About minimal surface extremals of Kähler action

M. Pitkänen
Email: matpitka6@gmail.com.
http://tgdtheory.com/.

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Abstract

The addition of the volume term to Kähler action has very nice interpretation as a generalization of equations of motion for a world-line extended to a 4-D space-time surface. The field equations generalize in the same manner for 3-D light-like surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian, for 2-D string world sheets, and for their 1-D boundaries defining world lines at the light-like 3-surfaces. For 3-D light-like surfaces the volume term is absent. Either light-like 3-surface is freely choosable in which case one would have Kac-Moody symmetry as gauge symmetry or that the extremal property for Chern-Simons term fixes the gauge.

The known non-vacuum extremals are minimal surface extremals of Kähler action and it might well be that the preferred extremal property realizing SH quite generally demands this. The addition of the volume term could however make Kähler coupling strength a manifest coupling parameter also classically when the phases of Λ and α_K are same. Therefore quantum criticality for Λ and α_K would have a precise local meaning also classically in the interior of space-time surface. The equations of motion for a world line of U(1) charged particle would generalize to field equations for a "world line" of 3-D extended particle.

The conjecture is that α_K has zeros of zeta as its spectrum of critical values. If so all preferred extremals are minimal surface extremals of Kähler action. In the following the two options are compared. Also the implications of minimal surface property for conservation laws and for the possibility of solving field equations exactly using the analogy with 2-D minimal surfaces is considered.

1 Introduction

The addition of the volume term to Kähler action has very nice interpretation as a generalization of equations of motion for a world-line extended to a 4-D space-time surface [K14, K13]. The field equations generalize in the same manner for 3-D light-like surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian, for 2-D string world sheets, and for their 1-D boundaries defining world lines at the light-like 3-surfaces. For 3-D light-like surfaces the volume term is absent. Either light-like 3-surface is freely choosable in which case one would have Kac-Moody symmetry as gauge symmetry or that the extremal property for Chern-Simons term fixes the gauge.

All the known non-vacuum extremals are minimal surface extremals of Kähler action [K1, K12] and it might well be that the preferred extremal property realizing SH quite generally demands this. CP_2 type vacuum extremals are also minimal surfaces if one assumes that the M^4 projection is light-like geodesic rather than only geodesic line. The addition of the volume term could however make Kähler coupling strength a manifest coupling parameter also classically when the phases of Λ and α_K are same. Therefore quantum criticality for Λ and α_K would have a precise local meaning also classically in the interior of space-time surface. The equations of motion for a world line of U(1) charged particle would generalize to field equations for a "world line" of 3-D extended particle.

This is an attractive idea consistent with standard wisdom but one can invent strong objections against it in TGD framework.

1. The conjecture is that α_K has zeros of zeta as its spectrum of critical values [L2]. If so then all preferred extremals are minimal surface extremals of Kähler action for a real value of cosmological constant Λ.

2. All known non-vacuum extremals of Kähler action are minimal surfaces and the minimal surface vacuum extremals of Kähler action become non-vacuum extremals. This suggest that preferred extremals are minimal surface extremals of Kähler action so that the two dynamics would apparently decouple. Minimal surface extremals are analogs for geodesics in the case of point-like particles: one might say that one has only gravitational interaction. This conforms with SH stating that gauge interactions at boundaries (orbits of partonic 2-surfaces and 2-surfaces at the ends of CD) correspond classically to the gravitational dynamics in the space-time interior.

   Note that at the boundaries of the string world sheets at light-like 3-surfaces the situation is different: one has equations of motion for geodesic line coupled to induce Kähler gauge potential and gauge coupling indeed appears classically as one might expect! For string world sheets one has only the topological magnetic flux term and minimal surface equation in string world sheet. Magnetic flux term gives the Kähler coupling at the boundary.

3. Decoupling would allow to realize number theoretical universality [K11] since the field equations would not depend on coupling parameters at all. It is very difficult to imagine how the solutions could be expressible in terms of rational functions with coefficients in algebraic extension of rationals unless α_K and Λ have very special relationship. If they have different phases, minimal surface extremals of Kähler action are automatically implied. If the values of α_K correspond to complex zeros of Riemann ζ, also Λ should have same complex phase, in order to have genuine classical coupling. This looks somewhat un-natural but cannot be excluded.

   The most natural option is that Λ is real and α_K corresponds to zeros of zeta. For non-trivial zeros the phases are different and decoupling occurs. For trivial zeros Λ and α_K differ by imaginary unit so that again decoupling occurs.

4. One can argue that the decoupling makes it impossible to understand coupling constant evolution. This is not the case. The point is that the classical charges assignable to super-symplectic algebra are sums over contributions from Kähler action and volume term and therefore depend on the coupling parameters. Their vanishing conditions for sub-algebra and its commutator with entire algebra give boundary conditions on preferred extremals so that coupling constant evolution creeps in classically!

   Quantum classical correspondence realized as the condition that the eigenvalues of fermionic charge operators are equal to the classical charges brings in the dependence of quantum charges on coupling parameters. Since the elements of scattering matrix are expected to involve as building bricks the matrix elements of super-symplectic algebra and Kac-Moody algebra of isometry charges, one expectes that discrete coupling constant evolution creeps in also quantally via the boundary conditions for preferred extremals.

5. Decoupling has also implications for the conservation laws and leads to the proposal for a generalization of 2-D minimal surface extremals based on generalization of holomorphy.

In the sequel Options I (effective decoupling of Kähler action and volume term) and II (their coupling) are considered. Also the possible implications of the assumption that the preferred extremals are minimal surface extremals of Kähler action are discussed.

2 Does the presence of cosmological constant term makes Kähler coupling strength a genuine coupling constant classically?

Although the above arguments seem to kill the idea that the dynamics of Kähler action and volume term could couple in space-time interior, one can compare this view (Option II) with the view based on complete decoupling (Option I).

1. For Option I the coupling between the two dynamics could be induced just by the condition that the space-time surface becomes an analog of geodesic line by arranging its interior so that the U(1) force vanishes! This would generalize Chladni mechanism (see `http://tinyurl.com/j9rsyqd`)! The interaction would be present but be based on going to the nodal surfaces! Also the dynamics of string world sheets is similar: if the string sheets carry vanishing W boson classical fields, em charge is well-defined and conserved. One would also avoid the problems produced by large coupling constant between the two-dynamics present already at the classical level. At quantum level the fixed point property of quantum critical couplings would be the counterparts for decoupling.

