About the structure of Dirac propagator in TGD

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Abstract

In this article the notion of a fermion propagator in the TGD framework is discussed. It is found that the construction is much more than a mere computational challenge. There are two alternative approaches. Fermionic propagation could correspond a) to 8-D propagation in H between points belonging to the space-time surface or its sub-manifold or b) to a 4-D- or lower-dimensional propagation at the space-time level for the induced spinor fields as analog of massless propagation.

For the option a), the separate conservation of baryon and lepton numbers requires fixed H-chirality so that the spinor mode is sum of products of M^4 and CP_2 spinors with fixed M^4 and CP_2 chiralities whose product is +1 or -1. This suggests that M^4 propagation is massless. It came as a total surprise that the propagation of color modes in the conventional sense is not possible in length scales above CP_2 scale. The M^4 part of the propagator for virtual masses above the mass of the color partial wave is of the standard form but for virtual masses below it the progator is its conformal inversion. The connection with color confinement is highly suggestive.

For light-like fermion lines at light-like partonic orbits, there are good reasons to expect that the condition $s_1 = s_2$ is satisfied and implies that the propagation from s_1 is possible to only a discrete set of points s_2. Also this has direct relevance for the understanding of color confinement and more or less implies the intuitively deduced TGD based model for elementary fermions as 1-dimensional geometric objects.

Although the option b) need not provide a realistic propagator, it could provide a very useful semiclassical picture for propagation along fermion lines. If the condition $s_1 = s_2$ is assumed, fermionic propagation along light-like geodesics of H is favored and in accordance with the model for elementary particles. This allows a classical space-time picture of particle massivation by p-adic thermodynamics and color confinement.

Also the interpretational and technical problems related to the construction of 4-D variants of super-conformal representations having spinor modes as ground states and to the p-adic thermodynamics are discussed.

Contents

1 Introduction

In this article the identification and calculation of the fermion propagator $S_F(h_1, h_2)$ in $H = M^4 \times CP_2$ is discussed in detail. Also the definition of fermion propagator at space-time surface is considered.

1.1 Basic expectations

Let us summarize the basic expectations first. The following view is only one of the many variants and found by trial and error and must be taken with reservations.

1. The TGD view of fundamental interactions [L12, L13] differs from that of the standard model. The notion of color differs from that of QCD and electroweak gauge potentials correspond to the components of the induced spinor connection of CP_2 [K4, K5]. A strong correlation between mass, color quantum numbers and electroweak quantum numbers at the fundamental level corresponding to space-time surfaces is predicted [L17]. The standard view is expected to emerge as an approximation at the QFT limit.

   An important constraint is that the end points of the propagator are restricted to the space-time surface X^4 or its lower dimensional sub-manifold. Second important point is that in M^4 the propagation is between identical M^4 chiralities by the separate conservation of lepton and quark quantum numbers.

2. Two kinds of propagators can be imagined.

   (a) The propagation in H is a geometric analog of off-mass shell propagation as off-space-time surface propagation. There is hierarchy of geometric objects with dimensions 1,2,3,4 and for 1-D fermion lines the geometric costrains on propagation are very powerful.

   (b) The notion of induced spinor structure suggests that it might make sense to speak of propagation along the space-time surface as analog of on-mass-shell propagation.

   The induced Dirac equation in X^4 is analogous to the massless Dirac equation. This suggests that the propagation is along light-like geodesics of H, whose projection is a geodesic circle of CP_2, could be a good semiclassical description. This would give a classical geometric correlate for the fermion propagation even if the space-time propagator might not be needed.

What can on say about H-propagation.

1. A well-defined H chirality implies that the M^4 part of the propagator is a massless propagator. The first guess is that the virtual fermions are associated with the masses $p^2 = \Lambda_n^2$, Λ_n^2 is the eigenvalue of the square of CP_2 Dirac operator, as predicted by the H Dirac equation in H. The fermion propagator should have poles at these masses. A direct calculation indeed shows that the coefficient of massless M^4 propagator has these poles.

2. In CP_2 degrees of freedom, a well-defined H chirality implies that the H propagator $D_F(m_1, s_1), (m_2, s_2))$ is reduced to a bilocal inner product for the two modes with the same CP_2 chirality defined by the propagator factor associated with the propagator.

3. For the propagator between fermionic color partial waves, there is an integration over the CP_2 projection at both ends of the propagator line, say 3-surfaces at the boundaries of causal diamond (CD). This strongly suggests that this inner product tends to vanish for 2-D and 3-D geometric objects.

   For 1-D light-like fermion lines identifiable as intersections of 3-D light-like partonic orbits and string world sheets, the situation changes. There is strong correlation between the ends of the fermion line. Even more, for the end points of the partonic lines, $s_1 = s_2$ is true in the case if the partonic line is light-like M^4 geodesic so that the outcome is the local inner product over CP_2 projection as 2-D partonic surface: the projections are identical at the ends of the line. It can also be a light-like geodesic of $M^4 \times S^1 \subset M^\times CP_2$ belonging to the partonic orbit.

4. There is also an argument [L16] supporting the view that the interactions can be assigned with string world sheets and fermion lines.

   (a) Consider space-time surfaces X^4 and Y^4 having an interpretation as analogs of Bohr orbits for 3-D particle-like objects. The interactions between X^4 and Y^4 can be assigned with fermion fields in their intersection. In the generic case, the intersection is a discrete set of points and one can argue that this does not give rise to any interactions.

   (b) The Hamilton-Jacobi structure [L7] of X^4 is induced from that of $M^4 \times CP_2$ and the the coordinates of X^4 can be locally chosen to be coordinates (u, v, w) of M^4 or (u, v, ξ_1) for $M^4 \times CP_2$. If the Hamilton-Jacobi structures associated with X^4 and Y^4 are equivalent, X^4 and Y^4 can have overlapping projections to the (u, v) plane. By hypercomplex analyticity, the functions (f_1, f_2) depend on u or v only. If this coordinate is u then v effectively disappears from the dynamics and can consider the situation for $u = constant$ slices of H, X^4 and Y^4 with dimensions 7, 3, and 3 respectively. The intersection of 3-D slices has dimension $D = 7 - 3 - 3 = 1$ having interpretation as a fermion line. The intersection $X^4 \cap Y^4$ has dimension 1+1=2 having interpretation as a string world sheet, which is however effectively 1-D because v is not a dynamical degree of freedom.

(c) The interpretation of the string world sheet is as a quantum coherent interaction region in the interaction of X^4 and Y^4. Since v has no dynamical role, it effectively reduces to the fermion lines identifiable as 1-D intersections of string world sheets and light-like partonic orbits.

1.1.1 How do the physical color quantum numbers emerge?

It must be taken into account that physical fermions are either color singlets or triplets, not arbitrarily high color partial waves satisfying only the triality condition $t = 1$ for quarks and $t = 0$ for leptons.

This is achieved in two ways.

1. Neutrino-antineutrino $\nu_L \overline{\nu}_R$ pair screens the electroweak charges above weak boson Compton length and could also screen the anomalous parts of color charges.

2. The solutions Dirac equation are ground states of conformal representations in which the action of a conformal scaling generator generates higher excitations, which can have color quantum numbers which add to those of color partial waves to give rise to color singlets and triplets.

p-Adic thermodynamics [L5] describes the thermodynamics of the scaling generator, and gives excellent predictions for the particle masses so that conformal algebra is involved in any case.

For the proposed form of the H propagator, the construction of the quantum states separates completely from that for the propagator. The thermal masses predicted by p-adic thermodynamics are not visible in the propagator. The masses of color partial waves are also invisible at the energies considerably below the CP_2 mass scale about 10^{-4} Planck masses.

1.1.2 How could the confinement scale emerge?

In QCD, confinement occurring on some M^4 scale, the confinement scale, is a fundamental concept. Free quarks are the basic dynamical entities below the confinement scale and transform to hadrons above this scale.

1. In attempts to understand this, the space-time representation of H propagator $D_F(m_1, s_1), (m_2, s_2))$ for the massa propagator M^4 degrees of freedom should be used (see).

   One can consider two options depending on whether the propagation occurs in H or in X^4. In the case of H propagation the expectation is that it is effectively massless for all color partial waves for the momenta considerably smaller than CP_2 mass scale. The total surprise was that there is no color propagation in the ordinary sense in scales longer than CP_2 scale. The M^4 propagator is its conformal inversion in long scales.

2. It is reasonable to expect that the p-adic length scale determines the scale at which the confinement occurs. This is where the algebraic geometry of the spacetime surface comes into play.

Here holography= holomorphy vision suggests a concrete approach.

1. In the holography= holomorphy vision [L11, L15], space-time surfaces correspond to the roots of polynomial pair $f = (f_1, f_2) : H \rightarrow C^2$ of generalized complex coordinates of H (one hypercomplex coordinate and 3 complex coordinates). They allow as dynamic symmetries the maps $g = (g_1, g_2) : C^2 \rightarrow C^2$ defined as functional compositions $f \rightarrow g \circ f$.

   One can define prime polynomial pairs f as those which do not allow a decomposition $f = g \circ h$. The prime polynomial pairs $g = (g_1, g_2)$ can be defined similarly. The pairs (g_1, Id) are of special interest and in this case the prime polynomials $g_{1,p} \equiv g_p$ have prime degree p.

2. One should study the H propagator $D_F(m_1, s_1), (m_2, s_2))$ for the polynomials g_p and the functional iterates $g_p^{\circ k} \circ f$. The integration over CP_2 projections of the end points gives a bilocal inner product of the modes.

The functional iterates of g_p involve classical non-determinism identifiable as p-adic non-determinism [L18] so that the correlation between color partial waves descreases as the number k of iteration steps increases. This would make the amplitude for a propagation over too long scales impossible. p-Adic length scale could correspond to this scale. Note that very large primes p are involved: for the electron one has $p = M_{127} = 2^{127} - 1$.

The confinement scale can be identified as a p-adic length scale for quarks or hadrons, which is determined by the thermal mass predicted by p-adic thermodynamics. The larger the value of p, the longer the confinement scale.

Concerning the option based on X^4 propagation two observations can be made.

1. The possibly existing X^4 propagator would be the analog of a massless propagator since the Dirac equation for induced spinor fields in X^4 is a generalization of the massless Dirac equation. This notion makes sense also for the lower dimensional surfaces of X^4 and 1-D light-like curves are natural candidates for fermion lines.

2. For the propagation along the 1-dimensional along fermion lines identified as light-like M^4 geodesics along parton surfaces, the condition $s_1 = s_2$ is strongly suggestive so that there would be no propagation in CP_2. This is not true for higher-dimensional objects.

There are two basic questions to be answered. Is the p-adic mass scale determined by the spectrum of CP_2 masses or by p-adic thermodynamics? Do the spectra of CP_2 masses and of p-adic thermal masses appear at the level of propagator or only at the level of initial and final states of propagation?

1. A natural assumption is that the masses of quarks and leptons appear in the propagator depend on the spectrum of color excitations $m^2 = \lambda_n^2$ for the fermions involved ($\nu_L \overline{\nu}_R$ for the second option). Direct calculation indeed shows that it appears as poles in the H propagator. The M^4 propagator is however effectively massless for masses scales considerably below the CP_2 mass scale.

2. p-Adic thermodynamics, in which the basic parameters characterizing the particle are the p-adic prime p and the p-adic temperature $T_p = 1/n$, works excellently so that it seems that CP_2 mass spectrum does not make itself visible in the propagator. In p-adic thermodynamics, a large $p \simeq 2^k$, simplifies enormously and something similar can be expected now as well. This separation between the structure of states and propagator would be a huge simplification.

