Quantum Adeles

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Abstract

Quantum arithmetics provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations. p-Adic numbers have p-adic pinary expansions $\sum a_n p^n$ satisfying $a_n < p$. of powers p^n to be products of primes $p_1 < p$ satisfying $a_n < p$ for ordinary p-adic numbers. One could map this expansion to its quantum counterpart by replacing a_n with their counterpart and by canonical identification map $p \rightarrow 1/p$ the expansion to real number. This definition might be criticized as being essentially equivalent with ordinary p-adic numbers since one can argue that the map of coefficients a_n to their quantum counterparts takes place only in the canonical identification map to reals.

One could however modify this recipe. Represent integer n as a product of primes l and allow for l all expansions for which the coefficients a_n consist of primes $p_1 < p$ but give up the condition $a_n < p$. This would give 1-to-many correspondence between ordinary p-adic numbers and their quantum counterparts.

It took time to realize that $l < p$ condition might be necessary in which case the quantization in this sense - if present at all - could be associated with the canonical identification map to reals. It would correspond only to the process taking into account finite measurement resolution rather than replacement of p-adic number field with something new, hopefully a field. At this step one might perhaps allow $l > p$ so that one would obtain several real images under canonical identification.

One can however imagine a third generalization of number concept. One can replace integer n with n-dimensional Hilbert space and sum $+$ and product $\times$ with direct sum $\oplus$ and tensor product $\otimes$ and introduce their co-operations, the definition of which is highly non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. The construction would also generalize the notion of algebraic holography and provide self-referential cognitive representation of mathematics.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the imbedding space.

Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. One could interpret $\times_q$ and $+_q$ and their co-algebra operations as 3-vertices for number theoretical Feynman diagrams describing algebraic identities $X = Y$ having natural interpretation in zero energy ontology. The two vertices have direct counterparts as two kinds of basic topological vertices in quantum TGD (stringy vertices and vertices of Feynman diagrams). The definition of co-operations would characterize quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers. One prediction is that all loops can be eliminated from generalized Feynman diagrams and diagrams are in projective sense invariant under permutations of incoming (outgoing legs).

1 Introduction

Quantum arithmetics [18] is a notion which emerged as a possible resolution of long-lasting challenge of finding mathematical justification for the canonical identification mapping p-adics to reals.

1.1 What quantum p-adics could be?

The basic idea is that p-adic numbers could have quantum counterparts. This idea has developed through several twists and turns and involved moments of almost despair.

1.1.1 The first attempts

The first attempts where based on the replacement of p-adic numbers with quantum p-adics in the hope that the arithmetics could be lifted to quantum level.

1. The earlier work with quantum arithmetics [18] suggests a modification of p-adic numbers by replacing the coefficients a_n p-adic pinary expansions with their quantum counterparts $(a_n)_q$ allowing the coefficients a_n of prime powers to be integers not divisible by p and involving only primes $l < p$ in the prime decomposition (for $l > p$ the quantum counterpart can be negative). $a_n > p$ is allowed for the "interesting but risky" and $a_n < p$ is required for "less-interesting but safe" option.

2. For the "interesting" option the assignment of quantum integer to a given p-adic integer is not unique. A natural looking but not absolutely necessary constraint is that the assignment respects the decomposition of the p-adic integer to powers of prime. With this assumption the construction of quantum integers would reduce to that for primes l. The quantum counterpart of $l > p$ is not unique if the coefficients of powers of p can be larger than p. There exists preferred quantum counterpart obtained by assuming that $a_n < p$. Restricting the consideration to these quantum integers gives just p-adic integers if one regards quantum map $n \to n_q$ and canonical identification as unrelated notions.

3. Quantum p-adic integers for the "interesting option" could be in some sense to p-adic integers what the integers in the extension of number field are for the number field and attempts to identify quantum Galois group for given prime were made. The attempt to define basic arithmetic operations for quantum p-adics led however to difficulties and motivated to assign to the conjecture quantum Galois group wave functions so that the quantum sum and product would be defined for the wave functions assigned for the quantum p-adic integers. This option looked also too complex to be fundamental. Also the question whether this option gives rise to a generalization of number field, remained open, and no natural identification of quantum Galois group was found.

Eventually I was forced to ask whether it would be wiser to be conservative and concentrate on the "less-interesting" option and try to make it more interesting. Could the emergence 1-to-many correspondence between ordinary and quantum p-adics be something totally unrelated to the construction of quantum p-adics? Could it emerge in the quantum map $n \to n_q$ taking into account the effects of finite measurement resolution and meaning symmetry breaking: the different p-adic expansions of n allowing the coefficient a_n of p^n to be integers divisible only by primes $l < p$ but having also values $a_n > p$ would be mapped to different quantum p-adic numbers. If this were the case, quantum p-adics must mean something else than was thought first.

1.1.2 The replacement of numbers with sequences of arithmetic operations and integers with Hilbert spaces

The first attempt to solve the problems related to the definition of $+_q$ and $\times_q$ was inspired by zero energy ontology and led to a replacement of numbers with sequences of arithmetic operations describable by analogs of Feyman diagrams. The comparison with generalized Feynman diagrams allowed to realize how "less-interesting" option could become "interesting": numbers could be replaced with Hilbert spaces and all the conditions would be trivially satisfied!

1. The notion of generalized Feynman diagram suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted. This kind of reduction to tree diagrams is an old proposal that I gave up as too "romantic" [3] but which re-emerged from zero energy ontology where the assumption that also internal lines (wormhole throats) are massless and on shell although the sign of energy can be negative, poses extremely powerful kinematical constraints reducing the number of Feynman diagrams. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands.

2. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to $\otimes$ and $\oplus$ obeying also associativity and distributivity and could quantum arithmetics for Hilbert spaces apply to quantum TGD? If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces and quantum states - and in the vertices the incoming states fuse to a direct sum $\oplus$ or tensor product $\otimes$!

3. One could assign to integer n a multiple covering defined by the state basis of n-dimensional Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests cyclic group Z_n. The non-trivial quantum Galois group would thus emerge also for the "less-interesting" but non-risky option so that the conservative approach might work after all!

4. The Hilbert spaces in question could represent physical states - in p-adic context one could speak about cognitive representations. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of imbedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the imbedding space. Hilbert spaces can be identified as function spaces associated with the discrete point sets of the covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

5. Quantum arithmetics would be arithmetics of Hilbert spaces and of states assigned to them. This arithmetics allows also extension to rationals and algebraic numbers, and even the Hilbert space variants of algebraic complex numbers, quaternions and octonions can be considered. Also quantum adeles can be defined in terms of Hilbert spaces. These generalization are expected to be crucial for the understanding of generalized Feynman diagrams.

1.2 Quantum TGD and Hilbert adeles

Irrespective of whether the isomorphism holds true quantum adeles - if they exist - could provide a very powerful tool also for the formulation of quantum TGD and realize the old intuition that AGG is a symmetry group of quantum TGD [8] .

1. The innocent TGD inspired question posed already earlier is whether the fusion of real and various p-adic physics together could be realized in terms of adeles providing - if not anything else - an ingenious book keeping device allowing to do real physics and all p-adic physics simultaneously by replacing the whole stuff by single letter A! Now however replaced with A_q.

2. The function spaces associated with quantum adeles decompose to tensor products of function spaces associated with the completions of rationals and one can speak about rational entanglement between different number fields. Rational entanglement can be generalized to algebraic entanglement when one replaces rationals with their algebraic extension and primes with corresponding primes. Could it be that this rational/algebraic entanglement is the rational/algebraic suggested to characterize living matter and to which one can assign negative entanglement entropy having interpretation as a measure for genuine information?

3. The basic vision of TGD inspired quantum bio-physics is that life resides in the intersection of real and p-adic worlds in which rational/algebraic entanglement is natural. One can argue that rational and algebraic entanglement are unstable and that it cannot be realized in any system - living or not. The objection is that Negentropy Maximization Principle (NMP [10]) favors the generation of negentropic entanglement and once formed between two material systems described by real numbers is stable. Could it be that the mechanism producing this kind of entanglement is the necessary rational/algebraic entanglement between different number fields - between matter and mind one might say - and that quantum jumps transforming p-adic space-time sheets to real ones generates rational/algebraic entanglement between systems consisting of matter. Intention transforming to action would be the interpretation for this process.

4. The construction of generalized Feynman diagrams leads to a picture in which propagator lines give rise to expressions in various p-adic number fields and vertices naturally to multi-p-adic expressions involving p-adic primes of incoming lines. This picture has also natural generalization to quantum variants of p-adic numbers and the expressions are eventually mapped to real numbers by canonical identification induced by $p \to 1/p$ for quantum rationals appearing in various lines and in vertices of the generalized Feynman diagram. This construct would naturally to a tensor product of state spaces assignable to different p-adic primes and also reals so that M-matrix elements would be naturally in this tensor product. Note that the function space associated with (quantum) adeles is naturally tensor product of functions spaces associated with Cartesian factors of the adele ring with rationals defining the entanglement coefficients. All this of course generalizes by replacing rationals by their algebraic extensions.

2 Earlier attempts to construct quantum arithmetics

Quantum arithmetics [18] provides a possible resolution of a long-lasting challenge of finding a mathematical justification for the canonical identification mapping p-adics to reals playing a key role in TGD - in particular in p-adic mass calculations [12].

In [18] several options for quantum arithmetics were discussed. Common feature of all options is that products of integers are mapped to products of quantum integers achieved by mapping primes l to quantum primes $l_q = (q^l - q^{-l})/(q - q^{-1})$, $q = exp(i\pi/p)$.

In the case of sum one could pose the condition that quantum sums are images of ordinary sums: in this case (option I) one obtains something reducing to ordinary p-adic numbers and $l \to l_q$ accompanies canonical identification $p \to 1/p$ mapping p-adic rationals to reals.

Option II gives up the condition that quantum sum is induced by p-adic sum and assumes that l_q generate act as generators of Kac-Moody type algebra defined by powers p^n such that sum is sum is completely analogous to that for Kac-Moody algebra: $a + b = \sum_n a_n p^n + \sum b_n p^n = \sum_n (a_n + b_n) p^n$.

In this chapter a third and much more general option is discussed. In order to give the needed context, the options discussed in [18] are however briefly discussed first.

2.1 Quantum arithmetics

The starting point idea was that quantum arithmetics maps products of integers to products of quantum integers. It has turned out that this need not be the case for the sum and even in the case of product one can ask whether the assumption is necessary. For Option I sum and product are respected but this option is more or less equivalent with p-adic numbers. For Option II the images of primes generate Kac-Moody type algebra and sums are not mapped to sums and the number of elements of quantum algebra is larger than that of p-adic number field. Also in this case one can consider option giving up the condition that products are mapped to products.

2.1.1 Are products mapped to products?

The first question is whether products are mapped to products.

1. The multiplicative structure of ordinary integers is respected in the map taking ordinary integers to quantum integers:

$$n = kl \rightarrow n_q = k_q l_q \ . \tag{2.1}$$

   This is guaranteed if the map is induced by the map of ordinary primes to quantum primes. This means that one decomposes n to a product of primes l and maps $l \rightarrow l_q$. For primes $l < p$ the map reads as $l \rightarrow l_q = (q^l - \overline{q}^{-l})/(q - \overline{q})$, $q = exp(i\pi/p)$ and gives positive number. For $l > p$ this need not be the case and for primes $l > p$ one expands l as $l = \sum l_m p^n$, $l_m < p$, and expresses l_m as product of primes $l < p$ mapped to l_q each to obtain $l_{mq} \geq 0$. Non-negativity is important in the modelling of Shnoll effect by a deformation of probability distribution $P(n)$ by replacing the argument n by quantum integers and the parameters of the distribution by quantum rationals [1].

2. One could of course consider giving up the condition that products are mapped to products. In this case one would simply expess n as $n = \sum n_k p^k$ and map n_k to n_{qk} by using its prime decompositions. Therefore product would be mapped to product only for integers $n < p$ with product smaller than p.

2.1.2 Are sums mapped to sums?

Second question is about whether quantum map commutes with sum. There are two options.

1. For Option I also the sum of quantum integers is well-defined and induces sum of the quantum rationals. Therefore the sum $+_q$ for quantum integers would reflect the summation of ordinary integers:

$$n = k + l \rightarrow n_q = k_q +_q l_q \ . \tag{2.2}$$

   Option I can be interpreted in terms of ordinary p-adic integers and therefore it will not be discussed in the following.

2. For option II one gives up the condition for the sum. This means that p-adic numbers are replaced with a ring of quantum p-adics generated by the the images l_q of primes $l < m$, where m defines the quantum phase. In other words, one forms all possible products and sums of the these generators and also their negatives. The sum is defined as the complete analog of sum for Kac-Moody algebras: $a + b = \sum a_n m^n + \sum b_n m^n = \sum (a_n + b_n) m^n$ and obviously differs from m-adic sum. The general element of algebra is $x = \sum x_n m^n$, where one has

$$x_n = \sum_{\{n_i\}} N(\{n_i\}) \prod_i x_i^{n_i} \ , \ x_i = p_{i,q}, \ p_i < m \ , \ q = exp(i\pi/m) \ .$$

Here $N(\{n_i\})$ is integer. $m = p$ gives what might be called quantum p-adic numbers. Note that also zeroth order term giving rise to integers as constant term of polynomials is also present. The map would produc integers from zeroth order terms so that skeptic could see the construction too complex.

One has what could be regarded as analog of polynomial algebra with coefficients of polynomials given by integers. Note that the coefficients can be also negative since quantum map combined with canonical identification maps -1 to -1: canonical identification mapping -1 to $(p-1)_q(1+p+p^2...)$ would give only non-negative real numbers. If one wants that also the images under canonical identification form a field (so that $-x$ for given x belongs to the system) one must assume that -1 is mapped to -1. Also the condition that one obtains classical groups requires this. One can form also rationals of this algebra as ratios of this kind of polynomials and a subset of them projects naturally to p-adic rationals.

3. One can project quantum integers for Option II to p-adic numbers by mapping the the products of powers of generators l_q, $l < m$ to products of ordinary p-adic primes $l < m$ in the sums defining the coefficients in the expansion in powers of m to ordinary p-adic integers. This projection defines a structure analogous to a covering space for p-adic numbers. The covering contains infinite number of elements since also the negatives of generators are allowed in the construction. The covering by elements with positive coefficients of m^n is finite.

4. Quantum p-adics form a ring but do they form a field? This seems to be the case since quantum p-adics are very much analogous to a function field for which the argument of function is defined by integer characterizing the powers of p in quantum pinary expansion. One would have the analogy of function field in the set of integers. This means that one can indeed speak of quantum rationals M/N which can be mapped to reals by $I(M/N) = I(M)/I(N)$.

2.1.3 About the choice of the quantum parameter q

Some comments about the quantum parameter q are in order.

1. The basic formula for quantum integers in the case of quantum groups is

$$n_q = \frac{q^n - \overline{q}^n}{q - \overline{q}} \ . \quad (2.3)$$

Here q is *any* complex number. The generalization respective the notion of primeness is obtained by mapping only the primes p to their quantum counterparts and defining quantum integers as products of the quantum primes involved in their prime factorization.

$$\begin{aligned} p_q &= \frac{q^p - \overline{q}^p}{q - \overline{q}} \\ n_q &= \prod_p p_q^{n_p} \ \text{for} \ \ n = \prod_p p^{n_p} \ . \end{aligned} \quad (2.4)$$

2. In the general case quantum phase is complex number with magnitude different from unity:

$$q = exp(\eta)exp(i\pi/m) \ . \quad (2.5)$$

The quantum map is 1-1 for a non-vanishing value of η and the limit $m \to \infty$ gives ordinary integers. It seems that one must include the factor making the modulus of q different from unity if one wants 1-1 correspondence between ordinary and quantum integers guaranteing a unique definition of quantum sum. In the p-adic context with $m = p$ the number $exp(\eta)$ exists as an ordinary p-adic number only for $\eta = np$. One can of course introduce a finite-dimensional extension of p-adic numbers generated by $e^{1/k}$.

3. The root of unity must correspond to an element of algebraic extension of p-adic numbers. Here Fermat's theorem $a^{p-1}\ mod\ p = 1$ poses constraints since $p-1$:th root of unity exists as ordinary p-adic number. Hence $m = p-1$:th root of unity is excluded. Also the modulus of q must exist either as a p-adic number or a number in the extension of p-adic numbers.

4. If q reduces to quantum phase, the $n = 0, 1, -1$ are fixed points of $n \to n_q$ for ordinary integers so that one could say that all these numbers are common tointegers and quantum integers for all values of $q = exp(i\pi/m)$. For p-adic integers $-1 = (p-1)(1+p+p^2+..$ is problematic. Should one use direct formula mapping it to -1 or should one map the expansion to $(p-1)_q(1+p+p^2+....)$? This option looks more plausible.

   (a) For the first option the images under canonical can have both signs and can form a field. For the latter option would obtain only non-negative quantum p-adics for ordinary p-adic numbers. They do not form a field. For algebraic extensions of p-adics by roots of unity one can obtain more general complex numbers as quantum images. For the latter option also the quantum p-adic numbers projecting to a given prime l regarded as p-adic integer form a finite set and correspond to all expansions $l = \sum l_k p^k$ where l_k is product of powers of primes $p_i < p$ but one can have also $l_k > p$.

   (b) Quantum integers containing only the $O(p^0)$ term in the binary expansion for a sub-ring. Corresponding quantum rationals could form a field defining a kind of covering for finite field $G(p, 1)$.

   (c) The image $I(m/n)$ of the pinary expansion of p-adic rational is different from $I(m)/I(n)$. The formula $m/n \to I(m)/I(n)$ is the correct manner to define canonical identification map. In this case the real counterparts of p-adic quantum integers do not form the analog of function fields since the numbers in question are always non-negative.

5. For p-adic rationals the quantum map reads as $m/n \to m_q/n_q$ by definition. But what about p-adic transcendentals such as e^p? There is no manner to decompose these numbers to finite primes and it seems that the only reasonable map is via the mapping of the coefficients x_n in $x = \sum x_n p^n$ to their quantum adic counterparts. It seems that one must expand all quantum transcendentals having as a signature non-periodic pinary expansion to quantum p-adics to achieve uniqueness. Second possibility is to restrict the consideration to rational p-adics. If one gives up the condition that products are mapped to products, one can map $n = n_k p^k$ to $n_q = \sum n_{kq} p^k$. Only the products of p-adic integers $n < p$ smaller than p would be mapped to products.

6. The index characterizing Jones inclusion [5] [6] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

To sum up, one can imagine several options and it is not clear which option is the correct one. Certainly Option I for which the quantum map is only part of canonical identification is the simpler one but for this option canonical identification respects discrete symmetries only approximately. The model for Shnoll effect requires only Option I. The notion of quantum integer as defined for Opion II imbeds p-adic numbers to a much larger structure and therefore much more general than that proposed in the model of Shnoll effect [1] but gives identical predictions when the parameters characterizing the probability distribution $f(n)$ correspond contain only single term in the p-adic power expansion. The mysterious dependence of nuclear decay rates on physics of solar system in the time scale of years reduces to similar dependence for the parameters characterizing $f(n)$. Could this dependence relate directly to the fact that canonical identification maps long length scale physics to short length scales physics. Could even microscopic systems such as atomic nuclei give rise to what might be called "cognitive representations" about the physics in astrophysical length scales?

2.2 Summary: the three options for quantum p-adics

I have proposed two alternative definitions for quantum integers.

1. Option I is that quantum integers are in 1-1 correspondence with ordinary p-adic integers and the correspondence is obtained by the replacement of the coefficients of the pinary expansion with their quantum counterparts. In this case quantum p-adic integers would inherit the sum and product of ordinary p-adic integers. This is the conservative option and certainly works but is equivalent with the replacement of canonical identification with a map replacing coefficients of powers of p with their quantum counterparts. This option has a m-adic generalization corresponding to the expansion of m-adic numbers in powers of integer m with coefficients $a_n < m$. As a special case one has $m = p^N$. The quantum map would contain the interesting physics.

