What could 2-D minimal surfaces teach about TGD?

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Matti Pitkänen

orcid:0000-0002-8051-4364.
email: matpitka6@gmail.com,
url: http://tgdtheory.com/public_html/,
address: Valtatie 8, as. 2, 03600, Karkkila, Finland.

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Abstract

In the TGD Universe space-time surfaces within causal diamonds (CDs) are fundamental objects.

1. $M^8 - H$ duality means that one can interpret the space-time surfaces in two manners: either as an algebraic surface in complexified M^8 or as minimal surfaces in $H = M^4 \times CP_2$. $M^8 - H$ duality maps these surfaces to each other.
2. Minimal surface property holds true outside the frame spanning minimal surface as 4-D soap film and since also extremal of Kähler action is in question, the surface is analog of complex surface. The frame is fixed at the boundaries of the CD and dynamically generated in its interior. At frame the isometry currents of volume term and Kähler action have infinite divergences which however cancel so that conservation laws coded by field equations are true. The frames serve as seats of non-determinism.
3. At the level of M^8 the frames correspond to singularities of the space-time surface. The quaternionic normal space is not unique at the points of a d-dimensional singularity and their union defines a surface of CP_2 of dimension $d_c = 4 - D < d$ defining in H a blow up of dimension d_c.

In this article, the inspiration provided by 2-D minimal surfaces is used to deepen the TGD view about space-time as a minimal surface and also about $M^8 - H$ duality and TGD itself.

1. The properties of 2-D minimal surfaces encourage the inclusion of the phase with a vanishing cosmological constant Λ phase. This forces the extension of the category of real polynomials determining the space-time surface at the level of M^8 to that of real analytic functions. The interpretation in the framework of consciousness theory would be as a kind of mathematical enlightenment, transcendence also in the mathematical sense.
2. $\Lambda > 0$ phases associated with real polynomials as approximations of real analytic functions would correspond to a hierarchy of inclusions of hyperfinite-factors of type II_1 realized as physical systems and giving rise to finite cognition based on finite-D extensions of rationals and corresponding extensions of p-adic number fields.
3. The construction of 2-D periodic minimal surfaces inspires a construction of minimal surfaces with a temporal periodicity. For $\Lambda > 0$ this happens by gluing copies of minimal surface and its mirror image together and for $\Lambda = 0$ by using a periodic frame.

   A more general engineering construction using different basic pieces fitting together like legos gives rise to a model of logical thinking with thoughts as legos. This also allows an improved understanding of how $M^8 - H$ duality manages to be consistent with the Uncertainty Principle (UP).
4. At the physical level, one gains a deeper understanding of the space-time correlates of particle massivation and of the TGD counterparts of twistor diagrams. Twistor lift predicts M^4 Kähler action and its Chern-Simons implying CP breaking. This part is necessary in order to have particles with non-vanishing momentum in the $\Lambda = 0$ phase.

Contents

1 Introduction

In the quantum TGD based on zero energy ontology (ZEO) space-time surfaces within causal diamonds (CDs) are fundamental objects [L11, L17]. $M^8 - H$ duality plays a central role: the earlier views can be found in [L2, L3, L4] and the recent view in [L12, L13, L16] differing in some aspects from the earlier view. $M^8 - H$ duality means that one can interpret the space-time surfaces in two manners: either as an algebraic surfaces in complexified M^8 or as minimal surfaces in $H = M^4 \times CP_2$ [L17]. $M^8 - H$ duality maps these surfaces to each other.

The twistor lift of TGD is another key element [K2, K3]. It replaces space-time surfaces with their 6-D twistor spaces represented as 6-D surfaces in the product of twistor spaces assignable to M^4 and CP_2 and having an induced twistor structure. This implies dimensional reduction of a 6-D Kähler action to a sum of a 4-D Kähler action and volume term having interpretation in terms of cosmological constant Λ. Kähler structure exists only for the twistor spaces of M^4 and CP_2 [A1] so that the theory is unique.

Each extension of rationals (EQ) corresponds to a different value $\Lambda > 0$. For $\Lambda = 0$, the finite-D extension of rationals determined by real polynomials would be replaced with real analytic functions or subset of them.

Whether $\Lambda = 0$ can be accepted physically, will be one of the key topics of this article. At the level of adelic theory of cognition [L6, L5] this question boils down to the question whether cognition is always finite and related to finite-D extensions of rationals of whether also infinite-D extensions and transcendence can be allowed.

1.1 Basic notions

$M^8 - H$ duality and twistor lift of TGD are the basic notions relevant for what follows and its is appropriate to discuss them briefly.

1.1.1 Space-time surfaces at the level of M^8

The recent view of $M^8 - H$ duality [L12, L13, L16] deserves a brief summary.

At M^8 level, space-time surfaces can be regarded as algebraic 4-surfaces in complexified M^8 having interpretation as complexified octonions. The dynamical principle states that the normal space of the space-time surface at each point is associative and therefore quaternionic. The space-time surfaces are determined by the condition that the real part of an octonionic polynomial obtained as an algebraic continuation of a real polynomial with rational coefficients vanishes.

This gives a complex surface which is minimal surface from which one takes a real part by projecting to real part of complexified M^8: it is not clear whether it is minimal surface of M^8. Minimal surface property is the geometric analog of a massless d'Alembert equation [L1, L9].

Also real analytic functions can be considered [L12, L13] but this leads to infinite-D extensions of rationals in the adelization requiring that also the p-adic counterparts of the space-time surfaces exist. Whether this phase which would correspond to $\Lambda = 0$, can be accepted physically, will be one of the key topics in the sequel.

The conditions defining the space-time surfaces are exactly solvable and the conjecture is that these surfaces are minimal surfaces by their holomorphy (the induced metric of the space-time surface does not however play any role and its role is taken by the complexification number theoretic octonion norm which is real valued for the real projections) [L12, L13, L16].

1.1.2 Space-time surfaces at the level of $H = M^4 \times CP_2$

At the level of $H = M^4 \times CP_2$, space-time surfaces are preferred extremals (PEs) of a 6-D Kähler action fixed by the twistor lift of TGD [K3]. The existence of the twistor lift makes TGD unique since only the twistor spaces of $T(M^4)$ and $T(CP_2)$ have the needed Kähler structure [?]. The 6-D twistor space $T(X^4)$ of the space-time surface X^4 is represented as a 6-surface X^6 in $T(M^4) \times T(CP_2)$. $T(X^4)$ has S^2 as fiber and X^4 as base. The twistor structure of $T(X^4)$ is induced from the product of twistor structures of $T(M^4)$ and $T(CP_2)$. The S^2 bundle structure of X^6 requires dimensional reduction and dimensionally reduced 6-D Kähler action consists of a volume term having an interpretation in terms of length scale dependent cosmological constant Λ and 4-D Kähler action.

Physically "preferred" means holography: to a given 3-surface at the either boundary of CD one can assign a unique space-time surface as an analog of Bohr orbit. This assumption is very probably too strong: the number of Bohr orbits is finite and the dynamically determined frames of the space-time surface would characterize the non-determinism [L17]. "Preferred" has several mathematical meanings, which are conjectured to be equivalent.

One of those meanings is that space-time surfaces simultaneous extremals of both volume term and Kähler action and field equations reduce almost everywhere to the analogs of the conditions satisfied by complex surfaces of complex manifolds. Note that the field equations express local conservation laws for the isometries of $H = M^4 \times CP_2$ and are in this sense hydrodynamic.

The field equations for preferred extremals do not depend on coupling parameters. This expresses quantum criticality and reduces the number of solutions dramatically as required by the fact that at the level the field equations are algebraic rather than differential equations.

Space-time surfaces are therefore minimal surfaces everywhere except at singularities, which are lower-dimensional surfaces. At singularities they are satisfied only for the entire action. The divergences of the isometry currents for the volume term and Kähler action would have delta function singularities, which must cancel each other to guarantee conservation laws.

