Intentionality, Cognition, and Number Theory

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Abstract

The identification of p-adic physics as physics of cognition and intention suggests strongly connections between cognition, intentionality, and number theory. The new idea is that also real transcendental numbers can appear in the extensions of p-adic numbers which must be assumed to be finite-dimensional at least in the case of human cognition. This idea, when combined with a more precise model for how intentions are transformed to actions, leads to a series of number theoretical conjectures. Also new insights about the number theoretical origin of the universal dynamics of conformally invariant critical systems emerge. The earlier approaches to the proof of Riemann hypothesis can be understood in a unified manner and the assumption that Riemann Zeta exists in all number fields when finite extensions are allowed for p-adic numbers leads to the view that that the zeros of Riemann Zeta correspond to the universal number theoretically quantized spectrum of scaling momenta associated with critical conformally invariant systems.

1 Introduction

TGD [1, 2] inspired theory of consciousness [3, 4] leads to the identification of p-adic physics as physics of cognition and intention (for a summary about the recent situation see the article "Time, Space-time, and Consciousness" [5]). This identification leads to rather fascinating new ideas concerning the characterization of intentional systems.

The basic ingredient is the new view about numbers: real and p-adic number fields are glued together like pages of a book along common rationals representing the rim of the book. This generalizes to the extensions of p-adic number fields and the outcome is a complex fractal book like structure containing books within books. This holds true also for manifolds and one ends up to the view about many-sheeted space-time realized as 4-surface in 8-D generalized imbedding space and containing both real and p-adic space-time sheets. The transformation of intention to action corresponds to a quantum jump in which p-adic space-time sheet is replaced with a real one.

One implication is that the rationals having short distance p-adically are very far away in real sense. This implies that p-adically short temporal and spatial distances correspond to long real distances and that the evolution of cognition proceeds from long to short temporal and spatial scales whereas material evolution proceeds from short to long scales. Together with p-adic non-determinism due the fact that the integration constants of p-adic differential equations are piecewise constant functions this explains the long range temporal correlations and apparent local randomness of intentional behavior. The failure of the real statistics and its replacement by p-adic fractal statistics for time series defined by varying number N of measurements performed during a fixed time interval T allows very general tests for whether the system is intentional and what is the p-adic prime p characterizing the "intelligence quotient" of the system. The replacement of $log(p_n)$ in the formula $S = -\sum_n p_n log(p_n)$ of Shannon entropy with the logarithm of the p-adic norm $|p_n|_p$ of the rational valued probability allows to define a hierarchy of number theoretic information measures which can have both negative and positive values.

Since p-adic numbers represent a highly number theoretical concept one might expect that there are deep connections between number theory and intentionality and cognition. The discussions with Uwe Kämpf in CASYS$'$2003 conference in Liege indeed stimulated a bundle of ideas allowing to develop a more detailed view about intention-to-action transformation and to disentangle these connections. These discussions made me aware of the fact that my recent views about the role of extensions of p-adic numbers are perhaps too limited. To see this consider the following arguments.

a) Pure p-adic numbers predict only p-adic length scales proportional to $p^{n/2}l$, l CP_2 length scale about 10^4 Planck lengths, $p \simeq 2^k$, k prime or power of prime. As a matter fact, all positive integer values of k are possible. This is however not enough to explain all known scale hierarchies. Fibonacci numbers $F_n : F_n + 1 = F_n + F_{n-1}$ behave asymptotically like $F_n = kF_{n-1}$, k solution of the equation $k^2 = k + 1$ given by $k = \Phi = (1 + \sqrt{5})/2 \simeq 1.6$. Living systems and self-organizing systems represent a lot of examples about scale hierarchies coming in powers of the Golden Mean $\Phi = (1 + \sqrt{5})/2$. According to Selvam [6] also meteorological phenomena involve spiral waves characterized by Golden Mean.

