components.

   Remark: In TGD only the roots of P_2 are interesting. The roots of iterates are inverse iterates of roots of P_2.

   Could one understand the size of CD and its evolution during the iteration of P_2?

1. Consider first the situation for real time $t = r$ and real polynomials. Since the boundary of CD contains only the roots $t = r_n$, the simplest guess is that the size of CD corresponds to the largest root of $P_2^{\circ N}$. The size of CD would increase in the iterations. The inverse images of the roots approach to Julia set so that the real counterpart of Julia set is important for understanding the asymptotic situation. Mandelbrot set defines the coefficient values for which iteration does not lead to infinity.

2. The situation is essentially the same for complexified time. The size of CD would correspond to the modulus for the largest of the iterate root and increases during iteration. The size of CD approaches to that for a point in Julia set.

2.5.2 Could the iteration lead to a stationary size of CD?

One can represent an objection to the idea that quantum iteration of P_2 could be more than an approximation.

1. Suppose that the size of CD is determined by the maximum for the iterates of the roots of P_2. Suppose that the parameters c are fixed and belong to Mandelbrot set $M(p_k)$. For given c there is therefore an upper for $\tau = 2r$ given by $r = r_{max}(c, p_k)$ for the Fatou set $F_c(p_k)$. One gets stuck to fixed τ since maximal root cannot become larger than $r_{max}(c)$ in the iteration. Note that in this situation the number of roots of $P_2^{\circ k}$ increases and if they corresponds to "special moments in the life of self", this could lead to quantum criticality and occurrence of BSFR.

2. Fluctuations of τ in the sequences of SSFRs is possible if superpositions of iterates are allowed. This could cause BSFR would occur and eventually second BSFR would eventually lead to the original situation. If P_2 is not modified, the iteration continues and one is still at criticality. BSFR soon occurs and same repeats itself.

Is this situation acceptable? Maybe - I have considered the possibility that the size of CD remains below some upper bound [L12, L8]. The selves such as our mental images could continue to live in the geometric past and memories would be communications with them. Or should one get rid of this situation? How?

1. Assume that SSFR creates a superposition of iterates with varying values of parameters c belonging to the Mandelbrot set $M(P_2)$. The value of $r_{max}(c, p_k)$ depends on c and it is possible to increase the value of τ in statistical sense if SSFR selects the values of c suitably. The value of L would be however given by maximal root and would remain below the maximum r_{max} of $r_{max}(c, p_k)$ in $M(P_2)$ if c belongs to $M(P_2)$. $\tau = 2L$ would remain below the maximum for the size of $J_c(P_2)$ in $M(P_2)$. One would get stuck if this size is finite, which is the case if $r_{max}(c, p_k)$ is bounded as function of c and p_k?

   Is $r_{max}(c, p_k)$ bounded? The polynomials with given degree of can have arbitrarily large roots and critical points in the same extension of rationals. Therefore it might be possible to avoid getting stuck if there is no restriction on the size of the roots of P_2 in the superposition over different values of c.

2.5.3 When death occurs and can self have a childhood?

I hope that talking about death and reincarnation does not irritate the reader too much. I use these terms as precisely defined technical terms applying universally. There are two extreme options for what happens to the former passive boundary in BSFR. The real situation could be between these two.

1. The first shift after reincarnation is to geometric past so that CD size increases.

2. The first shift is towards the former active boundary so that the size of CD decreases at least to the size of CD when the iteration of P_2 began. The reincarnated self would have "childhood" and would start from scratch so to say.

Consider $P_1 P_2$ option. Suppose that time evolution is induced by iteration of either polynomial and maximal root defines the size of the size of CD. What happens to P_1?

1. Could the new functional iteration start from where it stopped in previous re-incarnation: if P_1 is n:th functional power of Q ($P_1 = Q^{\circ n}$), the first step would corresponds to $P_1 \rightarrow Q \circ P_1$. This conservative option does not quite correspond to the idea that one starts from scratch.

