Langlands Program and TGD

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Abstract

Number theoretic Langlands program can be seen as an attempt to unify number theory on one hand and theory of representations of reductive Lie groups on one hand. So called automorphic functions to which various zeta functions are closely related define the common denominator. Geometric Langlands program tries to achieve a similar conceptual unification in the case of function fields. This program has caught the interest of physicists during last years.

TGD can be seen as an attempt to reduce physics to infinite-dimensional Kähler geometry and spinor structure of the "world of classical worlds" (WCW). If TGD can be regarded also as a generalized number theory, it is difficult to escape the idea that the interaction of Langlands program with TGD could be fruitful. I of course hasten to confess that I am not number theorists nor group theorists and that the following considerations are just speculations inspired by TGD.

More concretely, TGD leads to a generalization of number concept based on the fusion of reals and various p-adic number fields and their extensions implying also a generalization of manifold concept, which inspires the notion of number theoretic braid crucial for the formulation of quantum TGD. TGD leads also naturally to the notion of infinite primes and rationals. The identification of Clifford algebra of WCW in terms of hyper-finite factors of type II_1 in turn inspires further generalization of the notion of imbedding space and the idea that quantum TGD as a whole emerges from number theory. The ensuing generalization of the notion of imbedding space predicts a hierarchy of macroscopic quantum phases characterized by finite subgroups of $SU(2)$ and by quantized Planck constant. All these new elements serve as potential sources of fresh insights.

1. The Galois group for the algebraic closure of rationals as infinite symmetric group?

The naive identification of the Galois groups for the algebraic closure of rationals would be as infinite symmetric group S_∞ consisting of finite permutations of the roots of a polynomial of infinite degree having infinite number of roots. What puts bells ringing is that the corresponding group algebra is nothing but the hyper-finite factor of type II_1 (HFF). One of the many avatars of this algebra is infinite-dimensional Clifford algebra playing key role in Quantum TGD. The projective representations of this algebra can be interpreted as representations of braid algebra B_∞ meaning a connection with the notion of number theoretical braid.

2. Representations of finite subgroups of S_∞ as outer automorphisms of HFFs

Finite-dimensional representations of $Gal(\overline{Q}/Q)$ are crucial for Langlands program. Apart from one-dimensional representations complex finite-dimensional representations are not possible if S_∞ identification is accepted (there might exist finite-dimensional l-adic representations). This suggests that the finite-dimensional representations correspond to those for finite Galois groups and result through some kind of spontaneous breaking of S_∞ symmetry.

1. Sub-factors determined by finite groups G can be interpreted as representations of Galois groups or, rather infinite diagonal imbeddings of Galois groups to an infinite Cartesian power of S_n acting as outer automorphisms in HFF. These transformations are counterparts of global gauge transformations and determine the measured quantum numbers of gauge multiplets and thus measurement resolution. All the finite approximations of the representations are inner automorphisms but the limit does not belong to S_∞ and is therefore outer. An analogous picture applies in the case of infinite-dimensional Clifford algebra.
2. The physical interpretation is as a spontaneous breaking of S_∞ to a finite Galois group. One decomposes infinite braid to a series of n-braids such that finite Galois group acts in each n-braid in identical manner. Finite value of n corresponds to IR cutoff in physics in the sense that longer wave length quantum fluctuations are cut off. Finite measurement resolution is crucial. Now it applies to braid and corresponds in the language of new quantum measurement theory to a sub-factor $\mathcal{N} \subset \mathcal{M}$ determined by the finite Galois group G implying non-commutative physics with complex rays replaced by $\mathcal{N}$ rays. Braids give a connection to topological quantum field theories, conformal field theories (TGD is almost topological quantum field theory at parton level), knots, etc..
3. TGD based space-time correlate for the action of finite Galois groups on braids and for the cutoff is in terms of the number theoretic braids obtained as the intersection of real partonic 2-surface and its p-adic counterpart. The value of the p-adic prime p associated with the parton is fixed by the scaling of the eigenvalue spectrum of the

modified Dirac operator (note that renormalization group evolution of coupling constants is characterized at the level free theory since p-adic prime characterizes the p-adic length scale). The roots of the polynomial would determine the positions of braid strands so that Galois group emerges naturally. As a matter fact, partonic 2-surface decomposes into regions, one for each braid transforming independently under its own Galois group. Entire quantum state is modular invariant, which brings in additional constraints.

4. Braiding brings in homotopy group aspect crucial for geometric Langlands program. Another global aspect is related to the modular degrees of freedom of the partonic 2-surface, or more precisely to the regions of partonic 2-surface associated with braids. $Sp(2g, R)$ (g is handle number) can act as transformations in modular degrees of freedom whereas its Langlands dual would act in spinorial degrees of freedom. The outcome would be a coupling between purely local and and global aspects which is necessary since otherwise all information about partonic 2-surfaces as basic objects would be lost. Interesting ramifications of the basic picture about why only three lowest genera correspond to the observed fermion families emerge.

3. Correspondence between finite groups and Lie groups

The correspondence between finite and Lie group is a basic aspect of Langlands.

1. Any amenable group gives rise to a unique sub-factor (in particular, compact Lie groups are amenable). These groups act as genuine outer automorphisms of the group algebra of S_∞ rather than being induced from S_∞ outer automorphism. If one gives up uniqueness, it seems that practically any group G can define a sub-factor: G would define measurement resolution by fixing the quantum numbers which are measured. Finite Galois group G and Lie group containing it and related to it by Langlands correspondence would act in the same representation space: the group algebra of S_∞, or equivalently configuration space spinors. The concrete realization for the correspondence might transform a large number of speculations to theorems.

2. There is a natural connection with McKay correspondence which also relates finite and Lie groups. The simplest variant of McKay correspondence relates discrete groups $G \subset SU(2)$ to ADE type groups. Similar correspondence is found for Jones inclusions with index $\mathcal{M} : \mathcal{N} \leq 4$. The challenge is to understand this correspondence.

   (a) The basic observation is that ADE type compact Lie algebras with n-dimensional Cartan algebra can be seen as deformations for a direct sum of n SU(2) Lie algebras since SU(2) Lie algebras appear as a minimal set of generators for general ADE type Lie algebra. The algebra results by a modification of Cartan matrix. It is also natural to extend the representations of finite groups $G \subset SU(2)$ to those of $SU(2)$.

   (b) The idea would that is that n-fold Connes tensor power transforms the direct sum of n SU(2) Lie algebras by a kind of deformation to a ADE type Lie algebra with n-dimensional Cartan Lie algebra. The deformation would be induced by non-commutativity. Same would occur also for the Kac-Moody variants of these algebras for which the set of generators contains only scaling operator L_0 as an additional generator. Quantum deformation would result from the replacement of complex rays with $\mathcal{N}$ rays, where $\mathcal{N}$ is the sub-factor.

   (c) The concrete interpretation for the Connes tensor power would be in terms of the fiber bundle structure $H = M^4_\pm \times CP_2 \to H/G_a \times G_b$, $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2, C) \times SU(3)$, which provides the proper formulation for the hierarchy of macroscopic quantum phases with a quantized value of Planck constant. Each sheet of the singular covering would represent single factor in Connes tensor power and single direct $SU(2)$ summand. This picture has an analogy with brane constructions of M-theory.

4. Could there exist a universal rational function giving rise to the algebraic closure of rationals?

One could wonder whether there exists a universal generalized rational function having all units of the algebraic closure of rationals as roots so that S_∞ would permute these roots. Most naturally it would be a ratio of infinite-degree polynomials.

With motivations coming from physics I have proposed that zeros of zeta and also the factors of zeta in product expansion of zeta are algebraic numbers. Complete story might be that non-trivial zeros of Zeta define the closure of rationals. A good candidate for this function is given by $(\xi(s)/\xi(1-s)) \times (s-1)/s)$, where $\xi(s) = \xi(1-s)$ is the symmetrized variant of ζ function having same zeros. It has zeros of zeta as its zeros and poles and product expansion in terms of ratios $(s-s_n)/(1-s+s_n)$ converges everywhere. Of course, this might be too simplistic and might give only the algebraic extension involving the roots of unity given by $exp(i\pi/n)$. Also products of these functions with shifts in real argument might be considered and one could consider some limiting procedure containing very many factors in the product of shifted ζ functions yielding the universal rational function giving the closure.

5. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is entire infinity of infinities in number theoretical sense. Any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

Be it as it may, the expressive power of HFF:s seem to be absolutely marvellous. Together with the notion of infinite rational and generalization of number concept they might unify both mathematics and physics!

1 Introduction

Langlands program [?]s an attempt to unify number theory and representation theory of groups and as it seems all mathematics. About related topics I know frustratingly little at technical level. Zeta functions and theta functions [?, ?] and more generally modular forms [?]re the connecting notion appearing both in number theory and in the theory of automorphic representations of reductive Lie groups. The fact that zeta functions have a key role in TGD has been one of the reasons for my personal interest.

The vision about TGD as a generalized number theory [?]ives good motivations to learn the basic ideas of Langlands program. I hasten to admit that I am just a novice with no hope becoming a master of the horrible technicalities involved. I just try to find whether the TGD framework could allow new physics inspired insights to Langlands program and whether the more abstract number theory relying heavily on the representations of Galois groups could have a direct physical counterpart in TGD Universe and help to develop TGD as a generalized number theory vision. After these apologies I however dare to raise my head a little bit and say aloud that mathematicians might get inspiration from physics inspired new insights.

The basic vision is that Langlands program could relate very closely to the unification of physics as proposed in TGD framework [?] TGD can indeed be seen both as infinite-dimensional geometry, as a generalized number theory involving several generalizations of the number concept, and as an algebraic approach to physics relying on the unique properties of hyper finite factors of type II_1 so that unification of mathematics would obviously fit nicely into this framework. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and sub-factors, and the notion of infinite prime, inspired a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

1.1 Langlands Program Very Briefly

Langlands program [?]tates that there exists a connection between number theory and automorphic representations of a very general class of Lie groups known as reductive groups (groups whose all representations are fully reducible). At the number theoretic side there are Galois groups characterizing extensions of number fields, say rationals or finite fields. Number theory involves also so called automorphic functions to which zeta functions carrying arithmetic information via their coefficients relate via so called Mellin transform $\sum_n a_n n^s \rightarrow \sum_n a_n z^n$ [?]

Automorphic functions, invariant under modular group $SL(2,Z)$ or subgroup $\Gamma_0(N) \subset SL(2,Z)$ consisting of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \quad , \quad c \; mod \; N = 0 \quad .$$

emerge also via the representations of groups $GL(2,R)$. This generalizes also to higher dimensional groups $GL(n,R)$. The dream is that all number theoretic zeta functions could be understood in terms of representation theory of reductive groups. The highly non-trivial outcome would be possibility to deduce very intricate number theoretical information from the Taylor coefficients of these functions.

Langlands program relates also to Riemann hypothesis and its generalizations. For instance, the zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [?] in this case Riemann hypothesis has been proven.

It must be emphasized that algebraic 1-dimensionality is responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [?] In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

One might also conjecture that Langlands duality for Lie groups reflects some deep duality on physical side. For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in the framework of YM theories and string models. In particular, Witten proposes that electric-magnetic duality which indeed relates gauge group and its dual, provides a physical correlate for the Langlands duality for Lie groups and could be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [?] Interestingly, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT. In this chapter it will be proposed that super-symmetry might correspond to the Langlands duality in TGD framework.

1.2 Questions

Before representing in more detail the TGD based ideas related to Langlands correspondence it is good to summarize the basic questions which Langlands program stimulates.

1.2.1 Could one give more concrete content to the notion of Galois group of algebraic closure of rationals?

The notion of Galois group for algebraic closure of rationals $Gal(\overline{Q}/Q)$ is immensely abstract and one can wonder how to make it more explicit? Langlands program adopts the philosophy that this group could be defined only via its representations. The so called automorphic representations constructed in terms of adeles. The motivation comes from the observation that the subset of adeles consisting of Cartesian product of invertible p-adic integers is a structure isomorphic with the maximal abelian subgroup of $Gal(\overline{Q}/Q)$ obtained by dividing $Gal(\overline{Q}/Q)$ with its commutator subgroup. Representations of finite abelian Galois groups are obtained as homomorphisms mapping infinite abelian Galois group to its finite factor group. In this approach the group $Gal(\overline{Q}/Q)$ remains rather abstract and adeles seem to define a mere auxiliary technical tool although it is clear that so called l-adic representations for Galois groups are are natural also in TGD framework.

This raises some questions.

1. Could one make $Gal(\overline{Q}/Q)$ more concrete? For instance, could one identify it as an infinite symmetric group S_∞ consisting of finite permutations of infinite number of objects? Could one imagine some universal polynomial of infinite degree or a universal rational function resulting as ratio of polynomials of infinite degree giving as its roots the closure of rationals?

2. S_∞ has only single normal subgroup consisting of even permutations and corresponding factor group is maximal abelian group. Therefore finite non-abelian Galois groups cannot be represented via homomorphisms to factor groups. Furthermore, S_{infty} has only infinite-dimensional non-abelian irreducible unitary representations as a simple argument to be discussed later shows.

What is highly non-trivial is that the group algebras of S_∞ and closely related braid group B_∞ define hyper-finite factors of type II_1 (HFF). Could sub-factors characterized by finite groups G allow to realize the representations of finite Galois groups as automorphisms p HFF? The interpretation would be in terms of "spontaneous symmetry breaking" $Gal(\overline{Q}/Q) \rightarrow G$. Could it be possible to get rid of adeles in this manner?

3. Could one find a concrete physical realization for the action of S_∞? Could the permuted objects be identified as strands of braid so that a braiding of Galois group to infinite braid group B_∞ would result? Could the outer automorphism action of Galois group on number theoretic braids defining the basic structure of quantum TGD allow to realize Galois groups physically as Galois groups of number theoretic braids associated with subset of algebraic points defined by the intersection of real and p-adic partonic 2-surface? The requirement that mathematics is able to represent itself physically would provide the reason for the fact that reality and various p-adicities intersect along subsets of rational and algebraic points only.

1.2.2 Could one understand the correspondences between the representations of finite Galois groups and reductive Lie groups?

Langlands correspondence involves a connection between the representations of finite-dimensional Galois groups and reductive Lie groups.

1. Could this correspondence result via an extension of the representations of finite groups in infinite dimensional Clifford algebra to those of reductive Lie groups identified for instance as groups defining sub-factors (any compact group can define a unique sub-factor)? If Galois groups and reductive groups indeed have a common representation space, it might be easier to understand Langlands correspondence.

2. Is there some deep difference between between general Langlands correspondence and that for $GL(2, F)$ and could this relate to the fact that subgroups of $SU(2)$ define sub-factors with quantized index $\mathcal{M} : \mathcal{N} \leq 4$.

3. McKay correspondence [**?**]elates finite subgroups of compact Lie groups to compact Lie group (say finite sub-groups of SU(2) to ADE type Lie-algebras or Kac-Moody algebras). TGD approach leads to a general heuristic explanation of this correspondence in terms of Jones inclusions and Connes tensor product. Could sub-factors allow to understand Langlands correspondence for general reductive Lie groups as both the fact that any compact Lie group can define a unique sub-factor and an argument inspired by McKay correspondence suggest.

1.2.3 Could one unify geometric and number theoretic Langlands programs?

There are two Langlands programs: algebraic Langlands program and geometric one [**?**]ne corresponding to ordinary number fields and function fields. The natural question is whether and how these approaches could be unified.

1. Could the discretization based on the notion of number theoretic braids induce the number theoretic Langlands from geometric Langlands so that the two programs could be unified by the generalization of the notion of number field obtained by gluing together reals with union of reals and various p-adic numbers fields and their extensions along common rationals and algebraics. Certainly the fusion of p-adics and reals to a generalized notion of number should be essential for the unification of mathematics.

2. Could the distinction between number fields and function fields correspond to two kinds of sub-factors corresponding to finite subgroups $G \subset SU(2)$ and SU(2) itself leaving invariant the elements of imbedded algebra? This would obviously generalize to imbeddings of Galois groups to arbitrary compact Lie group. Could gauge group algebras contra Kac Moody algebras be a possible physical interpretation for this. Could the two Langlands programs correspond to two kinds of ADE type hierarchies defined by Jones inclusions? Could minimal conformal field theories with finite number of primary fields correspond to algebraic

Langlands and full string theory like conformal field theories with infinite number of primary fields to geometric Langlands? Could this difference correspond to sub-factors defined by disrete groups and Lie groups?

3. Could the notion of infinite rational [?]e involved with this unification? Infinite rationals are indeed mapped to elements of rational function fields (also algebraic extensions of them) so that their interpretation as quantum states of a repeatedly second quantized arithmetic super-symmetric quantum field theory might provide totally new mathematical insights.

1.2.4 Is it really necessary to replace groups $GL(n,F)$ with their adelic counterparts?

If the group of invertible adeles is not needed or allowed then a definite deviation from Langlands program is implied. It would seem that multiplicative adeles (ideles) are not favored by TGD view about the role of p-adic number fields. The l-adic representations of p-adic Galois groups corresponding to single p-adic prime l emerge however naturally in TGD framework.

1. The 2×2 Clifford algebra could be easily replaced with its adelic version. A generalization of Clifford algebra would be in question and very much analogous to $GL(2,A)$ in fact. The interpretation would be that real numbers are replaced with adeles also at the level of imbedding space and space-time. This interpretation does not conform with the TGD based view about the relationship between real and p-adic degrees of freedom. The physical picture is that H is 8-D but has different kind of local topologies and that spinors are in some sense universal and independent of number field.

2. WCW spinors define a hyper-finite factor of type II_1. It is not clear if this interpretation continues to make sense if configuration space spinors (fermionic Fock space) are replaced with adelic spinors. Note that this generalization would require the replacement of the group algebra of S_{infty} with its adelic counterpart.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L4]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L5].

2 Basic Concepts And Ideas Related To The Number Theoretic Langlands Program

The basic ideas of Langlands program are following.

1. $Gal(\overline{Q}/Q)$ is a poorly understood concept. The idea is to define this group via its representations and construct representations in terms of group $GL(2,A)$ and more generally $GL(n,A)$, where A refers to adeles. Also representations in any reductive group can be considered. The so called automorphic representations of these groups have a close relationship to the modular forms [A7], which inspires the conjecture that n-dimensional representations of $Gal(\overline{Q}/Q)$ are in 1-1 correspondence with automorphic representations of $GL(n,A)$.

2. This correspondence predicts that the invariants characterizing the n-dimensional representations of $Gal(\overline{Q}/Q)$ *resp.* $GL(n,A)$ should correspond to each other. The invariants at Galois sides are the eigenvalues of Frobenius conjugacy classes Fr_p in $Gal(\overline{Q}/Q)$. The non-trivial implication is that in the case of l-adic representations the latter must be algebraic numbers. The ground states of the representations of $Gl(n,R)$ are in turn eigen states of so called Hecke operators $H_{p,k}$, $k = 1,..,n$ acting in group algebra of $Gl(n,R)$. The eigenvalues of Hecke operators for the ground states of representations must correspond to the eigenvalues of Frobenius elements if Langlands correspondence holds true.

3. The characterization of the K-valued representations of reductive groups in terms of Weil group W_F associated with the algebraic extension K/F allows to characterize the representations in terms of homomorphisms of Weil group to the Langlands dual $G_L(F)$ of $G(F)$.

2.1 Correspondence Between N-Dimensional Representations Of $Gal(\overline{F}/F)$ And Representations Of $Gl(N, A_F)$ In The Space Of Functions In $Gl(N, F)\backslash Gl(N, A_F)$

The starting point is that the maximal abelian subgroup $Gal(Q^{ab}/Q)$ of the Galois group of algebraic closure of rationals is isomorphic to the infinite product $\hat{Z} = \prod_p Z_p^\times$, where $Z_p^\times$ consists of invertible p-adic integers [A22].