The singular surfaces can be wormhole throats as boundaries of CP_2 type extremals at which the signature of the induced metric changes, partonic 2-surfaces acting as analogs of vertices at which light-like partonic orbits representing the lines of generalized Feynman (or twistor) diagram meet, and string world sheets having light-like boundaries at partonic orbits.

Also 3-D singularities are predicted and could be associated to time= constant hyperplanes of M^4, which in M^8 picture are associated with the roots of the polynomials determining space-time region: I have christened these roots "very special moments in the life of self" [L8]. The roots define 6-spheres as universal special solutions and they intersect future light-cone along $t = r_n$ hyper-plane. It is possible to glue different solutions together along these planes so that they can serve as loci of classical non-determinism.

The singular surfaces are analogous to the frames of soap films [L17]: part of them are fixed and at the boundaries of CD and part of them are dynamically generated. Classical conservation laws for the isometry currents expressing field equations pose strong conditions on what can happen in vertices.

1.1.3 $M^8 - H$ correspondence for the singularities

By $M^8 - H$ correspondence, the singular surfaces of $X^4 \subset H$ correspond to the singularities of the pre-image at the level of M^8. For the singularities $X^4 \subset M^8$ the quaternionic normal space of X^4 is not unique at points of a $d < 4$ dimensional surface but is replaced with a union of quaternionic normal spaces labelled by the points of sub-manifold of CP_2 for which the dimension is $d_c = 4 - d$. At the level of H, the singular points blow-up to d_c-dimensional surfaces. What happens for the normal space at a puncture of 3-space serves as a good analog.

In particular, the deformation of a CP_2 type extremal as a singularity corresponds to an image of a 1-D singularity with $(d = 1, d_c = 3)$ and $d_c = 3$-dimensional blow up. The properties of CP_2 type extremals suggest the 1-D curve is light-like curve for mere Kähler action and light-like geodesic for the Kähler action plus volume term.

These situations correspond to $\Lambda = 0$ and $\Lambda > 0$, where Λ is length scale dependent cosmological constant as coefficient of the volume term of action.

1.1.4 Membrane like structures as particularly interesting singularities

Membrane-like structures appear in all length scales from soap bubbles to large cosmic voids and it would be nice if they were fundamental objects in the TGD Universe. The Fermi bubble in the galactic center is an especially interesting membrane-like structure also from the TGD point of view as also the membrane- like structure presumably defining the analog of horizon for the TGD counterpart of a blackhole. Cell membrane is an example of a biological structure of this kind. I have however failed to identify candidates for the membrane-like structures.

An especially interesting singularity would be a static 3-D singularity $M^1 \times X^2$ with a geodesic circle $S^1 \subset CP_2$ as a local blow-up.

1. The simplest guess is a bubble-like structure as a product $M^1 \times S^2 \times S^1 \subset M^4 \times CP_2$. The problem is that a soap bubble is not a minimal surface: a pressure difference between interior and exterior of the bubble is required so that the trace of the second fundamental form is constant. Quite generally, closed 2-D surfaces cannot be minimal surfaces in a flat 3-space since the vanishing curvature of the minimal surface forces the local saddle structure.

2. A correlation between M^4 and CP_2 degrees of freedom is required. In order to obtain a minimal surface, one must achieve a situation in which the S^2 part of the second fundamental form contains a contribution from a geodesic circle $S^1 \subset CP_2$ so that its trace vanishes. A simple example would correspond to a soap bubble-like minimal surface with M^4 projection $M^1 \times X^2$, which has having geodesic circle S^1 as a local CP_2 projection, which depends on the point of $M^1 \times X^2$.

3. The simplest candidate for the minimal surface $M^1 \times S^2 \subset M^4$. One could assign a geodesic circle $S^1 \subset CP_2$ to each point of S^2 in such a manner that the orientation of $S^1 \subset CP_2$ depends on the point of S^2.

4. A natural simplifying assumption is that one has $S^1 \subset S^2_1 \subset CP_2$, where S^2_1 is a geodesic sphere of CP_2 which can be either homologically trivial or non-trivial. One would have a map $S^2 \rightarrow S^2_1$ such that the image point of point of S^2 defines the position of the North pole of S^2_1 defining the corresponding geodesic circle as the equatorial circle.

   The maps $S^2 \rightarrow S^2_1$ are characterized by a winding number. The map could also depend on the time coordinate for M^1 so that the circle S^1 associated with a given point of S^1 would rotate in S^2_1. North pole of S^2_1 defining the corresponding geodesic circle as an equatorial circle. These maps are characterized by a winding number. The map could also depend on the time coordinate for M^1 so that the circle S^1 associated with a given point of S^1 would rotate in S^2_1.

   The minimal surface property might be realized for maximally symmetric maps. Isometric identification using map with winding number $n = \pm 1$ is certainly the simplest imaginable possibility.

Large voids of size scale or order 10^8 light years forming honeycomb like structures are rather mysterious objects, or rather non-objects. The GRT based proposal is that the formation of gravitational bound states leads to these kinds of structures in general relativity but I do now know how convincing these arguments really are.

One should answer two questions: what are these voids and why do they form these lattice-like structures?

One explanation of large voids is based on the TGD based view about space-time as a 4-surface in $H = M^4 \times CP_2$.

1. Space-time surfaces have M^4 projection, which is 4-D for what I call Einsteinian space-times. At this limit general relativity is expected to be a good approximation for the field theory limit of TGD.

   However, the M^4 projection can be also 3-D , 2-D or 1-D. In these cases one has what looks like a membrane, string, or point-like particle. All these options are realized. The simplest membranes would look like $M^1 \times S^2 \times S^1$, S^1 a geodesic circle of CP_2, which depends on a point of $M^1 \times S^2$ defining the M^4 projection. Only this assumption allows us to have a minimal surface. Varying S^1 creates the analog of pressure difference making soap films possible. I discovered this quite recently although the existence of membrane like entities was almost obvious from the beginning.

   Small perturbations tend to thicken the dimension of M^4 projection to 4 but the deformed objects are in an excellent approximation still 3-D, 2-D or 1-D.

2. Large voids could be really voids in a good idealization! Even 4-D space-time would be absent! The void would be the true vacuum. It should be noticed that matter as smaller objects, say cosmic strings thickened to flux tubes, would in turn have galaxies as tangles, which in turn would have stars as tangles. The TGD counterparts of blackholes would be dense flux tube spaghettis filling the entire volume.

3. What is remarkable that membranes are everywhere: large voids, blackhole horizons, Fermi bubbles, cell membranes, soap bubbles, bubbles in water, shock wave fronts, etc....

What could then give rise to the lattice like structures formed from voids? Here TGD suggests a rather obvious solution.

1. The lattices could correspond to tessellations of the 3-D hyperbolic space H^3 for which cosmic time coordinate identified as light-cone proper time is constant. H^3 allows an infinite number of tessellations whereas Euclidean 3-space allows a relatively small number of lattices.

   There is even empirical evidence for these tessellations. Along the same line of sight there are several sources of light and the redshifts are quantized. One speaks of God's fingers. This is what any tessellation of cosmic voids would predict: cosmic redshift would define effective distance. Of course also tessellations in smaller scales can be considered.

2. Also ordinary atomic lattices could involve this kind of tessellations with atomic nuclei at the centers of the unit cells as voids. The space between nucleus and atom would literally be empty, even 4-D space-time would be absent!

3. Also the TGD inspired model for genetic code [L15] involves a particular tessellation of H^3 realized at the magnetic body (MB) of a biological system and realizing genetic code. This leads to the conjecture that genetic code is universal and does not characterize only living matter. It would be induced to the space-time surface in the sense that part of tessellation would define a tessellation at the space-time surface. At the level of dark matter at MB, 1-D DNA could also have 2-D and even 3-D analogs, even in ordinary living matter!

1.2 Key questions

The basic question to be discussed in the following is what the general ideas about 2-D minimal surfaces can teach about minimal surfaces in M^8 and H, and more generally, about quantum TGD.

