The twistor space of $H = M^4 \times CP_2$ allows Lagrangian 6-surfaces: what does this mean physically?

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Abstract

This article was inspired by the article "A note on Lagrangian submanifolds of twistor spaces and their relation to superminimal surfaces" of Reinier Storm. For curiosity, I decided to look at Lagrangian surfaces in the twistor space of $H = M^4 \times CP_2$. The 6-D Kähler action of the twistor space existing only for $H = M^4 \times CP_2$ gives by a dimensional reduction rise to 6-D analog of twistor space assitable to a space-time surface. In the dimensional reduction the action reduces to 4-D Kähler action plus a volume term characterized by a dynamically determined cosmological constant Λ.

One can identify space-time surfaces, which are Lagrangian minimal surfaces and therefore have a vanishing Kähler action. If the Kähler structure of M^4 is non-trivial as strongly suggested by the notion of twistor space, these vacuum extremals are products $X^2 \times Y^2$ of Lagrangian string world sheet X^2 and 2-D Lagrangian surface Y^2 of CP_2, and are deterministic so that they allow holography. As minimal surfaces they allow a generalization of holography= holomorphy principle: now the holomorphy is not induced from that of H but by 2-D nature of X^2 and Y^2. Therefore holography=holomorphy principle generalizes.

Λ can vanish and in this case the dimensionally reduced action equals Kähler action. In this case, vacuum extremals are in question and symplectic transformations generate a huge number of these surfaces, which in general are not minimal surfaces. Holography= holomorphy principle is not however lost. $\Lambda = 0$ sector contains however only classical vacua and also the modified gamma matrices appearing in the modified Dirac action vanish so that this sector contributes nothing to physics.

Contents

1 Introduction

I received from Tuomas Sorakivi a link to the article "A note on Lagrangian submanifolds of twistor spaces and their relation to superminimal surfaces" [A3] (see this). The author of the article is Reinier Storm from Belgium.

The abstract of the article tells roughly what it is about.

In this paper a bijective correspondence between superminimal surfaces of an oriented Riemannian 4-manifold and particular Lagrangian submanifolds of the twistor space over the 4-manifold is proven. More explicitly, for every superminimal surface a submanifold of the twistor space is constructed which is Lagrangian for all the natural almost Hermitian structures on the twistor space. The twistor fibration restricted to the constructed Lagrangian gives a circle bundle over the superminimal surface. Conversely, if a submanifold of the twistor space is Lagrangian for all the natural almost Hermitian structures, then the Lagrangian projects to a superminimal surface and is contained in the Lagrangian constructed from this surface. In particular this produces many Lagrangian submanifolds of the twistor spaces and with respect to both the Kähler structure as well as the nearly Kähler structure. Moreover, it is shown that these Lagrangian submanifolds are minimal submanifolds.

The article examines 2-D minimal surfaces X^2 in the 4-D space X^4 assumed to have twistor space. From superminimality which looks somewhat peculiar assumption, it follows that in the twistor space of X^4 (assuming that it exists) there is a Lagrangian surface, which is also a minimal surface. Superminimality means that the normal spaces of the 2-surface form a 1-D curve in the space of all normal spaces, which for the Euclidian signature is the 4-D Grassmannian $SO(4)/SO(2) \times SO(2) = S^2 \times S^2$ ($SO(1,3)/SO(1,1) \times SO(2)$ for M^4). Superminimal surface is therefore highly flattened. Of course, already the minimal surface property favours flatness.

It is interesting to examine the generalization of the result to TGD because the interpretation for Lagrange manifolds, which are vacuum extremals for the Kähler action with a vanishing induced symplectic form, has remained open. Certainly, they do not fulfill the holomorphy=holography assumption, i.e. they are not surfaces for which the generalized complex structure in H induces a corresponding structure at 4-surface.

Superminimal surfaces look like the opposite of holomorphic minimal surfaces (this turned out to be an illusion!). In TGD, they give a huge vacuum degeneracy and non-determinism for the pure Kähler action, which has turned out to be mathematically undesirable. The cosmological constant Λ, which follows from twistoralization, was thought to correct the situation.

