Intentionality, Cognition, and Physics as Number theory or Space-Time Point as Platonia

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Abstract

In this chapter a braid of ideas inspired by the work with topological quantum computation and ideas about mathematical cognition is discussed.

a) The first bundle of ideas relates to quantum TGD and emerged when I learned about braid groups and type II_1 factors of von Neumann algebras. The connection with infinite-dimensional Clifford algebra of configuration space led to the idea about the realization of quantum geometry at the level of configuration space without non-commutative coordinates. Classical quantum correspondence led to very concrete view about how join along boundaries bonds defining braids serve as correlates for quantum bound state formation and bound state entanglement. Even a physicist's "proof" of four-color theorem emerges as an outcome.

The non-integer quantization of dimensions for effective II_1 tensor factors implies quantization of Planck constant: the values of h are given $h(n) = [2log(2)/log(r)] \times h$, where r is the dimension of the effective II_1 tensor factor. The spectrum of r contains continuum $r \geq 4$ and discrete spectrum $r = B_n = 4cos^2(\pi/n)$ $(n \geq 3)$ below it. B_n is so called Beraha number. The interpretation in terms of discrete bound states and continuum of unbound states is a suggestive, and in fact a sensible, interpretation. For $n = 3$ with $r = 1$ Planck constant becomes infinite and this corresponds to extremely quantal regime. Planck constant would approach zero at the limit $r \to \infty$. TGD approach mildly suggests that $r \geq 4$ is not possible.

b) Physics as a generalized number theory is the most ambitious dream inspired by TGD approach. The dimensions 4 and 8 for space-time and imbedding space led for years ago to the idea that space-time surface is in some sense maximal associative (that is quaternionic) sub-manifold of octonionic space. The problem was to understand why the compactification of octonion space to $H = M^4_+ \times CP_2$ does mean and here the observation that CP_2 parameterizes different quaternionic planes in the space of octonions is crucial.

c) The idea about p-adic physics as physics of cognition and intentionality inspired the generalization of a number concept so that reals and various extensions of p-adic numbers are glued together along common rationals and form a book like structure with the rim of book being represented by rationals. This in turn inspired the vision that p-adicization of physics should correspond to an algebraic continuation of rational physics to various number fields.

d) p-Adic continuation leads typically to the need of finite-dimensional extensions of p-adic numbers and also transcendental extensions are needed (e^p is ordinary p-adic number and defines finite-dimensional transcendental extension). This inspired the idea that the evolution of mathematical consciousness corresponds to the gradual increase of

both p and dimension of extension of p-adics and inspired various number theoretical conjectures relating different transcendentals to each other. The first form of these conjectures turned out to be wrong but in this chapter the conjectures are already much more realistic being minimal conjectures guaranteing the universality of physics.

The p-adication problem led to quite unexpected developments in the understanding of space-time correlates of mathematical cognition. The challenge is to locate the Platonia of mathematical ideas at space-time level, that is to identify space-time correlates of algebraic structures, manipulations and equations. The arguments evolved in the following manner.

a) The p-adicization of the vacuum functional defined as an exponent of Kähler function requires that the exponent proportional to the inverse of Kähler coupling constant converges for most primes p. The observation that this is highly improbable led the question whether infinite primes might be of help in the problem. The notion of infinite primes, integers and rationals generalized to complex, quaternionic and octonionic case was one of the first deep ideas inspired by TGD inspired theory of consciousness. The basic observation was that infinite primes correspond to quantum states of an arithmetic quantum field theory second quantized repeatedly.

The key observation was that the multiplication of inverse Kähler coupling strength with a power of quantity $Y = X/(1+X)$, where X is defined as product of all finite primes, and by powers of more general quantities $Y(n/m) = (n/m)X/Q(n/m)$ (n is integer and m square free integer), where $\Pi(n/m)$ is infinite rational solves the algebraic continuation problem. In real sense the numbers Y are units but p-adically their p-adic norm is $1/p$ for all primes except those dividing n and m.

b) The unit-in-real-sense property means that these numbers define an infinite-dimensional extension of rational numbers differing from ordinary rationals in no manner in the real context. The conclusion is that in TGD Universe space-time and imbedding space points are like the monads of Leibniz having infinitely complex structure. Since infinite primes, and their complex, quaternionic, and octonionic counterparts can represent quantum states of entire Universe, Universe is an algebraic hologram in the strongest sense that one can imagine.

c) Since this structure is not visible at the level of real number based physics, the interpretation is as space-time correlate for mathematical cognition. The free algebra generated by products and sums of infinite primes can be seen as the Mother of All Algebraic Structures allowing representation of any smaller structure. In particular, quantum states and quantum entanglement is representable and all kinds of algebraic rules can be represented using entanglement of the algebra elements representing algebra elements.

d) The paths of points in space-time define paths in this algebra and p-adic continuity for all primes implies the conservation of topological energy encountered in arithmetic quantum field theories and implying that the ordinary rational defined by the algebra element is a constant of motion. These paths are correlates for algebraic manipulations forming themselves an algebra with respect to local multiplication (analogous to gauge group multiplication).

A hierarchy of paths (paths of $d = 1$ objects, $d = 2$ and $d = 3$ objects) defining $d-$dimensional surfaces giving rise to analogs of local gauge algebras result. In TGD Universe the maximal dimension of this kind of structure is $d = 4$. This probably has some deep algebraic meaning. At classical level these structures have interpretation as abstractions of the rules obeyed by algebraic manipulations using a collection of examples defined by a pile of d-dimensional manipulation sequences. The algebraic manipulations correspond to Feynmann diagram like structures with algebra and co-algebra operations having particle fusion and creation as their physical analogs.

1 Introduction

In this chapter a braid of ideas inspired by the work with topological quantum computation and ideas about mathematical cognition is discussed.

a) The first bundle of ideas relates to quantum TGD and emerged when I learned about braid groups and type II_1 factors of von Neumann algebras. The connection with infinite-dimensional Clifford algebra of configuration space led to the idea about the realization of quantum geometry at the level of configuration space without non-commutative coordinates. Classical quantum correspondence led to very concrete view about how join along boundaries bonds defining braids serve as correlates for quantum bound state formation and bound state entanglement. Even a physicist's "proof" of four-color theorem emerges as an outcome.

The non-integer quantization of dimensions for effective II_1 tensor factors implies quantization of Planck constant: the values of h are given $h(n) = [2log(2)/log(r)] \times h$, where r is the dimension of the effective II_1 tensor factor. The spectrum of r contains continuum $r \geq 4$ and discrete spectrum $r = B_n = 4cos^2(\pi/n)$ $(n \geq 3)$ below it. B_n is so called Beraha number. The interpretation in terms of discrete bound states and continuum of unbound states is a suggestive, and in fact a sensible, interpretation. For $n = 3$ with $r = 1$ Planck constant becomes infinite and this corresponds to extremely quantal regime. Planck constant would approach zero at the limit $r \rightarrow \infty$. TGD approach mildly suggests that $r \geq 4$ is not possible.

b) Physics as a generalized number theory is the most ambitious dream inspired by TGD approach. The dimensions 4 and 8 for space-time and imbedding space led for years ago to the idea that space-time surface is in some sense maximal associative (that is quaternionic) sub-manifold of octonionic space. The problem was to understand why the compactification of octonion space to $H = M^4_+ \times CP_2$ does mean. In this chapter a concrete idea about how space-time surfaces can be seen as surfaces in both the space of octonions and in H is discussed. The idea relies crucially on the observation that CP_2 parameterizes different quaternionic planes in the space of octonions just like $S^2 = CP_1$ parameterizes different complex planes in the space of quaternions.

c) The idea about p-adic physics as physics of cognition and intentionality inspired the generalization of a number concept so that reals and various extensions of p-adic numbers are glued together along common rationals and form a book like structure with the rim of book being represented by rationals. This in turn inspired the vision that p-adicization of physics should correspond to an algebraic continuation of rational physics to various number fields (the first approach was based on topological continuation and led to conflict with symmetries). The observation that p-adic norms define a hierarchy of number theoretic entropies which can be also negative and can serve as genuine information measures was a closely related idea.

d) p-Adic continuation leads typically to the need of finite-dimensional extensions of p-adic numbers and also transcendental extensions are needed (e^p is ordinary p-adic number and defines finite-dimensional transcendental extension). This inspired the idea that the evolution of mathematical consciousness corresponds to the gradual increase of both p and dimension of extension of p-adics and inspired various number theoretical conjectures relating different transcendentals to each other. The first form of these conjectures turned out to be wrong but in this chapter the conjectures are already much more realistic being minimal conjectures guaranteing the universality of physics.

One outcome is a sharpening of the conjecture that the values of so called polyzeta functions (including Riemann Zeta) at integer valued points are all transcendental numbers to a conjecture that these numbers are rationally proportional to powers of π determined by the degree of polyzeta. This would guarantee the possibility to algebraically continue the notion of quantum groups and related algebraic structures (in particular, conformal quantum field theories) to all p-adic numbers fields.

The p-adication problem led to quite unexpected developments in the understanding of space-time correlates of mathematical cognition. The challenge is nothing less than to locate the Platonia of mathematical ideas at space-time level, that is to identify space-time correlates of algebraic manipulations and equations. Also one should identify the space-time correlates of imagined algebraic structures and understand how they differ from real ones. The arguments evolved in the following manner.

a) The p-adicization of the vacuum functional defined as an exponent of Kähler function requires that the exponent proportional to the inverse of Kähler coupling constant converges for most primes p. The observation that this is highly improbable led the question whether infinite primes might be of help in the problem. The notion of infinite primes, integers and rationals generalized to complex, quaternionic and octonionic case was one of the first deep ideas inspired by TGD inspired theory of consciousness. The basic observation was that infinite primes correspond to quantum states of an arithmetic quantum field theory second quantized repeatedly. The precise role of infinite primes however remained unclear. It seems that the notion of infinitesimal inspired by addition is not very useful. Rather, the number theoretic interpretation suggests that the correct algebraic operation is multiplication.

Indeed, the key observation was that the multiplication of inverse Kähler coupling strength with a power of quantity $Y = X/(1+X)$, where X is defined as product of all finite primes, and by powers of more general quantities $Y(n/m) = (n/m)X/Q(n/m)$ (n is integer and m square free integer), where $Q(n/m)$ is infinite rational solves the problem. In real sense the numbers Y are units but p-adically their p-adic norm is $1/p$ for all primes except those dividing n and m. Note that also quaternionic and octonionic variants of infinite rationals appear in TGD.

b) The unit-in-real-sense property means that these numbers define an infinite-dimensional extension of rational numbers differing from ordinary rationals in no manner in the real context. The conclusion is that in TGD Universe space-time and imbedding space points are like the monads of Leibniz having infinitely complex structure. Since infinite primes, and their complex, quaternionic, and octonionic counterparts can represent quantum states of entire Universe, Universe is an algebraic hologram in the strongest sense that one can imagine.

c) Since this structure is not visible at the level of real number based physics, the interpretation is as space-time correlate for mathematical cognition. The free algebra generated by products and sums of infinite primes can be seen as the Mother of All Algebraic Structures allowing representation of any smaller structure. In particular, quantum states and quantum entanglement is representable and all kinds of algebraic rules can be represented using entanglement of the algebra elements representing algebra elements.

d) The paths of points in space-time define paths in this algebra and p-adic continuity for all primes implies the conservation of topological energy encountered in arithmetic quantum field theories and implying that the ordinary rational defined by the algebra element is a constant of motion. These paths are correlates for algebraic manipulations forming themselves an algebra with respect to local multiplication (analogous to gauge group multiplication).

A hierarchy of paths (paths of $d = 1$ objects, $d = 2$ and $d = 3$ objects) defining $d-$dimensional surfaces giving rise to analogs of local gauge algebras result. In TGD Universe the maximal dimension of this kind of structure is $d = 4$. This probably has some deep algebraic meaning and probably relates also the special properties of 3- and 4-dimensional topologies as well as with the non-renormalizability of $d > 4$-dimensional quantum field theories, and obviously also with the dimension of quaternions as a maximal associative sub-algebra of octonions. At classical level these structures have interpretation as abstractions of the rules obeyed by algebraic manipulations using a collection of examples defined by a pile of d-dimensional manipulation sequences. Thus algebraic manipulations of manipulations of ... define themselves algebras up to $d = 4$. The algebraic manipulations correspond to Feynmann diagram like structures with algebra and co-algebra operations having particle fusion and creation as their physical analogs. This might mean that elementary particle reactions represent a blood and flesh realization for mathematical cognition.

2 Braid group, von Neumann algebras, quantum TGD, and formation of bound states

The article of Vaughan Jones in [7] discusses the relation between knot theory, statistical physics, and von Neumann algebras. The intriguing results represented stimulate concrete ideas about how to understand the formation of bound states quantitatively using the notion of join along boundaries bond. All mathematical results represented in the following discussion can be found in [7] and in the references cited therein so that I will not bother to refer repeatedly to this article in the sequel.

2.1 Factors of von Neumann algebras

Von Neumann algebras M are algebras of bounded linear operators acting in Hilbert space. These algebras contain identity, are closed with respect to Hermitian conjugation, and are topologically complete. Finite-dimensional von Neuman algebras decompose into a direct sum of algebras M_n, which act essentially as matrix algebras in Hilbert spaces $\mathcal{H}_{nm}$, which are tensor products $C^n \otimes \mathcal{H}_m$. Here $\mathcal{H}_m$ is an m-dimensional Hilbert space in which M_n acts trivially. m is called the multiplicity of M_n.

A factor of von Neumann algebra is a von Neumann algebra whose center is just the scalar multiples of identity. The algebra of bounded operators in an infinite-dimensional Hilbert space is certainly a factor. This algebra decomposes into "atoms" represented by one-dimensional projection operators. This kind of von Neumann algebras are called type I factors.