By introducing the ring of adeles one can transform this result to a slightly different form. Adeles are defined as collections $((f_p)_{p \in P}, f_\infty)$, P denotes primes, $f_p \in Q_p$, and $f_\infty \in R$, such that $f_p \in Z_p$ for all p for all but finitely many primes p. It is easy to convince oneself that one has $A_Q = (\hat{Z} \otimes_Z Q) \times R$ and $Q^\times \backslash A_Q = \hat{Z} \times (R/Z)$. The basic statement of abelian class field theory is that abelian Galois group is isomorphic to the group of connected components of $F^\times \backslash A_F^\times$.

This statement can be transformed to the following suggestive statement:

1) *1-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(1, A_F)$ in the space of functions defined in $GL(1, F)\backslash GL(1, A_F)$.*

The basic conjecture of Langlands was that this generalizes to n-dimensional representations of $Gal(\overline{F}/F)$.

2) *The n-dimensional representations of $Gal(\overline{F}/F)$ correspond to representations of $GL(n, A_F)$ in the space of functions defined in $GL(n, F)\backslash GL(n, A_F)$.*

This relation has become known as Langlands correspondence.

It is interesting to relate this approach to that discussed in this chapter.

1. In TGD framework adeles do not seem natural although p-adic number fields and l-adic representations have a natural place also here. The new view about numbers is of course an essentially new element allowing geometric interpretation.

2. The irreducible representations of $Gal(\overline{F}, F)$ are assumed to reduce to those for its finite subgroup G. If $Gal(\overline{F}, F)$ is identifiable as S_∞, finite dimensional representations cannot correspond to ordinary unitary representations since, by argument to be represented later, their dimension is of order order $n \to \infty$ at least. Finite Galois groups can be however interpreted as a sub-group of outer automorphisms defining a sub-factor of $Gal(\overline{Q}, Q)$ interpreted as HFF. Outer automorphisms result at the limit $n \to \infty$ from a diagonal imbedding of finite Galois group to its n^{th} Cartesian power acting as automorphisms in S_∞. At the limit $n \to \infty$ the imbedding does not define inner automorphisms anymore. Physicist would interpret the situation as a spontaneous symmetry breaking.

3. These representations have a natural extension to representations of $Gl(n, F)$ and of general reductive groups if also realized as point-wise symmetries of sub-factors of HFF. Continuous groups correspond to outer automorphisms of group algebra of S_∞ not inducible from outer automorphisms of S_{infty}. That finite Galois groups and Lie groups act in the same representation space should provide completely new insights to the understanding of Langlands correspondence.

4. The l-adic representations of $Gal(\overline{Q}/Q)$ could however change the situation. The representations of finite permutation groups in R and in p-adic number fields $p < n$ are more complex and actually not well-understood [A9]. In the case of elliptic curves [A22] (say $y^2 = x^3 + ax + b$, a, b rational numbers with $4a^3 + 27b^2 \neq 0$) so called first etale cohomology group is Q_l^2 and thus 2-dimensional and it is possible to have 2-dimensional representations $Gal(\overline{Q}/Q) \to GL(2, Q_l)$. More generally, l-adic representations σ of of $Gal(\overline{F}/F) \to GL(n, \overline{Q_l})$ is assumed to satisfy the condition that there exists a finite extension $E \subset \overline{Q}_l$ such that σ factors through a homomorphism to $GL(n, E)$.

   Assuming $Gal(\overline{Q}/Q) = S_\infty$, one can ask whether l-adic or adelic representations and the representations defined by outer automorphisms of sub-factors might be two alternative manners to state the same thing.

2.1.1 Frobenius automorphism

Frobenius automorphism is one of the basic notions in Langlands correspondence. Consider a field extension K/F and a prime ideal v of F (or prime p in case of ordinary integers). v decomposes into a product of prime ideals of K: $v = \prod w_k$ if v is unramified and power of this if not. Consider unramified case and pick one w_k and call it simply w. Frobenius automorphisms Fr_v is by definition the generator of the the Galois group $Gal(K/w, F/v)$, which reduces to Z/nZ for some n.

Since the decomposition group $D_w \subset Gal(K/F)$ by definition maps the ideal w to itself and preserves F point-wise, the elements of D_w act like the elements of $Gal(O_K/w, O_F/v)$ (O_X denotes integers of X). Therefore there exists a natural homomorphism $D_w : Gal(K/F) \rightarrow Gal(O_K/w, O_F/v)$ ($= Z/nZ$ for some n). If the inertia group I_w identified as the kernel of the homomorphism is trivial then the Frobenius automorphism Fr_v, which by definition generates $Gal(O_K/w, O_F/v)$, can be regarded as an element of D_w and $Gal(K/F)$. Only the conjugacy class of this element is fixed since any w_k can be chosen. The significance of the result is that the eigenvalues of Fr_p define invariants characterizing the representations of $Gal(K/F)$. The notion of Frobenius element can be generalized also to the case of $Gal(\overline{Q}/Q)$ [A22]. The representations can be also l-adic being defined in $GL(n, E_l)$ where E_l is extension of Q_l. In this case the eigenvalues must be algebraic numbers so that they make sense as complex numbers.

Two examples discussed in [A22] help to make the notion more concrete.

1. For the extensions of finite fields $F = G(p,1)$ Frobenius automorphism corresponds to $x \rightarrow x^p$ leaving elements of F invariant.

2. All extensions of Q having abelian Galois group correspond to so called cyclotomic extensions defined by polynomials $P_N(x) = x^N + 1$. They have Galois group $(Z/NZ)^\times$ consisting of integers $k < n$ which do not divide n and the degree of extension is $\phi(N) = |Z/NZ^\times|$, where $\phi(n)$ is Euler function counting the integers $n < N$ which do not divide N. Prime p is unramified only if it does not divide n so that the number of "bad primes" is finite. The Frobenius equivalence class Fr_p in $Gal(K/F)$ acts as raising to p^{th} power so that the Fr_p corresponds to integer $p\ mod\ n$.

2.1.2 Automorphic representations and automorphic functions

In the following I want to demonstrate that I have at least tried to do my home lessons by trying to reproduce the description of [A22] for the route from automorphic adelic representations of $GL(2,R)$ to automorphic functions defined in upper half-plane.

1. Characterization of the representation

The representations of $GL(2,Q)$ are constructed in the space of smooth bounded functions $GL(2,Q)\backslash GL(2,A) \rightarrow C$ or equivalently in the space of $GL(2,Q)$ left-invariant functions in $GL(2,A)$. A denotes adeles and $GL(2,A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of compact subgroup K of $GL(2,A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best understanding $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c\ mod\ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL(2,Z_p)$. For ramified primes K_p consists of $SL(2,Z_p)$ matrices with $c \in p^{n_p}Z_p$. Here p^{n_p} is the divisor of conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ ,$$

where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) .$$

3. The center $A^\times$ of $GL(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \to C$ is a character providing a multiplicative representation of $A^\times$.

4. Also the so called cuspidality condition

$$\int_{Q\backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g)du = 0$$

is satisfied [A22]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

The groups $gl(2,R)$, $O(2)$ and $GL(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL(2,Q)\backslash GL(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is conductor [A22]. This means enormous simplification since one gets ride of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL(2,Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) , \quad c \ mod \ N = 0.$$

$_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL(2,Z)$ so that also $SL(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \to \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL(2,R)$ to upper half-plane $H_u = SL(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A22]. For the discrete series representation π giving square integrable representation in $SL(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \quad , \quad X_+ v_\infty = 0 \ .$$

The latter module is in turn generated by the lowest weight vector

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) v_\infty \ .$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau)$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) \quad = \quad \sum_{n=0}^{\infty} a_n q^n \ . \tag{2.1}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$ which which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

2.1.3 Hecke operators

Spherical Hecke algebra (which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL(2, Z_p)$ bi-invariant functions on $GL(2, Q_p)$ with respect to convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators. The normalizations can be chosen so that the second eigenvalue equals to unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A22].

The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators have also representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmueller parameters of algebraic Riemann surface are affected by absolute Galois group. This induces Sl(2g, Z) transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroup of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

2.2 Some Remarks About The Representations Of $Gl(N)$ And Of More General Reductive Groups

The simplest representations of $Gl(n,R)$ have the property that the Borel group B of upper diagonal matrices is mapped to diagonal matrices consisting of character ξ which decomposes to a product of characters χ_k associated with diagonal elements b_k of B defining homomorphism

$$b_k \rightarrow sgn(b)^{m(k)}|b_k|^{ia_k}$$

to unit circle if a_k is real. Also more general, non-unitary, characters can be allowed. The representation itself satisfies the condition $f(bg) = \chi(b)f(g)$. Thus n complex parameters a_k defining a reducible representation of $C^\times$ characterize the irreducible representation.

In the case of $GL(2,R)$ one can consider also genuinely two-dimensional discrete series representations characterized by only single continuous parameter and the previous example represented just this case. These representations are square integrable in the subgroup $SL(2,R)$. Their origin is related to the fact that the algebraic closure of R is 2-dimensional. The so called Weil group W_R which is semi-direct product of complex conjugation operation with $C^\times$ codes for this number theoretically. The 2-dimensional representations correspond to irreducible 2-dimensional representations of W_R in terms of diagonal matrices of $Gl(2,C)$.

In the case of $GL(n,R)$ the representation is characterized by integers n_k: $\sum n_k = n$ characterizing the dimensions $n_k = k = 1,2$ of the representations of W_R. For $Gl(n,C)$ one has $n_k = 1$ since Weil group W_C is obviously trivial in this case.

In the case of a general reductive Lie group G the homomorphisms of W_R to the Langlands dual G_L of G defined by replacing the roots of the root lattice with their duals characterize the automorphic representations of G.

The notion of Weil group allows also to understand the general structure of the representations of $GL(n,F)$ in $GL(n,K)$, where F is p-adic number field and K its extension. In this case Weil group is a semi-direct product of Galois group of $Gal(K/F)$ and multiplicative group $K^\times$. A very rich structure results since an infinite number of extensions exists and the dimensions of discrete series representations.

The deep property of the characterization of representations in terms of Weil group is functoriality. If one knows the homomorphisms $W_F \rightarrow G$ and $G \rightarrow H$ then the composite homomorphism defines an automorphic representation of H. This means that irreps of G can be passed to those of H by homomorphism [A21] .

3 TGD Inspired View About Langlands Program

In this section a general TGD inspired vision about Langlands program is described. If is of course just a bundle of physics inspired ideas represented in the hope that real professionals might find some inspiration. The fusion of real and various p-adic physics based on the generalization of the number concept, the notion of number theoretic braid, hyper-finite-factors of type II_1 and their sub-factors, and the notion of infinite prime, lead to a new view about how to represent finite Galois groups and how to unify the number theoretic and geometric Langlands programs.

3.1 What Is The Galois Group Of Algebraic Closure Of Rationals?

Galois group is essentially the permutation group for the roots of an irreducible polynomial. It is a subgroup of symmetric group S_n, where n is the degree of polynomial. One can also imagine the notion of Galois group $Gal(\overline{Q}/Q)$ for the algebraic closure of rationals but the concretization of this notion is not easy.

3.1.1 $Gal(\overline{Q}/Q)$ as infinite permutation group?

The maximal abelian subgroup of $Gal(\overline{Q}/Q)$, which is obtained by dividing with the normal subgroup of even permutations, is identifiable as a product of multiplicative groups $Z_p^\times$ of invertible p-adic integers $n = n_0+pZ$, $n_0 \in \{1,..p-1\}$ for all p-adic primes and can be understood reasonably via its isomorphism to the product $\hat{Z} = \prod_p Z_p$ of multiplicative groups Z_p of invertible p-adic integers, one factor for each prime p [A21, A5, A22].

Adeles [A1] are identified as the subring of $(\hat{Z} \otimes_Z Q) \times R$ containing only elements for which the elements of Q_p belong to Z_p except for a finite number of primes so that the number obtained can be always represented as a product of element of $\hat{Z}$ and point of circle R/Z: $A = \hat{Z} \times R/Z$. Adeles define a multiplicative group $A^\times$ of ideles and $GL(1, A)$ allow to construct representations $Gal(Q^{ab}/Q)$.

It is much more difficult to get grasp on $Gal(\overline{Q}/Q)$. The basic idea of Langlands program is that one should try to understand $Gal(\overline{Q}/Q)$ through its representations rather than directly. The natural hope is that n-dimensional representations of $Gal(\overline{Q}/Q)$ could be realized in $GL(n, A)$.

1. $Gal(\overline{Q}/Q)$ as infinite symmetric group?

One could however be stubborn and try a different approach based on the direct identification $Gal(\overline{Q}/Q)$. The naive idea is that $Gal(\overline{Q}/Q)$ could in some sense be the Galois group of a polynomial of infinite degree. Of course, for mathematical reasons also a rational function defined as a ratio of this kind of polynomials could be considered so that the Galois group could be assigned to both zeros and poles of this function. In the generic case this group would be an infinite symmetric group S_∞ for an infinite number of objects containing only permutations for subsets containing a finite number of objects. This group could be seen as the first guess for $Gal(\overline{Q}/Q)$.

S_∞ can be defined by generators e_m representing permutation of $m^{th}$and $(m+1)^{th}$ object satisfying the conditions

$$\begin{aligned} e_m e_m &= e_n e_m \ \text{ for } \ |m-n| > 1 , \\ e_n e_{n+1} e_n &= e_n e_{n+1} e_n e_{n+1} \ \text{ for } \ n = 1, ..., n-2 \ , \\ e_n^2 &= 1 \ . \end{aligned} \tag{3.1}$$

By the definition S_∞ can be expected to possess the basic properties of finite-dimensional permutation groups. Conjugacy classes, and thus also irreducible unitary representations, should be in one-one correspondence with partitions of n objects at the limit $n \to \infty$. Group algebra defined by complex functions in S_∞ gives rise to the unitary complex number based representations and the smallest dimensions of the irreducible representations are of order n and are thus infinite for S_∞. For representations based on real and p-adic number based variants of group algebra situation is not so simple but it is not clear whether finite dimensional representations are possible.

S_n and obviously also S_∞ allows an endless number of realizations since it can act as permutations of all kinds of objects. Factors of a Cartesian and tensor power are the most obvious possibilities for the objects in question. For instance, S_n allows a representation as elements of rotation group $SO(n)$ permuting orthonormalized unit vectors e_i with components $(e_i)^k = \delta_i^k$. This induces also a realization as spinor rotations in spinor space of dimension $D = 2^{d/2}$.

2. Group algebra of S_∞ as HFF

The highly non-trivial fact that the group algebra of S_∞ is hyper-finite factor of type II_1 (HFF) [A4] suggests a representation of permutations as permutations of tensor factors of HFF interpreted as an infinite power of finite-dimensional Clifford algebra. The minimal choice for the finite-dimensional Clifford algebra is $M^2(C)$. In fermionic Fock space representation of infinite-dimensional Clifford algebra e_i would induce the transformation $(b^\dagger_{m,i}, b^\dagger_{m,i+1}) \to (b^\dagger_{m,i+1}, b^\dagger_{m,i})$. If the index m is lacking, the representation would reduce to the exchange of fermions and representation would be abelian.

3. Projective representations of S_∞ as representations of braid group B_∞

S_n can be extended to braid group B_n by giving up the condition $e_i^2 = 1$ for the generating permutations of the symmetric group. Generating permutations are represented now as homotopies exchanging the neighboring strands of braid so that repeated exchange of neighboring strands induces a sequence of twists by π. Projective representations of S_∞ could be interpreted as representations of B_∞. Note that odd and even generators commute mutually and for unitary representations either of them can be diagonalized and are represented as phases $exp(i\phi)$ for braid group. If $exp(i\phi)$ is not a root of unity this gives effectively a polynomial algebra and the polynomials subalgebras of these phases might provide representations for the Hecke operators also forming commutative polynomial algebras.

The additional flexibility brought in by braiding would transform Galois group to a group analogous to homotopy group and could provide a connection with knot and link theory [A23] and topological quantum field theories in general [A34]. Finite quantum Galois groups would generate braidings and a connection with the geometric Langlands program where Galois groups are replaced with homotopy groups becomes suggestive [A22, A20].

4. What does one mean with S_∞?

There is also the question about the meaning of S_∞. The hierarchy of infinite primes suggests that there is an entire infinity of infinities in number theoretical sense. After all, any group can be formally regarded as a permutation group. A possible interpretation would be in terms of algebraic closure of rationals and algebraic closures for an infinite hierarchy of polynomials to which infinite primes can be mapped. The question concerns the interpretation of these higher Galois groups and HFFs. Could one regard these as local variants of S_∞ and does this hierarchy give all algebraic groups, in particular algebraic subgroups of Lie groups, as Galois groups so that almost all of group theory would reduce to number theory even at this level?

3.1.2 The group algebra of Galois group of algebraic closure of rationals as hyper-finite factor of type II_1

The most natural framework for constructing unitary irreducible representations of Galois group is its group algebra. In the recent case this group algebra would be that for S_∞ or B_∞ if braids are allowed. What puts bells ringing is that the group algebra of S_∞ is a hyper-finite factor of type II_1 isomorphic as a von Neumann algebra to the infinite-dimensional Clifford algebra [A4], which in turn is the basic structures of quantum TGD whose localized version might imply entire quantum TGD. The very close relationship with the braid group makes it obvious that same holds true for corresponding braid group B_∞. Indeed, the group algebra of an infinite discrete group defines under very general conditions HFF. One of these conditions is so called amenability [A2]. This correspondence gives hopes of understanding the Langlands correspondence between representations of discrete Galois groups and the representations of $GL(n, F)$ (more generally representations of reductive groups).

Thus it seems that WCW spinor s (fermionic Fock space) could naturally define a finite-dimensional spinor representation of finite-dimensional Galois groups associated with the number theoretical braids. Inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors realize the notion of finite measurement resolution and give rise to finite dimensional representations of finite groups G leaving elements of $\mathcal{N}$ invariant. An attractive idea is that these groups are identifiable as Galois groups.

The identification of the action of G on $\mathcal{M}$ as homomorphism $G \to Aut(\mathcal{M})$ poses strong conditions on it. This is discussed in the thesis of Jones [C1] which introduces three algebraic invariants for the actions of finite group in hyperfinite-factors of type II_1, denoted by $\mathcal{M}$ in the sequel. In general the action reduces to inner automorphism of $\mathcal{M}$ for some normal subgroup $H \subset G$: this group is one of the three invariants of G action. In general one has projective representation for H so that one has $u_{h_1}u_{h_2} = \mu(h_1, h_2)u_{h_1h_2}$, where $\mu(h_1)$ is a phase factor which satisfies cocycle conditions coming from associativity.

1. The simplest action is just a unitary group representation for which $g \in G$ is mapped to a unitary operator u_g in $\mathcal{M}$ acting in $\mathcal{M}$ via adjoint action $m \to u_g m u_g^\dagger = Ad(u_g)m$. In this case one has $H = G$. In this case the fixed point algebra does not however define a factor and there is no natural reduction of the representations of $Gal(\overline{Q}/Q)$ to a finite subgroup.

2. The exact opposite of this situation outer action of G mean $H = \{e\}$. All these actions are conjugate to each other. This gives gives rise to two kinds of sub-factors and two kinds of representations of G. Both actions of Galois group could be realized either in the group or braid algebra of $Gal(\overline{Q}/Q)$ or in infinite dimensional Clifford algebra. In neither case the action be inner automorphic action $u \to gug^\dagger$ as one might have naively expected. This is crucial for circumventing the difficulty caused by the fact that $Gal(\overline{Q}/Q)$ identified as S_∞ allows no finite-dimensional complex representation.