By allowing the extensions of p-adics by algebraic numbers one ends up to the idea that also the length scales coming as powers of x, where x is a unit of algebraic extension analogous to imaginary unit, are possible. One would however expect that the generalization of the p-adic length scale hypothesis alone would predict only the powers $\sqrt{x}p^{n/2}$ rather than $x^k p^{n/2}$, $k = 1, 2, ...$ Perhaps the purely kinematical explanation of these scales is not possible and genuine dynamics is needed. For sinusoidal logarithmic plane waves the harmonics correspond to the scalings of the argument by powers of some scaling factor x. Thus the powers of Golden Mean might be associated with logarithmic sinusoidal plane waves.

b) Physicist Hartmuth Mueller has developed what he calls Global Scaling Theory [7] based on the observation that powers of e (Neper number) define preferred length scales. These powers associate naturally with the nodes of logarithmic sinusoidal plane waves and correspond to various harmonics (matter tends to concentrate on the nodes of waves since force vanishes at the nodes). Mueller talks about physics of number line and there is great temptation to assume that deep number theory is indeed involved. What is troubling from TGD point of view that Neper number e is not algebraic. Perhaps a more general approach allowing also transcendentals must be adopted.

c) Classical mathematics, such as the theory of elementary functions, involves few crucially important transcendentals such as e and π. This might reflect the evolution of cognition: these numbers should be cognitively and number theoretically very special. The numbers e and π appear also repeatedly in the basic formulas of physics. They however look p-adically very troublesome since it has been very difficult to imagine a physically acceptable generalization of such simple concepts as exponent function, trigonometric functions, and logarithm resembling its real counterpart by allowing only the extensions of p-adic numbers based on algebraic numbers.

These considerations stimulate the question whether, besides the extensions of p-adics by algebraic numbers, also the extensions of p-adic numbers involving π and e and other transcendentals might be needed. The intuitive expectation motivated by the finiteness of human intelligence is that these extensions should have finite algebraic dimensions, and it indeed turns out that this is possible under some conditions which can be formulated as very general number theoretical conjectures.

Second question is whether there might be some dynamical mechanism allowing to understand the hierarchy of scalings coming in powers of some preferred transcendentals and algebraic numbers like Golden Mean. Conformal invariance implying that the system is characterized by a universal spectrum of scaling momenta for the logarithmic counterparts of plane waves seems to provide this mechanism. This spectrum is determined by the requirement that it exists for both reals and all p-adic number fields assuming that finite-dimensional extensions are allowed in the latter case. The spectrum corresponds to the zeros of the Riemann Zeta if Zeta is required to exist for all number fields in the proposed sense, and a lot of new understanding related to Riemann hypothesis emerges and allows to develop further the previous TGD inspired ideas about how to prove Riemann hypothesis [8, 9].

2 General ideas about number theoretic aspects of cognition and intentionality

The following two ideas serve as guide lines in the attempt to relate cognition, intentionality and number theory to each other so that number theory would allow to construct a more detailed view about the realization of intentionality and cognition. As a matter fact, the general ideas about intention and cognition in turn generate very general number theoretical conjectures.

a) Real and p-adic number fields form a book like structure with pages represented by number fields glued together along rationals forming the rim of the book. For the extensions of p-adic numbers further common points result and the book becomes fractal if all possible extensions are allowed. This picture generalizes to the level of the imbedding space and allows to see space-time surfaces as consisting of real and p-adic space-time sheets belonging to various extensions of these numbers. This generalized view about numbers gives hopes about an un-ambigious definition of what some number, say e, appearing in an extension of p-adic numbers really means.

b) The first new idea is roughly that the discovery of notion of any algebraic or transcendental number x (such as Φ or e) involves a quantum jump in which there is generated a p-adic space-time sheet for which the existing finite-dimensional extension of p-adic numbers is replaced by a finite-dimensional extension involving also x. Also some higher powers of the number are involved. For instance, for e $p-1$ powers are necessarily needed (e^p exists p-adically).

c) The p-adic-to-real transition serving as a correlate for the transformation of intention to action is most probable if the number of common rational valued points for the p-adic and real space-time sheet is high. The requirement of real and p-adic continuity and even smoothness however forces upper and lower p-adic length scale cutoffs so that common points are in certain length scale range.

d) The points of M^4_+ with integer valued Minkowski coordinates using CP_2 length related fundamental length scale as a basic unit is a good guess for the subset of M^4_+ defining the rational points of the M^4_+ involved. CP_2 coordinates as functions of M^4_+ coordinates should be rational or belong to some finite-dimensional extension of p-adics. Of course, also rational points of M^4_+ are possible, and the evolution of cognition should correspond to the increase of the algebraic dimension of the extension.

e) A very powerful hypothesis is that the p-adic and real functions have the same analytic form besides coinciding at the chosen rational points defining the p-adic pseudo constant involved. Since the pseudo constant defines the corresponding real function in rational points, there are indeed good hopes that the transformation of p-adic intention to real action is possible. This assumption favors functions which allow at some point (most naturally origin) a Taylor series with rational valued Taylor coefficients.