1.2.1 Uncertainty Principle and M^8-H duality

The interpretation of M^8 as analog of momentum space [L12, L13] meant a breakthrough in the understanding of M^8-H duality but created also a problem. How can one guarantee that M^8-H duality is consistent with Uncertainty Principle (UP)? The surfaces to which one can assign well defined momentum in M^8 should correspond to the analogs of plane waves in H and geometrically to periodic surfaces.

The fact that at the level of M^8 the surfaces are algebraic surfaces defined by polynomials with rational coefficients poses therefore a problem. Periodicity requires trigonometric functions. The introduction of real anlytic functions with rational Taylor coeffients would force the introduction of infinite-D extensions of rationals and make this possible. This is however in conflict with the idea about the finiteness of cognition forming the basic principle of adelic physics [L6, L7].

1.2.2 Is the category of polymials enough?

Is it possible to have periodic minimal surfaces at the level of H or at the level of both M^8 and H without leaving the category polynomials?

1. Could the non-local character of the M^8-H duality in CP_2 degrees freedom miraculously give rise to periodic functions at the level of H? Or should one perhaps modify M^8-H duality itself to achieve this [L16].

2. Periodic frames assignable to light-like curves in M^8 as light-like curves would allow to achieve periodicity in the same manner as for helicoid but this requires the extension of the category of real polynomials to real analytic functions in M^8. One could even give up the assumption about a Taylor expansion with rational coefficients and assume that the coefficients belong to some possibly transcendental extension of rationals. This option would make sense in $\Lambda=0$ phase.

3. Or could geometry come in rescue of algebra? Could one construct periodic surfaces both at the level of M^8 and H purely geometrically by gluing minimal surfaces together to form repeating patterns as is done for 2-D minimal surfaces? This option could work in $\Lambda>0$ phases: smoothness at the junctions would be given up but local conservation laws would hold true for the entire action rather than for volume term and Kähler action separately.

If transcendental extensions are allowed, they would naturally contain some maximal root $e^{1/n}$ and its powers. The induced extension of p-adics is finite-D since e^p is an ordinary p-adic number. Logarithms of $log(k)$, $1\leq k\leq p$, and their powers are needed to define p-adic logarithm for given p. The outcome is an infinite-D extension. Also π and its powers are expected to belong to the minimal transcendental extension.

It came as a surprise to me that is not known whether e and π are algebraically independent over rationals, that is whether a polynomial equation $P(x,y)=0$ with rational coefficients is true for $(x,y)=(\pi,e)$ (`https://cutt.ly/xmyL23W`.) This would imply that π belongs to the extension defined by the polynomial $P(y, e)$ in an extension of rationals by e. Same would be true in the corresponding finite-D extensions of p-adic numbers. The algebraic independence of π and e would have rather dramatic implications for the TGD view about cognition. That π and e are algebraically independent follows from a more general conjecture by Schanuel and `https://cutt.ly/ImyL1YJ`).

1.2.3 Is also $\Lambda > 0$ phase physically acceptable?

Can one allow also $\Lambda = 0$ phase for the action. In this case the action reduces to mere Kähler action defined by M^4 and CP_2 Kähler forms analogous to self-dual covariantly constant $U(1)$ gauge fields? Could one see $\Lambda = 0$ phase as an analog of Higgs=0 phase?

In this phase the category of rational functions would expand to a category of real analytic functions and infinite extensions of rationals containing transcendental numbers would be unavoidable and allow light-like curves as frames instead of piecewise light-like geodesics.

One could argue that since the evolution of mathematical consciousness has led to the notion transcendentals and transcendental functions, they must be realized also at the level of space-time surfaces.

One can invent objections against the $\Lambda = 0$ phase for which Kähler action has only CP_2 part and serving at the same time as arguments for the necessity of M^4 part.

1. For a mere CP_2 Kähler action, the CP_2 type extremals representing building bricks of elementary particles become vacuum extremals and are lost from the spectrum. However, also the M^4 part of Kähler action predicted by the twistor lift gives rise to Chern-Simons (C-S) term assignable to the light-like 3-surface X^3_L as the orbit of partonic 2-surface and one can assign a momentum to X^3_L. The boundary conditions guaranteeing momentum conservation make possible momentum exchange between interior and X^3_L.

2. CP_2 Kähler action has a huge vacuum degeneracy since space-time surfaces with 2-D Lagrangian manifold as a CP_2 projection are vacuum extremals. $\Lambda > 0$ eliminates most of these extremals. Also the M^4 part of Kähler action, which vanishes for canonically imbedded M^4, implies that most vacuum extremals of CP_2 Kähler action cease to be extremals even for $\Lambda = 0$.

While writing the first version of this article I had not realized that what the correct form for the Kähler property in M^4 case is.

1. Suppose for definiteness the simplest option that the M^4 Kähler form are associated with the decomposition $M^4 = M^2 \times E^2$. A more general decomposition corresponds to Hamilton-Jacobi structure in which the distributions for $M^2(x)$ and $E^2(x)$ orthogonal to each other are integrable and define slicings of M^4 [L18].

2. The naive guess was that $J^2 = -g$ condition must be satisfied. This implies that the M^2 part of Kähler form of $M^4 = M^2 \times E^2$ decomposition has an electric part, which is imaginary so that the energy density is of form $-E^2 + B^2$ ($= 0$ for M^4). For instance, solutions of $M^2 \times Y^2$, where Y^2 is any Lagrangian manifold of CP_2 would have negative energy for $\Lambda = 0$. Even worse, Kähler gauge potential would be imaginary and the modified Dirac equation would be non-hermitian.

3. The problem disappears by noticing that the M^2 by its signature has hypercomplex rather than complex structure, which means that the counterpart of the imaginary unit satisfies $e^2 = 1$ rather than $i^2 = -1$. This allows a real Kähler electric field and the situation is the same as in Maxwell's theory.

2 About 2-D minimal surfaces

A brief summary about 2-D minimal surfaces and questions raised by them in TGD framework is in order. One can classify minimal surfaces to those without frame and with frame.

2.1 Some examples of 2-D minimal surfaces

The following examples about minimal surfaces are collected from the general Wikipedia article about minimal surface `https://cutt.ly/Hn673ry`) and various other Wikipedia articles. This article gives also references to articles (for instance the article "The classical theory of minimal surfaces" of Meeks and Perez [A5]) and textbooks discussing minimal surfaces , see for instance [A4]. Also links to online sources are given. "Touching Soap Films - An introduction to minimal surfaces" `https://cutt.ly/dmwMnJ7`) serves as a general introduction to minimal surfaces). There is also a gallery of periodic minimal surfaces (`https://cutt.ly/RmwMQ49`), which is of special interest from the TGD point of view.

1. *Minimal surfaces without frame*

In E^3 frameless minimal surfaces have an infinite size and are often glued from pieces, which asymptotically approach a flat plane.

Catenoid (`https://cutt.ly/in675Z6`) is obtained by a rotation of a catenoid, which is the form of the chain spanned between poles of equal height in the gravitational field of Earth. Catenoid has two planes as asymptotics and is obtained from torus by adding two punctures. Costa's minimal surface (`https://cutt.ly/in65wyP`) is obtained from torus by adding a single puncture and its second end looks like a catenoid.

Frameless minimal surfaces in E^3 allow also lattice-like structures. Schwarz minimal surface (`https://cutt.ly/dn65rJm`) is an example about minimal giving rise to 3-D lattice like structure. These surfaces have minimal genus $g = 3$.

In compact spaces closed minimal surfaces are possible and some quite surprising results hold true, see the popular article "*Math Duo Maps the Infinite Terrain of Minimal Surfaces*" (`http://tinyurl.com/yyetb7c7`). These surfaces have area proportional to volume of the embedding space and the explanation is that these surfaces fill the volume densely [A2, A3].

2. *Minimal surfaces with lattice like structure*

There exists also minimal surfaces with lattice-like structure.