2. For Option II the coupling is of conventional form. When cosmological constant is small as in the scale of the known Universe, the dynamics of Kähler action is perturbed only very slightly by the volume term. The alternative view is that minimal surface equation has a very large perturbation proportional to the inverse of Λ so that the dynamics of Kähler action could serve as a controller of the dynamics defined by the volume term providing a small push or pull now and then. Could this sensitivity relate to quantum criticality and to the view about morphogenesis relying on Chladni mechanism in which field patterns control the dynamics with charged flux tubes ending up to the nodal surfaces of (Kähler) electric field [L4]? Magnetic flux tubes containing dark matter would in turn control and serve as template for the dynamics of ordinary matter.

Could the possible coupling of the two dynamics suggest any ideas about the values of α_K and Λ at quantum criticality besides the expectation that cosmological constant is proportional to an inverse of p-adic prime [K14]?

1. Number theoretic vision suggests the existence of preferred extremals represented by rational functions with rational or algebraic coefficients in preferred coordinates. For Option I one has preferred extremals of Kähler action which are minimal surfaces so that there is no coupling and no constraints on the ratio of couplings emerges: even better, both dynamics are independent of the coupling. All known non-vacuum extremals of Kähler action are indeed also minimal surfaces. For Option II the ratio of the coefficients $\Lambda/8\pi G$ and $1/4\pi\alpha_K$ should be rational or at most algebraic number. One must be however very cautious here: the minimal option allowed by strong form of holography is that the rational functions of proposed kind emerge only at the level of partonic 2-surfaces and string world sheets.

2. I have proposed that that the inverse of Kähler coupling strength has spectrum coming as zeros of zeta or their imaginary parts [L2]. The phases of complexified $1/\alpha_K$ and $\Lambda/2G$ must be same in order to avoid the decoupling of Kähler action and minimal surface term implying minimal surface extremals of Kähler action.

   This conjecture is consistent with the rational function property only if α_K and vacuum energy density ρ_{vac} appearing as the coefficient of volume term are proportional to the same possibly transcendental number with proportionality coefficient being an algebraic or rational number.

If the phases are not identical (say Λ is real and one allows complex zeros) one has Option I and effective decoupling occurs. The coupling (Option2)) can occur for the trivial zeros of zeta if the volume term has coefficient $i\Lambda/8\pi G$ rather than $\Lambda/8\pi G$ to guarantee same phase as for $1/4\pi\alpha_K$. The coefficient $i\Lambda/8\pi G$ would give in Minkowskian regions large real exponent of volume and this looks strange. In this case also number theoretical universality might make sense but SH would be broken in the sense that the space-time surfaces would not be analogous to geodesic lines.

3. At quantum level number theoretical universality requires that the exponent of the total action defining vacuum functional reduces to the product of roots of unity and exponent of integer existing in finite-dimensional extension of p-adic numbers. This would suggest that total action reduces to a number of form $q_1 + iq_2\pi$, q_i rational number, so that its exponent is of the required form. Whether this can conform with the properties of zeros of zeta and properties of extremals is not clear.

ZEO suggests deep connections with the basic phenomenology of particle physics, quantum consciousness theory, and quantum biology and one can look the situation for both these options.

1. Option I: Decoupling of the dynamics of Kähler action and volume term in space-time interior for all values of coupling parameters.
2. Option II: Coupling of dynamics for trivial zeros of zeta and $\Lambda \to i\Lambda$.

Particle physics perspective.

Consider a typical particle physics experiment. There are incoming and outgoing free particles moving along geodesics, these particles interact, and emanate as free particles from the interaction volume. This phenomenological picture does not follow from quantum field theory but is put in by hand, in particular the idea about interaction couplings becoming non-zero is involved. Also the role of the observer remains poorly understood.

The motion of incoming and outgoing particles is analogous to free motion along geodesic lines with particles generalized to 3-D extended objects. For both options these would correspond to the preferred extremals in the complement of CD within larger CD representing observer or measurement instrument. Decoupling would take place. In interaction volume interactions are "coupled on" and particles interact inside the volume characterized by causal diamond (CD). What could be the TGD view translation of this picture?

1. For Option I one would still have decoupling and the interpretation would be in terms of twistor picture in which one always has also in the internal lines on mass shell particles but with complex four-momenta. In TGD framework the momenta would be always complex due to the contribution of Euclidian regions defining the lines of generalized scattering diagrams. As explained coupling constant evolution can be understood also in this case and also classical dynamics depends on coupling parameters via the boundary conditions. The transitory period (control action) leading to the decoupled situation would be replaced by state function reduction, possibly to the opposite boundary.
2. For Option II the transitory period would correspond to the coupling between the two classical dynamics and would take place inside CD after a phase transition identifiable as "big state function reduction" to time reversed mode. The problem is that in the interacting phase α_K would not have a value approximately equal to the U(1) coupling strength of weak interactions [L2] so that the physical picture breaks down.

Quantum measurement theory in ZEO

1. For Option I state preparation and state function reduction would be in symmetric role. Also now there would be inherent asymmetry between zero energy states and their time reversals. With respect to observer the time reversed period would be invisible being in geometric past.

2. For Option II state preparation for CD would correspond to a phase transition to a time reversed phase labelled by a trivial zero of zeta and $\Lambda \rightarrow i\Lambda$. In state function reduction to the original boundary of CD a phase transition to a phase labelled by non-trivial zero of zeta would occur and final state of free particles would emerge. The phase transitions would thus mean hopping from the critical line of zeta to the real axis and back and change the values of α_K and possibly Λ. There would be strong breaking of time reversal symmetry.

   One cannot of course take this large asymmetry as an adhoc assumption: it should be induced by the presence of larger CD, which could also affect quite generally the values of α_K and Λ (having also a spectrum of values).

TGD inspired theory of consciousness

What happens within sub-CD could be fundamental for the understanding of directed attention and sensory-motor cycle.

1. The target of directed attention would correspond to the volume of CD - call it c - within larger CD - call it C representing the observer - attendee having c as part of its perceptive field. c would correspond also to sub-self giving rise to a mental image of C.