2. Option II based on the identification of quantum p-adics as an analog of Kac-Moody algebra with powers p^n in the same role as the powers z^n for Kac-Moody algebra. The two algebras have identical rules for sum and multiplication, and one does not require the arithmetics to be induced from the p-adic arithmetics (as assumed originally) since this would lead to a loss of associativity in the case of sum. Therefore the quantum counterparts of primes $l \neq p$ generate the algebra. One can also make the limitation $l < p^N$ to the generators. The quantum counterparts of p-adic integers are identified as products of quantum counterparts for the primes dividing them. The counterparts of in the map of integers to quantum integers are $0, 1, -1$ are $, 0, 1, -1$ as is easy to see. The number of quantum integers projecting to same p-adic integer is infinite. For $p = 2$ quantum integers reduce to Z_2 since primes are mapped to ± 1 under quantum map. For $p = 3$ one obtains powers of 2_q. As p increase the structure gets richer. One can define rationals in this algebra as pairs of quantum integers not divisible with each other. At the limit when the quantum phase approaches to unit, quantum integers approach to ordinary ones and ordinary arithmetics results.

3. One can consider also quantum m-adic option with expansion $l = \sum l_k m^k$ in powers of integer m with coefficients decomposable to products of primes $l < m$. This option is consistent with p-adic topology for primes p divisible by m and is suggested by the inclusion of hyper-finite factors [6] characterized by quantum phases $q = exp(i\pi/m)$. Giving up the assumption that coefficients are smaller than m gives what could be called quantum covering of m-adic numbers. For this option all quantum primes l_q are non-vanishing. Phases $q = exp(i\pi/m)$ characterize Jones inclusions of hyper-finite factors of type II_1 assumed to characterize finite measurement resolution.

The definition of quantum p-adics discussed in this chapter replaces integers with Hilbert spaces of same dimension and $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$. Also co-product and co-sum must be introduced and assign to the arithmetics quantum dynamics, which leads to proposal that sequences of arithmetic operations can be interpreted arithmetic Feynman diagrams having direct TGD counterparts. This procedure leads to what might be called quantum mathematics or Hilbert mathematics since the replacement can be made for any structure such as rationals, algebraic numbers, reals, p-adic numbers, even quaternions and octonions. Even set theory has this kind of generalization. The replacement can be made also repeatedly so that one obtains a hierarchy of structures very similar to that obtained in the construction of infinite primes by a procedure analogous to repeated second quantization. One possible interpretation is in terms of a hierarchy of logics of various orders. Needless to say this definition is the really deep one and actually inspired by quantum TGD itself. In this picture the quantum p-adics as they are defined here would relate to the canonical identification map to reals and this map would apply also to Hilbert p-adics.

3 Hilbert p-adics, Hilbert adeles, and TGD

One can imagine also a third generalization of the number concept. One can replace integer n with n-dimensional Hilbert space and sum and product with direct sum and tensor product and introduced their co-operations, the definition of which is non-trivial. This procedure yields also Hilbert space variants of rationals, algebraic numbers, p-adic number fields, and even complex, quaternionic and octonionic algebraics. Also adeles can be replaced with their Hilbert space counterparts. Even more, one can replace the points of Hilbert spaces with Hilbert spaces and repeat this process, which is very similar to the construction of infinite primes having interpretation in terms of repeated second quantization. This process could be the counterpart for construction of n^{th} order logics and one might speak of Hilbert or quantum mathematics. It would also generalize the notion of algebraic holography.

This vision emerged from the connections with generalized Feynman diagrams, braids, and with the hierarchy of Planck constants realized in terms of coverings of the imbedding space. Hilbert space generalization of number concept seems to be extremely well suited for the purposes of TGD. For instance, generalized Feynman diagrams could be identifiable as arithmetic Feynman diagrams describing sequences of arithmetic operations and their co-operations. The definition of co-operations would define quantum dynamics. Physical states would correspond to the Hilbert space states assignable to numbers.

3.1 Could the notion of Hilbert mathematics make sense?

After having worked one month with the iea I found myself in a garden of branching paths and realized that something must be wrong. Is the idea about quantum p-adics a disgusting fix idee or is it something deeper?

The successful manner to make progress in this this kind of situation has been the combination of existing firmly established ideas with the newcomer. Could the attempt to relate quantum p-adics to generalized Feynman graphs, infinite primes, and hierarchy of Planck constants help?

Second good strategy is maximal simplification. In the recent case this encourages sticking to the most conservative option for which quantum p-adics are obtained from ordinary p-adics by mapping the coefficients of powers of p to quantum integers. This option has also a variant for which one has expansion in powers of p^N defining pinary cutoff. At the level of p-adic numbers different values of N make no difference but at the level of finite measurement resolution situation is different. Also quantum m-adicity would have natural interpretation in terms of measurement resolution rather than fundamental algebra.

3.1.1 Replacing integers with Hilbert spaces

Consider now the argument leading to the interpretation of p-adic integers as Hilbert space dimensions and the formulation of quantum p-adics as p-adic Hilbert spaces whose state basis defines a multiple covering of integer defining the dimension of the Hilbert space.

1. The notion of generalized Feynman diagram and zero energy ontology suggest suggests that of arithmetic Feynman diagram describing a sequence of arithmetic operations performed for a set of incoming integers and producing a set of outgoing integers. This approach indeed led to the discovery that integers could be replaced by Hilbert spaces.

2. The basic 3-vertices of the arithmetic Feynman diagram would be $\times_q$ and $+_q$ and their co-operations. The moves of Feynman diagrams leaving the amplitude invariant would code for associativity and distributivity. All loops could be eliminated by these moves and diagram transformed to a canonical tree diagram in which incoming *resp.* outgoing lines could be permuted.

3. Incoming lines would correspond to integers decomposing into products of primes and an attractive interpretation is that these primes correspond to braid strands for generalized Feynman diagrams.

4. The basic vertices in quantum TGD correspond to the stringy 3-vertex and 3-vertex for Feynman graphs. They correspond at Hilbert space level naturally to tensor product and direct sum. Could $\times_q$ and $+_q$ correspond to tensor product and direct sum obeying also associativity and distributivity?! If so, the integers characterizing the lines of arithmetic Feynman diagrams would correspond to Hilbert space dimensions - or rather, Hilbert spaces - and in vertices the incoming states fuse to direct sum of tensor product!

5. What this would mean is that one could assign to each p-adic integer a multiple covering defined by the state basis of the corresponding Hilbert space. This is just what one wants! The quantum Galois group would be subgroup of the permutation group permuting the elements of this basis. The analogy with covering spaces suggests just cyclic group. The non-trivial quantum Galois

group would emerge also for the "less-interesting" but non-risky option so that the conservative approach might work!

6. The Hilbert spaces in question could represent physical states - maybe cognitively in the p-adic context. It also turns out possible to relate these Hilbert spaces directly to the singular coverings of imbedding space associated with the hierarchy of Planck constants assigned with dark matter in TGD Universe. This gives a concrete content for the quantum Galois group as cyclic permutations of the sheets of the covering of the imbedding space and Hilbert spaces can be identified as function spaces associated with the discrete point sets of covering projected to the same point. Also a beautiful connection with infinite primes defining algebraic extensions of rationals emerges and infinite primes would characterize physical states by characterizing their dimensions of Hilbert spaces assignable to the incoming and outgoing lines.

This approach works for the ordinary p-adic integers. There is no need to allow coefficients $a_n > p$ ("interesting" option) in the expansion $\sum a_n p^n$ of p-adic numbers but still consisting of primes $l < p$. "Interesting" option would emerge as one takes finite measurement resolution into account by mapping the Hilbert spaces defining coefficients of Hilbert space pinary expansion with their quantum counterparts. More precisely.

1. At Hilbert space level pinary expansion of p-adic Hilbert space becomes direct sum $\oplus_n a_n \otimes p^n$. $a_n = \otimes_i p_i$, $p_i < p$, denotes tensor product of prime Hilbert spaces for which I use the same label as for p-adic numbers. p^n denotes Hilbert space with dimension p^n. In real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. Quantum p-adic numbers would be obtained by mapping the Hilbert space valued coefficients a_n of the to their quantum counterparts $(a_n)_q$, which are conjectured to allow precise definition in terms of inclusions of hyper-finite factors with Jones inclusions associated with the quantum counterpart of 2-D Hilbert space. The quantum map would reduce to the mapping of the tensor factors p_1 of a_n to $(p_1)_q$. Same would apply to quantum states. The map would be defined as $\oplus a_n \otimes p^n \rightarrow \oplus (a_n)_q \otimes p^n$, $(a_n)_q = \otimes_{p_1}(p_1)_q$. The map $p_1 \rightarrow (p_1)_q$ would take into account finite measurement resolution.

3. "Interesting" option would be obtained as follows. It is possible to express given p-adic number in many manners if one only requires that the coefficients a_n in the direct sum are tensor products of prime Hilbert spaces with dimension $p_1 < p$ but does not assume $a_n < p$. For instance, for $p = 3$ and $n = 8$ one has $8 = 2 \oplus 2\otimes$ or $8 = 2 \otimes 2 \otimes 2$. These representations are p-adically equivalent. Quantum map however spoils this equivalence. $2 \oplus 2 \otimes 3 \rightarrow 2_q \oplus 2_q \otimes 3$ and $8 = 2 \otimes 2 \otimes 2 \rightarrow 2_q \otimes 2_q \otimes 2_q$ are not same quantum Hilbert spaces. The "interesting" option would thus emerge as one takes into account the finite measurement resolution.

4. One could say that the quantum Hilbert spaces associated with a given p-adic Hilbert space form a covering space like structure. Quantum Galois group identified as a subgroup of permutations of these quantum Hilbert spaces need not make sense however.

After this lengthy motivating introduction I want to describe some details of the arithmetics of p-adic Hilbert spaces. This arithmetics is formally identical with the ordinary integer arithmetics. What is however interesting is that one can generalize it so that one obtains something that one could call Hilbert spaces of dimension which is negative, rational, algebraic, or even complex, and even quaternionic or octonionic. It might be necessary to have these generalizations if one wants full generality.

1. Consider first what might be called p-adic Hilbert spaces. For brevity I will denote Hilbert spaces in the same manner as p-adic numbers: reader can replace "n" with "H_n" if this looks more appropriate. p-Adic Hilbert spaces have direct sum expansions of form

$$n = \oplus_k a_k \otimes p^k \ .$$

All integers appearing in the formula can be also interpreted as Hilbert space dimensions. In the real context it is very natural to decompose real Hilbert spaces to tensor products of prime Hilbert spaces.

2. How to define Hilbert spaces with negative dimension? In p-Adic context this is not a problem. Hilbert space with dimension -1 is given by Hilbert spaces with dimension $(p-1)/(1-p) = (p-1)(1+p+p^2+...)$ converging p-adically and given by

$$-1 = \oplus_k (p-1) \otimes p^k \ .$$

In real context one must consider pairs of Hilbert spaces (m,n) and define equivalence $(m,n) = (m+k, n+k)$. In canonical representation Hilbert space with positive dimension m corresponds to $(m,0)$ and Hilbert spaces with negative dimension $-m$ to $(0,m)$. This procedure is familiar from the theory of vector bundles where one subtracts vector bundles and defines their negatives.

3. In p-adic context one can also define p-adic Hilbert spaces with rational dimension if the p-adic norm of the rational (m/n) is smaller than 1. This is achieved simply by the expansion

$$\frac{m}{n} = \oplus_k a_k \otimes p^k \ .$$

In real context tone can define Hilbert spaces with rational valued dimension just as one defines rational numbers - that is as pairs of Hilbert spaces (m,n) with equivalence $(m,n) \equiv (km, kn)$.

4. One can even define Hilbert spaces with dimensions in algebraic extensions of rationals.

   (a) Consider first the real case and the extension defined by Gaussian integers for which integers are of form $m+in \equiv (m,n)$. What is needed is just the product rule: $(m,n) \otimes (r,s) = (m \otimes r - \oplus(-n \otimes s), m \otimes s \oplus r \otimes n)$. This expression is completely well-defined in the p-adic context and also in real context if one accepts the proposed defined of integer Hilbert spaces as pairs of ordinary Hilbert spaces. For $Q(\sqrt{5})$ one would have $(m,n) \times (r,s) = (m \otimes r \oplus 5 \otimes n \otimes s, m \otimes s \oplus r \otimes n)$. In n-dimensional case one just replaces Hilbert spaces with n-multiple of ordinary Hilbert spaces and uses the multiplication rules.

   (b) In p-adic context similar approach works when the algebraic extension requires also extension of p-adic numbers. In p-adic context however many algebraic numbers can exist as ordinary p-adic numbers. For instance, for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as well as its Hilbert space counterpart. For quadratic extensions of $p > 2$-adic numbers the 4-D extension involving the addition of two square roots all square roots except that of p exist -adically.

3.1.2 Quantum Hilbert spaces and generalization to extensions of rationals

The map of p-adic integers to their quantum counterparts generalizes so that it applies to Hilbert spaces. This means that prime Hilbert spaces are mapped to the quantum counterparts. What this means is not quite obvious. Quantum groups appearing in the context of Jones inclusions lead to the emergence of quantum spinors that is quantum counterparts of 2-D Hilbert spaces. This suggest that more general inclusions lead to prime-dimensional quantum Hilbert spaces. The idea is simple: quantum matrix algebra M/N with quantum dimension $(2_q)^2$ is defined as a coset space of hyper-finite factor M and included factor $N \subset M$. This quantum matrix algebra acts in quantum spinor space of dimension 2_q. The generalization would introduce p_q-dimensional quantum Hilbert spaces.

A good test for the proposal is whether it generalizes naturally to algebraic extensions of rationals.

1. For algebraic extensions some ordinary primes split into products of primes associated with the extension. The problem is that for these algebraic primes the factors $exp(i\pi/P)$ fail to be algebraic numbers and finite roots of unity and its is not at all clear whether the naive generalization of the notion of quantum p-adic makes sense. This suggests that only the ordinary primes which do not split into products of primes of extension remain and one can define quantum p-adics only for these whereas the other primes correspond to ordinary algebraic extension of p-adic numbers. This would make algebraic extension of rationals the coefficient group of adele consisting of p-adic numbes fields associated with non-split primes only. Note that rationals or their extension would naturally appear as tensor factor of adeles meaning that their action can be thought to affect any of the factors of the adele.

2. For split primes the p-adic Hilbert spaces must be defined for their algebraic prime factors. The proposed procedure of defining Hilbert space counterparts for algebraic extensions of rationals provides a recipe for how to achieve this. These Hilbert spaces the quantum map would be trivial.

3. Hilbert space counterpart for the albebraic extension of rationals and of p-adics makes also sense. The Hilbert space assigned with integer which splits into primes of extension splits also to a tensor product of prime Hilbert spaces assignable with the extension. The splitting of integers and primes is highly analogous to the decomposition of hadron to quarks and gluons. This decomposition is not seen at the level of rationals reprsenting observed.

3.1.3 What about Hilbert spaces with real number valued dimension?

What Hilbert space variant of a real number could mean? What Hilbert space with dimension equal to arbitrary real number could mean? One can imagine two approaches.

1. The first approach is based on the map of Hilbert p-adics to real p-adics by a map used to map p-adic numbers to reals. The formula would be $\oplus_n a_n \otimes p^n \to \oplus (a_n)_q \otimes p^{-n}$. $(a_n)_q = \otimes_l l_q^{e_l}$, were l_q is quantum Hilbert space of prime dimension. Also the Hilbert space p^{-n} would be well-defined as a Hilbert rational defined as a pair of Hilbert spaces.

   For hyper-finite factors of type II_1 Hilbert spaces with continuous dimension emerge naturally. The reason is that the dimension of the Hilbert space is defined as quantum trace of identity operator characterized by quantum phase this dimension is finite and continuous. This allows a spectrum of sub-Hilbert spaces with continuously varying real dimension. The appearance of quantum Hilbert spaces in the canonical identification map conforms with this and even for dimension $0 < n < p$ gives rise to quantum Hilbert space with algebraic quantum dimension given as $n = \prod l_q^{e^l}$ for $n = \prod_l l^{e_l}$.

2. Second approach relies on the mimicry of the completion of ordinary rationals to real numbers. One can define Hilbert space analogs of rationals and algebraics by defining positive and negative rationals as pairs of Hilbert spaces with equivalence relation $(m,n) \equiv (m \oplus r, n \oplus r)$. Taking pairs of these pairs with equivalence relation $(M,N) \equiv (K \otimes M, K \otimes N)$ one obtains Hilbert spaces corresponding to rational numbers. Algebraic extensions are obtained similarly. By taking limits just in the same manner as for real numbers one would obtain Hilbert reals with transcendental dimensions. For instance, e could be defined as the limit of tensor power $(1 \oplus 1/n)^n$, $n \to \infty$.

Again one must remember that the co-vertices define the hard part of the problem and their definition means postulate of quantum dynamics. This would be the genuinely new element and transform mathematics to quantum physics. Every sequences of algebraic operations having a realization as Feynman diagram involving arithmetic operations as positive energy part of Feynman diagrams and co-operations as the negative energy part of diagram connected by single line.

It should not go un-noticed that the direct sum and tensor product decompositions of possibly infinite-dimensional Hilbert spaces are very essential for the interpretation. For infinite-dimensional Hilbert spaces these decompositions would be regarded as equivalent for an abstract definition of Hilbert space. In physical applications tensor product and direct sum representations have also very concrete physical content.

3.1.4 Hilbert calculus?

What this approach suggests is a generalization of calculus in both real and and p-adic context. The first thing to do is to define Hilbert functions as Hilbert space valued functions as $x \to f(x)$. This could be done formally by assigning to Hilbert space associated with point x Hilbert space associated with the point $f(x)$ for all values of x. Function could have representation as Taylor series or Laurent series with sum replaced with direct sum and products with tensor products. The correspondence $x \to f(x)$ would have as a counterpart the analog of Feynman diagram describing the Taylor series with final line defining the value $f(x)$. Also derivatives and integrals would be at least formally defined. This would requite separate diagram for every point x. One can consider also the possibility of more abstract definition of $f(x)$. For instance the set of coefficients $\{f_n\}$ in the Taylor series of f would defined a collection of Hilbert spaces.

One should be able to define also co-functions in terms of co-vertices. The value of co-function at point y would give all the values of x for which one has $f(x) = y$. Co-function would correspond to a quantum superposition of values of inverse function and to time reversed zero energy states. The breaking of time reversal would be inherent in the very definition of function as an arrow from one Hilbert set to another Hilbert set and typically the functions involved would be many-valued form beginning. Perhaps it would be better to speak from the beginning about relations between two sets rather than functions. The physical realization of Hilbert calculus would be obtained by assigning to incoming arguments represented as Hilbert space quantum states.

3.1.5 Quantum mathematics?

Could one transform entire mathematics to quantum mathematics - or would it be better to say Hilbert mathematics? Reader can decide. Consider first Hilbert set theory. The idea wold be to replace numbers with Hilbert spaces. This would give Hilbert structure. By replacing Hilbert spaces with their quantum counterparts characterized by quantum dimensions n_q one would obtain which might be called quantum Hilbert structure.

1. At the level of set theory this would mean replacement of sets with Hilbert sets. A set with n elements would correspond intuitively to n-dimensional Hilbert space. Therefore tHilbert sets would provide much more specific realization of set theory than abstract set theory in which the elements of set can be anything. For n-dimensional Hilbert space however the ordering of the elements of basis induces automatically the ordering of the elements of the set. Does the process of counting the elements of set corresponds to this ordering. Direct sum would be the counterpart of set theoretic union. One could construct natural numbers inductively as direct sums $(n+1) = n \oplus 1$. To be subset would correspond to sub-Hilbert space property. Intersection of two Hilbert sets would correspond to the direct sum of common direct summands. Also set difference and symmetric difference could be defined.

2. The set theoretic realization of Boolean logic would have Hilbert variant. This would mean that logical statements could be formulated using Hilbert variants of basic logical functions.

3. Cartesian product of sets would correspond to a tensor product of Hilbert spaces. This would bring in the notion of prime since Hilbert integers would have decomposition into tensor products of Hilbert primes. Note that here one can consider the symmetrization of tensor product modulo phase factor and this could give rise to bosonic and fermionic statistics and perhaps also to anyonic statistics when the situation is 2-dimensional as it indeed is for partonic 2-surfaces.