2.1 The case of elementary functions

Elementary functions like exp(x), log(1+x), cos(x), sin(x), are obviously favored by the previous considerations, in particular by the requirement of the form invariance of the function in p-adic-to-real transition. They indeed have p-adic Taylor expansion which converges for $|x|_p < 1$. The definition at integer valued points for which $x\ mod\ p = n$, $n = 0, 1, ..., p-1$, requires the introduction of an extension of p-adic numbers.

a) Neper number is obviously the simplest one and only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$. In case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

b) For logarithm function the numbers $log(n)$, $1 < n < p$ must exist in the extension. This requires the introduction of $log(q)$ for all primes $q < p$ appearing in the decomposition of numbers $n < p$ to a product of primes. Also the $p-1$ first powers of $log(q)$ are needed and the expansion of $log(q)^p$ in power series as $e^{log(log(q))p}$ is possible if also $log(log(q))$ and its powers up to $(p-1)^{th}$ one belong to the extension. This process leads to an infinite-dimensional extension unless there exists for any prime q a number n_q such that the n_q-fold logarithmic iterate $log(log(....log(q))..)$ is expressible in terms of algebraic numbers and transcendentals already generated in this manner. The simplest situation is that the iterate is a rational or algebraic number for any prime q.

Note that the number theoretical entropy associated with any p-adic prime for which the ordinary logarithm $log(p_n)$ is replaced by the logarithm of the p-adic norm of p_n, is proportional to a $log(p)$-factor so that the finiteness of the extension is natural also in this respect.

c) π appears naturally in the plane wave solutions of field equations $exp(in\pi x/L)$. This leads to the hypothesis that also π defines a finite-dimensional extension involving $p-1$ powers of π and a finite number of its logarithmic iterates and possibly some algebraic numbers.

2.2 Generalization of the conjecture

One can also consider a generalization of these conjectures.

a) Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

b) The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

What is highly interesting is that typically the basic differential equations deriving from the partial differential equations of physics by the separation of variables (hydrogen atom is basic example) produce functions of this kind if coupling constant parameters are rational numbers. This might be possible in TGD Universe since in TGD framework coupling constant evolution is replaced by a p-adic evolution for which coupling constants are piecewise constant functions (p-adic pseudo constants with vanishing derivative) of the length scale. If the iterated logarithms of $f(n)$ and any of its derivatives up to some order are algebraic numbers for some finite number of iterations, the extension should be finite according to the generalized conjecture.

These highly nontrivial number-theoretical conjectures are consistent with the idea that these numbers are in a well defined sense very simple and are first discovered in the evolution of mathematical cognition. Also the role of the elementary functions (rational functions, logarithm, exponential and trigonometric functions and their inverses) in the elementary integral calculus could be understood as reflecting the basic laws of cognitive evolution. Possibly also the functions appearing as solutions of differential equations of mathematical physics could be understood as reflections of the evolution of simplest forms of the mathematical cognition. I do not know whether mathematicians have proven or conjectured this kind of results.

2.3 Emergence of length scale hierarchies

These ideas lead to a more general view about how various length scale hierarchies emerge in TGD universe.

a) p-Adic length scales coming as powers of $\sqrt{2}$ involve no extensions of p-adics and result from 2-adic fractality alone. p-Adic length scales could however be involved also with dynamical hierarchies as will be found. It deserves to be noticed that $log(1+x)$ converges in the region $|x|<1$ in real context as also does the p-adic logarithm. At the scaled up point $1+u=(1+x)^n$ having values in the interval $(0,2^n)$ one has $log(1+u)=log((1+x)^n)=nlog(1+x)$. The possibility to continue logarithm from its fundamental converge region to larger values of argument by 2-adic scalings forces to ask whether p-adic length scale hypothesis might relate to the properties of logarithm.

b) More complex length scale hierarchies such as the length scale hierarchy coming as powers of Golden Mean $\Phi=(1+\sqrt{5})/2$ could be understood in terms of simple algebraic extensions of p-adic numbers and would correspond to a more complex cognition involving also dynamical aspects. Golden Mean appears again and again in self-organizing systems (consider for instance logarithmic spirals based on Golden Mean) in biology and even meteorology.