I had not however notice that the Kähler action, whose existence for $T(H) = T(M^4) \times T(CP_2)$ fixes the choice of H, gives a huge number of 6-D Lagrangian manifolds! Are they consistent with dimensional reduction, so that they could be interpreted as induced twistor structures? Can a complex structure be attached to them? Certainly not as an induced complex structure. Does the Lagrangian problem of Kähler action make a comeback? Furthermore, should one extend the very promising looking holography=holomorphy picture by allowing also Lagrangian 6-surfaces $T(H)$?

Do the Lagrangian surfaces of $T(H)$ have a physical interpretation, most naturally as vacuums? The volume term of the 4-D action characterized by the cosmological constant Λ does not allow vacuum extremals unless Λ vanishes. For the twistor lift Λ is however dynamic and can vanish! Do Lagrangian 6-surfaces in $T(H)$ correspond to 4-D minimal surfaces in H, which are vacuums and have a vanishing $\Lambda = 0$? Would even the original formulation of TGD be an exact part of the theory and not just a long-length-scale limit? And does one really avoid the original problem due to the huge non-determinism spoiling holography!

The question is whether the result presented in the article could generalize to the TGD framework even though the super-minimality assumption does not seem physically natural at first?

2 Lagrangian surfaces in the twistor space of $H = M^4 \times CP_2$

Let us consider the 12-D twistor space $T(H) = T(M^4) \times T(CP_2)$ and its 6-D Lagrangian surfaces having a local decomposition $X^6 = X^4 \times S^2$. Assume a twistor lift with Kähler action on $T(H)$. It exists only for $H = M^4 \times CP_2$ [L1, L2].

Let us first forget the requirement that these Lagrangian surfaces correspond to minimal surfaces in H. Consider the situation in which there is no generalized Kähler and symplectic structure in M^4.

One can actually identify Lagrangian surfaces in 12-D twistor space $T(H)$.

1. Since $X^6 = X^4 \times S^2$ is Lagrangian, the symplectic form for it must vanish. This is also true in S^2. Fibers S^2 together with $T(M^4)$ and $T(CP_2)$ are identified by an orientation-changing isometry. The induced Kähler form S^2 in the subset $X^6 = X^4 \times S^2$ is zero as the *sum* of these two contributions of different signs. If this sum appears in the 6-D Kähler action, its contribution to the 6-D Kähler action vanishes. Λ vanishes because the S^2 contribution to the 4-D action vanishes.

2. The 6-D Kähler action reduces in X^4 to the 4-D Kähler action plus, which was the original guess for the 4-D action. The problem is that in its original form, involving only CP_2 Kähler form, it involves a huge vacuum degeneracy. The CP_2 projection is a Lagrangian surface or its subset but the dynamics of M^4 projection is essentially arbitrary, in particular with respect to time. One obtains a huge number of different vacuum extremals. Since the time evolution is non-deterministic, the holography, and of course holography=holomorphy principle, is lost. This option is not physically acceptable.

How the situation changes when also M^4 has a generalized Kähler form that the twistor space picture strongly suggests, and actually requires.

1. Now the Lagrangian surfaces would be products $X^2 \times Y^2$, where X^2 and Y^2 are the Lagrangian surfaces of M^4 and CP_2. The M^4 projections of these objects look like string world sheets and in their basic state are vacuums.

   Furthermore, the situation is deterministic! The point is that X^2 is Lagrangian and highly fixed as such. In the previous case much more general surface M^4 projection, even 4-D, was Lagrangian. There is no loss of holography! Neither is the holography = holomorphy principle lost: by their 2-D character X^2 and Y^2 have a holomorphic structure.

   What is important is that these Lagrangian 4-surfaces of H are obtained also when Λ is non-vanishing. In this case they must be minimal surfaces. Physically this option means that one has Lagrangian strings.

2. For $\Lambda = 0$, the symplectic transformations of H produce new vacuum surfaces. If they are allowed, one might talk of symplectic phase. $J = 0$ phase gives rise to both classical and fermionic vacuum since the modified gamma matries vanish since they are propertional to vanishing canonical momentum currents. So that Lagrangian phase does not contribute to physics for $\Lambda = 0$. There are however non-vacuum extremals for which the induced Kähler field is non-vanishing (having induced complex structure).