4. What about sets of sets?

   (a) The elements of n-dimensional Hilbert space consist of numbers in some number field. By replacing these numbers with corresponding Hilbert spaces one would obtain Hilbert space of Hilbert spaces as a counterpart for sets of sets. One would have Hilbert space whose points are Hilbert spaces: Hilbert-Hilbert space!. This process could be continued indefinitely and would give rise to a hierarchy formed by Hilbert^n-spaces. This would be obviously something new and mean self-referential property. For Hilbert^n-spaces one would the points at n:th level of hierarchy with points of the number field involved and obtain a concrete realization. The construction of infinite primes involves formations of sets of rationals and sets of these sets, etc.... and would have also interpretation as formation of a hierarchy of Hilbert sets of sets of.....

   (b) Power set as set of subsets of set would be obtained from direct sum of Hilbert spaces, by replacing the points of each Hilbert space with corresponding Hilbert spaces.

   (c) One could define the analog of set theoretic intersection also for tensor products as the set of common prime Hilbert factors for two Hilbert sets. For ordinary integers defined as sets the intersection in this sense would correspond to the common prime factors. In Cartesian product the intersection would correspond to common Cartesian factors.

5. The completely new and non-trivial element bringing in the quantum dynamics is brought in by co-operations for union and intersection. The solution to the equation $f(x) = y$ could be represented as a number theoretic Feynman diagram in zero energy ontology. Positive energy part would correspond to y and diagram beginning from y would represent co-function of $f(x)$ identifiable as its inverse. Negative energy state would represent a quantum superposition of the values of x representing the solutions.

6. One can ask whether a Feynman diagrammatic representation for the statements like $\exists x \in A$ such that $f(x) = g(x)$ and $\forall x \in A f(x) = g(x)$ exists. One should be able to construct quantum state which is superposition of solutions to the condition $f(x) = g(x)$. If this state is non-vanishing the solution exists.

   This kind of statements are statements of first order logic involving existential quantifiers whereas the statements of predicate logic would correspond simply to a calculation of a value of function at given point. The hierarchy of Hilbert^n spaces brings in mind strongly the hierarchy of infinite primes assigned already earlier to a hierarchy of logics. Could the statements of n:th order logic require the use of Hilbert^n- spaces. The replacement of numbers with Hilbert spaces could correspond to formation of statements of first order logic. The individual quantum states satisfying the statement would represent the statements of predicate logic.

   The construction of infinite primes can be regarded as repeated second quantization in which the many particle states of the previous level become single particle states of the new level. Maybe also the hierarchy of Hilbert^n-spaces could be seen in terms of a hierarchy of second quantizations.

   Infinite primes lead to the notion of algebraic holography meaning that real point has infinitely rich number theoretical anatomy due to the existence of real units expressible as ratios of infinite integers reducing to real unit in real topology. The possibility to replace the points of space-time with Hilbert spaces and to continue this process indefinitely would realize the same idea.

3.1.6 Number theoretic Feynman diagrams

Could one imagine a number theoretical quantum dynamics in which integers are replaced with sequences of arithmetic operations? If numbers are replaced with Hilbert spaces and if one can assigns to each number a state of the Hilbert space accompanying it, this seems to be possible.

1. All algebraic functions would be replaced with their algebraic expressions, which would be interpreted as analogs of zero energy states in which incoming arguments would represent positive energy part and the result of operation outgoing state. This would also unify algebra and co-algebra thinking and the information about the values of the arguments of the function would not be forgotten in the operations.

2. The natural constraints on the dynamics would be trivial. In $+_q$ vertex a direct sum of incoming states and in $\times_q$ gives rise to tensor product. This also at the level of Hilbert spaces involved. The associativity and commutativity of direct sum and tensor product guarantee automatically the these properties for the vertices. The associativity and commutativity conditions are analogous to associativity conditions for 3-point functions of conformal field theories. Distributivity condition is something new. Co-product and co-sum obey completely analogous constraints as product and sum.

3. For product the total numbers of prime factors is conserved for each prime appearing in the product meaning that the total momenta $n_i log(p_i)$ are conserved separately for each prime in the process involving only products. This kind of conservation law is natural also for infinite primes and one can indeed map the simplest infinite primes at the lowest level analogous to free Fock states of bosons and fermions to ordinary rationals so that the addition of Galois degrees of freedom tentatively identified as cyclic permutations of the state basis for Hilbert space associated with given prime would give for a particle labelled by prime p additional internal degrees of freedom. In fact, one can illustrate infinite prime as in terms of two braids corresponding to the numerator and denominator of corresponding rational and the primes appearing in rationals take the role of braid strands. For $\times_q$ the conservation of quantum numbers would correspond

to conservation of representations. This guarantees commutativity and associativity of product. One can also allow co-product and co-sum and they obey completely analogous constraints as product and sum and they have counterparts at the level of Hilbert spaces two studied in the theory of quantum groups.

One can represent algebraic operations using the analogs of Feynman diagrams and there is an intriguing analogy with generalized Feynman diagrams which forces to ask whether the generalized Feynman diagrams of quantum TGD could be interpreted in terms of quantum counterparts algebraic equations transformed if one extends the number field to quaternions and their possibly existing p-adic counterparts.

1. Multiplicative and additive inverses - in the case that they exist - can be seen as kind of conjugation operations analogous to C and P which commute with each other. Their product $n \rightarrow -1/n$ could be seen as the analog of T if $CPT = 1$ is taken as identity. Co-product and co-sum would would be obtained from product and sum by CP or T.

2. One can represent the integer $X = X(\{n_k\})$ resulting from a sequence of algebraic operations $+_q$ and $\times_q$ performed for integers n_k appearing as inputs of a Feynman diagram having the value of X as outgoing line. $n_{+,k}$ represent incoming external lines and intermediate products of algebraic operations appear as internal "off-mass-shell" lines. $+_q$ and $\times_q$ represent the basic vertices. This gives only tree diagrams with single outgoing line representing the (quantum value) of X.

   Associativity and commutativity for $+_q$ resp. $\times_q$ would mean that the lines of diagram with 3 incoming particles and two vertices can be modified by permuting the incoming lines in all possible manners. Distributivity $a \times_q (b +_q c) = a \times_q b +_q a \times_q c$ does not correspond anything familiar from conformal field theories since the line representing a appears twice on the right hand side of the identity and there are 3 vertices whereas left hand side involves 2 vertices. In TGD framework the interpretation of the analogs of stringy decay vertices in terms of propagation along two different paths allows however to interpret these vertices as counterparts of $+_q$ whereas the TGD counterparts of vertices of Feynman diagrams would correspond to $\times_q$. $_+q$ would correspond at state space level to direct sum and $\times_q$ to tensor product.

3. The lines of Feynman diagrams are naturally replaced with braids - just as in quantum TGD. The decomposition of the incoming quantum rational $q = m/n$ to primes defines a braid with two colors of braid strands corresponding to the primes appearing in m and n so that a close connection with braid diagrams emerges. This of course raises the question whether one could allow non-trivial braiding operation for two braid strands represented by primes. Non-triviality would mean that $p_1p_2 = p_2p_1$ would not hold true only in projective sense so that the exchange would induce a phase factor. This would suggest that the commutativity of the basic operations - or at least multiplication - might hold true only apart from quantum phase factor. This would not be too surprising since quantum phases are the essence of what it is to be quantum integer.

4. The diagrammatical counterparts of co-operations are obtained by time reversal transforming incoming to outgoing lines and vice versa. If one adds co-products and sums to the algebraic operations producing X one obtains diagrams with loops. If ordinary algebraic rules generalizes the diagrams with loops must be transformable to diagrams without them by algebraic "moves". The simplification of arithmetic formulas that we learn in elementary school would correspond to a sequence of "moves" leading to a tree diagram with single internal line at the middle and representing $X = Y$. One can form also diagrams of form $X = Y = Z = ...$ just as in zero energy ontology.

5. In zero energy ontology a convenient manner to represent a identity $X = Y$ - call it a "quantum correlate for mathematical thought" - involving only sums and products and therefore no loops is as a tree diagram involving only two kinds of 3-vertices, namely $+_q$ and $\times_q$ and their co-algebra vertices representing time reversed processes. In zero energy ontology this kind of representation would correspond to either the condition $X/Y = 1$ or as $X - Y = 0$. In both cases one can say that the total quantum numbers would be conserved - that is net quantum numbers assignable to prime factors of X vanish for zero energy state. The diagram involves always single integral

line representing the identical values of X and Y. Line representing X would be preceded by a tree diagram involving only product and sum vertices and Y would involve only co-product and co-sum. For ordinary arithmetics every algebraic operation is representable in this kind of diagram, which suggests that infinite number of different diagrams involving loops are equivalent to this diagram with single internal line.

6. The resulting braid Feynman diagrammatics would obey extremely powerful rules due to the possibility of the "moves". All possible independent equations $X = Y$ would define the basis of zero energy states. In quantum TGD the breaking of time reversal invariance is unavoidable and means that only the positive or negative energy parts of the diagram can have well defined quantum numbers. The direct translation would be that the zero energy states correspond to sums over all diagrams for which either positive/negative energy part corresponds to given rationals and the negative/positive energy part of the state is superposition of states consisting of rationals. This would mean non-trivial U-matrix dictated by the coefficients of the superpositions and genuine arithmetic quantum dynamics.

3.2 Hilbert p-adics, hierarchy of Planck constants, and finite measurement resolution

The hierarchy of Planck constants assigns to the N-fold coverings of the imbedding space points N-dimensional Hilbert spaces. The natural identification of these Hilbert spaces would be as Hilbert spaces assignable to space-time points or with points of partonic 2-surfaces. There is however an objection against this identification.

1. The dimension of the local covering of imbedding space for the hierarchy of Planck constants is constant for a given region of the space-time surface. The dimensions of the Hilbert space assignable to the coordinate values of a given point of the imbedding space are defined by the points themselves. The values of the 8 coordinates define the algebraic Hilbert space dimensions for the factors of an 8-fold Cartesian product, which can be integer, rational, algebraic numbers or even transcendentals and therefore they vary as one moves along space-time surface.

2. This dimension can correspond to the locally constant dimension for the hierarchy of Planck constants only if one brings in finite measurement resolution as a pinary cutoff to the pinary expansion of the coordinate so that one obtains ordinary integer-dimensional Hilbert space. Space-time surface decomposes into regions for which the points have same pinary digits up to p^N in the p-adic case and down to p^{-N} in the real context. The points for which the cutoff is equal to the point itself would naturally define the ends of braid strands at partonic 2-surfaces at the boundaries of CD:s.

3. At the level of quantum states pinary cutoff means that quantum states have vanishing projections to the direct summands of the Hilbert spaces assigned with pinary digits p^n, $n > N$. For this interpretation the hierarchy of Planck constants would realize physically pinary digit representations for number with pinary cutoff and would relate to the physics of cognition.

One of the basic challenges of quantum TGD is to find an elegant realization for the notion of finite measurement resolution. The notion of resolution involves observer in an essential manner and this suggests that cognition is involved. If p-adic physics is indeed physics of cognition, the natural guess is that p-adic physics should provide the primary realization of this notion.

The simplest realization of finite measurement resolution would be just what one would expect it to be except that this realization is most natural in the p-adic context. One can however define this notion also in real context by using canonical identification to map p-adic geometric objets to real ones.

3.2.1 Does discretization define an analog of homology theory?

Discretization in dimension D in terms of pinary cutoff means division of the manifold to cube-like objects. What suggests itself is homology theory defined by the measurement resolution and by the fluxes assigned to the induced Kähler form.

1. One can introduce the decomposition of n-D sub-manifold of the imbedding space to n-cubes by $n-1$-planes for which one of the coordinates equals to its pinary cutoff. The construction works in both real and p-adic context. The hyperplanes in turn can be decomposed to $n-1$-cubes by $n-2$-planes assuming that an additional coordinate equals to its pinary cutoff. One can continue this decomposition until one obtains only points as those points for which all coordinates are their own pinary cutoffs. In the case of partonic 2-surfaces these points define in a natural manner the ends of braid strands. Braid strands themselves could correspond to the curves for which two coordinates of a light-like 3-surface are their own pinary cutoffs.

2. The analogy of homology theory defined by the decomposition of the space-time surface to cells of various dimensions is suggestive. In the p-adic context the identification of the boundaries of the regions corresponding to given pinary digits is not possible in purely topological sense since p-adic numbers do not allow well-ordering. One could however identify the boundaries sub-manifolds for which some number of coordinates are equal to their pinary cutoffs or as inverse images of real boundaries. This might allow to formulate homology theory to the p-adic context.

3. The construction is especially interesting for the partonic 2-surfaces. There is hierarchy in the sense that a square like region with given first values of pinary digits decompose to p square like regions labelled by the value $0, ..., p-1$ of the next pinary digit. The lines defining the boundaries of the 2-D square like regions with fixed pinary digits in a given resolution correspond to the situation in which either coordinate equals to its pinary cutoff. These lines define naturally edges of a graph having as its nodes the points for which pinary cutoff for both coordinates equals to the actual point.

4. I have proposed earlier [4] what I have called symplectic QFT involving a triangulation of the partonic 2-surface. The fluxes of the induced Kähler form over the triangles of the triangulation and the areas of these triangles define symplectic invariants, which are zero modes in the sense that they do not contribute to the line element of WCW although the WCW metric depends on these zero modes as parameters. The physical interpretation is as non-quantum fluctuating classical variables. The triangulation generalizes in an obvious manner to quadrangulation defined by the pinary digits. This quadrangulation is fixed once internal coordinates and measurement accuracy are fixed. If one can identify physically preferred coordinates - say by requiring that coordinates transform in simple manner under isometries - the quadrangulation is highly unique.

5. For 3-surfaces one obtains a decomposition to cube like regions bounded by regions consisting of square like regions and Kähler magnetic fluxes over the squares define symplectic invariants. Also Kähler Chern-Simons invariant for the 3-cube defines an interesting almost symplectic invariant. 4-surface decomposes in a similar manner to 4-cube like regions and now instanton density for the 4-cube reducing to Chern-Simons term at the boundaries of the 4-cube defines symplectic invariant. For 4-surfaces symplectic invariants reduce to Chern-Simons terms over 3-cubes so that in this sense one would have holography. The resulting structure brings in mind lattice gauge theory and effective 2-dimensionality suggests that partonic 2-surfaces are enough.

The simplest realization of this homology theory in p-adic context could be induced by canonical identification from real homology. The homology of p-adic object would the homology of its canonical image.

1. Ordering of the points is essential in homology theory. In p-adic context canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ map to reals induces this ordering and also boundary operation for p-adic homology can be induced. The points of p-adic space would be represented by n-tuples of sequences of pinary digits for n coordinates. p-Adic numbers decompose to disconnected sets characterized by the norm p^{-n} of points in given set. Canonical identification allows to glue these sets together by inducing real topology. The points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ having p-adic norms p^{-n} and p^{-n-1} are mapped to the same real point p^{-n} under canonical identification and therefore the points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ can be said to define the endpoints of a continuous interval in the induced topology although they have different p-adic norms. Canonical identification induces real homology to the p-adic realm. This suggests that one should include canonical identification to the boundary operation so that boundary operation would be map from p-adicity to reality.

2. Interior points of p-adic simplices would be p-adic points not equal to their pinary cutoffs defined by the dropping of the pinary digits corresponding p^n, $n > N$. At the boundaries of simplices at least one coordinate would have vanishing pinary digits for p^n, $n > N$. The analogs of $n-1$ simplices would be the p-adic points sets for which one of the coordinates would have vanishing pinary digits for p^n, $n > N$. $n-k$-simplices would correspond to points sets for which k coordinates satisfy this condition. The formal sums and differences of these sets are assumed to make sense and there is natural grading.

3. Could one identify the end points of braid strands in some natural manner in this cohomology? Points with $n \leq N$ pinary digits are closed elements of the cohomology and homologically equivalent with each other if the canonical image of the p-adic geometric object is connected so that there is no manner to identify the ends of braid strands as some special points unless the zeroth homology is non-trivial. In [19] it was proposed that strand ends correspond to singular points for a covering of sphere or more general Riemann surface. At the singular point the branches of the covering would co-incide.

   The obvious guess is that the singular points are associated with the covering characterized by the value of Planck constant. As a matter fact, the original assumption was that *all* points of the partonic 2-surface are singular in this sense. It would be however enough to make this assumption for the ends of braid strands only. The orbits of braid strands and string world sheet having braid strands as its boundaries would be the singular loci of the covering.

3.2.2 Does the notion of manifold in finite measurement resolution make sense?

A modification of the notion of manifold taking into account finite measurement resolution might be useful for the purposes of TGD.

1. The chart pages of the manifold would be characterized by a finite measurement resolution and effectively reduce to discrete point sets. Discretization using a finite pinary cutoff would be the basic notion. Notions like topology, differential structure, complex structure, and metric should be defined only modulo finite measurement resolution. The precise realization of this notion is not quite obvious.

2. Should one assume metric and introduce geodesic coordinates as preferred local coordinates in order to achieve general coordinate invariance? Pinary cutoff would be posed for the geodesic coordinates. Or could one use a subset of geodesic coordinates for $\delta CD \times CP_2$ as preferred coordinates for partonic 2-surfaces? Should one require that isometries leave distances invariant only in the resolution used?

3. A rather natural approach to the notion of manifold is suggested by the p-adic variants of symplectic spaces based on the discretization of angle variables by phases in an algebraic extension of p-adic numbers containing n^{th} root of unity and its powers. One can also assign p-adic continuum to each root of unity [7]. This approach is natural for compact symmetric Kähler manifolds such as S^2 and CP_2. For instance, CP_2 allows a coordinatization in terms of two pairs (P^k, Q^k) of Darboux coordinates or using two pairs $(\xi^k, \overline{\xi}^k)$, $k = 1, 2$, of complex coordinates. The magnitudes of complex coordinates would be treated in the manner already described and their phases would be described as roots of unity. In the natural quadrangulation defined by the pinary cutoff for $|\xi^k|$ and by roots of unity assigned with their phases, Kähler fluxes would be well-defined within measurement resolution. For light-cone boundary metrically equivalent with S^2 similar coordinatization using complex coordinates $(z, \overline{z})$ is possible. Light-like radial coordinate r would appear only as a parameter in the induced metric and pinary cutoff would apply to it.

3.2.3 Hierachy of finite measurement resolutions and hierarchy of p-adic normal Lie groups

The formulation of quantum TGD is almost completely in terms of various symmetry group and it would be highly desirable to formulate the notion of finite measurement resolution in terms of symmetries.

1. In p-adic context any Lie-algebra gG with p-adic integers as coefficients has a natural grading based on the p-adic norm of the coefficient just like p-adic numbers have grading in terms of their norm. The sub-algebra g_N with the norm of coefficients not larger than p^{-N} is an ideal of the algebra since one has $[g_M, g_N] \subset g_{M+N}$: this has of course direct counterpart at the level of p-adic integers. g_N is a normal sub-algebra in the sense that one has $[g, g_N] \subset g_N$. The standard expansion of the adjoint action $g g_N g^{-1}$ in terms of exponentials and commutators gives that the p-adic Lie group $G_N = exp(tpg_N)$, where t is p-adic integer, is a normal subgroup of $G = exp(tpg)$. If indeed so then also G/G_N is group, and could perhaps be interpreted as a Lie group of symmetries in finite measurement resolution. G_N in turn would represent the degrees of freedom not visible in the measurement resolution used and would have the role of a gauge group.

2. The notion of finite measurement resolution would have rather elegant and universal representation in terms of various symmetries such as isometries of imbedding space, Kac-Moody symmetries assignable to light-like wormhole throats, symplectic symmetries of $\delta CD \times CP_2$, the non-local Yangian symmetry, and also general coordinate transformations. This representation would have a counterpart in real context via canonical identification I in the sense that $A \to B$ for p-adic geometric objects would correspond to $I(A) \to I(B)$ for their images under canonical identification. It is rather remarkable that in purely real context this kind of hierarchy of symmetries modulo finite measurement resolution does not exist. The interpretation would be that finite measurement resolution relates to cognition and therefore to p-adic physics.