c) The hierarchy of distances defined by the nodes of logarithmic waves and coming as powers of e is central in the Global Scaling Theory of Hartmuth Mueller [7]. This length scale hierarchy might be seen as reflection of transcendental intentionality and mathematical cognition in cosmic length scales (very naturally since p-adically short length and time scales are very long in real sense)!

d) A lot of evidence has accumulated suggesting that the zeros of Riemann Zeta is very relevant for physics of fractal systems although the reason why is yet not well-understood. For some years ago I proposed that the numbers $p^{iy}=e^{ilog(p)y}$ for the values of y defined by the zeros $z=1/2+iy$ define rational numbers (being thus Pythagorean phases) or numbers in some finite algebraic extension of rationals. A more general conjecture implied by the above conjecture is that these phases belong to a finite algebraic extension of p-adics for every p so that the zeros of Zeta would make sense also for p-adic number fields if finitely extended.

With this assumption, the logarithmic waves $exp(iklog(x/x_0))$ for $x/x_0=n$ would hae values in a finite-dimensional extension if k corresponds to the zero of Riemann Zeta via $k=y$. This is consistent with the earlier conformal invariance motivated conjecture that $z=1/2+iy$ corresponds to an allowed value of the scaling momentum, defined by the zero of Riemann Zeta. These scaling momenta would be also in a preferred role as those appearing in the Fourier expansion based on the logarithmic waves at short wave length limit when boundary conditions are not important.

3 Logarithmic waves as scaling momentum eigen states

The logarithmic waves of Hartmuth Mueller [7] can be seen as scaling momentum eigen states obtained from ordinary plane waves which are momentum eigen states by replacing argument with its logarithm. By quantum criticality and conformal invariance of TGD Universe logarithmic waves are very natural in TGD framework. In particular they would be naturally associated with topological light rays (MEs, massless extremals).

For topological light rays the only reasonable counterpart for effectively standing waves is combination of positive and negative energy MEs (laser waves and their phase conjugates propagating in different directions of geometric time). No direct interference would be involved although the physical effects induced by the topological light ray and its phase conjugate might interfere and generate a standing wave. The mirror mechanism could be involved even in cosmic length and time scales and that Mueller's waves represent these and have interpretation as correlates of intentional quantal remote sensing over cosmological distances utilizing time mirror mechanism. The claimed coupling with biological matter, in particular DNA, would occur naturally since living matter should be full of population inverted many-sheeted lasers allowing the time reflected signal to be amplified.

It must be however emphasized that space-time sheets are by definition quantum coherence regions and behave like single units. Therefore the modulation of the space-time sheet sheet at a given point induces quantum jump replacing the entire space-time sheet with a new one so that an instantaneous communication over long, even cosmic, distances becomes possible. This is perhaps more in line with what Mueller claims. For instance, magnetic flux tubes could define linear structures realizing naturally the effectively one-dimensional spatial structures.

Intention and cognition are cosmic phenomena and it might be that p-adic counterparts of logarithmic waves transform to real ones. One should consider a concrete model for p-adic counterparts of MEs which correspond to scaling momentum eigen states and for how they are transformed to real counterparts in intentional action.

3.1 Long wave length logarithmic wave and standing wave boundary conditions

For long wave lengths boundary conditions are important. Scaling momentum eigen states can be constructed by regarding them as logarithmic counterparts of standing waves for a string of fixed length or counterpart of waves in a one-dimensional box satisfying periodic boundary conditions. The basic unit of momentum is $k_0 = \pi/L$ and $k_0 = 2\pi/L$ in these two cases.

3.1.1 Construction in real case

In the real case the construction goes as follows.

a) Fix the length scale unit l and maximal length scale L. Assume that a corresponds to ratio defining fractal scalings. It is natural to require that one has $L/l = a^n$ so that one has $log_a(L/l) = n$. Thus n full fractally scaled versions are involved in the structure defined by logarithmic wave. l would correspond naturally to some p-adic length scale which is near square root two power of CP_2 size.

