Positivity of $\mathcal{N} = 4$ scattering amplitudes from number theoretical universality?

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Abstract

With a motivation coming from explicit calculations Nima-Arkani Hamed and collaborators argue that positivity could be a quite general property of $calN = 4$ scattering amplitudes: as if the amplitudes were analogous to generalized volumes of higher-dimensional polyhedra. A related notion is positive Grassmannian. This article considers the possibility that the positivity of the scattering amplitudes and Grassmannians might follow from the condition of number theoretical universality demanding that scattering amplitudes can be algebraically continued from real (complex numbers) to p-adic number fields and vice versa. Complex phases correspond quite generally to roots of unity for an algebraic extension of p-adic numbers so that algebraic continuation is just identification for the phases: for their p-adic coefficients the situation is different. Note that discretization identifiable in terms of finite resolution, is unavoidable in p-adic sector. What about the mapping of the p-adic coefficients to reals and vice versa.

Besides direct algebraic continuation of reals through common rationals (or their algebraic extension), so called canonical identification mapping p-adics in a continuous manner to reals but not respecting smoothness and symmetries is involved. Canonical identification with cutoffs reflecting appropriate UV and IR resolutions appears also in the definition of p-adic space-time surface as "cognitive representation" of real space-time surface. What is important that canonical identification maps p-adic numbers to non-negative real numbers so that this map and its inverse require non-negativity on the real side. If canonical identification with cutoffs maps the ordinary p-adic numbers appearing in p-adic scattering amplitudes to reals and vice versa, p-adicizability requires positivity. Quite generally, complex scattering amplitude would be a superposition of numbers in an algebraic extension of p-adic numbers mapped to real amplitude such that the coefficientts of various algebraic numbers would be positive.

1 Introduction

Lubos (http://motls.blogspot.fi/2015/01/mysterious-positivity-of-amplituhedron.html) wrote a commentary about two new articles of Nima Arkani Hamed and collaborators. The first article is about the positivity (http://arxiv.org/abs/1412.8478) of the amplitudes in the amplituhedron [B2]. With motivation coming from explicit calculations it is argued that positivity could be a quite general property: as if the amplitudes were analogous to generalized volumes of higher-dimensional polyhedra. Twistor Grassmann approach has been restricted hitherto to planar amplitudes but in the second paper (http://arxiv.org/abs/1412.8478) [?] non-planar MHV amplitudes are discussed, and a suggestion is made that also these are expressible as positive sum of of differently ordered Parker-Taylor amplitudes.

The Grassmannian twistor approach considers also positive Grassmannians [B1]. This is not the same thing but most be closely related to the positivity of amplitudes. I recall that the core idea is however that one considers higher-D analogs of polygons. An ultrasimple representative for positive space polygon would be a triangle bounded by positive x- and y-axis and the line y=-x. x and y coordinates are positive.

What is of course certain is that the entire scattering amplitude cannot be positive since it is complex number. Rather, it must decompose into a product of "trivial" part determined by symmetries and non-trivial part which for some reason must be positive. What does this mean? Lubos considers this question in his posting: the idea is roughly that the real amplitude in question is an exponential and this guarantees positivity. Bundle theorist might speak about global everywhere non-vanishing section of vector bundle having geometric and topological meaning in algebraic geometry. Below I try to interpret positivity in terms of number theoretic arguments inspired by TGD. There is no need to emphasize that I can only admire the incredible technical skills of the advocates of twistor Grassmann approach and my comments are more philosophical than technical.

The idea is that the positivity of the scattering amplitudes and Grassmannians might follow from the condition of number theoretical universality demanding that scattering amplitudes can be algebraically continued from real (complex numbers) to p-adic number fields and vice versa. Complex phases correspond quite generally to roots of unity for an algebraic extension of p-adic numbers so that algebraic continuation is just identification for the phases: for their p-adic coefficients the situation is different. Note that discretization identifiable in terms of finite resolution, is unavoidable in p-adic sector. What about the mapping of the p-adic coefficients to reals and vice versa.