2. If P_1 can change, could one require that P_1 is replaced with a polynomial, which is minimal in the sense that it is not functional power of form $P_{1,new} = Q_{new}^{\circ n}$. Or could one even require that it is functional prime having prime valued degree: $n = p$. This would mean starting from scratch except that the algebraic extension of P_2 is fixed.

Probably these options represent only extreme situations. The most general option is that BSFR generates a state, which corresponds to a superposition of extensions of rationals characterized by polynomials $P_2 P_1$, P_2 fixed, and from these one is selected.

Suppose that L as the size of CD is minimal and thus given by the largest root of $P_2^{\circ N}$ in the filled Julia set, it is bound to increase in the iteration (this option is perhaps too deterministic). Under what conditions can BSFR occur? Can the re-incarnated self have childhood?

1. One can argue that L should be smaller than the sizes of Julia sets of both A and B since the iteration gives no roots outside Julia set. This would require iteration to stop when the largest root of P_2^{circN} exceeds the size of the Julia set of A. When applied to B this condition would prevent BSFRs in the opposite time direction would prevent the growth of CD and it would become stationary. This condition looks too deterministic.

2. This picture suggests that the unitary evolution preceding SSFR creates a superposition of iterates $P_2^{\circ N}$ and that the size of CD as outcome of SSFR is determined statistically as a maximal root for $P_2^{\circ N}$ selected in the iteration. N could also decrease. Since the density of roots increases, one would have a lot of choices for the maximal root and quantum criticality and fluctuations of the order of clock time $\tau = 2L$: the order of subjective time would not anymore correspond to that for clock time.

3. Could BSFR become only probable as L as the largest root for the iterate $P_2^{\circ N}$ has exceeded the size of Julia set of A? A quantum analogy with super-cooling comes in mind. The size of CD boundary at side A would contain more volume than needed to store the information provided by the roots r_n and bring no new "special moments in the life of self" at A side. At B side the density of these moments would eventually become large enough so that the reduction of the size of CD destroying part of these moments would mean only a loss of precision. Could this make death and re-incarnation with an opposite arrow of time probable?

   If $P_2^{\circ N}$ is achieved during the life cycle, the reduction in the size of CD in BSFR would reduce N to $N_1 < N$. For $P_1 = Q_1^M$ similar reduction of M to $M_1 < M$ would take place. If one returns to the situation when the iterated started, all new "special moments" are lost. Nothing would have been learned but one could start from scratch and live a childhood, as one might say.

   In the proposed picture - one of many - the opposite boundaries of CD would correspond to both short and long range quantum fluctuations. Could this observation be raised to a guiding principle: could one even say that the opposite boundaries of CD give holistic and reductionistic representations.

4. Do the roots of $P_2^{\circ N}$ belonging to filled Julia set approach the Julia set as N increases? Or are they located randomly inside Julia set? Indeed, the inverse iterate of a root of P_2 is larger than the root as one finds graphically. The $P_2^{\circ N}$ does the same for the roots $P_2^{\circ N}$. If this argument is correct, the density of the roots is largest near Julia set and near the maximum $L - t = L - r$ near the corner of CD.

5. The proposed picture is interesting from the point of view of consciousness theory. Action would be near the corner of CD in the sense that conscious experience would gain most of its content in Minkowskian sense here whereas larger smaller values of $L - r$.

   This does not mean paradox since the size of CD inreases and special moments already experienced are shifted to the future direction and would define the unchanging part - "soul" - of the next re-incarnation. This could be seen as wisdom gained in the previous life [L12].

6. Suppose that the approach to chaos in the iteration of P_2 indeed leads to death and re-incarnation. Can one avoid this or at least increase the span of life cycle? Could one start a new life by replacing P_2 with some polynomial Q_2 in the iteration so that the new iterates would be of form $Q_2^{\circ N_2} \circ P_2^{\circ N_1}$. If the replacement is done sufficiently early, the development of chaos might be delayed since reaching the boundary of Julia set of Q would require quite a many iterations if its largest root is larger than that for P_2. This is also true if the degree of Q_2 is small enough.

2.5.4 Unexpected observations about memories

Some comments about memories in the model of self based on iteration.

