Could one realize number theoretical universality for functional integral?

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June 20, 2019

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Abstract

Number theoretical vision relies on the notion of number theoretical universality (NTU). In fermionic sector NTU is necessary: one cannot speak about real and p-adic fermions as separate entities and fermionic anti-commutation relations are indeed number theoretically universal.

By supersymmetry NTU should apply also to functional integral over WCW (or its sector defined by given causal diamond CD) involved with the definition of scattering amplitudes. The expression for the integral should make sense in all number fields simultaneously. At first this condition looks horrible but the Kähler structure of WCW and the identification of vacuum functional as exponent of Kähler function, and the unique adelic properties of Neper number e give excellent hopes about NTU and also predict the general forms of the functional integral and of the value spectrum of Kähler action for preferred extremals.

1 What does one mean with functional integral?

The definition of functional integral in the "world of classical worlds" (WCW) is one of the key technical problems of quantum TGD [?] NTU states that the integral should be defined simultaneously in all number fields in the intersection of real and p-adic worlds defined by string world sheets and partonic 2-surfaces with WCW coordinates in algebraic extension of rationals and allowing by strong holography continuation to 4-D space-time surface. NTU is enormously powerful constraint and could help in this respect.

1. The first thing to notice is that path integral is not in question. Rather, the functional integral is analogous to Wiener integral and perhaps allows identification as a genuine integral in the real sector. In p-adic sectors algebraic continuation should give the integral and here number theoretical universality gives excellent hopes: the integral would have exactly the same form in real and p-adic sector and expressible solely in terms of algebraic numbers characterizing

algebraic extension and finite roots of e and roots of unity $U_n = exp(i2 \times \pi/n)$ in algebraic extension of p-adic numbers.

Since vacuum functional $exp(S_K)$ is exponent function such that S_K receives real/imaginary contributions from Euclidian/Minkowskian regions ($\sqrt{g}$ factor), the natural idea is that only rational powers e^q and roots of unity and phases $exp(i2\pi q)$ are involved and there is no dependence on p-adic prime p! This is true in the integer part of q is smaller than p so that one does not obtain e^{kp}, which is ordinary p-adic number and would spoil the number theoretic universality. This condition is not possible to satisfy for all values of p unless the value of Kähler function is smaller than 2. One might consider the possibility that the allow primes are above some minimum value.

2. What do we mean with functional integral? TGD is expected to be an integrable in some sense. In integrable quantum field theories functional integral reduces to a sum over stationary points of the action: typically only single point contributes - at least in good approximation.

   Vacuum functional is exponent of Kähler action and decomposes to a product of exponents of real contribution from Euclidian regions and imaginary contribution Minkowskian regions. Kähler function is identified as real part of Kähler action coming from the Euclidian region of space-time surface. In Minkowskian regions $\sqrt{g}$ is imaginary and Kähler action is imaginary having interpretation as analog of Morse function. Now saddle points must be considered. One can ask whether the Euclidian regions dictate the Minkowskian regions uniquely by boundary conditions. Strong form of holography suggests that partonic 2-surface and string world sheets in Minkowskian regions code Minkowskian regions apart from super-symplectic gauge symmetries.

   Preferred extremals satisfy extremely strong conditions. All classical Noether charges for a sub-algebra associated with super-symplectic algebra and isomorphic to the algebra itself vanish at both ends of CD. The conformal weights of this algebra are $n > 0$-ples of those for the entire algebra.

3. In TGD framework one is constructing zero energy states rather calculating the matrix elements of S-matrix in terms of path integral. This gives certain liberties but a natural expectation is that functional integral as a formal tool at least is involved.

   Could one *define* the functional integral as a discrete sum of contributions of standard form for the preferred extremals, which correspond to maxima in Euclidian regions and associated stationary phase points in Minkowskian regions? Could one assume that WCW spinor field is concentrated along single maximum/stationary point.

   Quantum classical correspondence suggests that in Cartan algebra isometry charges are equal to the quantal charges for quantum states expressible in number theoretically universal manner in terms of fermionic oscillator operators or WCW gamma matrices? Even stronger condition would be that classical correlation functions are identical with quantal ones for allowed space-time surfaces in the quantum superposition. Could the reduction to a discrete sum be interpreted in terms of a finite measurement resolution?

4. In quantum field theory Gaussian determinants produce problems because they are often poorly defined. In the recent case they could also spoil the NTU based on the exceptional properties of e. In the recent case however Gaussian determinant and metric determinant for Kähler metric cancel each other and this problem disappears. One could obtain just a sum over products of roots of e and roots of unity. Thus also Kähler structure seems to be crucial for the dream about NTU.

2 Concrete realization of NTU for functional integral

The special adelic properties of e raise hopes concerning the concrete realization of NTU. One obtains product of exponents of real and imaginary contributions to Kähler action.

1. The values of exponent of the real part of Kähler action should belong to an algebraic extension of rationals generated by a root $e^{1/r}$ for some finite value of r depending on algebraic extension of rationals involved. This extension is finite-dimensional p-adically since e^p is ordinary p-adic number. The exponent of the imaginary part of Kähler action should be root of unity and thus of form $exp(iq \times 2\pi)$, q rational. Also some roots $ep(i2\pi/n)$ of unity can belong to the algebraic extension involved. One would have roots of unity and their hyperbolic analogs natural also in p-adic group theory.

