Yangian Symmetry, Twistors, and TGD

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April 10, 2015

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Abstract

There has been impressive steps in the understanding of $\mathcal{N}=4$ maximally sypersymmetric YM theory possessing 4-D super-conformal symmetry. This theory is related by AdS/CFT duality to certain string theory in $AdS_5\times S^5$ background. Second stringy representation was discovered by Witten and is based on 6-D Calabi-Yau manifold defined by twistors. The unifying proposal is that so called Yangian symmetry is behind the mathematical miracles involved.

In the following I will discuss briefly the notion of Yangian symmetry and suggest its generalization in TGD framework by replacing conformal algebra with appropriate super-conformal algebras. Also a possible realization of twistor approach and the construction of scattering amplitudes in terms of Yangian invariants defined by Grassmannian integrals is considered in TGD framework and based on the idea that in zero energy ontology one can represent massive states as bound states of massless particles. There is also a proposal for a physical interpretation of the Cartan algebra of Yangian algebra allowing to understand at the fundamental level how the mass spectrum of n-particle bound states could be understood in terms of the n-local charges of the Yangian algebra.

Twistors were originally introduced by Penrose to characterize the solutions of Maxwell's equations. Kähler action is Maxwell action for the induced Kähler form of CP_2. The preferred extremals allow a very concrete interpretation in terms of modes of massless non-linear field. 4-D Minkowski space identifiable and CP_2 are the only 4-D spaces allowing twistor space with Kähler structure. This raises the hope that twistors would play fundamental role in TGD.

1 Introduction

Lubos Motl [B11] told for some time ago about last impressive steps in the understanding of $\mathcal{N}=4$ maximally supersymmetric YM theory (SYM) possessing 4-D super-conformal symmetry. This theory is related by AdS/CFT duality to certain string theory in $AdS_5\times S^5$ background. Second stringy representation was discovered by Witten and based on 6-D Calabi-Yau manifold defined by twistors.

In the following I will discuss briefly the notion of Yangian symmetry and suggest its generalization in TGD framework by replacing conformal algebra with appropriate super-conformal algebras. Also a possible realization of twistor approach and the construction of scattering amplitudes in terms of Yangian invariants defined by Grassmannian integrals is considered in TGD framework and based on the idea that in zero energy ontology one can represent massive states as bound states of massless particles. There is also a proposal for a physical interpretation of the Cartan algebra of Yangian algebra allowing to understand at the fundamental level how the mass spectrum of n-particle bound states could be understood in terms of the n-local charges of the Yangian algebra. The study of modified Dirac equation leads to a detailed proposal for the generators of Yangian algebras [K20]: the proposal is discussed also in this chapter.

Twistors were originally introduced by Penrose to characterize the solutions of Maxwell's equations. Kähler action is Maxwell action for the induced Kähler form of CP_2. The preferred extremals allow a very concrete interpretation in terms of modes of massless non-linear field. Both conformally compactified Minkowski space identifiable as so called causal diamond and CP_2 allow a description in terms of twistors.

1.1 Background

I am outsider as far as concrete calculations in $\mathcal{N}=4$ SUSY are considered and the following discussion of the background probably makes this obvious. My hope is that the reader had patience to not care about this and try to see the big pattern.

The developments began from the observation of Parke and Taylor [B12] that n-gluon tree amplitudes with less than two negative helicities vanish and those with two negative helicities have unexpectedly simple form when expressed in terms of spinor variables used to represent light-like momentum. In fact, in the formalism based on Grassmanian integrals the reduced tree amplitude for two negative helicities is just "1" and defines Yangian invariant. The article *Perturbative Gauge Theory As a String Theory In Twistor Space* [B15] by Witten led to so called Britto-Cachazo-Feng-Witten (BCFW) recursion relations for tree level amplitudes [B13, B8, B13] allowing to construct tree amplitudes using the analogs of Feynman rules in which vertices correspond to maximally helicity violating tree amplitudes (2 negative helicity gluons) and propagator is massless Feynman propagator for boson. The progress inspired the idea that the theory might be completely integrable meaning the existence of infinite-dimensional un-usual symmetry. This symmetry would be so called Yangian symmetry [K14] assigned to the super counterpart of the conformal group of 4-D Minkowski space.

Drumond, Henn, and Plefka represent in the article *Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory* [B9] an argument suggesting that the Yangian invariance of the scattering amplitudes ins an intrinsic property of planar $\mathcal{N}=4$ super Yang Mills at least at tree level.

The latest step in the progress was taken by Arkani-Hamed, Bourjaily, Cachazo, Carot-Huot, and Trnka and represented in the article *Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory* [B7]. At the same day there was also the article of Rutger Boels entitled *On BCFW shifts of integrands and integrals* [B3] in the archive. Arkani-Hamed *et al* argue that a full Yangian symmetry of the theory allows to generalize the BCFW recursion relation for tree amplitudes to all loop orders at planar limit (planar means that Feynman diagram allows imbedding to plane without intersecting lines). On mass shell scattering amplitudes are in question.

1.2 Yangian symmetry

The notion equivalent to that of Yangian was originally introduced by Faddeev and his group in the study of integrable systems. Yangians are Hopf algebras which can be assigned with Lie algebras as the deformations of their universal enveloping algebras. The elegant but rather cryptic looking definition is in terms of the modification of the relations for generating elements [K14]. Besides ordinary product in the enveloping algebra there is co-product Δ which maps the elements of the enveloping algebra to its tensor product with itself. One can visualize product and co-product is in terms of particle reactions. Particle annihilation is analogous to annihilation of two particle so single one and co-product is analogous to the decay of particle to two. Δ allows to construct higher generators of the algebra.

Lie-algebra can mean here ordinary finite-dimensional simple Lie algebra, Kac-Moody algebra or Virasoro algebra. In the case of SUSY it means conformal algebra of M^4- or rather its super counterpart. Witten, Nappi and Dolan have described the notion of Yangian for super-conformal algebra in very elegant and and concrete manner in the article *Yangian Symmetry in D=4 superconformal Yang-Mills theory* [B10]. Also Yangians for gauge groups are discussed.

In the general case Yangian resembles Kac-Moody algebra with discrete index n replaced with a continuous one. Discrete index poses conditions on the Lie group and its representation (adjoint representation in the case of $\mathcal{N}=4$ SUSY). One of the conditions conditions is that the tensor product $R\otimes R^*$ for representations involved contains adjoint representation only once. This condition is non-trivial. For $SU(n)$ these conditions are satisfied for any representation. In the case of $SU(2)$ the basic branching rule for the tensor product of representations implies that the condition is satisfied for the product of any representations.

Yangian algebra with a discrete basis is in many respects analogous to Kac-Moody algebra. Now however the generators are labelled by non-negative integers labeling the light-like incoming and outgoing momenta of scattering amplitude whereas in in the case of Kac-Moody algebra also negative values are allowed. Note that only the generators with non-negative conformal weight appear in the construction of states of Kac-Moody and Virasoro representations so that the extension to Yangian makes sense.

The generating elements are labelled by the generators of ordinary conformal transformations acting in M^4 and their duals acting in momentum space. These two sets of elements can be labelled by conformal weights $n=0$ and $n=1$ and and their mutual commutation relations are same as for Kac-Moody algebra. The commutators of $n=1$ generators with themselves are however something different for a non-vanishing deformation parameter h. Serre's relations characterize the difference and involve the deformation parameter h. Under repeated commutations the generating elements generate infinite-dimensional symmetric algebra, the Yangian. For $h=0$ one obtains just one half of the Virasoro algebra or Kac-Moody algebra. The generators with $n>0$ are $n+1$-local in the sense that they involve $n+1$-forms of local generators assignable to the ordered set of incoming particles of the scattering amplitude. This non-locality generalizes the notion of local symmetry and is claimed to be powerful enough to fix the scattering amplitudes completely.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://www.tgdtheory.fi/cmaphtml.html` [?]. Pdf representation of same files serving as a kind of glossary can be found at `http://www.tgdtheory.fi/tgdglossary.pdf` [?]. The topics relevant to this chapter are given by the following list.

- Generalized Feynman diagrams [?]
- The unique role of twistors in TGD [?]
- Twistors and TGD [?]

2 How to generalize Yangian symmetry in TGD framework?

As far as concrete calculations are considered, I have nothing to say. I am just perplexed. It is however possible to keep discussion at general level and still say something interesting (as I hope!). The key question is whether it could be possible to generalize the proposed Yangian symmetry and geometric picture behind it to TGD framework.

1. The first thing to notice is that the Yangian symmetry of $\mathcal{N} = 4$ SUSY in question is quite too limited since it allows only single representation of the gauge group and requires massless particles. One must allow all representations and massive particles so that the representation of symmetry algebra must involve states with different masses, in principle arbitrary spin and arbitrary internal quantum numbers. The candidates are obvious: Kac-Moody algebras [A2] and Virasoro algebras [A3] and their super counterparts. Yangians indeed exist for arbitrary super Lie algebras. In TGD framework conformal algebra of Minkowski space reduces to Poincare algebra and its extension to Kac-Moody allows to have also massive states.

2. The formal generalization looks surprisingly straightforward at the formal level. In zero energy ontology one replaces point like particles with partonic two-surfaces appearing at the ends of light-like orbits of wormhole throats located to the future and past light-like boundaries of causal diamond ($CD \times CP_2$ or briefly CD). Here CD is defined as the intersection of future and past directed light-cones. The polygon with light-like momenta is naturally replaced with a polygon with more general momenta in zero energy ontology and having partonic surfaces as its vertices. Non-point-likeness forces to replace the finite-dimensional super Lie-algebra with infinite-dimensional Kac-Moody algebras and corresponding super-Virasoro algebras assignable to partonic 2-surfaces.

3. This description replaces disjoint holomorphic surfaces in twistor space with partonic 2-surfaces at the boundaries of $CD \times CP_2$ so that there seems to be a close analogy with Cachazo-Svrcek-Witten picture. These surfaces are connected by either light-like orbits of partonic 2-surface or space-like 3-surfaces at the ends of CD so that one indeed obtains the analog of polygon.

What does this then mean concretely (if this word can be used in this kind of context)?

1. At least it means that ordinary Super Kac-Moody and Super Virasoro algebras associated with isometries of $M^4 \times CP_2$ annihilating the scattering amplitudes must be extended to a co-algebras with a non-trivial deformation parameter. Kac-Moody group is thus the product of Poincare and color groups. This algebra acts as deformations of the light-like 3-surfaces representing the light-like orbits of particles which are extremals of Chern-Simon action with the constraint that weak form of electric-magnetic duality holds true. I know so little about the mathematical side that I cannot tell whether the condition that the product of the representations of Super-Kac-Moody and Super-Virasoro algebras contains adjoint representation only once, holds true in this case. In any case, it would allow all representations of finite-dimensional Lie group in vertices whereas $\mathcal{N} = 4$ SUSY would allow only the adjoint.

2. Besides this ordinary kind of Kac-Moody algebra there is the analog of Super-Kac-Moody algebra associated with the light-cone boundary which is metrically 3-dimensional. The finite-dimensional Lie group is in this case replaced with infinite-dimensional group of symplectomorphisms of $\delta M^4_{+/-}$ made local with respect to the internal coordinates of the partonic 2-surface. Super-symplectic algebra is realized in terms of second quantized spinor fields and covariantly constant modes of right-handed neutrino. Symplectic group has as sub-group symplectic isometries and the Super-Kac-Moody algebra associated with this group and represented in terms of spinor modes localized to string world sheets plays also a key role in TGD.

3. The construction of TGD leads also to other super-conformal algebras and the natural guess is that the Yangians of all these algebras annihilate the scattering amplitudes.

4. Obviously, already the starting point symmetries look formidable but they still act on single partonic surface only. The discrete Yangian associated with this algebra associated with the closed polygon defined by the incoming momenta and the negatives of the outgoing momenta acts in multi-local manner on scattering amplitudes. It might make sense to speak about polygons defined also by other conserved quantum numbers so that one would have generalized light-like curves in the sense that state are massless in 8-D sense.

2.1 Is there any hope about description in terms of Grassmannians?

At technical level the successes of the twistor approach rely on the observation that the amplitudes can be expressed in terms of very simple integrals over sub-manifolds of the space consisting of k-dimensional planes of n-dimensional space defined by delta function appearing in the integrand. These integrals define super-conformal Yangian invariants appearing in twistorial amplitudes and the belief is that by a proper choice of the surfaces of the twistor space one can construct all invariants. One can construct also the counterparts of loop corrections by starting from tree diagrams and annihilating pair of particles by connecting the lines and quantum entangling the states at the ends in the manner dictated by the integration over loop momentum. These operations can be defined as operations for Grassmannian integrals in general changing the values of n and k. This description looks extremely powerful and elegant and -most importantly- involves only the external momenta.

The obvious question is whether one could use similar invariants in TGD framework to construct the momentum dependence of amplitudes.

1. The first thing to notice is that the super algebras in question act on infinite-dimensional representations and basically in the world of classical worlds assigned to the partonic 2-surfaces correlated by the fact that they are associated with the same space-time surface. This does not promise anything very practical. On the other hand, one can hope that everything related to other than M^4 degrees of freedom could be treated like color degrees of freedom in $\mathcal{N} = 4$ SYM and would boil down to indices labeling the quantum states. The Yangian conditions coming from isometry quantum numbers, color quantum numbers, and electroweak quantum numbers are of course expected to be highly non-trivial and could fix the coefficients of various singlets resulting in the tensor product of incoming and outgoing states.

2. The fact that incoming particles can be also massive seems to exclude the use of the twistor space. The following observation however raises hopes. The Dirac propagator for wormhole throat is massless propagator but for what I call pseudo momentum. It is still unclear how this momentum relates to the actual four-momentum. Could it be actually equal to it? The recent view about pseudo-momentum does not support this view but it is better to keep mind open. In any case this finding suggests that twistorial approach could work in in more or less standard form. What would be needed is a representation for massive incoming particles as bound states of massless partons. In particular, the massive states of super-conformal representations should allow this kind of description.

Could zero energy ontology allow to achieve this dream?

1. As far as divergence cancellation is considered, zero energy ontology suggests a totally new approach producing the basic nice aspects of QFT approach, in particular unitarity and coupling constant evolution. The big idea related to zero energy ontology is that all virtual particle particles correspond to wormhole throats, which are pairs of on mass shell particles. If their momentum directions are different, one obtains time-like continuum of virtual momenta and if the signs of energy are opposite one obtains also space-like virtual momenta. The on mass shell property for virtual partons (massive in general) implies extremely strong constraints on loops and one expect that only very few loops remain and that they are finite since loop integration reduces to integration over much lower-dimensional space than in the QFT approach. There are also excellent hopes about Cutkoski rules.

2. Could zero energy ontology make also possible to construct massive incoming particles from massless ones? Could one construct the representations of the super conformal algebras using only massless states so that at the fundamental level incoming particles would be massless and one could apply twistor formalism and build the momentum dependence of amplitudes using Grassmannian integrals.

   One could indeed construct on mass shell massive states from massless states with momenta along the same line but with three-momenta at opposite directions. Mass squared is given by $M^2 = 4E^2$ in the coordinate frame, where the momenta are opposite and of same magnitude. One could also argue that partonic 2-surfaces carrying quantum numbers of fermions and their superpartners serve as the analogs of point like massless particles and that topologically condensed fermions and gauge bosons plus their superpartners correspond to pairs of wormhole throats. Stringy objects would correspond to pairs of wormhole throats at the same space-time sheet in accordance with the fact that space-time sheet allows a slicing by string worlds sheets with ends at different wormhole throats and defining time like braiding.

The weak form of electric magnetic duality indeed supports this picture. To understand how, one must explain a little bit what the weak form of electric magnetic duality means.

1. Elementary particles correspond to light-like orbits of partonic 2-surfaces identified as 3-D surfaces at which the signature of the induced metric of space-time surface changes from Euclidian to Minkowskian and 4-D metric is therefore degenerate. The analogy with black hole horizon is obvious but only partial. Weak form of electric-magnetic duality states that the Kähler electric field at the wormhole throat and also at space-like 3-surfaces defining the ends of the space-time surface at the upper and lower light-like boundaries of the causal diamond is proportonial to Kähler magnetic field so that Kähler electric flux is proportional Kähler magnetic flux. This implies classical quantization of Kähler electric charge and fixes the value of the proportionality constant.

2. There are also much more profound implications. The vision about TGD as almost topological QFT suggests that Kähler function defining the Kähler geometry of the "world of classical worlds" (WCW) and identified as Kähler action for its preferred extremal reduces to the 3-D Chern-Simons action evaluted at wormhole throats and possible boundary components. Chern-Simons action would be subject to constraints. Wormhole throats and space-like 3-surfaces would represent extremals of Chern-Simons action restricted by the constraint force stating electric-magnetic duality (and realized in terms of Lagrange multipliers as usual).

   If one assumes that Kähler current and other conserved currents are proportional to current defining Beltrami flow whose flow lines by definition define coordinate curves of a globally defined coordinate, the Coulomb term of Kähler action vanishes and it reduces to Chern-Simons action if the weak form of electric-magnetic duality holds true. One obtains almost topological QFT. The absolutely essential attribute "almost" comes from the fact that Chern-Simons action is subject to constraints. As a consequence, one obtains non-vanishing four-momenta and WCW geometry is non-trivial in M^4 degrees of freedom. Otherwise one would have only topological QFT not terribly interesting physically.

Consider now the question how one could understand stringy objects as bound states of massless particles.

1. The observed elementary particles are not Kähler monopoles and there much exist a mechanism neutralizing the monopole charge. The only possibility seems to be that there is opposite Kähler magnetic charge at second wormhole throat. The assumption is that in the case of color neutral particles this throat is at a distance of order intermediate gauge boson Compton length. This throat would carry weak isospin neutralizing that of the fermion and only electromagnetic charge would be visible at longer length scales. One could speak of electro-weak confinement. Also color confinement could be realized in analogous manner by requiring the cancellation of monopole charge for many-parton states only. What comes out are string like objects defined by Kähler magnetic fluxes and having magnetic monopoles at ends. Also more general objects with three strings branching from the vertex appear in the case of baryons. The natural guess is that the partons at the ends of strings and more general objects are massless for incoming particles but that the 3-momenta are in opposite directions so that stringy mass spectrum and representations of relevant super-conformal algebras are obtained. This description brings in mind the description of hadrons in terms of partons moving in parallel apart from transversal momentum about which only momentum squared is taken as observable.

2. Quite generally, one expects for the preferred extremals of Kähler action the slicing of space-time surface with string world sheets with stringy curves connecting wormhole throats. The ends of the stringy curves can be identified as light-like braid strands. Note that the strings themselves define a space-like braiding and the two braidings are in some sense dual. This has a concrete application in TGD inspired quantum biology, where time-like braiding defines topological quantum computer programs and the space-like braidings induced by it its storage into memory. Stringlike objects defining representations of super-conformal algebras must correspond to states involving at least two wormhole throats. Magnetic flux tubes connecting the ends of magnetically charged throats provide a particular realization of stringy on mass shell states. This would give rise to massless propagation at the parton level. The stringy quantization condition for mass squared would read as $m^2 = n$ in suitable units for the representations of super-conformal algebra associated with the isometries. For pairs of throats of the same wormhole contact stringy spectrum does not seem plausible since the wormhole contact is in the direction of CP_2. One can however expect generation of small mass as deviation of vacuum conformal weight from half integer in the case of gauge bosons.

If this picture is correct, one might be able to determine the momentum dependence of the scattering amplitudes by replacing free fermions with pairs of monopoles at the ends of string and topologically condensed fermions gauge bosons with pairs of this kind of objects with wormhole throat replaced by a pair of wormhole throats. This would mean suitable number of doublings of the Grassmannian integrations with additional constraints on the incoming momenta posed by the mass shell conditions for massive states.