3. Concerning X^4 propagation, the semiclassical description as light-like H geodesics of the geodesic manifold $M^4 \times S^1 \subset M^4 \times CP_2$ could be considered as a model. The masslessness of H would no longer imply masslessness in M^4. One could say that p-adic thermal excitations deform the light-like geodesics of M^4 to those of $M^4 \times S^1 \subset H$. Also this suggests that the masses are not visible in the X^4 propagator as also the fact that they do not appear in the X^4 Dirac operator suggests.

1.2 Plan of the article

The plan of the article is as follows.

1. I have considered several approaches to the calculation of the fermion propagator and in the following I describe the most elegant looking view about the fermion propagation as propagation in H. Assuming a fixed H chirality, the propagator reduces in M^4 to the massless propagator in H and its coefficient has poles at the masses associated with the color partial waves. The fact that the propagation depends strongly on the dimension of the fundamental object, has powerful consequences and the existing intuitive picture about fundamental fermion lines as light-like geodesics at partonic 2-surfaces follows. In the case of fermion lines at partonic orbits, the condition $s_1 = s_2$ would make CP_2 propagation possible. One can also understand color confinement in this picture.

2. The 4-D propagation along X^4 involves induced spinor structure and my intuitive feeling is that also this perspective is important, at least as a semiclassical model. This would give a very concrete classical view of the propagation even if a 4-D propagator were not useful. In particular, the p-adic thermal massivation could be modelled semi-classically.

3. Also the problems related to the origin of p-adicity and interpretational problems of p-adic thermodynamics and to the generalized conformal invariance will be discussed in the Appendix.

2 Fermion propagator at the level of embedding space

One can consider three alternative views concerning what fermion propagation could mean.

1. Fermions propagate in H and virtuality in geometric sense could mean that they can leave the space-time surface during the propagation. Off mass-shell property would mean off-space-time surface.

2. Fermions propagate along the 4-D space-time surface X^4 as induced spinor fields and the propagator is defined by the analog of 4-D massless Dirac operator. Also the propagation along lower-dimensional surfaces of X^4, in particular fermion lines, is possible.

3. The propagation could also take place in the "world of classical worlds" (WCW). This would be the counterpart for the propagation as it is understood in string models. One must be cautious here since holography= holomorphy vision implies that there is no path integral.

Could the two interpretations for the fermion propagation be consistent? The space-time propagation as a 4-D propagation is geometrically much more restricted. Could the propagator code for the information about initial and final quantum states as a kind of quantum-classical correspondence?

2.1 Leptons and quarks as quantized free spinor fields in H

It is good to start with the description of H spinor fields and of the H Dirac equation. This topic is also discussed in Appendix A.

2.1.1 Leptons and quarks as H spinors with different H chiralities

Consider first the general theory.

1. The second quantized spinor field in $H = M^4 \times CP_2$ can be written as superpositions of modes multiplied with a creation or annihilation operator.

2. Suppose that the spinor fields of H have a definite H-chirality so that they are either leptonic or quark-like, not a superposition of the two: this would violate the superselection rule of charge and fermion number.

   One has

$$\Gamma_9 \Psi_\epsilon = \epsilon \Psi_\epsilon . \tag{2.1}$$

   $\epsilon = +1/-1$ is the H-chirality and quarks and leptons have opposite chiralities.

3. The Dirac equation is satisfied by the spinor modes of H. They are massless in H but not in M^4(!). This is essential for the twistorization and the divergence-free nature of the theory.

   The Dirac operator is

$$D = D(M^4) + D(CP_2), \tag{2.2}$$

where

$$D(M^4)) = \gamma^k p_k \tag{2.3}$$

is the Dirac operator in M^4 and

$$D(CP_2) = \Gamma^k D_k . \tag{2.4}$$

is the Dirac operator in CP_2. Here D_k is covariant derivative, which is defined by the spinor connection of CP_2 that encodes electroweak interactions. The Kähler gauge potential A_k for leptons is $3A_k$ and for quarks A_k. This explains the different electric charges of quarks and leptons.

Twistorialization suggests that M^4 allows the analog of Kähler structure [L7] whereas vielbein connection should be trivial by the flatness of the metric. Kähler form should satisfy the analog self-duality meaning that locally it is analogous to mutually orthogonal electric and magnetic fields of the same magnitude.

4. Quarks correspond to triality $t = 1$ partial waves and leptons to triality $t = 0$ partial waves (see Appendix A). These modes are not simply color triplets for physical quarks and color singlets for physical leptons. For physical fermions, a mechanism producing color triplets for quarks and singlets for leptons is needed.

   According to Appendix A, the color partial waves with different charges correspond to color partial waves characterized by an integer pair (p, q). p and q can thought of as giving the numbers of triplets and anti-triplets in the tensor containing the representation in question. The dimension of the color representation is $d(p, q) = (1/2)(p + 1)(1 + 1)(p + q + 2)$.

   In the case of leptons, one has representations (p, p) for neutrinos and $(3 + p, p)$ for charged leptons. The dimensions are $(p+1)^2(p+2)$ neutrinos and $(3+p+1)(3+p)(p+1)$ for charged leptons. The color representations are the same for both CP_2 chiralities which correspond to opposite M^4 chiralities for a fixed H chirality.

   In the case of quarks one has representations $(p+1, p)$ and $(p+4, p)$ for D *resp.* U type quarks and the corresponding dimensions are $1/2(p + 1)(p + 1)(2p + 3)$ *resp.* $(p + 6)(p + 1)(p + 3)$ The two CP_2 chiralities are transformed to each other by $D(CP_2)$: covariantly right-handed neutrino is an exception since it is annihilated by $D(CP_2)$.

5. The charge assignable to color partial wave is not a local notion characterized by a spinor index since $D(CP_2)$ mixes different isospin states, whose identification depends on the choice of the vierbein in CP_2. This is understandable since the choice of the CP_2 spinor basis is determined apart from a local $U(2)$ rotation. The same problem is encountered in standard model and solved by going to unitary gauge. One can ask whether it might be possible to define the counterpart of the unitary gauge as a global choice of CP_2 vierbein for which the CP_2 spinor is constant or covariantly constant for a given color partial wave.

2.1.2 Dirac equation in H

1. Dirac equation in H says the following

$$D\Psi_\epsilon = 0. \tag{2.5}$$

   Here $\epsilon = +1/-1$ refers to quark/lepton.

2. Solutions to Dirac equation can be constructed by taking a spinor Φ_ϵ, which is not a solution of Dirac equation, but satisfies the following conditions:

(a) H-chirality is well defined.

$$\Gamma_9 \Phi_\epsilon = \epsilon \Phi_\epsilon . \tag{2.6}$$

H-chirality ϵ is the product $\epsilon = \epsilon_1 \times \epsilon_2$ of M^4 chirality ϵ_1 and CP_2 chirality ϵ_2 . It is -1 if the M^4 and CP_2 chiralities are opposite and +1 if they are the same.
Consider first the possibility that Φ_ϵ is the tensor product of the M^4 spinor and the CP_2 spinor with the same or opposite chirality corresponding to quarks and leptons.

(b) Φ_ϵ is M^4 plane wave so that $D(M^4) = \gamma^k p_k$ is effectively true.

(c) Φ_ϵ satisfies the square of the Dirac equation but not the Dirac equation:

$$\begin{array}{l} D^2\Phi_\epsilon = 0 , \\ D(M^4)^2\Phi_\epsilon = p^2\Phi_\epsilon , \\ D^2(CP_2)\Phi_\epsilon = -\Lambda^2\Phi_\epsilon . \end{array} \tag{2.7}$$

An eigenmode of $D(M^4)^2$ and $D^2(CP_2)$ is therefore in question. We obtain a spectrum Λ_n^2, the details of which are not relevant in this context.
The mass is quantized:

$$p^2 = \Lambda_n^2 . \tag{2.8}$$

3. How to obtain the solution of Dirac's equation Ψ_ϵ? It is obtained by operating on the spinor Φ_ϵ with the Dirac operator $D = D(M^2) + D(CP_2)$:

$$\Psi_\epsilon = D\Phi_\epsilon . \tag{2.9}$$

The Dirac equation holds

$$D\Psi = D^2\Psi_0 = 0 . \tag{2.10}$$

A couple of comments regarding the chiralities are in order.

1. The spinor Φ_ϵ has well-defined M^4 and CP_2 chiralities, but for the spinor $D\Phi_\epsilon$ the chiralities are not well-defined.

2. Although Φ_ϵ can be chosen as the product of the $\Psi = \psi \otimes \phi$ definite M^4 of the spinor with chirality ψ and the spinor with chirality ϕ of the definition CP_2, then $D\Psi$ is a superposition

$$D(M^4)\psi \otimes \phi + \psi \otimes D(CP_2)\phi , \tag{2.11}$$

where the H-chiralities have changed because $D(M^4)$ ($D(CP_2)$) changes the chirality of M^4 (CP_2). It can be said that both M^4 and CP_2 chiralities are mixed when the mass is non-zero.

3. It turns out that only a right-handed neutrino can have a massless state.

2.2 Momentum space propagator in H

The first option is that the propagation takes place in H for the spinor modes of H in such a way that only the endpoints of the propagator $D_F(h_1, h_2)$ are restricted to the space-time surface X^4. One could say that virtuality in geometric sense for the fermions means that they can leave the space-time surface. The propagation along space-time surface already discussed would mean the analog of on-mass-shell propagation.

2.2.1 Massless Dirac propagator in M^4

The M^4 case provides a kind of role model. For momentum basis, the inverse of the Dirac operator can be solved algebraically and space-time representation involves integration over virtual momenta with $p^2 \neq 0$.

The momentum space - and space-time representations of the massless Dirac propagator can be found in the book "An introduction to quantum field theory" by Peskin and Schröder (edition 1995, p. 660) (see this) and are given by

1. The representation of the massless Dirac propagator in momentum space is given by

$$\tilde{S}_F(p) = \frac{i\gamma \cdot p}{p^2 + i\varepsilon} . \tag{2.12}$$

   Here the $1/(x + i\varepsilon)$ is a shorthand notation for $1/x \mp i\pi\delta(x)$.

2. The space-time representation of the massless Dirac propagator is obtained from the momentum representation $D_F(p) = ip^k\gamma_k/(p^2 + i\varepsilon)$ as Fourier transform

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i\gamma \cdot p}{p^2 + i\varepsilon} = -\frac{i}{2\pi^2}\left(\frac{\gamma \cdot (x-y)}{|x-y|^4}\right) . \tag{2.13}$$

   This form conforms can be deduced from the scaling invariance.

   As one might guess, the propagator has a part, which is a delta function concentrated on the boundary of a double light-cone. The reader can verify this by calculating the 4-D volume integral of $S_F(m = x - y)$ by using a standard definition of delta function in terms of 4-D volume integral and the basic rules for integrating delta functions.

   It is advantageous to use spherical coordinates (a, η, θ, ϕ) related to linear Minkowski coordinates m^k by the equations

$$m^0 = a \times cosh(\eta) , \quad m^1 = a \times sinh(\eta)cos(\theta) , \quad m^2 = a \times sinh(\eta)sin(\theta)cos(\phi) \quad m^3 = a \times sinh(\eta)sin(\theta)sin$$

   $a^2 = m^k m_k$ defines the light-cone proper-time. The line element of the M^4 metric is that for empty Robertson-Walker cosmology $ds^2 = da^2 - a^2(dr^2/(1 + r^2) - r^2 d\Omega^2)$. The 4-D volume element is $dV_4 = a^3 r^2 dadrsin(\theta)d\theta d\phi$.

   In the integrand of the volume integral, the dependence on a disappears. The integral of the spatial part of $m^k\gamma_k$ over angle coordinates vanishes. The contribution of $m^0\gamma_0$ gives two delta functions corresponding to $m^0 \pm r_M$ non-vanishing at the light-cone boundary.