2. Quite generally, the self and time-reversed self could be seen as sensory input and motor response (Libet's findings). Directed attention would define the sensory input and sub-self could react to it by dying and re-incarnating as time-reversed subself. The two selves would correspond to sensory input and motor action following it as a reaction. Motor reaction would be sensory mental image in reversed time direction experienced by time reversed self. Only the description for the reaction would differ for the two options.

   The motor action would be time-reversed sensory perception for Option I. For Option II motor action would correspond to a different phase in which Kähler action and volume term couple classically.

TGD inspired quantum biology

The free geodesic line dynamics with vanishing $U(1)$ Kähler force indeed brings in mind the proposed generalization of Chladni mechanism generating nodal surfaces at which charged magnetic flux tubes are driven [L4].

1. For Option I the interiors of all space-time surfaces would be analogous to nodal surfaces and "big" state function reductions would correspond to transition periods between different nodal surfaces. The decoupling would be dynamics of avoidance and could highly analogous to Chladni mechanism.

2. For Option II the phase labelled by trivial zeros of zeta would correspond to period during which nodal surfaces are formed. This view about state function reduction and preparation as phase transitions in ZEO would provide classical description for the transition to the phase without direct interactions.

To sum up, it seems that the complete decoupling of the two dynamics (Option I) is favored by both SH, realization of preferred extremal property (perhaps as minimal surface extremals of Kähler action, number theoretical universality, discrete coupling constant evolution, and generalization of Chladni mechanism to a dynamics of avoidance.

3 About minimal surface extremals of Kähler action

If the spectrum for the critical value of Kähler coupling strength is complex - say given by the complex zeros of zeta [L2] - the preferred extremals of Kähler action are minimal surfaces. This means that they satisfy simultaneously the field equations associated with two variational principles.

3.1 Conservation laws for the minimal surface extremals of Kähler action

Consider first the basic conservation laws.

1. Complex value of α_K means that conserved quantities are complex: this brings strongly in mind twistor approach in which the internal lines of the scattering amplitudes are also massless but four-momenta are complex [K14]. The value of cosmological constant is assumed to be real. There are two separate local conservations laws associated with the volume term and Kähler action respectively in both Minkowskian and Euclidian regions. This need not mean separate global conservation laws in Minkowskian and Euclidian regions. If there is non canonical momentum current between Minkowskian (M) and Euclidian (E) space-time regions the real and imaginary parts of conserved quantum numbers correspond schematically to the sums

$$\begin{aligned} Re(Q) &= Re(\frac{1}{\alpha_K})Q_K(E) + Im(\frac{1}{\alpha_K})Q_K(M) + \rho_{vac}Q_V(M) \ , \\ Im(Q) &= Im(\frac{1}{\alpha_K})Q_K(E) + Re(\frac{1}{\alpha_K})Q_K(M) \ . \end{aligned} \tag{3.1}$$

   Here the subscripts $_V$ and $_K$ refer to the volume term and Kähler action respectively.

2. If the canonical momentum current vanishes there both real and imaginary parts decompose to two separately conserved parts.

$$\begin{aligned} Re(Q_1) &= Re(\frac{1}{\alpha_K})Q_K(E) \ , \quad Re(Q_2) = Im(\frac{1}{\alpha_K})Q_K(M) + \rho_{vac}Q_V(M) \ , \\ Im(Q_1) &= Im(\frac{1}{\alpha_K})Q_K(E) \ , \quad Im(Q_2) = Re(\frac{1}{\alpha_K})Q_K(M) \ . \end{aligned} \tag{3.2}$$

   This looks strange and the natural assumption is that canonical momentum currents can flow between the Euclidian and Minkowskian regions and boundary conditions equate the components of normal currents at both sides.

3.2 Are minimal surface extremals of Kähler action holomorphic surfaces in some sense?

I have considered several ansätze for the general solutions of the field equations for the preferred extremals. One proposal is that preferred extremals as 4-surfaces of imbedding space with octonionic tangent space structure have quaternionic tangent space or normal space (so called $M^8 - H$ duality [K6]). Second proposal is that preferred extremals can be seen as quaternion analytic [A1] surfaces [K10, K15] [L1]. Third proposal relies on a fusion of complex and hyper-complex structures to what I call Hamilton-Jacobi structure [K7, K12]. In Euclidian regions this would correspond to complex structure. Twistor approach [K14] suggests that the condition that the twistor lift of the space-time surface to a 6-D surface in the product of twistor spaces of M^4 and CP_2 equals to the twistor space of CP_2. This proposal is highly interesting since twistor lift works only fr $M^4 \times CP_2$. The intuitive picture is that the field equations are integrable and all these views might be consistent.

Preferred extremals of Kähler action as minimal surfaces would be a further proposal. Can one make conclusions about general form of solutions assuming that one has minimal surface extremals of Kähler action?

In $D = 2$ case minimal surfaces are holomorphic surfaces or they hyper-complex variants and the imbedding space coordinates can be expressed as complex-analytic functions of complex coordinate or a hypercomplex analog of this. Field equations stating the vanishing of the trace $g\alpha\beta H^k_{\alpha\beta}$ if the second fundamental form $H^k_{\alpha\beta} \equiv D_\alpha \partial_\beta h^k$ are satisfied because the metric is tensor of type $(1,1)$ and second fundamental form of type $(2,0) \oplus (2,0)$. Field equations reduce to an algebraic identity and functions involved are otherwise arbitrary functions. The constraint comes from the condition that metric is of form $(1,1)$ as holomorphic tensor.

This raises the question whether this finding generalizes to the level of 4-D space-time surfaces and perhaps allows to solve the field equations exactly in coordinates generalizing the hypercomplex coordinates for string world sheet and complex coordinates for the partonic 2-surface.

The known non-vacuum extremals of Kähler action are actually minimal surfaces. The common feature suggested already earlier to be common for all preferred extremals is the existence of generalization of complex structure.