The so called type II_1 factors and type III factors came as a surprise even for Murray and von Neumann. II_1 factors are infinite-dimensional and analogs of the matrix algebra factors M_n. They allow a trace making possible to define an inner product in the algebra. The trace defines a generalized dimension for any subspace as the trace of the corresponding projection operator. This dimension is however continuous and in the range $[0,1]$: the finite-dimensional analog would be the dimension of the sub-space divided by the dimension of $\mathcal{H}_n$ and having values $(0, 1/n, 2/n, ..., 1)$. II_1 factors are isomorphic and there exists a minimal "hyper-finite" II_1 factor is contained by every other II_1 factor.

Just as in the finite-dimensional case, one can to assign a multiplicity to the Hilbert spaces where II_1 factors act on. This multiplicity, call it $dim_M(\mathcal{H})$ is analogous to the dimension of the Hilbert space tensor factor $\mathcal{H}_m$, in which II_1 factor acts trivially. This multiplicity can have all positive real values. Quite generally, von Neumann factors of type I and II_1 are in many respects analogous to the coefficient field of a vector space.

2.2 Sub-factors

Sub-factors $N \subset M$, where N and M are of type II_1 and have same identity, can be also defined. The observation that M is analogous to an algebraic extension of N motivates the introduction of index $|M:N|$, which is essentially the dimension of M with respect to N. This dimension is an analog for the complex dimension of CP_2 equal to 2 or for the algebraic dimension of the extension of p-adic numbers.

The following highly non-trivial results about the dimensions of the tensor factors hold true.

a) If $N \subset M$ are II_1 factors and $|M:N| < 4$, there is an integer $n \geq 3$ such $|M:N| = r = 4cos^2(\pi/n)$, $n \geq 3$.

b) For each number $r = 4cos^2(\pi/n)$ and for all $r \geq 4$ there is a sub-factor $R_r \subset R$ with $|R:R_r| = r$.

One can say that M effectively decomposes to a tensor product of N with a space, whose dimension is quantized to a certain algebraic number r. The values of r corresponding to $n = 3, 4, 5, 6...$ are $r = 1, 2, 1+\Phi \simeq 2.61, 3, ...$ and approach to the limiting value $r = 4$. For $r \geq 4$ the dimension becomes continuous.

An even more intriguing result is that by starting from $N \subset M$ with a projection e_N: $M \rightarrow N$ one can extend M to a larger II_1 algebra $\langle M, e_N\rangle$ such that one has

$$
\begin{aligned}
|\langle M, e_N\rangle : M| &= |M:N| \ , \\
tr(xe_N) &= |M:N|^{-1} tr(x) \ , \ x \in M \ .
\end{aligned} \tag{1}
$$

One can continue this process and the outcome is a tower of II_1 factors $M_i \subset M_{i+1}$ defined by $M_1 = N$, $M_2 = M$, $M_{i+1} = \langle M_i, e_{M_{i-1}}\rangle$. Furthermore, the projection operators $e_{M_i} \equiv e_i$ define a Temperley-Lieb representation of the braid algebra via the formulas

$$
\begin{aligned}
e_i^2 &= e_i \ , \\
e_i e_{i\pm 1} e_i &= \tau e_i \ , \ \tau = 1/|M:N| \\
e_i e_j &= e_j e_i \ , \ |i-j| \geq 2 \ .
\end{aligned} \tag{2}
$$

Temperley Lieb algebra will be discussed in more detail later. Obviously the addition of a tensor factor of dimension r is analogous with the addition of a strand to a braid.

The hyper-finite algebra R is generated by the set of braid generators $\{e_1, e_2, ...\}$ in the braid representation corresponding to r. Sub-factor R_1 is obtained simply by dropping the lowest generator e_1, R_2 by dropping e_1 and e_2, etc..

2.3 II_1 factors and the spinor structure of infinite-dimensional configuration space of 3-surfaces

The following observations serve as very suggestive guidelines for how one could interpret the above described results in TGD framework.

a) The discrete spectrum of dimensions $1, 2, 1+\Phi, 3, ..$ below $r < 4$ brings in mind the discrete energy spectrum for bound states whereas the for $r \geq 4$ the spectrum of dimensions is analogous to a continuum of unbound states. The fact that r is an algebraic number for $r < 4$ conforms with the vision that bound state entanglement corresponds to entanglement probabilities in an extension of rationals defining a finite-dimensional extension of p-adic numbers for every prime p.

b) The discrete values of r correspond precisely to the angles ϕ allowed by the unitarity of Temperley-Lieb representations of the braid algebra with $d = -\sqrt{r}$. For $r \geq 4$ Temperley-Lieb representation is not unitary since $cos^2(\pi/n)$ becomes formally larger than one (n would become imaginary and continuous). This could mean that $r \geq 4$, which in the generic case is a transcendental number, represents unbound entanglement, which in TGD Universe is not stable against state preparation and state function reduction processes.

c) The formula $tr(xe_N) = |M : N|^{-1}tr(x)$ is completely analogous to the formula characterizing the normalization of the link invariant induced by the second Markov move in which a new strand is added to a braid such that it braids only with the leftmost strand and therefore does not change the knot resulting as a link closure. Hence the addition of a single strand seems to correspond to an introduction of an r-dimensional sub-factor to II_1 factor.

In TGD framework the generation of bound state has the formation of (possibly braided) join along boundaries bonds as a space-time correlate and this encourages a rather concrete interpretation of these findings. Also the I_1 factors themselves have a nice interpretation in terms of the configuration space spinor structure.

1. The interpretation of II_1 factors in terms of Clifford algebra of configuration space

The Clifford algebra of an infinite-dimensional Hilbert space defines a II_1 factor. The counterparts for e_i would naturally correspond to the analogs of projection operators $(1+\sigma_i)/2$ and thus to operators of form $(1+\Sigma_{ij})/2$, defined by a subset of sigma matrices. The first guess is that the index pairs are $(i, j) = (1, 2), (2, 3), (3, 4), ...$ The dimension of the Clifford algebra is 2^N for N-dimensional space so that $\Delta N = 1$ would correspond to $r = 2$ in the classical case and to one qubit. The problem with this interpretation is $r > 2$ has no physical interpretation: the formation of bound states is expected to reduce the value of r from its classical value rather than increase it.

One can however consider also the sequence $(i,j) = (1, 1+k), (1+k, 1+2k), (1+2k, 1+3k), ...$ For $k = 2$ the reduction of r from $r = 4$ would be due to the loss of degrees of freedom due to the formation of a bound state and $(r = 4, \Delta N = 2)$ would correspond to the classical limit resulting at the limit of weak binding. The effective elimination of the projection operators from the braid algebra would reflect this loss of degrees of freedom. This interpretation could at least be an appropriate starting point in TGD framework.

In TGD Universe physical states correspond to configuration space spinor fields, whose gamma matrix algebra is constructed in terms of second quantized free induced spinor fields defined at space-time sheets. The original motivation was the idea that the quantum states of the Universe correspond to the modes of purely classical free spinor fields in the infinite-dimensional configuration space of 3-surfaces (the world of classical worlds) possessing general coordinate invariant (in 4-dimensional sense!) Kähler geometry. Quantum information-theoretical motivation could have come from the requirement that these fields must be able to code information about the properties of the point (3-surface, and corresponding space-time sheet). Scalar fields would treat the 3-surfaces as points and are thus not enough. Induced spinor fields allow however an infinite number of modes: according to the naive Fourier analyst's intuition these modes are in one-one correspondence with the points of the 3-surface. Second quantization gives much more. Also non-local information about the induced geometry and topology must be coded, and here quantum entanglement for states generated by the fermionic oscillator operators coding information about the geometry of 3-surface provides enormous information storage capacity.

In algebraic geometry also the algebra of the imbedding space of algebraic variety divided by the ideal formed by functions vanishing on the surface codes information about the surface: for instance, the maximal ideals of this algebra code for the points of the surface (functions of imbedding space vanishing at a particular point). The function algebra of the imbedding space indeed plays a key role in the construction of the configuration space-geometry besides second quantized fermions.

The Clifford algebra generated by the configuration space gamma matrices at a given point (3-surface) of the configuration space of 3-surfaces could be regarded as a II_1-factor associated with the local tangent space endowed with Hilbert space structure (configuration space Kähler metric). The counterparts for e_i would naturally correspond to the analogs of projection operators $(1+\sigma_i)/2$ and thus operators of form $(G_{\overline{A}B} \times 1 + \Sigma_{\overline{A}B})$ formed as linear combinations of components of the Kähler metric and of the sigma matrices defined by gamma matrices and contracted with the generators of the isometries of the configuration space. The addition of single complex degree of freedom corresponds to $\Delta N = 2$ and $r = 4$ and the classical limit and would correspond to the addition of single braid. $(r < 4, \Delta N < 2)$ would be due to the binding effects.

$r = 1$ corresponds to $\Delta N = 0$. The first interpretation is in terms of strong binding so that the addition of particle does not increase the number of degrees of freedom. In TGD framework $r = 1$ might also correspond to the addition of zero modes which do not contribute to the configuration space metric and spinor structure but have a deep physical significance. $(r = 2, \Delta N = 1)$ would correspond to strong binding reducing the spinor and space-time degrees of freedom by a factor of half. $r = \Phi^2$ $(n = 5)$ *resp.* $r = 3$ $(n = 6)$ corresponds to $\Delta N_r \simeq 1.3885$ *resp.* $\Delta N_r = 1.585$. Using the terminology of quantum field theories, one might say that in the infinite-dimensional context a given complex bound state degree of freedom possesses anomalous real dimension $r < 2$. $r \geq 4$ would correspond to a unbound entanglement and increasingly classical behavior.

2. How to construct quantum geometry of configuration space?

The first impression is that the ordinary algebra of sigma matrices for a finite-dimensional Kähler metric does not to allow the proposed kind of thinning of degrees of freedom. Here it is good to recall the basic idea behind anomalous dimension: the trace of the projection operator is less than 2 for single complex degree of freedom in case of bound states. One should be able to realize this simple idea in terms of configuration space geometry in a mathematically respectable manner.

a) The first thing which comes into mind is to multiply the Kähler metric by a factor $d(M:N)^{-1}$ such that this part is not regarded as part of the metric. This factor would help to realize quantum classical correspondence by coding to the metric that this particular degree of freedom associated with a given 3-surface and corresponding space-time sheet correspond to an unbound or bound degree of freedom, the type of which is characterized by $d(M:N)$.

b) The justification for the anomalous dimensioning of the Kähler metric could come from the presence of infinite number of zero modes, which allow the presence of the zero mode dependent factor $\Omega(\overline{A}, B)$ multiplying the element $g_{\overline{A},B}$ to give the full metric $G_{\overline{A},B} = \Omega(\overline{A}, B) g_{\overline{A},B}$. Originally this factor was interpreted as a conformal factor but one could interpret it as an anomalous dimension matrix, which multiplies the metric in element-wise manner rather than via matrix multiplication. Hence it might be more appropriate to talk about the pair $(\Omega(\overline{A},B), g_{\overline{A},B})$ instead of $G_{\overline{A},B}$.

The expression of the components of the metric as anti-commutators of gamma matrices requires $\Gamma_A = \Omega(A)\gamma_A$, $\Omega(A) = \overline{\Omega}(\overline{A})$ implying

$$\Omega(\overline{A},B) = \Omega(\overline{A}) \times \Omega(B) \ .$$

With respect to the degrees of freedom which contribute to the metric Ω behaves as a constant since the dependence on zero modes does not contribute to the curvature by no means. Curvature tensor and other quantities would be calculated by dropping the factors first, calculating the quantity, and after that multiplying each index with Ω_A irrespective whether it is covariant or contra-variant index.

c) This process might sound ad hoc but there is a justification for it. It became obvious in a rather early stage that gamma matrices defined as vector fields Γ_k of configuration space are not useful objects and have a rather vague mathematical meaning. Rather, the gamma matrices of the configuration space, as they are defined in [A4] can be seen as counterparts for the contractions $\Gamma^A = j^{Ak}\Gamma_k$ and $\Gamma^{\overline{A}} = j^{A\overline{k}}\Gamma_{\overline{k}}$ of the Lie-algebra generators of isometries with the ordinary gamma matrices, and are thus coordinate invariant quantities. These operators act as fermionic generators of an infinite-dimensional super-algebra extending the Lie-algebra of isometries, and their anti-commutators define a representation of the metric as inner products of the isometry generators of the configuration space. The scaling factors Ω_A can thus be interpreted as zero mode dependent scalings of the super generators of the super-algebra.

d) The explanation for the presence of the braid group could be provided by the topology of the configuration space of 3-surfaces. n-particle sector consists of configurations of n 3-space sheets and is representable as $C_n = (C_1^n - D)/S_n$, where C_1 is single particle configuration space. The division by S_n is necessary because the configurations differing by a permutation of 3-space sheets cannot be distinguished. D represents the singular configurations consisting of n 3-sheets of which 2 or more are identical. One might perhaps exclude also surfaces which have $n > 0$-dimensional intersection but this does not change the argument.

This is however not quite what one wants since C_n is infinite-dimensional Kähler manifold rather than 2-dimensional. Each C_n should contain a 2-dimensional space of zero modes. If the light-like boundaries act as causal determinants then at least a sub-space of the configuration space consists of lightlike 3-surfaces and thus metrically two-dimensional. If one can somehow mark one or more points in the 2-dimensional section (choice of gauge) of the lightlike surface, the center of mass degrees of freedom for the lightlike boundary component would correspond to a two-dimensional surface.