2.2.2 Deduction of the momentum space propagator H

The separate conservation of lepton and quark numbers forces the counterpart of the massless propagator, not only in H but also in M^4.

1. The H propagator corresponds to the inverse of the massless Dirac operator of H. It is essential that the modes have a well-defined H-chirality meaning that lepton and quarks numbers are separately conserved.

2. One could argue that since M^4 masses for spinor modes are given by $p^2 = \Lambda_n^2$, where Λ_n^2 is the eigenvalue of $D(CP_2)^2$, each mode gives rise to a propagator with mass Λ_n^2. Since Λ_n corresponds to CP_2 mass scale, only the covariantly constant right-handed neutrino would propagate above CP_2 mass scale! The other modes would propagate only in the CP_2 length scale. Note however that in M^4 massive propagation along light-like geodesics reduces to massless propagation.

The second problem is that for this view the poles at $p^2 = \Lambda_n^2$ do not emerge and must be put in by hand as normalization factors for off-mass-shell states. The reason is that the $1/D^2$ factor disappears in the matrix elements of $D/D^2 + i\epsilon$ between the massive H spinor modes Ψ, which are of form $\Psi = D\Phi$. The only exception is the right-handed neutrino, which is covariantly constant in CP_2 degrees of freedom. Only the right-handed neutrino would progate.

The delicate difference with respect to the ordinary massive propagation in M^4 is that the ordinary massive propagator is given by $D_F = 1/(p^k\gamma_k + m) = (p^k\gamma_k - m)/(p^2 - m^2 + i\epsilon)$ by $p^2 - m^2 = (p^k\gamma_k + m)(p^k\gamma_k - m)$. In TGD, a propagator massive in M^4 sense is massless in the 8-D sense and one has $D_F = 1/D = D/(D^2 + i\epsilon)$.

3. What does the fixing the H-chirality implied by the separate conservation of quark and lepton numbers mean? The H-spinor is a combination of two spinors, where the M^4 and CP_2 chiralities ϵ_1 and ϵ_2 are well defined and their product $\epsilon = \epsilon_1\epsilon_2$ is +1 or -1 depending on whether it is a quark or a lepton. The combinations (1,-1) and (-1,1) and on the other hand (1,1) and (-1,-1) are possible.

   But the fixed M^4 chirality means that propagation is like that for a massless fermion in M^4! The mass spectrum for $D(P_2)$ does not show itself in the M^4 part of the propagator and the construction of states and of the propagator are completely separate problems. This picture also corresponds to the picture of QCD where massless quarks are assumed as an approximation.

4. What about the propagation in CP_2 degrees of freedom? It follows directly from the defining condition that the propagator for geometric objects with dimension higher than 1 involves a bilocal inner product: the small correlation between the end points and integration over the geometric objects at the end points tends to make it very small.

   1-D fermion lines at the partonic 2-surfaces are an exception in this case one expects $s_1 = s_2$ so that effectively one has delta function in CP_2 degrees of freedom. There is no propagation at all. The matrix elements of the propagator between quantum states constructed from the conformal representations having H spinor modes as ground states are extremely simple in this case and reduce to the same form as in QFTs for fermions with internal quantum numbers.

2.2.3 Calculation of the momentum space propagator in H

Consider now the computation of the matrix elements of the fermionic H propagator plane waves in M^4 and color partial waves in CP_2. In the case of color partial waves there is integration of the initial and final 3-surfaces or its sub-manifold at the boundaries of CD. The same is true in the space-time representation of M^4 propagator.

1. One can write the H propagator as

$$\frac{1}{D} = \frac{D}{D^2 + i\epsilon} = \frac{D_{M^4} + DCP_2}{p^2 + D_{CP_2}^2} \quad , \tag{2.14}$$

(2.15)

   where one effectively has

$$D_{M^4} = p^k\gamma_k \quad , \quad D_{CP_2}^2 = \Lambda_n^2 \quad . \tag{2.16}$$

2. The propagator can be expressed in two ways. Either by factoring out $1/D_{M^4}$:

$$\frac{1}{p^k\gamma_k + D_{CP_2}} = \frac{1}{p^k\gamma_k} \times \frac{1}{1 + (D_{CP_2}/p^k\gamma_k)} \quad , \tag{2.17}$$

or by factoring out $1/D_{CP_2}$:

$$\frac{1}{D} = \frac{1}{D_{CP_2}} \times \frac{1}{1 + (p^k\gamma_k/D_{CP_2})} \quad . \tag{2.18}$$

For both factorizations one can expand the second factor in the geometric series.

3. For the first option this gives

$$\frac{1}{D} = \frac{1}{p^k\gamma_k} \times \sum_k (-1)^k \times (\frac{D_{CP_2}}{p^k\gamma_k})^k \quad . \tag{2.19}$$

Since H chirality is fixed, only the even powers in the series have non-vanishing matrix elements. This gives

$$\frac{1}{D} = \frac{1}{p^k\gamma_k} \times \frac{1}{1-x} \quad , \quad x = \frac{\Lambda_n^2}{p^2} \quad . \tag{2.20}$$

This is a massless M^4 propagator with a coefficient, which is well defined for $p^2 \geq \Lambda_n^2$. For $p^2 = \Lambda_n^2$ the coefficient has a pole as expected. Massless propagation for colored fermions is possible but by Uncertainty Principle only in very short length scales. This is strongly suggestive of color confinement. One obtains

4. For the second option, a well-defined H chirality implies

$$\frac{1}{D} = \frac{1}{D_{CP_2}} \times \sum_k (-1)^k \times (\frac{p^k\gamma_k}{D_{CP_2}})^k \quad . \tag{2.21}$$

In this case the chirality condition implies that only the odd powers have non-vanishing matrix elements and one obtains

$$\frac{1}{D} = -p^k\gamma_k \times \sum_k (\frac{p^2}{\Lambda_n^2})^k = -p^k\gamma_k \times \frac{1}{1-y} \quad , \quad y = \frac{p^2}{\Lambda_n^2} = \frac{1}{x} \quad . \tag{2.22}$$

The coefficient is well defined for $p^2 \leq \Lambda_n^2$ and has a pole at $p = \Lambda_n^2$. What looks strange is that the propagator for small momenta is proportional $p^k\gamma_k$. There is no propagation of colored fermions in long length scales in the usual sense. Again the interpretation in terms of color confinement is highly suggestive.

5. The propagators for the two options are are related by inversion

$$p^k\gamma_k \rightarrow \frac{1}{p^k\gamma_k} \quad , \quad x = \frac{\Lambda_n^2}{p^2} \rightarrow \frac{1}{x} = y \tag{2.23}$$

2.2.4 Some consequences

This result has some highly non-trivial consequences.

1. The H propagation includes the constraint that the end points of the propagator belong to the space-time surface through the fact that h_1 and h_2 are on the same spacetime surface.

   For geometric objects with dimension higher than 1 the bilocal matrix elements involving integrations over the object tend to vanish. The light-like parton orbits are in a special role now since the partonic 2-surfaces at the ends of the parton orbit can have a large number of common points if their CP_2 projections are identical. This would explain why they are physically in an exceptional position. For fermion lines, the theory discretizes itself in CP_2 degrees of freedom! Discretization would not be an approximation but a prediction of the theory. This is the basic idea behind the identification of cognitive representation as a number theoretic discretization using points of the embedding space for which H coordinates make sense both as real numbers and numbers in an extension of p-adic numbers.

   The classical time evolution involves the failure of classical determinism identifiable as p-adic non-determinism [L15, L18]. There is a sequence of analogs of multifurcations involving non-determinism. In this sequence the number of common points possessed by the initial and final partonic orbits decreases and eventually the propagator amplitude becomes very small. This means color confinement.

2. H propagation of color fermions is possible only for very short length scales corresponding to virtual masses larger than the minimum value of Λ_n^2. Massless propagation in the usual sense is not possible in long length scales since the M^4 part of the progator equals the inversion $p^k\gamma_k$ of the ordinary massless propagator. The occurrence of inversion could be understood in terms of the generalized conformal invariance.

   From this it should be easy to understand what happens in quark and color confinement (gluons are bound states of quarks and antiquarks in TGD). Colored fermions do not propagate in the usual sense.

2.3 Coordinate space representation of the fermion propagator in H

The momentum representation of the propagator is well-defined but it is not at all obvious that the space-time counterpart $D_F(m_1, s-1, m_2, s_2)$ of the H propagator exists. It's possible non-existence might be interpreted as the impossibility to talk about the space-time propagation of color in terms of color confinement: space-time propagation makes sense only for color singlets.

The explicit expression of the ordinary massless Dirac propagator as a Fourier transform of the momentum space-propagator in M^4 can serve as a starting point:

$$
\begin{array}{ll}
S_F(m) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot m} \frac{i\gamma\cdot p}{p^2+i\varepsilon} & \\
= -\frac{i}{2\pi^2}\left(\frac{\gamma\cdot m}{|a|^4}\right) & = \frac{i}{4\pi^2}\left(\frac{\gamma\cdot\nabla}{a^2}\right) \ .
\end{array} \tag{2.24}
$$

Here $m = m_1 - m_2$ is the difference of M^4 points and $a = \sqrt{m^k m_k}$.

1. For $p^2 > \Lambda_n^2$, the replacement would be

$$
\frac{i}{p^k\gamma_k+\varepsilon} \rightarrow \frac{1}{ip^k\gamma_k+\varepsilon} \times \frac{1}{1+x} \ , \quad x = \frac{\Lambda_n^2}{p^2} \ .
$$

2. For $p^2 < \Lambda_n^2$, the replacement would be

$$
\frac{i}{p^k\gamma_k+\varepsilon} \rightarrow -ip^k\gamma_k \times \frac{1}{1+y} \ , \quad y = 1/x = \frac{p^2}{\Lambda_n^2} \ . \tag{2.25}
$$

   Note that the pole in the momentum space disappears now. In the second case it is formally present but the condition $p^2 > \Lambda_n^2$ does not allow it. This could be interpreted by saying that colored fermions do not appear as massless on-mass-shell states.

Does the integral over the momentum space make sense and how could one perform it? Could the fact that the geometric series is an alternating series help to achieve convergence? A very naive approach would be to replaced $i\gamma^k p_k$ with differential operator $\gamma^k\partial_k$ and p^2 with the d'Alembertian $\Box = m^{kl}\partial_k\partial_l$ and apply it to the space-time representation of massless Dirac propagator.

Consider first the physically interesting case $p^2 < \Lambda_n^2$.

1. One can argue that the factor $1/(1+y) = \sum_k (-1)^k y^k$ is obtained by applying the operator $1/(1+y) = \sum_k (-1)^k \Box^k/\Lambda_n^{2k}$ to the analog of space-time propagator defined by $\gamma\cdot\nabla \int exp(ip\cdot(m_1-m_2))dV_4 = \gamma\cdot\nabla\delta(m_1-m_2)$.

2. This gives rise to a highly singular alternating series

$$D_F(m) = \gamma\cdot\nabla\sum_k (-1)^k \Lambda_n^{-2k}\Box^k\delta(m_1-m_2) \ . \tag{2.26}$$

   This expression can be interpreted as a distribution. The basic formula is the definition of the delta function as a distribution satisfying $f(x)\delta(x)dx = f(0)$. This definition generalizes to the derivatives of the delta function by using partial integration so that the derivatives of delta function can be transformed to derivatives of $f(x)$ at $x=0$: the basic formula is $f(x)\delta(x)dx = -(df/dx)(0)$.

   In the case of matrix elements of the Dirac propagator, the action of $\gamma\cdot\nabla$ and $\Box$ must be transferred in this way to an action on initial and final fermion states. This effectively transforma the action of the propagator to the momentum representation which was the starting point. Therefore it seems that in this case the space-time representation is not useful as such.

What about the situation for $p^2 > \Lambda_n^2$?