2.2 Could zero energy ontology make possible full Yangian symmetry?

The partons in the loops are on mass shell particles have a discrete mass spectrum but both signs of energy are possible for opposite wormhole throats. This implies that in the rules for constructing loop amplitudes from tree amplitudes, propagator entanglement is restricted to that corresponding to pairs of partonic on mass shell states with both signs of energy. As emphasized in [B7], it is the Grassmannian integrands and leading order singularities of $\mathcal{N} = 4$ SYM, which possess the full Yangian symmetry. The full integral over the loop momenta breaks the Yangian symmetry and brings in IR singularities. Zero energy ontologist finds it natural to ask whether QFT approach shows its inadequacy both via the UV divergences and via the loss of full Yangian symmetry. The restriction of virtual partons to discrete mass shells with positive or negative sign of energy imposes extremely powerful restrictions on loop integrals and resembles the restriction to leading order singularities. Could this restriction guarantee full Yangian symmetry and remove also IR singularities?

2.3 Could Yangian symmetry provide a new view about conserved quantum numbers?

The Yangian algebra has some properties which suggest a new kind of description for bound states. The Cartan algebra generators of $n = 0$ and $n = 1$ levels of Yangian algebra commute. Since the co-product Δ maps $n = 0$ generators to $n = 1$ generators and these in turn to generators with high value of n, it seems that they commute also with $n \geq 1$ generators. This applies to four-momentum, color isospin and color hyper charge, and also to the Virasoro generator L_0 acting on Kac-Moody algebra of isometries and defining mass squared operator.

Could one identify total four momentum and Cartan algebra quantum numbers as sum of contributions from various levels? If so, the four momentum and mass squared would involve besides the local term assignable to wormhole throats also n-local contributions. The interpretation in terms of n-parton bound states would be extremely attractive. n-local contribution would involve interaction energy. For instance, string like object would correspond to $n = 1$ level and give $n = 2$-local contribution to the momentum. For baryonic valence quarks one would have 3-local contribution corresponding to $n = 2$ level. The Yangian view about quantum numbers could give a rigorous formulation for the idea that massive particles are bound states of massless particles.

2.4 What about the selection of preferred $M^2 \subset M^4$?

The puzzling aspect of the proposed picture is the restriction of the pseudo-momenta to M^2 and quite generally the the selection of preferred plane $M^2 \subset M^4$. This selection is one the key aspects of TGD but is not too well understood. Also the closely related physical interpretation of the 2-D pseudo-momenta in M^2 is unclear.

2.4.1 The avatars of $M^2 \subset M^4$ in quantum TGD

The choice of preferred plane $M^2 \subset M^4$ pops u again and again in quantum TGD.

1. There are very strong reasons to believe that the solutions of field equations for the preferred extremals assign M^2 to each point of space-time surface and the interpretation is as the plane of non-physical polarizations. One can also consider the possibility that M^2 depends on the point of space-time surface but that the different choices integrate to 2-D surface analogous to string world sheet - very naturally projection of stringy worlds sheets defining the slicing of the space-time surface.

2. The number theoretic vision- in particular M^8-H duality ($H = M^4 \times CP_2$) providing a purely number theoretic interpretation for the choice $H = M^4 \times CP_2$ - involves also the selection of preferred M^2. The duality states that the surfaces in H can be regarded equivalently as surfaces in M^8. The induced metric and Kähler form are identical as also the value of Kähler function. The description of the duality is following.

   (a) The points of space-time surface in $M^8 = M^4 \times E^4$ in M^8 are mapped to points of space-time surface in $M^4 \times CP_2$. The M^4 part of the map is just a projection.

   (b) CP_2 part of the map is less trivial. The idea is that M^8 is identified as a subspace of complexified octonions obtained by adding commutative imaginary unit, I call this sub-space hyper-octonionic. Suppose that space-time surface is hyper-quaternionic (in appropriate sense meaning that one can attach to its each point a hyper-quaternionic plane, not necessary tangent plane). Assume that it also contains a preferred hypercomplex plane M^2 of M^8 at each point -or more generally a varying plane M^2 planes whose distribution however integrates to form 2-surface analogous to string world sheet. The interpretation is as a preferred plane of non-physical polarizations so that basic aspect of gauge symmetry would have a number theoretic interpretation. Note that one would thus have a local hierarchy of octonionic, quaternionic, and complex planes.

   (c) Under these assumptions the tangent plane (if action is just the four-volume or its generalization in the case of Kähler action) is characterized by a point of $CP_2 = SU(3)/U(2)$

where $SU(3)$ is automorphism group of octonions respecting preferred plane M^2 of polarizations and $U(2)$ is automorphism group acting in the hyper-quaternionic plane. This point can be identified as a point of CP_2 so that one obtains the duality.

3. Also the definition of CDs and the proposed construction of the hierarchy of Planck constants involve a choice of preferred M^2, which corresponds to the choice of rest frame and quantization axis of angular momentum physically. Therefore the choice of quantization axis would have direct correlates both at the level of CDs and space-time surface. The vector between the tips of CD indeed defines preferred direction of time and thus rest system. Similar considerations apply in the case of CP_2.

4. Preferred M^2 -but now at this time at momentum space level - appears as the plane of pseudo-momenta associated with the generalized eigen modes of the Kähler-Dirac equation associated with Chern-Simons action. Internally consistency requires a restriction to this plane. This looks somewhat mysterious since this would mean that all exchanged virtual momenta would be in M^2 if the choice is same for all lines of the generalized Feynman graph. This would restrict momentum exchanges in particle reactions to single dimension and does not make sense. One must however notice that in the description of hadronic reactions in QCD picture one makes a choice of longitudinal momentum direction and considers only longitudinal momenta. It would seem that the only possibility is that the planes M^2 are independent for independent exchanged momenta. For instance, in $2 \rightarrow 2$ scattering the exchange would be in plane defined by the initial and final particles of the vertex. There are also good arguments for a number theoretic quantization of the momenta in M^2.

   The natural expectation from $M^8 - H$ duality is that the selection of preferred M^2 implies a reduction of symmetries to those of $M^2 \times E^6$ and $M^2 \times E^2 \times CP_2$. Could the equivalence of M^8 and H descriptions force the reduction of M^4 momentum to M^2 momentum implied also by the generalized eigen value equation for the Kähler-Dirac operator at wormhole throats?

2.4.2 The moduli space associated with the choice of M^2

Lorentz invariance requires that one must have moduli space of CDs with fixed tips defined as $SO(3,1)/SO(1) \times SO(2)$ characterizing different choices of M^2. Maximal Lorentz invariance requires the association of this moduli space to all lines of the generalized Feynman graph. It is easy to deduce that this space is actually the hyperboloid of 5-D Minkowski space. The moduli space is 4-dimensional and has Euclidian signature of the metric. This follows from the fact that $SO(3,1)$ has Euclidian signature as a surface in the four-fold Cartesian power $H(1,3)^4$ of Lobatchevski space with points identified as four time-like unit vectors defining rows of the matrix representing Lorentz transformation. This surface is defined by the 6 orthogonality conditions for the rows of the Lorentz transformation matrix constraints stating the orthogonality of the 4 unit vectors. The Euclidian signature fixes the identification of the moduli space as $H(1,5)$ having Euclidian signature of metric. The 10-D isometry group $SO(1,5)$ of the moduli space acts as symmetries of 5-D Minkowski space (note that the conformal group of M^4 is $SO(2,4)$. The non-compactness of this space does not favor the idea of wave function in moduli degrees of freedom.

Concerning the interpretation of pseudo-momenta it is best to be cautious and make only questions. Should one assume that M^2 for the exchanged particle is fixed by the initial and final momenta of the particle emitting it? How to fix in this kind of situation a unique coordinate frame in which the number theoretic quantization of exchanged momenta takes place? Could it be the rest frame for the initial state of the emitting particle so that one should allow also boosts of the number theoretically preferred momenta? Should one only assume the number theoretically preferred mass values for the exchanged particle but otherwise allow the hyperbolic angle characterizing the energy vary freely?

2.5 Does $M^8 - H$ duality generalize the duality between twistor and momentum twistor descriptions?

$M^8 - H$ duality is intuitively analogous to the duality of elementary wave mechanics meaning that one can use either x-space or momentum space to describe particles. M^8 is indeed the tangent space of H and one could say that $M^8 - H$ duality assigns to a 4-surface in H its "momentum" or tangent as a 4-surface in M^8. The more concrete identification of M^8 as cotangent bundle of H so that its points would correspond to 8-momenta: this very naive picture is of course not correct.

$M^8 - H$ duality suggests that the descriptions using isometry groups of $M^4 \times E^4$ and $M^4 \times CP_2$ -or as the special role of M^2 suggests - those of $M^2 \times E^6$ and $M^2 \times E^2 \times CP_2$ should be equivalent. The interpretation in hadron physics context would be that $SO(4)$ is the counterpart of color group in low energy hadron physics acting on strong isospin degrees of freedom and $SU(3)$ that of QCD description useful at high energies. $SO(4)$ is indeed used in old fashioned hadron physics when quarks and gluons had not yet been introduced. Skyrme model is one example.

The obvious question is whether the duality between descriptions based on twistors and momentum space twistors generalizes to $M^8 - H$ duality. The basic objection is that the charges and their duals should correspond to the same Lie algebra- or rather Kac-Moody algebra. This is however not the case. For the massless option one has $SO(2) \times SU(3)$ at H-side and $SO(2) \times SO(4)$ or $SO(6)$ and M^8 side.This suggests that $M^8 - H$ duality is analogous to the duality between descriptions using twistors and momentum space twistors and transforms the local currents J_0 to non-local currents J_1 and vice versa. This duality would be however be more general in the sense that would relate Yangian symmetries with different Kac-Moody groups transforming locality to non-locality and vice versa. This interpretation is consistent with the fact that the groups $SO(2) \times SO(4)$, $SO(6)$ and $SO(2) \times SU(3)$ have same rank and the standard construction of Kac-Moody generators in terms of exponentials of the Cartan algebra involves only different weights in the exponentials.

If $M^8 - H$ duality has something to do with the duality between descriptions using twistors and momentum space twistors involved with Yangian symmetry, it should be consistent with the basic aspects of the latter duality. The following arguments provide support for this.

1. $SO(4)$ should appear as a dynamical symmetry at $M^4 \times CP_2$ side and $SU(3)$ at M^8 side (where it indeed appears as both subgroup of isometries and as tangent space group respecting the choice of M^2. One could consider the breaking of $SO(4)$ to the subgroup corresponding to vectorial transformations and interpreted in terms of electroweak vectorial $SU(2)$: this would conform with conserved vector current hypothesis and partially conserved axial current hypothesis. The $U(1)$ factor assignable to Kähler form is also present and allows Kac-Moody variant and an extension to Yangian.

2. The heuristics of twistorial approach suggests that the roles of currents J_0 and their non-local duals J_1 in Minkowski space are changed in the transition from H description to M^8 description in the sense that the non-local currents J_1 in H description become local currents in 8-momentum space (or 4-momentum +strong isospin) in M^8 description and J_0 becomes non-local one. In the case of hadron physics the non-local charges assignable to hadrons as collections of partons would become local charges meaning that one can assign them to partonic 2-surfaces at boundaries of CDs assigned to M^8: this says that hadrons are the only possible final states of particle reactions. By the locality it would be impossible decompose momentum and strong isospin to a collection of momenta and strong isospins assigned to partons.

3. In H description it would be impossible to do decompose quantum numbers to those of quarks and gluons at separate uncorrelated partonic 2-surfaces representing initial and final states of particle reaction. A possible interpretation would be in terms of monopole confinement accompanying electroweak and color confinement: single monopole is not a particle. In $M^4 \times E^4$ monopoles must be also present since induced Kähler forms are identical. The Kähler form represents magnetic monopole in E^4 and breaks its translational symmetry and also selects unique $M^4 \times E^4$ decomposition.

4. Since the physics should not depend on its description, color should be confined also now. Indeed, internal quantum numbers should be assigned in M^8 picture to a wave function in $M^2 \times E^6$ and symmetries would correspond to $SO(1,1) \times SO(6)$ or - if broken- to those of $SO(1,1) \times G$, $G = SO(2) \times SO(4)$ or $G = SO(3) \times SO(3)$. Color would be completely absent in accordance with the idea that fundamental observable objects are color singlets. Instead

of color one would have $SO(4)$ quantum numbers and $SO(4)$ confinement: note that the rank of this group crucial for Kac-Moody algebra construction is same as that of $SU(3)$.

It is not clear whether the numbers of particle states should be same for $SO(4)$ and $SU(3)$. If so, quark triplet should correspond to doublet and singlet for strong vectorial isospin in M^8 picture. Gluons would correspond to $SU(2)_V$ multiplets contained by color octet and would therefore contain also other representations than adjoint. This could make sense in composite particle interpretation.

5. For $M^2 \times E^2$ longitudinal momentum and helicity would make sense and one could speak of massless strong isospin at M^8 side and massless color at H-side: note that massless color is the only possibility. For $M^2 \times SO(6)$ option one would have 15-D adjoint representation of $SO(6)$ decomposing as $3\times 3+3\times 1+1\times 3$ under $SO(3)\times SO(3)$. This could be interpreted in terms of spin and vectorial isospin for massive particles so that the multiplets would relate to weak gauge bosons and Higgs boson singlet and triplet plus its pseudo-scalar variant. For 4-D representation of $SO(6)$ one would have 2×2 decomposition having interpretation in terms of spin and vectorial isospin.

   Massive spin would be associated as a local notion with $M^2 \times E^3$ and would be essentially 5-D concept. At H side massive particle would make sense only as a non-local notion with four-momentum and mass represented as a non-local operator.

These arguments indeed encourage to think that $M^8 - H$ duality could be the analog for the duality between the descriptions in terms of twistors and momentum twistors. In this case the Kac-Moody algebras are however not identical since the isometry groups are not identical.

3 Appendix: Some mathematical details about Grasmannian formalism

In the following I try to summarize my amateurish understanding about the mathematical structure behind the Grassmann integral approach. The representation summarizes what I have gathered from the articles of Arkani-Hamed and collaborators [B6, B7]. These articles are rather sketchy and the article of Bullimore provides additional details [B4] related to soft factors. The article of Mason and Skinner provides excellent introduction to super-twistors [B9] and dual super-conformal invariance. I apologize for unavoidable errors.

Before continuing a brief summary about the history leading to the articles of Arkani-Hamed and others is in order. This summary covers only those aspects which I am at least somewhat familiar with and leaves out many topics about existence which I am only half-conscious.

1. It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda_a\tilde{\lambda}_{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle\lambda,\mu\rangle &= \epsilon_{ab}\lambda^a\mu^b \ , \\ \left[\tilde{\lambda},\tilde{\mu}\right] &= \epsilon_{a'b'}\lambda^{a'}\mu^{b'} \ , \\ p\cdot q &= \langle\lambda,\mu\rangle\left[\tilde{\lambda},\tilde{\mu}\right] \ , \ (q_{aa'} = \mu_a\tilde{\mu}_{a'}) \ . \end{aligned} \tag{3.1}$$

   If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a\tilde{\lambda}_{a'}}{\langle\mu,\lambda\rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a\tilde{\mu}_{a'}}{\left[\tilde{\mu},\tilde{\lambda}\right]} \ , \ \text{negative helicity} \ . \end{aligned} \tag{3.2}$$

In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

2. Tree amplitudes are considered and it is convenient to drop the group theory factor $Tr(T_1T_2\cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \quad (3.3)$$

When the sign of the helicities is changed $\langle ..\rangle$ is replaced with $[..]$.

3. The article of Witten [B15] proposed that twistor approach could be formulated as a twistor string theory with string world sheets "living" in 6-dimensional CP_3 possessing Calabi-Yau structure and defining twistor space. In this article Witten introduced what is known as half Fourier transform allowing to transform momentum integrals over light-cone to twistor integrals. This operation makes sense only in space-time signature $(2,2)$. Witten also demonstrated that maximal helicity violating (MHV) twistor amplitudes (two gluons with negative helicity) with n particles with $k+2$ negative helicities and l loops correspond in this approach to holomorphic 2-surfaces defined by polynomials defined by polynomials of degree $D=k-1+l$, where the genus of the surface satisfies $g\leq l$. AdS/CFT duality provides a second stringy approach to $\mathcal{N}=4$ theory allowing to understand the scattering amplitudes in terms of Wilson loops with light-like edges: about this I have nothing to say. In any case, the generalization of twistor string theory to TGD context is highly attractive idea and will be considered later.

4. In the article [B8] Cachazo, Svrcek, and Witten propose the analog of Feynman diagrammatics in which MHV amplitudes can be used as analogs of vertices and ordinary $1/P^2$ propagator as propagator to construct tree diagrams with arbitrary number of negative helicity gluons. This approach is not symmetric with respect to the change of the sign of helicities since the amplitudes with two positive helicities are constructed as tree diagrams. The construction is non-trivial because one must analytically continue the on mass shell tree amplitudes to off mass shell momenta. The problem is how to assign a twistor to these momenta. This is achieved by introducing an arbitrary twistor $\eta^{a'}$ and defining λ_a as $\lambda_a = p_{aa'}\eta^{a'}$. This works for both massless and massive case. It however leads to a loss of the manifest Lorentz invariance. The paper however argues and the later paper [B13, B13] shows rigorously that the loss is only apparent. In this paper also BCFW recursion formula is introduced allowing to construct tree amplitudes recursively by starting from vertices with 2 negative helicity gluons. Also the notion which has become known as BCFW bridge representing the massless exchange in these diagrams is introduced. The tree amplitudes are not tree amplitudes in gauge theory sense where correspond to leading singularities for which 4 or more lines of the loop are massless and therefore collinear. What is important that the very simple MHV amplitudes become the building blocks of more complex amplitudes.

5. The nex step in the progress was the attempt to understand how the loop corrections could be taken into account in the construction BCFW formula. The calculation of loop contributions to the tree amplitudes revealed the existence of dual super-conformal symmetry which was found to be possessed also by BCFW tree amplitudes besides conformal symmetry. Together these symmetries generate infinite-dimensional Yangian symmetry [B9].

6. The basic vision of Arkani-Hamed and collaborators is that the scattering amplitudes of $\mathcal{N}=4$ SYM are constructible in terms of leading order singularities of loop diagrams. These singularities are obtained by putting maximum number of momenta propagating in the lines of the loop on mass shell. The non-leading singularities would be induced by the leading singularities by putting smaller number of momenta on mass shell are dictated by these terms. A related idea serving as a starting point in [B6] is that one can define loop integrals

as residue integrals in momentum space. If I have understood correctly, this means that one an imagine the possibility that the loop integral reduces to a lower dimensional integral for on mass shell particles in the loops: this would resemble the approach to loop integrals based on unitarity and analyticity. In twistor approach these momentum integrals defined as residue integrals transform to residue integrals in twistor space with twistors representing massless particles. The basic discovery is that one can construct leading order singularities for n particle scattering amplitude with $k+2$ negative helicities as Yangian invariants $Y_{n,k}$ for momentum twistors and invariants constructed from them by canonical operations changing n and k. The correspondence $k = l$ does not hold true for the more general amplitudes anymore.

3.1 Yangian algebra and its super counterpart

The article of Witten [B10] gives a nice discussion of the Yangian algebra and its super counterpart. Here only basic formulas can be listed and the formulas relevant to the super-conformal case are given.