Physically marking could mean the presence of the wormhole throats associated with a handle and relate to the Teichmuller parameters classifying conformal structures. $n-1$ handles would correspond to the braid group B_n. At the limit of entire Universe the number of lightlike surfaces approaches infinity and the infinite-dimensional braid group would appear in a natural manner as a subgroup of the first homotopy group of the configuration space. Note that one can imagine also a process in which lightlike 3-surfaces exchange handles so that the full braid group could be in question. A fractal structure is involved since each metrically two-dimensional boundary component can contain arbitrary number of handles. The fractal structure could relate to the the infinite de-composality of II_1 factors.

The concrete finite-dimensional representations of the braid group could be based on the use of braids defined by join along boundaries bonds connecting the boundaries of two magnetic flux tubes and anyonic space-time sheets inside braids and sensing the braid action.

e) In quantum theory the commutators of Lie-algebra generators T_k are proportional to $i\hbar$. Hence the scaling $T_k \rightarrow \lambda T_k$ corresponds to the scaling $\hbar \rightarrow \lambda\hbar$. Therefore there is a temptation to interpret the anomalous dimensions as a genuine scaling

$$\begin{aligned} \hbar &\rightarrow d(M:N)^{-1} \times \hbar = \frac{2log(2)}{log(r)} \times \hbar \ , \\ d(M:N) &= \frac{log(M:N)}{2log(2)} \end{aligned} \qquad (3)$$

of $\hbar$ inducing the scalings $\Gamma_A \rightarrow d(M:N)^{-1/2}\Gamma_A$ and implying fractionization of various charges. $r=1$ would correspond to $\hbar=\infty$ and extremely quantal behavior. The limit $r \rightarrow 4$ would correspond to increasingly classical behavior.

The limit $r \rightarrow \infty$ would correspond to $\hbar \rightarrow 0$ and would be even more classical. The interpretation would be natural since $r \geq 4$ would correspond to unbound entanglement unstable in quantum jump and thus corresponds to an effectively classical behavior since macro-temporal quantum coherence is not possible. It must be however emphasized that various arguments disfavor $r \geq 4$. The somewhat surprising conclusion would be that Planck constant would be dynamical and that it is possible to characterize precisely how classical the system is on basis of the properties of the system's state, that is Kähler coupling constant in turn determined by the properties of the space-time sheet itself.

3. Quantization of Kähler coupling strength and dynamical Planck constant

The obvious question is how the dependence of the scaling factor $\Omega(A)$ on zero modes could come out naturally. The so called Kähler coupling strength α_K is the only coupling constant strength of TGD and is completely analogous to fine structure constant [A1]. The Kähler function defining the configuration space metric is defined as $K = S_K/16\pi\alpha_K$ and the exponent $exp(K)$ defines the vacuum functional of the theory mathematically precisely analogous to Boltzman exponent. α_K plays therefore the role of temperature and the hypothesis is that the possible values of α_K correspond to critical temperatures. α_K can differ for two space-time regions if they are separated by elementary particle horizons (light-like 3-surfaces) surrounding wormhole contacts which possess Euclidian signature of induced metric.

The natural expectation is that the dependence of the configuration space metric on zero modes comes the dependence of α_K on zero modes defining kind of classical environment for the quantum system (zero modes correspond to classical degrees of freedom in TGD based quantum measurement theory).

The enormous vacuum degeneracy of Kähler action allows to expect a large number of quantum critical states so that α_K should allow a rich spectrum of values and the challenge is to deduce this spectrum. From $\alpha_K = g_K^2/4\pi\hbar c$ one can deduce that α_K should scale as $1/\hbar$. This would mean that α_K would have at least the following values

$$\alpha_K(r) = \frac{d(M:N)}{2} \times \alpha_K(4) = \frac{log(r)}{2log(2)} \times \alpha_K(4) \ .$$

This behavior is consistent also with the $1/\alpha_K$-proportionality of the Kähler metric.

Neither number theoretical vision nor quantum criticality favor the values $\alpha_K(r)$, $r \geq 4$. The discrete values

$$\alpha_K = \frac{log(4cos^2(\pi/n))}{2log(2)} \times \alpha_K(4) \ , \ \ n \geq 3$$

would be however realized and characterize an entire series of quantum critical states. The large values of α_K (small value of $\hbar$) would correspond to large temperatures in the vacuum functional $exp(K) = exp(S_K/16\pi\alpha_K)$ and thus to de-coherence and classical behavior.

These values need not be the only ones that are possible. Indeed, I have proposed [A5] that α_K depends on the p-adic length scale $L(p)$ of the real space-time sheet logarithmically so that one would have a discrete coupling constant evolution with $\alpha_K(4) \propto 1/log(p)$ for large primes (see Appendix). This would predict $\alpha_K(p) \rightarrow 0$ in long length scales just as one might expect for $U(1)$ coupling constant. Also the above described evolution as function of n predicts $\alpha_K(r=1)) = 0$ but the limit $p \rightarrow \infty$ cannot obviously correspond to this. If $\hbar$ is a genuine dynamical variable this is as it should be since genuine dynamics for $\hbar$ requires that g_K and $\hbar$ are independent dynamical variables.

The obvious question concerns the interpretation of the evolution with respect to n. In the p-adic context increasing values of n mean increasing angular resolution and this in turn means increasingly higher-dimensional algebraic extensions of p-adic numbers needed to represent trigonometric functions. Hence the evolution with respect to n would be related to the increasing angular resolution and the couplings could be at least partially understood as reflecting different dimensions of the extensions. This is strictly the case if the values of n comes as powers of p for given prime p. Increasing angular resolution means shorter length scales and the basic properties of $U(1)$ coupling constant evolution require that the coupling constant should increase with n as it indeed does.

The dynamics associated with $\hbar$ might relate to the reported slow increase of the fine structure constant as a function of cosmic time [1, B1]. Since the dependence of α on $\hbar$ is in the lowest order same as that of α_K, fine structure constant should obey the same evolution as Kähler coupling strength with respect to n. The slow increase of the α could be understood if the value of n characterizing the state of the emitting system increases slowly during the cosmic evolution. Hence the dimension of the algebraic extension of p-adic numbers possibly associated with the emitting system increases gradually during cosmic evolution. Kind of cosmic evolution of intelligence would be basically in question.

To sum up, the presence of zero modes and the interpretation of gamma matrices as super-generators would allow to realize the counterpart of the non-commutative geometry in terms of classical infinite-dimensional Kähler geometry without introducing non-commutative coordinates. Essentially a scaling of $\hbar$ would be in question. An interesting question is whether one could relate the continuous dimension $4-\epsilon$ applied as a calculational trick in the dimensional regularization of quantum field theories to the spectrum of dimensions of effective tensor factors of von Neumann algebras and perhaps justify this approach in terms of the spinor structure of infinite-dimensional configuration space.

4. Number theoretical vision and II_1 factors

TGD inspired theory of consciousness has inspired number theoretical conjectures stating that the ratios like $log(p)\pi$, p prime, $log(\Phi)\pi$ are rational numbers extended by suitable root of e (see for a more detailed discussion in Appendix). The motivation comes from the hypothesis that the evolution of mathematical consciousness corresponds to the gradual emergence of finite-dimensional extensions of p-adic numbers based, not only on algebraic numbers, but also on numbers like e and roots of polynomials with coefficients in finite-dimensional extension containing algebraic numbers and roots of e. One can however easily demonstrate that π cannot belong to a finite-dimensional extensions of rationals.

An attractive sharpening of this hypothesis is that also the numbers $log(r)\pi$ besides $log(p)\pi$ are numbers in extension F of rationals containing e. This implies that $log(p)/log(2)$ and $log(r)/log(2)$ appearing in the expression of α_K are numbers in F. Since $log(p)/log(2)$ is a natural unit of information in TGD, information would be measured as F valued fractions of bit in TGD Universe. The motivation for this comes from the hypothesis that bound state entanglement correspond to entanglement probabilities, which are rational or extended rational numbers (the extension must define a finite extension of p-adic numbers so that it is possible to define number theoretic entanglement entropy allowing interpretation as a measure for conscious information).

The number theoretic vision could allow also to say a lot about the possible values of the Kähler coupling constant. The dream is that an algebraic continuation from the extensions of rational numbers defining finite extensions of p-adic numbers allows to define the theory in various number fields. The fulfillment of this dream requires that physically important quantities such as the exponent of Kähler function for CP_2 extremal and other fundamental extremals exist in a finite-dimensional extension of p-adic numbers. Certainly, the continuum $\alpha_K(r \geq 4)$ is not favored by the number theoretic vision. This point is discussed in the Appendix in more detail.

2.4 Space-time correlates for the hierarchy of II_1 sub-factors

By quantum classical correspondence the infinite-dimensional physics at the configuration space level should have definite space-time correlates. In particular, the dimension r should have some fractal dimension as a space-time correlate.

1. Quantum classical correspondence

Join along boundaries bonds serve as correlates for bound state formation. The presence of join along boundaries bonds would lead to a generation of bound states just by reducing the degrees of freedom to those of connected 3-surface. The bonds would constrain the two 3-surfaces to single space-like section of imbedding space.

This picture would allow to understand the difficulties related to Bethe-Salpeter equations for bound states based on the assumption that particles are points moving in M^4. The restriction of particles to time=constant section leads to a successful theory which is however non-relativistic. The basic binding energy would relate to the entanglement of the states associated with the bonded 3-surfaces. Since the classical energy associated with the bonds is positive, the binding energy tends to be reduced as r increases.

By spin glass degeneracy join along boundaries bonds have an infinite number of degrees of freedom in the ordinary sense. Since the system is infinite-dimensional and quantum critical, one expects that the number r of degrees freedom associated with a single join along boundaries bond is universal. Since join along boundaries bonds correspond to the strands of a braid and are correlates for the bound state formation, the natural guess is that $r = 4cos^2(\pi/n),\ n = 3, 4, 5, ...$ holds true. $r < 4$ should characterize both binding energy and the dimension of the effective tensor factor introduced by a new join along boundaries bond.

The assignment of 2 "bare" and $\Delta N \leq 2$ renormalized real dimensions to single join along boundaries bond is consistent with the effective two-dimensionality of anyon systems and with the very notion of the braid group. The picture conforms also with the fact that the degrees of freedom in question are associated with metrically 2-dimensional light-like boundaries (of say magnetic flux tubes) acting as causal determinants. Also vibrational degrees of freedom described by Kac-Moody algebra are present and the effective 2-dimensionality means that these degrees of freedom are not excited and only topological degrees of freedom coded by the position of the puncture remain.

$(r \geq 4, \Delta N \geq 2)$ would mean that the tensor factor associated with the join along boundaries bond is effectively more than 4-dimensional due to the excitation of the vibrational Kac-Moody degrees of freedom. The finite value of r would mean that most of theme are eliminated also now but that their number is so large that bound state entanglement is not possible anymore.

The introduction of non-integer dimension could be seen as an effective description of an infinite-dimensional system as a finite-dimensional system in the spirit of renormalization group philosophy. The non-unitarity of $r \geq 4$ Temperley-Lieb representations could mean that they correspond to unbound entanglement unstable against state function reduction and preparation processes. Since this kind of entanglement does not survive in quantum jump it is not representable in terms of braid groups.

2. $r < 4$ as the average dimension of CP_2 projection

On basis of the quantum classical correspondence one expects that r should appear as a fractal dimension at the space-time level. Since r varies in the range $1, .., 4$ and the dimension of the CP_2 projection of the space-time sheet for the absolute minima of Kähler action has the values $D = 1, 2, 3, 4$, the guess is that r defines an kind of average dimension for the CP_2 projection of the join along boundaries bond. Note that $D = 0$ corresponds to a canonically imbedded M^4_+ and would correspond to $n = 2, r = 0$ formally which however does not make sense. The problem of this interpretation is that it does not apply to $r \geq 4$ effective tensor factors and there is not obvious manner to modify the interpretation to achieve overall consistency. Of course, one can quite well consider the possibility that the continuum values for r are not possible at all: this would certainly be consistent with the criticality hypothesis.

One can try to build a detailed physical interpretation of various values of r. First of all, the fact that the values $n = p^k$, p prime are favored p-adically slightly disfavors $n = 6$ option but allows $n = 2, 3, 4, 5$. By previous arguments $r = 1$ *resp.* $r = 2$ corresponds to the addition of zero mode or of complex dimension contributing to Kähler metric whereas larger values of r correspond to $\Delta N = log(r)/log(2)$ and are thus anomalous dimensions.

a) $(r = 1, n = 3, \phi = 2\pi/12)$ would correspond to $D = 1$ vacuum extremals. The vanishing of field energy for $r = 1$ means a maximal binding energy. $r = 1$ means that the resulting tensor factor does not increase the dimension of algebra so that in this sense it does not change anything: this is consistent with the vacuum extremal property. $r = 1$ would most naturally correspond to a trivial tensor factor in the ordinary sense.

b) $(r = 2, n = 4, \phi = 2\pi/16)$ would relate to $D = 2$ magnetic flux tubes. Also Yang-Baxter braid group representation with $R_1 = exp(i2\pi/16)R$ could correspond to $D = 2$ magnetic flux tubes. Single strand would correspond to exactly single qubit, perhaps in the ordinary sense.

c) $(r = 3, n = 6, \phi = 2\pi/24)$ would relate to $D = 3$. Note that the magic dimension $d = 24$ of string theories appears here. This option is slightly disfavored by the p-adic argument favoring $n = p^m$. Now single strand would correspond to single fractal tribit.

d) Since fractal dimensions are associated with critical systems, ($r = \Phi+1 \simeq 2.61, n = 5, \phi = 2\pi/20$) could be interpreted as being associated with a critical system representing a phase transition between $D = 2$ and $D = 3$ phases. Criticality would also explain why r actually corresponds to an infinite ordinary dimension. These phases would appear with probabilities $p(D = 2) = 2 - \Phi \simeq .39$ and $p(D = 3) = \Phi - 1 \simeq .61$ with $p(D = 3)/p(D = 2) = \Phi$ in the critical system. Single strand would correspond to $log(\Phi + 1)/log(2) \simeq 1.389$ fractal qubits. Three strands would correspond to $\simeq 4.17$ fractal qubits.

e) The dimensions $r \geq 3$ would relate to the critical states between $D = 3$ and $D = 4$ systems with $p(D = 3) = 4sin^3(\pi/n)$, $n > 6$. These dimensions would represent an approach to chaos via a discrete sequence of phase transitions in which join along boundaries bonds become more and more energetic.