1. For Minkowskian regions this structure would correspond to what I have called Hamilton-Jacobi structure [K7, K12]. The tangent space of the space-time surface X^4 decomposes to local direct sum $T(X^4) = T(X^2) \oplus T(Y^2)$, where the 2-D tangent places $T(X^2)$ and $T(Y^2)$ define an integrable distribution integrating to a decomposition $X^4 = X^2 \times Y^2$. The complex structure is generalized to a direct some of hyper-complex structure in X^2 meaning that there is a local light-like direction defining light-like coordinate u and its dual v. $Y2$ has complex complex coordinate $(w, \overline{w})$. Minkowski space M^4 has similar structure. It is still an open question whether metric decomposes to a direct sum of orthogonal metrics assignable to X^2 and Y^2 or is the most general analog of complex metric in question. g_{uv} and $g_{w\overline{w}}$ are certainly non-vanishing components of the induced metric. Metric could allow as non-vanishing components also g_{uw} and $g_{v\overline{w}}$. This slicing by pairs of surfaces would correspond to decomposition to a product of string world sheet and partonic 2-surface everywhere.

   In Euclidian regions ne would have 4-D complex structure with two complex coordinates (z, w) and their conjugates and completely analogous decompositions. In CP_2 one has similar complex structure and actually Kähler structure extending to quaternionic structure. I have actually proposed that quaternion analyticity could provide the general solution of field equations.

2. Assuming minimal surface property the field equations for Kähler action reduce to the vanishing of a sum of two terms. The first term comes from the variation with respect to the induced metric and is proportional to the contraction

$$A = J^{\alpha}_{\ \gamma} J^{\gamma\beta} H^k_{\alpha\beta} \quad . \tag{3.3}$$

   Second term comes from the variation with respect to induced Kähler form and is proportional to

$$B = j^{\alpha} P^k_{\ s} J^s_{\ l} \partial_{\alpha} h^l \quad . \tag{3.4}$$

   Here P^k_l is projector to the normal space of space-time surface and $j^{\alpha} = D_{\beta} J^{\alpha\beta}$ is the conserved Kähler current.

   For the known extremals j vanishes or is light-like (for massless extremals) in which case A and B vanish separately.

3. An attractive manner to satisfy field equations would be by assuming that the situation for 2-D minimal surface generalizes so that minimal surface equations are identically satisfied. Extremal property for Kähler action could be achieved by requiring that energy momentum tensor also for Kähler action is of type $(1,1)$ so that one would have $A = 0$. This implies $j^{\alpha}\partial_{\alpha} s^k = 0$. This is true if j vanishes or is light-like as it is for the known extremals. In Euclidian regions one would have $j = 0$.

4. The proposed generalization is especially interesting in the case of cosmic string extremals of form $X^2 \times Y^2$, where $X^2 \subset M^4$ is minimal surface (string world sheet) and Y^2 is complex homologically non-trivial sub-manifold of CP_2 carrying Kähler magnetic charge. The generalization would be that the two transversal coordinates $(w, \overline{w})$ in the plane orthogonal to the string world sheet defining polarization plane depend holomorphically on the complex coordinates of complex surface of CP_2. This would transform cosmic string to flux tube.

5. There are also solutions of form $X^2 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2 with vanishing Kähler magnetic charge and their deformations with $(w, \overline{w})$ depending on the complex coordinates of Y^2 (see the slides "On Lagrangian minimal surfaces on the complex projective plane" at `http://tinyurl.com/jrhl6gy`). In this case Y^2 is not complex sub-manifold of CP_2 with arbitrary genus and induced Kähler form vanishes. The simplest choice for Y^2 would be as homologically trivial geodesic sphere. Because of its 2-dimensionality Y^2 has a complex structure defined by its induced metric so that solution ansatz makes sense also now.

3.3 Spherically symmetry stationary metric as minimal surface

Physical intuition and the experience with the vacuum extremals as models for GRT space-times suggests that Kähler charge is not important in the case of astrophysical objects like stars so that it might be possible to model them as minimal surfaces, which in the simplest situation have spherically symmetric metric analogous to Schwartschild solution. The vanishing of the induced Kähler form does not of course exclude the presence of electromagnetic fields. It must be of course emphasized that the assumption that single-sheeted space-time surface can model GRT-QFT limit based on many-sheeted space-time could be un-realistic.

At 90's I studied the imbeddings of Schwartschild-Nordström solution as vacuum extremals of Kähler action and found that the solution is necessarily electromagnetically charged [K7]. This property is unavoidable. The imbedding in coordinates (t, r, θ, ϕ) for X^4, (m^0, r, θ, ϕ) for M^4 and (Θ, Φ) for the trivial geodesic sphere S^2_{II} of CP_2 was not stationary as the first guess might be. m^0 relates to Schwartschild time and radial coordinate r by a shift $m^0 = \Lambda t + h(r)$. Without this shift the perihelion shift would be negligibly small.

One has $(cos(\Theta) = f(r), \Phi = \omega t + k(r))$. Also the dependence of Φ is not the first possibility to come in mind. The shifts $h(r)$ and $k(r)$ are such that the non-diagonal contribution g_{tr} to the induced metric vanishes. The question is whether one obtains spherically symmetric metric as a minimal surface.

3.3.1 General form of minimal surface equations

Consider first the minimal surface equations generally.

1. The field equations are analogous to massless wave equations for scalar fields defined by CP_2 coordinates having gravitational self coupling and also covariant derivative coupling due to the non-flatness of CP_2. One might therefore expect that the Newtonian gravitation based on Laplace equation in empty space-time regions follows as an approximation. Therefore also something analogous to Schwartschild metric is to be expected. Note that also massless extremals (MEs) are obtained as minimal surfaces so that also the topologically quantized counterparts of em and gravitational radiation emerge.

2. The general field equations can be written as vanishing of the covariant divergence for canonical momentum current $T^{k\alpha}$

$$\begin{aligned} D_\alpha(T^{k\alpha}\sqrt{g}) &= \partial_\alpha\left[T^{k\alpha}\sqrt{g}\right] + \left\{ {k \atop \alpha\ m} \right\} T^{m\alpha}\sqrt{g} = 0 \ , \\ T^{k\alpha} &= g^{\alpha\beta}\partial_\beta h^k \ , \\ \left\{ {k \atop \alpha\ m} \right\} &= \left\{ {k \atop l\ m} \right\} \partial_\alpha h^l \ . \end{aligned} \tag{3.5}$$

D_α is covariant derivative taking into account that gradient $\partial_\alpha h^k$ is imbedding space vector.