3. Matrix group G contains only elements of form $g = 1 + O(p^m)$, $m \geq 1$ and does not therefore involve matrices with elements expressible in terms roots of unity. These can be included by writing the elements of the p-adic Lie-group as products of elements of above mentioned G with the elements of a discrete group for which the elements are expressible in terms of roots of unity in an algebraic extension of p-adic numbers. For p-adic prime p p:th roots of unity are natural and suggested strongly by quantum arithmetics [18].

3.3 Quantum adeles

Before saying anything about Hilbert space adeles it is better to consider ordinary adeles.

1. Fusing reals and quantum p-adic integers for various values of prime p to Cartesian product $A_Z = R \times (\prod_p Z_p)$ gives the ring of integer adeles. The tensor product $Q \otimes_Z A_Z$ gives rise to rational adeles. $_Z$ means the equivalence $(nq, a) \equiv (q, na)$. This definition generalization to any number field including algebraic extensions of rationals. It is not quite clear to me how essential the presence of R as Cartesian factor is. One can define ideles as invertible adeles by inverting individual p-adic numbers and real number in the product. If the component in the Cartesian product vanishes, the component of inverse also vanishes.

2. The definition of a norm of adele is not quite straightforward.

   (a) The norm of quantum adeles defined as product of real and p-adic norms is motivated by the formula for the norm of rational numbers as the product of its p-adic norms. This definition of norm however looks non-physical and non-mathematical. For instance, it requires that all p-adic components of quantum adele are non-vanishing and most of them have norm equal to one and are therefore p-adic integers of norm one. This condition would also break general coordinate invariance at the level of quantum adelic imbedding space very strongly. Also for adelic spinors and adelic Hilbert space this condition is definitely non-sensical.

   (b) The physically acceptable norm for adeles should reflect the basic properties of p-adic norm for a given p-adic field in the product but should also have the characteristic property of Hilbert space norm that the norm squared is sum of the norms squared for the factors of the adele. The solution to these demands seems to be simple: map the p-adic number to its quantum counterpart in each factor and map this number to real number by canonical identification. After this form the real Hilbert space norm of the resulting element of infinite-dimensional real Hilbert space. This norm generalizes in a natural manner to linear spaces possessing adeles as components. Most importantly, for this norm the elements of adele having finite number of components have a non-vanishing norm and field property is possible.

Consider now what happens when one replaces p-adic integers with p-adic Hilbert spaces and p-adic numbers as components of the vectors of the Hilbert space.

1. As far as arithmetics is considered, the definition of Hilbert space adeles for p-adic number fields is formally the same as that of ordinary adeles. It of course takes time to get accustomed to think that rationals correspond to a pair of Hilbert spaces and their product is formulated for this pair.

2. p-Adic Hilbert spaces would be linear spaces with p-adic coefficients that is vectors with p-adic valued components. Inner product and norm would be defined by mapping the components of vectors to real/complex numbers by mapping them first to quantum p-adics and them to reals by canonical identification. Note that the attempts to define p-adic Hilbert spaces using p-adic norm or formal p-adic valued norm mapped to real number by canonical identification lead to difficulties since already in 2-D case the equation $x^2 + y^2 = 0$ has solution $y = \sqrt{-1}x$ for $p \ mod \ 4 = 1$ since in this case $\sqrt{-1}$ exists p-adically.

3. A possible problem relates to the fact that all p-adic numbers are mapped to non-negative real numbers under canonical identification if the coefficients a_n in the expansion $\sum_n a_n p^n$ consists of primes $l < p$ for which quantum counterpart is non-negative. For ordinary p-adic numbers orthogonal vectors in a given basis would be simply vectors with no common non-vanishing components. Does this mean the existence of a preferred basis with elements $(0, .., 0, 1, 0...)$ so that any other unitarily related basis would be impossible. Or should one introduce cyclic algebraic extension of p-adic numbers with n-elements $exp(i2\pi k/n)$ for which one obtains linear superposition and can form new unitarily related basis taking into account the restrictions posed by p-adicity. This option is suggested also by the identification of the Hilbert space as wave functions in the local singular covering of imbedding space. The phases form also in a natural manner cyclic group Z_n identifiable as quantum Galois group assignable to integer n and decomposing to a product of cyclic groups Z_{p_i}, $p_i|n$.

Also real numbers form a Cartesian factor of adeles. The question what Hilbert spaces with dimension equal to arbitrary real number could mean has been already discussed and there are two approaches to the problem: one based on canonical identification and quantum counterparts of p-adic numbers and one to a completion of Hilbert rationals.

4 Generalized Feynman diagrams as quantum arithmetic Feynman diagrams?

The idea that the generalized Feynman diagrams of TGD could have interpretation in terms of arithmetic QFT is not new but the quantum arithmetic Feynman diagrams give much more precise content to this idea.

1. The possibility to eliminate all loops is by "moves" is an old idea (briefly discussed in [3]), which I introduced as a generalization of the old fashioned s-t duality of string models. One motivation was of course the resulting cancellation of diverges. I however gave up this idea as too romantic [3]. The properties of the counterparts of twistor diagrams in zero energy ontology re-inspires this idea.

2. The basic question concerns the possible physical interpretation of the two kinds of 3-vertices and their co-vertices, which are also included and mean decomposition of incoming particle characterized by integer m to quantum superposition of two particle states characterized by integers n, p satisfying $m = n + p$ for the co-sum and $m = n \times p$ for co-product. The amplitudes of different state pairs n, p in fact determine the quantum dynamics and typically the irreversible dynamics leading from state with well-defined quantum number characterized by integers would be due to the presence of co-vertices meaning delocalization.

3. If quantum p-adic integers correspond to Hilbert spaces then the identifications $+_q = \oplus$ and $\times_q = \otimes$ become possible. The challenge is to fix uniquely their co-vertices and this procedure fixes completely number theoretic Feynman amplitudes. Quantum dynamics would reduce to co-arithmetics. Or should one say that mathematics could reduce to quantum dynamics?

4. $\times_q$ and $+_q$ alone look very quantal and the generalization of string model duality means that besides cyclic permutations arbitrary permutations of incoming *resp.* outgoing lines act as symmetries. The natural question is whether this symmetry generalizes to permutations of all lines. This of course if commutativity in strict sense holds true also for quantum arithmetics: it could be that it holds true only in projective sense. Distributivity has however no obvious interpretation in terms of standard quantum field theory. The arithmetics for integers would naturally reflect the arithmetics of Hilbert spaces dimensions induced by direct sum and tensor product

4.1 Quantum TGD predicts counterparts for $\times_q$ and $+_q$ vertices

Also quantum TGD allows two kinds of vertices identifiable in terms of the arithmetic vertices and this gives strong physical constraints on $+_q$ vertices.

1. First kind of vertices are the direct topological analogs of vertices of ordinary Feynman diagrams and there are good arguments suggesting that only 3-vertices are possible and would mean joining of 3 light-like 3-surfaces representing lines of generalized Feyman diagram along their 2-dimensional ends. At the these vertices space-time fails to be a manifold but 3-surface and partonic 2-surface are manifolds. These vertices correspond naturally to $\times_q$ or equivalently $\otimes$.

2. The vertices of second kind correspond to the stringy vertices, in particular the analog of stringy trouser vertices. The TGD based interpretation - different from stringy interpretation- is that no decay takes place for a particle: rather the same particles travels along different routes. These vertices correspond to four-surfaces, which are manifolds but 3-surfaces and partonic 2-surface fail to be manifolds at the vertex. There is a strong temptation to interpret $+_q$ - or equivalently $\oplus$ - as the counterpart of stringy vertices so that the two lines entering to $+_q$ would represent same incoming particle and should have in some sense same quantum numbers in the situation when the particle is an eigenstate of the quantum numbers in question? This would allow to understand the strange looking quantum distributivity and also to deduce what can happen in $+_q$ vertex.

3. What does the conservation of quantum numbers mean for quantum Galois quantum numbers identified in the proposed manner as quantum number associated with the cyclic groups assignable to the integers appearing in the vertex? For $\times_q$ vertex the answer is simple since tensor product is formed. This means that the number theoretic momentum is conserved. For direct sum one obtains direct sum of the incoming states and one cannot speak about conservation of quantum numbers since the final state does not possess well-defined quantum numbers.

4.2 How could quantum numbers of physical states relate to the number theoretic quantum numbers?

Quite generally, the above proposal would allow to represent all n-plets of rationals as zero energy states with either positive or negative arrow of time and one could assign to these states M-matrices as entanglement coefficients and define quantum jump as a sequence of two state function reductions occurring to states with opposite arrow of time. This kind of strong structural similarities with quantum TGD are hardly not a accident when one takes into account the connection with infinite primes and one could hope that zero energy states and generalized Feynman graphs could represent the arithmetics of Hilbert adeles with very dramatic consequences due to the arithmetic moves allowing to eliminate loops and permuted incoming lines without affecting the diagram except by a phase factor. The hierarchy of infinite primes suggests strongly the generalization of this picture since the resulting states would correspond only to the infinite integers at the lowest level of the hierarchy and identifiable in terms of free Fock states of super-symmetric arithmetic QFT.

The possible reduction of generalized Feynman diagrams to Hilbert adelic arithmetics raises several questions and one can try to proceed by requiring consistency with the earlier speculations.

1. How the quantum numbers like momentum, spin and various internal quantum numbers relate to the number theoretic quantum numbers $k = n2\pi/p$ defined only modulo p? The natural idea is that they find a representation in the number theoretical anatomy of the state so that these quantum numbers corresponds to waves with these momenta at the orbits of quantum Galois group. Momentum UV cutoff would have interpretation in terms of finite measurement resolution completely analogous to that encountered in condensed matter physics for lattice like systems. This would realize self-reference in the sense that cognitive part of the quantum state would represent quantum numbers characterizing the real part of the quantum state.

2. What about the quantum p-adics themselves characterizing incoming and out-going states in number theoretic vertices? There would be a conservation of number theoretical "momentum" characterized by logarithm of a rational in $\times_q$ vertex. Does this momentum have any concrete physical counterpart? Perhaps not since it would be associated with quantum p-adic degrees of freedom serving as correlates for cognition. In fact, the following argument suggest interpretation in terms of a finite dimension (finite by finite measurement resolution) of a Hilbert space associated with the orbit of a partonic 2-surface.

   (a) The prime factors of integer characterizing the orbit of a partonic 2-surface correspond naturally to braid strands for generalized Feynman diagrams. This suggests that the primes in question can be assigned with braid strands and would be indeed something new. The product of the primes associated with the particles entering $\times_q$ vertex would be same as the product of primes leaving this vertex. In the case of $+_q$ vertex the integer associated with each line would be same. One cannot identify these primes as p-adic primes since entire orbit of partonic 2-surface and therefore all braid strands are characterized by single common p-adic prime p.

   (b) Hilbert spaces with prime dimension are in a well-defined sense primes for tensor product, and any finite-dimensional Hilbert space decomposes into a product of prime Hilbert spaces. Hence the integer n associated with the line of a generalized Feynman diagram could characterize the dimension of the finite-dimensional Hilbert space (by finite measurement resolution) associated with it. The decomposition of n to prime factors would correspond to a decomposition of this Hilbert space to a tensor product of prime factors assignable to braid strands. This would define a direct Hilbert space counterpart for the decomposition of braid into braid strands and would be very natural physically and actually define the notion of elementarity. The basic selection rule for $\times_q$ vertex would be that the prime factors of incoming Hilbert spaces recombining to form Hilbert spaces of outgoing particles. For the $+_q$ incoming Hilbert spaces of dimensions n_1 and n_b would fuse to $n_1 + n_2$ dimensional direct sum. $a(b + c) = ab + ac$ would state that the tensor product with direct sum is sum of tensor products with direct summands. Therefore the two kind of vertices as well as corresponding vertices of quantum TGD would correspond to basic algebraic operations for finite-dimensional Hilbert spaces very natural for finite measurement resolution.

   (c) Could the different quantum versions of p-adic prime $l > p$ correspond to different direct sum decompositions of a Hilbert space with prime dimension to Hilbert spaces with prime dimensions appearing in the quantum pinary expansion in powers of p? The coefficients of powers of p defined as products of quantum primes $l < p$ would be quantum dimensions and reflect effects caused by finite measurement resolution whereas the powers of p would correspond to ordinary dimensions. This decomposition would correspond to a natural decomposition to a direct sum by some natural criterion related to finite measurement resolution. For instance, power p^n could correspond to n-ary p-adic length scale. The decomposition would take place for every strand of braid.

   The objection is that for algebraic extensions of rationals the primes of the extension can be algebraic number so that the corresponding Hilbert space dimension would be complex algebraic number. It seems that only the primes $l > p$ which do not split for the algebraic extension used (and thus label quantum p-adic number fields in the adele) can be considered as prime dimensions for the Hilbert spaces associated with braid strands. The latter option is more natural and would mean that the number theoretic evolution generating increasingly higher-dimensional algebraic extensions implies selection of both preferred p-adic primes and preferred prime dimensions for state spaces. One implication would be that the quantum Galois group assignable to given p-adic integer would in general be smaller for an algebraic extension of rationals than for rationals since only the non-splittable primes in its factorization would contribute to the quantum Galois group.

(d) As already discussed, the most plausible interpretation is that the pair of co-prime integers defining the quantum rational defines a pair of Hilbert space dimensions possibly assignable to fermions and bosons respectively. Interestingly, for the simplest infinite primes representing Fock states and mappable to rationals m/n the integers m and n could be formally associated with many-boson and many-fermion states.

(e) Because of multiplicative conservation law in $\times_q$ vertex quantum p-adic numbers does not have a natural interpretation as ordinary quantum numbers - say momentum components. The problem is that the momentum defined as logarithm of multiplicatively conserved number theoretic momentum would not be p-adic number without the introduction of an infinite-dimensional transcendental extension to guarantee the existence of logarithms of primes.

(f) If this vision is correct, the representation of ordinary quantum numbers as quantum Galois quantum numbers would be a representation in a state space formed by (quantum) state spaces of various quantum dimensions and thus rather abstract but quite possible in TGD framework. This is of course a huge generalization from the simple wave mechanical picture based on single Hilbert space but in spirit with abstract category-theoretical thinking about what integers are category-theoretically. The integers appearing as integers in the Cartesian factors of adeles would represent Hilbert space dimensions in the case of generalized Feynman diagrams. The arithmetic Feynman rules would be only a part of story: as such very abstract but made concrete by braid representation.

3. Note that the interpretation of $+$ and $\times$ vertices in terms of Hilbert space dimensions makes sense also in the real context whereas the further decomposition into direct sum in powers of p^n does not make sense anymore.

4.3 Number theoretical quantum numbers and hierarchy of Planck constants

What could be the TGD inspired physical interpretation of these mysterious looking Hilbert spaces possessing prime dimensions and having no obvious identification in standard physics context?

4.3.1 How the Hilbert space dimension relates to the value of Planck constant?

The first question is how the Hilbert space dimension assigned to a given line of a generalized Feynman diagram relates to the the value of Planck constant.

1. As already noticed, the decomposition of integer to primes would naturally correspond to its decomposition to braid strands to which one can assign Hilbert spaces of prime-valued dimension $D = l$ appearing as factors of integer n. This suggests a Hilbert space is defined by wave functions in a set B_n with n points,. This Hilbert space naturally decomposes into a tensor product of Hilbert spaces with Hilbert spaces associated with point sets B_l containing l of points with $l|n$.

2. The only space of this kind that comes in mind relates to the proposed hierarchy of (effective) Planck constants coming as integer multiples of ordinary Planck constant. For the simplest option Planck constant $\hbar_n = n\hbar_0$ would correspond to a local (singular) covering of the imbedding space due to the n-valuedness of the time derivatives of the imbedding space coordinates as function of canonical momentum densities which is due to the huge vacuum degeneracy of Kähler action.

3. The discrete group Z_n would act as a natural symmetry of the covering and would decomposes a $Z_n = \prod_{l|n} Z_l^{e_l}$ and the orbits of Z_l in the covering would define naturally the sets B_l. Given prime l in the decomposition would correspond to an l-fold covering of a braid strand and to a wave function in this space.

4. The proposal for the hierarchy of Planck constants assumes that different sheets of this singular covering degenerate to single sheet at partonic 2-surfaces at the ends of CD. Furthermore, the integers n would decompose to to products $n = n_1 n_2$ corresponding to directions of time-like braids along wormhole throat and along the space-like 3-surface at the end of CD defining by effective 2-dimensionality (strong form of holography) two space-time coordinates playing the role of time coordinate in the field equations for preferred extremals. Note that the information about the presence of covering would be carried at partonic 2-surfaces by the tangent space data characterized by the n_i-valued normal derivatives.

5. The simplest option is that Hilbert space dimension corresponds to Planck constant for a given line of generalized Feynman diagram. This would predict that in the multiplicative vertex also the values of Planck constants characterizing the numbers of sheets for many-sheeted coverings would satisfy the condition $n_3 = n_1 n_2$. The assumption that the multiplicative vertex corresponds to the gluing of incoming lines of generalized Feynman diagram together along their ends seems however to require $n_1 = n_2 = n_3$. Furthermore, the identification of Hilbert space dimension as Planck constant is also inconsistent with the vision about book like structure of the imbedding space explaining the darkness as relative darkness due to the fact that only particles with the same value of Planck constant can appear in the same vertex [6].

   The way out of the difficulty is to assume that the value of Planck constant $\hbar = n\hbar_0$ corresponds to $n = n_3 = n_1 n_2$ or has n_3 as a factor. For $n = n_3$ the states with Hilbert space dimensions n_1 and n_2 are invariant under cyclic groups Z_{n_2} and Z_{n_1} respectively. For n containing n_3 as a genuine divisor analogous conditions would hold true.

6. p-Adic prime p would make itself manifest in the further decomposition of the l-dimensional Hilbert spaces to a direct sum of sub-Hilbert spaces with dimensions given by the terms $l_{n,q}p^n$ in the expression of l as quantum integer. The fact that the only prime ideal for p-adic integers is pQ_p should relate to this. It is quite possible that this decomposition occurs only for the p-adic sectors of the Hilbert adelic imbedding space.

   What suggests itself is symmetry breaking implying the decomposition of the covering A_n of braid strand to subsets $A_{n,m}$ with numbers of elements given by $\#_{n,m} = l_m p^m$ with l_m divisible only by primes $p_1 < p$. Wave functions would be localized to the sets $A_{n,m}$, and inside $A_{n,m}$ one would have tensor product of wave functions localized into the sets A_l with $l < p$ and $l|l_m$.

   Hilbert space dimensions would be now quantum dimensions associated with the quantum phase $exp(i\pi/l)$: this should be due to the finite measurement resolution and relate to the fact that one has divided away the hyper-finite factor N from the factor $M \supset N$.

   The index characterizing Jones inclusion [5] [6] is given by $[M : N] = 4cos^2(2\pi/n)$ and corresponds to quantum dimension of $2_q \times 2_q$ quantum matrices. TGD suggest that a series of more general quantum matrix dimensions identifiable as indices of inclusions and given by $[M : N] = l_q^2$, $l < p$ prime and $q = exp(i\pi/n)$, corresponding to prime Hilbert spaces and $q = n$-adicity. Note that $l_q < l$ is in accordance with the idea about finite measurement resolution and for large values of p one would have $l_q \simeq l$.

If the above identification is correct, the conservation laws in $\times_q$ and $+_q$ vertices would give rather precise information about what can happen for the values of Planck constants in thes vertices. In $\times_q$ co-vertices Hilbert space-timensions would combine multiplicatively to give the common value of Planck constant and in $\oplus_q$ co-vertices additively. The phase transitions changing Planck constant, for instance for photons, are central for quantum TGD and the selection rules would not allow them only if they correspond to a formation of a Bose-Einstein condensate like state or its decay by $\times_q$- or $+_q$-vertex.

4.3.2 Could one identify the Hilbert space dimension as value of Planck constant?

It has been already seen that the identification of Hilbert space dimension with Planci constant it is not consistent with the idea that product vertex means that the lines of generalized Feynman graph are glued along their 2-D ends together. I did not however realize this when I wrote the first version of this section and I decided to keep the earlier discussion about the option for which Planck constants correspond to Hilbert space dimensions so that $n_3 = n_1 n_2$ holds true for Planck constants. The question was whether it could be consistent with the idea of dark matter as matter with non-standard value of Planck constant. By replacing "Planck constant" with "Hilbert space dimension" below one obtains a discussion giving information about the selection rules for Hilbert space dimensions.