3.1.1 Yangian algebra

Yangian algebra $Y(G)$ is associative Hopf algebra. The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [B10] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{3.4}$$

Besides this Serre relations are satisfied. These have more complex and read as

$$\begin{aligned}
&\left[J^{(1)A},\left[J^{(1)B},J^C\right]\right] + \left[J^{(1)B},\left[J^{(1)C},J^A\right]\right] + \left[J^{(1)C},\left[J^{(1)A},J^B\right]\right] \\
&\qquad = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM}\{J_D, J_E, J_F\} \ , \\
&\left[\left[J^{(1)A},J^{(1)B}\right],\left[J^C,J^{(1)D}\right]\right] + \left[\left[J^{(1)C},J^{(1)D}\right],\left[J^A,J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24} f^{AGL} f^{BEM} f_K^{CD} \\
&\qquad + f^{CGL} f^{DEM} f_K^{AB}) f^{KFN} f_{LMN}\{J_G, J_E, J_F\} \ .
\end{aligned} \tag{3.5}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A, B, C\}$ denotes the symmetrized product of three generators.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} are given as

$$J^{(1)A} \quad = \quad f_{BC}^A \sum_{i<j} J_i^B J_j^C \quad . \tag{3.6}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C \ per, \end{aligned} \tag{3.7}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

3.1.2 Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N} = 4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B10].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n = m$ is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$
\begin{aligned}
J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f^{C'}_{AB} \sum_{i<j} J_i^A J_j^B \\
&= g_{CC'} f^{C'}_{AB} g^{AA'} g^{BB'} \sum_{i<j} J^i_{A'} J^j_{B'} \ .
\end{aligned}
\tag{3.8}
$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

3.1.3 Generators of super-conformal Yangian symmetries

The explicit formula for the generators of super-conformal Yangian symmetries in terms of ordinary twistors is given by

$$
\begin{aligned}
j^A_B &= \sum_{i=1}^{n} Z_i^A \partial_{Z_i^B} \ , \\
j^{(1)A}_B &= \sum_{i<j} (-1)^C \left[Z_i^A \partial_{Z_j^C} Z_j^C \partial_{Z_j^B} \right] \ .
\end{aligned}
\tag{3.9}
$$

This formula follows from completely general formulas for the Yangian algebra discussed above and allowing to express the dual generators $j_N^{(1)}$ as quadratic expression of j_N involving structures constants. In this rather sketchy formula twistors are ordinary twistors. Note however that in the recent case the lattice is replaced with its finite cutoff corresponding to the external particles of the scattering amplitude. This probably corresponds to the assumption that for the representations considered only finite number of lattice points correspond to non-trivial quantum numbers or to cyclic symmetry of the representations.

In the expression for the amplitudes the action of transformations is on the delta functions and by partial integration one finds that a total divergence results. This is easy to see for the linear generators but not so for the quadratic generators of the dual super-conformal symmetries. A similar formula but with j^A_B and $j^{(1)A}_B$ interchanged applies in the representation of the amplitudes as Grassmann integrals using ordinary twistors. The verification of the generalization of Serre formula is also straightforward.

3.2 Twistors and momentum twistors and super-symmetrization

In [B9] the basics of twistor geometry are summarized. Despite this it is perhaps good to collect the basic formulas here.

3.2.1 Conformally compactified Minkowski space

Conformally compactified Minkowski space can be described as $SO(2,4)$ invariant (Klein) quadric

$$
T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0 \ . \tag{3.10}
$$

The coordinates (T,V,W,X,Y,Z) define homogenous coordinates for the real projective space RP^5. One can introduce the projective coordinates $X_{\alpha\beta} = -X_{\beta\alpha}$ through the formulas

$$
\begin{array}{lll}
X_{01} = W - V \ , & X_{02} = Y + iX \ , & X_{03} = \frac{i}{\sqrt{2}} T - Z \ , \\
X_{12} = -\frac{i}{\sqrt{2}}(T+Z) \ , & X_{13} = Y - iX \ , & X_{23} = \frac{1}{2}(V+W) \ .
\end{array}
\tag{3.11}
$$

The motivation is that the equations for the quadric defining the conformally compactified Minkowski space can be written in a form which is manifestly conformally invariant:

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} = 0 \; per. \tag{3.12}$$

The points of the conformally compactified Minkowski space are null separated if and only if the condition

$$\epsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta} = 0 \tag{3.13}$$

holds true.

3.2.2 Correspondence with twistors and infinity twistor

One ends up with the correspondence with twistors by noticing that the condition is equivalent with the possibility to expression $X_{\alpha\beta}$ as

$$X_{\alpha\beta} = A_{[\alpha}B_{\beta]} \ , \tag{3.14}$$

where brackets refer to antisymmetrization. The complex vectors A and B define a point in twistor space and are defined only modulo scaling and therefore define a point of twistor space CP_3 defining a covering of 6-D Minkowski space with metric signature $(2,4)$. This corresponds to the fact that the Lie algebras of $SO(2,4)$ and $SU(2,2)$ are identical. Therefore the points of conformally compactified Minkowski space correspond to lines of the twistor space defining spheres CP_1 in CP_3.

One can introduce a preferred scale for the projective coordinates by introducing what is called infinity twistor (actually a pair of twistors is in question) defined by

$$I_{\alpha\beta} = \begin{pmatrix} \epsilon^{A'B'} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{3.15}$$

Infinity twistor represents the projective line for which only the coordinate X_{01} is non-vanishing and chosen to have value $X_{01} = 1$.

One can define the contravariant form of the infinite twistor as

$$I^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} \ . \tag{3.16}$$

Infinity twistor defines a representative for the conformal equivalence class of metrics at the Klein quadric and one can express Minkowski distance as

$$(x-y)^2 = \frac{X^{\alpha\beta}Y_{\alpha\beta}}{I_{\alpha\beta}X^{\alpha\beta}I_{\mu\nu}Y^{\mu\nu}} \ . \tag{3.17}$$

Note that the metric is necessary only in the denominator. In twistor notation the distance can be expressed as

$$(x-y)^2 = \frac{\epsilon(A,B,C,D)}{\langle AB\rangle\langle CD\rangle} \ . \tag{3.18}$$

Infinite twistor $I_{\alpha\beta}$ and its contravariant counterpart project the twistor to its primed and unprimed parts usually denoted by $\mu^{A'}$ and λ^A and defined spinors with opposite chiralities.

3.2.3 Relationship between points of M^4 and twistors

In the coordinates obtained by putting $X_{01} = 1$ the relationship between space-time coordinates $x^{AA'}$ and $X^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} -\frac{1}{2}\epsilon^{A'B'}x^2 & -ix^{A'}_{\ \ B} \\ ix_A^{\ \ B'} & \epsilon_{A,B} \end{pmatrix} \ , \quad X^{\alpha\beta} = \begin{pmatrix} \epsilon_{A'B}\ {}'x^2 & -ix_{\ \ A'}^{B} \\ ix^A_{B'} & -\frac{1}{2}\epsilon^{AB}x^2 \end{pmatrix} \ , \tag{3.19}$$

If the point of Minkowski space represents a line defined by twistors (μ_U, λ_U) and (μ_V, λ_V), one has

$$x^{AC'} = i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)^{AC'}}{\langle UV\rangle} \tag{3.20}$$

The twistor μ for a given point of Minkowski space in turn is obtained from λ by the twistor formula by

$$\mu^{A'} \quad = \quad -ix^{AA'}\lambda_A \ . \tag{3.21}$$

3.2.4 Generalization to the super-symmetric case

This formalism has a straightforward generalization to the super-symmetric case. CP_3 is replaced with $CP_{3|4}$ so that Grassmann parameters have four components. At the level of coordinates this means the replacement $[W_I] = [W_\alpha, \chi_\alpha]$. Twistor formula generalizes to

$$\mu^{A'} = -ix^{AA'}\lambda_A \ , \quad \chi_\alpha = \theta^A_\alpha\lambda_A \ . \tag{3.22}$$

The relationship between the coordinates of chiral super-space and super-twistors generalizes to

$$(x, \theta) \quad = \quad \left(i\frac{(\mu_V\lambda_U - \mu_U\lambda_V)}{\langle UV\rangle}, \frac{(\chi_V\lambda_U - \chi_U\lambda_V)}{\langle UV\rangle}\right) \tag{3.23}$$

The above formulas can be applied to super-symmetric variants of momentum twistors to deduce the relationship between region momenta x assigned with edges of polygons and twistors assigned with the ends of the light-like edges. The explicit formulas are represented in [B9]. The geometric picture is following. The twistors at the ends of the edge define the twistor pair representing the region momentum as a line in twistor space and the intersection of the twistor lines assigned with the region momenta define twistor representing the external momenta of the graph in the intersection of the edges.

3.2.5 Basic kinematics for momentum twistors

The super-symmetrization involves replacement of multiplets with super-multiplets

$$\Phi(\lambda, \tilde{\lambda}, \eta) = G^+(\lambda, \tilde{\lambda}) + \eta_i\Gamma^a\lambda, \tilde{\lambda}) + \cdots + \epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d G^-(\lambda, \tilde{\lambda}) \ . \tag{3.24}$$

Momentum twistors are dual to ordinary twistors and were introduced by Hodges. The light-like momentum of external particle a is expressed in terms of the vertices of the closed polygon defining the twistor diagram as

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = \lambda_i\tilde{\lambda}_i \ , \quad \theta_i - \theta_{i+1} = \lambda_i\eta_i \ . \tag{3.25}$$

One can say that massless momenta have a conserved super-part given by $\lambda_i\eta_i$. The dual of the super-conformal group acts on the region momenta exactly as the ordinary conformal group acts on space-time and one can construct twistor space for dua region momenta.

Super-momentum conservation gives the constraints

$$\sum p_i = 0 \ , \quad \sum \lambda_i \eta_i = 0 \ . \tag{3.26}$$

The twistor diagrams correspond to polygons with edges with lines carrying region momenta and external massless momenta emitted at the vertices.

This formula is invariant under overall shift of the region momenta x_a^μ. A natural interpretation for x_a^μ is as the momentum entering to the the vertex where p_a is emitted. Overall shift would have interpretation as a shift in the loop momentum. x_a^μ in the dual coordinate space is associated with the line $Z_{a-1}Z_a$ in the momentum twistor space. The lines $Z_{a-1}Z_a$ and Z_aZ_{a+1} intersect at Z_a representing a light-like momentum vector p_a^μ.

The brackets $\langle abcd\rangle \equiv \epsilon_{IJKL}Z_a^IZ_b^JZ_c^KZ_d^L$ define fundamental bosonic conformal invariants appearing in the tree amplitudes as basic building blocks. Note that Z_a define points of 4-D complex twistor space to be distinguished from the projective twistor space CP_3. Z_a define projective coordinates for CP_3 and one of the four complex components of Z_a is redundant and one can take $Z_a^0 = 1$ without a loss of generality.

3.3 Brief summary of the work of Arkani-Hamed and collaborators

The following comments are an attempt to summarize my far from complete understanding about what is involved with the representation as contour integrals. After that I shall describe in more detail my impressions about what has been done.

3.3.1 Limitations of the approach

Consider first the limitations of the approach.

1. The basis idea is that the representation for tree amplitudes generalizes to loop amplitudes. On other words, the amplitude defined as a sum of Yangian invariants expressed in terms of Grassmann integrals represents the sum of loops up to some maximum loop number. The problem is here that shifts of the loop momenta are essential in the UV regularization procedure. Fixing the coordinates $x_1, \cdots, x_n$ having interpretation as momenta associated with lines in the dual coordinate space allows to eliminate the non-uniqueness due to the common shift of these coordinates.

2. It is not however not possible to identify loop momentum as a loop momentum common to different loop integrals unless one restricts to planar loops. Non-planar diagrams are obtained from a planar diagram by permuting the coordinates x_i but this means that the unique coordinate assignment is lost. Therefore the representation of loop integrands as Grassmann integrals makes sense only for planar diagrams. From TGD point of view one could argue that this is one good reason for restricting the loops so that they are for on mass shell particles with non-parallel on mass shell four-momenta and possibly different sign of energies for given wormhole contact representing virtual particle.

3. IR regularization is needed even in $\mathcal{N} = 4$ for SYM given by "moving out on the Coulomb branch theory" so that IR singularities remain the problem of the theory.

3.3.2 What has been done?

The article proposes a generalization of the BCFW recursion relation for tree diagrams of $\mathcal{N} = 4$ for SYM so that it applies to planar diagrams with a summation over an arbitrary number of loops.

1. The basic goal of the article is to generalize the recursion relations of tree amplitudes so that they would apply to loop amplitudes. The key idea is following. One can formally represent loop integrand as a contour integral in complex plane whose coordinate parameterizes the deformations $Z_n \rightarrow Z_n + \epsilon Z_{n-1}$ and re-interpret the integral as a contour integral with oppositely oriented contour surrounding the rest of the complex plane which can be imagined also as being mapped to Riemann sphere. What happens only the poles which correspond to

lower number of loops contribute this integral. One obtains a recursion relation with respect to loop number. This recursion seems to be the counterpart for the recursive construction of the loops corrections in terms of absorptive parts of amplitudes with smaller number of loop using unitarity and analyticity.

2. The basic challenge is to deduce the Grassmann integrands as Yangian invariants. From these one can deduce loop integrals by integration over the four momenta associated with the lines of the polygonal graph identifiable as the dual coordinate variables x_a. The integration over loop momenta can induce infrared divergences breaking Yangian symmetry. The big idea here is that the operations described above allow to construct loop amplitudes from the Yangian invariants defining tree amplitudes for a larger number of particles by removing external particles by fusing them to form propagator lines and by using the BCFW bridge to fuse lower-dimensional invariants. Hence the usual iterative procedure (bottom-up) used to construct scattering amplitudes is replaced with a recursive procedure (top-down). Of course, once lower amplitudes has been constructed they can be used to construct amplitudes with higher particle number.

3. The first guess is that the recursion formula involves the same lower order contributions as in the case of tree amplitudes. These contributions have interpretation as factorization of channels involving single particle intermediate states. This would however allow to reduce loop amplitudes to 3-particle loop amplitudes which vanish in $\mathcal{N}=4$ SYM by the vanishing of coupling constant renormalization. The additional contribution is necessary and corresponds to a source term identifiable as a "forward limit" of lower loop integrand. These terms are obtained by taking an amplitude with two additional particles with opposite four-momenta and forming a state in which these particles are entangled with respect to momentum and other quantum numbers. Entanglement means integral over the massless momenta on one hand. The insertion brings in two momenta x_a and x_b and one can imagine that the loop is represented by a branching of propagator line. The line representing the entanglement of the massless states with massless momentum define the second branch of the loop. One can of course ask whether only massless momentum in the second branch. A possible interpretation is that this state is expressible by unitarity in terms of the integral over light-like momentum.

4. The recursion formula for the loop amplitude $M_{n,k,l}$ involves two terms when one neglects the possibility that particles can also suffer trivial scattering (cluster decomposition). This term basically corresponds to the Yangian invariance of n arguments identified as Yangian invariant of $n-1$ arguments with the same value of k.

   (a) The first term corresponds to single particle exchange between particle groups obtained by splitting the polygon at two vertices and corresponds to the so called BCFW bridge for tree diagrams. There is a summation over different splittings as well as a sum over loop numbers and dimensions k for the Grassmann planes. The helicities in the two groups are opposite.

   (b) Second term is obtained from an amplitude obtained by adding of two massless particles with opposite momenta and corresponds to $n+2, k+1, l-1$. The integration over the light-like momentum together with other operations implies the reduction $n+2 \rightarrow n$. Note that the recursion indeed converges. Certainly the allowance of added zero energy states with a finite number of particles is necessary for the convergence of the procedure.

3.4 The general form of Grassmannian integrals

If the recursion formula proposed in [B7] is correct, the calculations reduce to the construction of N^kMHV (super) amplitudes. MHV refers to maximal helicity violating amplitudes with 2 negative helicity gluons. For N^kMHV amplitude the number of negative helicities is by definition $k+2$ [B6]. Note that the total right handed R-charge assignable to 4 super-coordinates η_i of negative helicity gluons can be identified as $R=4k$. BCFW recursion formula [B13, B13] allows to construct from MHV amplitudes with arbitrary number of negative helicities.

The basic object of study are the leading singularities of color-stripped n-particle N^kMHV amplitudes. The discovery is that these singularities are expressible in terms Yangian invariants

$Y_{n,k}(Z_1, \cdots, Z_n)$, where Z_i are momentum super-twistors. These invariants are defined by residue integrals over the compact $nk-1$-dimensional complex space $G(n,k) = U(n)/U(k) \times U(n-k)$ of k-planes of complex n-dimensional space. n is the number of external massless particles, k is the number negative helicity gluons in the case of N^kMHV amplitudes, and Z_a, $i = 1, \cdots, n$ denotes the projective 4-coordinate of the super-variant $CP^{3|4}$ of the momentum twistor space CP_3 assigned to the massless external particles is following. $Gl(n)$ acts as linear transformations in the n-fold Cartesian power of twistor space. Yangian invariant $Y_{n,k}$ is a function of twistor variables Z^a having values in super-variant $CP_{3|3}$ of momentum twistor space CP_3 assigned to the massless external particles being simple algebraic functions of the external momenta.

It is also possible to define N^kMHV amplitudes in terms of Yangian invariants $L_{n,k+2}(W_1, \cdots, W_n)$ by using ordinary twistors W_a and identical defining formula. The two invariants are related by the formula $L_{n,k+2}(W_1, \cdots, W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1, \cdots, Z_n)$. Here M^{tree}_{MHV} is the tree contribution to the maximally helicity violating amplitude for the scattering of n particles: recall that these amplitudes contain two negative helicity gluons whereas the amplitudes containing a smaller number of them vanish [B8]. One can speak of a factorization to a product of n-particle amplitudes with $k-2$ and 2 negative helicities as the origin of the duality. The equivalence between the descriptions based on ordinary and momentum twistors states the dual conformal invariance of the amplitudes implying Yangian symmetry. It has been conjectured that Grassmannian integrals generate all Yangian invariants.

The formulas for the Grassmann integrals for twistors and momentum twistors appearing in the expressions of N^kMHV amplitudes are given by following expressions.

1. The integrals $L_{n,k}(W_1, \cdots, W_n)$ associated with $N^{k-2}MHV$ amplitudes in the description based on ordinary twistors correspond to k negative helicities and are given by

$$\begin{aligned} L_{n,k}(W_1, \cdots, W_n) &= \frac{1}{Vol(GL(2))} \int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)} \times \\ &\times \prod_{\alpha=1}^{k} d^{4|4}Y_\alpha \prod_{i=1}^{n} \delta^{4|4}(W_i - C_{\alpha i}Y_\alpha) \ . \end{aligned} \tag{3.27}$$

   Here $C_{\alpha a}$ denote the $n \times k$ coordinates used to parametrize the points of $G_{k,n}$.

2. The integrals $Y_{n,k}(W_1, \cdots, W_n)$ associated with N^kMHV amplitudes in the description based on momentum twistors are defined as

$$Y_{n,k}(Z_1, \cdots, Z_n) = \frac{1}{Vol(GL(k))} \times \int \frac{d^{k\times n}C_{\alpha a}}{(1\cdots k)(2\cdots k+1)\cdots(n1\cdots k-1)} \times \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}Z_a) \ . \tag{3.28}$$

   The possibility to select $Z_a^0 = 1$ implies $\sum_k C_{\alpha k} = 0$ allowing to eliminate $C_{\alpha n}$ so that the actual number of coordinates Grassman coordinates is $nk-1$. As already noticed, $L_{n,k+2}(W_1, \cdots, W_n) = M^{tree}_{MHV} \times Y_{n,k}(Z_1, \cdots, Z_n)$. Momentum twistors are obviously calculationally easier since the value of k is smaller by two units.