1. Riemann described a one parameter of minimal surfaces with a 1-D lattice structure consisting of shelfs connected by catenoids (`https://cutt.ly/Pn65y3f`).

2. Scherk surfaces (`https://cutt.ly/3n65oeB`) are singly or doubly periodic. Scwartz surfaces (`https://cutt.ly/un65pCK`) are triply periodic structures defining 3-D lattices and have minimal genus $g = 3$. This kind of surfaces have been used to model condensed matter lattices. These surfaces have also hyperbolic counterparts.

3. *Minimal surfaces spanned by frames*

Minimal surfaces with frames allow to models soap films and are obtained as a solution of the Plateau's problem (`https://cutt.ly/7n65fgT`).

1. Helicoid (`https://cutt.ly/Wn65jgT`) represents a basic example of a simply periodic framed surface. Also helicoid involves transcendental functions. A portion of helicoid is locally isometric to catenoid.

2. Arbitrary curves can serve as frames with some mild restrictions. The minimal surface need not be unique. A given 2-D minimal surface is obtained in topological sense from a compact manifold by adding a puncture to represent boundaries defined by frames or the boundaries at infinity.

2.2 Some comments on 2-D minimal surfaces in relation to TGD

The study of the general properties of 2-D minimal surfaces from the TGD perspective suggest a generalization to the TGD framework and also makes possible a wider perspective about TGD itself.

1. *Frameless minimal surfaces in TGD framework*

Frameless minimal surfaces in E^3 have infinite sizes since they are locally saddle like. In TGD framework, the most interesting space-time surface are expected to be framed. Despite this frameless minimal surfaces are of interest.

1. In the TGD framework the minimal surfaces could extend to infinity in time-direction and remain finite in spatial directions. The asymptotically flat 2-plane could in TGD correspond to the simplest extremals of action: M^4 and "massless extremals" (MEs); surfaces $X^2 \times Y^2$ with X^2 a string world sheet and Y^2 complex manifold of CP_2; and CP_2 type extremals with 1-D light-like curve as CP_2 projection.

   Conservation laws do not allow M^4 even in principle unless the total angular momentum and color charges vanish. Various singularities could deform flat M^4 in close analogy with point and line charges.

2. In curved compact spaces also closed minimal surfaces are possible [A2, A3] (`http://tinyurl.com/yyetb7c7`). One can wonder whether CP_2 as a curved space might allow a volume-filling closed 2-D or 3-D minimal surfaces besides complex surfaces and minimal Lagrangian manifolds [L9]. For $\Lambda > 0$, only complex surfaces defined by polynomials in M^8 appear in PEs. It is difficult to see how this kind of exotic structure could define a physically interesting partonic 2-surface although formally one could consider a product of string world sheet and this kind of 2-surface.

2. Minimal surfaces with lattice structure

2-D minimal surfaces in E^3 allow lattice-like structures with dimensions 1, 2 and even 3. They are are interesting also in TGD framework.

1. Scwartz surface (`https://cutt.ly/un65pCK`), call it S, allows in the TGD framework a variant of form $M^1 \times S \times S^1$, where S^1 is a geodesic sphere. Same applies to all 2-D minimal surfaces allowing a lattice structure and could be in a central role in condensed matter physics according to TGD. Also hyperbolic variants of a lattice like structure expected to relate to the tesselations of hyperbolic 3-space can be considered and could play important role at the level of magnetic bodies (MBs) as indeed suggested [L15].

2. If $\Lambda = 0$ phase is physically acceptable, it would make possible light-like curves as frames and also lattice-like minimal surfaces with periodicity forced by that of the light-like curve assignable to to CP_2 type extremal as M^8 pre-image.

   Note that $\Lambda = 0$ phase relates to $\Lambda > 0$ phase by the breaking of conformal symmetry transforming light-like curves to light-like geodesics. The interpretation of $\Lambda = 0$ phase in terms of the emergence of continuous string world sheet degrees of freedom is attractive.

   Another interpretation would be based on the hierarchy of Jones inclusions of hyper-finite factors of type II_1 (HFFs). $\Lambda > 0$ phase would define the reduced configuration space ("world of calassical worlds" (WCW)) in finite measurement resolution defined by the included HFF representing measurement resolution and $\Lambda = 0$ phase as the factor without this reduction. The approximation of real analytic functions by polynomials of a given degree would define the inclusion. This sequence of approximations would be realized as genuine physical systems ,rather than only approximate descriptions of them.

3. For $\Lambda > 0$ allowing only polynomial function, periodic smooth minimal surfaces in M^8. The construction of Schwartz surface suggests how one can circumvent this difficulty.

   Schwartz surface defines a 3-D lattice obtained by gluing together analogs of unit cells. If a region of a minimal surface intersects orthogonally a plane, the gluing of this surface together with its mirror image gives rise to a larger minimal surface and one can construct an entire lattice-like system in this way. These surfaces are not smooth at the junctions.

   In the TGD framework, one would construct lattice in time direction and the gluing would occur at edges defined by 3-D $t = r_n$ planes ("very special moments in the life of self" [L8]). Local conservation laws as limits of field equations are enough and derivatives can be discontinuous at $t = r_n$ planes. The expected non-uniqueness of the gluing procedure would mean a partial failure of the strict classical determinism having a crucial role in the understanding of cognition in ZEO. This is discussed in [L17].

M^8-picture suggests a very concrete geometric recipe for constructing minimal surfaces periodic in time direction and this would make it possible to realize UP for M^8-H duality.

The general vision would be that $\Lambda > 0$ phases the periodic minimal surfaces can be constructed as piecewise smooth lattice-like structures in the category of real polynomials by using the gluing procedure whereas in $\Lambda = 0$ phase they correspond to smooth surfaces in the category of real analytic functions.

3. Minimal surfaces spanned by frames

Minimal surfaces spanned by frames are of special interest from TGD point of view.

1. In the TGD framework. Minimal surfaces are spanned by fixed frames at the boundary of CD and by dynamically generated frames in the interior of CD. The dynamically generated frames break strict determinism, which means that space-time surfaces as analogs of Bohr orbits becomes non-unique [L17] and holography (for its various forms see [L12, L13]) forced by the General Coordinate Invariance is not completely unique.

2. CP_2 type extremal in H would correspond to 1-D singularity in M^8 analogous to a frame assigned 2-D minimal surfaces. The physical picture suggests that this curve is a light-like curve for the Kähler action ($\Lambda = 0$) and a light-like geodesic for action involving also volume term ($\Lambda > 0$). In the first case the periodicity of the light-like curve could give rise to periodic minimal surfaces as generalization of helicoid. In the second case discretized variants could replace these curves.

3. For the minimal surfaces discussed above, polynomials are not enough for their construction and the examples involve transcendental functions like trigonometric, exponential and logarithmic functions in their definition.

   The same is expected to be true also in TGD. Should one leave the category of polynomials and allow all real analytic functions with rational Taylor coefficients? Or should one assume also the $\Lambda = 0$ phase making possible real analytic functions?

   As far as cognitive representations are involved, this would mean that cognition becomes infinite since the extensions of p-adic become infinite. Could $\Lambda = 0$ phase be associated with an expansion of consciousness, kind of enlightment, and relate to mathematical consciousness?

3 Space-time surfaces as 4-D minimal surfaces

Years after writing the original version of this article holography = holomorphy principle [L25] led to the conclusion that space-time surfaces as analogs of Bohr orbits for particles as 3-surfaces are minimal surfaces apart from singularities. Therefore the 4-D minimal surfaces became basic object of study.

3.1 Periodic minimal surfaces with periodicity in time direction

There are several motivations for the periodic minimal surfaces.

3.2 Consistency of M^8-H duality with Uncertainty Principle

Consistency of M^8-H duality with UP is one motivation.