2.5 Could binding energy spectra reflect the hierarchy of effective tensor factor dimensions?

If one takes completely seriously the idea that join along boundaries bonds are a correlate of binding then the spectrum of binding energies might reveal the hierarchy of the fractal dimensions $r(n)$. Hydrogen atom and harmonic oscillator have become symbols for bound state systems. Hence it is of interest to find whether the binding energy spectrum of these systems might be expressed in terms of the "binding dimension" $x(n) = 4 - r(n)$ characterizing the deviation of dimension from that at the limit of a vanishing binding energy. The binding energies of hydrogen atom are in a good approximation given by $E(n)/E(1) = 1/n^2$ whereas in the case of harmonic oscillator one has $E(n)/E_0 = 2n + 1$. The constraint $n \geq 3$ implies that the principal quantum number must correspond $n - 2$ in the case of hydrogen atom and to $n - 3$ in the case of harmonic oscillator.

Before continuing one must face an obvious objection. By previous arguments different values of r correspond to different values of $\hbar$. The value of $\hbar$ cannot however differ for the states of hydrogen atom. This is certainly true. The objection however leaves open the possibility that the states of the light-like boundaries of join along boundaries bonds correspond to reflective level and represent some aspects of the physics of, say, hydrogen atom.

In the general case the energy spectrum satisfies the condition

$$\frac{E_B(n)}{E_B(3)} = \frac{f(4-r(n))}{f(3)} \quad , \tag{4}$$

where f is some function. The simplest assumption is that the spectrum of binding energies $E_B(n) = E(n) - E(\infty)$ is a linear function of $r(n) - 4$:

$$\frac{E_B(n)}{E_B(3)} = \frac{4-r(n)}{3} = \frac{4}{3}sin^2(\frac{\pi}{n}) \rightarrow \frac{4\pi^2}{3} \times \frac{1}{n^2} \quad . \tag{5}$$

In the linear approximation the ratio $E(n+1)/E(n)$ approaches $(n/n+1)^2$ as in the case of hydrogen atom but for small values the linear approximation fails badly. An exact correspondence results for

$$\frac{E(n)}{E(1)} = \frac{1}{n^2} \quad ,$$

$$n = \frac{1}{\pi \arcsin\left(\sqrt{1-r(n+2)/4}\right)} - 2 \quad .$$

Also the ionized states with $r \geq 4$ would correspond to bound states in the sense that two particle would be constrained to move in the same space-like section of space-time surface and should be distinguished from genuinely free states when particles correspond to disjoint space-time sheets.

For the harmonic oscillator one express $E(n) - E(0)$ instead of $E(n) - E(\infty)$ as a function of $x = 4 - r$ and one would have

$$\frac{E(n)}{E(0)} = 2n + 1 \quad ,$$

$$n = \frac{1}{\pi \arcsin\left(\sqrt{1-r(n+3)/4}\right)} - 3 \quad .$$

In this case ionized states would not be possible due to the infinite depth of the harmonic oscillator potential well.

2.6 Four-color problem, II_1 factors, and anyons

The so called four-color problem can be phrased as a question whether it is possible to color the regions of a plane map using only four colors in such a manner that no adjacent regions have the same color (for an enjoyable discussion of the problem see [2]). One might call this kind of coloring complete. There is no loss of generality in assuming that the map can be represented as a graph with regions represented as triangle shaped faces of the graph. For the dual graph the coloring of faces becomes coloring of vertices and the question becomes whether the coloring is possible in such a manner that no vertices at the ends of the same edge have same color. The problem can be generalized by replacing planar maps with maps defined on any two-dimensional surface with or without boundary and arbitrary topology. The four-color problem has been solved with an extensive use of computer [3] but it would be nice to understand why the complete coloring with four colors is indeed possible.

There is a mysterious looking connection between four-color problem and the dimensions $r(n) = 4cos^2(\pi/n)$, which are in fact known as Beraha numbers in honor of the discoverer of this connection [4]. Consider a more general problem of coloring two-dimensional map using m colors. One can construct a polynomial $P(m)$, so called chromatic polynomial, which tells the number of colorings satisfying the condition that no neighboring vertices have the same color. The vanishing of the chromatic polynomial for an integer value of m tells that the complete coloring using m colors is not possible.

$P(m)$ has also other than integer valued real roots. The strange discovery due to Beraha is that the numbers $B(n)$ appear as approximate roots of the chromatic polynomial in many situations. For instance, the four non-integral real roots of the chromatic polynomial of the truncated icosahedron are very close to $B(5)$, $B(7)$, $B(8)$ and $B(9)$. These findings led Beraha to formulate the following conjecture. Let P_i be a sequence of chromatic polynomials for a graph for which the number of vertices approaches infinity. If r_i is a root of the polynomial approaching a well-defined value at the limit $i \rightarrow \infty$, then the limiting value of $r(i)$ is Beraha number.

A physicist's proof for Beraha's conjecture based on quantum groups and conformal theory has been proposed [4]. It is interesting to look for the a possible physical interpretation of 4-color problem and Beraha's conjecture in TGD framework.

a) In TGD framework $B(n)$ corresponds to a renormalized dimension for a 2-spin system consisting of two qubits, which corresponds to 4 different colors. For $B(n) = 4$ two spin 1/2 fermions obeying Fermi statistics are in question. Since the system is 2-dimensional, the general case corresponds to two anyons with fractional spin $B(n)/4$ giving rise to $B(n) < 4$ colors and obeying fractional statistics instead of Fermi statistics. One can replace coloring problem with the problem whether an ideal antiferro-magnetic lattice using anyons with fractional spin $B(n)/4$ is possible energetically. In other words, does this system form a quantum mechanical bound state even at the limit when the lengths of the edges approach to zero.

c) The failure of coloring means that there are at least two neighboring vertices in the lattice with the property that the spins at the ends of the same edge are in the same direction. Lattice defect would be in question. At the limit of an infinitesimally short edge length the failure of coloring is certainly not an energetically favored option for fermionic spins ($m = 4$) but is allowed by anyonic statistics for $m = B(n) < 4$. Thus one has reasons to expect that when anyonic spin is $B(n)/4$ the formation of a purely 2-anyon bound states becomes possible and they form at the limit of an infinitesimal edge length a kind of topological macroscopic quantum phase with a non-vanishing binding energy. That $B(n)$ are roots of the chromatic polynomial at the continuum limit would have a clear physical interpretation.

d) Only $B(n) < 4$ defines a sub-factor of von Neumann algebra allowing unitary Temperley-Lieb representations. This is consistent with the fact that for $m = 4$ complete coloring must exists. The physical argument is that otherwise a macroscopic quantum phase with non-vanishing binding energy could result at the continuum limit and the upper bound for r from unitarity would be larger than 4. For $m = 4$ the completely anti-ferromagnetic state would represent the ground state and the absence of anyon-pair condensate would mean a vanishing binding energy.

3 Intentionality, cognition, physics, and number theory

TGD leads to an ambitious program of reducing entire physics to number theory. There are several strands involved with this program.

a) The first goal is to reduce the classical dynamics of space-time surfaces to number theory. The idea is that the 4-dimensionality of space-time and 8-dimensionality of imbedding space make possible local octonionic structure and that space-time surfaces somehow define maximally associative, that is quaternionic sub-manifolds of octonionic space. The basic challenges to understand why $M^4_+ \times CP_2$ rather than octonions. During writing of this chapter a considerable progress occurred in this respect.

b) The identification of p-adic physics as physics of cognition and intention suggests strongly connections between cognition, intentionality, and number theory. The new idea is that also real transcendental numbers can appear in the extensions of p-adic numbers which must be assumed to be finite-dimensional at least in the case of human cognition. This idea, when combined with a more precise model for how intentions are transformed to actions, leads to a series of number theoretical conjectures. Also new insights about the number theoretical origin of the universal dynamics of conformally invariant critical systems emerge. The earlier approaches to the proof of Riemann hypothesis can be understood in a unified manner and the assumption that Riemann Zeta exists in all number fields when finite extensions are allowed for p-adic numbers leads to the view that that the zeros of Riemann Zeta correspond to the universal number theoretically quantized spectrum of scaling momenta associated with critical conformally invariant systems.

c) Infinite primes, integers and rationals represent a further strand in the evolution of ideas. These numbers correspond to states of second quantized arithmetic quantum field theory. Their precise role has however remained somewhat obscure. During writing of this chapter, it however became clear that they are necessary element of p-adicization. They define an infinite-dimensional extension of real, quaternion and octonion rationals by multiplicative units which is not seen at all at the level of real topology but is directly visible at the level of p-adic topologies. This leads to the idea that mathematical points are like Leinizian monads, and that the Platonia of mathematical ideas is represented in the structure of a space-time point. The infinite-dimensional free algebra generated by the generalized octonionic units defined by infinite primes is capable of representing any algebra in its structure. For these reasons this algebra is excellent candidate for a space-time correlate of cognition.

3.1 Why $M^4_+ \times CP_2$?

The octonionic interpretation of the 8-dimensional imbedding space is very attractive but would suggest that imbedding space consists of octonions. The challenge is to understand the compactification to CP_2 and Minkowskian signature of the metric.

Minkowski metric does not seem to be a problem. Since one can provide quaternions with Minkowski metric defined by the real part of hh (rather than $h\overline{h}$ one obtains M^4_+ by restricting this norm to be non-negative. In fact, as far as cosmology is considered, it is not possible to distinguish between M^4 and M^4_+ options and M^4 option would be more natural for quaternions. For M^4 option the quantum states of Universe have necessarily vanishing net quantum numbers. For M^4_+ option universe could have net quantum numbers but this option is less predictive and less elegant.

Concerning the compactification O to $M^4 \times CP_2$ the starting point is the idea that space-time surfaces are associative manifolds. What this means imbedding of space-time manifold to the algebra of octonions and an assignment of associative 4-algebra to each point. This algebra is a quaternionic sub-algebra H The system must be also integrable which means that the quaternionic sub-spaces H correspond to tangent spaces of the 4-surface.

The embeddings of the complex plane to quaternion space are parameterized by the sphere S^2. One can understand this from the fact that quaternion automorphism induces rotation and the $U(1)$ group defined by the "imaginary" part of given quaternion h commutes with h. The natural guess is that the imbeddings of H to O are labelled by the points of CP_2. If one fixes standard quaternion H sub-space of octonions, one can obtain new quaternionic sub-spaces by automorphisms $h \rightarrow oho^{-1}$, o an arbitrary octonion. Since o has 8 components and and since $SU(3)$ is the only 8-dimensional simple Lie group, it is easy to believe that the group of automorphisms is $SU(3)$. Since only the 4-dimensional sub-space is what matters, one must divide away the subgroup which leaves the quaternionic sub-space invariant. This group is $SU(2)$ corresponding to quaternionic automorphisms. As in the case of complex numbers, also the $U(1)$ factor of $U(2)$ acting as an isotropy group of h must be divided out so that $U(2)$ group emerges as expected and $CP_2 = SU(3)/U(2)$ is the space of quaternionic sub-spaces at a given point of 4-surface.

The associative imbeddings of four-surfaces to the space of octonions can be obtained locally from the standard imbedding of H by performing an $SU(3)$ twist at each point of H. The local quaternion coordinate h would correspond to the space-time coordinate and M^4 coordinate $m = oho^{-1}$ to the $SU(3)$ twisted quaternion coordinate. CP_2 coordinates $s(h)$ characterize the quaternion subspace at a given point. Also space-time regions for which CP_2 or M^4 projection has dimension smaller than 4 should fit in this picture. For instance, CP_2 imbedded in standard manner itself would correspond to a constant value m_0 of m. All coordinates h at all points of space-time surface would be twisted to the same constant value m_0. Surfaces of form $X^2 \times Y^2 \subset M^4_+ \times CP_2$ would correspond to twists for which 4-dimensional Q is twisted to 2-dimensional X^2 and tangent spaces form sub-space Y^2. Hence it seems that the proposal can deal also with the singular cases. Note that one can say that space-time surface can be regarded as a 4-surface in both O and $M^4_+ \times CP_2$.

Dreams would be fulfilled if the field equations would reduce to integrability conditions stating that the quaternionic planes defined uniquely by the CP_2 coordinates s corresponds to the tangent plane defined by the imbedding $h \rightarrow (m, s)$ and thus integrate to 4-surfaces in O. The integrability conditions state that the components $o(h)e_A o^{-1}(h)$, where e_A represent the standard quaternion basis vector fields are expressible as linear combinations of the tangent vector fields having M^4 part ∇m and CP_2 part ∇s. Using $m = o(h)qo^{-1}(h)$ one can write this condition explicitly by using the linearity of the $SU(3)$ action in the octonion basis e_A

$$o(h)e_Ao^{-1}(h) \;\; = \;\; R_A^B(h)e_B \equiv \hat{e}_A \;\; , \;\; A = 1,2,3,4, \;\; B = 1,...,8 \;\; . \quad (6)$$

$o(h)$ is not uniquely fixed by $s(h)$ since there is always a $U(2)$ gauge transformation involved. Interestingly, both electro-weak and color gauge group appear in the conditions.