2. NTU is realized if the exponents of Kähler action from Euclidian and Minkowskian regions are expressible in terms of roots e^q and $ep(i2\pi q)$, q rational. Further, the integer part of q in e^q must be smaller than p since otherwise one obtains term $exp(mp)$ which is ordinary p-adic number and breaks the universality. This is not a problem for the p-adic primes assignable to elementary particles. For instance, for electron one has $p = M_{127} = 2^{127} - 1 \sim 10^{38}$ and the value of Kähler action from Euclidian regions is definitely smaller than p. For small p-adic primes problems can emerge. For imaginary part one does not have any problems.

One must be however very cautious. There is also an alternative possibility considered earlier. For this option the exponent of Kähler function would be a power of p: $exp(K) = p^n$ for p-adic prime. The preferred extremals would decompose to classes labelled by p-adic prime p. For imaginary part of Kähler action from Minkowskian regions one would have $exp(S_K) = exp(i2\pi q)$, q rational. This would mean a weaker form of NTU appearing in p-adic mass calculations and might make sense for the sum over maxima of Kähler function or even for the sum over preferred extremals if they form a discrete set so that functional integral would reduces to a sum making sense both in real sense and p-adically. If so, the integration over WCW would reduce to a sum analogous to the partition function appearing in p-adic thermodynamics [K2] and p-adic thermodynamics could actually have interpretation in terms of this sum.

An attractive hypothesis is that the poles of the fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ consisting of $h = -s/2$, s zero of zeta, and pole $h = s = 1$ of zeta correspond to radial conformal weights for the generating elements of super-symplectic algebra. Combined with the hypothesis that the exponents p^s exist p-adically for a super-symplectic sub-algebra G_n, this leads to the hypothesis that the imaginary parts of zeros of Riemann zeta decompose to classes $C(p)$ labelled by primes such that p^{iy} is a root of unity in given $C(p)$ (see later section). Could the values for $exp(S_K)$ correspond to a subset of exponents of super-symplectic conformal weights for the super-symplectic algebra, whose generating elements have $h = -s/2$ and $h = 1$ as conformal weights? If so, the spectrum of S_K would be completely fixed in terms of Riemann zeta and the functional integral over WCW would reduce to p-adic thermodynamics for a given value of p! Also the spectrum of the Teichmueller parameters for conformal moduli of partonic 2-surfaces could be number theoretically universal in the same Riemannian sense so that the integration over moduli would reduce to a sum [K2].

Third option would be a hybrid of the two: $exp(K) = p^n e^q$, q rational.

To sum up, the resulting conditions state that

1. Kähler action from Minkowskian regions is product of rational number and 2π and is analogous to phase angle: $S_K = q \times 2\pi$.

2. Kähler action from Euclidian regions is rational number and analogous to "hyperbolic" angle: $S_K = -q$. Alternative possibility is that one has $K = -n/log(p)$ giving $exp(K) = p^{-n}$. For the hybrid option one has $S_K = -q - n/log(p)$ and $exp(K) = e^{-q}p^{-n}$.

3 Finite measurement resolution and breaking of algebraic universality

Number theoretical universality is certainly broken: consider only p-adic mass calculations predicting that mass scale depends on p-adic prime. Also for Kähler action strong form of number theoretical universality might fail for small p-adic primes since the value of the real part of Kähler action would be larger than than p. Should one allow this? What one actually means with number theoretical universality in the case of Kähler action?

Canonical identification is an important element of p-adic mass calculations and might also be needed to map p-adic variants of scattering amplitudes to their real counterparts. The breaking of number theoretical universality would take place, when the canonical real valued image of the p-adic scattering amplitude differs from the real scattering amplitude. The interpretation would be in terms of finite measurement resolution. By the finite measurement/cognitive resolution characterized by p one cannot detect the difference.

The simplest form of the canonical identification is $x = \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$. Product xy and sum $x+y$ do not in general map to product and sum in canonical identification. The interpretation would be in terms of a finite measurement resolution: $(xy)_R = x_R y_R$ and $(x+y)_R = x_R + y_R$ only modulo finite measurement resolution. p-Adic scattering amplitudes are obtained by algebraic continuation from the intersection by replacing algebraic number valued parameters (such as momenta) by general p-adic numbers. The real images of these amplitudes under canonical identification are in general not identical with real scattering amplitudes the interpretation being in terms of a finite measurement resolution.

What about the real value Kähler action when its value is larger than p? It does not make sense to map it to a p-adic number by the standard canonical identification. Rather, one must perform this map by using the modification of the canonical identification to $\sum x_n p^{nN} \rightarrow x_n p^{nN}$, where coefficients x_n are now in the interval $[0, p^N - 1]$. By choosing N to be large enough, the Kähler action for all 4-surfaces in the quantum superposition can be mapped to itself in canonical identification for given p. If the value of $N(p)$ is determined for a given quantum state in this manner, WCW integration would be universal process apart from possible breaking of number theoretical universality coming from the inner product of the fermionic states at ends of CD for a given 4-surface. All p-adic primes would be in democratic position.