1. M^8 is interpreted as an analog of momentum space. M^8-H correspondence must be consistent with UP. If M^8-H correspondence in M^4 degrees of freedom involves inversion of form $m^k \rightarrow \hbar_{eff} m^k/m^2$. [L12, L13, L16]. This solves the problem only partially. M^8-H correspondence should realize also the idea about plane wave as space-time counterpart of point in momentum space.

The first guess [L16] would be that the $X^4 \subset CD \subset M^8$ is mapped to a union of translates of images of CD by inverse of P^k, where is the total momentum assignable to CD. What I saw as a problem, was that this gives a lattice-like many-particle state rather than a single particle state as a counterpart of a plane wave.

If the momentum is space-like, this is indeed the case. Therefore I proposed that the image is a quantum superposition of translates rather than their union and represents an analog of plane wave. I failed to realize that this is not the case for time-like momentum since periodicity in time direction does not mean lattice as many-particle state.

A geometric correspondence for time-like momenta is possible after all! The problem is a concrete realization of this correspondence and here the geometric construction gluing together the analogs of unit cells to form a periodic structure in time direction suggests itself.

2. Quite concretely, one could take part of $X^4 \subset CD \subset M^8$ defining particle and construct a periodic surface with a period determined by the total time-like momentum assignable to this part of X^4. X^4 has a slicing by planes $e = e_n$ [L8] assignable to 6-branes with topology of S^6 defining universal special solutions of algebraic equations. Here e_n is a root of the real polynomial defining X^4.

   One could take a piece $[e_1, ..., e_k]$ of $X^4 \subset CD$ and glue it to its time reversal in M^8 to get a basic unit cell and fuse these unit cells together to obtain a periodic structure.

   The differences $e_i - e_j$, which for M^8 correspond to energy differences, are mapped by inversion to time differences $t_i - t_j$ in H. The order of magnitude for the p-adic length scale assignable to CD in question is the same as for the largest difference for the roots as conjectured on basis of the conjecture that the p-adic length scale correspond to a ramified prime of the extension dividing $|t_i - t_j|^2$ for some pair (i, j). The p-adic prime for CD need not however be a ramified prime and one can develop an argument for how it emerges [L17].

3. Rather remarkably, one can glue together portions $[t_1, ..t_r]$ and the mirror image of $[t_k, t_r]$, for any k. All possible sequences of this kind are possible! This suggests an analogy to logical reasoning: $[t_n, t_{n+1}]$ would represent a basic step $t_n \rightarrow t_{n+1}$ in the reasoning and one could combine these steps. Could this process serve as the geometric correlate for logical thought or as engineeering at the level of fundamenta interactions?

   The physicalists refusing to accept non-determinism at the fundamental level fail to realize that our technology relies on a fusion of deterministic processes and is therefore not consistent with strict determinism. Also computer programs consist of deterministic pieces.

4. There is still one open question. Does the construction of the time lattices occur only at the level of H or both at the level of M^8 and H? One can argue that the realization of the analog of inverse Fourier transform forces the construction at both sides.

3.3 Bohr orbitology for particles in terms of minimal surfaces

In TGD, space-time surfaces correspond to analogs of Bohr orbits. One should also have classical space-time analogs for ordinary bound states as Bohr orbits for particles. Atoms represent the basic example. In TGD Universe, Bohr model should be much more than mere semiclassical model. Also the geodesic orbits of particles in gravitational fields should have minimal surface analogs.

The Bohr orbits should be representable as parts of minimal surfaces identifiable as deformed CP_2 type extremals. There are two options to consider corresponding to $\Lambda = 0$ phase and to $\Lambda > 0$ phases.

1. $\Lambda = 0$ phase

$\Lambda = 0$ phase corresponds to a long length scale limit but general consideratons encourage its inclusion as a genuine phase. Its relation to $\Lambda > 0$ phases would be like the relation of real numbers to extensions of rationals and transcendental functions to polynomials.

1. For $\Lambda = 0$, CP_2 type extremals are vacuum extremals and correspond to 1-D singularities, which are light-like curves in M^8 blown up to orbits of wormhole contacts in H.

Light-like curve as an M^4 projection of Bohr orbit of this kind can give rise to "zitterbewegung" as a helical motion with average cm velocity $v < c$. The proposal for the TGD based geometric description of Higgs mechanism realizes this zitterbewegung of CP_2 type extremals for Kähler action. This makes it possible to assign to any particle orbit - be it Bohr orbit in an atom or a geodesic path in a gravitational field, an average of a light-like curve.

2. Light-likeness gives rise to Virasoro conditions emerging in the bosonic string theories. This served as a stimulus leading to the assignment of extended Kac-Moody symmetries to the light-like partonic orbits X^3 . The isometries of H define the extended Kac-Moody group. The generators of the Kac-Moody algebra depend on the complex coordinate z of the partonic 2-surface and on the light-like radial coordinate of X^3. Super-symplectic symmetries assigned to the light-like $\delta M^4_\pm \times CP_2$ and identified as isometries of WCW have an analogous structure [K1] [L14].

   The light-like orbits of the partonic 2-surfaces in H are connected by string world sheets. The interpretation could be that in $\Lambda = 0$ phase strings emerge as additional degrees of freedom.

3. For CP_2 part of Kähler action $\Lambda = 0$ CP_2 type extremals are vacua (this need not be the case for the deformations). The C-S term for CP_2 Kähler action carries no momentum and cannot contribute to momentum and cannot realize momentum conservation for deformed CP_2 type extremals.

   However, the C-S term for the M^4 part of Kähler action defines the partonic orbits as dynamical entities. If the projection of the deformation of CP_2 type extremal at the wormhole throat has M^4 projection with dimension $D = 3$, M^4 C-S term gives rise to non-vanishing momentum currents and the smooth light-orbit is consistent with the momentum conservation if boundary conditions are realized. What is remarkable that M^4 C-S term also gives rise to small CP breaking, whose origin is not understood in the standard model. The tiny C-S breaking term would be paramount for the existence of elementary particles!

The implications of this picture are rather profound. It could be possible to assign to any physical system rather detailed view about the minimal surfaces involved both at the level of H and M^8.

Could tachyonic states appear as parts of non-tachyonic states somewhat like tachyonic virtual particles appear in Feynman graphs?

1. The possibly existing periodic minimal surfaces with tachyonic total momenta would have an interpretation as lattice-like many-particle states. This excludes them as unphysical. In fact, one cannot construct tachyonic periodic minimal surfaces in the proposed way since the planes $t = t_n$ have time-like normal.

2. M^8 picture allows to interpret tachyonicity as a trick. In the M^8 picture the choice of $M^4 \subset M^8$ is in principle free. The mass squared of the particle depends on this choice since M^4 momentum is a projection of M^8 momentum to $M^4 \subset M^8$. For eigenstates of M^4 mass, one can rotate $M^4 \subset M^8$ in such a manner that the mass squared vanishes. For a superposition of states with different mass squared possible in ZEO this is not possible but one can choose M^4 so that mass squared is minimized. This gives rise to p-adic thermodynamics as a description for the mixing with heavier states.

   One could understand the tachyonic ground state as an effective description for the choice of M^4 in this manner.

2. $\Lambda > 0$ phase

For $\Lambda > 0$ only light-like geodesics are possible and this forces a modification of the above picture by replacing light-like curves with piece-wise light-like geodesics.

1. A discrete variant of zitterbewegung consisting of pieces of light-like geodesics is suggestive. The dynamics in stringy degrees of freedom would be almost frozen and completely dictated by the ends of the string. Discretized version of smooth dynamics would be in question. This kind of phenomenological model for hadronic strings has been proposed.

2. The change of the direction of the partonic orbit takes place in a vertex. In M^8 picture it is associated with a partonic 2-surface associated with a $t = r_n$ hyperplane at which several CP_2 type extremals meet at the level of H. These reactions could be seen as ordinary particle reactions.