One might think that the gradients of CP_2 coordinates s^k corresponds to four vectors orthogonal to the quaternion plane defined by s. This is not the case since the presence of $U(2)$ gauge degrees of freedom bring in also transformation in quaternion plane. It is just this part which is important for the integrability conditions and one can assign to the gradient of CP_2 coordinates the tangent space projection of the vector field

$$\partial_\mu s(h) \rightarrow T_k^A(h)\partial_\mu s^k \hat{e}_A \;\; . \quad (7)$$

Since e_A can be taken to be constant vector fields the integrability conditions follow by writing the rotated tangent vectors as linear combinations of parts coming from the gradients of M^4 coordinates and CP_2 coordinates

$$\hat{e}_A = C^\mu (\partial_\mu R R^{-1})_A^{\;C} \hat{e}_C + D_A^\mu T_k^C \partial_\mu s^k \hat{e}_C \;\; . \quad (8)$$

Here ∂_μ refers to h^μ and $C_A^\mu(h)$ are coefficients which must be fixed for the solutions of the integrability conditions. An alternative form is

$$\delta_A^{\;B} = C^\mu (\partial_\mu R R^{-1})_A^{\;B} + D_A^\mu T_k^B \partial_\mu s^k \;\; , \;\; A,B = 1,...,4 \;\; . \quad (9)$$

The basic question is how the presence of the induced metric and Kähler in the field equations can be consistent with this picture. Here analyticity might come in rescue and the known solution spectrum identified as absolute minima of Kähler action reduces to high degree to conditions involving only the most general properties of the induced metric. By quantum classical correspondence the solutions of field equations should asymptotically represent space-time correlates for asymptotic self-organization patterns which do not dissipate anymore. This inspires the highly successful hypothesis that Lorentz-Kähler 4-force $j^\mu J_{\mu\nu}$ vanishes (see the chapter "Basic Extremals of the Kähler action"). The Kähler current j^μ, which involves induced metric in its definition is either vanishing, light-like or proportional to a topological current whose divergence gives the instanton density $J \wedge J$ for the induced Kähler form of CP_2.

Note that the coefficients C^μ and D^μ_A contain a contravariant vector index and if these vector fields are naturally obtained from C_μ and $D_{\mu A}$, the induced metric might creep into the equations. Alternatively, one could *define* the covariant vector fields by using the induced metric to lower the indices. These considerations raise the hope that the fundamental assumption of quantum TGD could be reduced to a generalized number theory although it is important to fill in the details correctly.

3.2 Cognitive evolution and extensions of p-adic number fields

The first proposal to realize the idea that the discovery of a transcendental number corresponds to an emergence of a finite-dimensional p-adic extension containing the transcendental, was based on the hypothesis that numbers like e/π, $log(p)/\pi$, and $log(\Phi)/\pi$ could be rational numbers. This idea did not work as since π cannot belong to a finite-dimensional extension of p-adic numbers as will be demonstrated below. One can however develop a more general approach giving good hopes about p-adicization of TGD by algebraic continuation from rationals to reals and p-adic number field.

3.2.1 Should one allow also transcendentals in the extensions of p-adic numbers?

TGD [TGD, padTGD] inspired theory of consciousness [cbookI, cbookII] leads to the identification of p-adic physics as physics of cognition and intention (for a summary about the recent situation see also the article "Time, Space-time, and Consciousness" [10]). This identification leads to a rather fascinating new ideas concerning the characterization of intentional systems.

The basic ingredient is the new view about numbers: real and p-adic number fields are glued together like pages of a book along common rationals representing the rim of the book. This generalizes to the extensions of p-adic number fields and the outcome is a complex fractal book like structure containing books within books. This holds true also for manifolds and one ends up to the view about many-sheeted space-time realized as 4-surface in 8-D generalized imbedding space and containing both real and p-adic space-time sheets. The transformation of intention to action corresponds to a quantum jump in which p-adic space-time sheet is replaced with a real one.

One implication is that the rationals having short distance p-adically are very far away in real sense. This implies that p-adically short temporal and spatial distances correspond to long real distances and that the evolution of cognition proceeds from long to short temporal and spatial scales whereas material evolution proceeds from short to long scales. Together with p-adic non-determinism due the fact that the integration constants of p-adic differential equations are piecewise constant functions this explains the long range temporal correlations and apparent local randomness of intentional behavior. The failure of the real statistics and its replacement by p-adic fractal statistics for time series defined by varying number N of measurements performed during a fixed time interval T allows very general tests for whether the system is intentional and what is the p-adic prime p characterizing the "intelligence quotient" of the system. The replacement of $log(p_n)$ in the formula $S = -\sum_n p_n log(p_n)$ of Shannon entropy with the logarithm of the p-adic norm $|p_n|_p$ of the rational valued probability allows to define a hierarchy of number theoretic information measures which can have both negative and positive values.

Since p-adic numbers represent a highly number theoretical concept one might expect that there are deep connections between number theory and intentionality and cognition. The discussions with Uwe Kämpf in CASYS$'$2003 conference in Liege indeed stimulated a bundle of ideas allowing to develop a more detailed view about intention-to-action transformation and to disentangle these connections. These discussions made me aware of the fact that my recent views about the role of extensions of p-adic numbers are perhaps too limited. To see this consider the following arguments.

a) Pure p-adic numbers predict only p-adic length scales proportional to $p^{n/2}l$, l CP_2 length scale about 10^4 Planck lengths, $p \simeq 2^k$, k prime or power of prime. As a matter fact, all positive integer values of k are possible. This is however not enough to explain all known scale hierarchies. Fibonacci numbers $F_n : F_n + 1 = F_n + F_{n-1}$ behave asymptotically like $F_n = kF_{n-1}$, k solution of the equation $k^2 = k + 1$ given by $k = \Phi = (1 + \sqrt{5})/2 \simeq 1.6$. Living systems and self-organizing systems represent a lot of examples about scale hierarchies coming in powers of the Golden Mean $\Phi = (1 + \sqrt{5})/2$.

By allowing the extensions of p-adics by algebraic numbers one ends up to the idea that also the length scales coming as powers of x, where x is a unit of algebraic extension analogous to imaginary unit, are possible. One would however expect that the generalization of the p-adic length scale hypothesis alone would predict only the powers $\sqrt{x}p^{n/2}$ rather than $x^k p^{n/2}$, $k = 1, 2, ...$ Perhaps the purely kinematical explanation of these scales is not possible and genuine dynamics is needed. For sinusoidal logarithmic plane waves the harmonics correspond to the scalings of the argument by powers of some scaling factor x. Thus the powers of Golden Mean might be associated with logarithmic sinusoidal plane waves.

b) Physicist Hartmuth Mueller has developed what he calls Global Scaling Theory [11] based on the observation that powers of e (Neper number) define preferred length scales. These powers associate naturally with the nodes of logarithmic sinusoidal plane waves and correspond to various harmonics (matter tends to concentrate on the nodes of waves since force vanishes at the nodes). Mueller talks about physics of number line and there is great temptation to assume that deep number theory is indeed involved. What is troubling from TGD point of view that Neper number e is not algebraic. Perhaps a more general approach allowing also transcendentals must be adopted. Indeed, since e^p is ordinary p-adic number in R_p, a finite-dimension transcendental extension containing e exists.

c) Classical mathematics, such as the theory of elementary functions, involves few crucially important transcendentals such as e and π. This might reflect the evolution of cognition: these numbers should be cognitively and number theoretically very special. The numbers e and π appear also repeatedly in the basic formulas of physics. They however look p-adically very troublesome since it has been very difficult to imagine a physically acceptable generalization of such simple concepts as exponent function, trigonometric functions, and logarithm resembling its real counterpart by allowing only the extensions of p-adic numbers based on algebraic numbers.

d) Number theoretic entropies measured in bits are proportional to $log(p)/log(2)$. The idea that these entropies are rational fractions of bit is attractive and implies that $log(p)$ for all primes is proportional to the same transcendental number. This would mean that logarithm of the rational number field would be a transcendental multiple of finite-dimensional extension of rationals involving possibly e.

These considerations stimulate the question whether, besides the extensions of p-adics by algebraic numbers, also the extensions of p-adic numbers involving e, and perhaps even π and other transcendentals might be needed. The intuitive expectation motivated by the finiteness of human intelligence is that these extensions might have finite algebraic dimensions. On the other hand, if one is only interested in quantities derived from phases $exp(i2\pi/n)$, a finite-dimensional algebraic extension is enough. π is needed only if one wants to deal with say length of circle's circumference in the p-adic context, and one could argue that p-adic Riemann geometry is local and only about angles and infinitesimal distances.

Second question is whether there might be some dynamical mechanism allowing to understand the hierarchy of scalings coming in powers of some preferred transcendentals and algebraic numbers like Golden Mean. Conformal invariance implying that the system is characterized by a universal spectrum of scaling momenta for the logarithmic counterparts of plane waves seems to provide this mechanism. This spectrum is determined by the requirement that it exists for both reals and all p-adic number fields assuming that finite-dimensional extensions are allowed in the latter case. The spectrum corresponds to the zeros of the Riemann Zeta if Zeta is required to exist for all number fields in the proposed sense, and a lot of new understanding related to Riemann hypothesis emerges and allows to develop further the previous TGD inspired ideas about how to prove Riemann hypothesis [12, 13].

The following two ideas serve as guide lines in the attempt to relate cognition, intentionality and number theory to each other so that number theory would allow to construct a more detailed view about the realization of intentionality and cognition. As a matter fact, the general ideas about intention and cognition in turn generate very general number theoretical conjectures.

a) Real and p-adic number fields form a book like structure with pages represented by number fields glued together along rationals forming the rim of the book. For the extensions of p-adic numbers further common points result and the book becomes fractal if all possible extensions are allowed. This picture generalizes to the level of the imbedding space and allows to see space-time surfaces as consisting of real and p-adic space-time sheets belonging to various extensions of these numbers. This generalized view about numbers gives hopes about an unambigious definition of what some number, say e, appearing in an extension of p-adic numbers really means.

b) The first new idea is roughly that the discovery of notion of any algebraic or transcendental number x (such as Φ or e) involves a quantum jump in which there is generated a p-adic space-time sheet for which the existing finite-dimensional extension of p-adic numbers is replaced by a finite-dimensional extension involving also x. Also some higher powers of the number are involved. For instance, for e $p-1$ powers are necessarily needed (e^p exists p-adically).

c) The p-adic-to-real transition serving as a correlate for the transformation of intention to action is most probable if the number of common rational valued points for the p-adic and real space-time sheet is high. The requirement of real and p-adic continuity and even smoothness however forces upper and lower p-adic length scale cutoffs so that common points are in certain length scale range.

d) The points of M^4_+ with integer valued Minkowski coordinates using CP_2 length related fundamental length scale as a basic unit is a good guess for the subset of M^4_+ defining the rational points of the M^4_+ involved. CP_2 coordinates as functions of M^4_+ coordinates should be rational or belong to some finite-dimensional extension of p-adics. Of course, also rational points of M^4_+ are possible, and the evolution of cognition should correspond to the increase of the algebraic dimension of the extension.

e) A very powerful hypothesis is that the p-adic and real functions have the same analytic form besides coinciding at the chosen rational points defining the p-adic pseudo constant involved. Since the pseudo constant defines the corresponding real function in rational points, there are indeed good hopes that the transformation of p-adic intention to real action is possible. This assumption favors functions which allow at some point (most naturally origin) a Taylor series with rational valued Taylor coefficients.

3.2.2 Is e an exceptional transcendental?

Neper number is obviously the simplest one and only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$. In case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

a) Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

b) The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

c) The differential equation allows to develop $f(x)$ in power series, say in origin

$$f(x) = \sum f_n \frac{x^n}{n!}$$

such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

3.2.3 Some no-go theorems

Elementary functions like exp(x), log(1+x), cos(x), sin(x), are obviously favored by the previous considerations, in particular by the requirement of the form invariance of the function in p-adic-to-real transition. They indeed have p-adic Taylor expansion which converges for $|x|_p < 1$. The definition at integer valued points for which $x \ mod \ p = n$, $n = 0, 1, ..., p-1$, requires the introduction of an extension of p-adic numbers. The natural first guess is that this extension is finite-dimensional. Of course, this is just a hypothesis to be discussed and motivated by the idea that p-adic extensions reflect our own finite intelligence.

1. *Can powers of $log(p)$ define a finite-dimensional extension of p-adics?*

The number theoretical entropy associated with any p-adic prime for which the ordinary logarithm $log(p_n)$ is replaced by the logarithm of the p-adic norm of p_n, is proportional to a $log(p)$-factor. As already noticed, if bit is used as unit, then only the rationality of $log(p)/log(2)$ is needed and $log(p)$ need not correspond to a finite-dimensional extension of p-adics.

The first observation is that $log(1+x)$, $x = O(p)$ exists as an ordinary p-adic number and the logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ need be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. The problem is however that the powers of $log(m)$ and $log(p)$ might generate an infinite-dimensional extension of p-adic numbers.