The $4k$ delta functions reduce the number of integration variables of contour integrals from nk to $(n-4)k$ in the bosonic sector (the definition of delta functions involves some delicacies not discussed here). The n quantities $(m, \cdots m+k)$ are $k \times k$-determinants defined by subsequent columns from m to $m+k-1$ of the $k \times n$ matrix defined by the coordinates $C_{\alpha a}$ and correspond geometrically to the k-volumes of the k-dimensional parallel-pipeds defined by these column vectors. The fact that the scalings of twistor space coordinates Z_a can be compensated by scalings of $C_{\alpha a}$ deforming integration contour but leaving the residue integral invariant so that the integral depends on projective twistor coordinates only.

Since the integrand is a rational function, a multi-dimensional residue calculus allows to deduce the values of these integrals as residues associated with the poles of the integrand in a recursive manner. The poles correspond to the zeros of the $k\times k$ determinants appearing in the integrand or equivalently to singular lower-dimensional parallel-pipeds. It can be shown that local residues are determined by $(k-2)(n-k-2)$ conditions on the determinants in both cases. The value of the integral depends on the explicit choice of the integration contour for each variable $C_{\alpha a}$ left when delta functions are taken into account. The condition that a correct form of tree amplitudes is obtained fixes the choice of the integration contours.

For the ordinary twistors W the residues correspond to projective configurations in CP_{k-1}, or more precisely in the space $CP^n_{k-1}/Gl(k)$, which is $(k-1)n-k^2$-dimensional space defining the support for the residues integral. $Gl(k)$ relates to each other different complex coordinate frames for k-plane and since the choice of frame does not affect the plane itself, one has $Gl(k)$ gauge symmetry as well as the dual $Gl(n-k)$ gauge symmetry.

CP_{k-1} comes from the fact that C_{α_k} are projective coordinates: the amplitudes are indeed invariant under the scalings $W_i\rightarrow t_iW_i$, $C_{\alpha i}\rightarrow tC_{\alpha i}$. The coset space structure comes from the fact that $Gl(k)$ is a symmetry of the integrand acting as $C_{\alpha i}\rightarrow \Lambda_\alpha^{\ \beta}C_{\beta_i}$. This analog of gauge symmetry allows to fix k arbitarily chosen frame vectors $C_{\alpha i}$ to orthogonal unit vectors. For instance, one can have $C_{\alpha i}=\delta_{\alpha_i}$ for $\alpha=i\in 1,\cdots,k$. This choice is discussed in detail in [B6]. The reduction to CP_{k-1} implies the reduction of the support of the integral to line in the case of MHV amplitudes and to plane in the case of NMHV as one sees from the expression $d\mu=\prod_\alpha d^{4|4}Y_\alpha\prod_{i=1}^n\delta^{4|4}(W_i-C_{\alpha i}Y_\alpha)$. For $(i_1,\cdots,i_k)=0$ the vectors $i_1,..i_k$ belong to $k-2$-dimensional plane of CP_{k-1}. In the case of $NMHV$ (N^2MHV) amplitudes this translates at the level of twistors to the condition that the corresponding twistors $\{i_1,i_2,i_3\}$ ($\{i_1,i_2,i_3,i_4\}$) are collinear (in the same plane) in twistor space. This can be understood from the fact that the delta functions in $d\mu$ allow to express W_i in terms of $k-1$ Y_α: s in this case.

The action of conformal transformations in twistor space reduces to the linear action of $SU(2,2)$ leaving invariant Hermitian sesquilinear form of signature $(2,2)$. Therefore the conformal invariance of the Grassmannian integral and its dual variant follows from the possibility to perform a compensating coordinate change for $C_{\alpha a}$ and from the fact that residue integral is invariant under small deformations of the integration contour. The above described relationship between representations based on twistors and momentum twistors implies the full Yangian invariance.

3.5 Canonical operations for Yangian invariants

General l-loop amplitudes can be constructed from the basic Yangian invariants defined by N^kMHV amplitudes by various operations respecting Yangian invariance apart from possible IR anomalies. There are several operations that one can perform for Yangian invariants $Y_{n,k}$ and all these operations appear in the recursion formula for planar all loop amplitudes. These operations are described in [B7] much better than I could do it so that I will not go to any details. It is possible to add and remove particles, to fuse two Yangian invariants, to merge particles, and to construct from two Yangian invariants a higher invariant containing so called BCFW bridge representing single particle exchange using only twistorial methods.

3.5.1 Inverse soft factors

Inverse soft factors add to the diagram a massless collinear particles between particles a and b and by definition one has

$$O_{n+1}(a,c,b,\cdots) = \frac{\langle ab\rangle}{\langle ac\rangle\langle cb\rangle}O_n(a'b') \ . \tag{3.29}$$

At the limit when the momentum of the added particle vanishes both sides approach the original amplitude. The right-handed spinors and Grassmann parameters are shifted

$$\begin{aligned}\tilde{\lambda}'_a &= \tilde{\lambda}_a+\tfrac{\langle cb\rangle}{\langle ab\rangle}\tilde{\lambda}_c \ , \quad \tilde{\lambda}'_b=\tilde{\lambda}_b+\tfrac{\langle ca\rangle}{\langle ba\rangle}\tilde{\lambda}_c \ ,\\ \eta'_a &= \eta_a+\tfrac{\langle cb\rangle}{\langle ab\rangle}\eta_c \ , \quad \eta'_b=\eta_b+\tfrac{\langle ca\rangle}{\langle ba\rangle}\eta_c \ .\end{aligned} \tag{3.30}$$

There are two kinds of inverse soft factors.

1. The addition of particle leaving the value k of negative helicity gluons unchanged means just the re-interpretation

$$Y'_{n,k}(Z_1,\cdots,Z_{n-1},Z_n) \quad = \quad Y_{n-1,k}(Z_1,\cdots,Z_{n-1}) \tag{3.31}$$

without actual dependence on Z_n. There is however a dependence on the momentum of the added particle since the relationship between momenta and momentum twistors is modified by the addition obtained by applying the basic rules relating region super momenta and momentum twistors (light-like momentum determines λ_i and twistor equations for x_i and λ_i, η_i determine (μ_i, χ_i)) is expressible assigned to the external particles [B4]. Modifications are needed only for the new vertex and its neighbors.

2. The addition of a particle increasing k with single unit is a more complex operation which can be understood in terms of a residue of $Y_{n,k}$ proportional to $Y_{n-1,k-1}$ and Yangian invariant $[z_1\cdots z_5]$ with five arguments constructed from basic Yangian invariants with four arguments. The relationship between the amplitudes is now

$$Y'_{n,k}(..,Z_{n-1}Z_n,Z_1\cdots) \quad = \quad [n-2\ n-1\ n\ 1\ 2]\times Y_{n-1,k-1}(\cdots\hat{Z}_{n-1},\hat{Z}_1,\cdots)\ . \tag{3.32}$$

Here

$$[abcde] \quad = \quad \frac{\delta^{0|4}(\eta_a\langle bcde\rangle + cyclic)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}\ . \tag{3.33}$$

denoted also by $R(a,b,c,d,e)$ is the fundamental R-invariant appearing in one loop corrections of MHV amplitudes and will appears also in the recursion formulas. $\langle abcd\rangle$ is the fundamental super-conformal invariant associated with four super twistors defined in terms of the permutation symbol.

$\hat{Z}_{n-1}, \hat{Z}_1$ are deformed momentum twistor variables. The deformation is determined from the relationship between external momenta, region momenta and momentum twistor variables. $\hat{Z}^1$ is the intersection $\hat{Z}^1 = (n-2\ n-1\ 2)\cap(12)$ of the the line (12) with the plane $(n-2\ n-1\ 2)$ and $\hat{Z}^{n-1}$ the intersection $\hat{Z}^1 = (12n)\cap(n-2\ n-1)$ of the the line $(n-2\ n-1)$ with the plane $(12n)$. The interpretation for the intersections at the level of ordinary Feynman diagrams is in terms of the collinearity of the four-momenta involved with the underlying box diagram with parallel on mass shell particles. These result from unitarity conditions obtained by putting maximal number of loop momenta on mass shell to give the leading singularities.

The explicit expressions for the momenta are

$$\begin{aligned}
\hat{Z}^1 &\equiv (n-2\ n-1\ 2)\cap(12)Z_1 = \langle 2\ n-2\ n-1\ n\rangle + Z_2\langle n-2\ n-1\ n\ 1\rangle\ , \\
\hat{Z}^{n-1} &\equiv (12n)\cap(n-2\ n-1) = Z_{n-2}\langle n-2\ n-1\ n\ 2\rangle + Z_{n-1}\langle n\ 1\ 2\ n-2\rangle\ .
\end{aligned} \tag{3.34}$$

These intersections also appear in the expressions defining the recursion formula.

3.5.2 Removal of particles and merge operation

Particles can be also removed. The first manner to remove particle is by integrating over the twistor variable characterizing the particle. This reduces k by one unit. Merge operation preserves the number of loops but removes a particle particle by identifying the twistor variables of neighboring particles. This operation corresponds to an integral over on mass shell loop momentum at the level of tree diagrams and by Witten's half Fourier transform can be transformed to twistor integral.

The product

$$Y'(Z_1, \cdots Z_n) = Y_1(Z_1, \cdots Z_m) \times Y_2(Z_{m+1}, \cdots Z_n) \quad (3.35)$$

of two Yangian invariants is again a Yangian invariant. This is not quite trivial since the dependence of region momenta and momentum twistors on the momenta of external particles makes the operation non-trivial.

Merge operation allows to construct more interesting invariants from the products of Yangian invariants. One begins from a product of Yangian invariants (Yangian invariant trivially) represented cyclically as points of circle and identifies the last twistor argument of given invariant with the first twistor argument of the next invariant and performs integrals over the momentum twistor variables appearing twice. The soft k-increasing and preserving operations can be described also in terms of this operation for Yangian invariants such that the second invariant corresponds to 3-vertex. The cyclic merge operation applied to four MHV amplitudes gives NMHV amplitudes associated with on mass shell momenta in box diagrams. By applying similar operation to NMHV amplitudes and MHV amplitudes one obtains 2-loop amplitudes. In [B7] examples about these operations are described.

3.5.3 BCFW bridge

BCFW bridge allows to build general tree diagrams from MHV tree diagrams [B13, B13] and recursion formula of [B7] generalizes this to arbitrary diagrams. At the level of Feynman diagrams it corresponds to a box diagram containing general diagrams labeled by L and R and MHV and $\overline{MHV}$ 3-vertices ($\overline{MHV}$ 3-vertex allows expression in terms of MHV diagrams) with the lines of the box on mass shell so that the three momenta emanating from the vertices are parallel and give rise to a one-loop leading singularity.

At the level of Feynman diagrams BCFW bridge corresponds to so called "two-mass hard" leading singularities associated with box diagrams with light-like momenta at the four lines of the diagram [B6]. The motivation for the study of these diagrams comes from the hypothesis the leading order singularities obtained by putting as many particles as possible on mass shell contain the data needed to construct scattering amplitudes of $\mathcal{N} = 4$ SYM completely. This representation of the leading singularities generalizes to arbitrary loops. The recent article is a continuation of this program to planar amplitudes.

Also BCFW bridge allows an interpretation as a particular kind fusion for Yang invariants and involves all the basic operations. One starts from the amplitudes $Y^L_{n_L,k_L}$ and $Y^R_{n_R,k_R}$ and constructs an amplitude $Y'_{n_L+n_R,k_L+k_R+1}$ representing the amplitude which would correspond to a generalization of the MHV diagrams with the two tree diagrams connected by the MHV propagator (BCFW bridge) replaced with arbitrary loop diagrams. Particle "1" *resp.* "j+1" is added by the soft k-increasing factor to Y_{n_L+1,k_L+1} *resp.* Y_{n_R+1,k_R+1} giving amplitude with $n+2$ particles and with k-charge equal to $k_L + k_R + 2$. The subsequent operations must reduce k-charge by one unit. First repeated "1" and "j+1" are identified with their copies by k conserving merge operation, and after that one performs an integral over the twistor variable Z^I associated with the internal line obtained and reducing k by one unit. The soft k-increasing factors bring in the invariants $[n-1\ n\ 1\ I\ j+2]$ associated with Y_L and $[1\ I\ j+1\ j\ j-1]$ associated with Y_R. The integration contour is chosen so that it selects the pole defined by $\angle n-1\ n\ 1\ I\rangle$ in the denominator of $[n-1\ n\ 1\ I\ j+2]$ and the pole defined by $\langle 1\ I\ j+1\ j\rangle$ in the denominator of $[1\ I\ j+1\ j\ j-1]$.

The explicit expression for the BCFW bridge is very simple:

$$\begin{aligned} (Y_L \otimes_{BCFW} Y_R)(1, \cdots, n) &= [n-1\ n\ 1\ j\ j+1] \times Y_R(1, \cdots, j, I) Y_L(I, j+1, \cdots, n-1, \hat{n}) \ , \\ \hat{n} &= (n-1\ n) \cap (j\ j+1\ 1) \ , \quad I = (j\ j+1) \cap (n-1\ n\ 1) \ . \end{aligned} \quad (3.36)$$

3.5.4 Single cuts and forward limit

Forward limit operation is used to increase the number of loops by one unit. The physical picture is that one starts from say 1-loop amplitude and cuts one line by assigning to the pieces of the line opposite light-like momenta having interpretation as incoming and outgoing particles. The resulting amplitude is called forward limit. The only reasonable interpretation seems to be that the loop integration is expressed by unitarity as forward limit meaning cutting of the line carrying the loop momentum. This operation can be expressed in a manifestly Yangian invariant way as entangled removal of two particles with the merge operation meaning the replacement $Z_n \rightarrow Z_{n-1}$. Particle $n+1$ is added adjacent to A, B as a k-increasing inverse soft factor and then A and B are removed by entangled integration, and after this merge operation identifies $n+1$ and 1.

Forward limit is crucial for the existence of loops and for Yangian invariants it corresponds to the poles arising from $\langle (AB)_q Z_n(z) Z_1 \rangle\rangle$ the integration contour $Z_n + zZ_{n-1}$ around Z_b in the basic formula $M = \oint (dz/z) M_n$ leading to the recursion formula. A and B denote the momentum twistors associated with opposite light-like momenta. In the generalized unitarity conditions the singularity corresponds to the cutting of line between particles n and 1 with momenta q and $-q$, summing over the multiplet of stats running around the loop. Between particles n_2 and 1 one has particles $n-1$, n with momenta $q, -q$. $q = x_1 - x_n = -x_n + x_{n-1}$ giving $x_1 = x_{n-1}$. Light-likeness of q means that the lines $(71) = (76)$ and (15) intersect. At the forward limit giving rise to the pole Z_6 and Z_7 approach to the intersection point $(76) \cap (15)$. In a generic gauge theories the forward limits are ill-defined but in super-symmetric gauge theories situation changes.

The corresponding Yangian operation removes two external particles with opposite four-momenta and involves integration over two twistor variables Z_a and Z_b and gives rise to the following expression

$$\int_{GL(2)} Y(\cdots, Z_n, Z_A, Z_B, Z_1, \cdots) \ . \tag{3.37}$$

The integration over $GL(2)$ corresponds to integration over twistor variables associated Z_A and Z_B. This operation allows addition of a loop to a given amplitude. The line $Z_a Z_b$ represents loop momentum on one hand and the dual x-coordinate identified as momentum propagating along the line on the other hand.

The integration over these variables is equivalent to an integration over loop momentum as the explicit calculation of [B7] (see pages 12-13) demonstrates. If the integration contours are products in the product of twistor spaces associated with a and b the and gives lower order Yangian invariant as answer. It is however also possible to choose the integration contour to be entangled in the sense that it cannot be reduced to a product of integration contours in the Cartesian product of twistor spaces. In this case the integration gives a loop integral. In the removal operation Yangian invariance can be broken by IR singularities associated with the integration contour and the procedure does not produce genuine Yangian invariant always.

What is highly interesting from TGD point of view is that this integral can be expressed as a contour integral over $CP_1 \times CP_1$ combined with integral over loop momentum. If TGD vision about generalized Feynman graps in zero energy ontology is correct, the loop momentum integral is discretized to an an integral over discrete mass shells and perhaps also to a sum over discretized momenta and one can therefore avoid IR singularities.

3.6 Explicit formula for the recursion relation

Recall that the recursion formula is obtained by considering super-symmetric momentum-twistor deformation $Z_n \rightarrow Z_n + zZ_{n-1}$ and by integrating over z to get the identity

$$M_{n,k,l} \ = \ \oint \frac{dz}{z} \hat{M}_{n,k,l}(z) \ . \tag{3.38}$$

This integral equals to integral with reversed integration contour enclosing the exterior of the contour. The challenge is to deduce the residues contributing to the residue integral and the claim of [B7] is that these residues reduce to simple basic types.

1. The first residue corresponds to a pole at infinity and reduces the particle number by one giving a contribution $M_{n-1,k,l}(1, \cdots, n-1)$ to $M_{n,k,l}(1, \cdots, n-1, n)$. This is not totally trivial since the twistor variables are related to momenta in different manner for the two amplitudes. This gives the first contribution to the right hand side of the formula below.

2. Second pole corresponds to the vanishing of $\langle Z_n(z)Z_1Z_jZ_{j+1}\rangle$ and corresponds to the factorization of channels. This gives the second BCFW contribution to the right hand side of the formula below. These terms are however not enough since the recursion formula would imply the reduction to expressions involving only loop corrections to 3-loop vertex which vanish in $\mathcal{N}=4$ SYM.

3. The third kind of pole results when $\langle (AB)_q Z_n(z)Z_1\rangle$ vanishes in momentum twistor space. $(AB)_q$ denotes the line in momentum twistor space associated with q: th loop variable.

The explicit formula for the recursion relation yielding planar all loop amplitudes is obtained by putting all these pieces together and reads as

$$\begin{aligned} M_{n,k,l}(1,\cdots,n) &= M_{n-1,k,l}(1,\cdots,n-1) \\ &+ \sum_{n_L,k_L,l_L;j} [j\ j+1\ n-1\ n\ 1]\, M^R_{n_R,k_R,l_R}(1,\cdots,j,I_j)\times M^L_{n_L,k_L,l_L}(I_j,j+1,\cdots,\hat{n}_j) \\ &+ \int_{GL(2)} [AB\ n-1\ n\ 1]\, M_{n+2,k+1,n,k-1}(1,\cdots,\hat{n}_{AB},\hat{A},B)\ , \\ n_L &+ n_R=n+2\ ,\ \ k_L+k_R=k-1\ ,\ \ l_R+l_L=l\ . \end{aligned} \tag{3.39}$$

The momentum super-twistors are given by

$$\begin{aligned} \hat{n}_j &= (n-1\ n)\cap(j\ j+1\ 1)\ ,\quad I_j=(j\ j+1\ 1)\cap(n-1\ n\ 1)\ , \\ \hat{n}_{AB} &= (n-1\ n)\cap(AB\ 1)\ ,\quad \hat{A}=(AB)\cap(n-1\ n\ 1)\ . \end{aligned} \tag{3.40}$$

The index l labels loops in $n+2$-particle amplitude and the expression is fully symmetrized with equal weight for all loop integration variables $(AB)_l$. A and B are removed by entangled integration meaning that $GL(2)$ contour is chosen to encircle points where both points A,B on the line (AB) are located at the intersection of the line (AB) with the plane $(n-1\ n\ 1)$. GL(2) integral can be done purely algebraically in terms of residues.

In [B7] and [B4] explicit calculations for N^kMHV amplitudes are carried out to make the formulas more concrete. For N^1MHV amplitudes second line of the formula vanishes and the integrals are rather simple since the determinants are 1×1 determinants.