In p-adic thermodynamics number theoretic universality in the strong sense fails since thermal masses depend on p-adic mass scale. Number theoretical universality can be broken by the fermionic matrix elements in the functional integral so that the real scattering amplitudes differ from the canonical images of the p-adic scattering amplitudes. For instance, the elementary particle vacuum functionals in the space of Teichmueller parameters for the partonic 2-surfaces and string world sheets should break number theoretical universality [K1].

The recent view about the map of real preferred extremals to their p-adic counterparts does not demand discrete local correspondence assumed in the earlier proposal [K4]. One can however ask whether this kind of correspondence could make sense when restricted to string world sheets and partonic 2-surfaces and defined by a variant of canonical identification characterized by a minimal values of N (depending on p) allowing the exponent of the real counterpart of Euclidian Kähler action to be equivalent with the canonical image its p-adic variant. Discretization using points of imbedding space with coordinates in the algebraic extension of rationals characterizing the adele, gives hopes that the field equations for string world sheets can be satisfied. Whether this kind of map has any practical use, is of course another question.

4 What can one say about the value of Kähler coupling strength

These conditions give conditions on Kähler coupling strength $\alpha_K = g_K^2/4\pi$ ($\hbar = 1$)) identifiable as an analog of critical temperature. Quantum criticality of TGD would thus make possible number theoretical universality (or vice versa).

Consider first the option $K = q$ possible if roots of e belong to the extension of rationals.

1. In Euclidian regions the natural starting point is CP_2 vacuum extremal for which the maximum value of Kähler action is

$$S_K = \frac{\pi^2}{2g_K^2} = \frac{\pi}{8\alpha_K} \quad .$$

The condition reads $S_K = q$ if one allows roots of e in the extension. If one requires minimal extension of involving only e and its powers one would have $S_K = n$. One obtains

$$\frac{1}{\alpha_K} = \frac{8q}{\pi} \quad ,$$

where the rational $q = m/n$ can also reduce to integer. One cannot exclude the possibility that q depends on the algebraic extension of rationals defining the adele in question [K5].

For CP_2 type extremals the value of p-adic prime should be larger than $p_{min} = 53$. One can consider a situation in which large number of CP_2 type vacuum extremals contribute and in this case the condition would be more stringent. The condition that the action for CP_2 extremal is smaller than 2 gives

$$\frac{1}{\alpha_K} \leq \frac{16}{\pi} \simeq 5.09 \quad .$$

It seems there is lower bound for the p-adic prime assignable to a given space-time surface inside CD suggesting that p-adic prime is larger than $53 \times N$, where N is particle number.

This bound has not practical significance. In condensed matter particle number is proportional to $(L/a)^3$ - the volume divided by atomic volume. On basis p-adic mass calculations [K2] p-Adic prime can be estimated to be of order $(L/R)^2$. Here a is atomic size of about 10 Angstroms and R CP_2 "radius". Using $R \simeq 10^4 L_{Planck}$ this gives as upper bound for the size L of condensed matter blob a completely super-astronomical distance $L \leq a^3/R^2 \sim 10^{25}$ ly to be compared with the distance of about 10^{10} ly travelled by light during the lifetime of the Universe. For a blackhole of radius $r_S = 2GM$ with $p \sim (2GM/R)^2$ and consisting of particles with mass above $M \simeq \hbar/R$ one would obtain the rough estimate $M > (27/2) \times 10^{-12} m_{Planck} \sim 13.5 \times 10^3$ TeV trivially satisfied.

2. The physically motivated expectation from earlier arguments - not necessarily consistent with the recent ones - is that the value α_K is quite near to fine structure constant at electron length scale: $\alpha_K \simeq \alpha_{em} \simeq 137.035999074(44)$.

   The latter condition gives $n = 54 = 2 \times 3^3$ and $1/\alpha_K \simeq 137.51$. The deviation from the fine structure constant is $\Delta\alpha/\alpha = 3 \times 10^{-3} - .3$ per cent. For $n = 53$ one obtains $1/\alpha_K = 134.96$ with error of 1.5 per cent. For $n = 55$ one obtains $1/\alpha_K = 150.06$ with error of 2.2 per cent. Is the relatively good prediction could be a mere accident or there is something deeper involved?

What about Minkowskian regions? It is difficult to say anything definite. For cosmic string like objects the action is non-vanishing but proportional to the area A of the string like object and the conditions would give quantization of the area. The area of geodesic sphere of CP_2 is proportional to π. If the value of g_K is same for Minkowskian and Euclidian regions, $g_K^2 \propto \pi^2$ implies $S_K \propto A/R^2\pi$ so that $A/R^2 \propto \pi^2$ is required.

This approach leads to different algebraic structure of α_K than the earlier arguments [?].

1. α_K is rational multiple of π so that g_K^2 is proportional to π^2. At the level of quantum TGD the theory is completely integrable by the definition of WCW integration(!) [K5] and there are no radiative corrections in WCW integration. Hence α_K does not appear in vertices and therefore does not produce any problems in p-adic sectors.