3. For isometry currents $j^{A,k}$ (Killing vector fields)

$$T^{A,\alpha} = T^{\alpha k} h_{kl} j^{A,l} \tag{3.6}$$

the covariant divergence simplifies to ordinary divergence

$$\partial_\alpha \left[T^{A,\alpha}\sqrt{g}\right] = 0 \quad . \tag{3.7}$$

This allows to simplify the equations considerably.

3.3.2 Spherically symmetric stationary minimal surface

Consider now the spherically symmetric stationary metric representable as minimal surface.

1. In the following we consider only the region exterior to the surface defining the TGD counterpart of Schwartschild horizon and the possible horizon at which the signature of the induced metric. The first possibility is $g_{tt} = 0$ at horizon. If g_{rr} remains non-vanishing, the signature changes to Euclidian. If also $g_{rr} = 0$, both g_{tt} and g_{rr} can change sign so that one has a smooth variant of Schwartschild horizon.

   Second possibility is $g_{rr} = 0$ at radius r_E in the region below Schwartschild radius. At r_E the determinant of 4-metric would vanish and the signature of the induced metric would change to Euclidian.

2. The reduction to the conservation of isometry currents can be used for isometry current corresponding to the rotation $\Phi \to \Phi + \epsilon$ and time translation $m^0 \to m^0 + \epsilon$.

3. With the experience coming from the imbedding of Reissner-Nordström metric the ansatz is exactly the same and can be written as

$$m^0 = \Lambda t + h(r) \quad , \quad \Phi = \omega t + k(r) \quad , \quad u \equiv cos(\Theta) = u(r) \quad , \tag{3.8}$$

4. The condition $g_{tr} = 0$ gives

$$\Lambda \partial_r h = R^2 \omega sin^2(\Theta) \partial_r k = 0 \quad . \tag{3.9}$$

   This allows to integrate $h(r)$ in terms of $k(r)$.

5. The interesting components of the induced metric are

$$g_{tt} = \Lambda^2 - R^2\omega^2 sin^2(\Theta) \quad , \quad g_{rr} = -1 - R^2(\partial_r\Theta)^2 + \Lambda^2(\partial_r h)^2 \quad . \tag{3.10}$$

6. The field equations reduce to conservation laws for various isometry currents. Consider energy current and the current related to the $SO(3) \subset SU(3)$ rotation acting on Φ as shift (call this current isospin current). The stationary character of the induced metric implies that the field equations reduce to the conservation of the radial current for energy current and isospin current. These two equations fix the solution together with diagonality condition. One obtains the following equations

$$\partial_r(\partial_r h \times g^{rr}\sqrt{g}) = 0 \quad , \quad \partial_r(sin^2(\Theta)\partial_r k \times g^{rr}\sqrt{g}) = 0) \quad . \tag{3.11}$$

   These two equations can be satisfied simultaneously only if one has

$$\partial_r h \times g^{rr} r^2\sqrt{g_2} = A sin^2(\Theta)\partial_r k \times g^{rr} r^2 \sqrt{g_2} + B \quad , \quad g_2 \equiv -g_{tt}g_{rr} \quad . \tag{3.12}$$

Note the presence of constant B.

Second implication is

$$g^{rr}\partial_r h\sqrt{g_2} = \frac{C}{r^2} \quad , \quad g^{rr}sin^2(\Theta)\partial_r k\sqrt{g_2} = \frac{D}{r^2} \quad , \quad C = AD + B \quad . \tag{3.13}$$

By substituting the expressions for the metric one has

$$\partial_r h = \sqrt{-\frac{g_{rr}}{g_{tt}}} \times \frac{C}{r^2} \quad , \quad sin^2(\Theta)\partial_r k = \sqrt{-\frac{g_{rr}}{g_{tt}}} \times \frac{D}{r^2} \quad . \tag{3.14}$$

7. It is natural to look what one obtains in the approximation that the metric is flat expected to make sense at large distances. Putting $g_{tt} = -g_{rr} = 1$, one obtains

$$\partial_r h \simeq \frac{C}{r^2} \quad , \quad sin^2(\Theta)\partial_r k \simeq \frac{D}{r^2} \quad . \tag{3.15}$$

The time component of the induced metric is given by

$$g_{tt} = \Lambda^2 - R^2\omega^2 sin^2(\Theta) \simeq \Lambda^2 - \frac{D}{r^2\partial_r k} \quad . \tag{3.16}$$

This gives $1/r$ gravitational potential of a mass point if one has $\partial_r k \simeq E/r$ giving for $\Lambda = 1$

$$g_{tt} = 1 - \frac{r_S}{r} \quad , \quad r_S = 2GM = \frac{D}{E} \quad . \tag{3.17}$$

with the identification $r_S = 2GM = D/E$ inspired by the behavior of the Scwartschild metric. It seems that one can take $\Lambda = 1$ without a loss of generality.

8. Using $g_{tr} = 0$ condition this gives for h the approximate expression

$$\partial_r h \simeq \frac{D}{r^2} \quad , \quad D = \frac{R^2\omega^2}{\Lambda} \quad . \tag{3.18}$$

so that the field equations are consistent with the $1/r$ behavior of gravitational potential. The solution carries necessarily a non-vanishing Abelian electroweak gauge field.

9. The asymptotic behaviors of k and h would be

$$k \simeq k_0 log(\frac{r}{r_0}) \quad , \quad h \simeq h_0 - \frac{C}{r} \quad . \tag{3.19}$$

3.3.3 Two horizons and layered structure as basic prediction

A very interesting question is whether $g_{tt} = 0$ defines Schwartschild type horizon at which the roles of the coordinates t and r change or whether one obtains horizon at which the signature of the induced metric becomes Euclidian. The most natural option turns out to be Schwartschild like horizon at which the roles of time and radial coordinate are changed and second inner horizon at which g_{rr} changes sign again so that the induced metric has Euclidian signature below this inner horizon.