3. The first sub-factor is $\mathcal{M}^G \subset \mathcal{M}$ corresponding, where the action of G on $\mathcal{M}$ is outer. Outer action defines a fixed point algebra for all finite groups G. For $D = \mathcal{M} : \mathcal{N} < 4$ only finite subgroups $G \subset SU(2)$ would be represented in this manner. The index identifiable as the fractal dimension of quantum Clifford algebra having $\mathcal{N}$ as non-abelian coefficients is $D = 4cos^2(\pi/n)$. One can speak about quantal representation of Galois group. The image of Galois group would be a finite subgroup of $SU(2)$ acting as spinor rotations of quantum Clifford algebra (and quantum spinors) regarded as a module with respect to the included algebra invariant under inner automorphisms. These representations would naturally correspond to 2-dimensional representations having very special role for the simple reason that the algebraic closure of reals is 2-dimensional.

4. Second sub-factor is isomorphic to $\mathcal{M}^G \subset (\mathcal{M} \otimes L(H))^G$. Here $L(H)$ is the space of linear operators acting in a finite-dimensional representation space H of a unitary irreducible representation of G. The action of G is a tensor product of outer action and adjoint action. The index of the inclusion is $dim(H)^2 \geq 1$ [A32] so that the representation of Galois group can be said to be classical (non-fractal).

5. The obvious question is whether and in what sense the outer automorphisms represent Galois subgroups. According to [C1] the automorphisms belong to the completion of the group of inner automorphisms of HFF. Identifying HFF as group algebra of S_∞, the interpretation would be that outer automorphisms are obtained as diagonal embeddings of Galois group to $S_n \times S_n \times ....$ If one includes only a finite number of these factors the outcome is an inner automorphisms so that for all finite approximations inner automorphisms are in question. At the limit one obtains an automorphisms which does not belong to S_∞ since it contains only finite permutations. This identification is consistent with the identification of the outer automorphisms as diagonal embedding of G to an infinite tensor power of sub-Clifford algebra of Cl_∞.

This picture is physically very appealing since it means that the ordering of the strands of braid does not matter in this picture. Also the reduction of the braid to a finite number theoretical braid at space-time level could be interpreted in terms of the periodicity at quantum level. From the point of view of physicist this symmetry breaking would be analogous to a spontaneous symmetry breaking above some length scale L. The cutoff length scale L would correspond to the number N of braids to which finite Galois group G acts and corresponds also to some p-adic length scale.

One might hope that the emergence of finite groups in the inclusions of hyper-finite factors could throw light into the mysterious looking finding that the representations of finite Galois groups and unitary infinite-dimensional automorphic representations of $GL(n, R)$ are correlated by the connection between the eigenvalues of Frobenius element Fr_p on Galois side and eigenvalues of commuting Hecke operators on automorphic side. The challenge would be to show that the action of Fr_p as outer automorphism of group algebra of S_∞ or B_∞ corresponds to Hecke algebra action on configuration space spinor fields or in modular degrees of freedom associated with partonic 2-surface.

3.1.3 Could there exist a universal rational function having $Gal(\overline{Q}/Q)$ as the Galois group of its zeros/poles?

The reader who is not fascinated by the rather speculative idea about a universal rational function having $Gal(\overline{Q}/Q)$ as a permutation group of its zeros and poles can safely skip this subsection since it will not be needed anywhere else in this chapter.

1. Taking the idea about permutation group of roots of a polynomial of infinite order seriously, one could require that the analytic function defining the Galois group should behave like a polynomial or a rational function with rational coefficients in the sense that the function should have an everywhere converging expansion in terms of products over an infinite number of factors $z - z_i$ corresponding to the zeros of the numerator and possible denominator of a rational function. The roots z_i would define an extension of rationals giving rise to the entire algebraic closure of rationals. This is a tall order and the function in question should be number theoretically very special.

2. One can speculate even further. TGD has inspired the conjecture that the non-trivial zeros $s_n = 1/2 + iy_n$ of Riemann zeta [A14] (assuming Riemann hypothesis) are algebraic numbers and that also the numbers p^{s_n}, where p is any prime, and thus local zeta functions serving as multiplicative building blocks of ζ have the same property [K17]. The story would be perfect if these algebraic numbers would span the algebraic closure of rationals.

   The symmetrized version of Riemann zeta defined as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfying the functional equation $\xi(s) = \xi(1-s)$ and having only the trivial zeros could appear as a building block of the rational function in question. The function

$$f(s) = \frac{\xi(s)}{\xi(s+1)} \times \frac{s-1}{s}$$

   has non-trivial zeros s_n of ζ as zeros and their negatives as $-s_n$ as poles. There are no other zeros since trivial zeros as well as the zeros at $s = 0$ and $s = 1$ are eliminated. Using Stirling formula one finds that $\xi(s)$ grows as s^s for real values of $s \to \infty$. The growths of the numerator and denominator compensate each other at this limit so that the function approaches constant equal to one for $Re(s) \to \infty$.

   If $f(s)$ indeed behaves as a rational function whose product expansion converges everywhere it can be expressed in terms of its zeros and poles as

$$\begin{aligned} f(s) &= \prod_{n>0} A_n(s) \ , \\ A_n &= \frac{(s-s_n)(s-\overline{s}_n)}{(1+s-s_n)(1+s-\overline{s}_n)} \ . \end{aligned} \tag{3.2}$$

   The product expansion seems to converge for any finite value of s since the terms A_n approach unity for large values of $|s_n| = |1/2 + iy_n|$. $f(s)$ has $s_n = 1/2 + iy_n$ indeed has zeros and $s_n = -1/2 + iy_n$ as poles.

3. This proposal might of course be quite too simplistic. For instance, one might argue that the phase factors p^{iy} associated with the non-trivial zeros give only roots of unity multiplied by Gaussian integers. One can however imagine more complex functions obtained by forming products of $f(s)$ with its shifted variants $f(s+\Delta)$ with algebraic shift Δ in, say, the interval $[-1/2, 1/2]$. Some kind of limiting procedure using a product of this kind of functions might give the desired universal function.

3.2 Physical Representations Of Galois Groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by discretization of continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

3.2.1 Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [K6, K5, K19] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [K6, K5]. The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire WCW of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

1. It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

   One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [A16] the non-trivial normal subgroups are fintary alternating subgroup A_∞ and finitary symmetric group consisting if finitary permutations. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

2. The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [A13] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal imbedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

3.2.2 About the detailed definition of number theoretic braids

The work with hyper-finite factors of type II_1 (HFFs) combined with experimental input led to the notion of hierarchy of Planck constants interpreted in terms of dark matter [K10]. The hierarchy is realized via a generalization of the notion of imbedding space obtained by gluing infinite number of its variants along common lower-dimensional quantum critical sub-manifolds. These variants of imbedding space are characterized by discrete subgroups of SU(2) acting in M^4 and CP_2 degrees of freedom as either symmetry groups or homotopy groups of covering. Among other things this picture implies a general model of fractional quantum Hall effect.

3.2.3 The identification of number theoretic braids

To specify number theoretical criticality one must specify some physically preferred coordinates for $M^4 \times CP_2$ or at least $\delta M^4_{\pm} \times CP_2$. Number theoretical criticality requires that braid belongs to the algebraic intersection of real and p-adic variants of the partonic 2-surface so that number theoretical criticality reduces to a finite number of conditions. This is however not strong enough condition and one must specify further physical conditions.

1. What are the preferred coordinates for H?

What are the preferred coordinates of M^4 and CP_2 in which algebraicity of the points is required is not completely clear. The isometries of these spaces must be involved in the identification as well as the choice of quantization axes for given CD. In [K14] I have discussed the natural preferred coordinates of M^4 and CP_2.

1. For M^4 linear M^4 coordinates chosen in such manner that $M^2 \times E^2$ decomposition fixing quantization axes is respected are very natural. This restricts the allowed Lorentz transformations to Lorentz boosts in M^2 and rotations in E^2 and the identification of M^2 as hyper-complex plane fixes time coordinate uniquely. E^2 coordinates are fixed apart from the action of $SO(2)$ rotation. The rationalization of trigonometric functions of angle variables allows angles associated with Pythagorean triangles as number theoretically simplest ones.

2. The case of CP_2 is not so easy. The most obvious guess in the case of CP_2 the coordinates corresponds to complex coordinates of CP_2 transforming linearly under $U(2)$. The condition that color isospin rotations act as phase multiplications fixes the complex coordinates uniquely. Also the complex coordinates transforming linearly under $SO(3)$ rotations are natural choice for S^2 ($r_M = constant$ sphere at $\delta M^4_{\pm}$).

3. Another manner to deal with CP_2 is to apply number M^8-H duality. In M^8 CP_2 corresponds to E^4 and the situation reduces to linear one and $SO(4)$ isometries help to fix preferred coordinate axis by decomposing E^4 as $E^4 = E^2 \times E^2$. Coordinates are fixed apart the action of the commuting $SO(2)$ sub-groups acting in the planes E^2. It is not clear whether the images of algebraic points of E^4 at space-time surface are mapped to algebraic points of CP_2.

2. The identification of number theoretic braids

The identification of number theoretic braids is not by no means a trivial task [K22, K16]. As a matter fact, there are several alternative identifications and it seems that all of them are needed. Consider first just braids without the attribute "number theoretical".

1. Braids could be identified as lifts of the projections of X^3_l to the quantum critical sub-manifolds M^2 or S^2_I, $i = I, II$, and in the generic case consist of 1-dimensional strands in X^3_l These sub-manifolds are obviously in the same role as the plane to which the braid is projected to obtain a braid diagram. This requires that a unique identification of the slicing of space-time surfaces by 3-surfaces.

2. Braid points are always quantum critical against the change of Planck constant so that TQFT like theory characterizes the freedom remaining intact at quantum criticality. Quantum criticality in this sense need not have anything to do with the quantum criticality in the sense

that the second variation of Kähler action vanishes -at least for the variations representing dynamical symmetries in the sense that only the inner product $\int (\partial L_D/\partial h^k_\alpha)\delta h^k d^4x$ (L_D denotes Kähler-Dirac Lagrangian) without the vanishing of the integrand. This criticality leads to a generalization of the conceptual framework of Thom's catastrophe theory [K22].

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$.

3. It is not clear whether these three braids form some kind of trinity so that one of them is enough to formulate the theory or whether all of them are needed. Note also that one has quantum superposition over CDs corresponding to different choices of M^2 and the pair formed by S^2_I and S^2_{II} (note that the spheres are not independent if both appear). Quantum measurement however selects one of these choices since it defines the choice of quantization axes.

4. One can consider also more general definition. The extrema of Kähler magnetic field strength defined as coordinate invariant $\epsilon^{\alpha\beta}J_{\alpha\beta}$ at X^2 define in natural manner a discrete set of points defining the nodes of symplectic triangulation: note that this involves division with metric determinant in preferred coordinates. This set of extremals is same for all deformations of X^3_l allowed in the functional integral over symplectic group although the positions of points change. For preferred symplectically invariant light-like coordinate of X^3_l braid results. Also now geodesic spheres and M^2 would define the counterpart of the plane to which the braids are projected.

5. A physically attractive realization of the braids - and more generally- of slicings of space-time surface by 3-surfaces and string world sheets, is discussed in [K11] by starting from the observation that TGD defines an almost topological QFT of braids, braid cobordisms, and 2-knots. The boundaries of the string world sheets at the space-like 3-surfaces at boundaries of CDs and wormhole throats would define space-like and time-like braids uniquely.

   The idea relies on a rather direct translation of the notions of singular surfaces and surface operators used in gauge theory approach to knots [A24] to TGD framework. It leads to the identification of slicing by three-surfaces as that induced by the inverse images of $r = constant$ surfaces of CP_2, where r is $U(2)$ invariant radial coordinate of CP_2 playing the role of Higgs field vacuum expectation value in gauge theories. $r = \infty$ surfaces correspond to geodesic spheres and define analogs of fractionally magnetically charged Dirac strings identifiable as preferred string world sheets. The union of these sheets labelled by subgroups $U(2) \subset SU(3)$ would define the slicing of space-time surface by string world sheets. The choice of $U(2)$ relates directly to the choice of quantization axes for color quantum numbers characterizing CD and would have the choice of braids and string world sheets as a space-time correlate. $r = \infty$ points correspond to three homologically non-trivial geodesic spheres S^2 analogous to North and South poles of CP_2 and the projections to M^4 and S^2 define braid projections. Braid strands could be interpreted as orbits of Kähler charged particle in Kähler magnetic field and enclosing fractional Kähler flux.

   The beauty of this identification is that one starts from braids at the ends of space-time surface partonic 2-surfaces at boundaries of CD and from intersection of braid points and determines space-time surface and string world sheets from this data in accordance with holography and quantum classical correspondence. This picture conforms also with the recent view about Kähler-Dirac equation for which the construction of solutions leads to the notion of braid too.

Number theoretic braids would be braids which are number theoretically critical. This means that the points of braid in preferred coordinates are algebraic points so that they can be regarded as being shared by real partonic 2-surface and its p-adic counterpart obeying same algebraic equations.

3.2.4 Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyper-finite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [A25]. Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [A25] !

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [A2], group defines a unique sub-factor by outer action [A25]. It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

3.2.5 Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands [B1] [A22, A30], whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. Number theoretic braids belong to the intersection of real and p-adic partonic surface

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [K6]).

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r,z,w) = 0$. By substituting w as a polynomial $w = Q(z,r)$ of z and r this gives polynomial equation $P(r,z,Q(z,r)) = 0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and WCW spinor s.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y - x^2 = 0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Kähler-Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the Kähler-Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [K6, K1]. One can solve the roots of the the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

1. The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.

2. For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X^2_k of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X^2_k. This means separation of modular degrees of freedom to those assignable to X^2_k and to "center of mass" modular degrees of freedom assignable to the boundaries between X^2_k. Only the piece X^2_k associated with the k^{th} braid would be affected non-trivially by the Galois group of braid. The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [A7] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [A21, A22, A8]. This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [A8]. Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \to (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a manner that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [K4] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

3.2.6 Hierarchy of Planck constants and dark matter and generalization of imbedding space

Second hierarchy of candidates for Galois groups is based on the generalization of the notion of the imbedding space $H = M^4 \times CP_2$, or rather the spaces $H_\pm = M^4_\pm \times CP_2$ defining future and past light-cones inside H [K10]. This generalization is inspired by the quantization of Planck constant explaining dark matter as a hierarchy of macroscopically quantum coherent phases and by the requirement that sub-factors have a geometric representation at the level of the imbedding space and space-time (quantum-classical correspondence).

Galois groups could also correspond to finite groups $G_a \times G_b \subset SU(2) \times SU(2)$ ⊂SL(2, C)×$SU(3)$. These groups act as covering symmetries for the sectors of the imbedding space, which can be regarded as singular $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm/G_a \times G_b$ bundles containing orbifold points (fixed points of $G_a \times G_b$ or either of them. The copies of H with same G_a or G_b are glued together along $M^4_\pm$ or CP_2 factor and along common orbifold points left fixed by G_b or G_a. The group $G_a \times G_b$ plays both the role of both Galois group and homotopy group.

There are good reasons to expect that both these Galois groups and those associated with number theoretic braids play a profound role in quantum TGD based description of dark matter as macroscopically quantum coherent phases. For instance, G_a would appear as symmetry group of dark matter part of bio-molecules in TGD inspired biology [L2].

3.2.7 Question about representations of finite groups

John Baez made an interesting question in n-Category-Cafe [A15]. The question reads as follows:

Is every representation of every finite group definable on the field Q^{ab} obtained by taking the field Q of rational numbers and by adding all possible roots of unity?

Since every finite group can appear as Galois group the question translates to the question whether one can represent all possible Galois groups using matrices with elements in Q^{ab}.

This form of question has an interesting relation to Langlands program. By Langlands conjecture the representations of the Galois group of algebraic closure of rationals can be realized in the space of functions defined in $GL(n,F)\backslash GL(n,Gal(Q^{ab}/Q))$, where $Gal(Q^{ab}/Q)$ is the maximal Abelian subgroup of the Galois group of the algebraic closure of rationals. Thus one has group algebra associated with the matrix group for which matrix elements have values in $Gal(Q^{ab}/Q)$. Something by several orders of more complex than matrices having values in Q^{ab}.

Suppose that Galois group of algebraic numbers can be regarded as the permutation group S_∞ of infinite number of objects generated by permutations for finite numbers of objects and that its physically interesting representations reduce to the representations of finite Galois groups G with element $g \in G$ represented as infinite product $g \times g \times ...$ belonging to the completion of S_∞ and thus to the completion of its group algebra identifiable as hyper-finite factor of type II_1. This would mean number theoretic local gauge invariance in the sense that all elements of S_∞ would leave physical states invariant whereas G would correspond to global gauge transformations. These tensor factors would have as space-time correlates number theoretical braids allowing to represent the action of G.

What this has then to do with John's question and Langlands program? S_∞ contains any finite group G as a subgroup. If all the representations of finite-dimensional Galois groups could be realized as representations in $Gl(n,Q^{ab})$, same would hold true also for the proposed symmetry breaking representations of the completion of S_∞ reducing to the representations of finite Galois groups. There would be an obvious analogy with Langlands program using functions defined in the space $Gl(n,Q)\backslash Gl(n,Gal(Q^{ab}/Q))$. Be as it may, mathematicians are able to work with incredibly abstract objects! A highly respectful sigh is in order!

3.3 What Could Be The TGD Counterpart For The Automorphic Representations?

The key question in the following is whether quantum TGD could act as a general math machine allowing to realize any finite-dimensional manifold and corresponding function space in terms of configuration space spinor fields and whether also braided representations of Galois groups accompanying the braiding could be associated naturally with this kind of representations.

3.3.1 Some general remarks

Before getting to the basic idea some general remarks are in order.

1. WCW spinor fields would certainly transform according to a finite-dimensional and therefore non-unitary representation of $SL(2,C)$ which is certainly the most natural group involved and should relate to the fact that Galois groups representable as subgroups of SU(2) acting as rotations of 3-dimensional space correspond to sub-factors with $\mathcal{M}:\mathcal{N}\leq 4$.
2. Also larger Lie groups can be considered and diagonal imbeddings of Galois groups would be naturally accompanied by diagonal imbeddings of compact and also non-compact groups acting on the decomposition of infinite-dimensional Clifford algebra Cl_∞ to an infinite tensor power of finite-dimensional sub-Clifford algebra of form $M(2,C)^n$.
3. The basic difference between Galois group representation and corresponding Lie group representations is that the automorphisms in the case of discrete groups are automorphisms of S_∞ or B_∞ whereas for Lie groups the automorphisms are in general automorphisms of group algebra of S_∞ or B_∞. This could allow to understand the correspondence between discrete groups and Lie groups naturally.
4. Unitary automorphic representations are infinite-dimensional and require group algebra of $GL(n,F)$. Therefore WCW spinors - to be distinguished from WCW spinor fields- cannot realize them. WCW spinor field might allow the realization of these infinite-dimensional representations if groups themselves allow a finite-dimensional geometric realization of groups. Are this kind of realizations possible? This is the key question.

3.3.2 Could TGD Universe act as a universal math machine?

The questions are following. Could one find a representations of both Lie groups and their linear and non-linear representation spaces -and even more - of any manifold representable as a sub-manifold of some linear space in terms of braid points at partonic 2-surfaces X^2? What about various kinds of projective spaces and coset spaces? Can one construct representations of corresponding function spaces in terms of WCW spinor fields? Can one build representations of parameter groups of Lie groups as braided representations defined by the orbits of braid points in X^3_l? Note that this would assign to the representations of closed paths in the group manifold a representation of braid group and Galois group of the braid and might make it easier to understand the Langlands correspondence.