Besides direct algebraic continuation of reals through common rationals (or their algebraic extension), so called canonical identification mapping p-adics in a continuous manner to reals but not respecting smoothness and symmetries is involved. Canonical identification with cutoffs reflecting appropriate UV and IR resolutions appears also in the definition of p-adic space-time surface as "cognitive representation" of real space-time surface. What is important that canonical identification maps p-adic numbers to non-negative real numbers so that this map and its inverse require non-negativity on the real side. If canonical identification with cutoffs maps the ordinary p-adic numbers appearing in p-adic scattering amplitudes to reals and vice versa, p-adicizability requires positivity. Quite generally, complex scattering amplitude would be a superposition of numbers in an algebraic extension of p-adic numbers mapped to real amplitude such that the coefficientts of various algebraic numbers would be positive.

2 Does number theoretical universality require positivity?

Number theoretical universality of physics [K1] is one of the key principles of TGD and states that real physics must allow algebraic continuation to p-adic physics and vice versa [K2]. I suggest that number theoretical universality requires the positivity.

1. The p-adicization of the amplitudes requires positivity. The "non-trivial" factor of the amplitude is mapped to its p-adic counterpart by a variant of canonical identification mapping p-adics to *non-negative* reals and vice versa and is well-defined only for non-negative real numbers. Hence positivity. This condition could also explain why positive Grassmannians [B1] are needed: the preferred coordinates must be positive in order to have well-defined p-adic counterparts and vice versa.

   The basic reason for positivity is that p-adic numbers are not well-ordered. When one has two p-adic numbers with the same p-adic norm, one cannot tell which of them is the larger one. This makes it impossible to tell whether a given p-adic number is positive or negative and one cannot talk about p-adic boundaries. p-Adic line segment has no ends. As a consequence, definite integral is very difficult notion p-adically although the notion of integral function can be defined. p-Adic counterparts of differential forms are also far from trivial to define.

2. What about the symmetry determined parts of the amplitudes, which are typically analogs of partial waves depending on angular coordinates and are complex and cannot be positive? Also here one encounters a technical problem related to another conceptual challenge of p-adicization program: the notion of angle is not well-defined in p-adic context and one can talk only about discrete phases. One can define exponential function $exp(x)$ if x has p-adic norm smaller than 1 but it does not have the properties of the ordinary exponential function (it has p-adic norm 1 for all values of x). One can define also trigonometric functions with the help of $exp(ix)$ for $p\ mod\ 4 = 3$ formally but the trigonometric functions are not periodic. Something is missing.

   In the case of ordinary trigonometry the only way out is to perform algebraic extension of p-adic numbers by adding roots of unity representing phases associated with corresponding angles: phases replace angles. For instance, $U_n = exp(i2\pi/n)$ exists in an algebraic extension of p-adic numbers.

   In the case of hyperbolic geometry, one can add roots of e to obtain p-adic counterparts of $e^(1/n)$ (note that e^p is ordinary p-adic numbers so that these extensions are finite-dimensional algebraically and e is completely unique real transcendental in that it is algebraic number p-adically!): this extension allows to get the counterparts of ordinary exponential functions in extension.

   One can also multiply the points of discretization by p-adic numbers of norm smaller than 1 (or some negative power of p) to get something as near as possible to continuum. Hence discretization in both hyperbolic and trigonometric degrees of freedom by algebraic extension solves the problems quite generally for amplitudes defined in highly symmetric spaces. In Grassmann twistor approach one indeed considers projective spaces.

3. Consider as an example the p-adic counterpart of Euclidian 2-space with coordinates (ρ, ϕ). ρ is non-negative radial coordinate and has p-adic counterpart obtained by canonical identification or its variant. The values of ϕ are replaced with discrete phase factors U_n characterizing the values of ϕ coordinate. One has a collection of n rays emanating from origin instead of entire plane. One has infinite number of variants of E^2 labelled by n characterizing the angular resolution.

   The p-adicization of Cartesian representation of real Euclidian 2-space defined using (x, y) coordinates would give only the first quadrant since negative x and y have no p-adic counterparts. In both cases cognitive representations lose a lot of information for purely number theoretical reasons. Cognitive analog of Uncertainty Principle is suggestive.

4. Obviously General Coordinate Invariance is broken at the level of cognition which is actually not so surprising after all since the worlds in which mathematician has chosen to used Cartesian resp. spherical coordinates must differ in some delicate manner! Of course, the resulting discretized spaces are very different.

3 How should one p-adicize?

The p-adicization of various spaces and amplitudes is needed. p-Adicization means that one assigns to a real (or complex) number a p-adic variant by some rule. In the case of trigonometric and hyperbolic angles one can use discretization and algebraic extension but what about other kinds of coordinates? There are two guesses [K3].