4 Could the Grassmannian program be realized in TGD framework?

In the following the TGD based modification of the approach based on zero energy ontology is discussed in some detail. It is found that pseudo-momenta are very much analogous to region momenta and the approach leading to discretization of pseudo-mass squared for virtual particles - and even the discretization of pseudo-momenta - is consistent with the Grassmannian approach in the simple case considered and allow to get rid of IR divergences. Also the possibility that the number of generalized Feynman diagrams contributing to a given scattering amplitude is finite so that the recursion formula for the scattering amplitudes would involve only a finite number of steps (maximum number of loops) is considered. One especially promising feature of the residue integral approach with discretized pseudo-momenta is that it makes sense also in the p-adic context in the simple special case discussed since residue integral reduces to momentum integral (summation) and lower-dimensional residue integral.

4.1 What Yangian symmetry could mean in TGD framework?

The loss of the Yangian symmetry in the integrations over the region momenta x^a ($p^a=x^{a+1}-x^a$) assigned to virtual momenta seems to be responsible for many ugly features. It is basically the source of IR divergences regulated by “moving out on the Coulomb branch theory” so that IR singularities remain the problem of the theory. This raises the question whether the loss of Yangian symmetry is the signature for the failure of QFT approach and whether the restriction of loop momentum integrations to avoid both kind of divergences might be a royal road beyond QFT. In TGD framework zero energy ontology indeed leads to to a concrete proposal based on the vision that virtual particles are something genuinely real.

The detailed picture is of course far from clear but to get an idea about what is involved one can look what kind of assumptions are needed if one wants to realize the dream that only a finite number of generalized Feynman diagrams contribute to a scattering amplitude which is Yangian invariant allowing a description using a generalization of the Grassmannian integrals.

1. Assume the bosonic emergence and its super-symmetric generalization holds true. This means that incoming and outgoing states are bound states of massless fermions assignable to wormhole throats but the fermions can opposite directions of three-momenta making them massive. Incoming and outgoing particles would consist of fermions associated with wormhole throats and would be characterized by a pair of twistors in the general situation and in general massive. This allows also string like mass squared spectrum for bound states having fermion and anti-fermion at the ends of the string as well as more general n-particle bound states. Hence one can speak also about the emergence of string like objects. For virtual particles the fermions would be massive and have discrete mass spectrum. Also super partners containing several collinear fermions and anti-fermions at a given throat are possible. Collinearity is required by the generalization of SUSY. The construction of these states bring strongly in mind the merge procedure involving the replacement $Z^{n+1} \rightarrow Z^n$.

2. The basic question is how the momentum twistor diagrams and the ordinary Feynman diagrams behind them are related to the generalized Feynman diagrams.

   (a) It is good to start from a common problem. In momentum twistor approach the relationship of region momenta to physical momenta remains somewhat mysterious. In TGD framework one can assign to the space-like 3-surfaces at the ends of CD four-momenta obeying stringy mass squared formula and to the fermion lines at light-like partonic orbits massless virtual momenta. Residue integration over virtual momenta leaves massless on mass shell momenta. The identification of these momenta as the TGD counterpart of the region momentum x looks like a natural first guess.

   (b) The identification $x_{a+1} - x_a = p_a$ with p_a representing light-like physical four-momentum generalizes in obvious manner. Also the identification of the light-like momentum of the external parton as pseudo-momentum looks natural. What is important is that this does not require the identification of the pseudo-momenta propagating along internal lines of generalized Feynman diagram as actual physical momenta since pseudo-momentum just like x is fixed only apart from an overall shift. The identification allows the physical four-momenta associated with the wormhole throats to be always on mass shell and massless: if the sign of the physical energy can be also negative space-like momentum exchanges become possible.

   (c) The pseudo-momenta and light-like physical massless momenta at the lines of generalized Feynman diagrams on one hand, and region momenta and the light-like momenta associated with the collinear singularities on the other hand would be in very similar mutual relationship. Partonic 2-surfaces can carry large number of collinear light-like fermions and bosons since super-symmetry is extended. Generalized Feynman diagrams would be analogous to momentum twistor diagrams if this picture is correct and one could hope that the recursion relations of the momentum twistor approach generalize.

3. The discrete mass spectrum for four-momentum would in the momentum twistor approach mean the restriction of x to discrete mass shells, and the obvious reason for worry is that this might spoil the Grassmannian approach relying heavily on residue integrals and making sense also p-adically. It seems however that there is no need to worry. In [B7] the $M_{6,4,l=0}(1234AB)$ the integration over twistor variables z_A and z_B using "entangled" integration contour leads to 1-loop MHV amplitude N^pMHV, $p=1$. The parameterization of the integration contour is $z_A = (\lambda_A, x\lambda_A)$, $z_B = (\lambda_B, x\lambda_B)$, where x is the M^4 coordinate representing the loop

momentum. This boils down to an integral over $CP_1 \times CP_1 \times M^4$ [B7]. The integrals over spheres CP_1s are contour integrals so that only an ordinary integral over M^4 remains. The reduction to this kind of sums occurs completely generally thanks to the recursion formula.

4. The obvious implication of the restriction of the four-momenta x on massive mass shells is the absence of IR divergences and one might hope that under suitable assumptions one achieves Yangian invariance. The first question is of course whether the required restriction of x to mass shells in z_A and z_B or possibly even algebraic discretization of momenta is consistent with the Yangian invariance. This seems to be the case: the integration contour reduces to entangled integration contour in $CP_1 \times CP_1$ not affected by the discretization and the resulting loop integral differs from the standard one by the discretization of masses and possibly also momenta with massless states excluded. Whether Yangian invariance poses also conditions on mass and momentum spectrum is an interesting question.

5. One can consider also the possibility that the incoming and outgoing particles - in general massive and to be distinguished from massless fermions appearing as their building blocks- have actually small masses presumably related to the IR cutoff defined by the size scale of the largest causal diamond involved. p-Adic thermodynamics could be responsible for this mass. Also the binding of the wormhole throats can give rise to a small contribution to vacuum conformal weight possibly responsible for gauge boson masses. This would imply that a given n-particle state can decay to N-particle states for which N is below some limit. The fermions inside loops would be also massive. This allows to circumvent the IR singularities due to integration over the phase space of the final states (say in Coulomb scattering).

6. The representation of the off mass shell particles as pairs of wormhole throats with non-parallel four-momenta (in the simplest case only the three-momenta need be in opposite directions) makes sense and that the particles in question are on mass shell with mass squared being proportional to inverse of a prime number as the number theoretic vision applied to the Kähler-Dirac equation suggests. On mass shell property poses extremely powerful constraints on loops and when the number of the incoming momenta in the loop increases, the number of constraints becomes larger than the number of components of loop momentum for the generic values of the external momenta. Therefore there are excellent hopes of getting rid of UV divergences.

   A stronger assumption encouraged by the classical space-time picture about virtual particles is that the 3-momenta associated with throats of the same wormhole contact are always in same or opposite directions. Even this allows to have virtual momentum spectrum and non-trivial mass spectrum for them assuming that the three momenta are opposite.

7. The best that one can hope is that only a finite number of generalized Feynman diagrams contributes to a given reaction. This would guarantee that amplitudes belong to a finite-dimensional algebraic extension of rational functions with rational coefficients since finite sums do not lead out from a finite algebraic extension of rationals. The first problem are self energy corrections. The assumption tht the mass non-renormalization theorems of SUSYs generalize to TGD framework would guarantee that the loops contributing to fermionic propagators (and their super-counterparts) do not affect them. Also the iteration of more complex amplitudes as analogs of ladder diagrams representing sequences of reactions $M \to M_1 \to M_2 \cdots . \to N$ such that at each M_n in the sequence can appear as on mass shell state could give a non-vanishing contribution to the scattering amplitude and would mean infinite number of Feynman diagrams unless these amplitudes vanish. If N appears as a virtual state the fermions must be however massive on mass shell fermions by the assumption about on-mass shell states and one can indeed imagine a situation in which the decay $M \to N$ is possible when N consists of states made of massless fermions is possible but not when the fermions have non-vanishing masses. This situation seems to be consistent with unitarity. The implication would be that the recursion formula for the all loop amplitudes for a given reaction would give vanishing result for some critical value of loops.

Already these assumptions give good hopes about a generalization of the momentum Grassmann approach to TGD framework. Twistors are doubled as are also the Grassmann variables and there are wave functions correlating the momenta of the the fermions associated with the opposite wormhole throats of the virtual particles as well as incoming gauge bosons which have suffered massivation. Also wave functions correlating the massless momenta at the ends of string like objects and more general many parton states are involved but do not affect the basic twistor formalism. The basic question is whether the hypothesis of unbroken Yangian symmetry could in fact imply something resembling this picture. The possibility to discretize integration contours without losing the representation as residue integral quite generally is basic prerequisite for this and should be shown to be true.

4.2 How to achieve Yangian invariance without trivial scattering amplitudes?

In $\mathcal{N}=4$ SYM the Yangian invariance implies that the MHV amplitudes are constant as demonstrated in [B7]. This would mean that the loop contributions to the scattering amplitudes are trivial. Therefore the breaking of the dual super-conformal invariance by IR singularities of the integrand is absolutely essential for the non-triviality of the theory. Could the situation be different in TGD framework? Could it be possible to have non-trivial scattering amplitudes which are Yangian invariants. Maybe! The following heuristic argument is formulated in the language of super-twistors.

1. The dual conformal super generators of the super-Lie algebra $U(2,2)$ acting as super vector fields reducing effectively to the general form $J=\eta_a^K\partial/\partial Z_a^J$ and the condition that they annihilate scattering amplitudes implies that they are constant as functions of twistor variables. When particles are replaced with pairs of wormhole throats the super generators are replaced by sums J_1+J_2 of these generators for the two wormhole throats and it might be possible to achieve the condition

$$(J_1+J_2)M=0 \tag{4.1}$$

with a non-trivial dependence on the momenta if the super-components of the twistors associated with the wormhole throats are in a linear relationship. This should be the case for bound states.

2. This kind of condition indeed exists. The condition that the sum of the super-momenta expressed in terms of super-spinors λ reduces to the sum of real momenta alone is not usually posed but in the recent case it makes sense as an additional condition to the super-components of the the spinors λ associated with the bound state. This quadratic condition is exactly of the same general form as the one following from the requirement that the sum of all external momenta vanishes for scattering amplitude and reads as

$$X = \lambda_1\eta_1+\lambda_2\eta_2=0 \ . \tag{4.2}$$

The action of the generators $\eta_1\partial_{\lambda_1}+\eta_2\partial_{\lambda_2}$ forming basic building blocks of the super generators on $p_1+p_2=\lambda_1\tilde{\lambda}_1+\lambda_2\tilde{\lambda}_2$ appearing as argument in the scattering amplitude in the case of bound states gives just the quantity X, which vanishes so that one has super-symmetry. The generalization of this condition to n-parton bound state is obvious.

3. The argument does not apply to free fermions which have not suffered topological condensation and are therefore represented by CP_2 type vacuum extremal with single wormhole throat. If one accepts the weak form of electric-magnetic duality, one can circumvent this difficulty. The free fermions carry Kähler magnetic charge whereas physical fermions are accompanied by a bosonic wormhole throat carrying opposite Kähler magnetic charge and opposite electroweak isospin so that a ground state of string like object with size of order electroweak length scale is in question. In the case of quarks the Kähler magnetic charges need not be opposite since color confinement could involve Kähler magnetic confinement:

electro-weak confinement holds however true also now. The above argument generalizes as such to the pairs formed by wormhole throats at the ends of string like object. One can of course imagine also more complex hybrids of these basic options but the general idea remains the same.

Note that the argument involves in an essential manner non-locality, which is indeed the defining property of the Yangian algebra and also the fact that physical particles are bound states. The massivation of the physical particles brings in the IR cutoff.

4.3 Could recursion formula allow interpretation in terms of zero energy ontology?

The identification of pseudo-momentum as a counterpart of region momentum suggests that generalized Feynman diagrams could be seen as a generalization of momentum twistor diagrams. Of course, the generalization from $\mathcal{N} = 4$ SYM to TGD is an enormous step in complexity and one must take all proposals in the following with a big grain of salt. For instance, the replacement of point-like particles with wormhole throats and the decomposition of gauge bosons to pairs of wormhole throats means that naive generalizations are dangerous.

With this in firmly in mind one can ask whether the recursion formula could allow interpretation in terms of zero energy states assigned to causal diamonds (CDs) containing CDs containing $\cdots$. In this framework loops could be assigned with sub-CDs.

The interpretation of the leading order singularities forming the basic building blocks of the twistor approach in zero ontology is the basic source of questions. Before posing these questions recall the basic proposal that partonic fermions are massless but opposite signs of energy are possible for the opposite throats of wormhole contacts. Partons would be on mass shell but besides physical states identified as bound states formed from partons also more general intermediate states would be possible but restricted by momentum conservation and mass shell conditions for partons at vertices. Consider now the questions.

1. Suppose that the massivation of virtual fermions and their super partners allows only ladder diagrams in which the intermediate states contain on mass shell massless states. Should one allow this kind of ladder diagrams? Can one identify them in terms of leading order singularities? Could one construct the generalized Feynman diagrams from Yangian invariant tree diagrams associated with the hierarchy of sub-CDs and using BCFW bridges and entangled pairs of massless states having interpretation as box diagrams with on mass shell momenta at microscopic level? Could it make sense to say that scattering amplitudes are represented by tree diagrams inside CDs in various scales and that the fermionic momenta associated with throats and emerging from sub-CDs are always massless?

2. Could BCFW bridge generalizes as such and could the interpretation of BCFW bridge be in terms of a scattering in which the four on mass shell massless partonic states (partonic throats have arbitrary fermion number) are exchanged between four sub-CDs. This admittedly looks somewhat artificial.

3. Could the addition of 2-particle zero energy state responsible for addition of loop in the recursion relations and having interpretation in terms of the cutting of line carrying loop momentum correspond to an addition of sub-CD such that the 2-particle zero energy state has its positive and negative energy part on its past and future boundaries? Could this mean that one cuts a propagator line by adding CD and leaves only the portion of the line within CD. Could the reverse operation mean to the addition of zero energy “thermally entangled” states in shorter time and length scales and assignable as a zero energy state to a sub-CD. Could one interpret the Cutkosky rule for propagator line in terms of this cutting or its reversal. Why only pairs would be needed in the recursion formula? Why not more general states? Does the recursion formula imply that they are included? Does this relate to the fact that these zero energy states have interpretation as single particle states in the positive energy ontology and that the basic building block of Feynman diagrams is single particle state? Could one regard the unitarity as an identity which states that the discontinuity

of T-matrix characterizing zero energy state over cut is expressible in terms of $TT^\dagger$ and T matrix is the relevant quantity?

Maybe it is again dangerous to try to draw too detailed correspondences: after all, point like particles are replaced by partonic two-surfaces in TGD framework.

4. If I have understood correctly the genuine l-loop term results from $l-1$-loop term by the addition of the zero energy pair and integration over GL(2) as a representative of loop integral reducing $n+2$ to n and calculating the added loop at the same time [B7]. The integrations over the two momentum twistor variables associated with a line in twistor space defining off mass shell four-momentum and integration over the lines represent the integration over loop momentum. The reduction to $GL(2)$ integration should result from the delta functions relating the additional momenta to $GL(2)$ variables (note that GL(2) performs linear transformations in the space spanned by the twistors Z_A and Z_B and means integral over the positions of Z_A an Z_B). The resulting object is formally Yangian invariant but IR divergences along some contours of integration breaks Yangian symmetry.

   The question is what happens in TGD framework. The previous arguments suggests that the reduction of the the loop momentum integral to integrals over discrete mass shells and possibly to a sum over their discrete subsets does not spoil the reduction to contour integrals for loop integrals in the example considered in [B7]. Furthermore, the replacement of mass continuum with a discrete set of mass shells should eliminate IR divergences and might allow to preserve Yangian symmetry. One can however wonder whether the loop corrections with on mass shell massless fermions are needed. If so, one would have at most finite number of loop diagrams with on mass shell fermionic momenta and one of the TGD inspired dreams already forgotten would be realized.

4.4 What about unitarity?

The approach of Arkani-Hamed and collaborators means that loop integral over four-momenta are replaced with residue integrals around a small sphere $p^2=\epsilon$. This is very much reminiscent of my own proposal for a few years ago based on the idea that the condition of twistorialization forces to accept only massless virtual states [K14, K10]. I of course soon gave up this proposal as too childish.

This idea seems to however make a comeback in a modified form. At this time one would have only massive and quantized pseudo-momenta located at discrete mass shells. Can this picture be consistent with unitarity?

Before trying to answer this question one must make clear what one could assume in TGD framework.

1. Physical particles are in the general case massive and consist of collinear fermions at wormhole throats. External partons at wormhole throats must be massless to allow twistorial interpretation. Therefore massive states emerge. This applies also to stringy states.

2. The simplest assumption generalizing the childish idea is that on mass shell massless states for partons appear as both virtual particles and external particles. Space-like virtual momentum exchanges are possible if the virtual particles can consist of pairs of positive and negative energy fermions at opposite wormhole throats. Hence also partons at internal lines should be massless and this raises the question about the identification of propagators.

3. One can assign ordinary massless fermionic propagators with fermionic lines identified as boundaries. Spinor modes are localized at string world sheets from the condition that electromagnetic charge is well-defined for the modes of the induced spinor field. With the boundaries of string world sheets at partonic orbits one can assign 1-D Dirac action with its bosonic counterpart defined by line length defining its bosonic counterpart. Field equations give induced massless Dirac equation at light-like geodesic of imbedding space so that light-like 8-momentum characterizes the fermion lines.

4. This picture suggests natural correspondence with twistor diagrams and 8-D generalization of twistor Grassmannian approach [K14]. For instance, the region momentum appearing

in BCFW bridge defining effective propagator is in general massive although the underlying Feynman diagram would contain online massless momenta. In TGD framework massless lines of Feynman graphs associated with singularities would correspond to real momenta of massless fermions at wormhole throats. Also other canonical operations for Yangian invariants involve light-like momenta at the level of Feynman diagrams and would in TGD framework have a natural identification in terms of partonic momenta. Hence partonic picture would provide a microscopic description for the lines of twistor diagrams.

Let us assume being virtual particle means only that the discretized pseudo-momentum is on shell but massive whereas all real momenta of partons are light-like, and that negative partonic energies are possible. Can one formulate Cutkosky rules for unitarity in this framework? What could the unitarity condition

$$iDisc(T - T^\dagger) = -TT^\dagger$$

mean now? In particular, are the cuts associated with mass shells of physical particles or with mass shells of pseudo-momenta? Could these two assignments be equivalent?

1. The restriction of the partons to be massless but having both signs of energy means that the spectrum of intermediate states contains more states than the external states identified as bound states of partons with the same sign of energy. Therefore the summation over intermediate states does not reduce to a mere summation over physical states but involves a summation over states formed from massless partons with both signs of energy so that also space-like momentum exchanges become possible.

2. The understanding of the unitarity conditions in terms of Cutkosky rules would require that the cuts of the loop integrands correspond to mass shells for the virtual states which are also physical states. Therefore real momenta have a definite sign and should be massless. Besides this bound state conditions guaranteeing that the mass spectrum for physical states is discrete must be assumed. With these assumptions the unitary cuts would not be assigned with the partonic light-cones but with the mass shells associated of physical particles.

3. There is however a problem. The pseudo-momenta of partons associated with the external partons are assumed to be light-like and equal to the physical momenta.

   (a) If this holds true also for the intermediate physical states appearing in the unitarity conditions, the pseudo-momenta at the cuts are light-like and cuts must be assigned with pseudo-momentum light-cones. This could bring in IR singularities and spoil Yangian symmetry. The formation of bound states could eliminate them and the size scale of the largest CD involved would bring in a natural IR cutoff as the mass scale of the lightest particle. This assumption would however force to give up the assumption that only massive pseudo-momenta appear at the lines of the generalized Feynman diagrams.