2. This approach is consistent with the proposed formula relating gravitational constant and p-adic length scale. G/L_p^2 for $p = M_{127}$ would be rational power of e now and number theoretically universally. A good guess is that G does not depend on p. As found this could be achieved also if the volume of CP_2 type extremal depends on p so that the formula holds for all primes. α_K could also depend on algebraic extension of rationals to guarantee the independence of G on p. Note that preferred p-adic primes correspond to ramified primes of the extension so that extensions are labelled by collections of ramified primes, and the ramimified prime corresponding to gravitonic space-time sheets should appear in the formula for G/L_p^2.

3. Also the speculative scenario for coupling constant evolution could remain as such. Could the p-adic coupling constant evolution for the gauge coupling strengths be due to the breaking of number theoretical universality bringing in dependence on p? This would require mapping of p-adic coupling strength to their real counterparts and the variant of canonical identification used is not unique.

4. An interesting possibility is that coupling constants are algebraically universal (no dependence on number field). Even the value of α_K, although number theoretically universal, could depend on the algebraic extension of rationals defining the adele. In this case coupling constant evolution would reflect the evolution assignable to the increasing complexity of algebraic extension of rationals. The dependence of coupling constants on p-adic prime would be induced by the fact that so called ramified primes are physically favored and characterize the algebraic extension of rationals used.

5. One must also remember that the running coupling constants are associated with QFT limit of TGD obtained by lumping the sheets of many-sheeted space-time to single region of Minkowski space. Coupling constant evolution would emerge at this limit. Whether this evolution reflects number theoretical evolution as function of algebraic extension of rationals, is an interesting question.

For the option $exp(K) = p^{-n}$ considered in the earlier work the condition for α_K from the maximal action for CP_2 type vacuum extremal leads to

$$\frac{1}{\alpha_K} = \frac{8n}{\pi log(p)} \quad .$$

$n = 1$ is the most natural guess giving $exp(K) = p^{-1}$. α_K would be logarithmically running piecewise constant coupling constant but the renormalization flow would be discrete so that primes p would label a discrete set of critical temperatures. One would have a hierarchy of quantum criticalities with different values of $1/\alpha_K$ having interpretation as p-adic temperature. The earlier proposal for the identification of gravitational constant corresponds to the assumption that gravitation is mediated by flux tubes corresponding to $p = M_{127} = 2^{127} - 1$ assignable to electron and the value of Kähler couplings strength equals to fine structure constant in this scale.

5 Other applications of NTU

NTU in the strongest form says that all numbers involved at “basic level” (whatever this means!) of adelic TGD are products of roots of unity and of power of a root of e. This is extremely powerful physics inspired conjecture with a wide range of possible mathematical applications.

1. For instance, vacuum functional defined as an exponent of Kähler action for preferred externals would be number of this kind. One could define functional integral as adelic operation in all number fields: essentially as sum of exponents of Kähler action for stationary preferred extremals since Gaussian and metric determinants potentially spoiling NTU would cancel each other leaving only the exponent.

2. The implications of NTU for the zeros of Riemann zeta will be discussed in more detail below. Suffice it to say that the observations about Fourier transform for the distribution of loci of non-trivial zeros of zeta together with NTU leads to explicit proposal for the algebraic for of zeros of zeta. The testable proposal is that zeros decompose to disjoint classes $C(p)$ labelled by primes p and the condition that p^{iy} is root of unity in given class $C(p)$.

3. NTU generalises to all Lie groups. Exponents $exp(in_iJ_i/n)$ of lie-algebra generators define generalisations of number theoretically universal group elements and generate a discrete subgroup of compact Lie group. Also hyperbolic ”phases” based on the roots $e^{m/n}$ are possible and make possible discretized NTU versions of all Lie-groups expected to play a key role in adelization of TGD.

   NTU generalises also to quaternions and octonions and allows to define them as number theoretically universal entities. Note that ordinary p-adic variants of quaternions and octonions do not give rise to a number field: inverse of quaternion can have vanishing p-adic variant of norm squared satisfying $\sum_n x_n^2 = 0$.

NTU allows to define also the notion of Hilbert space as an adelic notion. The exponents of angles characterising unit vector of Hilbert space would correspond to roots of unity.

5.1 Super-symplectic conformal weights and zeros of Riemann zeta

Since fermions are the only fundamental particles in TGD one could argue that the conformal weight of for the generating elements of supersympelectic algebra could be negatives for the poles of fermionic zeta ζ_F. This demands $n > 0$ as does also the fractal hierarchy of supersymplectic symmetry breakings. The number theoretic universality of Riemann zeta in some sense is strongly suggested if adelic physics is to make sense.

For ordinary conformal algebras there are only finite number of generating elements ($-2 \leq n \leq 2$). If the radial conformal weights for the generators of g consist of poles of ζ_F, the situation changes. ζ_F is suggested by the observation that fermions are the only fundamental particles in TGD.