1. Consider only rationals or their algebraic extension and map only them to their p-adic counterparts in the needed extension of p-adic numbers. This correspondence is however extremely discontinuous since real numbers which are arbitrary distant can be arbitrary near p-adically and vice versa. What is nice that this map respects symmetries suggesting that one has symmetries below some rational cutoff defining measurement resolution. This conforms with the general philosophy about measurement resolution realized in terms of inclusions of hyper-finite factors realizing measurement resolution as analog of dynamical gauge symmetry.

2. Use canonical identification mapping p-adic numbers to p-adic numbers by canonical identification: $x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ is the first option. It indeed works only for non-negative real numbers! Canonical identification is continuous but does not respect differentiability nor symmetries. Direct identification via common rationals in turn does not respect continuity.

The resolution of the problems is a compromize based on the use of two cutoffs. In some length scale range above p-adic UV cutoff L_{UV} and scale L a direct correspondence between common rationals is assumed. Between L IR cutoff L_{IR} canonical identification is used. Outside the range $[L_{UV}, L_{IR}]$ there is no correspondence and conditions like smoothness dictate the details at both sides.

4 p-Adic space-time surfaces as cognitive maps of real ones and real space-time surfaces as correlates for realized intentions

One wants to talk about real topological invariants also in p-adic context: p-adic space-time surface should be a kind of cognitive representation of real space-time surface.

1. Space-time surfaces are extremals of Kähler action (real/p-adic) but have a discrete set of common rational points. The notion of p-adic manifold for which p-adic regions are mapped to real chart leafs (rather than to p-adic ones!) formalizes this concept and allows to avoid the problem that p-adic balls are either disjoint or nested so that the usual construction of manifold does not make sense. The problem that there exists an endless variety of coordinate choices and each would give different notion of p-adic manifold.

2. The cure comes from space-time as a surface property, and from symmetries allowing to induce manifold structure from the level of imbedding space the level of space-time surfaces. In TGD imbedding space is $M^4 \times CP_2$. CP_2 allows very natural p-adicizations by using complex coordinates transforming linearly under maximal sectiongroup of isometries and one obtains discrete variants of coset space with points labeled by phases in the algebraic extensions of p-adic numbers. One can also have a generalization in which each discrete point corresponds to a continuum of p-adic units.

   In the case of M^4 one has more options but if one requires that the coordinate which is mapped by canonical identification to its real counterpart, the natural choices is the cosmic coordinates assignable to the future or past light-cone of causal diamond. Light-cone proper time would correspond to non-negative coordinate and the remaining coordinates would be ordinary angles hyperbolic angle and mappable to phase factors and real exponential, which exists if one introduces finite number of roots of e and discretizes the hyperbolic angle. Note that p-adic causal diamond (CD) is p-adically non-trivial manifold requiring two chart leafs and this might deeply relate to the fact that state function reductions occur to either boundary of CD.

3. Surface property implies that the correspondence between real and p-adic space-time surfaces is induced from that between the corresponding imbedding spaces and thus dictated to high degree by symmetries. Preferred imbedding space coordinates and their discretizations induce coordinates and discretizations at space-time surfaces so that there is a huge reduction in the number of different but cognitively non-equivalent discretizations.

4. This could make possible in practice to define the notion of p-adic space-time surface as a cognitive map of real space-time surface and real space-surface as a realizaton of intention represented by p-adic space-time surface. The real extremal of Kähler action is mapped to p-adic one only in a given resolution. A sectionset of discrete points of the space-time surface is mapped to p-adic ones and vice versa and this discrete skeleton is continued to a p-adic extremal of Kähler action. The outcome is interpreted as cognitive representation or its inverse (transformation of intention to action) and need not be unique. The mapping taking p-adic skeleton to real one is up to some cutoff pinary digit just identification along common rationals respecting symmetry and above that canonical identification up to the highest allowed pinary digit.

To sum up, number theoretic universality condition for scattering amplitudes might help to understand the success of twistor Grassmann approach. The existence of p-adic variants of the amplitudes in finite algebraic extensions is a powerful constraint and I have argued that they are satisfied for the polylogarithms used. Positive Grassmannians and positivity of the amplitudes might be also seen as a manner to satisfy these constraints.

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