3. Another way to change the direction would be based on the interaction of parton with the interior degrees of freedom so that conservation laws are not lost. The interaction between the 3-D orbit of wormhole throat and interior is defined by the condition that normal components of the isometry currents of the total Kähler action are equal to the divergences of C-S currents the partonic orbit. For the M^4 part of C-S action only momentum currents are non-vanishing whereas for CP_2 only color currents are non-vanishing.

   At the turning points the normal current of the entire Kähler action - and the divergence of the isometry current for C-S part CP_2 type extremal must become non-vanishing and divergent but cancel each other. Local conservation laws hold true and one can speak of a momentum exchange between interior and wormhole throat. This picture applies also to color currents.

3. A connection with Higgs mechanism

The fact that zitterbewegung makes the particle effectively massive in long enough scales, suggests an analogy with the massivation by the Higgs mechanism.

1. The interactions between partonic orbits and the interior of the space-time surface are analogous to the interactions of particles with a Higgs field leading to the massivation as the Higgs field develops a vacuum expectation value.

2. M^4 Kähler form represents a constant self-dual Abelian gauge field. Although this field is not a scalar field, it is analogous to the vacuum expectation value of the Higgs field as far as its effects are considered.

4. A connection twistor diagrams and generalization of cognitive representations

Also a connection with twistor diagrams is suggestive. The light-like geodesic lines appearing as 1-D singularities in M^8 would correspond to light-like differences of the time-like momenta assignable to vertices. In H they are assignable with partonic 2-surfaces identifiable as boundaries of 3-D blow ups of 1-D singularities in M^8. In M^8, the graphs containing time-like momenta connected by singular lines would define analogs of twistor diagrams. Also at the level of H the lines connecting partonic 2-surfaces would be light-like as also the distances between them since the inversion map preserves light-likeness of the tangent curves.

This would pose additional conditions on cognitive representations.

1. The original proposal [**?**]as that cognitive representation consists of points of X^4 for which M^8 coordinates belong to the EQ associated with the polynomial considered. The expectation was that one has a generic situation so that this set is automatically finite.

   The explicit solution of the polynomial equations however led to a surprising finding was that the number of these points was a dense set for the space-time surfaces satisfying co-associativity conditions [L12, L13]. The second surprise was that co-associativity (associativity of normal space) is the only possible option.

2. The additional conditions guaranteeing that the cognitive representation consists of a finite number of objects, generalize it from a discrete set of points to a union of singularities with co-dimension $d_c = 4 - d$, $d = 1, 2, 3$.

   The vertices would be connected by $d = 1$ light-like singularities and belong to 2-D partonic 2-surfaces as $d = 2$ singularities at $t = r_n$ surfaces in turn defining $d = 3$ singularities. Also 2-D string world sheets having $d = 1$ singularities as boundaries would be included.

3. This would also generalize twistor diagrams as a frame holographically coding for the space-time surface as an analog of Bohr orbit. At the M^8 level, the definition of the parts of this structure would involve only parameters with values in EQ (say the end points of a light-like geodesic defining it).

3.4 Periodic self-organization patterns, minimal surfaces, and time crystals

Periodic self-organization patterns which die and are reborn appear in biology. Even after images, which die and reincarnate, form this kind of periodic pattern. Presumably these patterns would relate to the magnetic body (MB), which carries dark matter in the TGD sense and controls the biological body (BB) consisting of ordinary matter. The periodic patterns of MB represented as minimal surface would induce corresponding biological patterns.

The notion of time crystal [B2] (`https://cutt.ly/2n65x0k`) as a temporal analog of ordinary crystals in the sense that there is temporal periodicity, was proposed by Frank Wilczeck in 2012. Experimental realization was demonstrated in 2016-2017 [D1] but not in the way theorized by Wilczek. Soon also a no-go theorem against the original form of the time crystal emerged [B3] and motivated generalizations of the Wilzeck's proposal.

Temporal lattice-like structures defined by minimal surfaces would be obvious candidates for the space-time correlates of time crystals.

1. One must first specify what one means with time crystals. If the time crystal is a system in thermo-dynamic equilibrium, the basic thermodynamics denies periodic thermal equilibrium. A thermodynamical non-equilibrium state must be in question and for the experimentally realized time crystals periodic energy feed is necessary.

   Electrons constrained on a ring in an external magnetic field with fractional flux posed to an energy feed form a time crystal in the sense that due to the repulsive Coulomb interaction electrons form a crystal-like structure which rotates. This example serves as an illustration of what time crystal is.

2. Breaking of a discrete time translation symmetry of the energy feed takes place and the period of the time crystal is a multiple of the period of the energy feed. The periodic energy feed guarantees that the system never reaches thermal equilibrium. According to the Wikipedia article, there is no energy associated with the oscillation of the system. In rotating coordinates the state becomes time-independent as is clear from the example. What comes to mind is a dynamical generation of Galilean invariance applied to an angle variable instead of linear spatial coordinate.

3. Also the existence of isolated time crystals has been proposed assuming unusual long range interactions but have not been realized in laboratory.

Time crystals are highly interesting from the TGD perspective.

1. The periodic minimal surfaces constructed by gluing together unit cells would be time crystals in geometric sense (no thermodynamics) and would provide geometric correlates for plane waves as momentum eigenstates and for periodic self-organization patterns induced by the periodic minimal surfaces realized at the level of the magnetic body. It is difficult to avoid the idea that geometric analogs of time crystals are in question.

2. The hierarchy of effective Planck constants $h_{eff} = nh_0$ is realized at the level of MB. To preserve the values of h_{eff} energy feed is needed since h_{eff} tends to be reduced spontaneously. Therefore energy feed would be necessary for this kind of time crystals. In living systems, the energy feed has an interpretation as a metabolic energy feed.

   The breaking of the discrete time translation symmetry could mean that the period at MB becomes a multiple of the period of the energy feed. The periodic minimal surfaces related to ordinary matter and dark matter interact and this requires con-measurability of the periods to achieve resonance.

3. Zero energy ontology (ZEO) predicts that ordinary ("big") state function reduction (BSFR) involves time reversal [L11, L17]. The experiments of Minev et al [B1] [?] give impressive experimental support for the notion in atomic scales, and that SFR looks completely classical deterministic smooth time evolution for the observer with opposite arrow of time. Macroscopic quantum jump can occur in all scales but ZEO together with h_{eff} hierarchy takes care that the world looks classical! The endless debate about the scale in which quantum world becomes classical would be solely due to complete misunderstanding of the notion of time.

4. Time reversed dissipation looks like self-organization from the point of view of the external observer. A sub-system with non-standard arrow of time apparently extracts energy from the environment [L10]. Could this mechanism make possible systems in which periodic oscillations take place almost without external energy feed?

Could periodic minimal surfaces provide a model for this kind of system?

1. Suppose that one has a basic unit consisting of the piece $[t_1, .., t_k]$ and its time reversal glued together. One can form a sequence of these units.

   Could the members of these pairs be in states, which are time reversals of each other? The first unit would be in a self-organizing phase and the second unit in a dissipative phase. During the self-organizing period the system would extract part of the dissipated energy from the environment. This kind of state would be "breathing" [L27].

   There is certainly a loss of energy from the system so that a metabolic energy feed is required but it could be small. Could living systems be systems of this kind?

2. One can consider also more general non-periodic minimal surfaces constructed from basic building bricks fitting together like legos or pieces of a puzzle. These minimal surfaces could serve as models for thinking and language and behaviors consisting of fixed temporal patterns.

4 What the failure of classical non-determinism could mean for 4-D minimal surfaces?

In TGD, holography = holomorphy principle predicts that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces and defining the holographic data.

1. The Bohr orbits out to be 4-D minimal surfaces irrespective of the action principle as long as it is general coordinate invariant and constructible in terms of the induced geometry. 2-D minimal surfaces are non-deterministic in the sense that same frames span several minimal surfaces. One can expect that also in the 4-D case, non-determinism is unavoidable in the sense that the Bohr orbit-like 4-surfaces are spanned by 3-D "frames" as loci of non-determinism.