1. In $\times_q$-vertex the Planck constants for the outgoing particles would be smaller and factors of incoming Planck constant. In $\times_q$ co-vertex Planck constant would increase. I have considered analogous selection rules already earlier. $\times_q$ vertex does not allow the fusion of photons with the ordinary value of Planck constant to fuse to photons with larger value of Planck constant.

   By conservation of energy the frequency of a photon like state resulting in the fusion is given by $f = \sum n_k f_k / N_{out} \prod_k n_k$ for $\hbar_k = n_k \hbar_0$, where N_{in} and N_{out} are the numbers of quanta in the initial and final state. For a common incoming frequency $f_k = f_0$ this gives $f/f_0 = \sum_k n_k / (N_{out} \prod_k n_k)$. If one assumes that spin unit for photon increases to $\prod_k n_k \hbar_0$ and spins are parallel one obtains from angular momentum conservation $N_{out} \prod_k n_k = N_{in} \sum n_k$ giving $N_{out} = \prod_k n_k N_{in} / \sum n_k = n^{N_{in}} / N_{in} n$, which in turn gives $f/f_0 = 1/N_{in}$. This looks rather natural.

   In the presence of a feed of $r = \hbar/\hbar_0$¿1 particles $\times_q$ vertex could lead to a phase transition generating particles with large values of Planck constant. Large values of Planck constant are in a key role in TGD based model of living matter since Compton lengths and other quantum scales are proportional to $\hbar$ so that large values of $\hbar$ make possible macroscopic quantum phases. The phase transition leading to living matter could be this kind of phase transition in presence of feed of $r > 1$ particles.

2. For $+_q$ co-vertex $r = \hbar/\hbar_0$ could be additive and for incoming photons with same frequency and Planck constants $\hbar_k$ the outgoing state with Planck constant $\sum_k \hbar_k$ energy conservation is guaranteed if the frequency stays same. This vertex would allow the transformation of ordinary photons to photons with large Planck constant, and one could say that effectively the photons fuse to form single photon. This is consistent with the quantization of spin since the unit of spin increases. For this option the presence of particles with ordinary value of Planck constant would be enough to generate particles with $r > 1$ and this in turn could lead to a the phase transition generation living matter.

3. One can of course ask whether it should be $r - 1 = \hbar/\hbar_0 + 1$, which corresponds to the integer n. For this option the third particle of $+_q$ vertex with two incoming particles with ordinary Planck constant would have ordinary Planck constant. For $\times_q$ vertex containing two incoming particles with $r = n$, $n = 1$ ($n = 2$), also the third particle would have $n = 1$ ($n = 2$). $\times_q$ and $+_q$ vertices could not generate $n > 1$ particles from particles with ordinary Planck constant. The phase transition leading from inanimate to living matter would require $n > 1$ states as a seed (one has $2 + 2 \rightarrow 3$ for $_+q$ vertex). A quantum jump generating a CD containing this kind of particles could lead to this kind of situation.

4. These selection rules would mean a deviation of the earlier proposal that only particles with same values of Planck constant can appear in a given vertex [6]. This assumption explains nicely why dark matter identified as phases with non-standard value of Planck constant decouples from ordinary matter at vertices. Now this explanation would be modidifed. If $\times_q$ vertex contains two particles with $r = n + 1$ for $r = n$ option ($r = 1$ or 2 for $r = n + 1$ option), also the third particle has ordinary value of Planck constant so that ordinary matter effectively decouples from dark matter. For $+_q$ vertex the decoupling of the ordinary from dark matter occurs for $r = n+1$ option but not for $r = n$ option. Hence $r = n + 1$ could explain the virtual decoupling of dark and ordinary matter from each other. The assumption that Planck constant is same for all incoming lines and corresponds to $n_3 = n_1 n_2$ defines however much more plausible option.

4.3.3 What happens in phase transitions changing the value of Planck constant?

The phase transitions changing the value of Planck constant are in a central role in TGD inspired quantum biology. The typical phase transition of this kind would change the Planck constant of photon. This phase transition would formally correspond to a 2-vertex changing the value of Planck constant. Can one pose selection rules to the change of Planck constant? By the above assumptions both the incoming and outgoing line correspond to Hilbert space dimension which is a factor of the integer defining Planck constant. If the value of the Hilbert space dimension is not changed in the process, theincoming and outgoing Planck constants must have this dimension as a common factor.

4.4 What is the relation to infinite primes?

Already quantum p-adics would mean a dramatic generalization of number concept by assigning to rationals ane even algebraic numbers Hilbert spaces and their states. Quantum adeles would mean a further generalization of number concept by gluing together reals and Hilbert space variants of p-adic number fields.

TGD leads also to another generalization of number concept based on the hierarchy of infinite primes [13]. This generalization also leads to a generalization of real number in the sense that one can construct infinite number of real units as infinite rationals which reduce to units in real sense. This would mean that space-time point has infinitely complex number theoretic anatomy not visible at the level of real physics [14].

The possibly existing relationship between these generalizations is of course interesting. Infinite primes can be mapped to polynomial primes and this means that one can assign to them algebraic extensions of rationals and corresponding Galois groups and in [17] I discussed a conjecture that the elements of these Galois groups could be represented as symplectic flows assignable to braids which emerge naturally as counterparts of partonic 2-surfaces in finite measurement resolution. This would suggest a possible relationship.

The construction of infinite primes relies on the product $X = \prod_p p$ of finite primes interpreted physical as analog of Dirac vacuum with all negative energy states filled. Simplest infinite primes are constructed by kicking away fermions from this vacuum and by adding also bosons labeled by primes. One obtains also the analogs of bound states as infinite primes which can be mapped to irreducible polynomials. The roots of the polynomial code for the infinite prime and the algebraic extension. The infinite primes corresponding to n^{th} order polynomials decompose to products of n simplest infinite primes of algebraic extension so that the corresponding Galois group emerges naturally.

The construction can be repeated endlessly by taking the infinite primes of the existing highest level and forming the product X of them and repeating the process. What these means that the many-particle states of the previous level define single particle states of the new level. One can map these infinite primes to polynomial primes for polynomials of several variables. Also this hierarchy might allow generalization obtained by assigning to infinite primes the orbits of their Galois groups. The earlier considerations [11] suggest strongly a reduction of the description to the lowest level and involving only algebraic numbers.

4.4.1 What do we understand about infinite primes?

Let us first try to summarize what we understand about infinite primes. What seems very natural is the postulate that arithmetic QFT associated with infinite primes conserves multiplicative number theoretic momenta defined by ordinary primes with separate conservation law for each prime. This law would hold for $\times_q$ vertices very naturally whereas for $+_q$ vertices it would be broken. Recall that these two vertices correspond to the TGD counterparts of 3-vertices for Feynman diagrams and stringy diagrams respectively and also to tensor product and direct sum.

1. What seems clear is that infinite prime characterizes an algebraic extension of rationals (or of its extension) in the case that infinite primes is defined in terms of finite primes of extension. Infinite prime dictates also the p-adic primes which are possible and appear in the quantum adele assignable to infinite prime.

2. The integer exponents of ordinary primes appearing in the infinite and finite part of the simplest lowest level infinite prime could define infinite number of conserved number theoretic momenta, one for each prime p and having $log(p)$, p prime, as a unit. Separate conservation follows from the algebraic independence. These number theoretic momenta do not make sense p-adically, which means that in p-adic context the multiplicative form of the conservation law is the appropriate one. Therefore it is appropriate to speak of multiplicative momenta. Therefore the relationship with ordinary additively conserved momenta does not lok plausible.

   Arithmetic QFT interpretation allows also to interpret the numbers n_p in p^{n_p} as particle numbers assignable to bosonic quanta and fermionic quanta in the case of the simplest infinite primes with "small part" representing fermions kicked out from the Dirac sea possibly accompanied by bosonic quanta. The conservation law at $\times_q$ vertices would mean conservation of total particle numbers assignable to primes p.

3. For the simplest primes at the lowest level identifiable as linear polynomials with integer coefficients there are two separate integers defining number theoretic momenta. The first integer corresponds to the finite part of infinite prime and the second one to the finite part of the infinite prime to which one assigns number theoretic fermions. These two parts are separately conserved. Since the integers have no common prime factors, one can also speak about rational valued multiplicative number theoretic momentum. The physical interpretation for the absence of common factors would be that given mode cannot simultaneously containing and not contain fermionic excitation. For higher irreducible polynomials of order n interpreted in terms of bound states there are $n+1$ integers defining a collection of number theoretic momenta. For the representation as a monic polynomial one has a collection of n rational valued number theoretic momenta.

4. The notion of multiplicative number theoretic momentum generalizes.

   (a) At the second level of the hierarchy ordinary primes are replaced with prime polynomials $P_n(x)$ of single variable. At the n^{th} level they are prime polynomials $P_n(x_1,...,x_{n-1})$ of $n-1$ variables. The value of the number theoretic momentum at n^{th} s level can be said to be a polynomial $P_n(x_1,...,x_{n-1})$ rather than integer.

   (b) This looks very abstract but can be concretized. For instance, each coefficient of $P_n(x,y)$ at second level as polynomial of y defines a polynomial $P_k(x)$ at the first level and $P_k(x)$ is characterized by a collection of number theoretic momenta defined by its integer coefficients in the representation as a polynomial with integer coefficients. Therefore $P_k(x)$ can be identified as the collection of $k+1$ integer coefficients or k rational coefficients in the monic representation identified as number theoretic momenta for a k-particle state. $P_n(x,y)$ in turn corresponds to a collection of n many-particles states with i^{th} one containing k_i particles, $i=1,...n$. The interpretation in terms of n-braid with braid strands decomposing to k_i braid strands is natural and conforms with the fractality of TGD Universe.

   (c) This example allows to deduce the number theoretic interpretation of the polynomial at the n^{th} level and one can continue this abstraction hierarchy ad infinitum. Eventually each prime at a given level of hierarchy reduces to a collection of number theoretic momenta defined by ordinary integers grouped in a manner characterized by the infinite prime. Physically this would characterize how these number theoretic elementary particles group to particles at the first level, these to particles at second level, and so on.

   (d) The possibility to express the irreducible polynomial as a product of first order polynomials with zeros which algebraic numbers gives for the bound state a representation as free many-particle state but with number theoretic momenta which are algebraic rationals in algebraic extension of rationals. These number theoretic momenta can be also complex and therefore do not allow interpretation as Hilbert space dimensions. This decomposition is analogous to a decomposition of hadron to quarks. The rational coefficients expressible in terms of the roots of the polynomial code for Galois invariants analogous to the observables assignable to hadrons and accessible to the experimenter.

5. The basic conservation law of arithmetic QFT and of TGD would be that the multiplicative number theoretic momenta labelled by finite primes are separately conserved in $\times_q$ vertices but not in $+_q$ vertices. The conservation number theoretic quantum numbers allows the interpretation of Hilbert space dimensions in terms of the hierarchy of Planck constants, and this leads to a proposal that infinite primes code the pairs of finite integers with no common factors assignable to the pairs of time-like and space-like braid strands.

   If one takes seriously the notion of number theoretic fermion, one could assign to space-like braid strands only bosonic excitations and to time-like braid strands fermion and possibly also bosonic excitations. The interpretation could be in terms of the super-conformal algebras containing both fermionic and bosonic generators. The hierarchy of infinite primes would correspond to a hierarchy of braids containing lower level braids as their strands as suggested already earlier [17].

What would be new would be a concrete assignment of primes to braid strands and detailed identification in terms of time-like and space-like braids.

This kind of assignment would mean a rather dramatic step of progress in the understanding of the complexities of generalized Feynman diagrams. One not completely settled old question is what selects the p-adic prime assignable to given partonic 2-surface.

This is the stable looking part of the vision about infinite primes, and any attempt to relate it to quantum p-adics and quantum adeles should respect this picture.

4.4.2 Hyper-octonionic primes correspond to p-adic primes in extension of rationals

The earlier interpretation hyper-complex and appropriately defined quaternionic and octonionic generalizations is in terms of standard model quantum numbers [7]. It seems that also this identification survives under the selective pressures by new ideas but that one cannot replace hyper-complex primes with their infinite counterparts. Rather, hyper-complex prime generalizes p-adic prime as a preferred prime by replacing ordinary integers with hyper-complex integers. The definition of infinite primes in quaternionic and octonionic context is plagued by the problems caused by non-commutativity and associativity so that the conclusion is well-come.

1. The solutions of modified Dirac equation suggest the interpretation of the M^2 projections of four-momenta as "hyper-complex" primes or perhaps more realistically. their integer multiples. These momenta are conserved additively rather than multiplicatively at vertices to which $\times_q$ is assigned and only their exponents - naturally phase factors - would be conserved multiplicatively.

2. Could this identification generalize from hyper-octonionic primes to hyper-octonionic infinite primes? This does not seem to be the case. The multiplicative conservation in $\times_q$ vertices for number theoretic momenta is in conflict with additive conservation for ordinary quantum numbers. Additive conservation is also in conflict with interpretation in terms Hilbert space dimensions allowing concretization in terms of the hierarchy of Planck constants. Of course, hyper-complex Hilbert space dimension does not make sense either.

3. One must remember that there are many kinds of primes involved and a little list helps to see what the correct interpretation for hyper-complex primes could be.

   (a) There are the primes l appearing in the decomposition of infinite primes and having interpretation in terms of Hilbert space dimensions. The conservation of multiplicative number theoretical momenta is natural at $\times_q$ vertices.

   (b) There are the p-adic primes p, and on basis of p-adic mass calculations it is this prime to which it is natural to assign additively conserved momenta. p characterizes the "active" sector of adeles and therefore also the various quantum variants of the prime l in which quantum primes $p_1 < p$ appear as factors. p characterizes partonic 2-surface.

   (c) The Abelizanization of the quantum Galois group assignable to prime l decomposes into prime factors Z_{p_2} and the phases $exp(i2\pi/p_1)$ might provide cognitive representations in finite measurement resolution for various standard model quantum numbers.

4. The only reasonable interpretation seems to be that the hyper-complex momenta and possible other quantum numbers assignable to them correspond to p-adic prime p for rationals or for an algebraic extension of rationals to the ring hyper-complex rationals. The failure of field property implies that the inverse of hyper-complex number fails to exist when it defines a light-like vector of M^2. This has however a concrete physical interpretation and light-like hyper-complex momentum for a massless state is massless only when the momentum of the state transverse to M^2 vanishes so that also propagator defined by M^2 momentum diverges.

What the identification of M^2 momenta as hyper-complex integers really means, deserves some comments.

1. Suppose that particle's p-adic mass squared is of form $m^2 = np$ as predicted by p-adic mass calculations. Assume that m^2 corresponds to M^2 momentum squared with preferred M^2 characterizing given causal diamond CD. Assume also that total M^4 mass squared vanishes in accordance with the idea that all states - even those representing virtual particles - carried by wormhole throats are massless. In accordance with the adelic vision, assume that the prime p does not split in the algebraic extension of rationals used (simplest extension would be $Q[\sqrt{-1}]$). This requires $p\ mod\ 4 = 3$ in accordance with Mersenne prime hypothesis. The idea is that p does not split for ordinary algebraic extension but splits in the ring of hyper-complex numbers.

2. The preferred plane $M^2 \subset M^4$ corresponds to a preferred hyper-complex plane of complexified (by commuting imaginary unit i) hyper-octonionic space M^8. M^2-momentum has therefore purely number theoretic interpretation being due to the slitting of $M^2 = np$ to a product of hypercomplex integer $N = N_0 + eN_z$ and its conjugate $N_0 - eN_z$). The hyper-complex imaginary unit $e = iI$ satisfying $e^2 = 1$ and $I^2 = -i^2 = -1$ would correspond to z-axes of M^2. Here is I is the preferred octonionic imaginary unit and i an imaginary unit commuting with it. One could say that 2-D particle momentum emerges via the emergence of hyper-complex extension of rationals of their extension. This would also generalize to quaternions and one could say that M^4 momentum emerges via extension of rationals to hyper-quaternions.

3. M^2 momentum squared would satisfy $P_0^2 - P_z^2 = (P_0 - eP_z)(P_0 + eP_z) = np$. The prime p does not split in the algebraic extension of rationals used but splits in the ring of hyper-complex numbers. Assume first $n = 1$. In this case the splitting of $p\ mod\ = 3$ ($p\ mod\ = 1$) to $p = (p_0 + ep_z)(p_0 - ep_z)$ implies p_0 is even (odd) and p_z is odd (even). For $n > 1$ one must have $(n_0 - en_1)(n_0 + en_1) = n$ and similar conditions apply to n so that one would have for M^2 momentum $P_0 + eP_z = (n_0 \pm en_z)(p_0 \pm ep_z)$.

4. Momentum components are hyper-complex integer multiples of hyper-complex prime so that that the allowed momenta would form an ideal of hyper-complex numbers. This is mathematically very nice but might be quite too strong a condition physically although it is typically encountered in systems in which particle is enclosed in box. Now the box would correspond to CD with periodic boundary conditions at the ends of CD for the modified Dirac equation. One could consider also a weaker condition for with the integer n is replaced with a rational (m/n) such that neither m nor n contains p as a prime factor.

5. The peculiar looking prediction would be that M^2 momentum cannot be purely time-like. In other words, the particle cannot be at rest M^2. Observer for which CD defines the rest system could not perform a state function reduction leading to a situation in which the particle is at rest with respect to the observer! In fact, this kind of situation is encountered also for particle in box since boundary conditions do not allow constant mode. If one recalls that all particles would be massless in M^4 sense, this condition does not look so strange.

4.4.3 Infinite primes and Hilbert space dimensions

Arithmetic QFT picture would strongly suggests that the number theoretic momenta at the lowest level are conserved in $\times_q$ vertices at least. For $+_q$ vertices the conservation cannot hold true. The conservation could mean that the total number of powers of given prime in state is same for positive and negative energy states.

Of course, much richer spectrum of conservation laws can be imagined since one could require similar conservation laws also at the higher levels of hierarchy, where various number theoretic momenta correspond to numbers prime polynomials at lower level present in the state. The physical interpretation would be that the numbers of bound states particles are conserved meaning that these particles can be regarded as stable. On physical grounds this kind of conservation laws can be only approximate.

1. Could infinite primes label infinite-dimensional prime Hilbert spaces as finite primes do? Could the interpretation for the object $X = \prod_p p$ be in terms of a tensor product of all prime-dimensional Hilbert spaces. Infinite primes with positive finite part would have interpretation as direct sums of this space and finite integer-dimensional Hilbert space. When the finite part of

the infinite prime is negative the interpretation would not be so straightforward, and this option does not look attractive.

2. A much more plausible option is that infinite prime at the first level defines an algebraic extension of rationals (or of its extension) and that this gives rise to a collection of norm for algebraic extension induced by complex norm. As a matter fact, these points at which this norm vanishes might have interpretation as complex coordinates for a corresponding braid strand in n-strand bound state braid in preferred complex coordinates for the partonic 2-surface. A possible geometric interpretation for these points inspired by the notion of dessins d'enfant is that the partonic 2-surface as an abstract Riemann surface representable as a covering of sphere becomes singular at these points as several sheets of covering co-incide.

3. The infinite primes of the lowest level of the hierarchy formally representing Fock states of free bosons and fermions can be mapped to rationals. These rationals could define pairs of Hilbert space dimensions assignable to bosonic and fermionic parts of the state and could this allow identification as quantum p-adic integer in each sector of the adele and the identification in terms of integer dimension in the real sector of quantum adeles. The fact that the two integers have no common factors would only mean that given mode cannot both contain and not contain fermionic excitation.

   One could even consider the possibility of concrete assignment of the first dimension in terms of fermionic braid strands with bosonic excitations and second dimension in terms of purely bosonic braid strands. This interpretation is very natural since the super-conformal algebras creating states have both purely bosonic and purely fermionic generators. These braids could correspond to space-like and time-like (actually light-like) braids having their ends at partonic 2-surfaces.

   The Galois groups associated with primes appearing as factors of the primes would correspond naturally to additional internal degrees of freedom. This identification makes sense also for the infinite primes represented by irreducible polynomials since the coefficients of the polynomial representable in terms of the roots of polynomials define rationals having interpretation as number theoretic momenta. Therefore the interpretation in terms of Hilbert space dimensions makes sense when rationals are interpreted as pairs of dimensions for Hilbert spaces.