1. The conscious activity is at the corner of CD in middle of CD if the new roots define "special moments in the life of self" as conscious experiences. The roots r_n of $P_2^{\circ N}$ defining already experienced special moments shift to Minkowskian geometric future as CD increases in size. Subjective memories are in Minkowskian future and become in re-incarnation stable memories about previous life!

   Subjective memories from recent and previous life could be obtained by communications with geometric future and past involving time reflection of the signal so that the constraints due to the finite light velocity can be overcome.

   One can ask whether self can have "remember" or "anticipate" also external world. If this is possible then the "memories" are indeed from geometric past and "anticipations" from geometric future.

2. The view about subjective memories raises interesting speculations (to be made with tongue in cheek). Consider an unlucky theoretician who believes that he has discovered wonderful theory and has used his lifetime to develop it. Unfortunately, colleagues have not shown

a slightest to his theory. Although personal fame might not matter for him, he might be interested in knowing during his lifetime whether his life work will ever gain recognition. Is this possible in TGD Universe?

Suppose that dreams involve sub-selves representing signals to Minkowskian future and their time reflection inside CD (re-incarnation). If sub-selves near the boundary of CD are able to send time signals to geometric future they might get information about the external world, maybe even about what colleagues think about the theory of unlucky theoretician. Dreams might allow to receive this information indirectly. Dreams might even involve meetings with colleagues of geometricfuture and if their behavior is very respectful, unlucky theoretician might wonder whether his work might have been recognized or is this only wishful thinking!

3. Usually it is thought the recollection of past is not good idea. One can however argue that it communication not only with subjective past but also with objective future (the world external to personal CD). This would give information about the external world of geometric future and also increase the span the time scale of conscious experience and of temporal quantum coherence. This might helpful or a theoretician not interested in fashionable thinking only.

3 Can one define the analogs of Mandelbrot and Julia sets in TGD framework?

The stimulus to this contribution came from the question related to possible higher-dimensional analogs of Mandelbrot and Julia sets (see this). The notion complex analyticity plays a key role in the definition of these notions and it is not all clear whether one can define these analogs.

I have already earlier considered the iteration of polynomials in the TGD framework [**?**] suggesting the TGD counterparts of these notions. These considerations however rely on a view of $M^8 - H$ duality which is replaced with dramatically simpler variant and utilizing the holography=holomorphy principle [L16] so that it is time to update these ideas.

This principle states that space-time surfaces are analogous to Bohr orbits for particles which are 3-D surfaces rather than point-like particles. Holography is realized in terms of space-time surfaces which can be regarded as complex surfaces in $H = M^4 \times CP_2$ in the generalized sense. This means that one can give H 4 generalized complex coordinates and 3 such generalized complex coordinates can be used for the 4-surface. These surfaces are always minimal surfaces irrespective of the action defining them as its extermals and the action makes itself visible only at the singularities of the space-time surface.

3.1 Ordinary Mandelbrot and Julia sets

Consider first the ordinary Mandelbrot and Julia sets.

1. The simplest example of the situation is the map $f : z \to z^2 + c$. One can consider the iteration of f by starting from a selected point z and look for various values of complex parameter c whether the iteration converges or diverges to infinity. The interface between the sets of the complex c-plane is 1-D Mandelbrot set and is a fractal. One can generalize the iteration to an arbitrary rational function f, in particular polynomials.

2. For polynomials of degree n also consider $n - 1$ parameters c_i, $i = 1, ..., n$, to obtain $n - 1$ complex-dimensional analog of Mandelbrot set as boundaries of between regions where the iteration lead or does not lead to infinity. For $n = 2$ one obtains a 4-D set.

3. One can also fix the parameter c and consider the iteration of f. Now the complex z-plane decomposes to two a finite region with a finite number of components and its complement, Fatou set. The iteration does not lead out from the finite region but diverges in the complement. The 1-D fractal boundary between these regions is the Julia set.

3.2   Holography= holomorphy principle

The generalization to the TGD framework relies heavily on holography=holomorphy principle.