1. The coefficient $1/(1+x) = \sum (-1)^n x^n$ is replaced with

$$\frac{1}{1+(\Lambda_n^2/\Box)} \ . \tag{2.27}$$

   Note that the representation

$$= \frac{\Box}{1+(\Box/\Lambda_n^2)} = \Box\sum_k (-1)^k \frac{\Box^k}{\Lambda_n^{2k}} \ .$$

   would lead to the same result as in the first case.

2. The challenge is to define the action of $1/\Box$ and its powers.

   (a) $1/\Box$ is essentially a massless scalar propagator $D(m_1,m_2) = D(m_1-m_2)$ for a scalar field and its action is not local. $1/\Box = D(m_1-m_2)$ generates a spherical wave concentrated at the light-cone $a^2 = (m_1-m_2)^2 = 0$ (see this). The powers of $1/\Box$ correspond to a repeated non-local action of $D(m_1-m_2)$. This brings in mind Huygens principle stating that each point of the radiation field serves as a source of secondary wave but this is something different. Here it is not quite clear to me whether to identify $D(m_1-m_2)$ as a retarded, advanced or Feynman propagator with both contributions present but with opposite signs.

   (b) One can start from the explicit expression for the Dirac propagator proportional $D_{F,0} = \gamma\cdot\nabla(1/a^2)$ as the zeroth order contribution serving as a source for the first order contribution. n^{th} order term serves as a source for $(n+1)^{th}$ order contribution. One can utilize the commutativity of the Dirac operator and $1/\Box$ to simplify the calculation by applying first the operator $1/(1+\Lambda_n^2/\Box) = \sum_k (-1)^k(\Lambda^2/\Box)^k$ first to $1/a^2$ and $\gamma\cdot\nabla$ only after that.

(c) The first step assigns an expanding spherical light-wave to each point m_1. If D can be identified as retarded propagator, as zero energy ontology (ZEO) suggests, the first order contribution at point m is given by an integral over the past directed light cone LC_- for a retarded propagator with $D_{F,0}$ as a source. This contribution is of form $D_{F,1} = -\Lambda_n^2 \int_{LC_-} D(m-m_1)D_{F,0}dV$. In the n^{th} order this gives the formula

$$D_{F,n+1} = -\Lambda_n^2 \int_{LC_-} D(m-m_1)D_{F,n}dV \quad . \tag{2.28}$$

How to identify the integration measure dV? The natural identification of dV is as a scale invariant integration measure $dV = (dr/r)d\Omega$ associated with the coordinates of the light-cone boundary (r,θ,ϕ) is natural.

3 Fermion propagator for the induced spinor fields

One could also associated fermionic propagation with the induced fermion fields. This could be seen as the geometric analog for on-mass-shell propagation.

3.1 Is the fermion propagation along space-time surface a sensical notion?

For the modes of the induced spinor field the propagation would take place along the spacetime surface and possibly also along a related lower-dimensional surfaces such as a partonic 2-surface and light-like curves along it.

1. This 4-dimensional (or possibly lower-dimensional) propagator corresponds to the inverse of the Dirac operator for the induced gamma matrices as projection of the 8-D vector formed by H gamma matrices. Induced gamma matrices anticommute to the induced metric.

2. One can also consider modified gamma matrices determined as contractions of the canonical momentum currents of the classical action with the gamma matrices of H [K9, K7]. These do not anticommute to the induced metric but for them 4-D super-conformal symmetry would be exact and classical field equations would appear as a consistency condition guaranteeing hermiticity.

   For the induced gamma matrices super-conformal symmetry is violated at the vertices [L14], where the full action makes itself visible: outside the singularities minimal surface property consistent with volume action, giving rise to induced gamma matrices, holds true.

3. The beauty of this view (also that based on H propagator) is that the constructions of the propagator and of the physical states would be completely separated from each other. The propagator is the inverse of the Dirac operator for induced spinors. For H propagator the construction is easy but the situation need not be the same now.

4. One might hope that holography= holomorphy principle [L11, L15, L18] allows the explicit construction of the propagator using complex coordinates assignable to the Hamilton-Jacobi structure involving hypercomplex coordinates (u,v) and complex coordinates $(w,\overline{w})$ [L7]. Hypercomplex coordinates (u,v) would be the analogs of light-like coordinates and the Minkowskian signature would make the propagation possible.

   In the Euclidean situation the propagator would be bilinear in holomorphic and antiholomorphic modes $h_1 = h_2$ would be a delta function singularity. Now the situation is more complex but one might hope that Wick rotation for the second complex coordinate, transforming it to hypercomplex coordinate, could give the propagator.

   For light-like curves at partonic orbits identified as fermionic lines, the propagator can be constructed explicitly by a semiclassical argument.

The induced/modified Dirac equation is necessary for the construction of WCW gamma matrices and WCW Dirac equation as counterpart of generalized Super Virasoro conditions.

1. At the level of space-time surface induced spinor structure emerges and induced spinors are restrictions of H spinors to X^4. The guess is that induced spinors are holomorphic and annihilated by holomorphic gammas just as in string models. For string world sheets the same holds true but the modes depend only on the other hypercomplex coordinate u.

2. The problem is that the modes of H spinor fields are not holomorphic with respect to the complex coordinates of H. It seems that one could construct holomorphic solutions of the Dirac equation of H using generalized complex coordinates but they need not remain finite in the entire H. However, space-time surfaces X^4 correspond to sub-manifolds of $CD \subset H$ and do not include the entire CP_2 if they have time-like direction.

   If a holomorphic basis for H spinor fields and a hypercomplex coordinate and one complex coordinates of H as coordinates of X^4 makes sense, the problem could be solved. The restriction of $\Psi(h)$ to X^4 as $\Psi(h(x))$ would allow to express the fermionic oscillator operators at X^4 as superpositions of those of H.

3. It is however far from clear whether the two spinor bases for H are equivalent and whether one can relate the oscillator operators of the two bases to each other. For CD boundary conditions reduce the number of M^4 modes: for instance, for a particle in a box the momenta are quantized. What is the effect of giving up the compactness of CP_2 by allowing holes in CP_2? Could it increase the number of modes or could giving up the boundary conditions restrict the number of modes? Certainly it would allow holomorphic modes as analogs of massless modes.

   Sphere S^2 with one point removed represents a simplified example. Spherical harmonics $Y_{l,m}$ are solutions of $\nabla^2\psi = l(l+1)\psi$ and the powers of z^n are solutions of $\nabla^2\psi = 0$. Can these two bases be equivalent or should holomorphic solutions be counted as separate singular solutions? Could the spinor modes be seen as sections for which behavior is z^n at the upper hemisphere and z^{-n} at the lower hemisphere. At the equatorial circle the modes would coincide. Note that for both bases the component L_z of angular momentum is well defined for a given mode. In CP_2 the color isospin and hypercharge would be well-defined for both bases.

   The holomorphic modes are proportional to z^n factors to $\rho^n exp(in\phi)$ whereas there is an infinite number of different spherical harmonics Y_n^l with the same value of n. One can say that holomorphic modes correspond effectively to 1-D space whereas Y_n^l correspond to 2-D sphere. Same is true also in the case of CP_2 and M^4 so that the embedding space becomes effectively 4-dimensional. This is to be expected since holomorphy effectively halves the number of dimensions.

4. If this works, one can write for the second half of H spinor field Ψ as

   $$\Psi = \sum_n \Psi^n(h(u,v,\xi_1,\xi_2))a_n^\dagger = \sum_n a_n^\dagger \sum_m A_m^n(X^4)\Psi^m(u,w) = \sum_m \Psi^m(u,v)a_n^\dagger \quad . \tag{3.1}$$

   A similar formula would be also true for the conjugate. The oscillator operators of H spinor field and X^4 spinor field would be related by the formula

   $$b_{\dagger m} = \sum_n A_m^n(X^4)a_n^\dagger \quad . \tag{3.2}$$

   This would fix the anticommutation relations for the oscillators operators at X^4 and the computation of X^4 propagator as time order product for the fundamental fermion fields would be possible.

5. It does not seem necessary to restrict the consideration to a lower-dimensional submanifold of X^4 but also partonic orbits, string world sheets, and fermion lines are possible. On a fermion line, there would be only one light-like coordinate and everything would be simple.

One can criticize this approach.

1. In the case of Minkowski space it is clear what virtuality of the fermion means for the massless case: the explicit expression for the massless fermion propagator implies the massive virtual modes. For virtual modes with $p^2 \neq 0$ $p^k\gamma_k\Psi$ is nonvanishing.

2. What could the non-vanishing virtual momentum squared correspond to in the recent case? The condition $D\Psi = (\Gamma^k D_k + \Gamma^{\overline{k}} D_{\overline{k}}\Psi = 0$, reduces for the holomorphic modes, satisfying $D_{\overline{k}}\Psi = 0$, to the condition $\Gamma^k\Psi = 0$. An analogous condition is true for the antiholomorphic modes.

   $\Gamma^k\Psi \neq 0$ should be true for the virtual modes. This condition is consistent with the condition that H chirality is fixed but means mixing of the M^4 chiralities which is the signature of massivation due to the virtual mass.

3.2 A model for 4-D propagation as propagation along light-like geodesics of H

Quantum classical correspondence suggests that the 4-D Dirac equation for the induced spinor fields at the space-time surface has a physical meaning. The condition that no propagation in CP_2 occurs in H propagator suggests that the fermionic propagation in X^4 reduces to 1-D propagation along light-like geodesics of H and it is interesting to look for the consequences of this assumption.

3.3 Does the notion of fermion propagation along a light-like geodesic line of H make sense?

The fermionic propagation in H looks like a realistic option and massless H- propagation could be interpreted in terms of "off-space-time sheet" propagation as analogs of off-mass-shell propagation.

On the other hand, the induction of spinor connection [K9, K7, L8] makes CP_2 spinor fields dynamical and the induction of spinor fields can be seen as its natural fermionic counterpart. Could the solutions of the induced or modified Dirac equation as geometric analogs of on-mass shell states have a physical meaning? What makes them so attractive is the 4-D conformal invariance allowing us to construct them explicitly. Their analogs also appear in string models.

There is an entire dimensional hierarchy involved: besides 4-D propagation also propagation in dimensions $D = 1, 2, 3$ are possible and these modes have edge states (see this) as condensed matter analogs [L4]. As the matrix elements for the propagation of color fermion states are expected to be very small except for fermionic lines at intersections of string world sheets and light-like partonic 2-surfaces. In the 1-D case the construction of the analog of the propagator is simple. Although the propagator might not be a useful object, the classical view about space-time propagation could be very useful and justified by quantum classical correspondence.

1. 1-D H propagation between $h_1 = (m_1, s_1)$ and $h_2 = (m_2, s_2)$ is expected to be possible only when the condition $s_1 = s_2$ is true. The set of points s_2 satisfying this condition for a given s_1 is discrete. In ZEO, the H propagator defines a map between the 3-surfaces at the boundaries of CD. This suggests that 1-D propagation in X^4 is preferred. At partonic 2-surfaces with 2-D CP_2 projection, which does not change during the time evolution each points s_1 has at least one s_2. One can consider also lattice-like structures.

2. 8-D light-likeness and restriction to a fermion line suggests that this propagation is along a light-like curve of H belonging to the space-time surface. If this light-like curve is geodesic, then for for $s_1 \neq s_2$ it is a light-like geodesic associated with $M^4 \times S^1$, where S^1 is the unique geodesic circle connecting connecting $s_1 \neq s_2$. For $s_1 = s_2$ all geodesics belonging to the space-time surface are possible and even a 4-D set of geodesic circles of CP_2 is possible unless one poses additional conditions. For instance, 2-D CP_2 projection is such a constraint. In this kind of situation, one might speak of a resonance-like phenomenon.