   (b) On the other hand, if pseudo-momenta are not regarded as a property of physical state and are thus allowed to be massive for the real intermediate states in Cutkosky rules, the cuts at parton level correspond to on mass shell hyperboloids and IR divergences are absent.

5 Comparing twistor revolution with TGD revolution

Lubos Motl saved my Sunday by giving a link to an excellent talk by Nima Arkani-Hamed about the latest twistorial breakthroughs. Lubos Motl talks about "minirevolution" but David Gross uses a more appropriate expression "uprising". I would prefer to speak about revolution inducing at the sociological level a revolt. One must give up QFT in fixed space-time and string theory, and replace them with a theory whose name Nima guesses to be just "T".

For some time ago Lubos Motl told about the latest articles from Nima and collaborators: A Note on Polytopes for Scattering Amplitudes and Local Integrals for Planar Scattering Amplitudes.

Soon after this Lubos Motl gave a link to a video in which Witten talked about knot invariants. This talk was very inspiring and led to TGD based vision about how to calculate invariants of braids, braid cobordisms, and 2-knots in TGD framework and the idea that TGD could be seen as symplectic QFT for calculating these invariances among other things. Much of work was just translation of the basic ideas involved to TGD framework.

One crucial observation was that one can assign to the symplectic group of $\delta M^4_+ \times CP_2$ gerbe gauge potentials generalizing ordinary gauge potentials in terms of which one can define infinite number of classical 2-fluxes allowing to generalize Wilson loop to a Wilson surface. Most importantly, a unique identification for the decomposition of space-time surface to string world sheets identified as singularities of induce gauge fields and partonic 2-surfaces emerged and one can see the two decompositions as dual descriptions. TGD as almost topological QFT concretized to a symplectic QFT for knots, braids, braid cobordisms, and 2-knots. These ideas are documented in the chapter Knots and TGD of "TGD: Physics as Infinite-Dimensional Geometry" [K6]. I did not realize the obvious connection with twistor approach as I wrote the new chapter.

In his rather energetic lecture Nima emphasized how the Yangian symmetry originally discovered in 2-D QFTs, algebraic geometry, twistor theory, and string theory fuse to something bigger called "T". I realized that the twistorial picture developed in the earlier postings integrates nicely with the braidy vision inspired by Witten's talk and that one could understand in TGD framework why twistor description, Yangian symmetry of 2-D integrable systems, and algebraic geometry picture are so closely related. In particular, the dual conformal symmetries of twistor approach could be understood in terms of duality between partonic 2-surfaces and string world sheets expressing the strong form of holography. Also a generalization for the dual descriptions provided by super Wilson loop and ordinary scattering amplitude in $\mathcal{N} = 4$ SUSY in terms of Wilson sheets suggests itself among many other things. Also a rather obvious solution to the problem posed by non-planar diagrams to twistor approach suggests itself. Planar diagrams are simply not present and parton-string duality and huge symmetries of TGD give good reasons for why this should be the case.

5.1 The declaration of revolution by Nima from TGD point of view

At first look Nima's program is a declaration of revolution against all sacred principles. Nima dooms space-time, wants to get rid of QFT, does not even explicitly care about unitarity, and wants to throw Feynman diagrams to paper basket. Nima does not even respect string theory and sees it only as one particular- possibly not the best- manner to describe the underlying simplicity.

5.1.1 Give up space-time

In many respects I agree with Nima about the fate of space-time of QFT. I however see Nima's view a little bit exaggerated: one can perhaps compute scattering amplitudes without Minkowski space but one cannot translate the results of computations to the language of experiments without bringing in frequencies and wavelengths, classical fields, and therefore also space-time. Quantum classical correspondence: this is needed and this brings space-time unavoidably into the picture. Space-time surface serves as a dynamical correlate for quantum dynamics- generalized Bohr orbit required by General Coordinate Invariance and strong form of holography. The enormously important implication is absence of Feynman graphs in ordinary sense since their is no path integral over space-time surface but just single surface: the preferred extremal of Kähler action is enough (forgetting the delicacies caused by the failure of classical determinism in standard sense for Kähler action allowing to realize also the space-time correlates of quantum jump sequences).

Nima uses black hole based arguments to demonstrate that local observables are not operationally defined in neither gravitational theories nor quantum field theories and concludes that space-time is doomed. What would remain would be 4-D space-time regarded as a boundary of higher dimensional space-time (AdS/CFT correspondence). I think that this is quite too complex and that the reduction in degrees of freedom is much more radical: the landscape misery is after all basically due to the exponential inflation in the number of degrees of freedom due to the fatal mistake of making 10-D or 11-D target space dynamical.

What remains in TGD are boundaries of space-time surfaces at the upper and lower ends of causal diamonds $CD \times CP_2$ (briefly CD) and wormhole throats at which the signature of induced metric changes from Euclidian to Minkowskian (recall that Euclidian regions represent generalized Feynman diagrams). CD is essentially a representation of Penrose diagram which fits nicely with twistor approach. Strong form of holography implies that partonic 2-surfaces (or dual string world sheets) and 4-D tangent space data a them are enough as basic particle physics objects. The rest of space-time is needed to realized quantum classical correspondence essential for quantum measurement theory.

The basic message of TGD is that quantum superpositions of space-time surfaces are relevant for physics in all scales. Particles are the dynamical space-time quanta. There is however higher-dimensional space-time which is fixed and rigid $H = M^4 \times CP_2$ and is needed for the symmetries of the theory and guarantees the Kähler geometric existence of the world of classical worlds (WCW). This simplifies the situation enormously: instead of 10- or 11-D dynamical space-time one has just 4-D space-time and 2-D surfaces plus 4-D tangent space data. Holography is what we experience it to be: we see only 2-D surfaces. And physics is experimental science although some super string theorists might argue something else!

5.1.2 Give up fields

Nima argues also that fields are doomed too. I must say that I do not like this Planck length mysticism: it assumes quite too much and in TGD framework something new emerge already in CP_2 scale about 10^4 longer than Planck scale. According to Nima all this pain with Feynman diagrams would be due to the need to realize unitary representations of Poincare group in terms of fields. For massless particles one is forced to assume gauge invariance to eliminate the unphysical polarizations. Nima sees gauge invariance as the source of all troubles. Here I do not completely agree with Nima. The unitary time evolution in fixed space-time translated to the path integral over classical fields is what leads to the combinatorial nightmare of summing over Feynman diagrams and plagues also ϕ^4 theory. Amusingly, as Nima emphasizes all this has been known for 60 years. It is easy to understand that the possibility to realize unitarity elegantly using Feynman diagrams led to the acceptance of this approach as the only possible one.

In TGD framework the geometry of sub-manifolds replaces fields: the dynamics of partonic 2-surfaces identified as throats of light-like wormhole contacts containing fermions at them gives rise to bosons as bound states of fermions and anti-fermions. There is no path integral over space-time surfaces, just functional integral over partonic 2-surfaces so that path integral disappears. In zero energy ontology this means that incoming states are bound states of massless fermions and anti-fermions at wormhole throats and virtual states consist also of massless fermions but without the bound state constraint. This means horribly strong kinematic constraints on vertices defined by partonic 2-surfaces and UV finiteness and IR finiteness are automatic outcome of the theory. Massivation guaranteeing IR finiteness is consistent with massless-ness of fundamental particles since massive states are bound states of massless particles.

Nima talks also about emergence as something fundamental and claims that also space-time emerges. In TGD framework emergence has very concrete meaning. All particles are bound states of massless fermions and the additional purely bosonic degrees of freedom correspond to vibrational degrees of freedom for partonic 2-surfaces.

What is lacking from the program of Nima is the vision about physics as a geometry of worlds of classical worlds [K11] and physics as generalized number theory [K12]. This is what makes the higher-D imbedding space unique and allows the geometrization of quantum physics and identification of standard model symmetries as number theoretical symmetries. Infinite-dimensional geometry is unique just from the requirement that it exists!

5.2 Basic results of twistor approach from TGD point of view

The basic ideas of twistor approach are remarkably consistent with the basic picture of TGD.

5.2.1 Only on mass-shell amplitudes appear in the recursion formula

What is striking that the recursion formula of Nima and collaborators for the integrands of the planar amplitudes of $\mathcal{N} = 4$ SUSY involve only on mass shell massless particles in the role of intermediate states. This is in sharp conflict with not only Feynman diagrammatic intuition but also with the very path integral ideology motivated by the need to realize unitary time development.

As already mentioned, in ZEO (zero energy ontology) all states- both on mass shell and off mass shell are composites of massless states assigned to 2-D partonic surfaces. Path integral is indeed replaced with generalized Bohr orbits and one obtains only very few generalized Feynman diagrams. What remains is functional integral over 3-surfaces, or even less over partonic 2-surfaces with varying tangent space data.

A further simplification is that as a result of the dynamics of preferred extremals many particle states correspond to discrete sets of points at partonic 2-surfaces serving as the ends of orbits of braid strands and possibly also 2-knots and functional integral involves integral over different configurations of these points [K16]. The physical interpretation is as a realization of finite measurement resolution as a property of dynamics itself. The string word sheets are uniquely identified as inverse images under imbedding map of space-time surface to $H = M^4 \times CP_2$ of homologically non-trivial geodesic sphere of CP_2 defining homological magnetic monopole. Holography in its strongest sense states that all information about non-trivial 2-homology if space-time surface and knottedness of the string world sheets is coded to the data at partonic 2-surfaces. For details see [K6].

5.2.2 Twistors and algebraic geometry connection emerge naturally in TGD framework

$H = M^4 \times CP_2$ and the reduction of all on mass shell states to bound states of massless states imply that twistor approach is the natural description of scattering amplitudes in TGD framework.

What is new that one must convolute massless theories in the sense that opposite throats of CP_2 sized wormhole contacts carry massless states. This allows to get rid of IR divergencies and realized exact Yangian symmetry by a purely physical mechanism making particle states massive.

An important implication is that even photon, gluons, and graviton have small masses and that in TGD framework all components of Higgs field are eaten by electroweak gauge bosons. Also gluons have colored scalar and pseudo-scalar counterparts and already now there are some hints at LHC for pseudo-scalar gluons. The discovery of Higgs can of course kill this idea anytime.

The connection with twistors allows to understand how algebraic geometry of projective spaces emerges in TGD framework and one indeed ends up to an alternative formulation of quantum TGD with space-time surfaces in H replaced with holomorphic 6-surfaces of $CP_3 \times CP_3$, which are sphere bundles and there effectively 4-D. The equations determining the 6-surfaces are dictated by rather general constraints.

5.2.3 Dual descriptions in terms of QFT and strings

The connections of $\mathcal{N} = 4$ SUSY with 2-D integrable systems and the possibly of both stringy and QFT descriptions characterized by dual conformal symmetries giving rise to Yangian invariance reduce in TGD framework to the duality between descriptions based on string world sheets and partonic 2-surfaces.

1. The connection with string description emerges from the basic TGD in the sense that one can localize the solutions of the Kähler-Dirac equation [K16] at braid strands located at the light-like 3-D wormhole throats. Similar localization to string world sheets defined in the above described manner holds true in space-time interior. The solutions of the Kähler-Dirac equation localized to braid strands (and to string world sheets in space-time interior) are characterized by what I called pseudo momenta not directly identifiable as momenta. The natural identification is as region momenta of the twistor approach. Recall that the twistorialization of region momenta leads to the momentum twistor approach making dual conformal invariance manifest.

2. The strange looking localization of fermions at braid strands makes mathematically sense only because the classical dynamics of preferred extremals reduces to hydrodynamics such that the flow parameters for flow lines integrate to global coordinates. So called Beltrami flows are in question and mean that preferred extremals have interpretation as perfect fluid flows for which dissipation is minimal [K16]. This property implies also the almost topological QFT property of TGD meaning that Kähler action reduces to Chern-Simons action localized at light-like wormhole throats and space-like 3-surfaces at the ends of CDs.

3. The mathematical motivation on braid strands comes from the fact that this allows to avoid delta functions in the anti-commutators of fermionic oscillator operators at partonic 2-surfaces and therefore also the basic quadratic divergences of quantum field theories. Oscillator algebra has countable -perhaps even finite number- of generators and the loss of complete locality is in terms of finite measurement resolution. The larger the number of braid points selected at partonic 2-surface, the larger the number string world sheets and the higher the complexity of space-time surface. This obviously means a concrete realization of holography. The oscillator algebra has interpretation as SUSY algebra with arbitrarily large N fixed by the number of braid points. This SUSY symmetry is dynamical and badly broken. For right handed neutrino the breaking is smallest but also in this case the mixing of left- and right handed M^4 chiralities in Kähler-Dirac equation implies non-conservation of R-parity as well as particle massivation and also the absence of lightest stable SUSY partner, which means that one particular dark matter candidate is out of game.

4. The big difference between TGD and string models is that super generators do not correspond to Majorana spinors: this is indeed impossible for $M^4 \times CP_2$ since it would mean non-conservation of baryon and lepton numbers. I believed for a long time that stringy propagators emerge from TGD and the long standing painful question was what about stringy propagator defined by the inverse $1/G$ of the hermitian super generator in string models. In TGD $1/G$ cannot define stringy propagator since G carries fermion number. The reduction of strings to pairs of massless particles saves the situation and ordinary massless propagator for the counterparts of region momenta gives well defined propagators for on mass shell massless states! Stringy states reduce to bound states of massless particles in accordance with emergence philosophy. Nothing is scared these days!

5.2.4 Connection with integrable 2-D discrete systems

Twistor approach has revealed a striking connection between 2-D integrable systems and $\mathcal{N} = 4$ SUSY. For instance, one can calculate the anomalous dimensions of $\mathcal{N} = 4$ SUSY from an integrable model for spin chain in 2 dimensions without ever mentioning Feynman diagrams.

The description in terms of partonic 2-surfaces mean a direct connection with braids appearing in 2-D integrable thermodynamical systems and the description in terms of string world sheets means connection with integrable theories in 2-D Minkowski space. Both theories involve Yangian symmetry [A4] for which there exists a hierarchy of non-local conserved charged. Super-conformal invariance and its dual crucial for Yangian symmetry correspond to partonic 2-surfaces and string world sheets. The symmetry algebra is extended dramatically. In $\mathcal{N} = 4$ SUSY one has Yangian of conformal algebra of M^4. In TGD this algebra is generalized to include the super Kac-Moody algebra associated with isometries of the imbedding space, the super-conformal variant of the symplectic algebra of $\delta M^4 \times CP_2$, and also conformal transformations of M^4 mapping given boundary of CD to itself.

This allows also to understand and generalize the duality stating that QFT amplitudes for $\mathcal{N} = 4$ SUSY have interpretation as supersymmetric Wilson loops in dual Minkowski space. The ends of braid strands indeed define Wilson loops. In TGD framework work one must however generalize Wilson loops to Wilson sheets [K6] and the circulations of gauge potentials are replaced with fluxes of gerbe gauge potentials associated with the symplectic group of $\delta M^4_+ \times CP_2$. As noticed, dual conformal symmetries correspond to duality of partonic 2-surfaces and string world sheets implies by the 2-D holography for string world sheets.

5.3 Could planar diagrams be enough in the theory transcending $\mathcal{N} = 4$ SUSY?

Twistor approach as it appears in $\mathcal{N} = 4$ SYM is of course not the final solution.

1. $\mathcal{N} = 4$ SUSY is not enough for the purposes of LHC.

2. The extremely beautiful Yangian symmetry fails as one performs integration to obtain the scattering amplitudes and generates IR singularities. ZEO provides an elegant solution to

this problem by replacing physical on mass shell particles with bound states of massless particles. Also string like objects emerge as this kind of states.

3. Only planar diagrams allow to assign to the sum of Feynman diagrams a single integrand defining the twistor diagram. Something definitely goes wrong unless one is able to treat the non-planar diagrams. The basic problem is that one cannot assign common loop momentum variables to all diagrams simultaneously and this is due to the tricky character of Feynman diagrams. It is difficult to integrate without integrand!

The easy-to-guess question is whether the sum over the non-planar diagrams vanishes or whether they are just absent in a theory transcending $\mathcal{N}=4$ SUSY and QFTs. Let N denote the number of colors of the SUSY. For $N\rightarrow\infty$ limit with g^2N fixed only planar diagrams survive in this kind of theory and one obtains a string model like description as conjectured long time ago by t'Hooft [B14]. This argument led later to AdS/CFT duality.

The stringy diagrams in TGD framework could correspond to planar diagrams of $\mathcal{N}=4$ QFT. Besides this one would have s functional integral over partonic 2-surfaces.

1. The description would be either in terms of partonic 2-surfaces or string world sheets with both determined uniquely in terms of a slicing of space-time surface with physical states characterized in terms of string world sheets in finite measurement resolution.

2. $N\rightarrow\infty$ limit could in TGD framework be equivalent with two replacements. The color group with the infinite-D symplectic group of $\delta M^4_\pm\times CP_2$ and symplectic group and isometry group of H are replaced with their conformal variants.

3. Could $g^2N = constant$ be equivalent with the use of hyper-finite factors of type II_1 [K15] for which the trace of the unit matrix equals to 1 instead of $\mathcal{N}=\infty$. These factors characterize the spinor structure of WCW identifiable in terms of Clifford algebra defined by infinite-D fermionic oscillator algebra defined by second quantized fermions at partonic 2-surfaces.

5.4 Stringy variant of twistor Grassmannian approach

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in elegant manner. One can imagin two manners to circumvent this problem. The first one is modification of the notion of massless to masslessness in 8-D sense. One can indeed imagine an 8-D generalization of the twistor approach of Penrose based on the notion of octonionic spinor [K22]. The status of octonionic spinors remains uncertain.

One can also consider a stringy variant of twistor Grassmannian approach [K14] in which fundamental fermions (as opposed to elementary fermions) are massless. Since this approach looks more promising it is briefly summarized below.

1. The approach is motivated by the stringy picture of elementary particles forced by the well-definedness of em charge for the modes of induced spinor field, and the assumption that elementary particles can be seen as bound states of massless fermions associated with the orbits of string ends at light-like orbits of partonic 2-surfaces. It is quite possible that this localization is consistent with Kähler-Dirac equation only in the Minkowskian regions where the effective metric defined by Kähler-Dirac gamma matrices can be effectively 2-dimensional and parallel to string world sheet.

   This brings the desired purely physical IR cutoff expected to cancel IR divergences. The fermions are massless and on-shell, and one assigns the inverse of massless propagator to the line which corresponds to non-physical helicity. This picture follows from Feynman graph approach if one can perform residue integral over virtual fermion momenta.

2. In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute

to the Kähler action. Chern-Simons Dirac terms to which Kähler action reduces could be responsible for the breaking of CP and T symmetries as they appear in CKM matrix.

3. The quantum numbers characterizing zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse. Thermodynamics would naturally couple to the space-time geometry via the thermodynamical or quantum averages of the quantum numbers.

4. The basic vertex is essentially four-fermion vertex although two light-like momenta combine to form virtual bosonic wormhole contact with total four-momentum which can be also space-like. BCFW recursion formula is expected to hold for the diagrams when one interprets these lines as virtual bosons. This picture could be seen as reduction of bosonic lines fermion lines and replacement of point like elementary particles with stringy structures formed by pairs wormhole contacts (see fig. `http://www.tgdtheory.fi/appfigures/wormholecontact.jpg` or fig. 10 in the appendix of this book).

5. Contrary to the original over-optimistic assumptions the logarithmic UV divergences do not cancel unless on assumes stringy picture. This means that one assigns to the ends of the fermion line the analog of super-conformal propagator and its Hermitian conjugate. The analog of super-conformal propagator is defined by the inverse G/L_0 of super-generator $G^\dagger$. This assignment allows to circumvent the problem due to the fact that G carries fermion number in TGD framework.