1. In the recent formulation of TGD, holography required by the realization of General Coordinate Invariance is realized in terms of two functions f_1, f_2 of 4 analogs of generalized complex coordinates, one of them corresponds to the light-like (hypercomplex) M^4 coordinate for a surface $X^2 \subset M^4$ and the 3 complex coordinates to those of Y^2 orthogonal to X^2 and the two complex coordinates of CP_2.

   Space-time surfaces are defined by requiring the vanishing of these two functions: $(f_1, f_2) = (0, 0)$. They are minimal surfaces irrespective of the action as long it is general coordinate invariant and constructible in terms of the induced geometry.

2. In the number theoretic vision of TGD, $M^8 - H$-duality [L16] maps the space-time as a holomorphic surface $X^4 \subset H$ is mapped to an associative 4-surface $Y^4 \subset M^8$. The condition for holography in M^8 is that the normal space of Y^4 is quaternionic.

   In the number theoretic vision, the functions f_i are naturally rational functions or polynomials of the 4 generalized complex coordinates. I have proposed that the coefficients of polynomials are rationals or even integers, which in the most stringent approach are smaller than the degree of the polynomial. In the most general situation one could have analytic functions with rational Taylor coefficients.

   The polynomials $f_i = P_i$ form a hierarchy with respect to the degree of P_i, and the iteration defined is analogous to that appearing in the 2-D situation. The iteration of P_i gives a hierarchy of algebraic extensions, which are central in the TGD view of evolution as an increase of algebraic complexity. The iteratikon would also give a hierarchy of increasingly complex space-time surface and the approach to chaos at the level of space-time would correspond to approach of Mandelbrot or Julia set.

3. In the TGD context, there are 4-complex coordinates instead of 1 complex coordinate z. The iteration occurs in H and the vanishing conditions for the iterates define a sequence of 4-surfaces. The initial surface is defined by the conditions $(f_1, f_2) = 0$. This set is analogous to the set $f(z) = 0$ for ordinary Julia sets.

   One could consider the iteration as $(f_1, f_2) \rightarrow (f_1 \circ f_1, f_2 \circ f_2)$ continued indefinitely. One could also iterate only f_1 or f_2. Each step defines by the vanishing conditions a 4-D surface, which would be analogous to the image of the $z = 0$ in the 2-D iteration. The iterates form a sequence of 4-surfaces of H analogous to a sequence of iterates of z in the complex plane.

   The sequence of 4-surfaces also defines a sequence of points in the "world of classical worlds" (WCW) analogous to the sequence of points $z, f(z), \dots$. This conforms with the idea that 3-surface is a generalization of point-like particles, which by holography can be replaced by a Bohr orbit-like 4-surface.

4. Also in this case, one can see whether the iteration converges to a finite result or not. In the zero energy ontology (ZEO), convergence could mean that the iterates of X^4 stay within a causal diamond CD having a finite volume.

3.3   The counterparts of Mandelbrot and Julia sets at the level of WCW

What the WCW analogy of the Mandelbrot and Julia sets could look like?

1. Consider first the Mandelbrot set. One could start from a set of roots of $(f_1, f_2) = (c_1, c_2)$ equivalent with the roots of $(f_1 - c_1, f_2 - c_2) = (0, 0)$. Here c_1 and c_2 define complex parameters analogous to the parameter c of the Mandelbrot sent. One can iterate the two functions for all pairs (c_1, c_2). One can look whether the iteration converges or not and identify the Mandelbrot set as the critical set of parameters (c_1, c_2). The naive expectation is that this set is 3-D dimensional fractal.

2. The definition of Julia set requires a complex plane as possible initial points of the iteration. Now the iteration of $(f_1, f_2) = 0$ fixes the starting point (not necessarily uniquely since 3-D surface does not fix the Bohr orbit uniquely: this is the basic motivation for ZEO). The analogy with the initial point of iteration suggests that we can assume $(f_1, f_2) = (c_1, c_2)$ but this leads to the analog of the Mandelbrot set. The notions coincide at the level of WCW.