   Could one consider additional conditions? Light-like geodesics as basic geometric aspect of twistors has also a CP_2 counterpart. For the CP_2 counterpart for twistorialization [L18], the geodesic lines connecting a given point to infinity of the local Eguchi-Hanson coordinates [L8] defining a homologically non-trivial geodesic sphere is in a special role and would correspond to the twistor sphere of the CP_2 twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$. Could the limitation to this set of CP_2 circles make sense? The fact that two CP_2 geodesic circles always have an intersection point has a counterpart for M^4 twistors: the spheres of twistor bundles for two points connected by a light-like geodesic intersect.

3. One cannot exclude the alternative in which the propagation occurs along general light-like geodesics of X^4. Also in this case the condition $s_1 = s_2$ implies that the set of points s_2 connected to point s_1 is finite. This option will not be discussed in the sequel. Also in this case the resonance can take place when the CP_2 projection is a geodesic sphere.

These observations provide motivations for the study of the 1-D fermionic propagation for induced spinor fields.

3.4 A model in which the fermions move along light-like geodesics of $M^4 \times S^1 \subset M^4 \times CP_2$

It is interesting to study in more detail the assumption that the motion is along a light-like geodesic of $M^4 \times S^1$.

1. The light-like geodesic of H makes it possible to get from the point s_1 in CP_2 to point satisfying in general $s_2 \neq s_1$. In this case the velocity in M^4 is less than the speed of light: $\beta = v/c < 1$ and the M^4 mass is not effectively zero as it is for a light-like geodesics of M^4.

   It is also possible that the CP_2 projection of the particle orbit wraps n times around S^1 and returns to the original point in CP_2. This would indeed happen if the propagator in CP_2 degrees of freedom reduces to a delta function. At least this assumption is a very promising simplification since the length scale of CP_2 is very small compared to the length of the projection of the geodesic line in M^4.

2. The fermion can be thought of as moving along the light-like geodesic of $M^4 \times S^1$:

$$
\begin{aligned}
&(m^0 = t, m^3 \equiv z = \beta t), \Phi = \omega t, \\
&\beta = \tfrac{v}{c} = \sqrt{1 - R^2\omega^2} \leq 1.
\end{aligned}
\tag{3.3}
$$

   Here β is the particle's velocity in M^4.

   The projection of the particle orbit in CP_2 is a geodesic circle. The starting point and end point of the propagator are on this geodesic line.

3. Since the fermion is no longer moving along the light-like geodesic line of M^4, it no longer behaves like a massless particle in M^4 sense. Quantum classical correspondence suggests that this serves as a classical correlate for the mass of the particle resulting from p-adic thermodynamics.

   Unless β is very close to 1, i.e. unless the parameter ωR is very small, there will be an exponential damping caused by the thermal mass. If p-dic thermodynamics described the situation, a reasonable guess is that the parameter ωR is of the order of p-adic thermal mass using CP_2 mass as a unit (here $\hbar = 1$) [L5].

4. If the CP_2 propagator is approximated as or is a delta function then $s_1 = s_2$ can be realized if the particle wraps n times around S^1:

$$
\Delta\Phi = n \times 2\pi. \tag{3.4}
$$

   In this case a 4-D set of circles S^1 belonging to X^4 is possible and one can say that a resonance occurs. For CP_2 type extremals this could occur. This is also the case if the CP_2 projections is a geodesic sphere. The twistorial picture would suggest that the set of circles correspond to a homologically trivial geodesic sphere of CP_2 as an analog of the twistor sphere parametrizing light-like geodesics in M^4.

5. For M^4 the time coordinate difference in the M^4 propagator would be

$$t_2 - t_1 = \frac{n2\pi}{\omega} \quad z_2 - z_1 = \beta(t_2 - t_1) = \beta\frac{n2\pi}{\omega} . \tag{3.5}$$

This could actually be a good approximation.

Suppose that the model indeed provides a semiclassical geometric description for the p-adic thermal massivation.

1. One has $m_2 - m_1 = (t, \beta t)$. The difference $a^2 = (m_2 - m_1)^2$ is the square of the Lorentz invariant distance of M^4 and the parameter a should be of the order of the hadronic length scale L_H.

   This gives

$$\frac{a}{R} = \sqrt{(m_2 - m_1)^2} = n2\pi\beta . \tag{3.6}$$

   R is the CP_2 radius and the natural length unit. The result is amazingly simple. $a/R \sim L_H/R$ could be interpreted as the ratio of the hadron sizes scale to CP_2 radius. This gives estimate for n:

$$n = \frac{L_H}{2\pi\beta R} \simeq \frac{L_H}{2\pi R} \sim 10^{15} , \tag{3.7}$$

   which gives $R/L_H \simeq 1/(2\pi n)$. The parameter $\omega R = \sqrt{1-\beta^2}$ is very small: one has

$$1 - \beta \simeq 1 - \frac{R^2\omega^2}{2} . \tag{3.8}$$

2. One can estimate the order of magnitude for ω and therefore also for $1 - \beta$. By quantum classical correspondence, ω should be of the order of the p-adic mass scale m_q of quark: $\omega \simeq m_q$. This gives

$$1 - \beta \simeq 1 - \frac{R^2 m_q^2}{2} . \tag{3.9}$$

   For β this gives and order of magnitude estimate relating β and n to each other:

$$1 - \beta \sim (\frac{m_H}{m_q})^2 (2\pi n)^{-2} . \tag{3.10}$$

   The order of magnitude is 10^{-32} for $m_H/m_q = 1$.

   The p-adic probabilities for the lowest excitations appearing in p-adic thermodynamics are extremely small and of this order of magnitude. This guarantees that the p-adic thermal mass coming from excitations with CP_2 mass scale is of order particle mass.

   p-Adic length scale hypothesis $p \simeq 2^k$ gives $m_q R \sim R/L_p \sim 1/\sqrt{p} = 2^{-k/2}$. Therefore one has

$$\omega R \simeq m_q R = \frac{R}{L_p} \sim 2^{-k/2} . \tag{3.11}$$

   All these estimates are rough order of magnitude estimates.

3. The expressions for the initial and final points of the light-like geodesic must be placed to the space-time representation of the M^4 propagator for which a concrete formula must be given as a Fourier transform. The CP_2 points are in complex coordinates $s_2 = (Rsin(\omega Rt), 0, 0, 0,)$ and $s_1 = (0, 0, 0, 0)$

   For $s_1 = s_2$ this gives

$$\frac{a}{R} = n \times 2\pi\beta \simeq n \times 2\pi . \tag{3.12}$$

   The relevant parameter would be n, which would be very large and determined by the p-adic length-scale hypothesis.

3.5 Could the propagator for the induced spinor fields along a light-like geodesic of H make sense?

It is not at all clear whether the propagator in X^4 or along light-like geodesics of H is a useful concept. It is however interesting to see whether one could define and calculate the propagator for the Dirac operator associated with the induced spinor fields. There is dimensional hierarchy: 4-D propagator, 2-D propagator at string world sheets, and 1-D propagator at fermion lines. Perhaps also 3-D propagator at partonic orbits. Holography= holomorphy vision could make it possible to solve the propagator in terms of holomorphic solutions of the modified Dirac equation.

In 1-D situation the condition $s_1 = s_2$ looks natural since the light-like geodesics of H defining fermion lines are very simple. Could this be possible in this case?

1. The induced gamma matrix Γ_t is defined as a projection of the 8-vector defined by H gamma matrices to the light-like geodesic ($m^0 = t, m^3 = \beta t, \Phi = \omega t$):

$$\Gamma_t = \gamma_0 + \beta\gamma_z + \omega R\gamma_\phi , \quad \gamma_\phi^2 = -1 . \tag{3.13}$$

2. One could identify the fermionic lines as light-like $M^4 \times S^1$ geodesics. The modified Dirac equation $\Gamma^t D_t \Psi = 0$ is proportional to $1/g_{tt}$, which diverges. This suggests that $D_t\Psi = 0$ or $\Gamma_t\Psi = 0$ as a counterpart of the Dirac equation or both of them are true for fermionic on-mass-shell states.

   The condition $\Gamma_t\Psi = 0$ as the analog of $\gamma^k p_k \Psi = 0$ would fix a definite H chirality correlating M^4 and CP_2 chiralities and should hold true also for off-mass-shell states. The off-mass shell property would correspond to $D_t\Psi = p\Psi$, where p is the counterpart of mass squared for off-mass-shell mass.

3. The analog of the momentum space propagator associated with a point pair connected by a light-like H geodesic would be

$$D_F = \frac{p\Gamma_t}{p^2 + i\epsilon} . \tag{3.14}$$

   Here p is the analog of momentum. D_F would be contracted by H spinor mode at both ends of the light-like H geodesic connecting the points of H. These modes would be on-mass-shell-modes as solutions of the Dirac equation of H, just like the end points of the Feynman graphs. The spectrum of the Dirac operator of H would not appear at all in the propagator, which would mean an enormous simplification. This picture is also consistent with the view based on the conformal algebra.

4. The spatial representation of the propagator would be given by a Fourier transform as

$$D_F(t_1, t_2) = \frac{\Gamma_t}{2\pi} \int dp \frac{exp(ip(t_1 - t_2)/\omega)}{p + i\epsilon} . \tag{3.15}$$

The ends of the propagator lines satisfy $t_1 - t_2 = t = n\omega R$. For momenta $p = k\omega/n$, $k \in Z$, the phase factors would be equal to 1 and a constructive interference would take place.

One can argue that the sum over these momenta gives the dominating contribution to the propagator. One can also argue that only these momenta are allowed by the boundary conditions at the ends of the propagator line.

This contribution would be

$$D_F(t_1, t_2) = \frac{\Gamma_t}{2\pi} \sum_k \frac{1}{k\omega/n + i\epsilon} = \frac{n}{\omega}\frac{\Gamma_t}{2\pi}\zeta(1). \quad (3.16)$$

The sum diverges but could be interpreted as the value of Riemann zeta $\zeta(s) = \sum n^{-s}$ for $s = 1$.

The matrix element of D_F is between two H spinor modes which define the ground states for the super-conformal representations associated with the physical fermions.

1. The H spinor modes are estimated at points $h_1 = (m_1, s_1)$ and $h_2 = (m_2, s_2)$ connecting by the circle S^1 and $m_1^0 = t$ satisfies $t = n\omega 2\pi R + \phi_1$, where $\phi_1 \in \{0, 2\pi\}$ is negligible as compared to $n\omega 2\pi R$.

   One can fix the M^4 points m_1 and m_2 apart from the constraint $(t \simeq n\omega 2\pi R, z = \beta t)$ but must integrate over CP_2 points s_1 and s_2.

2. The geodesic circle S^1 is determined by the pair (s_1, s_2) of CP_2 points so that the $\omega R\gamma_\phi$, where γ_ϕ is induced gamma matrix

$$\Gamma_\phi = \Gamma_k \partial_\phi s^k \frac{d\phi}{dt}, \quad \frac{d\phi}{dt} = \omega. \quad (3.17)$$

   Here Γ_k are CP_2 gamma matrices and are proportional to R.

3. The matrix element between eigenstates of H-chirality in sum of parts involving $\gamma_a \equiv \gamma_0 + \beta\gamma_z$ and $\gamma_b \equiv \omega\Gamma_\phi$.

   (a) γ_b is proportional Γ_ϕ depends explicitly on the point pair (s_1, s_2) so that this matrix element correlates points s_1 and s_2.

   (b) The matrix element of γ_a involves the contraction $\overline{\Phi}_{\epsilon_1}(s_1)\Phi_{\epsilon_2}(s_2)$. If the integrations were over the entire CP_2, the outcome would vanish. However, the constraint that the points h_1 and h_2 belong to the same space-time surface does not allow the integration over both CP_2:s and the constraint that the points are at the ends of the light-like curve at the partonic orbit and the ends of the fermion line resides at partonic 2-surfaces, implies further restrictions. The integrations are over the partonic 2-surfaces X^2 and Y^2 using the measure $\sqrt{g_2}$ defined by the induced metric. This gives to the matrix element and dependence on the genus of partonic 2-surface essential for the family replication phenomenon [K1, K3] [L6].