First some no-go theorems inspired by wishful conjectures (professional number theorists must regard me as an idiot!).

a) $log(p) = q/t$, where t is a transcendental number, say π, cannot hold true. The reason is that the rationality of $log(p_1)/log(p_2) = q_1/q_2 = r/s$ implies that $p_1^s = p_2^r$ in contradiction with the prime number property of p_1 and p_2.

b) $log(q)$, q prime, cannot correspond to a finite dimensional extension of R_p in the sense that a finite power of $log(q)$ would be a rational number. Assume that this is the case, i.e. $(log(q))^{m_{p,q}} = x_{p,q}$, where $x_{p,q}$ is an ordinary p-adic number in R_p, and assume that e belongs to extension. For definiteness let us assume $|x_{p,q}| < 1$ and write

$$q = exp(log(q)) = \sum_n log(q)^n/n! = \sum_{k=0}^{m-1} c_k log(q)^k \ \ , \ \ c_k = \sum_n \frac{x_{p,q}^n}{(k+nm_{p,q})!} \ \ .$$

The righthand side gives m terms corresponding to the m powers of $log(q)$ and only the lowest term can be non-vanishing and equals to q. The convergence of series requires that $x_{p,q}$ has p-adic norm smaller than one. This however implies that lowest order term has p-adic norm equal to one. For $q = p$ this leads to contradiction since one would have $p = 1 + O(p)$. For $|x_{p,q}|_p \geq 1$ the argument fails since the expansion does not make sense. For $q = exp(p^k log(q))$, k sufficiently large, the expansion exists and in this case one as $q^{p^k} = 1 + O(p)$, which for $q = p$ gives a contradiction.

c) One might hope that $log(p)$ belongs to an extension containing e or its root, or in the most general case root of a polynomial with coefficients which belongs to an extension of rationals by e and algebraic numbers. For instance, the ansatz $log(p) = e^{q_1(p)} q_2(p)$ with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes, would guarantee that logarithms belong to a finite-dimensional extension. There are no problems with the prime property as is clear from the expression

$$p_1 = p_2^{[exp(q_1(p_1)-q_1(p_2)]\times \frac{q_2(p_1)}{q_2(p_2)}} \ .$$

From the assumption it follows that the exponent cannot reduce to a rational number.

Unfortunately the ansatz does not work! One can write

$$p_1 = exp\left(e^{q_1(p_1)}q_2(p_1)\right)$$

and for those primes p_2 whose positive power divides $q_2(p_1)$, one can expand the exponential in a converging power series in powers of a root of e, and one obtains that ordinary p-adic number is expressible as a non-trivial combination of powers of a root of e.

e) Obviously one must give up hopes for obtaining a finite-dimensional extension for the logaritms. One might however hope that $log(p)/log(2)$ is always rational in order that p-adic entropy would be always rational multiple of bit. This is achieved if one has

$$log(p) \ = \ e^{q_1(p)}q_2(p) \times t \ , \ q_2(p_1) \neq q_2(p_2) \ \text{for} \ p_1 \neq p_2 \qquad (10)$$

such that t is a transcendental number other than root of e so that one does not get contradiction by exponentiating both sides of the above equation. This ansatz does not lead to any obvious contradictions. For instance, power of π is a reasonable candidate and for physical reasons $t = 1/\pi$ is a favored value of t.

3. π cannot belong to a finite-dimensional extension of p-adic numbers

A simple argument excludes the possibility that π could belong to some finite-dimensional extension $\pi = \sum c_n e_n$. If this is the case one can write $exp(ip^k\pi) = -1$ as a converging Taylor expansion in powers of p for high enough value of k, and the coefficients of all e_n expect $e_0 = 1$ must vanish. Since the terms in this series come in powers of p it is highly implausible that they could sum up to zero. In fact, even the coefficient of $e_0 = 1$ has wrong sign. By considering more general numbers $exp(iq\pi)$ one obtains that the expansion in terms of e_i equals to the expression of phase in infinite number of different algebraic extensions. Thus it seems obvious that π cannot belong to a finite extension.

3.2.4 Does the integration of complex rational functions lead to rationals extended by a root of e and powers of π?

These cold showers suggest that the best one might hope is that the numbers like $log(p)$ and $log(\Phi)$ could be proportional to some power π with a coefficient which belongs to a finite extension of p-adic numbers containing e. This might make it possible to continue the theory to p-adic context and also make very strong predictions.

The elementary differential and integral calculus provides important hints for as how to proceed. Derivation takes rational functions to rational functions unlike integration since the integrals of $1/x$ and and $1/(1+x^2)$ give $log(x)$ and $arctan(x)$ leading outside the realm of rational numbers. One can go to complex plane and consider the integrals of complex rational functions with complex rational coefficients and here one encounters integrals over closed curves and between two points. The rational approach is to consider rational complex plane, and first restrict to Gaussian integers which allow primes.

i) The first observation is that residy calculus for rational functions gives always integrals which are of form $2\pi iq$, q a rational number.

ii) The integral $I = \int_a^b dz/z$, $a = m_1 + in_1$, $b = m_2 + in_2$ in turn gives

$$I = log(a/b) = \frac{1}{2}\left(log(m_2^2 + n_2^2) - log(m_1^2 + n_1^2)\right)$$

$$+i(arctan(n_2/m_2) - arctan(n_1/m_1)) \ .$$

a) The strongest hypothesis would be that logarithm and arctan are also rationally proportional to π so that all integrals of this kind lead to an infinite-dimensional transcendental extension of p-adic numbers containing π. The strong hypothesis cannot be correct. Consider arcus tangent as an example. $arctan(m/n) = r\pi/s$ would imply $tan(r\pi/s) = m/n$, and this cannot hold true since it would imply that s:th powers of Gaussian integer $n+im$ would give an ordinary integer. This would be also true for Gaussian primes and the decomposition of Gaussian integers as products of Gaussian primes would become non-unique. There is this kind of uniqueness but this is due the units $exp(i\pi/4)$ and its powers. Indeed, $arctan(1) = \pi/4$ and proportional to π.

b) One can overcome this difficulty by replacing the ansatz with

$$arctan(q) = e^{q_1(q)}q_2\pi$$

such that $q_1(q)$ is non-vanishing for $q \neq \pm 1 \pm i$ corresponding to the units of Gaussian primes. This ansatz is completely analogous to the ansatz for

$log(p)$. The beauty of this ansatz would be that the imaginary parts for the integral of $1/(z-z_0)$ between complex rational points would be proportional to π irrespective of whether the integration is over a closed or open curve. The real parts of complex integrals in turn would be proportional to $1/\pi$ of $log(p) \propto 1/\pi$ ansatz holds true.

The requirement that complex integrals are powers of π could also mean quantization of topology in TGD framework. For instance, the conformal equivalence classes of Riemann surfaces of genus g are represented by period integrals of 1-forms defining elements of cohomology group H^1 over the circles representing the elements of homology group H_1. Restricting the cohomology to a rational cohomology, the periods with standard normalization would be quantized to complex rationals multiplied by a power of π. For surfaces characterized by a given power of π one might perhaps perform the p-adicization finite-dimensionally by suitable normalizations by powers of π.

3.2.5 Why should one have $p = q_1 exp(q_2)/\pi$?

There are good physical arguments suggesting that $log(p)$ should be proportional to $1/\pi$.

a) π appears naturally in the plane wave solutions of field equations $exp(in\pi u)$, $u = x/L)$. These phases are well defined in a finite-dimensional algebraic extension if x/L is rational. One can however consider also logarithmic plane waves

$$exp(iku), \quad u = log(x/L) \ ,$$

and ask under what conditions they are well defined and in particular, under what conditions the real/imaginary parts of these plane waves can have zeros at $u = e^n$ required by Mueller's hypothesis [11]. Mueller's hypothesis implies that $exp(ikn)$ has zeros so that $k = q\pi$ must hold true. Thus one obtains essentially ordinary plane waves.

If one has $u = q_1 e^n$, q_1 rational, one obtains also the exponential $exp(iq\pi log(q_1))$. From the point of view of p-adicization program it would be very nice if also this exponent would exist p-adically. This is guaranteed if one has

$$log(p) = \frac{q_1(p) exp\left[q_2(p)\right]}{\pi}$$

for every prime p. One can write

$$exp(iq\pi u) = exp\left[iqq_1(p)exp(q_2(p))\right] \ .$$

The exponential exists for those primes p_1 for which the exponent is divisible by a positive power of p_1. This means quantization conditions favoring selected primes p_1 or alternatively scaling momenta q. An easy manner to satisfy these conditions is to assume that q is a multiple of a power of p.

c) Besides Mueller's hierarchy in powers of e there are also p-adic hierarchies and the hierarchies associated with Golden Mean and one can look whether these hierarchies are obtained for suitable logarithmic waves. For $u = x/L = mp^n$ the scaling wave reads

$$exp(iku) = exp\left[iknlog(p)\right] exp\left[iklog(m)\right] \quad .$$

For $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$ the existence of nodes for the the first factor requires $k = q\pi^2 exp\left[-q_2(p)\right]$. The second factor exists only for $m = 1$ so that nodes are possible only at $u = p^n$.

Note that $k = q\pi$ for e so that these length scale hierarchies are distinguishable number theoretically. This assumption implies that also the second exponential of product can exist in a finite-dimensional algebraic extension and can have even nodes. For the hierarchy defined by powers of Golden Mean the assumption $log(\Phi) = q_1 q exp(q_2)/\pi$ would lead to similar conclusions. Again one must leave door open for more general power of π.

3.2.6 p-Adicization of vacuum functional of TGD and infinite primes

A further input comes from TGD. The basic challenge is to continue the exponent $exp(K)$ of the Kähler function to p-adic number fields. K can be expressed as

$$K = \frac{S_K}{16\pi\alpha_K} \quad ,$$

where α_K is so called Kähler coupling strength and $S_K = \int J_{\mu\nu}J^{\mu\nu}\sqrt{g}d^4x$ is Kähler action, which is essentially the Maxwell action for the induced Kähler form. The dream is that an algebraic continuation from the extensions of rational numbers defining finite extensions of p-adic numbers allows to define the theory in various number fields. The fulfillment of this dream requires that physically important quantities such as the exponent of Kähler function for CP_2 extremal and other fundamental extremals exist in a finite-dimensional extension of p-adic numbers.

1. *What is the value of Kähler coupling strength?*

The value of Kähler coupling strength is analogous to a critical temperature and can have only discrete values.

a) The discrete p-adic evolution of the Kähler coupling strength follows from the requirement that gravitational coupling constant is renormalization group invariant [A6]. When combined with the requirement that the exponent of CP_2 action is a power of prime, the argument would give

$$\frac{1}{\alpha_K(p)} = \frac{4}{\pi} log(K^2) \ , \ K^2 = \prod_{q=2,3,...23} q \times p$$

with $\alpha_K(p = M_{127}) \simeq 136.5585$ and $\alpha/\alpha_K \simeq .9965$. Note that M_{127} corresponds to electron length scale. If the action is a rational fraction of CP_2 action, and the extension of p-adic numbers is by an appropriate root of p is enough to guarantee the existence of the Kähler function.

b) One can consider also an alternative ansatz based on the requirement that Kähler function is a rational number rather than a logarithm of a power of integer K^2. This requires an extension of p-adic numbers involving some root of e and a finite number of its powers. S_R must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of CP_2 type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of e or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of e and algebraic numbers.

Since CP_2 action is rationally proportional to π^2, the exponent is rational if $4\pi\alpha_K$ satisfies the same condition. If the conjecture $log(p) = q_1(p)exp\left[q(p_2)\right]/\pi$ holds , then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $Q \simeq 4/\pi$. The presence of $log(K^2)$, K^2 product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor.

This gives for the Kähler function the expression

$$K = Q\left[q_1(p)exp\left[q_2(p)\right] + \sum_i q_1(q_i)exp\left[q_2(q_i)\right]\right]\frac{S}{S_{CP_2}} \ . \qquad (11)$$

$exp(K)$ exists p-adically only provided that K has p-adic norm smaller than one. For given p this poses strong conditions unless one assumes that the condition $S/S_{CP_2} = p^n r$, r rational. In the case of many-particle state of

CP_2 extremals this would mean that particle number is divisible by a power of p.

For single CP_2 extremal, the fact that p cannot divide $q_1(p)$ means that either Q contains a power of p or the sum of terms is proportional to a power of p. Obviously this condition is extremely strong and allows only very few primes. One might wander whether this could provide the first principle explanation for p-adic length scale hypothesis selecting primes $p \simeq 2^k$, k integer, and with prime power powers being preferred.

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from α is .2 per cent (of course, α_K need not equal to α and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1 exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and might lead to an improved understanding of the origin of the mystery prime $k = 137$. Of course, one must be extremely cautious with the numerics. For instance, one could replace 137/107 with the ratio of $137/log(M_{107}$ and in this case the M_{107} would become an "easy" prime.

3.2.7 A connection with Riemann hypothesis

The considerations of the preceding subsection led to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. These approaches are summarized in [A8]. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

1. First approach

In this approach (see the preprint in [12] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [13]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. Second approach

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \quad .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by a positive power of the ratio $Y = X/(1+X)$, where $X = \prod p_i$ is the product of all primes, would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

a) At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short ($k < -1/2$) or long distances ($k > -1/2$).

b) Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

c) The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

3.2.8 Polyzetas, braids, and TGD

Already Riemann defined also functions which he called polyzetas [5](the role of polyzetas are discussed in a wider context in [8]). Polyzeta $\zeta(z_1, ..., z_n)$ is defined via the sum

$$\zeta(z_1, ..., z_k) = \sum_{n_1 > n_2 > ... > n_k \geq 1} \prod_i n_i^{-z_i} \tag{12}$$

The values $\zeta(n_1, ..., n_k)$ of polyzetas appear as values of the integrals

$$I_{\epsilon_1,....,\epsilon_n}) = \int_{0<t_1<...t_n<1} \omega_{\epsilon_1}(t_1) \wedge .... \wedge \omega_{\epsilon_n}(t_n) \tag{13}$$

where one has $\epsilon_i \in \{0,1\}$, $\epsilon_0 = 1$, $\epsilon_n = 0$, $\omega_0(t) = dt/t$, $\omega_1(t) = dt/(1-t)$. For instance, one has $\zeta(n) = I_{1,0,...,0}$ for n-dimensional integral. Polyzetas satisfy identifies following directly from their defining representations. For instance, the identity

$$\zeta(a)\zeta(b) = \zeta(a+b) + \zeta(a+b) + \zeta(a,b) + \zeta(b,a)$$

holds true. These identities relate only polyzetas for which the total "momentum" $K = n_1 + ...n_k$ is same. In fact, polyzetas form an Abelian graded algebra graded by the value of K satisfying $U_{K_1}U_{K_2} \subset U_{K_1+K_2}$. Note that the identities are true for other than integer values of k_i and one can consider also more general gradings, say the one based on half odd integer values of K_i suggests by TGD based interpretation of the real part of z_k as a real part of a complex conformal weight [A8].