1. Riemann Zeta $\zeta(s) = \prod_p (1/(1-p^{-s})$ identifiable formally as a partition function $\zeta_B(s)$ of arithmetic boson gas with bosons with energy $log(p)$ and temperature $1/s = 1/(1/2+iy)$ should be replaced with that of arithmetic fermionic gas given in the product representation by $\zeta_F(s) = \prod_p (1+p^{-s})$ so that the identity $\zeta_B(s))/\zeta_F(s) = \zeta_B(2s)$ follows. This gives

$$\frac{\zeta_B(s)}{\zeta_B(2s)} \quad .$$

$\zeta_F(s)$ has zerosat zeros s_n of $\zeta(s)$ and at the pole $s = 1/2$ of $zeta(2s)$. $\zeta_F(s)$ has poles at zeros $s_n/2$ of $\zeta(2s)$ and at pole $s = 1$ of $\zeta(s)$.

The spectrum of $1/T$ would be for the generators of algebra $\{(-1/2+iy)/2, n > 0, -1\}$. In p-adic thermodynamics the p-adic temperature is $1/T = 1/n$ and corresponds to "trivial" poles of ζ_F. Complex values of temperature does not make sense in ordinary thermodynamics. In ZEO quantum theory can be regarded as a square root of thermodynamics and complex temperature parameter makes sense.

2. If the spectrum of conformal weights of the generating elements of the algebra corresponds to poles serving as analogs of propagator poles, it consists of the "trivial" conformal $h = n > 0$- the standard spectrum with $h = 0$ assignable to massless particles excluded - and "non-trivial" $h = -1/4 + iy/2$. There is also a pole at $h = -1$.

Both the non-trivial pole with real part $h_R = -1/4$ and the pole $h = -1$ correspond to tachyons. I have earlier proposed conformal confinement meaning that the total conformal weight for the state is real. If so, one obtains for a conformally confined two-particle states corresponding to conjugate non-trivial zeros in minimal situation $h_R = -1/2$ assignable to N-S representation.

In p-adic mass calculations ground state conformal weight must be $-5/2$ [K2]. The negative fermion ground state weight could explain why the ground state conformal weight must be tachyonic $-5/2$. With the required 5 tensor factors one would indeed obtain this with minimal conformal confinement. In fact, arbitrarily large tachyonic conformal weight is possible but physical state should always have conformal weights $h > 0$.

3. $h = 0$ is not possible for generators, which reminds of Higgs mechanism for which the naive ground states corresponds to tachyonic Higgs. $h = 0$ conformally confined massless states are necessarily composites obtained by applying the generators of Kac-Moody algebra or super-symplectic algebra to the ground state. This is the case according to p-adic mass calculations [K2], and would suggest that the negative ground state conformal weight can be associated with super-symplectic algebra and the remaining contribution comes from ordinary super-conformal generators. Hadronic masses whose origin is poorly understood could come from super-symplectic degrees of freedom. There is no need for p-adic thermodynamics in super-symplectic degrees of freedom.

5.2 Dyson's comment about Fourier transform of Riemann zeta and general formula for the zeros of zeta

1. Zeros of zeta and primes as 1-D quasicrystals?

Dyson's comment about Fourier transform of Riemann Zeta [A1] (`http://tinyurl.com/hjbfsuv`) is interesting from the point of NTU for Riemann zeta.

1. The numerical calculation of Fourier transform for the imaginary parts iy of zeros $s = 1/2+iy$ of zeta shows that it is concentrated at discrete set of frequencies coming as $log(p^n)$, p prime. This translates to the statement that the zeros of zeta form a 1-dimensional quasicrystal, a discrete structure Fourier spectrum by definition is also discrete (this of course holds for ordinary crystals as a special case). Also the logarithms of powers of primes would form a quasicrystal, which is very interesting from the point of view of p-adic length scale hypothesis. Primes label the "energies" of elementary fermions and bosons in arithmetic number theory, whose repeated second quantization gives rise to the hierarchy of infinite primes [K3]. The energies for general states are logarithms of integers.

2. Powers p^n label the points of quasicrystal defined by points $log(p^n)$ and Riemann zeta has interpretation as complex analog of partition function for boson case with this spectrum. Could p^n label also the points of the dual lattice defined by iy.

3. The existence of Fourier transform for points $log(p_i^n)$ for any vector y_a in class $C(p)$ of zeros labelled by p requires $p_i^{iy_a}$ to be a root of unity inside $C(p)$. This could define the sense in which zeros of zeta are universal. This condition also guarantees that the factor $n^{-1/2-iy}$ appearing in zeta at critical line are number theoretically universal ($p^{1/2}$ is problematic for Q_p: the problem might be solved by eliminating from p-adic analog of zeta the factor $1-p^{-s}$.

   (a) One obtains for the pair (p_i, s_a) the condition $log(p_i)y_a = q_{ia}2\pi$, where q_{ia} is a rational number. Dividing the conditions for (i, a) and (j, a) gives

   $$p_i = p_j^{q_{ia}/q_{ja}}$$

   for every zero s_a so that the ratios q_{ia}/q_{ja} do not depend on s_a. From this one easily deduce $p_i^M = p_j^N$, where M and N are integers so that one ends up with a contradiction.