A About the solutions of Dirac equation in H

This section serves as an Appendix and re-represents information appearing in [K3]. First the TGD view of electroweak and color interactions is compared with the standard view. After that the general solutions of the Dirac equation in H are discussed.

A.1 How does the TGD view of standard model interactions differ from the standard model view?

TGD based vision standard model interactions differs in several respects from the standard view.

1. In TGD, elementary particles correspond to closed monopole flux tubes as analogs of hadronic strings connecting two Minkowskian space-time sheets by Euclidean wormhole contacts. The light-like orbits of wormhole throats (partonic orbits) carry fermions and antifermions at light curves located at light-like 3-surfaces, which define interfaces between Minkowskian string world regions and Euclidean regions identified as deformed CP_2 type extremals.

2. The basic difference at the level of H spinor fields is that color quantum numbers are not spin-like but are replaced with color partial waves in CP_2. Color degrees of freedom are analogous to the rotational degrees of freedom of a rigid body. An infinite number of color partial waves emerges for both quarks and leptons. In TGD, color and electroweak degrees of freedom are strongly correlated as is also clear from the fact that color symmetries correspond to the non-broken symmetries as isometries of CP_2 and electroweak symmetries correspond to the holonomies of CP_2, which are automatically broken gauge symmetries.

   The spectrum of color partial waves in H is different for U and D type quarks and for charged leptons and neutrinos. The triality of the partial wave is zero for leptons and 1 *resp.* -1 for quarks *resp.* antiquarks. At the level of fundamental fermions, which do not correspond as such to fermions as elementary particles, there is a strong violation of isospin symmetry.

The physical states are constructed using p-adic thermodynamics [K3, K1] [L5] for the scaling generator L_0 of the conformal symmetries extended to the space-time level and involve the action of Kac-Moody type algebras. The basic challenge of the state construction of the physical states is to obtain physical states with correct color quantum numbers.

1. General irrep of $SU(3)$ is labelled by a pair (p,q) of integers, where p *resp.* q corresponds intuitively to the number of quarks *resp.* antiquarks. The dimension of the representation is $d(p,q) = (1/2)(p+1)(q+1)(p+q+2)$.

   The spinors assignable to left and right handed neutrino correspond to representations of color group of type (p,p), where the integers and only right-handed neutrino allows singlet $(0,0)$ as covariantly constant CP_2 spinor mode. $(1,1)$ corresponds to octet 8. Charged leptons allow representations of type $(3+p,p)$: $p=0$ corresponds to decuplet 10. Note that $(0,3)$ corresponds to $\overline{10}$.

   Quarks correspond to irreps of type obtained from leptons by adding one quarks that is replacing $(p+3,p)$ with $(p+4,p)$ ($p=0$ gives $d=20$) or (p,p) with $(p+1,p)$ ($p=1$ gives $d=42$). Antiquarks are obtained by replacing $(p,p+3)$ replaced with $(p,p+4)$ and (p,p) with $(p,p+1)$.

2. Physical leptons (quarks) are color singlets (triplets). One can imagine two ways to achieve this.

   Option I: The conformal generators act on the ground state defined by the spinor harmonic of H. Could the tensor product of the conformal generators with spinor modes give a color singlet state for leptons and triplet state for quarks? The constraint that Kac-Moody type generators annihilate the physical states, realizing conformal invariance, might pose severe difficulties.

   In fact, TGD leads to the proposal that there is a hierarchical symmetry breaking for conformal half-algebras containing a hierarchy of isomorphic sub-algebras with conformal weights coming as multiplets of the weights of the entire algebra. This would make the gauge symmetry of the subalgebra with weights below given maximal weight to a physical symmetry.

   Option II: The proposal is that the wormhole throats also contain pairs of left- and right-handed neutrinos guaranteeing that the total electroweak quantum numbers of the string-like closed monopole flux tube representing hadron vanishes. This would make the weak interactions short-ranged with the range determined by the length of the string-like object.

One must study the tensor products of $\nu_L\overline{\nu}_R$ and $\overline{\nu_L}\nu_R$ states with the leptonic (quark) spinor harmonic to see whether it is possible to obtain singlet (triplet) states. The tensor product of a neutrino octet with a neutrino type spinor contains a color singlet. The tensor product $8 \otimes 8 = 1 + 8_A + 8_S + 10 + \overline{10} + 27$ contains $\overline{10}$ and its tensor product with 10 for quark contains a color triplet.

The ground states for the Super Virasoro representations correspond to spinor harmonics in $M^4 \times CP_2$ characterized by momentum and color quantum numbers. The correlation between color and electro-weak quantum numbers is wrong for the spinor harmonics and these states would be also hyper-massive. One can imagine two mechanisms.

1. The super-symplectic generators could allow to build color triplet states having negative vacuum conformal weights, and their values are such that p-adic massivation is consistent with the predictions of the earlier model differing from the recent one in the quark sector. In the following the construction and the properties of the color partial waves for fermions and bosons are considered. The discussion follows closely to the discussion of [A1] .

2. In the concrete model for elementary particles a pair of left- and right-handed neutrino associated with the monopole flux tube screens weak isospin above weak boson Compton length. This pair could be in such color partial waves that the outcome is leptonic color singlet and quark color triplet.

In the sequel only the color partial waves in H are discussed. This reduces to the discussion of the eigenstates of CP_2 d'Alembertian whose eigestates the spinor harmonics are.

A.2 General construction of solutions of Dirac operator of H

The construction of the solutions of massless spinor and other d'Alembertians in $M^4_+ \times CP_2$ is based on the following observations.

1. d'Alembertian corresponds to a massless wave equation $M^4 \times CP_2$ and thus Kaluza-Klein picture applies, that is M^4_+ mass is generated from the momentum in CP_2 degrees of freedom. This implies mass quantization:

$$M^2 = M_n^2 \ , \tag{A.1}$$

where M_n^2 are eigenvalues of CP_2 Laplacian. Here of course, ordinary field theory is considered. In TGD the vacuum weight changes mass squared spectrum.

2. In order to get a respectable spinor structure in CP_2 one must couple CP_2 spinors to an odd integer multiple of the Kähler gauge potential. Leptons and quarks correspond to $n = 3$ and $n = 1$ couplings respectively. The spectrum of the electromagnetic charge comes out correctly for leptons and quarks.

3. Right handed neutrino is covariantly constant solution of CP_2 Laplacian for $n = 3$ coupling to Kähler gauge potential whereas right handed "electron" corresponds to the covariantly constant solution for $n = -3$. From the covariant constancy it follows that all solutions of the spinor Laplacian are obtained from these two basic solutions by multiplying with an appropriate solution of the scalar Laplacian coupled to Kähler gauge potential with such a coupling that a correct total Kähler charge results. Left handed solutions of spinor Laplacian are obtained simply by multiplying right handed solutions with CP_2 Dirac operator: in this operation the eigenvalues of the mass squared operator are obviously preserved.

4. The remaining task is to solve scalar Laplacian coupled to an arbitrary integer multiple of Kähler gauge potential. This can be achieved by noticing that the solutions of the massive CP_2 Laplacian can be regarded as solutions of S^5 scalar Laplacian. S^5 can indeed be regarded as a circle bundle over CP_2 and massive solutions of CP_2 Laplacian correspond to the

solutions of S^5 Laplacian with $exp(is\tau)$ dependence on S^1 coordinate such that s corresponds to the coupling to the Kähler gauge potential:

$$s = n/2 \ . \tag{A.2}$$

Thus one obtains

$$D_5^2 \ = \ (D_\mu - iA_\mu \partial_\tau)(D^\mu - iA^\mu \partial_\tau) + \partial_\tau^2 \tag{A.3}$$

so that the eigen values of CP_2 scalar Laplacian are

$$m^2(s) \ = \ m_5^2 + s^2 \tag{A.4}$$

for the assumed dependence on τ.

5. What remains to do, is to find the spectrum of S^5 Laplacian and this is an easy task. All solutions of S^5 Laplacian can be written as homogenous polynomial functions of C^3 complex coordinates Z^k and their complex conjugates and have a decomposition into the representations of $SU(3)$ acting in natural manner in C^3.

6. The solutions of the scalar Laplacian belong to the representations $(p, p+s)$ for $s \geq 0$ and to the representations $(p+|s|, p)$ of $SU(3)$ for $s \leq 0$. The eigenvalues $m^2(s)$ and degeneracies d are

$$\begin{aligned} m^2(s) \ &= \ \frac{2\Lambda}{3}[p^2 + (|s|+2)p + |s|] \ , \ \ p > 0 \ , \\ d \ &= \ \frac{1}{2}(p+1)(p+|s|+1)(2p+|s|+2) \ . \end{aligned} \tag{A.5}$$

Λ denotes the "cosmological constant" of CP_2 ($R_{ij} = \Lambda s_{ij}$).

A.2.1 Solutions of the leptonic spinor Laplacian

Right handed solutions of the leptonic spinor Laplacian are obtained from the asatz of form

$$\nu_R = \Phi_{s=0}\nu_R^0 \ , \tag{A.6}$$

where u_R is covariantly constant right handed neutrino and Φ scalar with vanishing Kähler charge. Right handed "electron" is obtained from the ansatz

$$e_R = \Phi_{s=3} e_R^0 \ , \tag{A.7}$$

where e_R^0 is covariantly constant for $n = -3$ coupling to Kähler gauge potential so that scalar function must have Kähler coupling $s = n/2 = 3$ a in order to get a correct Kähler charge. The d'Alembert equation reduces to

$$\begin{aligned} (D_\mu D^\mu - (1-\epsilon)\Lambda)\Phi = -m^2\Phi \ , \\ \epsilon(\nu) = 1 \ , \quad \epsilon(e) = -1 \ . \end{aligned} \tag{A.8}$$

The two additional terms correspond to the curvature scalar term and $J_{kl}\Sigma^{kl}$ terms in spinor Laplacian. The latter term is proportional to Kähler coupling and of different sign for ν and e, which explains the presence of the sign factor ϵ in the formula.

Right handed neutrinos correspond to (p,p) states with $p \geq 0$ with mass spectrum

$$\begin{aligned} m^2(\nu) &= \frac{m_1^2}{3}\left[p^2+2p\right] \ , \ p \geq 0 \ , \\ m_1^2 &\equiv 2\Lambda \ . \end{aligned} \tag{A.9}$$

Right handed "electrons" correspond to $(p,p+3)$ states with mass spectrum

$$m^2(e) = \frac{m_1^2}{3}\left[p^2+5p+6\right] \ , \ p \geq 0 \ . \tag{A.10}$$

Left handed solutions are obtained by operating with CP_2 Dirac operator on right handed solutions with one exception: the action of the Dirac operator on the covariantly constant right handed neutrino ($(p=0,p=0)$ state) annihilates it.

A.2.2 Quark spectrum

Quarks correspond to the second conserved H-chirality of H-spinors. The construction of the color partial waves for quarks proceeds along similar lines as for leptons. The Kähler coupling corresponds to $n=1$ (and $s=1/2$) and right handed U type quark corresponds to a right handed neutrino. U quark type solutions are constructed as solutions of form

$$U_R = u_R\Phi_{s==1} \ , \tag{A.11}$$

where u_R possesses the quantum numbers of covariantly constant right handed neutrino with Kähler charge $n=3$ ($s=3/2$). Hence Φ_s has $s=-1$. For D_R one has

$$D_R = d_r\Phi_{s=2} \ . \tag{A.12}$$

d_R has $s=-3/2$ so that one must have $s=2$. For U_R the representations $(p+1,p)$ with triality one are obtained and $p=0$ corresponds to color triplet. For D_R the representations $(p,p+2)$ are obtained and color triplet is missing from the spectrum ($p=0$ corresponds to $\overline{6}$).