A professional mathematician - if she still continues reading - might regard the following argument as rather pathetic poor man's argument but I want to be honest and demonstrate my stupidity openly.

1. The n braid points represent points of $\delta H=\delta M^4_\pm\times CP_2$ so that braid points represent a point of $7n$-dimensional space $\delta H^n/S_n$. $\delta M^4_\pm$ corresponds to E^3 with origin removed but $E^{2n}/S_n=C^n/S_n$ can be represented as a sub-manifold of $\delta M^4_\pm$. This allows to almost-represent both real and complex linear spaces. E^2 has a unique identification based on $M^4=M^2\times E_2$ decomposition required by the choice of quantization axis. One can also represent the spaces $(CP_2)^n/S_n$ in this manner.

2. The first - and really serious - problem is caused by the identification of the points obtained by permuting the n coordinates: this is of course what makes possible the braiding since braid group is the fundamental group of $(X^2)^n$. Could the quantum numbers at the braid points act as markers distinguishing between them so that one would effectively have E^{2n}? Could the fact that the representing points are those of imbedding space rather than X^2 be of significance? Second - less serious - problem is that the finite size of CD allows to represent only a finite region of E^2. On the other hand, ideal mathematician is a non-existing species and even non-ideal mathematician can imagine the limit at which the size of CD becomes infinite.

3. Matrix groups can be represented as sub-manifolds of linear spaces defined by the general linear group $Gl(n,R)$ and $Gl(n,C)$. In the p-adic pages of the imbedding space one can realize also the p-adic variants of general linear groups. Hence it is possible to imbed any real (complex) Lie group to E^{2n} (C^n), if n is chosen large enough.

4. WCW spinor fields restricted to the linear representations spaces or to the group itself represented in this manner would allow to realize as a special case various function spaces, in particular groups algebras. If WCW spinor fields satisfy additional symmetries, projective spaces and various coset spaces can be realized as effective spaces. For instance CP_2 could be realized effectively as $SU(3)/U(2)$ by requiring $U(2)$ invariance of the WCW spinor fields in $SU(3)$ or as C^3/Z by requiring that WCW spinor field is scale invariant. Projective spaces might be also realized more concretely as imbeddings to $(CP_2)^n$.

5. The action of group element $g = exp(Xt)$ belonging to a one-parameter sub-group of a non-compact linear group in a real (complex) linear representation space of dimension m could be realized in a subspace of E^{2n}, $m < 2n$ (C^n, $m \leq n$), as a flow in X^3_l taking the initial configuration of points of representation space to the final configuration. Braid strands - the orbits of points p_i defining the point p of the representation manifold under the action of one-parameter subgroup- would correspond to the points $exp(Xu)(p)$, $0 \leq u \leq t$. Similar representation would work also in the group itself represented in a similar manner.

6. Braiding in X^3_l would induce a braided representation for the action of the one parameter subgroup. This representation is not quite the same thing as the automorphic representation since braiding is involved. Also trivial braid group representation is possible if the representation can be selected freely rather than being determined by the transformation properties of fermionic oscillator operator basis in the braiding.

7. An important prerequisite for math machine property is that the wave function in the space of light-like 3-surfaces with fixed ends can be chosen freely. This is the case since the degrees of freedom associate with the interior of light-like 3-surface X^3_l correspond to zero modes assignable to Kac-Moody symmetries [K7, K19]. Dicretization seems however necessary since functional integral in these degrees of freedom is not-well defined even in the real sense and even less so p-adically. This conforms with the fact that real world mathematical representations are always discrete. Quantum classical correspondence suggests the dynamics represented by X^3_l correlates with the quantum numbers assigned with X^2 so that Boolean statements represented in terms of Fermionic Fock states would be in one-one correspondence with these wave functions.

Besides representing mathematical structures this kind of math machine would be able to perform mathematical deductions. The fermionic part of the state zero energy state could be interpreted as a quantum super-position of Boolean statement $A_i \rightarrow B_i$ representing various instances of the general rule $A \rightarrow B$. Only the statements consistent with fundamental conservation laws would be possible. Quantum measurements performed for both positive and negative energy parts of the state would produce statements. Performing the measurement of the observable $O(A \rightarrow B)$ would produce from a given state a zero energy state representing statement $A \rightarrow B$. If the measurement of observable $O(C \rightarrow D)$ affects this state then the statement $(A \rightarrow B) \rightarrow (C \rightarrow D)$ cannot hold true. For $A = B$ the situation reduces to simpler logic where one tests truth value of statements of form $A \rightarrow B$. By increasing the number of instances in the quantum states generalizations of the rule can be tested.

3.4 Super-Conformal Invariance, Modular Invariance, And Langlands Program

The geometric Langlands program [A22, A20] deals with function fields, in particular the field of complex rational analytic functions on 2-dimensional surfaces. The sheaves in the moduli spaces of conformal blocks characterizing the n-point functions of conformal field theory replaces automorphic functions coding both arithmetic data and characterizing the modular representations of $GL(n)$ in number theoretic Langlands program [A22]. These moduli spaces are labelled both by moduli characterizing the conformal equivalence class of 2-surface, in particular the positions of punctures, in TGD framework the positions of strands of number theoretic braids, as well as the moduli related to the Kac-Moody group involved.

3.4.1 Transition to function fields in TGD framework

According to [A22] conformal field theories provide a very promising framework for understanding geometric Langlands correspondence.

1. That the function fields on 2-D complex surfaces would be in a completely unique role mathematically fits nicely with the 2-dimensionality of partons and well-defined stringy character of anti-commutation relations for induced spinor fields. According to [A22] there are not even conjectures about higher dimensional function fields.

2. There are very direct connections between hyper-finite factors of type II_1 and topological QFTs [A34, A23], and conformal field theories. For instance, according to the review [H1] [A25] Ocneacnu has show that Jones inclusions correspond in one-one manner to topological quantum field theories and TGD can indeed be regarded as almost topological quantum field theory (metric is brought in by the light-likeness of partonic 3-surfaces). Furthermore, Connes has shown that the decomposition of the hierarchies of tensor powers $\mathcal{M} \otimes_{\mathcal{N}} .... \otimes_{\mathcal{N}} M$ as left and right modules to representations of lower tensor powers directly to fusion rules expressible in terms of 4-point functions of conformal field theories [A25].

In TGD framework the transition from number fields to function fields would not be very dramatic.

1. Suppose that the representations of $SL(n, R)$ occurring in number theoretic Langlands program can indeed be realized in the moduli space for conformal equivalence classes of partonic 2-surface (or, by previous arguments, moduli space for regions of them with fixed boundaries). This means that representations of local Galois groups associated with number theoretic braids would involve global data about entire partonic 2-surface. This is physically very important since it otherwise discretization would lead to a loss of the information about dimension of partonic 2-surfaces.

2. In the case of geometric Langlands program this moduli space would be extended to the moduli space for n-point functions of conformal field theory defined at these 2-surfaces containing the original moduli space as a subspace. Of course, the extension could be present also in the number theoretic case. Thus it seems that number theoretic and geometric Langlands programs would utilize basic structures and would differ only in the sense that single braid would be replaced by several braids in the geometric case.

3. In TGD Kac-Moody algebras would be also present as well as the so called super-symplectic algebra [K6] related to the isometries of "the world of classical worlds" (the space of light-like 3-surfaces) with generators transforming according to the irreducible representations of rotation group $SO(3)$ and color group $SU(3)$. It must be emphasized that TGD view about conformal symmetry generalizes that of string models since light-like 3-surfaces (orbits of partons) are the basic dynamical objects [K6].

3.4.2 What about more general reductive groups?

Langlands correspondence is conjectured to apply to all reductive Lie groups. The question is whether there is room for them in TGD Universe. There are good hopes.

1. *Pairs formed by finite Galois groups and Lie groups containing them and defining sub-factors*

Any amenable (in particular compact Lie) group acting as outer automorphism of $\mathcal{M}$ defines a unique sub-factor $\mathcal{N} \subset \mathcal{M}$ as a group leaving the elements of $\mathcal{N}$ invariant. The representations of discrete subgroups of compact groups extended to representations of the latter would define natural candidates for Langlands correspondence and would expand the repertoire of the Galois groups representable in terms of unique factors. If one gives up the uniqueness condition for the sub-factor, one can expect that almost any Lie group can define a sub-factor.

2. *McKay correspondences and inclusions*

The so called McKay correspondence assigns to the finite subgroups of SU(2) extended Dynkin diagrams of ADE type Kac-Moody algebras. McKay correspondence also generalizes to the discrete subgroups of other compact Lie groups q [A35]. The obvious question is how closely this correspondence between finite groups and Lie groups relates with Langlands correspondence.

The principal graphs representing concisely the fusion rules for Connes tensor products of $\mathcal{M}$ regarded as $\mathcal{N}$ bi-module are represented by the Dynkin diagrams of ADE type Lie groups for $\mathcal{M} : \mathcal{N} < 4$ (not all of them appear). For index $\mathcal{M} : \mathcal{N} = 4$ extended ADE type Dynkin diagrams labelling Kac-Moody algebras are assigned with these representations.

I have proposed that TGD Universe is able to emulate almost any ADE type gauge theory and conformal field theory involving ADE type Kac-Moody symmetry and represented somewhat misty ideas about how to construct representations of ADE type gauge groups and Kac-Moody groups using many particle states at the sheets of multiple coverings $H \to H/G_a \times G_b$ realizing the idea about hierarchy of dark matters already mentioned. Also vertex operator construction also distinguishes ADE type Kac-Moody algebras in a special position.

It is possible to considerably refine this conjecture picture by starting from the observation that the set of generating elements for Lie algebra corresponds to a union of triplets $\{J_i^{\pm}, J_i^3\}$, $i = 1, ..., n$ generating SU(2) sub-algebras. Here n is the dimension of the Cartan sub-algebra. The non-commutativity of quantum Clifford algebra suggests that Connes tensor product can induce deformations of algebraic structures so that ADE Lie algebra could result as a kind of deformation of a direct sum of commuting SU(2) Lie (Kac-Moody) algebras associated with a Connes tensor product. The physical interpretation might in terms of a formation of a bound state. The finite depth of $\mathcal{N}$ would mean that this mechanism leads to ADE Lie algebra for an n-fold tensor power, which then becomes a repetitive structure in tensor powers. The repetitive structure would conform with the diagonal imbedding of Galois groups giving rise to a representation in terms of outer automorphisms.

This picture encourages the guess that it is possible to represent the action of Galois groups on number theoretic braids as action of subgroups of dynamically generated ADE type groups on configuration space spinors. The connection between the representations of finite groups and reductive Lie groups would result from the natural extension of the representations of finite groups to those of Lie groups.

3. *What about Langlands correspondence for Kac-Moody groups?* vm

The appearance of also Kac-Moody algebras raises the question whether Langlands correspondence could generalize also to the level of Kac-Moody groups or algebras and whether it could be easier to understand the Langlands correspondence for function fields in terms of Kac-Moody groups as the transition from global to local occurring in both cases suggests.

3.4.3 Could Langlands duality for groups reduce to super-symmetry?

Langlands program involves dualities and the general structure of TGD suggests that there is a wide spectrum of these dualities.

1. A very fundamental duality would be between infinite-dimensional Clifford algebra and group algebra of S_∞ or of braid group B_∞. For instance, one can ask could it be possible to map this

group algebra to the union of the moduli spaces of conformal equivalence classes of partonic 2-surfaces. HFFs consists of bounded operators of a separable Hilbert space. Therefore they are expected to have very many avatars: for instance there is an infinite number sub-factors isomorphic to the factor. This seems to mean infinite number of manners to represent Galois groups reflected as dualities.

2. Langlands program involves the duality between reducible Lie groups G and its Langlands dual having dual root lattices. The interpretation for this duality in terms of electric-magnetic duality is suggested by Witten [A30]. TGD suggests an alternative interpretation. The super symmetry aspect of super-conformal symmetry suggests that bosonic and fermionic representations of Galois groups could be very closely related. In particular, the representations in terms of WCW spinor s and in terms of modular degrees of freedom of partonic 2-surface could be in some sense dual to each other. Rotation groups have a natural action on WCW spinor s whereas symplectic groups have a natural action in the moduli spaces of partonic 2-surfaces of given genus possessing symplectic and Kähler structure. Langlands correspondence indeed relates $SO(2g+1,R)$ realized as rotations of WCW spinor s and $Sp(2g,C)$ realized as transformations in modular degrees of freedom. Hence one might indeed wonder whether super-symmetry could be behind the Langlands correspondence.

3.5 What Is The Role Of Infinite Primes?

Infinite primes primes at the lowest level of the hierarchy can be represented as polynomials and as rational functions at higher levels. These in turn define rational function fields. Physical states correspond in general to infinite rationals which reduce to unit in real sense but have arbitrarily complex number theoretical anatomy [K18], [L1, L3].

3.5.1 Does infinite prime characterize the l-adic representation of Galois group associated with given partonic 2-surface

Consider first the lowest level of hierarchy of infinite primes [K18]. Infinite primes at the lowest level of hierarchy are in a well-defined sense composites of finite primes and correspond to states of super-symmetric arithmetic quantum field theory. The physical interpretation of primes appearing as composites of infinite prime is as characterizing of the p-adic prime p assigned by the Kähler-Dirac action to partonic 2-surfaces associated with a given 3-surface [K22, K6].

This p-adic prime could naturally correspond to the possible prime associated with so called l-adic representations of the Galois group(s) associated with the p-adic counterpart of the partonic 2-surface. Also the Galois groups associated with the real partonic 2-surface could be represented in this manner. The generalization of moduli space of conformal equivalence classes must be generalized to its p-adic variant. I have proposed this generalization in context of p-adic mass calculations [K4].

It should be possible to identify WCW spinor s associated with real and p-adic sectors if anti-commutations relations for the fermionic oscillator operators make sense in any number field (that is involve only rational or algebraic numbers). Physically this seems to be the only sensible option.

3.5.2 Could one assign Galois groups to the extensions of infinite rationals?

A natural question is whether one could generalize the intuitions from finite number theory to the level of infinite primes, integers, and rationals and construct Galois groups and there representations for them. This might allow alternative very number theoretical approach to the geometric Langlands duality.

1. The notion of infinite prime suggests that there is entire hierarchy of infinite permutation groups such that the N_∞ at given level is defined as the product of all infinite integers at that level. Any group is a permutation group in formal sense. Could this mean that the hierarchy of infinite primes could allow to interpret the infinite algebraic sub-groups of Lie groups as Galois groups? If so one would have a unification of group theory and number theory.

2. An interesting question concerns the interpretation of the counterpart of hyper-finite factors of type II_1 at the the higher levels of hierarchy of infinite primes. Could they relate to a

hierarchy of local algebras defined by HFF? Could these local algebras be interpreted in terms of direct integrals of HFFs so that nothing essentially new would result from von Neumann algebra point of view? Would this be a correlate for the fact that finite primes would be the irreducible building block of all infinite primes at the higher levels of the hierarchy?

3. The transition from number fields to function fields is very much analogous to the replacement of group with a local gauge group or algebra with local algebra. I have proposed that this kind of local variant based on multiplication by of HFF by hyper-octonion algebra could be the fundamental algebraic structure from which quantum TGD emerges. The connection with infinite primes would suggest that there is infinite hierarchy of localizations corresponding to the hierarchy of space-time sheets.

4. Perhaps it is worth of mentioning that the order of S_∞ is formally $N_\infty = \lim_{n\rightarrow\infty} n!$. This integer is very large in real sense but zero in p-adic sense for all primes. Interestingly, the numbers $N_\infty/n+n$ behave like normal integers in p-adic sense and also number theoretically whereas the numbers $N_\infty/n+1$ behave as primes for all values of n. Could this have some deeper meaning?

3.5.3 Could infinite rationals allow representations of Galois groups?

One can also ask whether infinite primes could provide representations for Galois groups. For instance, the decomposition of infinite prime to primes (or prime ideals) assignable to the extension of rationals is expected to make sense and would have clear physical interpretation. Also (hyper-)quaternionic and (hyper-)octonionic primes can be considered and I have proposed explicit number theoretic interpretation of the symmetries of standard model in terms of these primes. The decomposition of partonic primes to hyper-octonionic primes could relate to the decomposition of parton to regions, one for each number theoretic braid.

There are arguments supporting the view that infinite primes label the ground states of super-conformal representations [K6, K18]. The question is whether infinite primes could allow to realize the action of Galois groups. Rationality of infinite primes would imply that the invariance of ground states of super-conformal representations under the braid realization of $Gal(\overline{Q}/Q)$ of finite Galois groups. The infinite prime as a whole could indeed be invariant but the primes in the decomposition to a product of primes in algebraic extension of rationals need not be so. This kind of decompositions of infinite prime characterizing parton could correspond to the above described decomposition of partonic 2-surface to regions X^2_k at which Galois groups act non-trivially. It could also be that only infinite integers are rational whereas the infinite primes decomposing them are hyper-octonionic. This would physically correspond to the decomposition of color singlet hadron to colored partons [K18].

3.6 Could Langlands Correspondence, Mckay Correspondence And Jones Inclusions Relate To Each Other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [A25]. For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence q [A35] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

3.6.1 About McKay correspondence

McKay correspondence q [A35] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows q [A35].

1. Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.

2. Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j-1/2$, and $j+1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j+1/2, j+1, ...$

   One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n ($SO(2n+1)$), C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M}:\mathcal{N}<4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [A11].

1. The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2

2. The classification of simply laced semisimple Lie groups.

3. The classification of finite sub-groups of the 3-dimensional rotation group.

4. The classification of simple singularities. In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.

5. The classification of tame quivers.

3.6.2 Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

1. The so called principal graphs characterizing $\mathcal{M}:\mathcal{N}=4$ Jones inclusions for $G=SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n \geq 4$.

2. $\mathcal{M}:\mathcal{N}<4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n ($SU(n)$), D_{2n} ($SO(2n)$), and E_6 and E_8. Thus D_{2n+1} ($SO(2n+2)$) and E_7 are not allowed. For instance, for $G=S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

1. The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N} \subset \mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has $\mathcal{N}' \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}^1 \subset ...$ and $C = \mathcal{M}' \cap \mathcal{M} \subset \mathcal{M}' \cap \mathcal{M}^1 \subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}' \cap \mathcal{M}^k \subset \mathcal{N}' \cap \mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras Templieb assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [A17].

2. The information provided by infinite Bratteli diagram can be coded by a much simpler bi-partite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

1. It is natural to decompose the Connes tensor powers q [A35] $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ to irreducible $\mathcal{M}-\mathcal{M}$, $\mathcal{N}-\mathcal{M}$, $\mathcal{M}-\mathcal{N}$, or $\mathcal{N}-\mathcal{N}$ bi-modules. If $\mathcal{M}:\mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.

2. If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M}-\mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N}-\mathcal{N}$ representations resulting in the decomposition of $\mathcal{M}-\mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

3.6.3 A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [K10] is that in $\mathcal{M}:\mathcal{N} < 4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M}:\mathcal{N} = 4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \to H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

1. McKay correspondence for subgroups of G ($\mathcal{M}:\mathcal{N}=4$) *resp.* its variants ($\mathcal{M}:\mathcal{N}<4$) and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.

2. Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

1. The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2) \otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$ module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.

2. This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of WCW spinor s extend to the representations of SU(2) naturally.

3. Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \rightarrow H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

WCW spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between $(\mathcal{M} : \mathcal{N} = 4)$ and $(\mathcal{M} : \mathcal{N} < 4)$ cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in $(\mathcal{M} : \mathcal{N} = 4)$ case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \rightarrow G_a \times G_b$ bundle structure would be $H \rightarrow H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups q [A35]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

7. Flavor groups of hadron physics as a support for HFF?

The deformation assigning to an n-fold tensor power of representations of Lie group G with k-dimensional Cartan algebra a representation of a Lie group with nk-dimensional Cartan algebra could be also seen as a dynamically generated symmetry. If quantum measurement is characterized by the choice of Lie group G defining measured quantum numbers and defining Jones inclusion characterizing the measurement resolution, the measurement process itself would generate these dynamical symmetries. Interestingly, the flavor symmetry groups of hadron physics cannot be justified from the structure of the standard model having only electro-weak and color group as fundamental symmetries. In TGD framework flavor group $SU(n)$ could emerge naturally as a fusion of n quark doublets to form a representation of $SU(n)$.

3.6.4 Conformal representations of braid group and a possible further generalization of McKay correspondence

Physically especially interesting representations of braid group and associated Temperley-Lieb-Jones algebras (TLJ) are representations provided by the n-point functions of conformal field theories studied in [A33]. The action of the generator of braid group on n-point function corresponds to a duality transformation of old-fashioned string model (or crossing) represented as a monodromy relating corresponding conformal blocks. This effect can be calculated. Since the index $r = \mathcal{M} : \mathcal{N}$ appears as a parameter in TLJ algebra, the formulas expressing the behavior of n-point functions under the duality transformation reveal also the value of index which might not be easy to calculate otherwise.

Note that in TGD framework the arguments of n-point function would correspond to the strands of the number theoretic braid and thus to the points of the geodesic sphere S^2 associated with the light-cone boundary $\delta M^4_\pm$. The projection to the geodesic sphere of CP_2 projection would be same for all these strands.

WZW model for group G and Kac-Moody central charge k quantum phase is discussed in [A33]. The non-triviality of braiding boils to the fact that quantum group G_q defines the effect of braiding operation. Quantum phase is given as $q = exp(i\pi/(k + C(G))$, where $C(G)$ is the value of Casimir operator in adjoint representation. The action of the braid group generator reduces to the unitary matrix relating the basis defined by the tensor product of representations of G_q to the basis obtained by application of a generator of the braid group. For n-point functions of primary fields belonging to a representation D of G, index is the square of the quantum dimension $d_q(D)$ of the corresponding representation of G_q. Hence each primary field correspond to its own inclusion of HFF, which corresponds to $n \to \infty$-point function.

The result could have been guessed as the dimension of quantum Clifford algebra emerging naturally in inclusion when HFF is represented as an infinite tensor power of $M(d(D), C)$. For $j = 1/2$ representation of $SU(2)$ standard Jones inclusions with $r < 4$ are obtained. The resulting inclusion is irreducible ($\mathcal{N}' \cap \mathcal{M} = C$, where $\mathcal{N}'$ is the commutator of $\mathcal{N}'$). Using the representation of HFF as infinite tensor power of $M(2, C)$ the result would not be so easy to understand.

The mathematical challenge would be to understand how the representations HFF as an infinite tensor power of $M(n, C)$ relate to each other for different values of n. It might be possible to understand the relationship between different infinite tensor power representations of HFF by representing $M(n_1, C)$ as a sub-algebra of a tensor power of a finite tensor power of $M(n_2, C)$. Perhaps a detailed construction of the maps between representations of HFF as infinite tensor power of $M(n, C)$ for various values of n could reveal further generalizations of McKay correspondence.

3.7 Technical Questions Related To Hecke Algebra And Frobenius Element

3.7.1 Frobenius elements

Frobenius element Fr_p is mapped to a conjugacy class of Galois group using the decomposition of prime p to prime ideals in the algebraic extension K/F.

1. At the level of braid group Frobenius element Fr_p corresponds to some conjugacy class of Galois group acting imbedded to S_n (only the conjugacy equivalence class is fixed) and thus can be mapped to an element of the braid group. Hence it seems possible to assign to Fr_p an element of infinitely cyclic subgroup of the braid group.

2. One can always reduce in given representation the element of given conjugacy class to a diagonal matrix so that it is possible to chose the representatives of Fr_p to be commuting operators. These operators would act as a spinor rotation on quantum Clifford algebra elements defined by Jones inclusion and identifiable as element of some cyclic group of the group G defining the sub-factor via the diagonal embedding.

3. Fr_p for a given finite Galois group G should have representation as an element of braid group to which G is imbedded as a subgroup. It is possible to chose the representatives of Fr_p so that they commute. Could one chose them in such a manner that they belong to the commuting subgroup defined by even (odd) generators e_i? The choice of representatives for Fr_p for various Galois groups must be also consistent with the hierarchies of intermediate extensions of rationals associated with given extension and characterized by subgroups of Galois group for the extension.

3.7.2 How the action of commutative Hecke algebra is realized in hyper-finite factor and braid group?

One can also ask how to imbed Hecke algebra to the braid algebra. Hecke algebra for a given value of prime p and group $GL(n, R)$ is a polynomial algebra in Hecke algebra generators. There is a fundamental difference between Hecke algebra and Frobenius element Fr_p in the sense that Fr_p has finite order as an element of finite Galois group whereas Hecke algebra elements do not except possibly for representations. This means that Hecke algebra cannot have a representation in a finite Galois groups.

Situation is different for braid algebra generators since they do not satisfy the condition $e_i^2 = 1$ and odd and even generators of braid algebra commute. The powers of Hecke algebra generators would correspond to the powers of basic braiding operation identified as a π twist of neighboring strands. For unitary representations eigenvalues of e_i are phase factors. Therefore Hecke algebra might be realized using odd or even commuting sub-algebra of braid algebra and this could allow to deduce the Frobenius-Hecke correspondence directly from the representations of braid group. The basic questions are following.

1. Is it possible to represent Hecke algebra as a subalgebra of braid group algebra in some natural manner? Could the infinite cyclic group generated by braid group image of Fr_p belong represent element of Hecke algebra fixed by the Langlands correspondence? If this were the case then the eigenvalues of Frobenius element Fr_p of Galois group would correspond to the eigen values of Hecke algebra generators in the manner dictated by Langlands correspondence.

2. Hecke operators $H_{p,i}$, $i = 1, .., n$ commute and expressible as two-side cosets in group $GL(n, Q_p)$. This group acts in $\mathcal{M}$ and the action could be made rather explicit by using a proper representations of $\mathcal{M}$ (note however that physical situation can quite well distinguish between various representations). Does the action of the Hecke sub-algebra fixed by Hecke-Frobenius correspondence co-incide with the action of Frobenius element Fr_p identified as an element of braid sub-group associated with some cyclic subgroup of the Galois group identified as a group defining the sub-factor?

4 Langlands Conjectures And The Most Recent View About TGD

Langlands program [A5, A22, A20] relies on very general conjectures about a connection between number theory and harmonic analysis relating the representations of Galois groups with the representations of certain kinds of Lie groups to each other. Langlands conjecture has many forms and it is indeed a conjecture and many of them are imprecise since the notions involved are not sharply defined.

Peter Woit noticed that Edward Frenkel had given a talk with rather interesting title "What do Fermat's Last Theorem and Electro-magnetic Duality Have in Common?" [A12] ? I listened the talk and found it very inspiring. The talk provides bird's eye of view about some basic aspects of Langlands program using the language understood by physicist. Also the ideas concerning the connection between Langlands duality and electric-magnetic duality generalized to S-duality in the context of non-Abelian gauge theories and string theory context and developed by Witten and Kapustin [A30] and followers are summarized. In this context $D = 4$ and twisted version of $\mathcal{N} = 4$ SYM familiar from twistor program and defining a topological QFT appears.

For some years ago I made my first attempt to understand what Langlands program is about and tried to relate it to TGD framework [K12]. At that time I did not really understand the motivations for many of the mathematical structures introduced. In particular, I did not really understand the motivations for introducing the gigantic Galois group of algebraic numbers regarded as algebraic extension of rationals.

1. Why not restrict the consideration to finite Galois groups [A3] or their braided counterparts (as I indeed effectively did [K12])? At that time I concentrated on the question what the enormous Galois group of algebraic numbers regarded as algebraic extension of rationals could mean, and proposed that it could be identified as a symmetric group consisting of permutations of infinitely many objects. The definition of this group is however far from trivial. Should one allow as generators of the group only the permutations affecting only finite number of objects or permutations of even infinite number of objects?

   The analogous situation for the sequences of binary digits would lead to a countable set of sequence of binary digits forming a discrete set of finite integers in real sense or to 2-adic integers forming a 2-adic continuum. Something similar could be expected now. The physical constraints coming the condition that the elements of symmetric group allow lifting to braidings suggested that the permutations permuting infinitely many objects should be periodic meaning that the infinite braid decomposes to an infinite number of identical N-braids and braiding is same for all of them. The p-adic analog would be p-adic integers, which correspond to rationals having periodical expansion in powers of p. Braids would be therefore like pinary digits. I regarded this choice as the most realistic one at that time. I failed to realize the possibility of having analogs of p-adic integers by general permutations. In any case, this observation makes clear that the unrestricted Galois group is analogous to a Lie group in topology analogous to p-adic topology rather than to discrete group. Neither did I realize that the Galois groups could be finite and be associated with some other field than rationals, say a Galois group associated with the field of polynomials of n-variable with rational coefficients and with its completion with coefficients replaced by algebraic numbers.

2. The ring of adeles [A1] can be seen as a Cartesian product of non-vanishing real numbers $R_\times$ with the infinite Cartesian product $\prod Z_p$ having as factors p-adic integers Z_p for all values of prime p. Rational adeles are obtained by replacing R with rationals Q and requiring that multiplication of rational by integers is equivalent with multiplication of any Z_p with rational. Finite number of factors in Z_p can correspond to Q_p: this is required in to have finite adelic norm defined as the product of p-adic norms. This definition implicitly regards rationals as common to all number fields involved. At the first encounter with adeles I did not realize that this definition is in spirit with the basic vision of TGD.

   The motivation for the introduction of adele is that one can elegantly combine the algebraic groups assignable to rationals (or their extensions) and all p-adic number fields or even more general function fields such as polynomials with some number of argument at the same time as a Cartesian product of these groups as well as to finite fields. This is indeed needed if one wants to realize number theoretic universality which is basic vision behind physics as generalized number theory vision. This approach obviously means enormous economy of thought irrespective of whether one takes adeles seriously as a physicist.

In the following I will discuss Taniyama-Shimura-Weil theorem and Langlands program from TGD point view.

4.1 Taniyama-Shimura-Weil Conjecture From The Perspective Of TGD

4.1.1 Taniyama-Shimura-Weil theorem

It is good to consider first the Taniyama -Shimura-Weil conjecture [A10] from the perspective provided by TGD since this shows that number theoretic Langlands conjecture could be extremely useful for practical calculations in TGD framework.

1. Number theoretical universality requires that physics in real number field and various p-adic number fields should be unified to a coherent hole by a generalization of the notion of number: different number fields would be like pages of book intersecting along common rationals. This would hold true also for space-time surfaces and imbedding space but would require some preferred coordinates for which rational points would determined the intersection of real and p-adic worlds. There are good reasons for the hypothesis that life resides in the intersection of real and p-adic worlds.

   The intersection would correspond at the level of partonic 2-surfaces rational points of these surfaces in some preferred coordinates, for which a finite-dimensional family can be identified on basis of the fundamental symmetries of the theory. Allowing algebraic extensions one can also consider also some algebraic as common points. In any case the first question is to count the number of rational points for a partonic 2-surface.

   2-dimensional Riemann surfaces serve also as a starting point of number theoretic Langlands problem and the same is true for the geometric Langlands program concentrating on Riemann surfaces and function fields defined by holomorphic functions.

2. The number theoretic side of Taniyama-Shimura-Weil (TSW briefly) theorem for elliptic surfaces, which is essential for the proof of Fermat's last theorem, is about counting the integer (or equivalently rational) points of the elliptic surfaces

   $$y^2 = x^3 + ax + b \ , \ \ a, b \in Z \ .$$

   The theorem relates number theoretical problem to a problem of harmonic analysis, which is about group representations. What one does is to consider the above Diophantine equation modulo p for all primes p. Any solution with finite integers smaller than p defines a solution in real sense if $mod \ \ p$ operation does not affect the equations. Therefore the existence of a finite number of solutions involving finite integers in real sense means that for large enough p the number a_p of solutions becomes constant.

3. On harmonic analysis one studies so called modular forms $f(\tau)$, where τ is a complex coordinate for upper half plane defining moduli space for the conformal structures on torus. Modular forms have well defined transformation properties under group $Gl_2(R)$: the action is defined by the formula $\tau \rightarrow (a\tau + b)/(c\tau + d)$. The action of $Gl_2(Z)$ or its appropriate subgroup is such that the modular form experiences a mere multiplication by a phase factor: $D(hk) = c(h,k)D(h)D(k)$. The phase factors obey cocycle conditions $D(h,k)D(g,hk) = D(gh,k)D(g,h)$ guaranteeing the associativity of the projective representation.

   Modular transformations are clearly symmetries represented projectively as quantum theory indeed allows to do. The geometric interpretation is that one has projective representations in the fundamental domain of upper plane defined by the identification of the points differing by modular transformations. In conformally symmetric theories this symmetry is essential.

Fundamental domain is analogous to lattice cell. One often speaks of cusp forms: cusp forms vanish at the boundary of the fundamental domain defined as the quotient of the upper half plane by a subgroup -call it Γ of the modular group $Sl_2(Z)$. The boundary corresponds to $Im(\tau) \rightarrow \infty$ or equivalently $q = exp(i2\pi\tau) \rightarrow 0$.

Remark: In TGD framework modular symmetry says that elementary particle vacuum functionals are modular invariants. For torus one has the above symmetry but for Riemann surface with higher genus modular symmetries correspond to a subgroup of $Sl_{2g}(Z)$.

4. One can expand the modular form as Fourier expansion using the variable $q = exp(i2\pi\tau)$ as

$$f(\tau) = \sum_{n>0} b_n q^n \quad .$$

$b_1 = 1$ fixes the normalization. $n > 0$ in the sum means that the form vanishes at the boundary of the fundamental domain associated with the group Γ. The TSW theorem says that for prime values $n = p$ one has $b_p = a_p$, where a_p is the number of mod p integer solutions to the equations defining the elliptic curve. At the limit $p \rightarrow \infty$ one obtains the number of real actual rational points of the curve if this number is finite. This number can be also infinite. The other coefficients b_n can be deduced from their values for primes since b_n defines what is known as a multiplicative character in the ring of integers implying $b_{mn} = b_m b_n$ meaning that b_n obeys a decomposition analogous to the decomposition of integer into a product of primes.

The definition of the multiplicative character is extremely general: for instance it is possible to define quantum counterparts of multiplicative characters and of various modular forms by replacing integers with quantum integers defined as products of quantum primes for all primes except one -call it p_0, which is replaced with its inverse: this definition of quantum integer appears in the deformation of distributions of integer valued random variable characterized by rational valued parameters and is motivated by strange findings of Shnoll [K2]. The interpretation could be in terms of TGD based view about finite measurement resolution bringing in quantum groups and also preferred p-adic prime naturally.

5. TSW theorem allows to prove Fermat's last theorem: if the latter theorem were wrong also TSW theorem would be wrong. What also makes TSW theorem so wonderful is that it would allow to count the number of rational points of elliptic surfaces just by looking the properties of the automorphic forms in $Gl_2(R)$ or more general group. A horrible looking problem of number theory is transformed to a problem of complex analysis which can be handled by using the magic power of symmetry arguments. This kind of virtue does not matter much in standard physics but in quantum TGD relying heavily on number theoretic universality situation is totally different. If TGD is applied some day the counting of rational points of partonic surfaces is everyday practice of theoretician.

4.1.2 How to generalize TSW conjecture?

The physical picture of TGD encourages to imagine a generalization of the Tanyama-Shimura-Weil conjecture.

1. The natural expectation is that the conjecture should make sense for Riemann surfaces of arbitrary genus g instead of $g = 1$ only (elliptic surfaces are tori). This suggests that one should one replace the upper half plane representing the moduli space of conformal equivalence classes of toric geometries with the $2g$-dimensional (in the real sense) moduli space of genus g conformal geometries identifiable as Teichmüller space.

   This moduli space has symplectic structure analogous to that of $g + g$-dimensional phase space and this structure relates closely to the cohomology defined in terms of integrals of holomorphic forms over the $g + g$ cycles which each handle carrying two cycles. The moduli are defined by the values of the holomorphic one-forms over the cycles and define a symmetric matrix Ω_{ij} (modular parameters), which is modular invariant [K4]. The modular parameters related $Sp_2g(Z)$ transformation correspond to same conformal equivalence class.

If Galois group and effective symmetry group G are representable as symplectic flows at the light-like boundary of $CD(\times CP_2)$, their action automatically defines an action in the moduli space. The action can be realized also as a symplectic flow defining a braiding for space-like braids assignable to the ends of the space-time surface at boundaries of CD or for time-like braids assignable to light-like 3-surfaces at which the signature of the induced metric changes and identified as orbits of partonic 2-surfaces analogous to black hole horizons.

2. It is possible to define modular forms also in this case. Most naturally they correspond to theta functions used in the construction of elementary particle functionals in this space [K4]. Siegel modular forms transform naturally under the symplectic group $Sp_{2g}(R)$ and are projectively invariant $Sp_{2g}(Z)$. More general moduli spaces are obtained by allowing also punctures having interpretation as the ends of braid strands and very naturally identified as the rational points of the partonic 2-surface. The modular forms defined in this extended moduli space could carry also information about the number of rational points in the same manner as the automorphic representations of $Gl_2(R)$ carry information about the number of rational points of elliptic curves.

3. How Tanyama-Shimura-Weil conjecture should be generalized? Also now one can consider power series of modular forms with coefficients b_n defining multiplicative characters for the integers of field in question. Also now the coefficients a_p could give the number of integer/rational points of the partonic 2-surface in mod p approximation and at the limit $p \to \infty$ the number of points a_p would approach to a constant if the number of points is finite.

4. The only sensical interpretation is that the analogs of elementary particle vacuum functionals [K4] identified as modular forms must be always restricted to partonic 2-surfaces having the same number of marked points identifiable as the end points of braid strands rational points. It also seems necessary to assume that the modular forms factorize to a products of two parts depending on Teichmüller parameters and positions of punctures. The assignment of fermionic and bosonic quantum numbers with these points conforms with this interpretation. As a special case these points would be rational. The surface with given number or marked points would have varying moduli defined by the conformal moduli plus the positions of the marked points. This kind of restriction would be physically very natural since it would mean that only braids with a given number of braid strands ending at fixed number of marked points at partonic 2-surfaces are considered in given quantum state. Of course, superpositions of these basis states with varying braid number would be allowed.

4.2 Unified Treatment Of Number Theoretic And Geometric Langlands Conjectures In TGD Framework

One can already now wonder what the relationship of the TGD view about number theoretic Langlands conjecture to the geometric Langlands conjecture could be?

1. The generalization of Taniyama-Shinamure-Weil theorem to arbitrary genus would allow to deduce the number of rational points already for finite but large enough values of p from the Taylor coefficients of an appropriate modular form. Is this enough for the needs of TGD? The answer is “No”. One must be able to count also numbers of “rational 2-surfaces” in the space of 2-surfaces and the mere generalization of TSW conjecture does not allow this. Geometric Langlands replacing rational points with “rational” surfaces is needed.

   If the geometric Langlands conjecture holds true in the spirit with TGD, it must allow to deduce the number of rational variants of of partonic 2-surfaces assignable to given quantum state defined to be a state with fixed number of braid strands for each partonic 2-surface of the collection. What is new is that collections of partonic 2-surfaces regarded as sub-manifolds of $M^4 \times CP_2$ are considered.