   (b) Dividing the conditions for (i, a) and (i, b) one obtains

   $$\frac{y_a}{y_b} = \frac{q_{ia}}{q_{ib}}$$

   so that the ratios q_{ia}/q_{ib} do not depend on p_i. The ratios of the imaginary parts of zeta would be therefore rational number which is very strong prediction and zeros could be mapped by scaling y_a/y_1 where y_1 is the zero which smallest imaginary part to rationals.

   (c) The impossible consistency conditions for (i, a) and (j, a) can be avoided if each prime and its powers correspond to its own subset of zeros and these subsets of zeros are disjoint: one would have infinite union of sub-quasicrystals labelled by primes and each p-adic number field would correspond to its own subset of zeros: this might be seen as an abstract analog for the decomposition of rational to powers of primes. This decomposition would be natural if for ordinary complex numbers the contribution in the complement of this set to the Fourier trasform vanishes. The conditions $(i, a)$and (i, b) require now that the ratios of zeros are rationals only in the subset associated with p_i.

For the general option the Fourier transform can be delta function for $x = log(p^k)$ and the set $\{y_a(p)\}$ contains N_p zeros. The following argument inspires the conjecture that for each p there is an infinite number N_p of zeros $y_a(p)$ in class $C(p)$ satisfying

$$p^{iy_a(p)} = u(p) = e^{\frac{r(p)}{m(p)}i2\pi} \quad ,$$

where $u(p)$ is a root of unity that is $y_a(p) = 2\pi(m(a) + r(p))/log(p)$ and forming a subset of a lattice with a lattice constant $y_0 = 2\pi/log(p)$, which itself need not be a zero.

In terms of stationary phase approximation the zeros $y_a(p)$ associated with p would have constant stationary phase whereas for $y_a(p_i \neq p))$ the phase $p^{iy_a(p_i)}$ would fail to be stationary. The phase e^{ixy} would be non-stationary also for $x \neq log(p^k)$ as function of y.

1. Assume that for $x = qlog(p)$, where q not a rational, the phases e^{ixy} fail to be roots of unity and are random implying the vanishing/smallness of $F(x)$.

2. Assume that for a given p all powers p^{iy} for $y \notin \{y_a(p)\}$ fail to be roots of unity and are also random so that the contribution of the set $y \notin \{y_a(p)\}$ to $F(p)$ vanishes/is small.

3. For $x = log(p^{k/m})$ the Fourier transform should vanish or be small for $m \neq 1$ (rational roots of primes) and give a non-vanishing contribution for $m = 1$. One has

$$\begin{aligned} F(x = log(p^{k/m}) &= \textstyle\sum_{1\leq a\leq N(p)} e^{k\frac{M(a,p)}{mN(p)}i2\pi} u(p) \quad , \\ u(p) &= e^{\frac{r(p)}{m(p)}i2\pi} \quad . \end{aligned}$$

   Obviously one can always choose $N(a,p) = N(p)$.

4. For the simplest option $N(p) = 1$ one would obtain delta function distribution for $x = log(p^k)$. The sum of the phases associated with $y_a(p)$ and $-y_a(p)$ from the half axes of the critical line would give

$$F(x = log(p^n)) \propto X(p^n) \equiv 2cos(n\frac{r(p)}{m(p)}2\pi) \quad .$$

   The sign of F would vary.

5. For $x = log(p^{k/m})$ the value of Fourier transform is expected to be small by interference effects if $M(a,p)$ is random integer, and negligible as compared with the value at $x = log(p^k)$. This option is highly attractive. For $N(p) > 1$ and $M(a,p)$ a random integer also $F(x = log(p^k)$ is small by interference effects. Hence it seems that this option is the most natural one.

6. The rational $r(p)/m(p)$ would characterize given prime (one can require that $r(p)$ and $m(p)$ have no common divisors). $F(x)$ is non-vanishing for all powers $x = log(p^n)$ for $m(p)$odd. For $p = 2$, also $m(2) = 2$ allows to have $|X(2^n)| = 2$. An interesting ad hoc ansatz is $m(p) = p$ or $p^{s(p)}$. One has periodicity in n with period $m(p)$ that is logarithmic wave. This periodicity serves as a test and in principle allows to deduce the value of $r(p)/m(p)$ from the Fourier transform.

What could one conclude from the data (`http://tinyurl.com/hjbfsuv`)?

1. The first graph gives $|F(x = log(p^k)|$ and second graph displays a zoomed up part of $|F(x = log(p^k)|$ for small powers of primes in the range $[2, 19]$. For the first graph the eighth peak $(p = 11)$ is the largest one but in the zoomed graphs this is not the case. Hence something is wrong or the graphs correspond to different approximations suggesting that one should not take them too seriously.

   In any case, the modulus is not constant as function of p^k. For small values of p^k the envelope of the curve decreases and seems to approach constant for large values of p^k (one has $x < 15$ $(e^{15} \simeq 3.3 \times 10^6)$.