Polyzetas appear to have a fundamental role in conformal quantum field theories. In particular, the so called graded Grothendieck-Teichmuller group acts as automorphisms of braid group completed to a Lie-group in the field of rationals, and has Lie algebra, whose generators are labelled by $\zeta(k)$, $k \geq 3$. It has turned out that polyzetas appear universally in the regularization of the Feynman diagrams of quantum field theories and GRT group is believed to act as an automorphism group in the spaces of quantum field theories transformable to each other by a change of coupling constants.

1. Conjecture about the values of polyzetas guaranteing the number theoretical universality of quantum algebras

Polyzetas have inspired several conjectures. One conjecture is that all polynomial relations between the values of polyzetas are of those which are known to follow from the manipulation of their series. This would mean that the values of polyzetas at integer arguments are transcendental numbers.

TGD suggests a strengthening of this conjecture. From $\zeta(2) = \pi^2/6$, and $\zeta(4) = \pi^4/90$, and more general result $\zeta(2n) \propto \pi^{2n}$, one could guess that that the values of polyzetas at the level $K = n$ are proportional of form $q\pi^n$. This would sharpen the conjecture about the transcendental nature of polyzeta values and would also resemble the conjecture $y = q\pi$ for the zeros of ζ (here also power of e might be involved). In particular, it would reduce the identities between polyzetas to the field of rationals and allow to deduce a lot of information about the values.

As following argument shows, this conjecture would imply that quantum groups can be continued algebraically in a straightforward manner to various p-adic number fields ([6] is an excellent reference).

a) Bi-algebra A is an algebra possessing besides multiplication with unit also co-multiplication $\Delta : A \rightarrow A \otimes A$ and co-unit $\epsilon : A \rightarrow C$. Both act as algebra homomorphisms and co-unit acts as the inverse of co-multiplication: $(\epsilon \otimes Id)\Delta = (Id \otimes \epsilon)\Delta = Id$. Comultiplication $A \otimes A \rightarrow A \otimes A \times A$ satisfies the co-associativity constraint $(Id \otimes \Delta)\Delta = (\Delta \otimes Id)\Delta$. Clearly, co-multiplication is kind of a "time reversal" of multiplication.

b) For quasi-bi-algebras co-unit is acts as an inverse of co-multiplication only modulo algebra homomorphism: $(\epsilon \otimes Id)\Delta(a) = lal^{-1}$, $(Id \otimes \epsilon)\Delta(a) = rar^{-1}$. Also associativity holds true also only modulo algebra homomorphism: $(Id \otimes \Delta)\Delta = \Phi((\Delta \otimes Id)\Delta)\Phi^{-1}$. Φ is an element of $A \otimes A \otimes A$ and known as an associator.

c) The so called Drinfeld associator $\Phi(A, B)$ is a universal function of two non-commutative arguments and relates very closely to Φ. $\Phi(A, B)$ can be expressed as Taylor series with respect to A and B with coefficients which are rational multiples of numbers $a_n = \zeta(n)/(i2\pi)^n$. If the numbers a_n are rationals, the Drinfeld associator allows the continuation of quasi-bialgebras to various p-adic number fields so that they become universal structures. This is true at least if one allows the use of the group of units defined by infinite rationals and defining extension of rational numbers equivalent with ordinary rationals with respect to real norm (see the Appendix).

2. Questions

In TGD framework the momentum K plays the role of conformal weight (scaling momentum) and this inspires the generalization of the conjecture to half odd integer values of d allowed also by conformal invariance. In this case also square roots of rationals would be possible. For all values of z the conjecture cannot hold true since this would imply that ζ is a product of rational function of z and π^z. In any case it would seem that important numbers relating to polyzeta functions could belong to a infinite-dimensional extension of rationals by powers of π and be of utmost importance in quantum theory.

This raises some questions.

a) Could GRT group appear as an automorphism group of some structure in TGD? Each complex degree of freedom of configuration space of three surfaces corresponds to single complex variable and to a 4-dimensional tensor factor of configuration space spinor field, and I have already discussed the possibility that each topological degree of freedom of this kind corresponds to a strand of a braid. Hence GRT group would act as automorphisms of the braid group acting as symmetries of the configuration space spinor fields and as already explained the braid group could relate directly to the topology of the configuration space.

As already proposed, punctures could also correspond to the positions of the second end for handles of the lightlike boundaries. Since the 3-surfaces related by a permutation of complex coordinates are equivalent, configuration space would be like C^n/S_n at the limit $n \rightarrow \infty$. This would mean that the infinite-dimensional braid group acts as a subgroup of the first homotopy group of the configuration space.

b) Polyzetas appear at integer values of argument. Integer values of the argument, and more generally rational values with some cutoff, appear also in the algebraic continuation of functions from the field of rationals to various number fields. Could polyzetas relate closely to the continuation of the configuration space spinor fields from rationals to various p-adic number fields? Could polyzetas with weight K divided by π^K define a hierarchy of universal quantum states appearing in the construction of S-matrix also in TGD?

3.3 Infinite primes, p-adicization, and the physics of cognition

The continuation of rational physics to p-adic number fields poses deep technical problems and the so called infinite primes (see the chapter "Quaternions, Octonions, and Infinite Primes", which were one of the first mathematical discoveries inspired by the work with TGD inspired theory of consciousness might provide an elegant general solution to these difficulties. Infinite primes might also allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time. Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving as Mother of all algebras. The units of the algebra multiplying ordinary rational numbers are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure.

3.3.1 Could infinite primes appear in the p-adicization of the exponent of Kähler action?

The difficulties related to the p-adic continuation of Kähler function to an arbitrary p-adic number field and the fact that infinities are every day life in quantum field theory bring in mind infinite primes discussed in [A7].

Infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1 + X,\ X = \prod_i p_i$, where product is over all finite primes. The factor $Y = X/(1+X)$ is in the real sense equivalent with 1. In p-adic sense it has norm $1/p$ for every prime. Thus one could multiply Kähler function by Y or its positive power in order to guarantee that the continuation to p-adic number fields exists for all primes. Of course, these states might differ physically in p-adic sense from the states having $Y = 1$. Thus it would seem that the physics of cognition could differentiate between states which are in real sense equivalent.

More general infinite primes are of form $\Pi = nX/m + n$, such that $m = \prod_i q_i$ and $n = \prod_i p_i^{n_i}$ have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion in each mode labelled by q_i and n_i bosons labelled in modes labelled by p_i. Also positive powers of the ratio $Y = X/\Pi$, Π some infinite prime, are possible as a multiplier of the Kähler function. In the real sense this ratio would correspond to the ratio m/n.

If this picture is correct, infinite primes would emerge naturally in the p-adicization of the theory. Since octonionic infinite primes could correspond to the states of a super-symmetric quantum field theory more or less equivalent with TGD, the presence of infinite primes could make it possible to code the quantum physical state to the vacuum functional via coupling constant renormalization.

One could also consider the possibility of defining functions like $exp(x)$ and $log(1+x)$ p-adically by replacing x with Yx without introducing the algebraic extension. The series would converge for all values of x also p-adically and would be in real sense equivalent with the function. This trick would apply to a very general class of Taylor series having rational coefficients. One could also say that p-adic physics allowing infinite primes would be very similar to real physics.

The fascination of infinite primes is that the ratios of infinite primes which are ordinary rational numbers in the real sense could code the particle number content of a super-symmetric arithmetic quantum field theory. For the octonic version of the theory natural in the TGD framework these states could represent the states of a real Universe. Universe would be an algebraic hologram in the sense that space-time points, something devoid of any structure in the standard view, could code for the quantum states of possible Universes!

The simplest manner to realize this scenario is to consider an extension of rational numbers by the multiplicative group of real units obtained from infinite primes and powers of X. Real number 1 would code everything in its structure! This group is generated as products of powers of $Y(m/n) = (m/n) \times [X/\Pi(m/n)]$ which is a unit in the real sense. Each $Y(m/n)$ would define a subgroup of units and the power of $Y(m/n)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of p with the prime factors of m and n forming an exception and being reflected in p-adic physics of cognition, Universe would "feel" its real or imagined state with its every point, be it a point of space-time surface, of imbedding space, or of configuration space.

3.3.2 Infinite primes and p-adic physics as physics of cognition

There is an objection against the notion of finite-p p-adic evolution of the Kähler coupling strength. Logical consistency would suggest that configuration space sectors D_P are labelled by infinite primes P. The formula for the Kähler coupling strength however depends on finite p-adic prime p. The substitution of an infinite-p p-adic prime into the formula implies an infinite value for $1/\alpha_K$ in a complete accordance with the intuition provided by the renormalization theory. The question is whether infinite-p p-adic coupling constant evolution could effectively reduce to finite-p p-adic coupling constant evolution and if it does, how this is possible. One can imagine two possible answers to the question.

First of all, the sector D_P (P denotes infinite prime) of the configuration space could correspond to the subgroup of units of rational numbers defined by the unit $Y(n/m) = (n/m) \times X/\Pi(n/m)$ at the level of space-time sheet of the particle. At the level of space-time geometry this would mean that the rational values of space-time coordinates would possess this group of units. Also sectors D_Q characterized by infinite rationals are possible and correspond to the products of powers of $Y(n/m)$ for various rationals n/m. Every space-time point would be characterized by the subgroup of the group generated by the units defined by ratios $(n/m)X/Pi(n/m)$. Also the units associated with the higher levels of the infinite hierarchy of infinite primes could be present so that the group of units would be really huge!

1. *The physics of infinite primes as physics of cognition?*

The first possibility is that infinite primes make them visible only in the physics of cognition. Infinite rationals would give rise to a group of multiplicative units of rational numbers multiplying also the inverse of the Kähler coupling strength. The evolution of coupling constant as a function of infinite primes would not be seen at all at the level of real physics but would make itself manifest in the physics of cognition since the possibility to continue the exponent of Kähler function from rationals to p-adics depends strongly on this factor in case of primes associated with the rational numbers characterizing it. The infinities of quantum field theories might however reflect the presence of this multiplicative unit.

The factor 137/107 could be understood in terms of the ratio

$$\frac{X}{\Pi(107/137)}$$

meaning that the bosonic mode $k = 107$ and fermionic mode $k = 137$ are populated. The presence of this factor along would imply the convergence of the exponent of Kähler action as p-adic power series for almost all primes. The length scales corresponding to hadrons ($k = 107$) and atoms ($k = 137$) would be cognitively very special. Number 24 is familiar from string models but appears also in the context of elementary particle vacuum functionals (see the chapter "Elementary Particle Vacuum Functionals"), which suggest that the fermionic occupation numbers might relate to the elementary particle vacuum functional somehow.

2. Do infinite primes contribute also to the physics of matter?

The second option is that the evolution of infinite primes is visible also at the level of real physics. The value of the Kähler coupling strength is formally zero for infinite-p p-adics. One could however consider the possibility that the Kähler coupling constant is proportional to the logarithm $log\,[X/\Pi(n/m)]$, which gives a perfectly well defined ordinary real number $log(m/n)$ in real sector. In quantum field theory the expression of this logarithm as a difference of logarithms of infinite numbers might correspond to a subtraction of infinity.

This would suggest how to understand the proposed formula for Kähler coupling strength as holding true for $m/n \simeq K^2 = 2 \times 3 \times .... \times 23 \times p$. The naive conclusion would be $(n, m) = (1, K^2)$. K^2 would represent the number theoretic analog of 10-fermion state for which the modes $q = 2, 3, ..., 23, p$ are populated. The cognitively simple space-time sheets of elementary particles might correspond to simple units of this kind.

$log\left[X/\Pi(1/K^2)\right]$ is however problematic from the point of view of p-adicization. For $k = 4/\pi$ option (the exponent for the action of CP_2 type extremal is integer K^2) the exponent of Kähler function can be written in the form

$$exp(K) = \left[\frac{X}{\Pi(1/K^2)}\right]^{\frac{S_K}{S_{CP_2}}} \quad .$$

For rational values of S_K/S_{CP_2} a finite-dimensional extension at the level of infinite primes would be required and this is not encouraging. For $k = 137/107$ factor of π appears in the exponent of Kähler function for CP_2 type extremal. Unless $log\left[X/\Pi(1/K^2)\right]\pi$ satisfies in some sense a condition analogous to $log(p)\pi = q_1 exp(q_2)$, the exponent does not exist unless one allows an extension containing the powers of π. These shortcomings encourage to think that the first option is the right one so that the physics of infinite primes would be physics of cognition coded directly into the structure of the points of space-time sheets.

3.3.3 The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions group becomes non-commutative and in case of unit octonions the group is replaced by a kind non-associative generalization of group. Non-commutativity means that one cannot tell how the products AB and BA of two infinite primes explicitly since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units is a free group. Same holds true in the case of octonionic units. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are rationals or complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \rightarrow 1_1 = A/I_1 \rightarrow A_1/I_2...$ where the ideal I_k is defined by $k:th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface would have same value for all points of A which differ by an element of the ideal. At the configuration space level, the configuration space spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i\Pi_i$.