2. Finite measurement resolution conjectured to be definable in terms of effective symmetry group G defined by the inclusion of hyper-finite factors of type II_1 [K21] (HFFs in the sequel) effectively replaces partonic 2-surfaces with collections of braid ends and the natural idea is that the orbits of these collections under finite algebraic subgroup of symmetry group

defining finite measurement resolution gives rise to orbit with finite number of points (point understood now as collection of rational points). The TGD variant of the geometric Langlands conjecture would allow to deduce the number of different collections of rational braid ends for the quantum state considered (one particular WCW spinor field) from the properties of automorphic form.

3. Quantum group structure is associated with the inclusions of HFFs, with braid group representations, integrable QFTs, and also with the quantum Yangian symmetry [A31, A29] suggested strongly by twistor approach to TGD. In zero energy ontology physical states define Lie-algebra and the multi-locality of the scattering amplitudes with respect to the partonic 2-surfaces (that is at level of WCW) suggests also quantum Yangian symmetry. Therefore the Yangian of the Kac-Moody type algebra defining measurement resolution is a natural candidate for the symmetry considered. What is important is that the group structure is associated with a finite-dimensional Lie group.

This picture motivates the question whether number theoretic and geometric Langlands conjecture could be realized in the same framework? Could electric-magnetic duality generalized to S-duality imply these dualities and bring in the TGD counterpart of effective symmetry group G in some manner. This framework would be considerably more general than the 4-D QFT framework suggested by Witten and Kapustin [A30] and having very close analogies with TGD view about space-time.

The following arguments support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The basic notions are following.

1. Zero energy ontology and the related notion of causal diamond CD (CD is short hand for the cartesian product of causal diamond of M^4 and of CP_2). This notion leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet.

2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.

3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [K21]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations ("symmetry group" hitherto). At space-time level the finite measurement resolution would be represented in terms of braids at space-time level which come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.

   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multilocality and therefore quantum Yangian algebra with multilocal generators is unavoidable.

   In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.

4. The natural identification of the associate of G is as quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized.

5. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

6. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [K7] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.

7. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program [A30] is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

4.2.1 Number theoretic Langlands conjecture in TGD framework

Number theoretic Langlands conjecture generalizes TSW conjecture to a duality between two kinds of groups.

1. At the number theoretic side of the duality one has an n-dimensional representation of Galois group for the algebraic numbers regarded as algebraic extension of rationals. In the more general case one can consider arbitrary number field identified as algebraic extension of rationals. One can assign to the number field its rational adele. In the case of rationals this brings in both real numbers and p-adic numbers so that huge amount of information can be packed to the formulas. For anyone who has not really worked concretely with number theory it is difficult to get grasp of the enormous generality of the resulting theory.

2. At the harmonic analysis side of the conjecture one has n-dimensional representation of possibly non-compact Lie group G and its Langlands dual ${}^LG$ appearing also in the non-Abelian form of electric-magnetic duality. The idea that electric-magnetic duality generalized to S-duality could provide a physical interpretation of Langlands duality is suggestive. $U(n)$ is self dual in Langlands sense but already for $G = SU(3)$ one has ${}^LG = SU(3)/Z_3$. For most Lie groups the Lie algebras of G and ${}^LG$ are identical but even the Lie algebras can be different.

$Gl_2(R)$ is replaced with any reductive algebraic group and in the matrix representation of the group the elements of the group are replaced by adeles of the discrete number field considered.

3. Langlands duality relates the representations of the Galois group in question to the automorphic representations of G. The action of the Lie group is on the argument of the modular form so that one obtains infinite-dimensional representation of G for non-compact G analogous to a unitary representation of Lorentz group. The automorphic forms are eigenstates of the Casimir operator of G. Automorphy means that a subgroup Γ of the modular group leaves the automorphic form invariant modulo phase factor.

4. The action of the modular transformation $\tau \rightarrow -1/\tau$ in the case of $Gl_2(R)$ replaces G with ${}^LG$. In the more general case (for the moduli space of Riemann surfaces of genus g possessing n punctures) the definition of the modular transformation induce the change $G \rightarrow^L G$ does not look obvious. Even the idea that one has only two groups related by modular transformation is not obvious. For electromagnetic duality with τ interpreted in terms of complexified gauge coupling strength this interpretational problem is not encountered.

4.2.2 Geometric Langlands conjecture in TGD framework

Consider next the geometric Langlands conjecture from TGD view point.

1. The geometric variant of Langlands conjecture replaces the discrete number field F (rationals and their algebraic extensions say) with function number field- say rational function with rational coefficients- for which algebraic completion defines the gigantic Galois group. Witten and Kapustin [A30] proposed a concrete vision about how electric-magnetic duality generalized to S-duality could allow to understand geometric Langlands conjecture.

2. By strong form of general coordinate invariance implying holography the partonic 2-surfaces and their 4-D tangent space data (not completely free probably) define the basic objects so that WCW reduces to that for partonic 2-surfaces so that the formulation of geometric Langlands conjecture for the local field defined by holomorphic rational functions with rational coefficients at partonic 2-surface might make sense.

3. What geometric Langlands conjecture could mean in TGD framework? The transition from space-time level to the level of world of classical worlds suggests that polynomials with rational functions with rational coefficients define the analog of rational numbers which can be regarded to be in the intersection real and p-adic WCW s. Instead of counting rational points of partonic 2-surface one might think of counting the numbers of points in the intersection of real and p-adic WCW s in which life is suggested to reside. One might well consider the possibility that a kind of volume like measure for the number of these point is needed. Therefore the conjecture would be of extreme importance in quantum TGD. Especially so if the intersection of real and p-adic worlds is dense subset of WCW just as rationals form a dense subset of reals and p-adic numbers.

4.2.3 Electric-magnetic duality in TGD framework

Consider first the ideas of Witten and Kapustin in TGD framework.

1. Witten and Kapustin suggest that electric-magnetic duality and its generalization to S-duality in non-abelian is the physical counterpart of $G \leftrightarrow^L G$ duality in geometric Langlands. The model is essentially a modification $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 2$ SUSY allowing this duality with Minkowski space replaced with a Cartesian product of two Riemann surfaces. In TGD framework M^4 would correspond naturally to space-time sheet allowing a slicing to string world sheets and partonic 2-surfaces. Witten and Kapustin call these 2-dimensional surfaces branes of type A and B with motivation coming from M-theory. The generalization of the basic dimensional formulas of S-duality to TGD framework implies that light-like 3-surfaces at which the signature of the induced metric changes and space-like 3-surfaces at the boundaries of CDs are analogs of brane orbits. Branes in turn would be partonic 2-surfaces. S-duality would be nothing but strong form of general coordinate invariance.

2. Witten and Kapustin introduce the notions of electric and magnetic eigen branes and formulate the duality as a transformation permuting these branes with each other. In TGD framework the obvious identification of the electric eigen branes are as string world sheets and these can be indeed identified essentially uniquely. Magnetic eigen branes would correspond to partonic 2-surfaces.

3. Witten and Kapustin introduce gauge theory with given gauge group. In TGD framework there is no need to introduce gauge theory description since the symmetry group emerges as the effective symmetry group defining measurement resolution. Gauge theory is expected to be only an approximation to TGD itself. In fact, it seems that the interpretation of G as Lie-group associated with Kac-Moody symmetry is more appropriate in TGD framework. This would mean generalization of 2-D sigma model to string model in moduli space. The action of G would not be visible in the resolution used.

4. Edward Frenkel represents the conjecture that there is mysterious 6-dimensional theory behind the geometric Langlands duality. In TGD framework this theory might correspond to twistorial formulation of quantum TGD using instead of $M^4 \times CP_2$ the space $CP_3 \times CP_3$ with space-time surfaces replaced by 6-D sphere bundles.

4.2.4 Finite measurement resolution realized group theoretically

The notion of finite measurement resolution allows to identify the effective symmetry groups G and ${}^LG$ in TGD framework. The most plausible interpretation of G is as Lie group giving rise to Kac-Moody type symmetry and assignable to a string model defined in moduli space of partonic 2-surfaces. By electric magnetic duality the roles of the string world sheet and partonic 2-surface can be exchanged provided the replacement $G \rightarrow G_L$ is performed. The duality means a duality of closed Euclidian strings and Minkowskian open strings.

1. The vision is that finite measurement resolution realized in terms of inclusions of HFFs corresponds to effective which is gauge or Kac-Moody type local invariance extended to quantum Yangian symmetry. A given finite measurement resolution would correspond to effective symmetry G giving rise to confinement so that the effective symmetry indeed remains invisible as finite measurement resolution requires. The finite measurement resolution should allow to emulate almost any gauge theory or string model type theory. This theory might allow super-symmetrization reducing to broken super-symmetries of quantum TGD generated by the fermionic oscillator operators at partonic 2-surfaces and string world sheets.

2. Finite measurement resolution implies that the orbit of the partonic 2-surface reduces effectively to a braid. There are two kinds of braids. Time-like braids have their ends at the boundaries of CD consisting of rational points in the intersection of real and p-adic worlds. Space-like braids are assignable to the space-like 3-surfaces at the boundaries of CD and their ends co-incide with the ends of time-like braids. The electric-magnetic duality says that the descriptions based using either kind of braids is all that is needed and that the descriptions are equivalent.

   The counterpart of $\tau \rightarrow -1/\tau$ should relate these descriptions. This need not involve transformation of effective complex Kähler coupling strength although this option cannot be excluded. If this view is correct the descriptions in terms of string world sheets and partonic 2-surfaces would correspond to electric and magnetic descriptions, which is indeed a very natural interpretation. This geometric transformation should replace G with ${}^LG$.

3. Finite measurement resolution effectively replaces partonic 2-surface with a discrete set of points and space-time surface with string world sheets or partonic 2-surfaces. The natural question is whether finite measurement resolution also replaces geometric Langlands and the "rational" intersection of real and p-adic worlds with number theoretic Langlands and rational points of the partonic 2-surface. Notice that the rational points would be common to the string world sheets and partonic 2-surfaces so that the duality of stringy and partonic descriptions would be very natural for fintie measurement resolution.

The basic question is how the symmetry group G emerges from finite measurement resolution. Are all Lie groups possible? Here the theory of Witten and Kapustin suggests guidelines.

1. What Witten and Kapustin achieve is a transformation of a twisted $\mathcal{N} = 4$ SUSY in $M^4 = \Sigma \times C$, where Σ is "large" as compared to Riemann surface C SUSY to a sigma model in Σ with values of fields in the moduli space of Higgs bundle defined in C. If one accepts the basic conjecture that at least regions of space-time sheets allow a slicing by string world sheets and partonic 2-surfaces one indeed obtains $M^4 = \Sigma \times C$ type structure such that Σ corresponds to string world sheet and C to partonic 2-surface.

   The sigma model -or more generally string theory- would have as a natural target space the moduli space of the partonic 2-surfaces. This moduli space woudl have as coordinates its conformal moduli and the positions of the punctures expressible in terms of the imbedding space coordinates. For M^4 coordinates only the part transversal to Σ would represent physical degree of freedom and define complex coordinate. Each puncture would give rise to two complex E^2 coordinates and 2 pairs of complex CP_2 coordinates. If one identifies the string world sheets as an inverse image of a homologically non-trivial geodesic sphere as suggested in [K11]. This would eliminate CP_2 coordinates as dynamical variables and one would have just n complex valued coordinates.

2. How to construct the Lie algebra of the effective symmetry group G defining the measurement resolution? If G is gauge group there is no obvious guess for the recipe. If G defines Kac-Moody algebra the situation is much better. There exists an extremely general construction allowing a stringy construction of Kac-Moody algebra using only the elements of its Cartan algebra with central extension defined by integer valued central extension parameter k. The vertex operators defining the elements of the complement of the Cartan algebra of complexified Kac-Moody algebra are ordered exponentials of linear combinations of the Cartan algebra generators with coefficient given by the weights of the generators, which are essentially the quantum numbers assignable to them as eigenvalues of Cartan algebra generators acting in adjoint representations.

   The explicit expression for the Kac-Moody generator as function of complex coordinate of Riemann sphere S^2 is

   $$J_\alpha(z) =: exp(\alpha \cdot \phi(z))) : \quad .$$

   $J_\alpha(z)$ represents a generator in the complement of Cartan algebra in standard Cartan basis having quantum numbers α and $\phi(z)$ represents the Cartan algebra generator allowing decomposition into positive and negative frequency parts. The weights α must have the same length ($(\alpha, \alpha) = 2$) meaning that the Lie group is simply laced. This representation corresponds to central extension parameter $k = 1$. In bosonic string models these operators are problematic since they represent tachyons but in the recent context this not a problem. The central extension parameter c for the associated Virasoro representation is also non-vanishing but this should not be a problem now.

3. What is remarkable that depending on the choice of the weights α one obtains a large number of Lie algebras with same dimension of Cartan algebra. This gives excellent hopes of realizing in finite measurement resolution in terms of Kac-Moody type algebras obtained as ordered exponentials of the operators representing quantized complex E^2 coordinates. Any complexified simply laced Lie group would define a Kac-Moody group as a chacterizer of finite measurement resolution. Simply laced groups correspond by MacKay correspondence finite subgroups of $SU(2)$, which suggests that only Galois groups representable as subgroups of $SU(2)$ can be realized using this representation. It however seems that free field representations can be defined for an arbitrary affine algebra: these representations are discussed by Edward Frenkel [A18].

4. The conformal moduli of the partonic 2-surface define part of the target space. Also they could play the role of conformal fields on string world sheet. The strong form of holography poses heavy constraints on these fields and the evolution of the conformal moduli could be

completely fixed once their values at the ends of string world sheets at partonic 2-suraces are known. Are also the orbits of punctures fixed completely by holography from initial values for "velocities" at partonic 2-surfaces corresponding to wormhole throats at which the signature of the metric changes? If this were the case, stringy dynamics would reduce to that for point like particles defined by the punctures. This cannot be true and the natural expectation is that just the finite spatial measurement resolution allows a non-trivial stringy dynamics as quantum fluctuations below the measurement resolution.

The assumption that electromagnetic charge is well-defined for the modes of the induced spinor field implies in the generic case that the modes are localized to 2-D surfaces carrying vanishing induced W fields and above weak scale also vanishing induced Z^0 field. This makes sense inside the Minkowskian regions at least. The boundaries of the string world sheets carrying fundamental fermions would define uniquely braids and their intersections with partonic 2-surfaces would define the braid points. The imbedding space coordinates of these points in preferred coordinates should be rational in the intersection of realities and p-adicities.

Finite measurement resolution would pose upper limit of the number of the string world sheets and thus to the fermion number of wormhole throat.

5. One can assign to the lightlike parton orbits at which the signature of the induced metric changes a 1-D Dirac action and its bosonic counterpart. The outcome spectrum of light-like 8-momenta and light-like geodesics with the direction of the 8.momenta. Since spinor modes are localized at string world sheets - at least in Minkowskian space-time regions - this term is actually localized at their 1-D boundaries. Finite measurement resolution would mean IR and UV cutoffs to the spectrum of p^k. IR cutoff would be due to the finite size of causal diamond (CD) and UV cutoff to the lower bound for the size of sub-CDS involved.

   Note that Kähler action contains also measurement interaction terms at the space-like ends of the space-time surface. They fix the values of some classical conserved quantities to be equal to their quantum counterparts for the space-time surfaces allowed in quantum superpositions [K22]. Also here finite measurement resolution is expected.

6. The electric-magnetic duality induces S-duality permuting G and ${}^LG$ and the roles of string world sheet as 2-D space-time and partonic 2-surface defining defining the target manifold of string model. The moduli spaces of string world sheets and partonic 2-surfaces are in very close correspondence as implied by the strong form of holography.

4.2.5 How Langlands duality relates to quantum Yangian symmetry of twistor approach?

The are obvious objections against the heuristic considerations represented above.

1. One cannot restrict the attention on single partonic 2-surface or string world sheet. It is the collection of partonic 2-surfaces at the two light-like boundaries of CD and the string world sheets which define the geometric structure to which one should assign both the representations of the Galois group and the collection of world sheets as well as the groups G and ${}^LG$. Therefore also the group G defining the measurement resolution should be assigned to the entire structure and this leaves only single option: G defines the quantum Yangian defining the symmetry of the theory. If this were not complicated enough, note that one should be also able to take into account the possibility that there are CDs within CDs.

2. The finite measurement resolution should correspond to the replacement of ordinary Lie group with something analogous to quantum group. In the simplest situation the components of quantum spinors cease to commute: as a consequence the components correlate and the dimension of the system is reduced to quantum dimension smaller than the algebraic dimension $d = 2$. Ordinary (p, q) wave mechanics is a good example about this: now the dimension of the system is reduced by a factor two from the dimension of phase space to that of configuration space.

3. Quantum Yangian algebra is indeed an algebra analogous to quantum group and according to MacKay did not receive the attention that it received as a symmetry of integrable systems because quantum groups became the industry [A31]. What can one conclude about the quantum Yangian in finite measurement resolution. One can make only guesses and which can be defended only by their internal consistency.

   (a) Since the basic objects are 2-dimensional, the group G should be actually span Kac-Moody type symplectic algebra and Kac-Moody algebra associated with the isometries of the imbedding space: this conforms with the proposed picture. Frenkel has discussed the relations between affine algebras, Langlands duality, and Bethe ansatz already at previous millenium [A19].

   (b) Finite measurement resolution reduces the partonic 2-surfaces to collections of braid ends. Does this mean that Lie group defining quantum Yangian group effectively reduces to something finite-dimensional? Or does the quantum Yangian property already characterize the measurement resolution as one might conclude from the previous argument? The simplest guess is that one obtains quantum Yangian containing as a factor the quantum Yangian associated with a Kac-Moody group defined by a finite-D Lie group with a Cartan algebra for which dimension equals to the total number of ends of braid strands involved. Zero energy states would be singlets for this group. This identification conforms with the general picture.

   (c) There is however an objection against the proposal. Yangian algebra contains a formal complex deformation parameter h but all deformations are equivalent to $h = 1$ deformation by a simple re-scaling of the generators labelled by non-negative integers trivial for $n = 0$ generators. Is Yangian after all unable to describe the finite measurement resolution. This problem could be circumvented by replacing Yangian with so called (twisted) quantum Yangian characterized by a complex quantum deformation parameter q. The representations of twisted quantum Yangians are discussed in [A29].

   (d) The quantum Yangian group should have also as a factor the quantum Yangian assigned to the symplectic group and Kac-Moody group for isometries of H with M^4 isometries extended to the conformal group of M^4. Finite measurement resolution would be realized as a q-deformation also in these degrees of freedom.

   (e) The proposed identification looks consistent with the general picture but one can also consider a reduction of continuous Kac-Moody type algebra to its discrete version obtained by replacing partonic 2-surfaces with the ends of braid strands as an alternative.

4. The appearance of quantum deformation is not new in the context of Langlands conjecture. Frenkel has proposed Langlands correspondence for both quantum groups [A26], and finite-dimensional representations of quantum affine algebras [A27].

4.2.6 The representation of Galois group and effective symmetry group as symplectic flow

Langlands duality involves both the Galois group and effective gauge or Kac-Moody groups G and ${}^LG$ extended to quantum Yangian and defining the automorphic forms and one should understand how these groups emerge in TGD framework.

1. What is the counterpart of Galois group in TGD? It need not be the gigantic Galois group of algebraic numbers regarded as an extension of rationals or algebraic extension of rationals. Here the proposal that infinite primes, integers and rationals are accompanied by collections of partonic 2-surfaces is very natural. Infinite primes can be mapped to irreducible polynomials of n variables and one can construct a procedure which assigns to infinite primes a collection of Galois groups. This collection of Galois groups characterizes a collection of partonic 2-surfaces.