2. According to the first graph $|F(x)|$ decreases for $x = klog(p) < 8$, is largest for small primes, and remains below a fixed maximum for $8 < x < 15$. According to the second graph the amplitude decreases for powers of a given prime (say $p = 2$). Clearly, the small primes and their powers have much larger $|F(x)|$ than large primes.

There are many possible reasons for this behavior. Most plausible reason is that the sums involved converge slowly and the approximation used is not good. The inclusion of only 10^4 zeros would show the positions of peaks but would not allow reliable estimate for their intensities.

1. The distribution of zeros could be such that for small primes and their powers the number of zeros is large in the set of 10^4 zeros considered. This would be the case if the distribution of zeros $y_a(p)$ is fractal and gets "thinner" with p so that the number of contributing zeros scales down with p as a power of p, say $1/p$, as suggested by the envelope in the first figure.

2. The infinite sum, which should vanish, converges only very slowly to zero. Consider the contribution $\Delta F(p^k,p_1)$of zeros not belonging to the class $p_1 \neq p$ to $F(x = log(p^k)) = \sum_{p_i} \Delta F(p^k,p_i)$, which includes also $p_i = p$. $\Delta F(p^k,p_i)$, $p \neq p_1$ should vanish in exact calculation.

   (a) By the proposed hypothesis this contribution reads as

   $$\begin{aligned}\Delta F(p,p_1) &= \sum_a cos\left[X(p^k,p_1)(M(a,p_1)+\frac{r(p_1)}{m(p_1)})2\pi)\right] \quad .\\ X(p^k,p_1) &= \frac{log(p^k)}{log(p_1)} \quad .\end{aligned}$$

   Here a labels the zeros associated with p_1. If p^k is "approximately divisible" by p^1 in other words, $p^k \simeq np_1$, the sum over finite number of terms gives a large contribution since interference effects are small, and a large number of terms are needed to give a nearly vanishing contribution suggested by the non-stationarity of the phase. This happens in several situations.

   (b) The number $\pi(x)$ of primes smaller than x goes asymptotically like $\pi(x) \simeq x/log(x)$ and prime density approximately like $1/log(x) - 1/log(x)^2$ so that the problem is worst for the small primes. The problematic situation is encountered most often for powers p^k of small primes p near larger prime and primes p (also large) near a power of small prime (the envelope of $|F(x)|$ seems to become constant above $x \sim 10^3$).

   (c) The worst situation is encountered for $p = 2$ and $p_1 = 2^k - 1$ - a Mersenne prime and $p_1 = 2^{2^k} + 1$, $k \leq 4$ - Fermat prime. For $(p,p_1) = (2^k,M_k)$ one encounters $X(2^k,M_k) = (log(2^k)/log(2^k-1)$ factor very near to unity for large Mersennes primes. For $(p,p_1) = (M_k,2)$ one encounters $X(M_k,2) = (log(2^k-1)/log(2) \simeq k$. Examples of Mersennes and Fermats are $(3,2),(5,2),(7,2),(17,2),(31,2),(127,2),(257,2),...$ Powers 2^k, $k = 2,3,4,5,7,8,..$ are also problematic.

   (d) Also twin primes are problematic since in this case one has factor $X(p = p_1+2,p_1) = \frac{log(p_1+2)}{log(p_1)}$. The region of small primes contains many twin prime pairs: (3,5), (5,7), (11,13), (17,19), (29,31),....

   These observations suggest that the problems might be understood as resulting from including too small number of zeros.

3. The predicted periodicity of the distribution with respect to the exponent k of p^k is not consistent with the graph for small values of prime unless the periodic $m(p)$ for small primes is large enough. The above mentioned effects can quite well mask the periodicity. If the first graph is taken at face value for small primes, $r(p)/m(p)$ is near zero, and $m(p)$ is so large that the periodicity does not become manifest for small primes. For $p = 2$ this would require $m(2) > 21$ since the largest power $2^n \simeq e^{15}$ corresponds to $n \sim 21$.

To summarize, the prediction is that for zeros of zeta should divide into disjoint classes $\{y_a(p)\}$ labelled by primes such that within the class labelled by p one has $p^{iy_a(p)} = e^{(r(p)/m(p))i2\pi}$ so that has $y_a(p) = [M(a,p) + r(p)/m(p))]2\pi/log(p)$.

2. More precise view about zeros of zeta

Recall that number theoretical universality in TGD framework leads to the conjecture that the non-trivial zeros of zeta can be divided into classes $C(p)$ labelled by primes p such that for zeros y in given class $C(p)$ the phases p^{iy} are roots of unity. The impulse leading to the idea came from an argument of Dyson referring to the evidence that the Fourier transform for the locus of non-trivial zeros of zeta is a distribution concentrated on powers of primes.

There is a very interesting blog post by Mumford (`http://tinyurl.com/zemw27o`), which leads to much more precise formulation of the idea and improved view about the Fourier transform hypothesis: the Fourier transform must be defined for all zeros, not only the non-trivial ones and trivial zeros give a background term allowing to understand better the properties of the Fourier transform.