6. What is of special interest is that M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. For CP_2 the twistor space has interpretation as the space $SU(3)/U(1) \times U(1)$ for the choices of quantization axes for color quantum numbers. This kind of twistor space can be assigned even with WCW but it is not clear what the physical interpretation and mathematical role of this twistor space is.

5.5 Motives and twistors

Nima mentions at the end of his talk motives [B1]. I know about this abstract branch of algebraic geometry only that it is an attempt to build a universal cohomology theory, which in turn is an algebraic approach to topology allowing to linearize highly non-linear situations encountered typically in algebraic geometry where topology is replaced with holomorphy which is must more stringent property and allows richer structures.

1. Physics as generalized number theory vision involving also fusion of real and p-adic number fields to a larger super structure brings algebraic geometry to the core of TGD. The partonic 2-surfaces allowing interpretation as inhabitants of the intersection of real and p-adic worlds serve as correlates for living matter in TGD Universe. They are algebraic surfaces allowing in preferred coordinates a representation in terms of polynomials with rational coefficients. Motives would be needed to understand the cohomology of these surfaces. One encounters all kinds of problems such as counting the number of rational points in the intersection of p-adic and real variants of the surface and for algebraic surfaces this reduces to the counting of rational points for real 2-surface about which algebraic geometers know a lot of. For instance, surfaces of form $x^n + y^n + z^n = 0$ for $n \geq 3$ appearing in Fermat's theorem are child's play since they allow only origin as a common point.

2. As cautiously concluded in [K6], the intersection form for string world sheets defines a representation of the second relative homology of space-time surface and by Poincare duality also second cohomology. "Relative" is with respect to ends of space-time at the boundaries of CDs and light-like wormhole throats. The intersection form characterizing the collection of self-intersection points at which the braid strands are forced to go through each other is almost enough to characterize connected 4-manifolds topologically by Donaldson theorem [A1].

3. String world sheets define a violent unknotting procedure based on reconnections for braid strands- basic stringy vertex for closed strings- and in this manner knot invariant in the same manner as the recursion allowing to calculate the value of Jones polynomial for a given knot. Quantum TGD gives as a by-product rise to a symplectic QFT describing braids, their cobordisms, and 2-knots. It would not be surprising if the M-matrix elements would have also interpretation as symplectic covariants providing information about the topology of the space-time surface. The 2-braid theory associated with space-time surface would also characterize its topology just as ordinary knots can characterize topology of 3-manifolds.

To sum up, TGD suggests a surprisingly stringy but at the same time incredibly simple generalization of string model in which the discoveries made possible by the twistor approach to $\mathcal{N}=4$ SUSY find a natural generalization. Nima has realized that much more than a mere discovery of computational recipes is involved and indeed talks about T-theory. I feel that the lonely "T" is desperately yearning for the company of "G" and "D" !

5.6 Reducing non-planar diagrams to planar ones by a generalization of algorithm for calculating knot invariants?

I have been listening some lectures in Strings 2001. The lectures related to progress in the calculation of gauge theory and super-gravity amplitudes are really electrifying: one really feels the sparking enthusiasm of the speakers. Besides twistor revolution there is also other amazing progress taking place in QFT side.

At this morning I started to listen the talk of Henrik Johansson about Lie algebra structures in YM and gravitational amplitudes. I have already earlier written about the finding that there is a symmetry between kinematical numerators of the amplitudes involving polarizations and momenta on one hand and color factors on the other hand, and that one can in well defined sense express gravitational scattering amplitudes in terms of squares of YM amplitudes. This holds true for on mass shell amplitudes. The reduction of the gravitational amplitudes to squares of YM amplitudes would be incredible simplification: even 3-graviton off mass shell vertex contains about 100 terms! As a matter fact, gravitation is a gauge theory too with gauge group replaced with Poincare group so that it would not be totally surprising that this kind of duality between kinematics would hold true.

This duality is not however the topic of this posting. As Johansson was explaining the Jacobi identity for the kinematical Lie algebra I got Eureka experience. What the kinematic Jacobi identity states is following:

The numerator for four-point amplitude with twisted legs in s-channel is expressible as a **difference** *of planar s- and t-channel amplitudes.*

If you did not get the association to twistor program already from this sentence, recall that the basic problem of twistor approach are non-planar diagrams. For them one cannot order the loop momenta in such a manner that the ordering would be universal and depend only on the number of loops as it is for planar diagrams without crossings. Hence one is not able to combine all diagrams to single integrand and this is related to the tricks one is forced to apply to make the loop integrals finite: same identification of loop momenta for all diagrams is not possible if one wants finiteness.

What one needs for a generalizaton of twistor approach to apply to non-planar diagrams is a universal identification of the loop momenta by cancelling all crossings: the amplitude itself need not be equal to the difference of the amplitudes obtained by reconnecting in two manners but could be something more general. This operation would be performed for internal lines only. For external lines it tells that the amplitudes changes possible sign when external lines are permuted. For braid statistics a more phase factor would result.

The duality of old-fashioned string models says that the difference of s- and t-channel amplitudes vanishes so that one can say that amplitudes with twisted legs vanish. Also at large N (number of colors) limit of $\mathcal{N}=4$ SUSY these differences vanish and YM theory behaves like string theory and planar twistor approach should give exact answers at this limit. In TGD framework the effective replacement of gauge group with infinite-dimensional symplectic group could have the same effect. But what about finite values of N in super YM theories?

Could one generalize the twistor approach so that one could calculate **all** *amplitudes by recursion- not only the planar ones?*

Alert reader has of course answered already but I try to explain for non-specialists (with me included). If one has worked with braids and knots, one realizes that the expression for the amplitude as difference of planar amplitudes is analogous to what you get in elementary un-knotting operation for braids annihilating one crossing in the knot diagram! In the process you form the difference of two possible reconnections at the crossing point. If you interpret the process as time evolution, it corresponds to two vertices in which interiors of strings touch each other and reconnect in a new manner. In the construction of Jones polynomial as a knot invariant the repeated application of these un-twisting operations eventually leads to un-knot and you get as an outcome the knot invariant. Also non-planar Feynman diagram is like a knot diagram and the outcome of similar procedure should consists of only planar amplitudes.

For Feynman diagrams one cannot distinguish between upper and lower crossings of the lines. This could be interpreted by saying that both crossings give the same contribution. This is the case if untwisting gives the difference of numerators in both color and momentum degrees of freedom so that the signs cancel and the integrals of both contributions are identical despite the fact that the propagator denominators are not identical. The most general outcome would be a term proportional to the sum of the four planar contributions and one could perhaps treat the situation using twistorial methods. Proportionality coefficient could depend on dimensionless Lorentz scalars constructed from the incoming momenta of the sub-diagram with crossing and dictated to high degree by conformal invariance. Professional could probably demonstrate in five minutes that the conjecture cannot hold true.

Especially, if you have written N times "Quantum TGD as almost topological QFT..." you get at the large N limit the vibe in your spine. Because the combinatorics of an almost topological QFT must be that of a topological QFT and because braids are basic building brick of TGD amplitudes, it should be possible to reduce all non-planar amplitudes -both those of TGD and those of $\mathcal{N} = 4$ SUSY and even other gauge theories - by a repeated un-twisting to planar amplitudes. A generalization of the basic algorithm of knot theory would become part of twistorial Feynman diagrammatics and could perhaps also be used to *define* the integrand including also the loops with crossings!

If the proposal can be realized in some sense, the rules for calculating the twistor amplitudes would be simple.

1. You - or your knot theoretical friend- must first patiently unknot the Feynman diagrams involved by eliminating all twists using the basic formula allowing to express twisted sub-amplitude with a difference of un-twisted sub-amplitudes. You might even dream that he gives you explicit formulas for the outcome to get rid of your continual requests for help.

2. At the end of the day you get just planar diagrams and you can apply the general recursive formulas of Nima and others working for all numbers of external particles and all numbers of loops to get the *integrand*, which you should be able to integrate.

3. Unfortunately you are not! But you can knock the door of Goncharov and ask whether he could kindly perform the integral using his magic Symbolic Integration Machine [A5] about which Anastasia Volovich tells in her talk "Symbolifying N=4 SUSY Scattering Amplitudes".

Is this idea just a passign daydream? Or morning dream- my hungry cat forced me to wake up at 3 a'clock so that I might be hallucinating in half-sleeping state. A specialist could immediately tell where this crazy idea of Europe's (if not World's) worst Feynman diagrammatician fails.

5.7 Langlands duality, electric-magnetic duality, S-duality, finite measurement resolution, and quantum Yangian symmetry

The arguments represented in the chapter "Langlands program and TGD" [K7] support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The reader is warmly recommended to consult to this chapter for a more detailed representation.

What is important is that the discussion improves considerably the understanding about TGD proper. Same can be said about other attempts to apply TGD approach to the problems of modern mathematics to which topological quantumf field theories have been applied [K13, K17, K7]. In particular, a connection of Langlands conjectures and Yangian symmetry emerges. The group G *resp.* its Langlands dual ${}^LG$ would define what might be called twisted quantum Yangian associated with G *resp.* ${}^LG$. The Lie group G *resp.* ${}^LG$ corresponds to the description of TGD in terms of partonic 2-surfaces *resp.* string world sheets made possible by strong form of holography in turn implied by strong form of general coordinate invariance implying also electric-magnetic duality and S-duality. Another new result is the identification of the gauge group G as a group defining the measurement resolution in the approach based on hyperfinite factors of type II_1 and proposal for the concrete representation of the corresponding Kac-Moody algebra. A further unexpected outcome are S-dual descriptions of TGD in terms of open string world sheets and partonic 2-surfaces in the moduli spaces of each other. Besides TGD based view about space-time, zero energy ontology and the notion of finite measurement resolution are the basic new notions as compared with the approach of Witten and Kapustin [A6] to the geometric Langlands duality.

1. Zero energy ontology (ZEO) and the related notion of causal diamond CD (CD is a short hand for the cartesian product of causal diamond of M^4 and of CP_2). ZEO leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet. These notions are central in the recent view about TGD. One can assign to the partonic 2-surfaces a conformal moduli space having as additional coordinates the positions of braid strand ends (punctures). By electric-magnetic duality this moduli space must correspond closely to the moduli space of string world sheets.

2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.

3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [K15]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations. At space-time level the finite measurement resolution would be represented in terms of braids at space-time level. The braids come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.

   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multi-locality and therefore quantum Yangian algebra with multilocal generators is unavoidable. Also ZEO forces multi-locality since zero energy states defining orthonormal M-matrices are define multi-local Kac-Moody type algebra with integer powers of $S-$matrix defining the exponent of phase factor assignable with power z^n in the loop algebra generator.

4. In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.

5. The natural identification of the associate of G is quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized (note that in string models and conformal field theories this realization of vertex operators in terms of free fields is of comparable importance).

6. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

7. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [K4] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.

8. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

5.8 About the structure of the Yangian algebra

The attempt to understand Langlands conjecture in TGD framework led to a completely unexpected progress in the understanding of the Yangian symmetry expected to be the basic symmetry of quantum TGD and the following vision suggesting how conformal field theory could be generalized to four-dimensional context is a fruit of this work.

The structure of the Yangian algebra is quite intricate and in order to minimize confusion easily caused by my own restricted mathematical skills it is best to try to build a physical interpretation for what Yangian really is and leave the details for the mathematicians.

1. The first thing to notice is that Yangian and quantum affine algebra are two different quantum deformations of a given Lie algebra. Both rely on the notion of R-matrix inducing a swap of braid strands. R-matrix represents the projective representations of the permutation group for braid strands and possible in 2-dimensional case due to the non-commutativity of the first homotopy group for 2-dimensional spaces with punctures. The R-matrix $R_q(u,v)$ depends on complex parameter q and two complex coordinates u,v. In integrable quantum field theories in M^2 the coordinates u,v are real numbers having identification as exponentials representing Lorenz boosts. In 2-D integrable conformal field theory the coordinates u,v have interpretation as complex phases representing points of a circle. The assumption that the coordinate parameters are complex numbers is the safest one.

2. For Yangian the R-matrix is rational whereas for quantum affine algebra it is trigonometric. For the Yangian of a linear group quantum deformation parameter can be taken to be equal to one by a suitable rescaling of the generators labelled by integer by a power of the complex quantum deformation parameter q. I do not know whether this true in the general case. For the quantum affine algebra this is not possible and in TGD framework the most interesting values of the deformation parameter correspond to roots of unity.

5.8.1 Slicing of space-time sheets to partonic 2-surfaces and string world sheets

The proposal is that the preferred extremals of Kähler action are involved in an essential manner the slicing of the space-time sheets by partonic 2-surfaces and string world sheets. Also an analogous slicing of Minkowski space is assumed and there are infinite number of this kind of slicings defining what I have called Hamilton-Jaboci coordinates [K1]. What is really involved is far from clear. For instance, I do not really understand whether the slicings of the space-time surfaces are purely dynamical or induced by special coordinatizations of the space-time sheets using projections to special kind of sub-manifolds of the imbedding space, or are these two type of slicings equivalent by the very property of being a preferred extremal. Therefore I can represent only what I think I understand about the situation.

1. What is needed is the slicing of space-time sheets by partonic 2-surfaces and string world sheets. The existence of this slicing is assumed for the preferred extremals of Kähler action [K1]. Physically the slicing corresponds to an integrable decomposition of the tangent space of space-time surface to 2-D space representing non-physical polarizations and 2-D space representing physical polarizations and has also number theoretical meaning.

2. In zero energy ontology the complex coordinate parameters appearing in the generalized conformal fields should correspond to coordinates of the imbedding space serving also as local coordinates of the space-time surface. Problems seem to be caused by the fact that for string world sheets hyper-complex coordinate is more natural than complex coordinate. Pair of hyper-complex and complex coordinate emerge naturally as Hamilton-Jacobi coordinates for Minkowski space encountered in the attempts to understand the construction of the preferred extremals of Kähler action.

   Also the condition that the flow lines of conserved isometry currents define global coordinates lead to the to the analog of Hamilton-Jacobi coordinates for space-time sheets [K1]. The physical interpretation is in terms of local polarization plane and momentum plane defined by local light-like direction. What is so nice that these coordinates are highly unique and determined dynamically.

3. Is it really necessary to use two complex coordinates in the definition of Yangian-affine conformal fields? Why not to use hyper-complex coordinate for string world sheets? Since the inverse of hyper-complex number does not exist when the hyper-complex number is light-like, hyper-complex coordinate should appear in the expansions for the Yangian generalization of conformal field as positive powers only. Intriguingly, the Yangian algebra is "one half" of the affine algebra so that only positive powers appear in the expansion. Maybe the hyper-complex expansion works and forces Yangian-affine instead of doubly affine structure. The appearance of only positive conformal weights in Yangian sector could also relate to the fact that also in conformal theories this restriction must be made.

4. It seems indeed essential that the space-time coordinates used can be regarded as imbedding space coordinates which can be fixed to a high degree by symmetries: otherwise problems with general coordinate invariance and with number theoretical universality would be encountered.

5. The slicing by partonic 2-surfaces could (but need not) be induced by the slicing of CD by parallel translates of either upper or lower boundary of CD in time direction in the rest frame of CD (time coordinate varying in the direction of the line connecting the tips of CD). These slicings are not global. Upper and lower boundaries of CD would definitely define analogs of different coordinate patches.

5.8.2 Physical interpretation of the Yangian of quantum affine algebra

What the Yangian of quantum affine algebra or more generally, its super counterpart could mean in TGD framework? The key idea is that this algebra would define a generalization of super conformal algebras of super conformal field theories as well as the generalization of super Virasoro algebra. Optimist could hope that the constructions associated with conformal algebras generalize: this includes the representation theory of super conformal and super Virasoro algebras, coset construction, and vertex operator construction in terms of free fields. One could also hope that the classification of extended conformal theories defined in this manner might be possible.

1. The Yangian of a quantum affine algebra is in question. The heuristic idea is that the two R-matrices - trigonometric and rational- are assignable to the swaps defined by space-like braidings associated with the braids at 3-D space-like ends of space-time sheets at light-like boundaries of CD and time like braidings associated with the braids at 3-D light-like surfaces connecting partonic 2-surfaces at opposite light-like boundaries of CD. Electric-magnetic duality and S-duality implied by the strong form of General Coordinate Invariance should be closely related to the presence of two R-matrices. The first guess is that rational R-matrix is assignable with the time-like braidings and trigonometric R-matrix with the space-like braidings. Here one must or course be very cautious.

2. The representation of the collection of Galois groups associated with infinite primes in terms of braided symplectic flows for braid of braids of.... braids implies that there is a hierarchy of swaps: swaps can also exchange braids of...braids. This would suggest that at the lowest level of the braiding hierarchy the R-matrix associated with a Kac-Moody algebra permutes two braid strands which decompose to braids. There would be two different braided variants of Galois groups.

3. The Yangian of the affine Kac-Moody algebra could be seen as a 4-D generalization of the 2-D Kac-Moody algebra- that is a local algebra having representation as a power series of complex coordinates defined by the projections of the point of the space-time sheet to geodesic spheres of light-cone boundary and geodesic sphere of CP_2.

4. For the Yangian the generators would correspond to polynomials of the complex coordinate of string world sheet and for quantum affine algebra to Laurent series for the complex coordinate of partonic 2-surface. What the restriction to polynomials means is not quite clear. Witten sees Yangian as one half of Kac-Moody algebra containing only the generators having $n \geq 0$. This might mean that the positivity of conformal weight for physical states essential for the construction of the representations of Virasoro algebra would be replaced with automatic positivity of the conformal weight assignable to the Yangian coordinate.

5. Also Virasoro algebra should be replaced with the Yangian of Virasoro algebra or its quantum counterpart. This construction should generalize also to Super Virasoro algebra. A generalization of conformal field theory to a theory defined at 4-D space-time surfaces using two preferred complex coordinates made possible by surface property is highly suggestive. The generalization of conformal field theory in question would have two complex coordinates and conformal invariance associated with both of them. This would therefore reduce the situation to effectively 2-dimensional one rather than 3-dimensional: this would be nothing but the effective 2-dimensionality of quantum TGD implied by the strong form of General Coordinate Invariance.

6. This picture conforms with what the generalization of $D = 4$ $\mathcal{N} = 4$ SYM by replacing point like particles with partonic 2-surfaces would suggest: Yangian is replaced with Yangian of quantum affine algebra rather than quantum group. Note that it is the finite measurement resolution alone which brings in the quantum parameters q_1 and q_2. The finite measurement resolution might be relevant for the elimination of IR divergences.

5.8.3 How to construct the Yangian of quantum affine algebra?

The next step is to try to understand the construction of the Yangian of quantum affine algebra.

1. One starts with a given Lie group G. It could be the group of isometries of the imbedding space or subgroup of it or even the symplectic group of the light-like boundary of $CD \times CP_2$ and thus infinite-dimensional. It could be also the Lie group defining finite measurement resolution with the dimension of Cartan algebra determined by the number of braid strands.

2. The next step is to construct the affine algebra (Kac-Moody type algebra with central extension). For the group defining the measurement resolution the scalar fields assigned with the ends of braid strands could define the Cartan algebra of Kac-Moody type algebra of this group. The ordered exponentials of these generators would define the charged generators of the affine algebra.

   For the imbedding space isometries and symplectic transformations the algebra would be obtained by localizing with respect to the internal coordinates of the partonic 2-surface. Note that also a localization with respect to the light-like coordinate of light-cone boundary or light-like orbit of partonic 2-surface is possible and is strongly suggested by the effective 2-dimensionality of light-like 3-surfaces allowing extension of conformal algebra by the dependence on second real coordinate. This second coordinate should obviously correspond to the restriction of second complex coordinate to light-like 3-surface. If the space-time sheets allow slicing by partonic 2-surfaces and string world sheets this localization is possible for all 2-D partonic slices of space-time surface.