2. How the Galois group is realized and how the symmetry group G realization finite measurement resolution is realized. How the finite-dimensional representations of Galois group lift to the finite-dimensional representations of G. The proposal is that Galois group is lifted

to its braided counterpart just like braid group generalizes the symmetric group. One can speak about space-like and time-like braidings so that one would have two different kind of braidings corresponding to stringy and partonic pictures and it might be possible to understand the emergence of G and ${}^LG$. The symplectic group for the boundary of CD define the isometries of WCW and by its infinite-dimensionality it is unique candidate for realizing representation of any group as its subgroup. he braidings are induced by symplectic flows.

3. Obviously also the symmetry groups G and ${}^LG$ should be realized as symplectic flows in appropriate moduli spaces. There are two different symplectic flows corresponding to space-like and time-like braids so that G and ${}^LG$ can be different and might differ even at the level of Lie algebra. The common realization of Galois group and symmetry group defining measurement resolution would imply Langlands duality automatically. The electric magnetic duality would in turn correspond to the possibility of two kinds of braidings. It must be emphasized that Langlands duality would be something independent of electric-magnetic duality and basically due to the realization of group representations as projective representations realized in terms of braidings. Note that also the automorphic forms define projective representations of G.

Why should the finite Galois group (possibly so!) correspond to Lie group G as it does in number theoretic Langlands correspondence?

1. The dimension of the representation of Galois group is finite and this dimension would correspond to the finite dimension for the representation of G defined by the finite-dimensional space in which G acts. This space is very naturally the moduli space of partonic 2-surfaces with n punctures corresponding to the n braid ends. A possible additional restriction is that the end points of braids are only permuted under the action of G. If the representations of the Galois group indeed automatically lift to the representations of the group defining finite measurement resolution, then Langlands duality would follow automatically.

2. The group G would correspond to the Galois group in very much the same manner as finite subgroups of $SU(2)$ correspond to simply laced Lie groups in MacKay correspondence [A6]. This would generalize Mc Kay correspondence to much more general theorem holding true for the inclusions of HFFs.

An interesting open question is whether one should consider representations of the collection of Galois groups assignable to the construction of zeros for polynomials associated with infinite prime or the gigantic Galois group assignable to algebraic numbers. The latter group could allow naturally p-adic topology. The notion of finite measurement resolution would strongly suggest that one should consider the braided counterpart of the finite Galois group. This would give also a direct connection with the physics in TGD Universe. Langlands correspondence would be basic physics of TGD Universe.

4.2.7 The practical meaning of the geometric Langlands conjecture

This picture seems to lead naturally to number theoretic Langlands conjecture. What geometric Langlands conjecture means in TGD Universe?

1. What it means to replace the braids with entire partonic 2-surfaces. Should one keep the number of braid strands constant and allow also non-rational braid ends? What does the number of rational points correspond at WCW level? How the automorphic forms code the information about the number of rational surfaces in the intersection?

2. Quantum classical correspondence suggests that this information is represented at space-time level. Braid ends characterize partonic 2-surfaces in finite measurement resolution. The quantum state involves a quantum super position of partonic 2-surfaces with the same number of rational braid strands. Different collections of rational points are of course possible. These collections of braid ends should be transformed to each other by a discrete algebraic subgroup of the effective symmetry group G. Suppose that the orbit for a collection of n braid end points contains N different collections of braid points.

One can construct irreps of a discrete subgroup of the symmetry group G at the orbit. Could the number N of points at the orbit define the number which could be identified as the number of rational surfaces in the intersection in the domain of definition of a given WCW spinor field defined in terms of finite measurement resolution. This would look rather natural definition and would nicely integrate number theoretic and geometric Langlands conjectures together. For infinite primes which correspond to polynomials also the Galois groups of local number fields would also entire the picture naturally.

3. One can of course consider the possibility of replacing them with light-like 3-D surfaces or space-like 3- surfaces at the ends of causal diamonds but this is not perhaps not essential since holography implies the equivalence of these identifications. The possible motivation would come from the observations that vanishing of two holomoprhic functions at the boundary of CD defines a 3-D surface.

4.2.8 How TGD approach differs from Witten-Kapustin approach?

The basic difference as compared to Witten-Kapustin approach [A30] is that the moduli space for partonic 2-surfaces replaces in TGD framework the moduli space for Higgs field configurations. Higgs bundle defined as a holomorphic bundle together with Higgs field is the basic concept. In the simplest situations Higgs field is not a scalar but holomorphic 1-form at Riemann surface Y (analog of partonic 2-surface) related closely to the gauge potential of $M^4 = C \times Y$ whose components become scalars in spontaneous compactification to C. This is in complete analogy with the fact that the values of 1-forms defining the basis of cohomology group for partonic 1-surface for cycles defining the basis of 1-homology define conformal moduli.

A possible interpretation is in terms of geometrization of all gauge fields and Higgs field in TGD framework. Color and electroweak gauge fields are geometrized in terms of projections of color Killing vectors and induced spinor connection. Conformal moduli space for the partonic 2-surface would define the geometrization for the vacuum expectation value of the Higgs field.

One can even argue that dynamical Higgs is not consistent with the notion that the modulus characterizes entire 2-surfaces. Maybe the introducing of the quantum fluctuating part of Higgs field is not appropriate. Also the fact, that for Higgs bundle Higgs is actually 1-form suggests that something might be wrong with the notion of Higgs field. Concerning Higgs the recent experimental situation at LHC is critical: it might well turn out that Higgs boson does not exist. In TGD framework the most natural option is that Higgs like particles exist but all of them are "eaten" by gauge bosons meaning that also photon, gluons possess a small mass. Something analogous to the space of Higgs vacuum expectation values might be however needed and this something could correspond to the conformal moduli space. In TGD framework the particle massivation is described in terms of p-adic thermodynamics and the dominant contribution to the mass squared comes from conformal moduli. It might be possible to interpret this contribution as an average of the contribution coming from geometrized Higgs field.

One challenge is to understand whether the moduli spaces assignable to partonic 2-surfaces and with string world sheets are so closely related that they allow the analog of mirror symmetry of the super-string models relating 6-dimensional Calabi-Yau manifolds. For Calabi-Yau: s the mirror symmetry exchanges complex and Kähler structures. Could also now something analogous make sense.

1. Strong form of general coordinate invariance and the notion of preferred extremal implies that the collection of partonic 2-surfaces fixes the collection of string world sheets (these might define single connected sheet as a connected sum). This alone suggests that there is a close correspondence between moduli spaces of the string world sheets and of partonic 2-surfaces.

2. One problem is that space-time sheets in the Minkowskian regions have hyper-complex rather than complex structure. The analog of Kähler form must represent hypercomplex imaginary unit and must be an antisymmetric form multiplied by the complex imaginary unit so that its square equals to the induced metric representing real unit.

3. How the moduli defined by integrals of complex 1-forms over cycles generalize? What one means with cycles now? How the handle numbers g_i of handles for partonic 2-surfaces reveal

themselves in the homology and cohomology of the string world sheet? Do the ends of the string world sheets at the orbits of a given partonic 2-surface define curves which rotate around the handles and is the string would sheet a connected structure obtained as topological sum of this kind of string world sheets. Does the dynamics for preferred extremals of Kähler dictate this?

In the simplest situation (abelian gauge theory) the Higgs bundle corresponds to the upper half plane defined by the possible values of the inverse of the complexified coupling strength

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \quad .$$

Does the transformation for τ defined in this manner make sense?

1. The vacuum functional is the product of exponent of imaginary Kähler action from Minkowskian regions and exponent of real Kähler action from Euclidian regions appears as an exponent proportional to this kind of parameter. The weak form of electric-magnetic duality reduces Kähler action to 3-D Chern-Simons terms at light-like wormhole throats plus possible contributions not assignable to wormhole throats. This realizes the almost topological QFT property of quantum TGD and also holography and means an enormous calculational simplification. The complexified Kähler coupling strength emerges naturally as the multiplier of Chern-Simons term if the latter contributions are not present.

2. There is however no good reason to believe that string world sheets and partonic two-surface should correspond to the values of τ and $-1/\tau$ for a moduli space somehow obtained by gluing the moduli spaces of string worlds sheets and partonic 2-surfaces. More general modular symmetries for τ seem also implausible in TGD framework. The weak form of electric magnetic duality leads to the effective complexification of gauge coupling but there is no reason to give up the idea about the quantum criticality implying quantization of Kähler coupling strength.

3. From the foregoing it is clear that the identification of G as a Kac-Moody type group extended to quantum Yangian and assignable to string model in conformal moduli space is strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

4. The notion of finite measurement resolution suggesting strongly quantum group like structure is what distinguishes TGD approach from Witten's approach and from the foregoing it is clear that the identification of G as a group defining Kac-Moody type group assignable to string model in conformal moduli space and further extended to quantum Yangian is the strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

5. In his lecture Edward Frenkel explains that the recent vision about the conformal moduli is as parameters analogous to gauge coupling constants. It might well be that the moduli could take the role of gauge couplings. This might allow to have a fresh view to the conjecture that the lowest three genera are in special role physically because all these Riemann surfaces are hyper-elliptic (this means global Z_2 conformal symmetry) and because for higher genera elementary particle vacuum functionals vanish for hyper-elliptic Riemann surfaces [K4].

To sum up, the basic differences seem to be due to zero energy ontology, finite measurement resolution, and the identification of space-time as a 4-surface implying strong form of general coordinate invariance implying electric-magnetic and S-dualities implying also the replacement of Higgs bundle with the conformal moduli space.

4.3 About The Structure Of The Yangian Algebra

The attempt to understand Langlands conjecture in TGD framework led to a completely unexpected progress in the understanding of the Yangian symmetry expected to be the basic symmetry of quantum TGD and the following vision suggesting how conformal field theory could be generalized to four-dimensional context is a fruit of this work.

The structure of the Yangian algebra is quite intricate and in order to minimize confusion easily caused by my own restricted mathematical skills it is best to try to build a physical interpretation for what Yangian really is and leave the details for the mathematicians.

1. The first thing to notice is that Yangian and quantum affine algebra are two different quantum deformations of a given Lie algebra. Both rely on the notion of R-matrix inducing a swap of braid strands. R-matrix represents the projective representations of the permutation group for braid strands and possible in 2-dimensional case due to the non-commutativity of the first homotopy group for 2-dimensional spaces with punctures. The R-matrix $R_q(u,v)$ depends on complex parameter q and two complex coordinates u,v. In integrable quantum field theories in M^2 the coordinates u,v are real numbers having identification as exponentials representing Lorenz boosts. In 2-D integrable conformal field theory the coordinates u,v have interpretation as complex phases representing points of a circle. The assumption that the coordinate parameters are complex numbers is the safest one.

2. For Yangian the R-matrix is rational whereas for quantum affine algebra it is trigonometric. For the Yangian of a linear group quantum deformation parameter can be taken to be equal to one by a suitable rescaling of the generators labelled by integer by a power of the complex quantum deformation parameter q. I do not know whether this true in the general case. For the quantum affine algebra this is not possible and in TGD framework the most interesting values of the deformation parameter correspond to roots of unity.

4.3.1 Slicing of space-time sheets to partonic 2-surfaces and string world sheets

The proposal is that the preferred extremals of Kähler action are involved in an essential manner the slicing of the space-time sheets by partonic 2-surfaces and string world sheets. Also an analogous slicing of Minkowski space is assumed and there are infinite number of this kind of slicings defining what I have called Hamilton-Jaboci coordinates [K3]. What is really involved is far from clear. For instance, I do not really understand whether the slicings of the space-time surfaces are purely dynamical or induced by special coordinatizations of the space-time sheets using projections to special kind of sub-manifolds of the imbedding space, or are these two type of slicings equivalent by the very property of being a preferred extremal. Therefore I can represent only what I think I understand about the situation.

1. What is needed is the slicing of space-time sheets by partonic 2-surfaces and string world sheets. The existence of this slicing is assumed for the preferred extremals of Kähler action [K3]. Physically the slicing corresponds to an integrable decomposition of the tangent space of space-time surface to 2-D space representing non-physical polarizations and 2-D space representing physical polarizations and has also number theoretical meaning.

2. In zero energy ontology the complex coordinate parameters appearing in the generalized conformal fields should correspond to coordinates of the imbedding space serving also as local coordinates of the space-time surface. Problems seem to be caused by the fact that for string world sheets hyper-complex coordinate is more natural than complex coordinate. Pair of hyper-complex and complex coordinate emerge naturally as Hamilton-Jacobi coordinates for Minkowski space encountered in the attempts to understand the construction of the preferred extremals of Kähler action.

   Also the condition that the flow lines of conserved isometry currents define global coordinates lead to the to the analog of Hamilton-Jacobi coordinates for space-time sheets [K3]. The physical interpretation is in terms of local polarization plane and momentum plane defined by local light-like direction. What is so nice that these coordinates are highly unique and determined dynamically.

3. Is it really necessary to use two complex coordinates in the definition of Yangian-affine conformal fields? Why not to use hyper-complex coordinate for string world sheets? Since the inverse of hyper-complex number does not exist when the hyper-complex number is light-like, hyper-complex coordinate should appear in the expansions for the Yangian generalization of conformal field as positive powers only. Intriguingly, the Yangian algebra is "one half" of the affine algebra so that only positive powers appear in the expansion. Maybe the hyper-complex expansion works and forces Yangian-affine instead of doubly affine structure. The appearance of only positive conformal weights in Yangian sector could also relate to the fact that also in conformal theories this restriction must be made.

4. It seems indeed essential that the space-time coordinates used can be regarded as imbedding space coordinates which can be fixed to a high degree by symmetries: otherwise problems with general coordinate invariance and with number theoretical universality would be encountered.

5. The slicing by partonic 2-surfaces could (but need not) be induced by the slicing of CD by parallel translates of either upper or lower boundary of CD in time direction in the rest frame of CD (time coordinate varying in the direction of the line connecting the tips of CD). These slicings are not global. Upper and lower boundaries of CD would definitely define analogs of different coordinate patches.

4.3.2 Physical interpretation of the Yangian of quantum affine algebra

What the Yangian of quantum affine algebra or more generally, its super counterpart could mean in TGD framework? The key idea is that this algebra would define a generalization of super conformal algebras of super conformal field theories as well as the generalization of super Virasoro algebra. Optimist could hope that the constructions associated with conformal algebras generalize: this includes the representation theory of super conformal and super Virasoro algebras, coset construction, and vertex operator construction in terms of free fields. One could also hope that the classification of extended conformal theories defined in this manner might be possible.

1. The Yangian of a quantum affine algebra is in question. The heuristic idea is that the two R-matrices - trigonometric and rational- are assignable to the swaps defined by space-like braidings associated with the braids at 3-D space-like ends of space-time sheets at light-like boundaries of CD and time like braidings associated with the braids at 3-D light-like surfaces connecting partonic 2-surfaces at opposite light-like boundaries of CD. Electric-magnetic duality and S-duality implied by the strong form of General Coordinate Invariance should be closely related to the presence of two R-matrices. The first guess is that rational R-matrix is assignable with the time-like braidings and trigonometric R-matrix with the space-like braidings. Here one must or course be very cautious.

2. The representation of the collection of Galois groups associated with infinite primes in terms of braided symplectic flows for braid of braids of.... braids implies that there is a hierarchy of swaps: swaps can also exchange braids of...braids. This would suggest that at the lowest level of the braiding hierarchy the R-matrix associated with a Kac-Moody algebra permutes two braid strands which decompose to braids. There would be two different braided variants of Galois groups.

3. The Yangian of the affine Kac-Moody algebra could be seen as a 4-D generalization of the 2-D Kac-Moody algebra- that is a local algebra having representation as a power series of complex coordinates defined by the projections of the point of the space-time sheet to geodesic spheres of light-cone boundary and geodesic sphere of CP_2.

4. For the Yangian the generators would correspond to polynomials of the complex coordinate of string world sheet and for quantum affine algebra to Laurent series for the complex coordinate of partonic 2-surface. What the restriction to polynomials means is not quite clear. Witten sees Yangian as one half of Kac-Moody algebra containing only the generators having $n \geq 0$. This might mean that the positivity of conformal weight for physical states essential for the construction of the representations of Virasoro algebra would be replaced with automatic positivity of the conformal weight assignable to the Yangian coordinate.

5. Also Virasoro algebra should be replaced with the Yangian of Virasoro algebra or its quantum counterpart. This construction should generalize also to Super Virasoro algebra. A generalization of conformal field theory to a theory defined at 4-D space-time surfaces using two preferred complex coordinates made possible by surface property is highly suggestive. The generalization of conformal field theory in question would have two complex coordinates and conformal invariance associated with both of them. This would therefore reduce the situation to effectively 2-dimensional one rather than 3-dimensional: this would be nothing but the effective 2-dimensionality of quantum TGD implied by the strong form of General Coordinate Invariance.

6. This picture conforms with what the generalization of $D = 4$ $\mathcal{N} = 4$ SYM by replacing point like particles with partonic 2-surfaces would suggest: Yangian is replaced with Yangian of quantum affine algebra rather than quantum group. Note that it is the finite measurement resolution alone which brings in the quantum parameters q_1 and q_2. The finite measurement resolution might be relevant for the elimination of IR divergences.

4.3.3 How to construct the Yangian of quantum affine algebra?

The next step is to try to understand the construction of the Yangian of quantum affine algebra.

1. One starts with a given Lie group G. It could be the group of isometries of the imbedding space or subgroup of it or even the symplectic group of the light-like boundary of $CD \times CP_2$ and thus infinite-dimensional. It could be also the Lie group defining finite measurement resolution with the dimension of Cartan algebra determined by the number of braid strands.

2. The next step is to construct the affine algebra (Kac-Moody type algebra with central extension). For the group defining the measurement resolution the scalar fields assigned with the ends of braid strands could define the Cartan algebra of Kac-Moody type algebra of this group. The ordered exponentials of these generators would define the charged generators of the affine algebra.

   For the imbedding space isometries and symplectic transformations the algebra would be obtained by localizing with respect to the internal coordinates of the partonic 2-surface. Note that also a localization with respect to the light-like coordinate of light-cone boundary or light-like orbit of partonic 2-surface is possible and is strongly suggested by the effective 2-dimensionality of light-like 3-surfaces allowing extension of conformal algebra by the dependence on second real coordinate. This second coordinate should obviously correspond to the restriction of second complex coordinate to light-like 3-surface. If the space-time sheets allow slicing by partonic 2-surfaces and string world sheets this localization is possible for all 2-D partonic slices of space-time surface.

3. The next step is quantum deformation to quantum affine algebra with trigonometric R-matrix $R_{q_1}(u, v)$ associated with space-like braidings along space-like 3-surfaces along the ends of CD. u and v could correspond to the values of a preferred complex coordinate of the geodesic sphere of light-cone boundary defined by rotational symmetry. It choice would fix a preferred quantization axes for spin.

4. The last step is the construction of Yangian using rational R-matrix $R_{q_2}(u, v)$. In this case the braiding is along the light-like orbit between ends of CD. u and v would correspond to the complex coordinates of the geodesic sphere of CP_2. Now the preferred complex coordinate would fix the quantization axis of color isospin.

These arguments are of course heuristic and do not satisfy any criteria of mathematical rigor and the details could of course change under closer scrutinity. The whole point is in the attempt to understand the situation physically in all its generality.