The CP_2 contributions to masses are given by the formula

$$\begin{aligned} m^2(U,p) &= \frac{m_1^2}{3}\left[p^2+3p+2\right] \ , \ p \geq 0 \ , \\ m^2(D,p) &= \frac{m_1^2}{3}\left[p^2+4p+4\right] \ , \ p \geq 0 \ . \end{aligned} \tag{A.13}$$

Left handed quarks are obtained by applying Dirac operator to right handed quark states and mass formulas and color partial wave spectrum are the same as for right handed quarks.

The color contributions to p-adic mass squared are integer valued if $m_0^2/3$ is taken as a fundamental p-adic unit of mass squared. This choice has an obvious relevance for p-adic mass calculations since canonical identification does not commute with a division by integer. More precisely, the images of number xp in canonical identification has a value of order 1 when x is a non-trivial rational whereas for $x=np$ the value is n/p and extremely is small for physically interesting primes. This choice does not however affect the spectrum of massless states but can affect the spectrum of light states in case of electro-weak gauge bosons.

B Foundational questions

There are several foundational questions involved. What is the origin of the p-adicity? What is the justification of p-adic thermodynamics and what does it have a counterpart at quantum level? What are the poorly understood aspects of p-adic thermodynamics? What is the understanding of generalized superconformal representations and possible interpretational and technical problems.

B.1 Questions related to the p-adic thermodynamics

The nice feature of both approaches is that the propagators carry a minimum amount of information about the quantum states at the 3-surfaces to which the end points of the propagator are associated. Therefore the construction of physical states with correct total color quantum numbers do not affect the propagators. One also circumvents the problem how to take into account the massivation by p-adic thermodynamics at the level of the fermionic propagators.

This construction says nothing about how physical fermions are constructed and how the get their masses via p-adic thermodynamics.

B.1.1 The origin of p-adic thermodynamics

The understanding of the origin of p-adic thermodynamics is the first challenge.

1. In p-adic thermodynamics, the scaling generator L_0 takes the role of energy as a generator of time translations. Ordinary 2-D conformal invariance requires that physical states are annihilated by L_0. All states created by L_0 would correspond to gauge degrees of freedom. This cannot be the case in p-adic thermodynamics.

   In TGD, the conformal algebras are replaced by half-algebras [L9], which allow an infinite hierarchy of sub-algebras isomorphic to the full algebra, which means that a finite-dimensional sub-algebra transforms from gauge algebra to a dynamical symmetry algebra and p-adic thermodynamics applies to states in this finite-dimensional state space.

2. In the 2-D version of the p-adic thermodynamics, the ground state is necessarily tachyonic with a negative conformal weight representing the mass squared. The physical ground state with a vanishing mass squared is obtained by applying conformal generators with a positive conformal weight. Where do the generators with tachyonic conformal weight come from?

   (a) p-Adic thermodynamics in its recent formulation [L5] answers this question by extending the conformal algebra from dimension 2 to dimension 4 so that one has hypercomplex and complex coordinate and there are conformal weights associated with both of them. The 4-dimensional conformal invariance is realized in holography= holomorphy (H-H) vision involving the notion of Hamilton-Jacobi structure in an essential way [L7]. There are conformal weights associated with hypercomplex *resp.* complex coordinates and the corresponding conformal weights are non-negative *resp.* negative. Their sum vanishes for massless ground states.

   (b) $M^8 - H$ duality [L10, L18] provides further insights to the tachyon problem. In M^8, the momentum space M^4 corresponds to a quaternionic subspace of octonions realized as the normal space of point y of the 4-surfaces $Y^4 \subset M^8$. The point y carries the fermion. The 8-D mass squared always vanishes but the M^4 part is in general non-vanishing. For some points $y \in Y^4$ the normal space M^4 is such that the E^4 part of the 8-momentum vanishes and the particle is massless in M^4 sense. This is the counterpart for the cancellation of the two conformal weights at the level of H.

B.1.2 Questions related to the origin of p-adicity

One can also make ontological questions related to p-adicity.

1. What is the origin of the p-adicity? H-H vision [L15, L18] has led to a considerable progress in the attempts to understand the origin of p-adicity and the p-adic length scale hypothesis. The notion of functional p-adic numbers suggests how the p-adicization and adelization emerge. One can even ask whether it is necessary to introduce the p-adic and adelic variants of the embedding space in this approach.

2. p-Adic thermodynamics is a statistical description and must be an approximation to a genuine quantal description. Quantum TGD can be formally regarded as a square root of thermodynamics at a quantum critical temperature defined by the Kähler coupling strength [K6, K2]. Could it make sense to speak of a square root of p-adic thermodynamics in which thermal probabilities correspond to quantal probabilities. Thermal state would be replaced by a

quantum state. Very naively, thermodynamic weights would be replaced with their complex square roots, that is ordinary square roots involving a phase factor.

B.1.3 Challenging the p-adic view of cognition

The progress in the mathematical understanding of TGD has repeatedly challenged various working hypotheses. At this time, the progress in the understanding of holography= holomorphy vision challenged the proposed p-adic view of cognition and intention and raised the question whether ordinary p-adics could emerge from what I call functional p-adics as a generalization of p-adics [L15].

1. In the TGD based vision about p-adic physics as physics of cognition, the discretizations defining the intersection of realities and p-acidicites as real and p-adic space-time surfaces define cognitive representations consisting of points of $H = M^4 \times CP_2$ with preferred H coordinates in an extension of rationals. One can challenge this picture for CP_2 since it is not a linear space. For M^8 identifiable as octonions with the inner product defined as the real part of the octonionic product, the situation is linear and the objection can be circumvented since there exist highly unique linear coordinates. The need for preferred coordinates raises however as an objection.

2. One of the motivations for the p-adic vision about cognition was the inherent non-determinism of p-adic differential equations. Could p-adic view of cognition and imagination explain the non-determinism of imagination? Could the realization of an intention as action correspond to a quantum transition in which a p-adic counterpart of real space-time surfaces, obeying the same defining equations, is replaced with its real counterparts.

   The basic objection is that conservation laws might not allow this unless the conserved quantities are in an intersection of reals and the extension of p-adic numbers induced by an algebraic extension of rationals. If one accepts zero energy ontology [L1], in which quantum states are zero energy states identified as pairs of states with opposite total quantum numbers (this expresses classical conservation laws), this objection might be circumvented.

Holography = holomorphy vision, which in fact forces zero energy ontology, suggests a re-analysis of these proposals.

1. Space-time surfaces for which the Taylor coefficients appearing in function pairs (f_1, f_2) appearing in holomorphy= holography vision are in an extension of rationals and can be regarded as real numbers or to belong corresponding extensions p-adics. This is the case also for cognitive representations as the discretizations of space-time surfaces defined by points of H with coordinates in an extension of rationals. Are the p-adic $\rightarrow$ real transitions really needed? The question about the meaning of the intention-to action transition however remains. Is the p-adic variant of the space-time surface replaced with the real variant?

2. Is p-adic physics really something fundamental or does it emerge at the level of WCW? Holography= holomorphy vision leads to a generalization of p-adics to functional p-adics. Functional p-adic primes correspond to polynomials with prime degree p and functional p-adic number would be a power series of a polynomial with degree p with coefficients which have degree smaller than p. Functional p-adics would have space-time surfaces as representations.

   The functional p-adic numbers can be mapped by very-many-to-one morphism to p-adics [L15] by assigning to the functional p-adic number defined in terms of the degrees of the polynomials involved. This would provide a long-sought-for explanation for the p-adic length scale hypothesis and give a direct connection with the vision about cognitive hierarchies. It would also explain why small primes 2 and 3 appearing in the p-adic length scale hypothesis are special: for these primes the roots of the polynomial can be solved analytically. What is also nice that functional p-adics do not require any preferred coordinates and do not pose any restrictions on the coefficients of f_1 and f_2 since only the degree of the polynomial matters.

3. In this framework it is not clear whether ordinary p-adics are needed as a fundamental concept: are only the real variants of H and M^8 needed? Could ordinary p-adics emerge

from the functional p-adics represented at WCW level as real space-time surfaces. Could the crucial p-adic non-determinism correspond to the classical non-determinism at the level of real space-time?

4. But what could the transformation of intention to action mean in this framework? Is it associated with any small state function reduction (SSFR) in the degrees of freedom corresponding to the classical and p-adic non-determinism replacing superposition of real space-time surfaces with a new one?

   Intention means an idea about the future, about a goal. Intention means time directedness. Zero energy ontology replaces 3-surfaces with 4-surfaces with analogs of Bohr orbits as slightly non-deterministic space-time surfaces obeying holography. Could this 4-D character of the fundamental geometric objects be enough to explain intentionality.

B.2 Two problems related to generalized superconformal representations

The generalizations of super-conformal representation involved two problems.

B.2.1 A problem with super-conformal representations

What about the higher conformal excitations giving rise to the p-adic thermal mass? There does not seem to be any point or need to modify the massless propagators since they are associated with the ground state of the super-conformal representations. There is however a problem.

In string models the super generators G_n of Ramond representation and $G_{n+1/2}$ of Neveu-Schwarz representation define the analog of the Dirac operator. The anticommutators $\{G_m, G_n^\dagger\}$ and $\{G_{m+1/2}, G_{n+1/2}^\dagger\}$ give Virasoro generators L_n. In the Ramond representation one has $G_0 = p^k\gamma_k + G_{vib,0}$. The Ramond representation therefore contains the ordinary Dirac operator acting in cm degrees of freedom.

In pa-dic thermodynamics [K3] [L5] based on the TGD view of conformal invariance [L9], the Majorana option is not possible. Since the ordinary gamma matrices do not carry a fermion number, the Ramond formula of G would involve generators with a vanishing fermion number and non-vanishing fermion number. Same applies also to G_n.

1. A possible solution of the problem is that super generators and their Hermitian conjugates define two kinds of supergenerators. For Ramond representation, this would also include the counterparts of ordinary gamma matrices. The algebra creating the states would involve only non-negative conformal weights and possess a fractal hierarchy of included sub-algebras isomorphic to the entire algebra.

2. Could G_0 is non-hermitian so that both G_0 *resp.* $G_0^\dagger$ carrying a fermion number 1 *resp.* -1 are allowed. The counterparts of M^4 gamma matrices would be analogous to fermionic creation and annihilation operators, which would correspond to a complex structure in center-or mass degrees of freedom.

   WCW gamma matrices [L9] correspond to Noether super charges defined by the classical action principle related by supersymmetry to the bosonic Noether charges and carry a well-defined fermion number and they are matrix elements between the zero modes of the induced spinor modes and H spinor fields. The constant spinor modes could correspond to the counterparts of the ordinary gamma matrices. Therefore H spinor fields need not be Majorana.

   The Hamilton-Jacobi structure [L7] for M^4 based on hyper-complex and complex structure would make the generalized complexification of M^4 possible. The conjugation for complex coordinate and its hypercomplex counterpart would correspond to the hermitian conjugation for the operators γ_u and γ_w producing $\gamma_u^\dagger = \gamma_v$ and $\gamma_w^\dagger = \gamma_{\overline{w}}$ for both leptonic and quark-like representations. These gamma matrices would act as quantum operators.

B.2.2 Could one speak of the propagation of color singlets at WCW level?

There are 3 kinds of spinor fields and Dirac operators corresponding to three levels: embedding space H, spacetime surface X^4 and the "world of classical worlds" (WCW). A clear distinction must be made between these.