1. Unless one has $g_{tt}g_{rr} = C \neq 0$ ($C = -1$ holds true in Scwhartschild-Nordström metric) the surface $g_{tt} = 0$ - if it exists - defines a light-like 3-surface identifiable as horizon at which the signature of the induced metric changes. The conditions $g_{tt} = 0$ gives

$$\Lambda^2 - R^2\omega^2(1-u^2) = 0 \ . \tag{3.20}$$

giving

$$0 < sin^2(\Theta) = 1 - u^2 \quad = \quad \frac{\Lambda^2}{R^2\omega^2} < 1 \ . \tag{3.21}$$

For $\Lambda = 1$ this condition implies that ω is a frequency of order of the inverse of CP_2 radius R. Note that $g_{tt} = 0$ need mean change of the metric signature to Euclidian if the analog of Schwarschild horizon is in question.

2. $g_{tt} = 0$ surface is light-like surface if g_{rr} has non-vanishing and finite value at it. g_{rr} could diverges at this surface guaranteeing $g_{tt}g_{rr} > 0$. The quantities $\partial_r h$ and $sin^2(\Theta)\partial_r k$ are proportional to $\sqrt{g_{rr}/g_{tt}}$, which diverges for $g_{tt} = 0$ unless also g_{rr} vanishes so that also these derivatives would diverge. The behavior of g_{rr} at this surface is

$$g_{rr} = -1 - R^2\tfrac{(\partial_r u)^2}{1-u^2} + \Lambda^2(\partial_r h)^2 \quad , \quad u \equiv cos(\Theta) \ . \tag{3.22}$$

There are several options to consider.

   (a) Option I: The divergence of $(\partial_r h)^2$ as cause for the divergence of g_{rr} is out of question. If this quantity increases for small values of r, g_{rr} can change sign for with finite value of $\partial_r h$ and $u^2 < 1$ at some larger radius r_S analogous to Schwartschild radius. Since it is impossible to have two time-like directions also the sign of g_{tt} must change so that one would have the analog of Schwartschild horizon at this radius - call it r_S: $r_S = 2GM$ need not hold true. The condition $g_{tt} = 0$ at this radius fixes the value of $sin^2(\Theta)$ at this radius

$$sin^2(\Theta_S) \quad = \quad \frac{\Lambda^2}{R^2\omega^2} \ . \tag{3.23}$$

   If g_{rr} has finite value and is continuous, the metric has Euclidian signature in interior. If g_{rr} is discontinuous and changes sign as in the case of Schwartschild metric, one has counterpart of Scwartschild horizon without infinities. This option will be called Option I.

   (b) Second possibility giving rise to would be that u becomes equal 1. This is not consistent with $sin^2(\Theta_S) = 0$.

   (c) Option II: Voth g_{tt} and g_{rr} change their sign and vanish at r_S. This however requires both radial and time-like direction become null directions locally. Space-time surface would become locally metrically 2-dimensional at the horizon. This would conform with the idea of strong form of holography (SH) but it is not possible to have two different light-like directions simultaneously unless these directions are actually same. Mathematically it is certainly possible to have surfaces for which the dimension is locally reduced from the maximal one but it is difficult to visualize what this kind of metric reduction of local space-time dimension could mean. This option will be considered in what follows.

To sum up, g_{rr} changes sign at horizon. For Option I g_{rr} is finite and dis-continuous. For Option II g_{rr} vanishes and is continuous. Whether g_{rr} vanishes at horizon or not, remains open.

3. For Schwartschild-Nordström metric g_{rr} becomes infinite and changes sign at horizon. The change of the roles of g_{tt} and g_{rr} could for Option II take place smoothly so that both could become zero and change their sign at r_S. This would keep $\partial_r h$ and $sin^2(\Theta)\partial_r k$ finite. One would have the analog of the interior of Schwartschild metric.

   What happens at the smaller radii? The obvious constraint is that $sin^2(\Theta)$ remains below unity. If g_{rr}/g_{tt} remains bounded, the condition for $sin^2(\Theta)\partial_k$ however suggests that $sin^2(\Theta) = 1$ is eventually achieved. This is the case also for the imbedding of Schwartschild metric. Could this horizon correspond to a surface at which the signature of the metric changes? g_{rr} should becomes zero in order to obtain light-like surface. g_{rr} contains indeed a term proportional to $1/sin^2(\Theta)$ which diverges at $u = 1$ so that g_{rr} must change sign for second time already above the radius for $sin^2(\Theta) = 1$ if h and k behaves smoothly enough. At this radius - call it r_E - g_{tt} would be finite and the signature would become Euclidian below this radius.

   One would therefore have two special radii r_S and r_E and a layer between these radii. $r_S = 2GM$ need not hold true but is expected to give a reasonable order of magnitude estimate.

Is there any empirical evidence for the existence of two horizons? There is evidence that the formation of the recently found LIGO blackhole (discussed from TGD view point in [L3]) is not fully consistent with the GRT based model (see `http://tinyurl.com/zbbz58w`). There are some indications that LIGO blackhole has a boundary layer such that the gravitational radiation is reflected forth and back between the inner and outer boundaries of the layer. In the proposed model the upper boundary would not be totally reflecting so that gravitational radiation leaks out and gave rise to echoes at times .1 sec, .2 sec, and .3 sec. It is perhaps worth of noticied that time scale .1 sec corresponds to the secondary p-adic time scale of electron (characterized by Mersenne prime $M_{127} = 2^{127} - 1$). If the minimal surface solution indeed has two horizons and a layer like structure between them, one might at least see the trouble of killing the idea that it could give rise to repeated reflections of gravitational radiation.

The proposed model (see `http://tinyurl.com/zbbz58w`) assumes that the inner horizon is Schwarstchild horizon. TGD would however suggests that the outer horizon is the TGD counterpart of Schwartschild horizon. It could have different radius since it would not be a singularity of g_{rr} (g_{tt}/g_{rr} would be finite at r_S which need not be $r_S = 2GM$ now). At r_S the tangent space of the space-time surface would become effectively 2-dimensional for $g_{rr} = 0$: the interpretation in terms of strong holography (SH) has been already mentioned.

The condition that the normal components of the canonical momentum currents for Kähler action and volume term are finite implies that $g^{nn}\sqrt{g_4}$ is finite at both sides of the horizon. Also the weak form of electric magnetic duality for Kähler form requires this. This condition can be satisfied if g_{tt} and g_{nn} approach to zero in the same manner at both sides of the horizon. Hence it seems that strong form of holography in the horizon is forced by finiteness.