   For $\Lambda \neq 0$ Lagrangian surfaces which are non-vacuum extermals and only isometries are allowed as symmetries. One can say that symplectic symmetr breaks down to isometries. Irrespective of the value of Λ, the second phase with a induced complex structure would be present and give rise to color interactions and hadrons and probably also elementary particles. The interpretation of Lagrangian surfaces, which are string like entities, remains open.

3. In the Lagrangian phase induced Kähler form J and the induced color gauge fields vanish and it does not involve monopole fluxes. This phase might be called Maxwell phase. For $\Lambda \neq 0$ one would have two kinds of non-vacuum string like objects with string tension to which Λ contributes.

   Could the Lagrangian phase for $\Lambda \neq 0$ correspond to the Coulomb phase as the perturbative phase of the gauge theories, while the monopole flux tubes (large h_{eff} and dark matter) would correspond to the non-perturbative phase in which magnetic monopole fluxes are present? If so, there would be an analogy with the electric-magnetic duality of gauge theories although the two phases does not look like two equivalent descriptions of one and the same thing unless one restricts the consideration to fermions.

2.1 Can Lagrangian 4-surfaces be minimal surfaces?

I have not yet considered the question whether the Lagrangian surfaces can be minimal surfaces. For non-vanishing Λ they must be such but for $\Lambda = 0$ this need not be the case. One can of course ask whether this does matter at all for $\Lambda = 0$. In this case, one has only vacuum extremals and the modified gamma matrices are proportional to the canonical momentum currents, which vanish. Both bosonic and fermionic dynamics are trivial for $\Lambda = 0$. Therefore $\Lambda = 0$ does not give any physics.

In the theorem the minimal Lagrangian surfaces were superminimal surfaces. For super-minimal surfaces, a unit vector in the normal direction defines a very specific curve in normal space.

For a non-vanishing cosmological constant, the field equations for the Kähler action do not force the Lagrangian surfaces to be minimal surfaces. For $\Lambda \neq 0$ there exists a lot of minimal Lagrangian surfaces.

2.1.1 Lagrangian minimal surfaces in CP_2

Consider first the Lagrangian minimal surfaces in CP_2

1. In CP_2, a homologically trivial geodesic sphere is a minimal surface. Note that the geodesic spheres obtained by isometries are regarded here as equivalent. Also a $g = 1$ minimal Lagrangian surface (Clifford torus) in CP_2 is known.

2. There are many other minimal Lagrangian surfaces and second order partial differential equations for both Lagrangian and minimal Lagrangian surfaces are known (see this). In the article "A new look at equivariant minimal Lagrangian surfaces in CP_2 by Dorfmeister and Ma [A1] Lagrangian minimal surfaces in CP_2 are discussed and general partial differential equations for them are deduced.

   (a) An essential role is played by the used of complex coordinates in which the induced metric of X^2 is of form $ds^2 = e^u dz d\overline{z}$ and X^2 corresponds to immersion f.

   (b) The Lagrangian property makes it possible the lift of f and to an immersion defined to unit sphere $S^5 \subset C^3$ and therefore of X^2 to a surface in $S^5 \subset C^3$ defined by a complex triplet F. This allows to combine F, F_z and $F_{\overline{z}}$ to an orthgonal Hermitian tripet which can be can be replaced with a orthonormalized triplet $\mathcal{F} = (F, e^{-u/2}F_z, e^{-u/2}F_{\overline{z}})$.

   (c) At the next step minimal surface property is introduced. It translation to statement that

   $$\mathcal{F}_z = \mathcal{F}\mathcal{U} \ \ , \ \ \mathcal{F}_{\overline{z}} = \mathcal{F}\mathcal{N} \ \ .$$

   Here one has

   $$\mathcal{U} = \left(\begin{array}{lll} u_z/2 & 0 & e^u \\ e^{-u}\psi & -u_z/2 & 0 \\ 0 & -e^u/2 & 0 \end{array} \right)$$
   $$\mathcal{N} = \mathcal{U}^\dagger$$

   Here ψdz^3 is so called Hopf differential with ψ given by

   $$\psi = F_{zz}\overline{F_{\overline{z}}} \ \ .$$

   Clearly, $\mathcal{U}$ is the negative of the hermitian conjugate of $\mathcal{N}$. One can say that complex differentiation corresponds to the action of $SU(3)$ Lie algera generator so that $\mathcal{F}$ defines an element of $SU(3)$ loop group at X^2.