b) $u = log(x/l)$ is the new variable reducing logarithmic Fourier analysis to ordinary Fourier analysis if logarithmic plane waves are proportional to the factor $1/\sqrt{u}$. The requirement that scaling momentum eigen states correspond to complex scaling momenta allows also power of u as a factor. $\sqrt{u}$ is forced by the scaling invariance of the Fourier integral corresponds to vacuum scaling momentum $L_0 = -1/2$ familiar from the representations of Super conformal Algebras. $u_{max} = log(L/l) = log(a)n$ defines the length of the logarithmic box. $sin(u_{max})$ should vanish at the both ends of the box. One could also consider periodic boundary conditions.

c) The most general logarithmic plane waves are of form

$$\Psi_k = e^{ik\pi u/u_{max}} \quad .$$

For string like sine wave solution vanishes at the ends of the logarithmic string so that k must be an integer.

d) The positions of nodes for string like boundary conditions are at

$$u_m = (m/k) \times u_{max} = m \times (n/k) \times log(a), \quad m = 1, ..., k \quad .$$

Clearly, k nodes result. The nodes correspond to the values of $x = x_m$ given by

$$\frac{x_m}{l} = e^{(m/k)\times n \times log(a)} = a^{m\times n/k} \quad .$$

e) If one requires that n fractally scaled parts are present also in the architecture of the logarithmic plane wave, one must assume that k is a multiple of n:

$$k = r \times n \ ,$$

and one has

$$\frac{x_m}{l} = a^{m/r} \ .$$

This gives $a^{1/r}$ as the basic scaling ratio and the natural period coming from the convergence region of logarithm corresponds naturally to the basic structure.

3.1.2 Construction in the p-adic case

In p-adic context the existence of these expressions requires the introduction of extension of p-adics in order to carry out the construction of the logarithmic plane waves.

a) The existence of $log(n)$ is required so that the conjecture about logarithmic iterates $log(p)$ of primes must hold true.

b) For $a = e$ the situation is very simple. x_m/l exists p-adically if the extension contains the root $e^{n/k}$ and its powers. The simplest situation is obtained if one has $k = n$. This situation corresponds to the nodes of the logarithmic waves of Mueller. For k a higher multiple of n also roots of of e must be introduced in the extension and this means higher level cognition.

c) For $a \neq e$ also $1/log(a)$ appears in the exponent and one must assume more complex extension. $log(a)$ and its first $p-1$ powers must belong to extension. The same holds true for some number of iterated logarithms of $log(a)$ and for some number of iterations $log(log.....log(a))...)$ must satisfy the constraints stated earlier. Whether these conditions can be satisfied for say $a = \Phi$ (Golden Mean) is not at all clear. Certainly the logarithmic waves of Mueller are exceptional. A fascinating possibility is that also the Golden Mean Φ, which is maximally irrational in the sense that its continued fraction expansion converges with minimal rate, could represent a number for which some iterate of the logarithm satisfies the conjectured conditions.

d) Powers of $e^{ik\pi/u_{max}}$ should exist p-adically. For $a = e$ this requires only an algebraic extension allowing sines and cosines with argument values $k\pi/n$. These are indeed algebraic numbers. For $a \neq e$ the powers $exp(ik\pi/nlog(a))$ should exist p-adically and this might lead to an infinite-dimensional extension and explain why powers of primes p, Φ, e and possibly some other selected numbers are unique from the point of view of physics of cognition and intention. Note that the logarithmic iterates of $exp(ik\pi/nlog(a))$ are in the extension automatically if π belongs to the extension.

e) For $a = p$, p prime, the length scale hierarchy associated with the logarithmic plane waves corresponds to the p-adic length scale hierarchy. The existence of a finite extension based on $log(n)$ already guarantees that in this case one obtains finite extension of also the the numbers $e^{ik\pi/nlog(p)}$ belong to the extension.

The overall conclusion is that Mueller's logarithmic waves are completely exceptional in their simplicity. Logarithmic waves are involved also with the p-adic length scale hierarchies and with Golden Mean provided appropriate conditions formulated in terms of iterated logarithms of primes p and Golden Mean Φ hold true.