4. What about the infinite primes representing bound states and mappable to irreducible polynomials with rational coefficients and defining polynomial primes characterized by a collection of roots [11]. These roots define an algebraic extension of rationals and this suggests that the quantum adele associated with the infinite prime in question is defined accordingly. The infinite primes mappable to n^{th} order monic polynomials would have interpretation as many particle states consisting of single particle states which correspond to algebraic number rather than rational. The rational coefficients of the monic polynomial would define the rationals defining pairs of Hilbert space dimensions.

5. The natural identification for the Hilbert spaces in question would be in terms of the singular local coverings of imbedding space associated with the hierarchy of Planck constants suggested to emerge from the vacuum degeneracy of Kähler action. The integer n decomposing to primes would correspond to sub-braids labeled by prime factors l of n and consisting of l strands in the l-fold sub-covering.

The consistency with the quantum adeles would force the following highly speculative picture. Main justification comes from its internal consistency and consistency with generalize Feynman graphs.

1. Infinite prime (integer, rational) defines the algebraic extension used and the allowed quantum p-adic number fields contributing as factors to the corresponding quantum adele. p-Adic primes, which can be also algebraic primes if one starts from extension of rationals, by definition do not split in the algebraic extension. Infinite primes assignable to particle states obey the conservation of multiplicative number theoretic momenta and define naturally collections of pairs if Hilbert space dimensions assignable to the particles and decomposing to primes l assignable to braid strands. The integers characterizing the rational defining number theoretic momentum correspond to time-like and space-like braid strands and only the time- or space-like strand carries fermionic quantum numbers.

2. These Hilbert spaces have a natural interpretation in terms of the hierarchy of Planck constants realizable in terms of local singular coverings of the imbedding space forced by the enormous vacuum degeneracy of Kähler action.

3. Hyper-complex primes are identifiable as generalizations of p-adic primes and have nothing to do with infinite primes. They could code for standard model quantum number.

4. The quantum Galois quantum numbers assignable to primes l for given p-adic prime p and appearing in the infinite prime characterizing the state would provide a cognitive representation of the standard model quantum numbers.

5. Mersenne primes and primes near powers of 2 and $p = 2$ also should be selected as a p-adic prime in this manner.

6. The basic uncertain aspect of the scenario is whether the notion of quantum p-adic with coefficients in quantum pinary expansion satisfying only the condition $x_n < p^N$ for $N > 1$, with N dictated by the pinary cutoff, makes sense. Physically $N > 1$ is very natural generalization. Most of the preceding considerations remain intact even if $N = 1$ is the only internally consistent option. What is lost is the representation of quantum numbers using quantum Galois group and the crazy proposal that quantum Galois group could be isomorphic to AGG.

This is only the simplest possibility that I can imagine now and reader is encouraged to imagine something better!

4.4.4 The relationship between the infinite primes of TGD and of algebraic number theory

While preparing this chapter I experienced quite a surprise as I learned that something called infinite primes emerges in algebraic number theory [1]. Infinite primes in this sense looked first to me like a heuristic concept characterizing norms for algebraic extensions of rationals induced by the complex norm for the imbeddings of the extension to complex plane. The nomenclature is motivated by the analogy with p-adic norms defined by algebraic primes. It however turns out that there is a close connection with infinite primes at the first level of the hierarchy.

1. The embeddings (ring homomorphisms) of Galois extension to complex plane induce a collection of norms induced by the complex norm. The analogy with p-adic norms labelled by primes serves as a partial motivation for calling these norms infinite primes. The imbeddings are induced by the imbeddings of the roots of an irreducible monic polynomials $P_n(x) = x^n + ...$ with rational coefficients, which defines a polynomial prime so that infinite primes in the sense of algebraic number theory correspond to a polynomial primes.

2. The imbeddings (ring homomorphisms) of the extension of K in C an be defined to those reducing to imbeddings in R and those not. The imbeddings to R correspond in one-one manner to real roots and complex imbeddings come in pairs corresponding to complex root and its conjugate. The norm is defined as $|z - z_k|$, where z_k is the root. The number of imbeddings and therefore of norms is $r = r_1 + 2r_2$, where r is the the degree of the extension K/Q and also the degree of its Galois group for Galois extensions (defined by polynomials with rational coefficients).

3. Also in TGD framework the infinite primes at the lowest level of hierarchy can be mapped to irreducible monic polynomials of single variable: at n^{th} level polynomials of n variables is required. Now however also polynomials $P_1(x)$, whose roots are rationals and have interpretation in terms of free Fock states, are included. Note that the replacement of the variable z with $z - m/n$ shifts the roots of a monic polynomial by m/n so that the corresponding algebraic extension is not modified. For the simplest infinite primes the norm would correspond to $|z - m/n|$. Therefore infinite prime indeed characterizes the algebraic extension and its imbeddings and the "real" factor of quantum adeles is identifiable with this algebraic extension endowed with any of these norms.

4.5 What selects preferred primes in number theoretical evolution?

Preferred p-adic length scales seem to correspond to primes near powers of two, in particular Mersenne primes. The proposed explanation is that number theoretic evolution as emergence of higher-dimensional extensions of rationals and also of p-adics somehow selects Mersenne primes as fittest. But what fitness could mean? This is the question. The answer to the question might be banally simple. The fittest primes could be stable in the process of generation of algebraic extensions! Stability means very concretely that primes do not split into products of primes of the extension and therefore can define p-adic primes for quantum adeles! Number theoretic evolution by algebraic extensions would gradually kill p-adic primes.

The splitting to primes need not be unique (if it is one speaks of principal ideal domain). For instance, in $Q[\sqrt{-5}]$ for which factorization to algebraic primes is not unique (but is unique to prime ideals): $6 = 2 \times 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. In this kind of situation it is better to speak about prime ideals since this makes the splitting unique for what is known as Dedekind domains. The ideal class group characterizes the non-uniqueness of splitting to primes and consists of equivalence classes of fractional ideals (essentially integers defined by some fixed integer) under equivalence defined by multiplication by a rational of extension. The non-uniqueness of the factorization is characterized by so called ideal class group [4].

Are Mersenne primes especially stable against splitting to algebraic primes? Generally, for an especially large set of algebraic extensions, or for some special but physically important extensions? The cautious guess is that Mersenne primes could be special in the sense that the set of (physically relevant) algebraic extensions for which they do not split is especially large. This in turn would raise the infinite primes defining these special algebraic extensions in a special physical role. A possible physical interpretation for these infinite primes would be in terms of bound states. Therefore the stability of Mersenne primes could be translate to the stability of the bound states for which Mersenne primes are stable.

quadratic fields $Q[\sqrt{d}]$ are the simplest algebraic extensions of rationals since they correspond to second order prime polynomials and are also relatively well-studied so that one can look them at first. For $Q[\sqrt{d}]$ there are general results about the splitting of primes.

1. Quite generally, given prime p can be inert, split to a product of two distinct prime ideals, or can be ramified. The so called discriminant D characterizes the situation: for $d\ mod\ 4 = 1$ equals to $D = d$ and otherwise to $D = 4d$.

2. If p - say M_k - is an odd prime not dividing d, p splits only if one has

$$D\ \ mod\ \ p = x^2$$

   In this case one has $(D/p) = 1$, where (D/p) is Legendre symbol having values in the set $\{0, 1, -1\}$. $(D/p) = -1$ means stability of p against slitting.

   Legendre symbol is a multiplicative function in the set of integers D meaning that if p splits under D_1 and D_2 it splits also under D_1D_2 , and if p does not split under D_1 nor under D_2 it splits under D_1D_2. The multiplicative property implies $(4p_1/p) = (2/p)^2 \times (p_1/p) = (p_1/p)$. It is obviously enough to check whether the splitting occurs for primes p_1. Non-splitting prime p_1 gives rise to a set of non-splitting integers obtained by multiplying p_1 with any splitting prime. Also odd powers of non-splitting p_1 define this kind of sets.

3. Also the following properties of Legendre symbol are useful. One has $(D/p) = (p/D)$ if either $D\ mod\ 4 = 1$ or $p\ mod\ 4 = 1$ holds true. $D\ mod\ 4 = 3$ and $p\ mod\ 4 = 3$ one has $(D/p) = -(p/D)$. One has also $(-1/p) = (-1)^{(p-1)/2}$ and $(2/p) = (-1)^{(p^2-1)/8}$.

4. If the p-adic number fields, which do not allow $\sqrt{-1}$ as ordinary p-adic number are in special role then there might be hopes about the understanding of the special role of Mersenne primes. Mersenne primes are also stable for Gaussian integers and quadratic extensions $Q[\sqrt{\pm d}]$ of rationals defined by positive integers d, which are products $d = d_1d_2$ of two integers. d_1 factorizes to a product of primes $p_1\ mod\ 4 = 1$ splitting M_k, and d_2 is a product of an odd number of primes $p_1\ mod\ 4 = 1$ not splitting M_k.

5. One must also distinguish between the algebraic extensions of rationals and finite dimensional extensions of p-adic numbers (also powers e^k, $k < p$ define finite-dimensional extension). For instance, one can consider a quadratic extension $Q[\sqrt{-1}]$ for rationals defining similar extension for the allowed p-adic primes $p \ mod \ 4 = 3$ and fuse it with a quadratic extension $Q[\sqrt{d}]$ for which $d \ mod \ 4 = 1$ holds true. For adeles the extension of rationals and the extensions of p-adic numbers can be said to separate.

Some special examples are in order to make the situation more concrete.

1. A good example about physically very relevant quadratic extension is provided by Gaussian integers, which correspond to Galois extension $Q[\sqrt{-1}]$ [3]. $p = 2$ splits as $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and the splitting to primes is non-unique. The splitting to prime ideals is however unique so that $p = 2$ is not ramified.

   The primes $p \ mod \ 4 = 1$ split also as stated by Fermat's theorem of two squares. Mersenne primes satisfy $p \ mod \ 4 = 3$ but some additional criterion is needed to select them. Primes $p \ mod \ 4 = 3$ do not and cannot define p-adic primes appearing in quantum adele for Gaussian rationals. Note that for $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as p-adic number, which might cause problems in the p-adic formulation of quantum mechanics. These observations suggest that p-adic primes $p \ mod 4 = 1$ suffer extinction when $\sqrt{-1}$ emerges in the number theoretic evolution and only the primes $p \ mod 4 = 3$ remain. One could also start from the extension $Q[\sqrt{-1}]$ rather than rationals as the role of $\sqrt{-1}$ in quantum theory suggests so that the primes $p \ mod 4 = 3$ would be the only allowed quantum p-adic primes.

2. for $Q[\sqrt{2}]$ for which 2-adicity would not be possible. What happens for Mersenne primes? One can write $M_3 = 7 = (\sqrt{2} + 3)(-\sqrt{2} + 3)$ where $3 \pm \sqrt{2}$ is an algebraic integer as a root of a monic polynomial $P(x) = x^2 - 6x + 7$ so that the splitting of M_3 occurs in $Q[\sqrt{2}]$. Therefore it seems that the absence of $\sqrt{2}$ and allowance of 2-adicity is necessary for Mersenne-adicity. This conforms with the naive physical picture that the p-adic scales defined by Mersennes are in excellent approxiation n-ary 2-adic length scales.

   One should check whether the extension defined by $\sqrt{2}$ is somehow special as compared to the extensions defined by odd primes. Certainly the fact that this prime is the only even prime makes it rather special. It allows extension with $\sqrt{-1}$ and p-adic extension allowing all square roots except those of 2 is spanned by four square roots unlike similar extensions for other p-adic numbers fields which require only two square roots.

3. Suppose $D = p_1$ with $p_1 \ mod \ 4 = 1$. For $p = M_k$ quadratic resiprocity implies that the condition is equivalent with $M_k \ mod \ p_1 = x^2$. Neither the extensions $Q[\sqrt{p_1}]$ nor $Q[\sqrt{-p_1}]$ induce splitting of M_k for $p_1 \ mod \ 4 = 1$. For $M_3 = 7$ and $p_1 \in \{5, 13, 17\}$ no splitting of M_3 takes place but for $p_1 = 29$ splitting occurs. This suggests that there is no general rule guaranteing the stability of Mersenne primes in this case.

4. Suppose $D = p_1 \ mod \ 4 = 3$. One has $(4p_1, M_k) = (p_1, M_k)$ by the multiplicative character of the Legendre symbol. Quadratic resiprocity gives now $(p_1, M_k) = -(M_k, p_1)$ so that splitting occurs for M_k only if it does not occur for p_1. If splitting occurs for p_1 it does not occur for $-p_1$ and vice versa. $p_1 = 7$ and $M_2 = 3$ serve as a testing sample. One has $(3, 7) = 1$ so that the splitting of $M_2 = 3$ takes place for $Q[\sqrt{7}]$ but not for $Q(\sqrt{-7})$ and the splitting of $M_3 = 7$ takes place for $Q[\sqrt{-3}]$ but not for $Q(\sqrt{3})$.No obvious general rule seems to hold.

4.6 Generalized Feynman diagrams and adeles

The notion of Hilbert adeles seems to fit nicely with the recent view about generalized Feynman diagrams. The basic heuristic idea is the idea about fusion of physics in various number fields. p-Adic mass calculations lead to the conclusion that elementary particles are characterize by p-adic primes and inside hadron quarks obeying different effective or real p-adic topologies are present. One can speak about real and p-adic space-time sheets and real and p-adic spinors and also WCW has real and p-adic sectors. There is a hierarchy of algebraic extensions of rationals and presumably of also p-adic numbers. Even more general finite-dimensional extensions containing for instance Neper number e and its roots are also possible and involve extensions of p-adic numbers.

At the level of Feynman graphs this means that different lines correspond to different p-adic topologies and I have already proposed how this could give rise p-adic length scale hypothesis when the Feynman amplitudes in the tensor product of quantum variants p-adic number fields are mapped to reals by canonical identification [16]. Rational or even more general entanglement between different number fields would be essential.

The vertices of generalized Feynman diagrams for different incoming p-adic number fields could be multi-p p-adic objects in quantum sense involving powers expansions in powers of integer n decomposed to product of powers of quantum primes associated with its factors with coefficients not divisible by the factors. An alternative option is that vertices are rational numbers common to all number fields serving as entanglement coefficients. A third option is that they are real numbers in corresponding tensor factor. One should also formulate symmetries in p-adic sectors and the simplest option is that symmetries represented as affine transformations simply reduce to products of the symmetries in various p-adic sectors of the imbedding space.

The challenge is to formulate all this in a concise and elegant manner. It seems that adeles generalized to Hilbert adeles might indeed provide this formulation. The naive basic recipe would be extremely simple: whenever you have a real number, replace it with Hilbert adele. You can even replace the points of Hilbert spaces involved with corresponding Hilbert spaces! One could replace imbedding space, space-time surfaces, and WCW as well as imbedding space spinors and spinor fields and WCW spinors and spinor fields with the hierarchy of their Hilbert adelic counterparts obtaining in this manner what might be interpreted as cognitive representations.

5 Quantum Mathematics and Quantum Mechanics

Quantum Mathematics (QM) suggests that the basic structures of Quantum Mechanics (QM) might reduce to fundamental mathematical and metamathematical structures, and that one even consider the possibility that Quantum Mechanics reduces to Quantum Mathematics with mathematician included or expressing it in a concice manner: QM=QM!

The notes below were stimulated by an observation raising a question about a possible connection between multiverse interpretation of quantum mechanics and quantum mathematics. The heuristic idea of multiverse interpretation is that quantum state repeatedly branches to quantum states which in turn branch again. The possible outcomes of the state function reduction would correspond to different branches of the multiverse so that one could save keep quantum mechanics deterministic if one can give a well-defined mathematical meaning to the branching. Could quantum mathematics allow to somehow realize the idea about repeated branching of the quantum universe? Or at least to identify some analog for it? The second question concerns the identification of the preferred state basis in which the branching occurs.

Quantum Mathematics replaces numbers with Hilbert spaces and arithmetic operations + and $\times$ with direct sum $\oplus$ and tensor product $\otimes$.

1. The original motivation comes from quantum TGD where direct sum and tensor product are naturally assigned with the two basic vertices analogous to stringy 3-vertex and 3-vertex of Feynman graph. This suggests that generalized Feynman graphs could be analogous to sequences of arithmetic operations allowing also co-operations of $\oplus$ and $\otimes$.

2. One can assign to natural numbers, integers, rationals, algebraic numbers, transcendentals and their p-adic counterparts for various prime p Hilbert spaces with formal dimension given by the number in question. Typically the dimension of these Hilbert spaces in the ordinary sense is infinite. Von Neuman algebras known as hyper-finite factors of type II_1 assume as a convention that the dimension of basic Hilbert space is one although it is infinite in the standard sense of the word. Therefore this Hilbert space has sub-spaces with dimension which can be any number in the unit interval. Now however also negative and even complex, quaternionic and octonionic values of Hilbert space dimension become possible.

3. The decomposition to a direct sum matters unlike for abstract Hilbert space as it does also in the case of physical systems where the decomposition to a direct sum of representations of symmetries is standard procedure with deep physical significance. Therefore abstract Hilbert space is replaced with a more structured objects. For instance, the expansion $\sum_n x_n p^n$ of a

p-adic number in powers of p defines decomposition of infinite-dimensional Hilbert space to a direct sum $\oplus_n x_n \otimes p^n$ of the tensor products $x_n \otimes p^n$. It seems that one must modify the notion of General Coordinate Invariance since number theoretic anatomy distinguishes between the representations of space-time point in various coordinates. The interpretation would be in terms of cognition. For instance, the representation of Neper number requires infinite number of pinary digits whereas finite integer requires onlya finite number of them so that at the level of cognitive representations general coordinate invariance is broken.

Note that the number of elements of the state basis in p^n factor is p^n and $m \in \{0, ..., p-1\}$ in the factor x_n. Therefore the Hilbert space with dimension $p^n > x_n$ is analogous to the Hilbert space of a large effectively classical system entangled with the microscopic system characterized by x_n. p-Adicity of this Hilbert space in this example is for the purpose of simplicity but raises the question whether the state function reduction is directly related to cognition.

4. On can generalize the concept of real numbers, the notions of manifold, matrix group, etc... by replacing points with Hilbert spaces. For instance, the point $(x_1, .., x_n)$ of E^n is replaced with Cartesian product of corresponding Hilbert spaces. What is of utmost importance for the idea about possible connection with the multiverse idea is that also this process can be also repeated indefinitely. This process is analogous to a repeated second quantization since intuitively the replacement means replacing Hilbert space with Hilbert space of wave functions in Hilbert space. The finite dimension and its continuity as function of space-time point must mean that there are strong constraints on these wave functions. What does this decomposition to a direct sum mean at the level of states? Does one have super-selection rules stating that quantum inteference is possible only inside the direct summands?

5. Could one find a number theoretical counterpart for state function reduction and preparation and unitary time evolution? Could zero energy ontology have a formulation at the level of the number theory as earlier experience with infinite primes suggest? The proposal was that zero energy states correspond to ratios of infinite integers which as real numbers reduce to real unit. Could zero energy states correspond to states in the tensor product of Hilbert spaces for which formal dimensions are inverses of each other so that the total space has dimension 1?

5.1 Unitary process and state function reduction in ZEO

The minimal view about unitary process and state function reduction is provided by ZEO [2].

1. Zero energy states correspond to a superposition of pairs of positive and negative energy states. The M-matrix defining the entanglement coefficients is product of Hermitian square root of density matrix and unitary S-matrix, and various M-matrices are orthogonal and form rows of a unitary U-matrix. Quantum theory is square root of thermodynamics. This is true even at single particle level. The square root of the density matrix could be also interpreted in terms of finite measurement resolution.

2. It is natural to assume that zero energy states have well-defined single particle quantum numbers at the either end of CD as in particle physics experiment. This means that state preparation has taken place and the prepared end represents the initial state of a physical event. Since either end of CD can be in question, both arrows of geometric time identifiable as the Minkowski time defined by the tips of CD are possible.