3.3.4 The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures. Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical congition. I must admit that that it did not occur to me that Leibniz might have been right about his monads! The idealization is however in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One can say that each point represents an equation. The left hand side of the equation corresponds to the element of the free algebra defined by octonionic units. Consider as an example product of powers of $X/\Pi(Q_q)$ representing infinite quaternionic rationals. Equality sign corresponds to the evaluation of this expression by interpreting it as a real quaternionic rational number: real physics does the evaluation automatically. The information about the primes appearing as factors of the result is not however lost at cognitive level. Note that the analogs of quantum states represented by superpositions of the unit elements of the algebra A can be interpreted as equations defining them.

3.3.5 When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n + 1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [14] poses a question which at first sounds absurd. What might be the the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$\begin{aligned} Y_I &= \frac{\prod_i Y(q^I_{1i})}{\prod_i Y(q^I_{2i})} \ , \quad Y_F = \frac{\prod_i Y(q^F_{1i})}{\prod_i Y(q^F_{2i})} \ , \\ q^I_{ki} &= \frac{n^I_{k_i}}{m^I_{k_i}} \ , \qquad q^F_{ki} = \frac{n^F_{k_i}}{m^F_{k_i}} \ , \end{aligned} \tag{14}$$

Here $m^{\cdot}_{\cdot}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{\cdot}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{\cdot}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$\frac{\prod_i q^I_{1i}}{\prod_i q^I_{2i}} = \frac{\prod_i q^F_{1i}}{\prod_i q^F_{2i}} \ . \tag{15}$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labelled by the prime p is $E_p = log(p)$:

$$\begin{aligned} E^I &= \sum_i log(n^I_{1i}) - \sum_i log(n^I_{2i}) - \sum_i log(m^I_{1i}) + \sum_i log(m^I_{2i}) = \\ &= \sum_i log(n^F_{1i}) - \sum_i log(n^F_{2i}) - \sum_i log(m^F_{1i}) + \sum_i log(m^F_{2i}) = E^F \ . \end{aligned} \tag{16}$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

3.3.6 TGD inspired analog for d-algebras

Maxim Kontsevich has done deep work with quantizations interpreted as a deformation of algebraic structures and there are deep connections with this work and braid group [8]. In particular, the Grothendienck-Teichmueller algebra believed to act as automorphisms for the deformation structures acts as automorphisms of the braid group at the limit of infinite number of strands. I must admit that my miserable skills in algebra does not allow to go to the horrendous technicalities but occasionally I have the feeling that I have understood some general ideas related to this work. In his article "Operads and Motives in Deformation Quantization" Kontsevich introduces the notions of operad and d-algebras over operad. Without going to technicalities one can very roughly say that d-algebra is essentially d-dimensional algebraic structure, and that the basic conjecture of Deligne generalized and proved by Kontsevich states in its generalized form that $d+1$-algebras have a natural action in all d-algebras.

In the proposed extension of various rationals a notion resembling that of universal d-algebra to some degree but not equivalent with it emerges naturally. The basic idea is simple.

a) Points correspond to the elements of the assumed to be universal algebra A which in this sense deserves the attribute $d = 0$ algebra. By its universality A should be able to represent any algebra and in this sense it cannot correspond $d = 0$-algebra of Kontsevich defined as a complex, that is a direct sum of vector spaces V_n and possessing d operation $V_n \rightarrow V_{n+1}$, satisfying $d^2 = 0$. Each point of a manifold represents one particular element of 0-algebra and one could loosely say that multiplication of points represents algebraic multiplication. This algebra has various subalgebras, in particular those corresponding to reals, complex numbers and quaternions. One can say that sub-algebra is non-associative, non-commutative, etc.. if its real evaluation has this property.

b) Lines correspond to evolutions for the elements of A which are continuous with respect to real (trivially) and all p-adic number fields. The latter condition is nontrivial and allows to interpret evolution as an evolution conserving number theoretical analog of total energy. Universal 1-group would consist of curves along which one has the analog of group valued field (group being the group of generalized units) having values in the universal 0-group G. The action of the 1-group in 0-group would simply map the element of 0-group at the first end of the curve its value at the second end. Curves define a monoid in an obvious manner. The interpretation as a map to A allows pointwise multiplication of these mappings which generalizes to all values of d.

One could also consider the generalization of local gauge field so that there would be gauge potential defined in the algebra of units having values on A. This potential would define holonomy group acting on 0-algebra and mapping the element at the first end of the curve to its gauge transformed variant at the second end. In this case also closed curves would define non-trivial elements of the holonomy group. In fact, practically everything is possible since probably any algebra can be represented in the algebra generated by units.

c) Two-dimensional structures correspond to dynamical evolutions of one-dimensional structures. The simplest situation corresponds to 2-cubes with the lines corresponding to the initial and final values of the second coordinate representing initial and final states. One can also consider the possibility that the two-surface is topologically non-trivial containing handles and perhaps even holes. One interpret this cognitive evolutions represents 1-dimensional flow so that the initial points travel to final points. Obviously there is symmetry breaking involved since the second coordinate is in the role of time and this defines kind of time orientation for the surface.

d) The generalization to 4- and higher dimensional cases is obvious. One just uses d-manifolds with edges and uses their time evolution to define $d+1$-manifolds with edges. Universal 3-algebra is especially interesting from the point of view of braid groups and in this case the maps between initial and final elements of 2-algebra could be interpreted as braid operations if the paths of the elements along 3-surface are entangled. For instance field lines of Kähler gauge potential or of magnetic field could define this kind of braiding.

e) The d-evolutions define a monoid since one can glue two d-evolutions together if the outcome of the first evolution equals to the initial state of the second evolution. $d+1$-algebra also acts naturally in d-algebra in the sense that the time evolution $f(A \rightarrow B)$ assigns to the d-algebra valued initial state A a d-algebra valued final state and one can define the multiplication as $f(A \rightarrow B)C = B$ for $A = C$, otherwise the action gives zero. If time evolutions correspond to standard cubes one gets more interesting structure in this manner since the cubes differing by time translation can be identified and the product is always non-vanishing.

f) It should be possible to define generalizations of homotopy groups to what might be called "cognitive" homotopy groups. Effectively the target manifold would be replaced by the tensor product of an ordinary manifold and some algebraic structure represented in A. All kinds of "cognitive" homotopy groups would result when the image is cognitively non-contractible. Also homology groups could be defined by generalizing singular complex consisting of cubes with cubes having the hierarchical decomposition into time evolutions of time evolutions of... in some sub-algebraic structure of A. If one restricts time evolutions to sub-algebraic structures one obtains all kinds of homologies. For instance, associativity reduces 3-evolutions to paths in rational $SU(3)$ and since $SU(3)$ just like any Lie group has non-trivial 3-homology, one obtains nontrivial "cognitive" homology for 3-surfaces with non-trivial 3-homology.

The following heuristic arguments are inspired by the proposed vision about algebraic cognition and the conjecture that Grothendienck-Teichmueller group acts as automorphisms of Feynmann diagrammatics relating equivalent quantum field theories to each other.

a) The operations of $d+1$-algebra realized as time evolution of d-algebra elements suggests an interpretation as cognitive counterparts for sequences of algebraic manipulations in d-algebra which themselves become elements of $d+1$ algebra. At the level of paths of points the sequences of algebraic operations correspond to transitions in which the number of infinite primes defining an infinite rational can change in discrete steps but is subject to the topological energy conservation guaranteing the p-adic continuity of the process for all primes. Different paths connecting a and b represent different but equivalent manipulations sequences.

For instance, at $d = 2$ level one has a pile of these processes and this in principle makes it possible an abstraction to algebraic rules involved with the process by a pile of examples. Higher values of d in turn make possible further abstractions bringing in additional parameters to the system. All kinds of algebraic processes can be represented in this manner. For instance, multiplication table can be represented as paths assigning to an the initial state product of elements a and b represented as infinite rationals and to the final state their product ab represented as single infinite rational. Representation is of course always approximate unless the algebra is finite. All kinds abstract rules such as various commutative diagrams, division of algebra by ideal by choosing one representative from each equivalence class of A/I as end point of the path, etc... can be represented in this manner.

b) There is also second manner to represent algebraic rules. Entanglement is purely algebraic notion and it is possible to entangle the many-particle states formed as products of infinite rationals representing inputs of an algebraic operation A with the outcomes of A represented in the same manner such that the entanglement is consistent with the rule.

c) There is nice analogy between Feynmann diagrams and sequences of algebraic manipulations. Multiplication ab corresponds to a map $A \otimes A \rightarrow A$ is analogous to a fusion of elementary particles since the product indeed conserves the number theoretical energy. Co-algebra operations are time reversals of algebra operations in this evolution. Co-multiplication Δ assigns to $a \in A$ an element in $A \otimes A$ via algebra homomorphism and corresponds to a decay of initial state particle to two final state particles. It defines co-multiplication assign to $a \otimes b \in A \otimes A$ an element of $A \otimes A \rightarrow A \otimes A \otimes A$ and corresponds to a scattering of elementary particles with the emission of a third particle. Hence a sequence of algebraic manipulations is like a Feynmann diagram involving both multiplications and co-multiplications and thus containing also loops. When particle creation and annihilation are absent, particle number is conserved and the process represents algebra endomorphism $A \rightarrow A$. Otherwise a more general operation is in question. This analogy inspires the question whether particle reactions could serve as a blood and flesh representation for $d = 4$ algebras.

e) The dimension $d = 4$ is maximal dimension of single space-time evolution representing an algebraic operation (unless one allows the possibility that space-time and imbedding space dimensions are come as multiples of four and 8 discussed in [A9]). Higher dimensions can be effectively achieved only if several space-time sheets are used defining $4n$-dimensional configuration space. This could reflect some deep fact about algebras in general and also relate to the fact that 3- and 4-dimensional manifolds are the most interesting ones topologically.

References

[TGD] M. Pitkänen (1990), *Topological Geometrodynamics*. Internal Report HU-TFT-IR-90-4 (Helsinki University). http://www.physics.helsinki.fi/∼matpitka/tgd.html .

[padTGD] M. Pitkänen (1995), *Topological Geometrodynamics and p-Adic Numbers*. Internal Report HU-TFT-IR-95-5 (Helsinki University). http://www.physics.helsinki.fi/∼matpitka/padtgd.html.

[cbookI] M. Pitkänen (2001), *TGD inspired theory of consciousness with applications to bio-systems*. http://www.physics.helsinki.fi/∼matpitka/cbookI.html .

[cbookII] M. Pitkänen (2001) *Genes, Memes, Qualia, and Semitrance*, http://www.physics.helsinki.fi/∼matpitka/cbookII.html .

[1] J. K. Webb *et al* (2001), *Further Evidence for Cosmological Evolution of the Fine Structure Constant*, arXiv:astro-ph/0012539.

[2] John Baez, This Weeks's Finds in Mathematical Physics, Week22. http://math.ucr.edu/home/baez/week22.html.

[3] K. Appel and W.Haken (1989), *Every Planar Map is Four Colorable*, Contemporary Mathematics (American Mathematical Society), v. 98.

[4] H. Saleur (1990), *Zeroes of chromatic polynomials: a new approach to the Beraha conjecture using quantum groups*, Comm. Math. Phys. 132, 657.

[5] D. Zagier (1994), *Values of Zeta Functions and Their Applications*, First European Congress of Mathematics (Paris, 1992), Vol. II, Progress in Mathematics 120, Birkhauser, 497-512.

[6] C. Kassel (1995), *Quantum Groups*, Springer Verlag.

[7] C. N. Yang, M. L. Ge (1989), *Braid Group, Knot Theory, and Statistical Mechanics*, World Scientific.

[8] M. Kontsevich (1999), *Operads and Motives in Deformation Quantization*, arXiv: math.QA/9904055.

[9] A. Coghlan (2004), *Our genome 'reads' junk as well as genes*, New Scientist, 21 February.

[10] Articles about TGD inspired theory of consciousness in previous issues of Journal of Non-Locality and Remote Mental Interactions. http://www.emergentmind.org .

[11] H. Mueller, *Global Scaling*, http://www.dr-nawrocki.de/globalscalingengl2.html .

[12] M. Pitkänen (2002), *A Strategy for Proving Riemann Hypothesis*, math-arXiv.org/0111262.

[13] M. Pitkänen (2003), *A Strategy for Proving Riemann Hypothesis*, Acta Math. Univ. Comeniae, vol. 72.

[14] P. Cartier (2001), *A Mad Day's Work: From Grothendienck to Connes and Kontsevich: the Evolution of Concepts of Space and Symmetry*, Bulletin of the American Mathematical Society, Vol 38, No 4, pp. 389-408.

[A1] The chapters *Identification of the Configuration Space Kähler Function* of [TGD].

[A2] The chapters *Construction of Configuration Space Kähler Geometry from Symmetry Principles: Part I,II* of [TGD].

[A3] The chapter *Basic Extremals of Kähler Action* of [TGD].

[A4] The chapter *Configuration Space Spinor Structure* of [TGD].

[A5] The chapter *Construction of Quantum Theory* of [TGD].

[A6] The chapter *Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory* of [TGD].

[A7] The chapter *Quaternions, Octonions, and Infinite Primes* of [TGD].

[A8] The chapter *Riemann Hypothesis and Physics* of [TGD].

[A9] The chapter *TGD as a Generalized Number Theory* of [TGD].

[B1] The chapter *TGD and Astrophysics* of [padTGD].

[B2] The chapter *The Notion of Free Energy and Many-Sheeted Space-Time Concept* of [padTGD].

[C1] The chapter *Negentropy Maximization Principle* of [cbookI].

[C2] The chapter *Quantum Control and Coordination in Bio-Systems: part II* of [cbookI].

[D1] The chapter *Bio-Systems as Conscious Holograms* of [cbookII].

[D2] The chapter *Pre-Biotic Evolution in Many-Sheeted Space-Time* of [cbookII].