One should understand why it takes rather long time $T = .1$ seconds for radiation to travel forth and back the distance $L = r_S - r_E$ between the horizons. The maximal signal velocity is reduced for the light-like geodesics of the space-time surface but the reduction should be rather large for $L \sim 20$ km (say). The effective light-velocity is measured by the coordinate time $\Delta t = \Delta m^0 + h(r_S) - h(r_E)$ needed to travel the distance from r_E to r_S. The Minkowski time Δm^0_{-+} would be the from null geodesic property and $m^0 = t + h(r)$

$$\Delta m^0_{-+} = \Delta t - h(r_S) + h(r_E) \ , \quad \Delta t = \int_{r_E}^{r_S} \sqrt{\frac{g_{rr}}{g_{tt}}} dr \equiv \int_{r_E}^{r_S} \frac{dr}{c\#} \ . \tag{3.24}$$

Note that $c\#$ approaches zero at horizon if g_{rr} is non-vanishing at horizon.

The time needed to travel forth and back does not depend on h and would be given by

$$\Delta m^0 = 2\Delta t = 2\int_{r_E}^{r_S} \frac{dr}{c_\#} \ . \tag{3.25}$$

This time cannot be shorter than the minimal time $(r_s - r_E)/c$ along light-like geodesic of M^4 since light-like geodesics at space-time surface are in general time-like curves in M^4. Since .1 sec corresponds to about 3×10^4 km, the average value of $c_\#$ should be for $L = 20$ km (just a rough guess) of order $c\# \sim 2^{-11}c$ in the interval $[r_E, r_S]$. As noticed, $T = .1$ sec is also the secondary p-adic time assignable to electron labelled by the Mersenne prime M_{127}. Since g_{rr} vanishes at r_E one has $c_\# \to \infty$. $c_\#$ is finite at r_S.

There is an intriguing connection with the notion of gravitational Planck constant. The formula for gravitational Planck constant given by $h_{gr} = GMm/v_0$ characterizing the magnetic bodies topologically for mass m topologically condensed at gravitational magnetic flux tube emanating from large mass M [K4, K3, K8, K9]. The interpretation of the velocity parameter v_0 has remained open. Could v_0 correspond to the average value of $c_\#$? For inner planets one has $v_0 \simeq 2^{-11}$ so that the order of magnitude is same as for the the estimate for $c_\#$.

3.4 What about TGD inspired cosmology?

Before the discovery of the twistor lift TGD inspired cosmology has been based on the assumption that vacuum extremals provide a good estimate for the solutions of Einstein's equations at GRT limit of TGD [K7, K5] . One can find imbeddings of Robertson-Walker type metrics as vacuum extremals and the general finding is that the cosmological with super-critical and critical mass density have finite duration after which the mass density becomes infinite: this period of course ends before this. The interpretation would be in terms of the emergence of new space-time sheet at which matter represented by smaller space-time sheets suffers topological condensation. The only parameter characterizing critical cosmologies is their duration. Critical (over-critical) cosmologies having $SO3 \times E^3$ ($SO(4)$) as isometry group is the duration and the CP_2 projection at homologically trivial geodesic sphere S^2: the condition that the contribution from S^2 to g_{rr} component transforms hyperbolic 3-metric to that of E^3 or S^3 metric fixes these cosmologies almost completely. Sub-critical cosmologies have one-dimensional CP_2 projection.

Do Robertson-Walker cosmologies have minimal surface representatives? Recall that minimal surface equations read as

$$\begin{aligned} D_\alpha(g^{\alpha\beta}\partial_\beta h^k\sqrt{g}) &= \partial_\alpha\left[g^{\alpha\beta}\partial_\beta h^k\sqrt{g}\right] + \left\{ {k \atop \alpha\ m} \right\} g^{\alpha\beta}\partial_\beta h^m\sqrt{g} = 0 \ , \\ \left\{ {k \atop \alpha\ m} \right\} &= \left\{ {k \atop l\ m} \right\} \partial_\alpha h^l \ . \end{aligned} \tag{3.26}$$

Sub-critical minimal surface cosmologies would correspond to $X^4 \subset M^4 \times S^1$. The natural coordinates are Robertson-Walker coordinates, which co-incide with light-cone coordinates ($a = \sqrt{(m^0)^2 - r_M^2}, r = r_M/a, \theta, \phi$) for light-cone M^4_+. They are related to spherical Minkowski coordinates (m^0, r_M, θ, ϕ) by $(m^0 = a\sqrt{1+r^2}, r_M = ar)$. $\beta = r_M/m_0 = r/\sqrt{1+r^2}$ corresponds to the velocity along the line from origin (0,0) to (m^0, r_M). r corresponds to the Lorentz factor $\gamma\beta = \beta/\sqrt{1-\beta^2}$. The metric of M^4_+ is given by the diagonal form $[g_{aa} = 1, g_{rr} = a^2/(1+r^2), g_{\theta\theta} = a^2r^2, g_{\phi\phi} = a^2r^2sin^2(\theta)]$. One can use the coordinates of M^4_+ also for X^4.

The ansatz for the minimal surface reads is $\Phi = f(a)$. For $f(a) = constant$ one obtains just the flat M^4_+. In non-trivial case one has $g_{aa} = 1 - R^2(df/da)^2$. The g^{aa} component of the metric becomes now $g^{aa} = 1/(1 - R^2(df/da)^2)$. Metric determinant is scaled by $\sqrt{g_{aa}} = 1 \to \sqrt{1 - R^2(df/da)^2}$. Otherwise the field equations are same as for M^4_+. Little calculation shows that they are not satisfied unless one as $g_{aa} = 1$.

Also the minimal surface imbeddings of critical and over-critical cosmologies are impossible. The reason is that the criticality alone fixes these cosmologies almost uniquely and this is too much for allowing minimal surface property.

Thus one can have only the trivial cosmology M^4_+ carrying dark energy density as a minimal surface solution! This obviously raises several questions.