3.2 Are boundary conditions replaced by number theoretic existence requirements for short wavelength logarithmic waves?

Large values of the scaling momenta correspond to short wavelength contributions to the logarithmic waves superposing with the harmonic long range contributions. For the short wavelength part boundary conditions are irrelevant. In standard physics there is no obvious conditions replacing the boundary conditions at this limit. In recent case one can however wonder whether boundary conditions could be replaced by number theoretic existence requirement at points $u = log(s)$, s integer. In this case one case write the logarithmic plane waves simply as

$$\Psi = exp(iKu) \ .$$

By factorizing s to a product of primes the condition reduces to the existence requirement at prime valued points s: $s = p$. This boils down to requirement that the numbers p^{iK} exists in a finite-dimensional extension, and one ends up with the conjecture about logarithmic iterates of primes. The natural guess inspired by the idea that Riemann Zeta allows also p-adic counterparts is that the values of of the scaling momentum correspond to the zeros $z = 1/2 + iy$ of Riemann Zeta: $K = y$.

The condition is satisfied also for the integer multiples of K if it is satisfied for K so that scaling harmonics are possible. These harmonics cannot however satisfy the boundary conditions as is easy to see. Boundary conditions would require $Klog(u_{max}) = Knlog(a)$ to be an integer multiple of

π and imply that one has $y = (m/n) \times (1/log(a))$ for the zeros of Riemann Zeta. For $a = e$ this would mean that the zeros of Zeta would be complex rational numbers and the phase angles $ylog(p)$ would correspond to some rational numbers. Hence the ratios $log(p_1)/log(p_2)$ would be rational numbers, and this in turn implies $p_i = p_j^{q_{ij}}$, q_{ij} a rational number. From this one easily deduces that $p_i^r = p_j^s$ for some integers r and s, which obviously does not make sense.

3.3 A connection with Riemann hypothesis

There is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$. I have applied two basic strategies in my attempts to understand Riemann hypothesis. These approaches are summarized in the chapter "Number Theory and TGD: Riemann hypothesis" of [1]. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

Approach 1: In this approach (see the preprint in [8] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [9]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

Approach 2: The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1 - p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations allow to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also some transcendentals. This implies that also the quantities n^{iy} make sense for all number fields and one can develop Zeta in p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta is therefore defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

a) At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short ($k < -1/2$) or long distances ($k > -1/2$).

b) Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

c) The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

3.4 The minimal extension containing e, π and $log(p)$ and the universality of the Riemann Zeta

In the following the conditions guaranteing that the minimal extension of p-adic numbers containing both e, π, and $log(p)$ for any value of prime p is finite-dimensional, are studied. The crucial step leading to a partial identification of the extension is the physically motivated assumption that the phases p^{iy} appearing factors of Riemann Zeta exist as p-adic and even algebraic numbers at the zeros of Zeta, which are rational numbers themselves, so that number theoretic partition functions exist universally.

1. The existence of e requires a finite-dimensional transcendental extension

The existence of e and its powers is guaranteed if one introduces e and the powers $e^2, ..., e^{p-1}$ to the extension so that the minimal extension is p-dimensional. The power $e^p = \sum p^n/n!$ exists as a converging power series. The fact that exponentiation is the basic operation of Lie group theory suggests that the finite-dimensionality of this extension is of fundamental importance for the group theory applying the generalized notion of number.

2. The existence of logarithms of primes

If $log(q)$ exists for all primes q for R_p, it exists for all integers, rational numbers, and even products rational exponents of rational numbers by the basic properties of logarithm. In the p-adic topology one can define the logarithm $log(q)$, $q \neq p$ by noticing that by Fermat's little theorem one has

$$q^{d(q,p)} \ mod \ p \quad = \quad 1 \ , \tag{1}$$

for some $d(q,p)$, which is always a divisor of $p-1$. [Fermat's little theorem derives from the fact that the multiplicative group of the finite field $G(q,1)$ (integers modulo prime q) is a cyclic group of order $q-1$, whose elements have orders which are factors of $q-1$]. As a consequence, $log(p^{d(q,p)})$ exists p-adically for any q different from p and is defined by the usual Taylor expansion of $log(1+x)$. The trick needed to define the logarithm is to define the p-adic logarithm of $log(q)$ as

$$log(q) = \frac{1}{d(q,p)} log(q^{d(p,q)}) \ .$$

The remaining hard problem is how to define the p-adic logarithm $log(p)$ in R_p. The idea is that the ration $log(p)/\pi$ is rational number so that it is enough to define π p-adically.