   (d) The condition of integrability $(\mathcal{F}_z)_{\overline{z}} = (\mathcal{F}_{\overline{z}})_z$ gives

   $$\mathcal{U}_{\overline{z}} = -\mathcal{N}_z \ \ ,$$

   and the final equations

   $$u_{z\overline{z}} = e^{-2u}|\psi|^2 - e^u \ \ , \ \ \psi_{\overline{z}} = 0 \ \ .$$

   The Hopf differential is therefore a holomorphic function.

Since any stable stable minimal submanifold in CP_n is a complex submanifold, the Lagrangian minimal surfaces cannot be stable under general variations.

2.1.2 Lagrangian minimal surfaces in M^4

Consider next the situation in M^4.

1. In M^4, the plane M^2 is an example of a minimal surface, which is a Lagrangian surface. Are there others? Could Hamilton-Jacobi structures [L3] that also involve the symplectic form and generalized Kähler structure (more precisely, their generalizations) define Lagrangian surfaces in M^4?

2. The Lagrangian surfaces, and as a special case Lagrangian minimal surfaces in R^4 are discussed in [A2]. The result of the article can be phrased as follows.

   Let L be a simply connected domain in C. Then for any smooth conformal Lagrangian immersion $f : L \to R^4$, there exist smooth functions $\beta : L \to R/2\pi Z$, which is the Lagrangian angle, and $s_1, s_2 : L \to C$, not simultaneously vanishing, that satisfy the Dirac-type equation

   $$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\overline{z}} & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ \overline{s}_2 \end{pmatrix} = \begin{pmatrix} \overline{U} & 0 \\ 0 & -U \end{pmatrix} \begin{pmatrix} s_1 \\ \overline{s}_2 \end{pmatrix} \ .$$

   with complex potential $U = \partial_z \beta/2$. Conversely, given β and any solution (s_1, s_2) to the Dirac equation satisfying $(|s_1|^2 + |s_2|^2 \geq 0)$ gives rise to a conformal Lagrangian immersion given by

   $$f(z) = Re\left[\int^z exp(\beta J/2) \begin{pmatrix} s_1 \\ s_2 \\ -is_1 \\ is_2 \end{pmatrix}\right] \ , \ J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ .$$

   Here the 4×4 matrix J defines the standard symplectic structure.

3. When the Lagrange angle is constant, one obtains minimal Lagrangian immersion. Note that this in this case one has free massless Dirac equation.This suggests quantum classical correspondence in which the solutions of massless Dirac equation in M^4 correspond to Lagrangian minimal surfaces.

4. This solution is defined for Euclidian E^4 rather than M^4 but the analytic continuation to M^4 case should be straightforward. This requires an appropriate modification of J. In TGD one must consider the possibility, that Hamilton-Jacobi structures defines large number of non-quivalent Kähler- and symplectic structures for M^4. The naive guess is that J in the exponential is replaced with the matrix $J_{kl}\sigma^{kl}$ in order to obtain a more general solution.

In the case considered now, the Lagrangian surfaces in H would be products $X^2 \times Y^2$. Interestingly, in the 2-D case the induced metric always defines a holomorphic structure. Now, however, this holomorphic structure would not be the same as the one related to the holomorphic ansatz: it is induced from H.

2.2 So What?

These findings raise several questions related to the detailed understanding of TGD. Should one allow only non-vanishing values of Λ? This would allow minimal Langrangian surfaces $X^2 \times Y^2$ besides the holomorphic ansatz. The holomorphic structure due to the 2-dimensionality of X^2 and Y^2 means that holography=holomorphy principle generalizes.

If one allows $\Lambda = 0$, all Lagrangian surfaces $X^2 \times Y^2$ are allowed but also would have a holomorphic structure due to the 2-dimensionality of X^2 and Y^2 so that holography=holomorphy principle would generalize also now! Minimal surface property is obtained as a special case. Classically the extremals correspond to a vacuum sector and also in the fermionic sector modified Dirac equation is trivial. Therefore there is no physics involved.

Minimal Lagrangian surfaces are favored by the physical interpretation in terms of a geometric analog of the field particle duality. The orbit of a particle as a geodesic line (minimal 1-surface) generalizes to a minimal 4-surface and the field equations inside this surface generalizes massless field equations.

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