4.3.4 How 4-D generalization of conformal invariance relates to strong form of general coordinate invariance?

The basic objections that one can rise to the extension of conformal field theory to 4-D context come from the successes of p-adic mass calculations. p-Adic thermodynamics relies heavily on the properties of partition functions for super-conformal representations. What happens when one replaces affine algebra with (quantum) Yangian of affine algebra? Ordinary Yangian involves the original algebra and its dual and from these higher multi-local generators are constructed. In the recent case the obvious interpretation for this would be that one has Kac-Moody type algebra with expansion with respect to complex coordinate w for partonic 2-surfaces and its dual algebra with expansion with respect to hyper-complex coordinate of string world sheet.

p-Adic mass calculations suggest that the use of either algebra is enough to construct single particle states. Or more precisely, local generators are enough. I have indeed proposed that the multi-local generators are relevant for the construction of bound states. Also the strong form of general coordinate invariance implying strong form of holography, effective 2-dimensionality, electric-magnetic duality and S-duality suggests the same. If one could construct the states representing elementary particles solely in terms of either algebra, there would be no danger that the results of p-adic mass calculations are lost. Note that also the necessity to restrict the conformal weights of conformal representations to be non-negative would have nice interpretation in terms of the duality.

4.4 Summary And Outlook

It is good to try to see the relationship between Langlands program and TGD from a wider perspective and relate it to other TGD inspired views about problems of what I would call recent day physical mathematics. I try also to become (and remain!) conscious about possible sources of inconsistencies to see what might go wrong.

I see the attempt to understand the relation between Langlands program and TGD as a part of a bigger project the goal of which is to relate TGD to physical mathematics. The basic motivations come from the mathematical challenges of TGD and from the almost-belief that the beautiful mathematical structures of the contemporary physical mathematics must be realized in Nature somehow.

The notion of infinite prime is becoming more and more important concept of quantum TGD and also a common denominator. The infinite-dimensional symplectic group acting as the isometry group of WCW geometry and symplectic flows seems to be another common denominator. Zero energy ontology together with the notion of causal diamond is also a central concept. A further common denominator seems to be the notion of finite measurement resolution allowing discretization. Strings and super-symmetry so beautiful notions that it is difficult to imagine physics without them although super string theory has turned out to be a disappointment in this respect. In the following I mention just some examples of problems that I have discussed during this year.

Infinite primes are certainly something genuinely TGD inspired and it is reasonable to consider their possible role in physical mathematics.

1. The set theoretic view about the fundamentals of mathematics is inspired by classical physics. Cantor's view about infinite ordinals relies on set theoretic representation of ordinals and is plagued by difficulties (say Russel's paradox) [K18]. Infinite primes provide an alternative to Cantor's view about infinity based on divisility alone and allowing to avoid these problems. Infinite primes are obtained by a repeated second quantization of an arithmetic quantum field theory and can be seen as a notion inspired by quantum physics. The conjecture is that quantum states in TGD Universe can be labelled by infinite primes and that standard model symmetries can be understood in terms of octonionic infinite primes defined in appropriate manner.

   The replacement of ordinals with infinite primes would mean a modification of the fundamentals of physical mathematics. The physicists's view about the notion set is also much more restricted than the set theoretic view. Subsets are typically manifolds or even algebraic varieties and they allow description in terms of partial differential equations or algebraic equations.

   Boolean algebra is the quintessence of mathematical logic and TGD suggests that quantum Boolean algebra should replace Boolean algebra [K18]. The representation would be in terms of fermionic Fock states and in zero energy ontology fermionic parts of the state would define Boolean states of form $A \to B$. This notion might be useful for understanding the physical correlates of Boolean cognition and might also provide insights about fundamentals of physical mathematics itself. Boolean cognition must have space-time correlates and this leads to a space-time description of logical OR *resp.* AND as a generalization of trouser diagram of string models *resp.* fusion along ends of partonic 2-surfaces generalizing the 3-vertex of Feyman diagrammatics. These diagrams would give rise to fundamental logic gates.

2. Infinite primes can be represented using polynomials of several variables with rational coefficients [K18]. One can solve the zeros of these polynomials iteratively. At each step one can identify a finite Galois group permuting the roots of the polynomial (algebraic function in general). The resulting Galois groups can be arranged into a hierarchy of Galois groups and the natural idea is that the Galois groups at the upper levels act as homomorphisms of Galois groups at lower levels. A generalization of homology and cohomology theories to their non-Abelian counterparts emerges [K23]: the square of the boundary operation yields unit element in normal homology but now an element in commutator group so that abelianization yields ordinary homology. The proposal is that the roots are represented as punctures of the partonic 2-surfaces and that braids represent symplectic flows representing the braided counterpars of the Galois groups. Braids of braids of.... braids structrure of braids is inherited from the hierarchical structure of infinite primes.

   That braided Galois groups would have a representation as symplectic flows is exactly what physics as generalized number theory vision suggests and is applied also to understand Langlands conjectures. Langlands program would be modified in TGD framework to the study of the complexes of Galois groups associated with infinite primes and integers and have direct physical meaning.

The notion of finite measurement resolution realized at quantum level as inclusions of hyperfinite factors and at space-time level in terms of braids replacing the orbits of partonic 2-surfaces - is also a purely TGD inspired notion and gives good hopes about calculable theory.

1. The notion of finite measurement resolution leads to a rational discretization needed by both the number theoretic and geometric Langlands conjecture. The simplest manner to understand the discretization is in terms of extrema of Chern-Simons action if they correspond to "rational" surfaces. The guess that the rational surfaces are dense in the WCW just as rationals are dense in various number fields is probably quite too optimistic physically. Algebraic partonic 2-surfaces containing typically finite number of rational points having interpretation in terms of finite measurement resolution. Same might apply to algebraic surfaces as points of WCW in given quantum state.

2. The charged generators of the Kac-Moody algebra associated with the Lie group G defining measurement resolution correspond to tachyonic momenta in free field representation using ordered exponentials. This raises unpleasant question. One should have also a realization for the coset construction in which Kac-Moody variant of the symplectic group of $\delta M^4_\pm$ and Kac-Moody algebra of isometry group of H assignable to the light-like 3-surfaces (isometries at the level of WCW *resp.* H) define a coset representation. The actions of corresponding super Virasoro algebras are identical. Now the momenta are however non-tachyonic.

   How these Kac-Moody type algebras relate? From p-adic mass calculations it is clear that the ground states of super-conformal representations have tachyonic conformal weights. Does this mean that the ground states can be organized into representations of the Kac-Moody algebra representing finite measurement resolution? Or are the two Kac-Moody algebra like structures completely independent. This would mean that the positions of punctures cannot correspond to the H-coordinates appearing as arguments of symplectic and Kac-Moody algebra. The fact that the groups associated with algebras are different would allow this.

TGD is a generalization of string models obtained by replacing strings with 3-surfaces. Therefore it is not surprising that stringy structures should appear also in TGD Universe and the strong form of general coordinate invariance indeed implies this. As a matter fact, string like objects appear also in various applications of TGD: consider only the notions of cosmic string [K8] and nuclear string [K13]. Magnetic flux tubes central in TGD inspired quantum biology making possible topological quantum computation [K9] represent a further example.

1. What distinguishes TGD approach from Witten's approach is that twisted SUSY is replaced by string model like theory with strings moving in the moduli space for partonic 2-surfaces or string world sheets related by electric-magnetic duality. Higgs bundle is replaced with the moduli space for punctured partonic 2-surfaces and its electric dual for string world sheets. The new element is the possibility of trouser vertices and generalization of 3-vertex if Feynman diagrams having interpretation in terms of quantum Boolean algebra.

2. Stringy view means that all topologies of partonic 2-surfaces are allowed and that also quantum superpositions of different topologies are allowed. The restriction to single topology and fixed moduli would mean sigma model. Stringy picture requires quantum superposition of different moduli and genera and this is what one expects on physical grounds. The model for CKM mixing indeed assumes that CKM mixing results from different topological mixings for U and D type quarks [K15] and leads to the notion of elementary particle vacuum functional identifiable as a particular automorphic form [K4].

3. The twisted variant of $\mathcal{N} = 4$ SUSY appears as TQFT in many mathematical applications proposed by Witten and is replaced in TGD framework by the stringy picture. Supersymmetry would naturally correspond to the fermionic oscillator operator algebra assignable to the partonic 2-surfaces or string world sheet and SUSY would be broken.

When I look what I have written about various topics during this year I find that symplectic invariance and symplectic flows appear repeatedly.

1. Khovanov homology provides very general knot invariants. In [**?**] rephrased Witten's formulation about Khovanov homology as TQFT in TGD framework. Witten's TQFT is obtained by twisting a 4-dimensional $\mathcal{N} = 4$ SYM. This approach generalizes the original 3-D Chern-Simons approach of Witten. Witten applies twisted 4-D $\mathcal{N} = 4$ SYM also to geometric Langlands program and to Floer homology.

   TGD is an almost topological QFT so that the natural expectation is that it yields as a side product knot invariants, invariants for braiding of knots, and perhaps even invariants for 2-knots: here the dimension $D = 4$ for space-time surface is crucial. One outcome is a generalization of the notion of Wilson loop to its 2-D variant defined by string world sheet and a unique identification of string world sheet for a given space-time surface. The duality between the descriptions based on string world sheets and partonic 2-surfaces is central. I have not yet discussed the implications of the conjectures inspired by Langlands program for the TGD inspired view about knots.

2. Floer homology generalizes the usual Morse theory and is one of the applications of topological QFTs discussed by Witten using twisted SYM. One studies symplectic flows and the basic objects are what might regarded as string world sheets referred to as pseudo-holomorphic surfaces. It is now wonder that also here TGD as almost topological QFT view leads to a generalization of the QFT vision about Floer homology [K23]. The new result from TGD point of view was the realization that the naivest possible interpretation for Kähler action for a preferred extremal is correct. The contribution to Kähler action from Minkowskian regions of space-time surface is imaginary and has identification as Morse function whereas Euclidian regions give the real contribution having interpretation as Kähler function. Both contributions reduce to 3-D Chern-Simons terms and under certain additional assumptions only the wormhole throats at which the signature of the induced metric changes from Minkowskian to Euclidian contribute besides the space-like regions at the ends of the space-time surface at the light-like boundaries of CD.

3. Gromov- Witten invariants are closely related to Floer homology and their definition involves quantum cohomology in which the notion of intersection for two varieties is more general taking into account "quantum fuzziness". The stringy trouser vertex represent the basic diagram: the incoming string world sheets intersect because they can fuse to single string world sheet. Amazingly, this is just that OR in quantum Boolean algebra suggested by TGD. Another diagram would be *and* responsible for genuine particle reactions in TGD framework. There would be a direct connection with quantum Boolean algebra.

Number theoretical universality is one of the corner stones of the vision about physics as generalized number theory. One might perhaps say that a similar vision has guided Grothendieck and his followers.

1. The realization of this vision involves several challenges. One of them is definition of p-adic integration. At least integration in the sense of cohomology is needed and one might also hope that numerical approach to integration exists. It came as a surprise to me that something very similar to number theoretical universality has inspired also mathematicians and that there exist refined theories inspired by the notion of motive introduced by Groethendieck to to define universal cohomology applying in all number fields. One application and also motivation for taking motives very seriously is motivic integration which has found applications in in physics as a manner to calculate twistor space integrals defining scattering amplitudes in twistor approach to $\mathcal{N} = 4$ SUSY. The essence of motivic integral is that integral is an algebraic operation rather than defined by a measure. One ends up with notions like scissor group and integration as processing of symbols. This is of course in spirit with number theoretical approach where integral as measure is replaced with algebraic operation. The problem is that numerics made possible by measure seems to be lost.

2. The TGD inspired proposal for the definition of p-adic integral relies on number theoretical universality reducing the integral essentially to integral in the rational intersection of real and p-adic worlds. An essential role is played at the level of WCW by the decomposition of WCW to a union of symmetric spaces allowing to define what the p-adic variant of WCW is. Also this would conform with the vision that infinite-dimensional geometric existence is unique just from the requirement that it exists. One can consider also the possibility of having p-adic variant of numerical integration [K23].

Twistor approach has led to the emergence of motives to physics and twistor approach is also what gives hopes that some day quantum TGD could be formulated in terms of explicit Feynman rules or their twistorial generalization [K20].

1. The Yangian symmetry and its quantum counterpart were discovered first in integrable quantum theories is responsible for the success fo the twistorial approach. What distinguishes Yangian symmetry from standard symmetries is that the generators of Lie algebra are multilocal. Yangian symmetry is generalized in TGD framework since point like particles are replaced by partonic 2-surfaces meaning that Lie group is replaced with Kac-Moody group or its generalization. Finite measurement resolution however replaces them with discrete set of points defining braid strands so that a close connection with twistor approach and ordinary Yangian symmetry is suggestive in finite measurement resolution. Also the fact that Yangian symmetry relates closely to topological string models supports the expectation that the proposed stringy view about quantum TGD could allow to formulate twistorial approach to TGD.

2. The vision about finite measurement resolution represented in terms of effective Kac-Moody algebra defined by a group with dimension of Cartan algebra given by the number of braid strands must be consistent with the twistorial picture based on Yangians and this requires extension to Yangian algebra- as a matter to quantum Yangian. In this picture one cannot speak about single partonic 2-surface alone and the same is true about the TGD based generalization of Langlands program. Collections of two-surfaces and possibly also string world sheets are always involved. Multi-locality is also required by the basic properties of quantum states in zero energy ontology.

3. The Kac-Moody group extended to quantum Yangian and defining finite measurement resolution would naturally correspond to the gauge group of $\mathcal{N} = \triangle$ SUSY and braid points to the arguments of N-point functions. The new element would be representation of massive particles as bound states of massless particles giving hopes about cancellation of IR divergences and about exact Yangian symmetry. Second new element would be that virtual particles correspond to wormholes for which throats are massless but can have different momenta and opposite signs of energies. This implies that absence of UV divergences and gives hopes that the number of Feynman diagrams is effectively finite and that there is simple expression of twistorial diagrams in terms of Feynman diagrams [K20].

5 Appendix

5.1 Hecke Algebra And Temperley-Lieb Algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{5.1}$$

The algebra reduces to that for symmetric group for $t=1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and imbedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M} : \mathcal{N} < 4$ is given by the relations

$$\begin{aligned} e_{n+1}e_nen+1 &= e_{n+1} \\ e_ne_{n+1}e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ ,t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \tag{5.2}$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_nen+1 &= te_{n+1} \\ e_ne_{n+1}e_n &= te_n \ , \\ e_n^2 &= e_n = e_n^* \ , \ \ ,t = -\sqrt{\mathcal{M}:\mathcal{N}} = -2cos(\pi/n) \ \ ,n = 3,4,... \end{aligned} \tag{5.3}$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

5.2 Some Examples Of Bi-Algebras And Quantum Groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

5.2.1 Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n) \rightarrow A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a,b,c,d)$ to $M_2^{\otimes 2} = k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \ .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det-1=ad-bc-1=0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\left(\begin{array}{cc} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{array} \right) = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \otimes \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ ,$$

$$\left(\begin{array}{cc} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \ .$$

$$S\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = (ad-bc)^{-1} \left(\begin{array}{cc} d & -b \\ -c & a \end{array} \right) \ .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a,b,c,d of the algebra. For instance, one has $\Delta(a) = a \rightarrow a\otimes a + b\otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a,b,c,d by the relations

$$\begin{array}{rr} ba = qab \ , & db = qbd \ , \\ ca = qac \ , & dc = QCD \ , \\ bc = cb \ , & ad-da = (q^{-1}-1)bc \ , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

μ,η,Δ,ϵ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = det_q^{-1} \left(\begin{array}{cc} d & -qb \\ -q^{-1}c & a \end{array} \right) \ .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A,B,C,D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

5.2.2 Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+,X_-] = H \ , \quad [H,X_\pm] = \pm 2X_\pm \ . \tag{5.4}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned} &\Delta(J) = J\otimes 1 + 1\otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_{\pm}, H \ , \\ &S(1) = 1 \ , \qquad\qquad\qquad \epsilon(1) = 1 \ . \end{aligned} \tag{5.5}$$

The enveloping algebras of Borel algebras $U(B_{\pm})$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^{\star}$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^{\star}$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \to H^{\star}$ reads as

$$\begin{array}{ll} a\to a^{\star} \ , \quad ab = (ab)^{\star} = b^{\star}a^{\star} \ , & \\ 1\to 1 \ , \qquad H \to H^{\star} = hH \ , & X_+ \to (X_+)^{\star} = hX_- \ . \end{array} \tag{5.6}$$

The commutation relations of $U_q(sl(2)$ read as

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad [H, X_{\pm}] = \pm 2X_{\pm} \ . \tag{5.7}$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_{\pm}$:

$$\begin{aligned} \Delta(X_{\pm}) &= X_{\pm}\otimes q^{H/2} + q^{-H/2}\otimes X_{\pm} \ , \\ S(X_{\pm}) &= -q^{\pm 1}X_{\pm} \ . \end{aligned} \tag{5.8}$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \ . \tag{5.9}$$

When q is m: th root of unity the q-factorial $[n]_q!$ vanishes for $n\geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [A28].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x\to\Psi(x)$ $M_q(2)\to U_q^{\star}$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q\times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\begin{aligned} &\langle uv, x\rangle = \langle u\otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x\otimes y\rangle \ , \\ &\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u) \end{aligned}$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^\star$.

5.2.3 General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [A28]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:
i) A is indecomposable, that is does not reduce to a direct sum of matrices.
ii) $a_{ij} \leq 0$ holds true for $i < j$.
iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.
A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.
The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{cc} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}}e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}}e_jk_i \ , & e_if_j - f_je_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{5.10}$$

and so called Serre relations

$$\begin{array}{l} \sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} e_i^{1-a_{ij}-l}e_je_i^l = 0, \ \ i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}}(-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} f_i^{1-a_{ij}-l}f_jf_i^l = 0 \ , \ \ i \neq j \ . \end{array} \tag{5.11}$$

Here $q_i = q^{D_i}$ where one has $D_ia_{ij} = a_{ij}D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{5.12}$$
$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{5.13}$$
$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{5.14}$$
$$\tag{5.15}$$

The action of antipode S is defined as

$$S(e_i) = -e_ik_i^{-1} \ , \ \ S(f_i) = -k_if_i \ , \ \ S(k_i) = -k_i^{-1} \ . \tag{5.16}$$

5.2.4 Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [A28].

1. *Affine algebras*

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l + 1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras.

From the $(l+1)\times(l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l\times l$ Cartan matrix of A by dropping away 0: th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i=0,1,..,l)$ satisfying the relations of Eq. 5.11 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d,e_i]=\delta_{i0}e_i \ , \quad [d,f_i]=\delta_{i0}f_i, \quad [d,k_i]=0 \ . \tag{5.17}$$

with comultiplication $\Delta(d)=d\otimes 1+1\otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G})=\mathcal{G}\otimes C\left[t,t^{-1}\right] \ , \tag{5.18}$$

where $C\left[t,t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x\times P, y\otimes Q]=[x,y]\otimes PQ \ . \tag{5.19}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x\otimes P, y\otimes Q)=(x,y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a,b)=Res(\frac{da}{dt},b) \ , \tag{5.20}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n)=a_{-1}$. The two-cocycle satisfies the conditions

$$\begin{aligned}\Psi(a,b)&=-\Psi(b,a) \ , \\ \Psi([a,b],c)+\Psi([b,c],a)+\Psi([c,a],b)&=0 \ .\end{aligned} \tag{5.21}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l)\otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned}J_n^x &= x\otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]}+n\delta_{m+n,0}c \ .\end{aligned} \tag{5.22}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c=0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{5.23}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{5.24}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{5.25}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{5.26}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{5.27}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations $e_i, f_i, k_i, \ i > 0$.

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