3. The simplest identification of the U-matrix is as the unitary U-matrix relating to each other the state basis for which M-matrices correspond to prepared states at two opposite ends of CD. Let us assume that the preparation has taken place at the "lower" end, the initial state. State function reduction for the final state means that one measures the single particle observables for the "upper" end of CD. This necessarily induces the loss of this property at the "lower" end. Next preparation in turn induces localization in the "lower" end. One has a kind of time flip-flop and the breaking of time reversal invariance would be absolutely essential for the non-triviality of the process.

The basic idea of Quantum Mathematics is that M-matrix is characterized by Feynman diagrams representing sequences of arithmetic operations and their co-arithmetic counterparts. The latter ones give rise to a superposition of pairs of direct summands (factors of tensor product) giving rise to same direct sum (tensor product). This vision would reduce quantum physics to generalized number theory. Universe would be calculating and the consciousness of the mathematician would be in the quantum jumps performing the state function reductions to which preparations reduce.

Note that direct sum, tensor product, and the counterpart of second quantization for Hilbert spaces in the proposed sense would be quantum mathematics counterpart for set theoretic operations, Cartesian product and formation of the power set in set theory.

5.2 ZEO, state function reduction, unitary process, and quantum mathematics

State function reduction acts in a tensor product of Hilbert spaces. In the p-adic context to be discussed n the following $x_n \otimes p^n$ is the natural candidate for this tensor product. One can assign a density matrix to a given entangled state of this system and calculate the Shannon entropy. One can also assign to it a number theoretical entropy if entanglement coefficients are rationals or even algebraic numbers, and this entropy can be negative. One can apply Negentropy Maximization Principle to identify the preferred states basis as eigenstates of the density matrix. For negentropic entanglement the quantum jump does not destroy the entanglement.

Could the state function reduction take place separately for each subspace $x_n \otimes p^n$ in the direct sum $\oplus_n x_n \otimes p^n$ so that one would have quantum parallel state function reductions? This is an old proposal motivated by the many-sheeted space-time. The direct summands in this case would correspond to the contributions to the states localizable at various space-time sheets assigned to different powers of p defing a scale hierarhcy. The powers p^n would be associated with zero modes by the previous argument so that the assumption about independent reduction would reflect the super-selection rule for zero modes. Also different values of p-adic prime are present and tensor product between them is possible if the entanglement coefficients are rationals or even algebraics. In the formulation using adeles the needed generalization could be formulated in a straightforward manner.

How can one select the entangled states in the summands $x_n \otimes p^n$? Is there some unique choice? How do unitary process and state function reduction relate to this choice? Could the dynamics of Quantum Mathematics be a structural analog for a sequence of state function reductions taking place at the opposite ends of CD with unitary matrix U relating the state basis for which single particle states have well defined quantum numbers either at the upper or lower end of CD? Could the unitary process and state function reduction be identified solely from the requirement that zero energy states correspond to tensor products Hilbert spaces, which correspond to inverses of each other as numbers? Could the extension of arithmetics to include co-arithmetics make the dynamics in question unique?

5.3 What multiverse branching could mean?

Could QM allow to identify a mathematical counterpart for the branching of quantum states to quantum states corresponding to preferred basis? Could one can imagine that a superposition of states $\sum c_n \Psi_n$ in a direct summand $x_n \otimes p^n$ is replaced by a state for which Ψ_n belong to different direct summands and that branching to non-intefering sub-universes is induced by the proposed super-selection rule or perhaps even induces state function reduction? These two options seem to be equivalent experimentally. Could this decoherence process perhaps correspond to the replacement of the original Hilbert space characterized by number x with a new Hilbert space corresponding to number y inducing the splitting of $x_n \otimes p^n$? Could the interpretation of finite integers x_n and p^n as p-adic numbers $p_1 \neq p$ induce the decoherence?

This kind of situation is encountered also in symmetry breaking. The irreducible representation of a symmetry group reduces to a direct sum of representations of a sub-group and one has in practice super-selection rule: one does not talk about superpositions of photon and Z^0. In quantum measurement the classical external fields indeed induce symmetry breaking by giving different energies for the components of the state. In the case of the factor $x_n \otimes p^n$ the entanglement coefficients define the density matrix characterizing the preferred state basis. It would seem that the process of branching decomposes this state space to a direct sum 1-D state spaces associated with the eigenstates of the density matrix. In symmetry breaking superposition principle holds true and instead of quantum superposition for different orientations of "Higgs field" or magnetic field a localization selecting single orientation of the "Higgs field" takes place. Could state function reduction be analogous process?

Could non-quantum fluctuating zero modes of WCW metric apper as analogs of "Higgs fields". In this picture quantum superposition of states with different values of zero modes would not be possible, and state function reduction might take place only for entanglement between zero modes and non-zero modes.

5.4 The replacement of a point of Hilbert space with Hilbert space as a second quantization

The fractal character of the Quantum Mathematics is what makes it a good candidate for understanding the self-referentiality of consciousness. The replacement of the Hilbert space with the direct sum of Hilbert spaces defined by its points would be the basic step and could be repeated endlessly corresponding to a hierarchy of statements about statements or hierarchy of n^{th} order logics. The construction of infinite primes leads to a similar structure.

What about the step leading to a deeper level in hierarchy and involving the replacement of each point of Hilbert space with Hilbert space characterizing it number theoretically? What could it correspond at the level of states?

1. Suppose that state function reduction selects one point for each Hilbert space $x_n \otimes p^n$. The key step is to replace this direct sum of points of these Hilbert spaces with direct sum of Hilbert spaces defined by the points of these Hilbert spaces. After this one would select point from this very big Hilbert space. Could this point be in some sense the image of the Hilbert space state at previous level? Should one imbed Hilbert space $x_n \otimes p^n$ isometrically to the Hilbert space defined by the preferred state $x_n \otimes p^n$ so that one would have a realization of holography: part would represent the whole at the new level. It seems that there is a canonical manner to achieve this. The interpretation as the analog of second quantization suggest the identification of the imbedding map as the identification of the many particle states of previous level as single particle states of the new level.

2. Could topological condensation be the counterpart of this process in many-sheeted space-time of TGD? The states of previous level would be assigned to the space-time sheets topologically condensed to a larger space-time sheet representing the new level and the many-particle states of previous level would be the elementary particles of the new level.

3. If this vision is correct, second quantization performed by theoreticians would not be a mere theoretical operation but a fundamental physical process necessary for cognition! The above proposed unitary imbedding would imbed the states of the previous level as single particle states to the new level. It would seem that the process of second quantization, which is indeed very much like self-reference, is completely independent from state function reduction and unitary process. This picture would conform with the fact that in TGD Universe the theory about the Universe is the Universe and mathematician is in the quantum jumps between different solutions of this theory.

Returning to the motivating question: it seems that the endless branching of the states in multiverse interpretation cannot correspond to a repeated second quantization but could have interpretation as a decoherence identifiable as delocalization in zero modes. If state function is allowed, it corresponds to a localization in zero modes analogous to Higgs mechanism. The Quantum Mathematics realization for a repeated second quantization would represent a genuinely new kind of process which does not reduce to anything already known.

6 Speculations related to Hilbert adelization

This section contains further speculations related to realization of number theoretical universality in terms of Hilbert adeles and to the notion of number theoretic emergence. One can construct infinite hierarchy of Hilbert adeles by replacing the points of Hilbert spaces with Hilbert spaces repeatedly: this generalizes the repeated second quantization used to construct infinite primes and realizes also algebraic holography since the points of space have infinitely complex structure. There are strong restrictions on the values of coordinates of Hilbert space for the p-adic sectors of the adele and the number of state basis satisfying orthonormality conditions is very restricted: a good guess is that unitary transformations reduce to a permutation group and that its cyclic subgroup defines quantum Galois group. Also the Hilbert counterpart of real factor of adeles is present and in this case there are no such restrictions.

A logical use of terms is achieved if one refers by term "quantum Hilbert adele" to the adele obtained by replacing the Hilbert space coefficients $a_n < p$ of pinary expansions with their quantum Hilbert spaces. On the other hand the hierarchy of Hilbert adeles is very qunalta since it is analogous to a hierarchy of second quantizations so that Hilbert adeles could be also called quantum adeles. Reader can decide.

6.1 Hilbert adelization as a manner to realize number theoretical universality

Hilbert adelization is highly suggestive realization of the number theoretical universality. The very construction of adeles and their Hilbert counterparts is consistent with the idea that rational numbers are common to all completions of rationals. This suggests a generalization of the formalism of physics allowing to realize number theoretical universality in terms of adeles and their Hilbert counterparts. What this would mean the replacement of real numbers everywhere by adeles containing real numbers as one Cartesian factor. Field equations make sense for the adeles separately in each Cartesian factor.

If one can define differential calculus for the Hilbert reals and p-adics as seems to be the case, this abstraction might make sense. There seems to be no obvious objection for field property and the entire hierarchy of n-Hilbert spaces could be seen as a cognitive self-referential representation of the mathematical structure allowing perhap also physical realization if the structure is consistent with the general axioms.

Field equations would thus make sense also for an infinite hierarchy formed by Hilbert^n adeles. The fascinating conjecture is that quantum physics reduces to quantum mathematics and one might hope that TGD provides a realization for this physics because of its very strong ties with number theory.

6.1.1 Hilbert adelication at imbedding space level

The Hilbert adelization at the level of imbedding space makes senses if adelization works so that one can consider only adelization.

1. Could imbedding space coordinates regarded as adeles? In the p-adic sectors general coordinate invariance would require some preferred coordinate choices maybe unique enough by symmetry considerations. One can also consider a spontaneous breaking of GCI by cognitive representations. Adelization would code field equations in various p-adic number fields to single field equation for adeles and would not bring anything new.

2. What could field equations mean for Hilbert adeles? One could imagine that ordinary field equations as local algebraic statements are expressed separately at each point of space-time surface giving infinite number of equations of form $F^k(x) = 0$, where k labels imbeding space coordinates. Moving to the first level of hierarchy would mean that one replaces the points of Hilbert spaces involved with Hilbert spaces. The connection with the first order logic would suggest that the points of the Hilbert spaces representing points of imbedding space and space-time - in general infinite-dimensional for real and p-adic numbers - represent points of imbedding space and of space-time. This second quantization would transform infinite number of statements of predicate logic to a statement of first order logic.

   This certainly sounds hopelessly abstract and no-one would seriously consider solving field equations in this manner. But maybe mathematical thinking relying on quantum physics could indeed do it like this? At the next level of hierarchy one might dream of combining field equations for entire families of solutions of field equations to single equation and so on. Maybe these families could correspond to supports of WCW spinor fields in WCW. At the next level statements would be about families of WCW spinors fields and so on - ad infinitum. In fact, WCW spinors can be seen as quantum superpositions of logical statements in fermionic Fock space and WCW spinor fields would assign to WCW a direct sum of this kind of statements, one to each point of WCW.

This sounds infinitely infinite but one must remember that the sub-WCW consisting of surfaces expressible in terms of rational functions is discrete.

3. The conjecture that field equations reduce to octonion real-analyticity requires that octonions and quaternions make sense also p-adically. The problem is that the p-adic variants of octonions and quaternions do not form a field: the reason is that even the equation $x^2 + y^2 = 0$ can have solutions in p-adic number fields so that the inverses of quaternions and octonions, and even p-adic complex numbers need not make sense. The p-adic counterparts of quaternions and octonions however exist as a ring so that one could speak about polynomials and Taylor series whereas the definition of rationals and therefore rational functions would involve problems. Octonion real-analyticity and quaternion real-analyticity and therefore also space-time surfaces defined by polynomials or even by infinite Taylor series could make sense also for the p-adic variants of octonions and quaternions.

Could imbedding space spinors be regarded as adelic and even Hilbert adelic spinors? Again the problems reduce to the adelic level.

1. Adelization could be perhaps seen as a convenient book keeping device allowing to encapsulate the infinite number of physics in various quantum p-adic number fields to single physics. Hilbert adelic structures could however provide much deeper realization of physics as generalized number theory. One can indeed ask whether the action of the p-adic quantum counterparts of various symmetries could representable in the quantum quantum Galois groups for Hilbert adeles: these groups might reduce to cyclic groups and might relate to cyclic coverings of imbedding space at the level of physics.

   The minimal interpretation would be as a cognitive representation of quantum numbers of physical states at the first "material" level of hierarchy using the number theoretic Hilbert space anatomy of the point to achieve the representation. The representative capacity would be infinite for transcendental numbers with infinite number of pinary digits and finite for rational numbers. For real unit if would be miminal and zero could not represent anything. Quantum entanglement would be possible for tensor product coefficients and quantum superposition would be possible due to direct sum of pinary digits.

2. Imbedding space spinor fields could be regarded as Cartesian products (direct sums) of spinor fields in real and various p-adic imbedding spaces having values in the same number field. Also the induced metric and spinor connection would correspond to Cartesian product rather than tensor product. The isometries of the imbedding space would have matrix representation in terms of adeles on the adelic omponents of spinors and imbedding space coordinates.

6.1.2 Hilbert adelication at the level of WCW

What about quantum TGD at the level of WCW? Could Hilbert adelication apply also at this level? Could one use the same general recipes to adelize? The step from adele to the hierarchy of Hilbert adeles does not seem to be a conceptual problem and the basic problem is to understand what adele means.

1. Could WCW described in terms of generalized number theory? Could adelic WCW be defined as the Cartesian product of real WCW and p-adic WCWs? The observations about dessins d'enfant [2] [19] suggest that the description of WCW could be reduced to the description in terms of orbits of algebraic 2-surfaces identified as partonic 2-surfaces at the boundaries of CDs (also the 4-D tangent space data at them codes for physics).

2. For a Cartesian product of finite-dimensional spaces spinors are formed as tensor products associated with with the Cartesian factors. Adelic WCW is Cartesian sum of real and p-adic variants. Could Hilbert adelic WCW spinors be identified as a tensor product of WCW spinors defined in the Hilbert adelic variant of WCW. This would conform with the physical vision that real and p-adic physics (matter and cognition) correspond to tensor factors of a larger state space. Furthermore, spinos generalizes scalar functions and the function space for adele valued functions with adelic argument forms in a natural manner tensor product of function spaces for various completions of reals. Note that one can speak about rational quantum entanglement since rational numbers are common to all the Cartesian factors.

3. Could also the moduli space of conformal equivalence classes of partonic 2-surfaces be regarded as adele in the sense that Teichmueller parameters from adele. This requires that the Teichmueller space of conformal equivalence classes of Riemann surfaces corresponds to the p-adic version of real Teichmueller space: this has been actually assumed in p-adic mass calculations [5, 9].

   One could start from the observation that algebraic Riemann surfaces are dense in the space of all Riemann surfaces. This means that the algebraic variant of Teichmueller space is able to characterize the conformal equivalence classes. What happens when one adds the Riemann surfaces for which the coefficients of the Belyi function and rational functions defining are allowed to be in real or p-adic completion of rationals. A natural guess is that completion of the algebraic variant of Teichmueller space results in this manner. If this is argument makes sense then adelic moduli space makes sense too.

   There are however technical delicacies involved. Teichmueller parameters are defined as values of 1-forms for the homology generators of Riemann surface. What does one mean with the values of these forms when one has a surface containing only algebraic points and ordinary integral is not well-defined? Also in the p-adic context the definition of the integral is problematic and I have devoted a lot of time and energy to this problem (see for instance [17]). Could the holomorphy of these forms help to define them in terms of residue calculus? This option looks the most plausible one.

   What about the partial well-ordering of p-adic numbers induced by the map $n \rightarrow n_q$ combined with canonical identification: could this allow an elegant notion of integration by using the partial well-ordering. Note that one cannot say which of the numbers 1 and $-(p-1)\sum n = 1^{\infty} p^n$ is bigger in this ordering, and this induces similar problem for all p-adic integers which have finite number of pinary digits.

6.1.3 Problems to solutions and new questions

Usually one becomes fully conscious of a problem only after one has found the solution of the problem. The vision about Hilbert adeles - as a matter fact, already adeles- solves several nasty nuisances of this kind and I have worked hardly to prevent these problems from running off under the rug.

1. What one means with integer -1 is not a problem for p-adic mathematics. It becomes a problem for physical interpretation when one must relate real and p-adic physics to each other since canonical identification maps p-adic numbers to non-negative reals. This leads to problems with Hilbert space inner product but algebraic extensions of p-adic numbers by roots of unity allow to define p-adic Hilbert spaces but it seems that the allowed state basis are very restricted since the number of unitary isometries of Hilbert space is restricted dramatically by number theoretical existence requirement. The optimistic interpretation would that full quantum superposition is highly restricted in cognitive sectors by the condition of number theoretic existence.

2. What one means with complex p-adics is second problem. $\sqrt{-1}$ exists p-adically for $p \ mod \ 4 = 1$ so that one cannot introduce it via algebraic extension of p-adics in this case. This is a problem of p-adic quantum mechanics. Allowance of only p-adic primes p which do not split for the extension containing imaginary unit seems to be a general solution of the problem.

3. p-Adic counterparts of quaternions, and octonions do not exist for the simple reason that the p-adic norm can be vanishing even for p-adic complex number for p-adic fields allowing $\sqrt{-1}$. This problem can be circumvented by giving up the requirement that one has number field.

4. The norm for adeles exist as a product of real and norm and p-adic norms but is not physical. Also the assignment of Hilbert space structure to adeles is problematic. Canonical identification combined with $n \rightarrow n_q$ allows the mapping p-adic components of adele to real numbers and this allows to define natural inner product and norm analogous to Hilbert space norm for adeles and their Hilbert counterparts.

5. p-Adic numbers are not well ordered. This implies that difficulties with the definition of integral since definite integral relies heavily on well-orderness of reals. Canonical identification suggests that quantum p-adics are well ordered: $a < b$ holds true if it holds true for the images under canonical identification. This gives hopes about defining also definite integral. For integrable

functions the natural definition of quantum p-adic valued integral would be by using substitution for integral function. One - and rather ugly - option is to define the integral as ordinary real integral for the canonical image of the quantum p-adic valued function. This because this image is not expected to be smooth in real sense even if p-adic function is smooth.

6. p-Adic integration is plagued also by the problem that already for rational integrals one obtains numbers like $log(n)$ and π and is forced to introduce infinite-dimensional extension of p-adic numbers. For $log(n)$ one could restrict the consideration to p-adic primes p satisfying $n\ mod\ p = 1$ but this looks like a trick. Could this difficulty be circumvented somehow for p-adic numbers? The only possibility that one can imagine would be canonical identification map combined with $n \rightarrow n_q$ and the interpretation of integral as a real number.

   This could provide also the trick to interpret the integrals involving powers of π possible emerging from Feynman diagrams in sensible manner. All integrals can be reduced with the use of Laurent series to integrals of powers of x so that integral calculus would exist in analytic sense for analytic functions of quantum p-adic numbers.

7. What does one mean with the p-adic counterpart of CP_2 or more generally, with the p-adic counterpart of any non-linear manifold? What does one mean with the complex structure of p-adic CP_2 for $p\ mod\ 4 = 1$? Should one restrict the consideration to $p\ mod\ 4 = 3$? What does one mean with groups and coset spaces? One can inceed have a satisfactory looking definition based on algebraic extensions and effective discretization by introducing roots of unity replacing complex phases as continuous variables [17].

   One could consider two options.

   (a) Could the p-adic counterpart of real $M^4 \times CP_2$ be M^8? The objection is that algebraic groups are however fundamental for mathematics and typically non-linear manifolds. Therefore there are excellent motivations for their (Hilbert) adelic existence. Projective spaces are in turn central in algebraic geometry and in this spirit one might hope that CP_2 could have non-trivial p-adic counterpart defined as quantum p-adic projective space.

   (b) Another option accepts that adeles contain only those p-adic number fields as Cartesian factors for which the prime does not split. This excludes automatically $p\ mod\ 4 = 1$ if $\sqrt{-1}$ is present from the beginning in the algebraic extension of rationals defining the adeles. What happens if one does not assume this. Does CP_2 degenerate to real projective spae RP_2? What happens to M^4 if regarded as a Cartesian product of hyper-complex numbers and complex numbers. Does it reduce to M^2. Could the not completely well understood role of M^2 in quantum TGD relate to this kind of reduction?

The new view raises also questions challenging previous basic assumptions.