3. Mandelbrot and Julia sets and their generalizations are critical in a well-defined sense. Whether iteration could be relevant for quantum dynamics is of course an open question. Certainly it could correspond to number theoretic evolution in which the dimension of the algebraic extension rapidly increases. For instance, one could one consider a WCW spinor field as a wave function in the set of converging iterates. Quantum criticality would correspond to WCW spinor fields restricted to the Mandelbrot or Julia sets.

Could the 3-D analogs of Mandelbrot and Julia sets correspond to the light-like partonic orbits defining boundaries between Euclidean and Minkowskian regions of the space-time surface and space-time boundaries? Can the extremely complex fractal structure as sub-manifold be consistent with the differentiability essential for the induced geometry? Could light-likeness help here.

3.4 Do the analogs of Mandelbrot and Julia sets exist at the level of space-time?

Could one identify the 3-D analogs of Mandelbrot and Julia sets for a given space-time surface? There are two approaches.

1. The parameter space (c_1, c_2) for a given initial point h of H for iterations of $f_1 - c_1, f_2 - c_2)$ defines a 4-D complex subspace of WCW. Could one identify this subset as a space-time surface and interpret the coordinates of H as parameters? If so, there would be a duality, which would represent the complement of the Fatou set (the thick Julia set) defined as a subset of WCW as a space-time surface!

2. One could also consider fixed points of iteration for which iteration defines a holomorphic map of space-time surface to itself. One can consider generalized holomorphic transformations of H leaving X^4 invariant locally. If they are 1-1 maps they have interpretation as general coordinate transformations. Otherwise they have a non-trivial physical effect so that the analog of the Julia set has a physical meaning. For these transformations one can indeed find the 3-D analog of Julia set as a subset of the space-time surface. This set could define singular surface or boundary of the space-time surface.

3.5 Could Mandelbrot and Julia sets have 2-D analogs in TGD?

What about the 2-D analogs of the ordinary Julia sets? Could one identify the counterparts of the 2-D complex plane (coordinate z) and parameter space (coordinate c).

1. Hamilton-Jacobi structure defines what the generalized complex structure is [L15] and defines a slicing of M^4 in terms of integrable distributions of string world sheets and partonic 2-surfaces transversal or even orthogonal to each other. Partonic 2-surface could play the role of complex plane and string world sheet the role of the parameter space or vice versa.

   Partonic 2-surfaces *resp.* and string world sheet having complex *resp.* hyper-complex structures would therefore be in a key role. $M^8 - H$ duality maps these surfaces to complex *resp.* co-complex surfaces of octonions having Minkowskian norm defined as number theoretically as $Re(o^2)$.

2. In the case of Julia sets, one could consider generalized holomorphic transformations of H mapping X^4 to itself as a 4-surface but not reducing to 1-1 maps. If f_2 (f_1) acts trivially at the partonic 2-surface Y^2 (string world sheet X^2), the iteration reduces to that for f_1 (f_2). Within string world sheets and partonic 2-surfaces the iteration defines Julia set and its hyperbolic analog in the standard way. One can argue that string world sheets and partonic 2-surfaces should correspond to singularities in some sense. Singularity could mean this fixed point property.

The natural proposal is that the light-like 3-surfaces defining boundaries between Euclidean and Minkowskian regions of the space-time surface define light-like orbits of the partonic 2-surface. And string world sheets are minimal surfaces having light-like 1-D boundaries at the partonic 2-surface having physical interpretation as world-lines of fermions.

One could also iterate only f_1 or f_2 allow the parameter c of the initial value of f_1 to vary. This would give the analog of Mandelbrot set as a set of 2-D surfaces of H and it might have dual representation as a 2-surface.

3. The 2-D analog of the Mandelbrot set could correspond to a set of 2-surfaces obtained by fixing a point of the string world sheet X^2. Also now one could consider holomorphic maps leaving the space-time surface locally but not acting 1-1 way. The points of Y^2 would define the values of the complex parameter c remaining invariant under these maps. The convergence of the iteration of f_1 in the same sense as for the Mandelbrot fractal would define the Mandelbrot set as a critical set. For the dual of the Mandelbrot set X^2 and Y^2 would change their roles.

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