The propagation of fundamental fermions in H was already considered and supports color confinement: colored modes do not propagate on-mass-shell except in CP_2 mass scales. Could the propagation of the physical states identifiable as color singlet bound states of fundamental fermions be describable at the level of WCW?

1. Bound states of fundamental fermions are color singlets. The lowest states are massless but p-adic thermodynamics generates thermal mass for them. At quantum level, one expects that the square root of p-adic thermodynamics provides a proper description. A lot remains to be understood.

   The problem is that superposition of states with different masses violates Lorentz invariance in the standard ontology. Zero energy ontology (ZEO) could be of help here: there would be time-like quantum entanglement and Lorentz invariance would not be violated.

2. WCW Dirac equation would correspond to the super Virasoro conditions and for the states of generalize superconformal/super symplectic representations. WCW gamma matrices correspond to Noether super charges defined by the classical action principle related by supersymmetry to the bosonic Noether charges. They also include the counterparts of ordinary gamma matrices as entities carrying fermion numbers. This solves the already mentioned problem caused by the fact that spinor fields are not Majorana.

3. Is it possible to speak of the analog of propagation for WCW spinor fields or is this necessary at all? The analog of the fermionic propagator would correspond to the operator defined by the Super Virasoro charge bilinear in the counterparts of the analogs of Kac-Moody generators and WCW gamma matrices and therefore carries a fermion number. It is difficult to understand how this could make sense physically without giving up fermion number conservation. Its square corresponds to the Virasoro scaling generator conserving fermion number. Note that there are also two scaling generators and also this complicates the situation further. It would seem that their sum annihilates the ground states as massless states.

B.3 Does 8-dimensionality imply 8-dimensional relativity?

One cannot avoid the question whether 8-dimensionality implies an 8-D generalization of relativity.

1. The completely new element is that at M^8 level M^4 as normal space is not fixed but dynamical: could one speak of relativity for mass squared? The Lorentz boosts of this relativity could correspond to the Lorentz group $SO(1,7)$ of M^8. Also G_2 as a subgroup affects the M^4 mass squared values since it does not leave quaternionic subspace M^4 invariant although it does not affect the energy.

2. Masslessness in the 8-D sense at M^8 level forces us to challenge the assumption that the particles always have a well-defined 4-D mass squared. As a matter of fact, p-adic thermodynamics and its possible quantum variant as a square root of thermodynamics [K6, K2] already does this. Could one think that mass is more like a transversal or longitudinal momentum squared and depends on the state of motion characterized by the normal space of Y^4?

3. What about color quantum numbers: could 8-D relativity apply even to them. This would change dramatically the views of color. In M^8, $SO(4)$ acting on $E^4 \subset M^8$ is the counterpart of $SU(3)$ acting in $CP_2 \subset H$ and also as a subgroup $SU(3) \subset G_2$ of octonion automorphisms leaving invariant the octonion unit defining M^4 time direction in the normal space. M^4 as a quaternionic normal space of Y^4 varies and therefore also the subgroup $SU(3) \subset G_2$. Suppose that one replaces normal space-with a new one by deforming Y^4 such that the point $y \in E^4$ is not affected. What happens?

   I have asked whether $SO(4) - SU(3)$ duality between hadron physics and QCD type physics could correspond to $M^8 - H$ duality [L2, L3, L10, L18]. Could $M^8 - H$ duality map the

2 Cartan quantum numbers of $SO(4)$ representations appearing in M^8 spinor harmonics to those of $SU(3)$ representations of H spinor harmonics. Could the representations of $SU(3)$ be restricted to the $U(2)$ subgroup of the covering group of $SO(4)$ in accordance with the left-right asymmetry of the standard model?

Suppose that $SO(1,7)$ and also G_2 transformations affect the $SO(4)$ representations. If so, then by $M^8 - H$ duality in the proposed form they also affect the $SU(3)$ color partial waves for H spinors. By color confinement this is however not easy to observe.

B.4 Is the proposed picture consistent with $M^8 - H$ duality

M^8-H duality [L10, L18] plays a central role in TGD. Therefore one can also worry about whether octotwistors, which for physical states must reduce in some sense to quaternionic twistors, are a well-defined notion. The octonionic spinors must reduce to quaternionic spinors for the physical states in some sense. Massless Dirac equation must be satisfied and the analogs of the chirality conditions for H, M^4 and CP_2 must make sense.

In the sequel the massless octonionic Dirac equation will be discussed.

1. Octonionic and quaternionic spinors are very much analogous to the 2-spinors appearing in the definition of twistors since γ^0 in the Dirac equation is replaced with a real octonion unit. The masslessness condition is obtained when Minkowski inner product is defined as the real part for the product of octonions. The counterparts of sigma matrices are identified as quaternions and octonions and matrix representations for the quaternionic units are not used. This allows the introduction of a commuting imaginary unit i doubling the number of degrees of freedom and replacing octospinors with their complexified counterparts.

2. The octonionic Dirac equation looks like the ordinary Dirac equation but with gamma matrices replaced with octonionic units. The quaternionic Dirac equation involves quaternionic units but it is essential that they are not represented as matrices. This allows the introduction of imaginary uni i commuting with the quaternionic and octonionic units and implies double of the degrees of freedom so that one can have analogs of complex spinors.

   The octonionic units are analogs of Pauli sigma matrices and the first problem is caused by the lacking anticommutativity of the real unit with other octonion units. The Dirac equation however makes sense also in this case.

3. The 8-D masslessness condition must correspond to the condition that the real part of the square of Dirac operator on spinors vanishes. For momentum eigenstates this gives the usual algebraic conditions for masslessness.

4. H spinors have a defined H-chirality guaranteeing separate conservation of quark and lepton numbers. H-chirality ϵ is a product $\epsilon = \epsilon_1\epsilon_2$ of M^4 and CP_2 chiralities. Alls these chiralities should be definable also at the level of M^8. Also the octonionic Dirac equation for H spinors should be consistent with the chirality condition.

   (a) The decomposition of octonion units to quaternion units $\{1, I_k | k = 1,2,3\}$ and co-quaternion units $I_4\{1, I_k | k = 1,2,3\}$ suggests the identification of the counterpart of Γ_9. The matrix Γ_9 is defined as the product of H gamma matrices satisfies $\Gamma_9^2 = -1$, anticommutes with H gamma matrices. H chirality corresponds to the eigenvalue of $i\Gamma_9$ equal to $\epsilon = \pm 1$. The eigen spinors with chirality ϵ are of the form $(1 + \epsilon i\Gamma_9)\Psi_0$.

   The spinors with fixed H-chirality are tensor products of spinors of fixed M^4 chirality and CP_2 chirality and the product of these chiralities defines H-chirality.

   (b) The operator iI_4, satisfying $(iI_4)^2 = 1$, is a good guess for the counterpart of Γ_9 for the octonionic spinors. The octospinors with a fixed M^8 chirality ϵ should be of the form $(1 + \epsilon iI_4)\Psi_0$. It is easy to check that for an octospinor of form $\Psi_\epsilon = \Psi_a + I_4\Psi_b$ having a fixed chirality ϵ, one obtains

$$\Psi_b = i\epsilon\Psi_a \tag{B.1}$$

so that the spinor is determined completely by its quaternionic part. Perhaps this might be regarded as a realization of quaterniocity.

(c) One can decompose also the quaternionic spinors to two parts corresponding to the decomposition to complex subspace spanned by $\{1, I_1\}$ and co-complex subspace spanned by $\{I_2, I_3\}$. This allows us to define M^4 chirality and its E^4 counterpart.

5. Octonionic Dirac equation for the momentum eigenstates can be decomposed to a sum of quaternionic and co-quaternionic parts

$$D\Psi = (p_1^k I_k + I_4 p_2^k I_k)\Psi = 0 \ . \tag{B.2}$$

The real part of D^2 gives the Minkowskian mass shell condition $p_1^2 - p_2^2 = 0$.

The Dirac equation for the $\Psi_\epsilon = (1 + i\epsilon I_4)\Psi_0$ gives

$$\begin{array}{l} D\Psi_0 = (p_1^k I_k + I_4 p_2^k I_k)\Psi_0 + \tilde{D}\tilde{\Psi}_0 = 0 \ . \\ \tilde{D} = \tilde{p}_1^k I_k + I_4 \tilde{p}_2^k I_k) \ . \end{array} \tag{B.3}$$

Tilde means a conjugation of quaternionic imaginary units. This gives two separate equations? Are they consistent? By multiplying the equation with tildes by $1 = -I_2^2$ from left and transporting the second I_4 through the equation to right, one obtains the equation $-I_2(p_1^k I_k + I_4 p_2^k I_k)\Psi_0 I_5 = 0$. The two equations are therefore consistent.

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REFERENCES

Mathematics

[A1] Pope CN. Eigenfunctions and $Spin^c$ Structures on CP_2, 1980.

Books related to TGD

[K1] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K2] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Topological Geometrodynamics: Overview: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdoverview2.* Available at: https://tgdtheory.fi/pdfpool/vNeumannnew, 2023.

[K3] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K4] Pitkänen M. New Physics Predicted by TGD: Part I. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys1.pdf, 2023.

[K5] Pitkänen M. New Physics Predicted by TGD: Part II. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys2.pdf, 2023.

[K6] Pitkänen M. Number theoretic vision, Hyper-finite Factors and S-matrix. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/UandM.pdf, 2023.

[K7] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K8] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds". In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwsymm.pdf, 2023.

[K9] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

Articles about TGD

[L1] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L2] Pitkänen M. A critical re-examination of M^8-H duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L3] Pitkänen M. A critical re-examination of M^8-H duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L4] Pitkänen M. TGD and Condensed Matter. https://tgdtheory.fi/public_html/articles/TGDcondmatshort.pdf., 2021.

[L5] Pitkänen M. Two objections against p-adic thermodynamics and their resolution. https://tgdtheory.fi/public_html/articles/padmass2022.pdf., 2022.

[L6] Pitkänen M. About the TGD based views of family replication phenomenon and color confinement. https://tgdtheory.fi/public_html/articles/emuanomaly.pdf., 2023.

[L7] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[L8] Pitkänen M. Reduction of standard model structure to CP_2 geometry and other key ideas of TGD. https://tgdtheory.fi/public_html/articles/cp2etc.pdf., 2023.

[L9] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[L10] Pitkänen M. A fresh look at M^8-H duality and Poincare invariance. https://tgdtheory.fi/public_html/articles/TGDcritics.pdf., 2024.

[L11] Pitkänen M. About Langlands correspondence in the TGD framework. https://tgdtheory.fi/public_html/articles/Frenkel.pdf., 2024.

[L12] Pitkänen M. TGD as it is towards the end of 2024: part I. https://tgdtheory.fi/public_html/articles/TGD2024I.pdf., 2024.

[L13] Pitkänen M. TGD as it is towards the end of 2024: part II. https://tgdtheory.fi/public_html/articles/TGD2024II.pdf., 2024.

[L14] Pitkänen M. What gravitons are and could one detect them in TGD Universe? https://tgdtheory.fi/public_html/articles/whatgravitons.pdf., 2024.

[L15] Pitkänen M. A more detailed view about the TGD counterpart of Langlands correspondence. https://tgdtheory.fi/public_html/articles/Langlands2025.pdf., 2025.

[L16] Pitkänen M. Gödel,Lawvere and TGD. https://tgdtheory.fi/public_html/articles/Gtgd.pdf., 2025.

[L17] Pitkänen M. The violation of isospin symmetry in strong interactions and .511 MeV anomaly: evidence for TGD view of quark color. https://tgdtheory.fi/public_html/articles/isospinbreak.pdf., 2025.

[L18] Pitkänen M. Twistors and holography= holomorphy vision. https://tgdtheory.fi/public_html/articles/HHtwistor.pdf., 2025.