2. At these 3-surfaces minimal surface property fails, the derivatives of the embedding space coordinates are discontinuous and the second fundamental form diverges. Also the generalized holomorphy fails. The failure of smooth structure caused by the edge in 4-D case can give rise to an exotic smooth structure.

3. One can also say the singularities act as sources for the analog of massless field equations defined by the vanishing of the trace of the second fundamental form and this justifies the identification of the singularities as vertices in the construction of the scattering amplitudes.

4. In the TGD inspired theory of consciousness, classical non-determinism gives rise to geometric correlates of cognition and intentionality and the loci of non-determinism serve as memory seats. Free will is not in conflict with classical determinism and the basic problem of quantum measurement theory finds a solution in zero energy ontology.

5. The proposal is that the classical non-determinism corresponds to the non-determinism of p-adic differential equations. In fact, TGD leads to a generalization of p-adic number fields to their functional counterparts and they can be mapped to p-adic number fields by category-theoretical morphism. This generalization allows us to understand the p-adic length scale hypothesis which is central in TGD.

The study of the non-determinism for 2-D minimal surfaces could serve as a role model in the attempts to understand non-determinism for 4-D minimal surfaces. What can one say about the geometric aspects of classical non-determinism in the case of 2-D minimal surfaces? Here Google Gemini provides help and one obtains a surprisingly detailed summary and its also possible to make further questions. Here I summarize briefly what Google says.

4.1 The classical non-determinism of 2-D minimal surfaces

The 2-D minimal surface spanned by a given frame (a closed, non-intersecting, simple wire loop or collection of them in 3D space) is generally non-unique. While the existence of at least one minimal surface (a surface of zero mean curvature with vanishing trace of the second fundamental form) is guaranteed, a single frame can bound multiple, and sometimes even a continuum of, distinct minimal surfaces. Here is a breakdown of the uniqueness of minimal surfaces.

1. Many frames, particularly non-convex ones, can span several distinct minimal surfaces. A classic example is two coaxially aligned circles, which can bound two different catenoid surfaces (a wider and a narrower one) or two separate disks.

2. In certain cases, a given curve can bound a continuous family of minimal surfaces, a phenomenon often observed in physical soap film experiments.

3. Uniqueness is achieved only under specific conditions.

   (a) Convex projection: If a closed Jordan curve Γ has a one-to-one orthogonal projection onto a convex planar curve, then Γ bounds a unique minimal disk, which is a graph over that plane.
   (b) Small total curvature: A smooth Jordan curve with a total curvature less than or equal to 4π bounds a unique minimal disk.
   (c) Sufficiently close to a plane: A C^2-Jordan curve that is sufficiently close to a plane curve in the C^2-topology bounds a unique minimal disk.

4. Stability vs. sbsolute uniqueness: A minimal surface is "stable" if small perturbations increase its area. Often, a frame may bound multiple minimal surfaces, but only one is the absolute, global minimum, while others are unstable or local minima.Plateau's Problem: The classical problem asks for the surface of minimum area, which exists, but is not always unique.

Summary: While soap film experiments often produce a single, stable minimal surface, the boundary value problem can have multiple solutions. Uniqueness is the exception, not the rule, and depends strongly on the geometric "convexity" of the framing wire.

4.2 What could one conclude about the space-time surfaces as minimal surfaces?

The above Google summary helps to make guesses about the naive generalization of these findings in the 4-D situation.

How unique is the minimal surface spanning a given frame?

One can go to Google and pose the question "How unique is the minimal surface spanning a given frame?". One obtains a nice summary and can ask additional questions. The following considerations are inspired by this question.

1. In the case of ordinary minimal surfaces, it is enough that there exists a plane for which the minimal surface is representable as a graph of a map and the projection of the frame to the plane is convex, i.e. any of its points can be connected by a line inside the curve defined by the projection. An essential assumption is that the 2-D surface is representable locally as a graph over a plane. Curves whose plane projection has an interior, which is non-convex (not all interior points can be connected by a curve in the interior) can also lead to a failure of determinism. Cusp catastrophe, defined in terms of roots of a polynomial of degree 3, is a 2-D example of non-convexity. Note that the cusp is 3-sheeted.

2. Consider the general meaning of convexity for objects of dimension d in linear spaces with dimension $d+1$. One considers a projection of the object with dimension d (say frame to a higher-dimensional space. For minimal surfaces, the object is the frame of dimension $d=1$ and the space has dimension $d=3$. For Riemannian manifolds straight lines can be identified as geodesic lines. Planes could be generalized to geodesic manifolds.

The convexity criterion has a straightforward analog when the embedding space is 8-D $H = M^4 \times CP_2$ and minimal surface is 4-D space-time surface X^4.

1. The projection of the 3-D frame, defining the holographic data or a locus of non-determinism defining secondary holographic data, to some 4-D submanifold analogous to the plane should be convex. The surface should be also representable as a graph of a map from the 4-D manifold to H. One could consider projections of the frame X^3 to all geodesic submanifolds G_4 of dimension $D = 4$. $G_4 \in \{M^4, E^3 \times S^1, E^2 \times S^2\}$, where S^1 and S^2 are geodesic manifolds of CP_2 appear as candidates.

   For physically most interesting cases CP_2 projection has at least dimension 2 so that $E^2 \times S^2$ is of special interest. Could one choose G_4 to be holomorphic sub-manifolds? If hypercomplex holomorphy does not matter, this would leave only 2-D M^4 projection. Is it enough to consider $G_4 = E^2 \times S^2$? Situation would resemble that for ordinary minimal surfaces. Could one consider the convexity of the E^2 and S^2 projections?

2. Convexity: the points of X^3 can be connected by geodesic lines. Should they be space-like or could also light-like partonic orbits serve as loci of non-determinism. What about 3-surfaces inside CP_2 representing a wormhole contact at which two parallel Minkowskian space-time sheets meet?

3. The convexity criterion should be satisfied for all frames defined by 3-D singularities assumed to be given.

4. If the 3-D frame corresponding to the roots of $f_1 = 0, f_2 = 0$ is manysheeted over G^4, the projection contains several overlapping regions corresponding to the roots. One does not have a single convex region. This is one source of non-determinism.

5. Note: If the projection to M^4 is bounded by genus $g > 0$ surface, the M^4 projection is not convex. Now however CP_2 comes to rescue. Consider as an example a cosmic string $X^1 \times S^2$, where X^1 is convex and space-like. If the CP_2 projection is $g > 0$ surface, the situation is the same. Could this relate to the instability of higher genera. Would it be induced by classical non-determinism?

4.2.1 What could be the role of generalized holomorphy?

The failure of holomorphy implies singularities identified as loci of auxiliar holomorphic data and seats of non-determinism.

1. Often the absolute minimum is unique. The degeneracy of the absolute minimum would mean additional symmetry. This kind of additional symmetry in the case of Bohr orbits of electrons in an atom corresponds to rotational symmetry implying that the orbit can be in any plane going through the origin.

2. How does this relate to $f = (f_1, f_2) = 0$ conditions has as roots the space-time surface as a generalized complex submanifold of H? Each solution corresponds to a collection of the roots for these conditions and each root corresponds to a space-time region. Two or more roots are identical at the 3-D interfaces of the roots. Each root defines a region of some geodesic submanifold of H defining local generalized complex coordinates of X^4 as a subset of corresponding H coordinates in this region. Separate solutions would be independent collections of the roots. Two roots co-incide at at the 3-D interfaces between roots. Cusp catastrophe gives a good 2-D illustration.

3. 3-D singularities as analogs of frames correspond to the frames of 4-D "soap films". Since derivatives are discontinuous, the singularities correspond to edges of the space-time and would define defects of the standard smooth structure. This would give rise to an exotic smooth structures.

4. The non-determinism should correspond to the branching of the space-time surfaces at the singularities X^3 giving rise to alternative Bohr orbits. There is analogy with bifurcations,

in particular with shock waves and bifurcations could correspond to the underlying 2-adicity and relate to the p-adic length scale hypothesis.