1. Could $\Lambda = 0$ case for which action reduces to Kähler action provide vacuum extremals provide single-sheeted model for Robertson-Walker cosmologies for the GRT limit of TGD for which many-sheeted space-time surface is replaced with a slightly curved region of M^4? Could $\Lambda = 0$ correspond to a genuine phase present in TGD as formal generalization of the view of

mathematicians about reals as $p = \infty$ p-adic number suggest. p-Adic length scale would be strictly infinite implying that $\Lambda \propto 1/p$ vanishes.

2. Second possibility is that TGD is quantum critical in strong sense. Not only 3-space but the entire space-time surface is flat and thus M^4_+. Only the local gravitational fields created by topologically condensed space-time surfaces would make it curved but would not cause smooth expansion. The expansion would take as quantum phase transitions reducing the value of $\Lambda \propto 1/p$ as p-adic prime p increases. p-Adic length scale hypothesis suggests that the preferred primes are near but below powers of 2 $p \simeq 2^k$ for some integers k. This led for years ago to a model for Expanding Earth [K2].

3. This picture would explain why individual astrophysical objects have not been observed to expand smoothly (except possibly in these phase transitions) but participate cosmic expansion only in the sense that the distance to other objects increase. The smaller space-time sheets glued to a given space-time sheet preserving their size would emanate from the tip of M^4_+ for given sheet.

4. RW cosmology should emerge in the idealization that the jerk-wise expansion by quantum phase transitions and reducing the value of Λ (by scalings of 2 by p-adic length scale hypothesis) can be approximated by a smooth cosmological expansion.

One should understand why Robertson-Walker cosmology is such a good approximation to this picture. Consider first cosmic redshift.

1. The cosmic recession velocity is defined from the redshift by Doppler formula.

$$z = \frac{1+\beta}{1-\beta} - 1 \simeq \beta = \frac{v}{c} \ . \tag{3.27}$$

In TGD framework this should correspond to the velocity defined in terms of the coordinate r of the object.

Hubble law tells that the recession velocity is proportional to the proper distance D from the source. One has

$$v = HD \ , \quad H = (\frac{da/dt}{a}) = \frac{1}{\sqrt{g_{aa}}a} \ . \tag{3.28}$$

This brings in the dependence on the Robertson-Walker metric.

For M^4_+ one has $a = t$ and one would have $g_{aa} = 1$ and $H = 1/a$. The experimental fact is however that the value of H is larger for non-empty RW cosmologies having $g_{aa} < 1$. How to overcome this problem?

2. To understand this one must first understand the interpretation of gravitational redshift. In TGD framework the gravitational redshift is property of observer rather than source. The point is that the tangent space of the 3-surface assignable to the observer is related by a Lorent boost to that associated with the source. This implies that the four-momentum of radiation from the source is boosted by this same boost. Redshift would mean that the Lorentz boost reduces the momentum from the real one. Therefore redshift would be consistent with momentum conservation implied by Poincare symmetry.

g_{aa} for which a corresponds to the value of cosmic time for the observer should characterize the boost of observer relative to the source. The natural guess is that the boost is characterized by the value of g_{tt} in sufficiently large rest system assignable to observer with t is taken to be M^4 coordinate m^0. The value of g_{tt} fluctuates do to the presence of local gravitational fields. At the GRT limit g_{aa} would correspond to the average value of g_{tt}.

3. There is evidence that H is not same in short and long scales. This could be understood if the radiation arrives along different space-time sheets in these two situations.

4. If this picture is correct GRT description of cosmology is effective description taking into account the effect of local gravitation to the redshift, which without it would be just the M^4_+ redshift.

Einstein's equations for RW cosmology [K7, K5] should approximately code for the cosmic time dependence of mass density at given slightly deformed piece of M^4_+ representing particular sub-cosmology expanding in jerkwise manner.

1. Many-sheeted space-time implies a hierarchy of cosmologies in different p-adic length scales and with cosmological constant $\Lambda \propto 1/p$ so that vacuum energy density is smaller in long scale cosmologies and behaves on the average as $1/a^2$ where a characterizes the scale of the cosmology. In zero energy ontology given scale corresponds to causal diamond (CD) with size characterized by a defining the size scale for the distance between the tips of CD.

2. For the comoving volume with constant value of coordinate radius r the radius of the volume increases as a. The vacuum energy would increase as a^3 for comoving volume. This is in sharp conflict with the fact that the mass decreases as $1/a$ for radiation dominated cosmology, is constant for matter dominated cosmology, and is proportional to a for string dominated cosmology.

   The physical resolution of the problem is rather obvious. Space-time sheets representing topologically condensed matter have finite size. They do not expand except possibly in jerkwise manner but in this process Λ is reduced - in average manner like $1/a^2$.

   If the sheets are smaller than the cosmological space-time sheet in the scale considered and do not lose energy by radiation they represent matter dominated cosmology emanating from the vertex of M^4_+. The mass of the co-moving volume remains constant.

   If they are radiation dominated and in thermal equilibrium they lose energy by radiation and the energy of volume behaves like $1/a$.

   Cosmic strings and magnetic flux tubes have size larger than that the space-time sheet representing the cosmology. The string as linear structure has energy proportional to a for fixed value of Λ as in string dominated cosmology. The reduction of Λ decreasing on the average like $1/a^2$ implies that the contribution of given string is reduced like $1/a$ on the average as in radiation dominated cosmology.

3. GRT limit would code for these behaviours of mass density and pressure identified as scalars in GRT cosmology in terms of Einstein's equations. The time dependence of g_{aa} would code for the density of the topologically condensed matter and its pressure and for dark energy at given level of hierarchy. The vanishing of covariant divergence for energy momentum tensor would be a remnant of Poincare invariance and give Einstein's equations with cosmological term.

4. Why GRT limit would involve only the RW cosmologies allowing imbedding as vacuum extremals of Kähler action? Can one demand continuity in the sense that TGD cosmology at $p \to \infty$ limit corresponds to GRT cosmology with cosmological solutions identifiable as vacuum extremals? If this is assumed the earlier results are obtained. In particular, one obtains the critical cosmology with 2-D CP_2 projection assumed to provide a GRT model for quantum phase transitions changing the value of Λ.

If this picture is correct, TGD inspired cosmology at the level of many-sheeted space-time would be extremely simple. The new element would be many-sheetedness which would lead to more complex description provided by GRT limit. This limit would however lose the information about many-sheetedness and lead to anomalies such as two Hubble constants.

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