1. Could adeles and their octonionic counterpart allow to understand the origin of commutative complexification for quaternions and octonions in number theoretic vision about TGD? How could the commutative imaginary unit emerge number theoretically?

2. One must also reconsider $M^8 - M^4 \times CP_2$ duality. For instance, could M^8 be the natural choice in p-adic sectors and $M^4 \times CP_2$ in the real sector?

3. The preferred extremals of Kähler action are conjectured to be quaternionic in some sense. There are two proposals for what this means. Could it be that the sense in which the space-time surfaces are quaternionic depends on whether the surface is real or quantum p-adic?

4. The idea that rationals are in the intersection of reals and p-adics is central in the applications of TGD. How does this vision change? For $p = 2$ quantum rationals in the sense that pinary coefficients are quantum integer, are ordinary rational numbers. For $p > 2$ the pinary coefficients are in general mapped to algebric numbers involving l_q, $0 < l < p$. The common points with reals would in general algebraic numbers.

6.1.4 Do basic notions require updating in the Hilbert adelic context?

In the adelic context one must take a fresh look to what one means with phrases like "imbedding space" and "space-time surfaces". The phrase "space-time surface as a preferred extremal of Kähler action" might be quite too strong a statement in adelic context and could actually make sense only in the real sector of the quantum adelic imbedding space. Also the phrase "p-adic variant of $M^4 \times CP_2$" might involve un-necessarily strong implicit assumptions since for p-adic integers one has automatically the counterparts of compactness even for M^8. The proposed identification of the quantum p-adic numbers as Hilbert p-adic quantum numbers reduces the question to whether p-adic counterparts of various structures exist or are needed as such.

1. We "know" that the real imbedding space must be $M^4 \times CP_2$. What about p-adic counterpart of the imbedding space? Is it really possible to have a p-adic counterpart of CP_2 or could non-linearity destroy this kind of hopes? Are there any strong reasons for having the counterpart of $M^4 \times CP_2$ in p-adic sectors? Could one have $M^4 \times CP_2$ only in real sector and M^8 in p-adic sectors. Complex structure of CP_2 requires $p \ mod \ 4 = 3$. This is not a problem if one assumes that adeles contain only the p-adic primes which do not split in the extension of rationals containing imaginary unit. Definition as coset space $CP_2 = SU(3)/U(2)$ is one possible manner to proceed and seems to work also.

   One can also wonder whether octonion real-analyticity really makes sense for $M^4 \times CP_2$ and its p-adic variants. The fact that real analyticity makes sense for S^2 suggests that it does. In any case, octonion real-analyticity would make life very easy for p-adic sectors if regarded as octonionic counterpart of M^8 rather than $M^4 \times CP_2$.

2. If the p-adic factors are identified as linear spaces with M^8 regarded as sub-space of the ring of complexified p-adic octonions, octonion real-analyticity for polynomial functions with rational coefficients could replace field equations in the ring formed by Z_p. Note however that octonion real-analyticity requires the Wick rotation mapping to ordinary octonions, the identification of the 4-surface from the vanishing of the imaginary part of the octonion real-analytic function, and map back to Minkowski space by Wick rotation. This is well-defined procedure used routinely in quantum field theories but could be criticized as mathematically somewhat questionable. One could consider also the definition of Minkowski space inner product as real part of $z_1 z_2$ for quaternions and use similar formula for octonions. This would give Minkowski norm squared for $z_1 = z_2$.

   Linear space would also allow to realize the idea that partonic 2-surfaces are in some sense trivial in most sectors reducing to points represented most naturally by the tips of causal diamonds (CDs). For p-adic sectors CP_2 would be replaced with E^4 and for most factors M_p^8 the partonic 2-surfaces would reduce to the point $s = 0$ of E^4 representing the origin of coordinates in which E^4 rotations act linearly.

3. The conjecture is that preferred extremals correspond to loci for the zeros of the imaginary or real part of octonion real-analytic function. Is this identification really necessary? Could it be that in the *real* sector the extremals correspond to quaternionic 4-surfaces in the sense that they have quaternionic tangent spaces? And could the identification as loci for the zeros of the imaginary or real part of oction real-analytic function be the sensible option in the p-adic sectors of the adelic imbedding space: in particular if these sectors correspond to octonionic M^8. If this were the case, $M^8 - M^4 \times CP_2$ duality would have a meaning differing from the original one and would relate the real sector of adelic imbedding space to its p-adic sectors in manner analogous to the expression of real rational as a Cartesian product of powers of p-adic primes in various sectors of adele.

My cautious conclusion is that the earlier vision is correct: $M^4 \times CP_2$ makes sense in all sectors.

6.2 Could number theoretic emergence make sense?

The observations made in this and previous sections encourage to ask whether some kind of number theoretic emergence could make sense. One would end up step by step from rationals to octonions by performing algebraic extensions and completions. At some step also the attribute "Hilbert" would lead to a further abstraction and relate closely to the evolution of cognition. This would mean something like follows.

Rationals → algebraic extensions → algebraic numbers → completions of rationals to reals and p-adics → completions of algebraic 2-surfaces to real and p-adic ones in algebraic extensions reals and classical number fields→ hierarchy of Hilbert variants of these structures as their cognitive reprsentations.

The Maximal Abelian Galois group (MAGG) for rationals is isomorphic to the multiplicative group of ideles and involves reals and various p-adic number fields. How could one interpret the Hilbert variant of this structure. Could some kind of physical and cognitive evolution lead from rationals to octonions and eventually to Universe according to TGD? Could it be that the gradual emergence of algebraic numbers and AGG (Absolute Galois Group defined as Galois group of algebraic numbers as extension of rationals) brings in various completions of rationals and further extensions to quaternions and octonions and symmetry groups like SU(2) acting as automorphisms of quaternions as extension of reals and $SU(3) \subset G_2$ where G_2 acts as Galois for the extension of octonions as extension of reals?

6.2.1 Objections against emergence

The best manner to develop a new idea is by inventing objections against it. This applies also to the notion of algebraic emergence. The objections actually allow to see the basic conjectures about preferred extremals of Kähler action in new light.

1. Algebraic numbers emerge via extensions of rationals and complex numbers via completion of algebraic numbers. But can higher dimensions really emerge? This is possible but only when they correspond to those of classical number fields: reals, quaternions, and octonions. This is enough in TGD framework. Adelization could lead to the emergence of real space-time and its p-adic variants. Completion of solutions of algebraic equations to p-adic and real number fields is natural. Also the extensions of reals and complex numbers to quaternions and octonions are natural and could be seen as emergence.

2. All algebraic Riemann surfaces are compact but the reverse of this does not hold true. Partonic 2-surfaces are fundamental in TGD framework. Once the induced metric of the compact partonic 2-surface is known, one can regard it as a Riemann surface. Only if it is algebraic surface, the action of Galois group on it is well-defined as an action on the algebraic coefficients appearing in rational functions defining the surface. This is consistent with the basic vision about life as something in the intersection of real and p-adic worlds and therefore having as correlates algebraic partonic 2-surfaces. The non-algebraic partonic 2-surfaces are naturally present and if they emerge they must do so via completion to reals occurring also at adelic level.

   All partonic 2-surfaces allow a representation as projective varieties in CP_3 which forces again the question about possible connection with twistors.

   Representation as algebraic projective varieties in say CP_3 does not imply this kind of representation in $\delta CD \times CP_2$. This kind of representation can make sense for 3-surfaces consisting of light like geodesics emanating from the tip of the CD. If one wants to obtain 2-surfaces one must restrict light-like radial coordinate r to be a real function of complex variables so that the 2-surface cannot be algebraic surface defined as a null locus of holomorphic functions unless r is taken to be a constant equal to algebraic number. Note that the light rays of 3-D light-cone are parametrized by S^2, which corresponds to $CP_1 \subset CP_3$. This kind of partonic 2-surfaces might correspond to maxima for Kähler function.

3. Could one do without the non-algebraic partonic 2-surfaces? This is not the case if one believes on the notion of number theoretic entanglement entropy which can be negative for rational or even algebraic entanglement and presumably also for its quantum variant. Non-algebraic partonic 2-surfaces would naturally correspond to reals as a Cartesian factor of adeles. All partonic 2-surfaces which do not allow a representation as algebraic surfaces would belong to this factor of adelic imbedding space. The ordinary real number based physics would prevail in this sector and entanglement in this sector would be in generic case real so that ordinary definition of entropy would work. In quantum p-adic sectors entanglement probabilities would

be quantum rational (in the sense of $n \to n_q$) and the generalization of number theoretic entanglement entropy should make sense. Completion must be taken as would be part of the emergence.

Could imbedding space spinors really emerge? The dimension of the space of imbedding space spinors is dictated by the dimension of the imbedding space. Therefore it is difficult to image how 8+8-complex-dimensional spinors could emerge from spinors in the set of algebraic numbers since these spinors are naturally 2-dimensional for algebraic numbers which are geometrically 2-dimensional. Does this mean that one must introduce algebraic octonions and their complexifications from the very beginning? Not necessarily.

1. The idea that also the imbedding space spinors emerge algebraically suggests that imbedding space spinors in p-adic sectors are octonionic (p-adic octonions form a ring but this might be enough). In real sector both interpretations might make sense and have been considered [15]. For octonionic spinors ordinary gamma matrices are replaced with the analogs of gamma matrices obtained as tensor products of sigma matrices having quaternionic interpretation and of octonionic units. For these gamma matrices SO(1,7) as vielbein group is replaced with G_2. Physically this corresponds to the presence of a preferred time direction defined by the line connecting the tips of CD. It would seem that $SO(1,7)$ must be assigned with the ordinary imbedding space spinors assignable to the reals as a factor of quantum adeles. The relationship between the ordinary and octonionic imbedding space spinors is unclear. One can however ask whether the p-adic spinors in various factors of adelic spinors could correspond to the octonionic modification of gamma matrices so that these spinors would be 1-D spinors algebraically extended to octonionic spinors.

2. Also quaternionic spinors make sense and could emerge in a well-defined sense. The basic conjecture is that the preferred extremals of Kähler action are quaternionic surfaces in some sense. This could mean that the octonionic tangent space reduces to quaternionic one at each point of the space-time surface. This condition involves partial derivatives and these make sense for p-adic number fields . The "real" gamma matrices would be ordinary gamma matrices. In p-adic sectors at least octonion real-analyticity would be the natural condition allowing to identify quaternionic 4-surfaces [14] if one allows only Taylor series expansions.

6.2.2 Emergence of reals and p-adics via quantum adeles?

MAGG (Maximal Abelian Galois Group) brings in reals and various p-adic number fields although one starts from algebraic numbers as maximal abelian extension of rationals. Does this mean emergence?

1. Could one formulate the theory by starting from algebraic numbers? The proposal that octonion real-analytic functions can be used to define what quaternionicity looks sensible for quantum p-adic space-time surfaces. For real space-time surfaces octonion real-analyticity might be an unrealistic condition and quaternionicity as the condition that octonionic gamma matrices generate quaternionic algebra in the tangent space looks more plausible alternative. Quantum p-adic space-time surfaces would be naturally algebraic but in real context also non-algebraic space-time surfaces and partonic 2-surfaces are possibe. In real sector partial differential equations would prevail and in quantum p-adic sectors algebraic equations would dictate the dynamics.

2. The p-adic variants of quaternions and octonions do not exist as fields. The vanishing of the sum of Euclidian norm for quaternions and octonions for p-adic octonions and quaternions makes it impossible to define p-adic quaternion and octonionic fields. There are also problems due to the fact that $\sqrt{-1}$ exists as p-adic number for $p \ mod \ 4 = 1$.

3. The notion of quaternionic space-time surface requires complexified octonions with additional imaginary unit i commuting with octonionic imaginary units I_k. Space-time surfaces are identified as surfaces in the sub-space of complexified octonions of form $o_0 + i \sum o_k I^k$. Could i relate to the algebraic extensions of rationals and could complexified quantum p-adic imbedding spaces have complex coordinates $x + iy$?

4. Polynomial equations with real algebraic coefficients make sense even if adeles where not a field and one can assign to the roots of polynomials with quaternionic and octonionic argument Galois group if one restricts to solution which reduce to complex solutions in some complex plane defined by preferred imaginary unit. For quaternions Galois group consist of rotations in SO(3) acting via adjoint action combined with AAG. For octonions Galois group consists of G_2 elements combined with AAG. $SU(3)$ leaves the preferred imaginary unit invariant and $U(2)$ the choice of quaternionic plane. Are there any other solutions of polynomial equations than those reducing to complex plane?

6.2.3 Is it really necessary to introduce p-adic space-time sheets?

The (Hilbert) adelization of imbedding space, space-time, and WCW as well as spinors fields of imbedding space and WCW would be extremely elegant manner to realize number theoretic universality. One must however keep the skeptic attitude. The definition of p-adic imbedding space and space-time surfaces is not free of technical problems. The replacement of $M^4 \times CP_2$ with M^8 in p-adic sectors could help solve these problems. The conservative approach would be based on giving up p-adicization in imbedding space degrees of freedom. It is certainly not an imaginative option but must be considered as a manner to gain additional insights.

1. p-Adic mass calculations do not mention anything about the p-adicization of space-time sheets unless one wants to answer the question what is the concrete realizations of various conformal algebras. Only p-adic and adelic interpretation of conformal weights would be needed. Adelic interpretation of conformal weights makes sense. The replacement $n \rightarrow n_q$ (interpreted originally as quantum p-adicization) brings in only $O(p^2)$ corrections which are typically extremely small in elementary particle scales.

2. Is the notion of p-adic or Hilbert p-adic (Hilbert adelic) spinor field in imbedding space absolutely necessary? If one has p-adic spinors one must have also p-adic spinor connection. This does not require p-adic imbedding space and space-time surface if one restricts the consideration to algebraic points and if the components of connection are algebraic numbers or even rational numbers and allow p-adic interpretation. This assumption is however in conflict with the universality of adelization.

3. What about Hilbert adelic WCW spinor fields. They are needed to give both p-adic and real quantum states. These fields should have adelic values. Their arguments could be algebraic partonic surfaces. There would be no absolute need to perform completions of algebraic partonic 2-surfaces although this would be very natural on basis of number theoretical universality.

4. p-Adic space-time sheets are identified as correlates of intention and cognition. Transformation of intention to action as leakage from p-adic to real sector of imbedding space. This idea provides strong support for p-adic space-time. But could one assume only that the quantum states are p-adic or quantum p-adic but that space-time is real? Does it mean only that the WCW spinor field or zero energy state assignable to light-like 3-surface or partonic 2-surface is Hilbert adelic. Quantum transitions between states for which initially WCW spinor field is p_i-adic and in the final state p_f-adic. Only the number field for WCW spinors would change in the transition. One could say that partonic 2-surface is p-adic if the value of WCW spinor field assigned with it is p-adic. This idea does not look attractive and is in complete conflict with the adelization idea.

5. The vision about life in the intersection of real and p-adic worlds is very attractive. The p-adicization of algebraic surfaces is very natural as completion meaning that one just solves the algebraic equations using series in powers of p. Imaginary unit is key number of quantum theory and the fact that $\sqrt{-1}$ exists for $p \ mod \ 4 = 1$ is potential problem for p-adic quantum mechanics. For these primes also splitting occurs in the ring of Gaussian integers. For quantum adeles this problem disappears if one allows only the p-adic number fields for which p does not slit in algebraic extension (now Gaussian rationals).

7 Appendix: Some possibly motivating considerations

The path to the idea that quantum adeles could represent algebraic numbers originated from a question having no obvious relation to quantum p-adics or quantum adeles and I will proceed in the following by starting from this question.

Function fields are much simpler objects to handle than rationals and their algebraic extensions. In particular, the objects of function fields have inverses and inverse is well defined also for sum of elements. This is not true in the ring of adeles. This is the reason why geometric Langlands is easier than the number theoretic one. Also the basic idea of Langlands correspondence is that it is possible to translate problems of classical number theory (rationals and their extensions) to those involving functions fields. Could it be possible to represent the field of rationals as a function field in some sense? Quantum arithmetics gives a slight hope that this might be possible.

7.1 Analogies between number theoretic and function field theoretic ramification

Consider first the analogies between number theoretic and geometric ramification (probably trivialities for professionals but not for a physicist like me!). The relationship between number theoretic and geometric ramification is interesting and mathematician could of course tell a lot about it. My comments are just wonderings of a novice.

1. The number theoretic ramification takes place for the primes of number field when it is extended. If one knows the roots of the polynomials involved with the rational function $f(z)$ defining Belyi function one knows the coefficient field F of polynomial and its algebraic extension K and can deduce the representations of ordinary primes as products of those of F and of the primes of the coefficient field F as products of those of K. In particular, one can find the ramified primes of ordinary integers and of integers of F.

2. The ramification however occurs also for ordinary integers and means that their decomposition to primes involves higher powers of some primes: $n = \prod_l l^{e_l}$ with $e_l > 1$ for some primes l dividing n. Could one introduce an extension of some ring structure in which ordinary primes would be analogous to the primes in the extension of rationals?

3. Geometric ramification takes place for polynomials decomposing to products of first order monomials $P(z) = z - z_k$ with roots which are in algebraic extension of coefficients. The polynomials can however fail to be irreducible meaning that they have multiple roots. For multiple roots one obtains a ramified zero of a root and for Belyi functions these critical points correspond to zeros which are ramified when the degree is larger than zero. The number theoretic ramification implies that the polynomials involved have several algebraic roots and when they coincide, a geometric ramification takes place. Degeneration of roots of polynomial implies ramification.

4. Ordinary integers clearly correspond to the space of polynomials and the integers, which are not square free are analogous to polynomials with multiple roots. The ramification of prime in the extension of rationals and also the appearance of higher powers of p in non-square free integer is analogous to the degeneration of roots of polynomial.

7.2 Could one assign analog of function field to integers and analogs prime polynomials to primes?

Could one assign to integer (prime) a map analogous to (prime) polynomial? Prime polynomial can be labeled by its zero and polynomial by its zeros. What kind of maps could represent ordinary primes and integers. What could be the argument of this kind of maps and do zeros of these map label them? What could be the ring in which the counterparts of polynomials are defined?

Could quantum arithmetics [18] help to answer these questions?

1. Quantum arithmetics involves the map $f_q : n = \prod_{l|n} l^{e_l} \rightarrow n_q = \prod_{l|n} l_q^{e_l}$, where l are primes in the prime decomposition of n and quantum primes s $l_q = (q^l - q^{-ln})/(q - q^{-1})$ are defined by the phase $q = exp(i\pi/p)$, where p is the preferred prime. Note that one has $p_q = 0$ and

$(p+1)_q = -1$. Note also that one has $q = exp(i\pi/p)$ rather than $q = exp(i2\pi/p)$ (as in the earlier version of article). This is necessary to get the denominator correctly also for $p = 2$ and to make quantum primes l_q non-negative for $l < p$. Under $n \rightarrow n_q$ all integers n divisible by p are mapped to zero. This would suggest that the counterparts of prime polynomials are the maps f_q, $q = q_p$ and that the analogs of polynomials are products $\prod_p f_{q_p}$ defined in some sense.

2. The more conventional view about quantum integers defines analogous map as $n \rightarrow n_q = (q^n - q^{-n})/(q - q^{-1})$. Choosing $q = exp(i\pi/p)$ one finds also now that integers divisible by p are mapped to zero. By finding the primes for which n is mapped to zero one finds the prime decomposition of n. Now one does not however have a decomposition to a product of quantum primes as above. Similar statement is of course true also for the above definition of quantum decomposition: the maps $n \rightarrow n_q$ are analogous to polynomials and primes are analogous to the zeros of these polynomials.

3. One can also consider $q = exp(i\pi/m)$ and used decomposition primes which are smaller than m. This would give non-vanishing quantum integers. They would correspond to quantum q-adicity with $q = m$ integer: q-adic numbers do not form a field. q could be even rational. As a special case these numbers give rise to multi-p p-adicity. The Jones inclusions of hyperfinite factors of type II_1 [6] suggests that also these quantum phases should be considered. The index $[M : N] = 4cos^2(2\pi/n)$ of the inclusion would correspond to quantum matrix dimension 2_q^2, for $q = exp(i\pi/n)$ corresponding to quantum 2-spinors so that quantum dimension p_q could be interpreted as dimension of p-dimensional quantum Hilbert space.

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