There would be several kinds of edges of X^4 associated with the same X^3. The non-representability of the singularity X^3 as a graph $P(X^3) \rightarrow X^3$, where $P(X^3)$ is the projection of the singularity to G_4 should be essential. Also the non-convexity of the region bounded by $P(X^3)$ in G_4 matters.

5. The volumes of the minimal surfaces spanning a given frame need not be the same and the absolute minimum for the volume, or more generally classical action, could be in the special role. The original proposal indeed was that absolute minima are physically special.

   If dynamical symmetries are involved, the extrema can be degenerate. The minimal surfaces are analogs of Bohr orbits and in atomic physics Bohr orbits have degeneracy due to the fact they can be in arbitrary plane: this corresponds to the choice of the quantization axis of angular momentum.

   Could the symmetries for the 3-D "frames" induce this kind of degeneracy? Could Galois groups act as symmetries? This would give connection between the view of cognition as an outcome of classical non-determinism and the number theoretic view of cognition relying on Galois groups.

5 About the justification for the holography = holomorphy vision and related ideas

The recent view of Quantum TGD [L20, L22, L21, L25, L26, L23] has emerged from several mathematical discoveries.

1. Holography = holomorphy principle (HH) reduces classical field equations at the Minkowskian regions of the space-time surface to algebraic roots $f = (f1, f2) = (0, 0)$ of two functions which are analytic functions of 4 generalized complex coordinates of $H = M^4 \times CP_2$ involving 3 complex coordinates and one hypercomplex coordinate of M^4.

2. Space-time surface as an analog of Bohr orbit is minimal surface, which means that it generalized the notion of geodesic line in the replacement of point-like particle with 3-surface and that the non-linear analogs of massless field equations are satisfied by H coordinates so that analog of particle-wave duality is realized geometrically.

3. Minimal surface property holds true independently of the classical action as long as it is general coordinate invariant and constructible in terms of the induced geometry. This strongly suggests the existence of a number theoretic description in which the value of action as analog of effective action becomes a number theoretic invariant.

4. The minimal surface property fails at 3-D singularities at which derivatives of the embedding coordinates are discontinuous and the components of the second fundamental form have delta function divergences so that its trace as local acceleration and an analog of the Higgs field, diverges.

   These discontinuities give rise to defects of smooth structure and in 4-D case an exotic smooth structure emerges and makes possible description of fermion pair creation (boson emission) although the fermions are free particles. Fermions and also 3-surfaces turn backwards in time. This is possible only in dimension D=4.

One can criticize this picture as too heuristic and of the lack of explicit examples. I am grateful for Marko Manninen, a member of our Zoom group, who raised this question. In the following I try to make it clear that the outcome is extremely general and depends only on the very general aspects of what generalized holomorphy means. I hope that colleagues would realize that the TGD approach to theoretical physics is based on general mathematical principles and refined conceptualization: this approach is the diametric opposite of, say, the attempt to understand physics by performing massive QCD lattice calculations. Philosophical and mathematical thinking, taking empirical findings seriously, dominates rather than pragmatic model building and heavy numerics.

5.1 H-H principle and the solution of field equations

Consider first how H-H leads to an exact solution of the field equations in Minkowskian regions of the space-time surface (the solution can be found also in Euclidean regions).

1. The partial differential equations, which are extremely non-linear, reduce by generalized H-H to algebraic equations in which one has contractions of holomorphic tensors of different type vanishing identically if one has roots of $f = (f_1, f_2) = (0, 0)$. f_1 and f_2 and generalized analytic functions of generalized complex coordinates of H.

   This means a huge simplification since the Riemannian geometry reduces to algebraic geometry and partial differential equations reduce to local algebraic equations.

2. There are two kinds of induced gauge fields: induced metric and induced gauge potentials, Kähler gauge potential for the Kähler action. The variation with respect to induced metric gives a contraction of two holomorphic 2-tensors to the field equations. The variation with respect to gauge potential gives contraction of two holomorphic vector fields. The contractions are between tensors/vectors of different types and vanish identically.

   (a) Consider the metric first. The contraction is between the energy momentum tensor of type (1,-1)+(-1,1) and the second fundamental form of type (1,1)+(-1,-1). Here 1 refers to a complex coordinate and -1 to its conjugate as tensor index. These contractions vanish identically.

   The vanishing of the trace of the second fundamental form occurs independently of the action and gives minimal surface except at singularities.

   (b) Consider next the induced gauge potentials. In this case one has contraction of vector fields of different type (of type (1)and (-1) and also now the outcome is vanishing. In the case of more general action, such as volume + Kähler action, one also has a contraction of light-like Kähler current with a light-like vector field which vanishes too. The light-like Kähler current is non-vanishing for what I call "massless extremals". This miracle reflects the enormous power of generalized conformal invariance.

3. For more general actions these results are probably true too but there I have no formal proof. If higher derivatives are involved one obtains higher derivatives of the second fundamental form which are of type (1,1,...,1) contracted with tensors which have mixed indices.

   Actions containing higher derivatives might be excluded by the requirement that only delta function singularities for the trace of the second fundamental form defining the analog of the Higgs field are possible.

4. The result has analog already in ordinary electrodynamics in 2-D systems. The real and imaginary parts of an analytic function satisfy the field equations except at poles and cuts define the point charges and line charges. Also in string models the same occurs.

Concerning explicit examples, I used 8 years after my thesis to study exact solutions of field equations of TGD [**?**, **?**]. The solutions that I found were essentially action independendent and had interpretation as minimal surfaces.

5.2 Singularities as analogs of poles of analytic functions

Consider now the singularities.

1. The singularities 3-surfaces at which the generalized analyticity fails for (f_1, f_2): they are analogs of poles and zeros for analytic functions. At 3-D singularities the derivatives of H coordinates are discontinuous and the trace of the second fundamental form has a delta function singularity. This gives rise to edge.

Singularities are analogous to poles of analytic functions and correspond to vertices and also to loci of non-determinism serving as seats of conscious memories.

2. At singularities the entire action contributes to the field equations which express conservation laws of classical isometry charges. Note that the trace of the second fundamental form defines a generalized acceleration and behaves like a generalization of the Higgs field with respect to symmetries.

   Outside singularities the analog of massless geodesic motion with a vanishing acceleration occurs and the induced fields are formally massless. At singularities there is an infinite acceleration so that particles perform 8-D Brownian motion.

3. Singularities as edges correspond to defects of the standard smooth structure as edges of space-time surface analogous to the frames of a soap film. The dependence of the loci of singularities on the classical action is expected from the condition that the field equations stating conservation laws are true for the entire action.

   It is possible that exotic smooth structure is at least partially characterized by the classical action having interpretation as effective action. For a mere volume action singularities might not be possible: if this is true it would correspond to the analog of massless free theory without fermion pair creation. In this case, the trace of the second fundamental form should vanish although its components should have delta function divergences.

   This makes it possible to interpret fermionic Feynman diagrams geometrically as Brownian motion of 3-D particles in H [L24, L26, L23]. In particular, fermion pair creation (and also boson emission) corresponds to 3-surface and fermion lines turning backwards in time.

4. The physical interpretation generalizes the interpretation in classical field theories, where charges are point-like. In massless field theories, charges as singularities serve as sources of fields. The trace of the second fundamental form vanishes almost everywhere (minimal surface property) stating that the analog of the charge density, serving as a source of massless field defined for H coordinates, vanishes except at the singularities. The generalized Higgs field defines the source concentrated to 3-D singularities.

5. Classical non-determinism is an essential assumption. Already 2-D minimal surfaces allow non-determinism and soap films spanned by a given frame provide a basic example. The geomeric conditions under which non-determinism is expected, are known and can be generalized to 4-D context. Google LLM gives detailed information about the non-determinism in 2-D case and I have discussed the generalization to 4-D case in [L19] [K4].

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