Mumford essentially begins from Riemann's "explicit formula" in von Mangoldt's form.

$$\sum_p \sum_{n\geq 1} log(p)\delta_{p^n}(x) = 1 - \sum_k x^{s_k-1} - \frac{1}{x(x^2-1)} \quad ,$$

where p denotes prime and s_k a non-trivial zero of zeta. The left hand side represents the distribution associated with powers of primes. The right hand side contains sum over cosines

$$\sum_k x^{s_k-1} = 2\frac{\sum_k cos(log(x)y_k)}{x^{1/2}} \quad ,$$

where y_k ithe imaginary part of non-trivial zero. Apart from the factor $x^{-1/2}$ this is just the Fourier transform over the distribution of zeros.

There is also a slowly varying term $1 - \frac{1}{x(x^2-1)}$, which has interpretation as the analog of the Fourier transform term but sum over trivial zeros of zeta at $s = -2n$, $n > 0$. The entire expression is analogous to a "Fourier transform" over the distribution of all zeros. Quasicrystal is replaced with union on 1-D quasicrystals.

Therefore the distribution for powers of primes is expressible as "Fourier transform" over the distribution of both trivial and non-trivial zeros rather than only non-trivial zeros as suggested by numerical data to which Dyson [A1] referred to (`http://tinyurl.com/hjbfsuv`). Trivial zeros give a slowly varying background term large for small values of argument x (poles at $x = 0$ and $x = 1$ - note that also $p = 0$ and $p = 1$ appear effectively as primes) so that the peaks of the distribution are higher for small primes.

The question was how can one obtain this kind of delta function distribution concentrated on powers of primes from a sum over terms $cos(log(x)y_k)$ appearing in the Fourier transform of the distribution of zeros.

Consider $x = p^n$. One must get a constructive interference. Stationary phase approximation is in terms of which physicist thinks. The argument was that a destructive interference occurs for given $x = p^n$ for those zeros for which the cosine does not correspond to a real part of root of unity as one sums over such y_k: random phase approximation gives more or less zero. To get something nontrivial y_k must be proportional to $2\pi \times n(y_k)/log(p)$ in class $C(p)$ to which y_k belongs. If the number of these y_k:s in $C(p)$ is infinite, one obtains delta function in good approximation by destructive interference for other values of argument x.

The guess that the number of zeros in $C(p)$ is infinite is encouraged by the behaviors of the densities of primes one hand and zeros of zeta on the other hand. The number of primes smaller than real number x goes like

$$\pi(x) = N(primes < x) \sim \frac{x}{log(x)}$$

in the sense of distribution. The number of zeros along critical line goes like

$$N(zeros < t) = (t/2\pi) \times log(\frac{t}{2\pi})$$

in the same sense. If the real axis and critical line have same metric measure then one can say that the number of zeros in interval T per number of primes in interval T behaves roughly like

$$\frac{\#(zeros < T)}{\#(primes < T)} = log(\frac{T}{2\pi}) \times \frac{log(T)}{2\pi}$$

so that at the limit of $T \to \infty$ the number of zeros associated with given prime is infinite. This asumption of course makes the argument a poor man's argument only.

3. Zeros of zeta and TGD

What this speculative picture from the point of view of TGD?

1. A possible formulation for number theoretic universality for the poles of fermionic Riemann zeta $\zeta_F = \zeta(s)/\zeta(2s)$ could be as a condition that is that the exponents $p^{ks_a(p)/2} = p^{k/4}p^{iky_a(p)/2}$ exist in a number theoretically universal manner for the zeros $s_a(p)$ for given p-adic prime p and for some subset of integers k. If the proposed conditions hold true, exponent reduces $p^{k/4}e^{k(r(p/m(p)i2\pi}$ requiring that k is a multiple of 4. The number of the non-trivial generating elements of super-symplectic algebra in the monomial creating physical state would be a multiple of 4. These monomials would have real part of conformal weight -1. Conformal confinement suggests that these monomials are products of pairs of generators for which imaginary parts cancel.

2. Quasi-crystal property might have an application to TGD. The functions of light-like radial coordinate appearing in the generators of supersymplectic algebra could be of form r^s, s zero of zeta or rather, its imaginary part. The eigenstate property with respect to the radial scaling rd/dr is natural by radial conformal invariance.

   The idea that arithmetic QFT assignable to infinite primes is behind the scenes in turn suggests light-like momenta assignable to the radial coordinate have energies with the dual spectrum $log(p^n)$. This is also suggested by the interpretation of ζ as square root of thermodynamical partition function for boson gas with momentum $log(p)$ and analogous interpretation of ζ_F.

   The two spectra would be associated with radial scalings and with light-like translations of light-cone boundary respecting the direction and light-likeness of the light-like radial vector. $log(p^n)$ spectrum would be associated with light-like momenta whereas p-adic mass scales would characterize states with thermal mass. Note that generalization of p-adic length scale hypothesis raises the scales defined by p^n to a special physical position: this might relate to ideal structure of adeles.

3. Finite measurement resolution suggests that the approximations of Fourier transforms over the distribution of zeros taking into account only a finite number of zeros might have a physical meaning. This might provide additional understand about the origins of generalized p-adic length scale hypothesis stating that primes $p \simeq p_1^k$, p_1 small prime - say Mersenne primes - have a special physical role.

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