2. The existence of $log(p)$ in R_p

The second hard problem is how to define $log(p)$ in R_p. Here physical intuition and Riemann Zeta come in a rescue. Riemann Zeta can be regarded as a product of partition functions $Z_p = 1/(1-p^{-1/2}p^{-iy})$. It is natural to require that these functions exist at the zeros of $s = 1/2 + iy$ Zeta and can be continued to their neighborhood in all number fields. This of course does not yet guarantee the existence of Riemann Zeta at $Re(s) = 1/2$ axis since the product need not converge p-adically.

Thus p^{-iy}, p prime, should exist p-adically for every zero of ζ and for every prime p. One can write the exponent in the form $exp(-ilog(p)y)$. The simplest assumption is that for every p and y one has

$$log(p) \times y \quad = \quad q(p,y) \times \pi \ , \tag{2}$$

where $q(p,y)$ is rational number.

Under this assumption one has

$$p^{-iy} = e^{-iq(p,y)\pi} \ .$$

The exponent can be expanded into a p-adic power series for any value of p for $|q(p,y)| < p^2$. For $|q(p,y)| = p^n$, $n > p$ the exponent can be expressed as an algebraic number in a p^n extension of p-adic numbers. The strongest condition is that the p-adic partition functions can be interpreted as real partition functions at the zeros of ζ so that the exponent p^{-iy} must allow interpretation as an ordinary algebraic number. In this case one must have $|q(p,y)| > p$.

The general idea is that ζ exists universally for some subset of rational numbers. If one assumes that the zeros y are rational numbers, this subset is spanned by all linear combinations of zeros y with integer coefficients and are expected to form a dense set of y-axis. The assumption means that the ratio of the transcendentals $log(p)$ and π is always rational:

$$log(p) \quad = \quad Q(p) \times \pi \ , \tag{3}$$

where $Q(p) = q(p,y)/y$ is a rational number. In particular, $log(p)$ could be defined in R_p using the existence of π in R_p so that the question reduces to the existence of π in a finite-dimensional extension.

3. The existence π

It is not absolutely necessary to have π in the extension. One can define an algebraic extension of p-adic numbers by requiring that the function $exp(i\pi x)$ is well defined for the rational values of $x = k/p^n$, $k = 0, ..., p^n - 1$ by using the basic formulas of the exponent function allowing to express

$cos(q\pi)$ and $sin(q\pi)$ as algebraic numbers. Around these values one can extend these functions to p-adic regime by powers series expansions. Therefore one can say that the p-adic variant of trigonometry and Fourier analysis exists even in the case that π does not exist in the extension. There is a preferred discretization of the unit circle to p^n parts. The requirement of orthogonality for angular momentum eigen states implies angular momentum quantization such that the values of J come as powers of p in R_p: $J = p^n$.

The next question is it possible to have also π. There is indeed an attractive but wrong argument suggesting that π does not require more than an extension containing $\sqrt{-1}$. π can defined as logarithm of -1: $\sqrt{-1}\pi \equiv log(-1)$. By writing

$$log(-1) = log\frac{(p-1)}{(1-p)}) = log(p-1) - log(1-p)$$

$$= \frac{1}{2}log[(p-1)^2] - log(1-p) = \frac{1}{2}log(1-2p+p^2) - log(1-p) \ ,$$

one can use converging p-adic series for the two logarithms involved and obtains

$$\pi = -\sqrt{-1} \times log(-1) = -\sqrt{-1}(p^2 - 2p^3 + ...) \ .$$

For $p \ mod \ 4 = 3$ a quadratic extension allowing $\sqrt{-1}$ is needed and one could perhaps say that π is imaginary in this case. For $p \ mod \ 4 = 1$ the imaginary unit $\sqrt{-1}$ exists as an ordinary p-adic number. The argument however fails since the assumption $e^{i\pi} = -1$ is not satisfied for the proposed candidate of π. Thus the only hope is that e/π is a rational number or that an analogous statement holds true for some higher logarithmic iterate of π.

To sum up, the hypothesis that the zeros of ζ are rational numbers; that the ratios $\pi/log(p)$ are rational numbers for all primes p; and that e/π is a rational number, would provide the simplest manner to realize the idea that the numbers e, π, and $log(p)$, which appear in the basic formulas of physics, are defined for finite-dimensional extensions of p-adic numbers. The scalings by powers of e and p emerge naturally for logarithmic waves: in this case it is however not absolutely necessary to have π and $log(p)$ in the extension. If the ratio $log(\Phi)/\pi$ is rational, also powers of Golden Mean can appear in the logarithmic waves.

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