3. The next step is quantum deformation to quantum affine algebra with trigonometric R-matrix $R_{q_1}(u,v)$ associated with space-like braidings along space-like 3-surfaces along the ends of CD. u and v could correspond to the values of a preferred complex coordinate of the geodesic sphere of light-cone boundary defined by rotational symmetry. It choice would fix a preferred quantization axes for spin.

4. The last step is the construction of Yangian using rational R-matrix $R_{q_2}(u,v)$. In this case the braiding is along the light-like orbit between ends of CD. u and v would correspond to the complex coordinates of the geodesic sphere of CP_2. Now the preferred complex coordinate would fix the quantization axis of color isospin.

These arguments are of course heuristic and do not satisfy any criteria of mathematical rigor and the details could of course change under closer scrutinity. The whole point is in the attempt to understand the situation physically in all its generality.

5.8.4 How 4-D generalization of conformal invariance relates to strong form of general coordinate invariance?

The basic objections that one can rise to the extension of conformal field theory to 4-D context come from the successes of p-adic mass calculations. p-Adic thermodynamics relies heavily on the properties of partition functions for super-conformal representations. What happens when one replaces affine algebra with (quantum) Yangian of affine algebra? Ordinary Yangian involves the original algebra and its dual and from these higher multi-local generators are constructed. In the recent case the obvious interpretation for this would be that one has Kac-Moody type algebra with expansion with respect to complex coordinate w for partonic 2-surfaces and its dual algebra with expansion with respect to hyper-complex coordinate of string world sheet.

p-Adic mass calculations suggest that the use of either algebra is enough to construct single particle states. Or more precisely, local generators are enough. I have indeed proposed that the multi-local generators are relevant for the construction of bound states. Also the strong form of general coordinate invariance implying strong form of holography, effective 2-dimensionality, electric-magnetic duality and S-duality suggests the same. If one could construct the states representing elementary particles solely in terms of either algebra, there would be no danger that the results of p-adic mass calculations are lost. Note that also the necessity to restrict the conformal weights of conformal representations to be non-negative would have nice interpretation in terms of the duality.

6 Twistor revolution and TGD

Lubos Motl wrote a nice summary about the talk of Nima Arkani Hamed about twistor revolution in Strings 2012 and gave also a link to the talk [B2]. It seems that Nima and collaborators are ending to a picture about scattering amplitudes which strongly resembles that provided by generalized Feynman diagrammatics in TGD framework

TGD framework is much more general than $\mathcal{N} = 4$ SYM and is to it same as general relativity for special relativity whereas the latter is completely explicit. Of course, I cannot hope that TGD view could be taken seriously - at least publicly. One might hope that these approaches could be combined some day: both have a lot to give for each other. Below I compare these approaches.

The recent approach below emerges from the study of preferred extremals of Kähler and solutions of the Kähler-Dirac equations so that it begins directly from basic TGD whereas the approaches hitherto have been based on general arguments and the precise role of right-handed neutrino has remained enigmatic. Chapters "Construction of quantum TGD: Symmetries" [K3] and "The recent vision about preferred extremals and solutions of the Kähler-Dirac equation" [K16] contain section explaining how super-conformal and Yangian algebras crucial for the Grassmannian approach emerge from the basic TGD.

6.1 The origin of twistor diagrammatics

In TGD framework zero energy ontology forces to replace the idea about continuous unitary evolution in Minkowski space with something more general assignable to causal diamonds (CDs), and S-matrix is replaced with a square root of density matrix equal to a hermitian l square root of density matrix multiplied by unitary S-matrix. Also in twistor approach unitarity has ceased to be a star actor. In p-Adic context continuous unitary time evolution fails to make sense also mathematically.

Twistor diagrammatics involves only massless on mass shell particles on both external and internal lines. Zero energy ontology (ZEO) requires same in TGD: wormhole lines carry parallelly moving massless fermions and anti-fermions. The mass shell conditions at vertices are enormously powerful and imply UV finiteness. Also IR finiteness follows if external particles are massive.

What one means with mass is however a delicate matter. What does one mean with mass? I have pondered 35 years this question and the recent view is inspired by p-adic mass calculations and ZEO, and states that observed mass is in a well-defined sense expectation value of longitudinal mass squared for all possible choices of $M^2 \subset M^4$ characterizing the choices of quantization axis for energy and spin at the level of "world of classical worlds" (WCW) assignable with given causal diamond CD.

The choice of quantization axis thus becomes part of the geometry of WCW. All wormhole throats are massless but develop non-vanishing longitudinal mass squared. Gauge bosons correspond to wormhole contacts and thus consist of pairs of massless wormhole throats. Gauge bosons could develop 4-D mass squared but also remain massless in 4-D sense if the throats have parallel massless momenta. Longitudinal mass squared is however non-vanishing and p-adic thermodynamics predicts it.

6.2 The emergence of 2-D sub-dynamics at space-time level

Nima et al introduce ordering of the vertices in 4-D case. Ordering and related braiding are however essentially 2-D notions. Somehow 2-D theory must be a part of the 4-D theory also at space-time level, and I understood that understanding this is the challenge of the twistor approach at this moment.

The twistor amplitude can be represented as sum over the permutations of n external gluons and all diagrams corresponding to the same permutation are equivalent. Permutations are more like braidings since they carry information about how the permutation proceeded as a homotopy. Yang-Baxter equation emerges and states associativity of the braid group. The allowed braidings are minimal braidings in the sense that the repetitions of permutations of two adjacent vertices are not considered to be separate. Minimal braidings reduce to ordinary permutations. Nima also talks about affine braidings which I interpret as analogs of Kac-Moody algebras meaning that one uses projective representations which for Kac-Moody algebra mean non-trivial central extension.

Perhaps the condition is that the square of a permutation permuting only two vertices which each other gives only a non-trivial phase factor. Lubos Motl suggests an alternative interpretation which would select only special permutations and cannot be therefore correct.

There are rules of identifying the permutation associated with a given diagram involving only basic 3-gluon vertex with white circle and its conjugate. Lubos Motl explains this "Mickey Mouse in maze" rule in his posting in detail: to determine the image $p(n)$ of vertex n in the permutation put a mouse in the maze defined by the diagram and let it run around obeying single rule: if the vertex is black turn to the right and if the vertex is white turn to the left. The mouse cannot remain in a loop: if it would do so, the rule would force it to run back to n after single full loop and one would have a fixed point: $p(n) = n$. The reduction in the number of diagrams is enormous: the infinity of different diagrams reduces to $n!$ diagrams!

What happens in TGD framework?

1. In TGD framework string world sheets and partonic 2-surfaces (or either or these if they are dual notions as conjectured) at space-time surface would define the sought for 2-D theory, and one obtains indeed perturbative expansion with fermionic propagator defined by the inverse of the Kähler-Dirac operator and bosonic propagator defined by the correlation function for small deformations of the string world sheet. The vertices of twistor diagrams emerge as braid ends defining the intersections of string world sheets and partonic 2-surfaces.

   String model like description becomes part of TGD and the role of string world sheets in X^4 is highly analogous to that of string world sheets connecting branes in $AdS^5 \times S^5$ of $\mathcal{N} = 4$ SYM. In TGD framework 10-D $AdS^5 \times S^5$ is replaced with 4-D space-time surface in $M^4 \times CP_2$. The meaning of the analog of AdS^5 duality in TGD framework should be understood. In particular, it could it be that the descriptions involving string world sheets on one hand and partonic 2-surfaces - or 3-D orbits of wormhole throats defining the generalized Feynman diagram- on the other hand are dual to each other. I have conjectured something like this earlier but it takes some time for this kind of issues to find their natural answer.

2. As described in the article, string world sheets and partonic 2-surfaces emerge directly from the construction of the solutions of the Kähler-Dirac equation by requiring conservation of em charge. This result has been conjectured already earlier but using other less direct arguments. 2-D "string world sheets" as sub-manifolds of the space-time surface make the ordering possible, and guarantee the finiteness of the perturbation theory involving n-point functions of a conformal QFT for fermions at wormhole throats and n-point functions for the deformations of the space-time surface. Conformal invariance should dictate these n-point functions to a high degree. In TGD framework the fundamental 3-vertex corresponds to joining of light-like orbits of three wormhole contacts along their 2-D ends (partonic 2-surfaces).

6.3 The emergence of Yangian symmetry

Yangian symmetry associated with the conformal transformations of M^4 is a key symmetry of Grassmannian approach. Is it possible to derive it in TGD framework?

1. TGD indeed leads to a concrete representation of Yangian algebra as generalization of color and electroweak gauge Kac-Moody algebra using general formula discussed in Witten's article about Yangian algebras (see the article).

2. Article discusses also a conjecture about 2-D Hodge duality of quantized YM gauge potentials assignable to string world sheets with Kac-Moody currents. Quantum gauge potentials are defined only where they are needed - at string world sheets rather than entire 4-D space-time.

3. Conformal scalings of the effective metric defined by the anti-commutators of the Kähler-Dirac gamma matrices emerge as realization of quantum criticality. They are induced by critical deformations (second variations not changing Kähler action) of the space-time surface. This algebra can be generalized to Yangian using the formulas in Witten's article (see the article).

4. Critical deformations induce also electroweak gauge transformations and even more general symmetries for which infinitesimal generators are products of U(n) generators permuting n modes of the Kähler-Dirac operator and infinitesimal generators of local electro-weak gauge transformations. These symmetries would relate in a natural manner to finite measurement resolution realized in terms of inclusions of hyperfinite factors with included algebra taking the role of gauge group transforming to each other states not distinguishable from each other.

5. How to end up with Grassmannian picture in TGD framework? This has inspired some speculations in the past. From Nima's lecture one however learns that Grassmannian picture emerges as a convenient parameterization. One starts from the basic 3-gluon vertex or its conjugate expressed in terms of twistors. Momentum conservation implies that with the three twistors λ_i or their conjugates are proportional to each other (depending on which is the case one assigns white or black dot with the vertex). This constraint can be expressed as a delta function constraint by introducing additional integration variables and these integration variables lead to the emergence of the Grassmannian $G_{n,k}$ where n is the number of gluons, and k the number of positive helicity gluons.

   Since only momentum conservation is involved, and since twistorial description works because only massless on mass shell virtual particles are involved, one is bound to end up with the Grassmannian description also in TGD.

6.4 The analog of AdS^5 duality in TGD framework

The generalization of AdS^5 duality of $\mathcal{N} = 4$ SYMs to TGD framework is highly suggestive and states that string world sheets and partonic 2-surfaces play a dual role in the construction of M-matrices. Some terminology first.

1. Let us agree that string world sheets and partonic 2-surfaces refer to 2-surfaces in the slicing of space-time region defined by Hermitian structure or Hamilton-Jacobi structure.

2. Let us also agree that *singular* string world sheets and partonic 2-surfaces are surfaces at which the *effective* metric defined by the anti-commutators of the Kähler-Dirac gamma matrices degenerates to effectively 2-D one.

3. Braid strands at wormhole throats in turn would be loci at which the *induced* metric of the string world sheet transforms from Euclidian to Minkowskian as the signature of induced metric changes from Euclidian to Minkowskian.

AdS^5 duality suggest that string world sheets are in the same role as string world sheets of 10-D space connecting branes in AdS^5 duality for $\mathcal{N} = 4$ SYM. What is important is that there should exist a duality meaning two manners to calculate the amplitudes. What the duality could mean now?

1. Also in TGD framework the first manner would be string model like description using string world sheets. The second one would be a generalization of conformal QFT at light-like 3-surfaces (allowing generalized conformal symmetry) defining the lines of generalized Feynman diagram. The correlation functions to be calculated would have points at the intersections of partonic 2-surfaces and string world sheets and would represent braid ends.

2. General Coordinate Invariance (GCI) implies that physics should be codable by 3-surfaces. Light-like 3-surfaces define 3-surfaces of this kind and same applies to space-like 3-surfaces. There are also preferred 3-surfaces of this kind. The orbits of 2-D wormhole throats at which 4-metric degenerates to 3-dimensional one define preferred light-like 3-surfaces. Also the space-like 3-surfaces at the ends of space-time surface at light-like boundaries of causal diamonds (CDs) define preferred space-like 3-surfaces. Both light-like and space-like 3-surfaces should code for the same physics and therefore their intersections defining partonic 2-surfaces plus the 4-D tangent space data at them should be enough to code for physics. This is strong form of GCI implying effective 2-dimensionality. As a special case one obtains singular string world sheets at which the effective metric reduces to 2-dimensional and singular partonic 2-surfaces defining the wormhole throats. For these 2-surfaces situation could be especially simple mathematically.

3. The guess inspired by strong GCI is that string world sheet -partonic 2-surface duality holds true. The functional integrals over the deformations of 2 kinds of 2-surfaces should give the same result so that functional integration over either kinds of 2-surfaces should be enough. Note that the members of a given pair in the slicing intersect at discrete set of points and these points define braid ends carrying fermion number. Discretization and braid picture follow automatically.

4. Scattering amplitudes in the twistorial approach could be thus calculated by using *any* pair in the slicing - or only either member of the pair if the analog of AdS^5 duality holds true as argued. The possibility to choose any pair in the slicing means general coordinate invariance as a symmetry of the Kähler metric of WCW and of the entire theory suggested already early: Kähler functions for difference choices in the slicing would differ by a real part of holomorphic function and give rise to same Kähler metric of "world of classical worlds" (WCW). For a general pair one obtains functional integral over deformations of space-time surface inducing deformations of 2-surfaces with only other kind 2-surface contributing to amplitude. This means the analog of stringy QFT: Minkowskian or Euclidian string theory depending on choice.

5. For singular string world sheets and partonic 2-surfaces an enormous simplification results. The propagators for fermions and correlation functions for deformations reduce to 1-D instead of being 2-D: the propagation takes place only along the light-like lines at which the string world sheets with Euclidian signature (inside CP_2 like regions) change to those with Minkowskian signature of induced metric. The local reduction of space-time dimension would be very real for particles moving along sub-manifolds at which higher dimensional space-time has reduced metric dimension: they cannot get out from lower-D sub-manifold. This is like ending down to 1-D black hole interior and one would obtain the analog of ordinary Feynman diagrammatics. This kind of Feynman diagrammatics involving only braid strands is what I have indeed ended up earlier so that it seems that I can trust good intuition combined with a sloppy mathematics sometimes works.

   These singular lines represent orbits of point like particles carrying fermion number at the orbits of wormhole throats. Furthermore, in this representation the expansions coming from string world sheets and partonic 2-surfaces are identical automatically. This follows from the fact that only the light-like lines connecting points common to singular string world sheets and singular partonic 2-surfaces appear as propagator lines!

6. The TGD analog of AdS^5 duality of $\mathcal{N} = 4$ SUSYs would be trivially true as an identity in this special case, and the good guess is that it is true also generally. One could indeed use integral over either string world sheets or partonic 2-sheets to deduce the amplitudes.

What is important to notice that singularities of Feynman diagrams crucial for the Grassmannian approach of Nima and others would correspond at space-time level 2-D singularities of the effective metric defined by the Kähler-Dirac gamma matrices defined as contractions of canonical momentum currents for Kähler action with ordinary gamma matrices of the imbedding space and therefore directly reflecting classical dynamics.

6.5 Problems of the twistor approach from TGD point of view

Twistor approach has also its problems and here TGD suggests how to proceed. Signature problem is the first problem.

1. Twistor diagrammatics works in a strict mathematical sense only for $M^{2,2}$ with metric signature (1, 1, -1, -1) rather than M^4 with metric signature (1, -1, -1, -1). Metric signature is wrong in the physical case. This is a real problem which must be solved eventually.

2. Effective metric defined by anti-commutators of the Kähler-Dirac gamma matrices (to be distinguished from the induced gamma matrices) could solve that problem since it would have the correct signature in TGD framework (see the article). String world sheets and partonic 2-surfaces would correspond to the 2-D singularities of this effective metric at which

the even-even signature (1, 1, 1, 1) changes to even-even signature (1, 1, -1, -1). Space-time at string world sheet would become locally 2-D with respect to effective metric just as space-time becomes locally 3-D with respect to the induced metric at the light-like orbits of wormhole throats. String world sheets become also locally 1-D at light-like curves at which Euclidian signature of world sheet in induced metric transforms to Minkowskian.

3. Twistor amplitudes are indeed singularities and string world sheets implied in TGD framework by conservation of em charge would represent these singularities at space-time level. At the end of the talk Nima conjectured about lower-dimensional manifolds of space-time as representation of space-time singularities. Note that string world sheets and partonic 2-surfaces have been part of TGD for years. TGD is of course to $\mathcal{N} = 4$ SYM what general relativity is for the special relativity. Space-time surface is dynamical and possesses induced and effective metrics rather than being flat.

Second limitation is that twistor diagrammatics works only for planar diagrams. This is a problem which must be also fixed sooner or later.

1. This perhaps dangerous and blasphemous statement that I will regret it some day but I will make it. Nima and others have not yet discovered that $M^2 \subset M^4$ must be there but will discover it when they begin to generalize the results to non-planar diagrams and realize that Feynman diagrams are analogous to knot diagrams in 2-D plane (with crossings allowed) and that this 2-D plane must correspond to $M^2 \subset M^4$. The different choices of causal diamond CD correspond to different choices of M^2 representing choice of quantization axes 4-momentum and spin. The integral over these choices guarantees Lorentz invariance. Gauge conditions are modified: longitudinal M^2 projection of massless four-momentum is orthogonal to polarization so that three polarizations are possible: states are massive in longitudinal sense.

2. In TGD framework one replaces the lines of Feynman diagrams with the light-like 3-surfaces defining orbits of wormhole throats. These lines carry many fermion states defining braid strands at light-like 3-surfaces. There is internal braiding associated with these braid strands. String world sheets connect fermions at different wormhole throats with space-like braid strands. The M^2 projections of generalized Feynman diagrams with 4-D "lines" replaced with genuine lines define the ordinary Feynman diagram as the analog of braid diagram. The conjecture is that one can reduce non-planar diagrams to planar diagrams using a procedure analogous to the construction of knot invariants by un-knotting the knot in Alexandrian manner by allowing it to be cut temporarily.

3. The permutations of string vertices emerge naturally as one constructs diagrams by adding to the interior of polygon sub-polygons connected to the external vertices. This corresponds to the addition of internal partonic two-surfaces. There are very many equivalent diagrams of this kind. Only permutations matter and the permutation associated with a given diagram of this kind can be deduced by the Mickey-Mouse rule described explicitly by Lubos Motl. A connection with planar operads is highly suggestive and also conjecture already earlier in TGD framework.

6.6 Could $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM be a part of TGD after all?

Whether right-handed neutrinos generate a supersymmetry in TGD has been a long standing open question. $\mathcal{N} = 1$ SUSY is certainly excluded by fermion number conservation but already $\mathcal{N} = 2$ defining a "complexification" of $\mathcal{N} = 1$ SUSY is possible and could generate right-handed neutrino and its antiparticle. These states should however possess a non-vanishing light-like momentum since the fully covariantly constant right-handed neutrino generates zero norm states. So called massless extremals (MEs) allow massless solutions of the Kähler-Dirac equation for right-handed neutrino in the interior of space-time surface, and this seems to be case quite generally in Minkowskian signature for preferred extremals. This suggests that particle represented as magnetic flux tube structure with two wormhole contacts sliced between two MEs could serve as a starting point in attempts to understand the role of right handed neutrinos and how $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM emerges at the level of space-time geometry. The following arguments inspired by the article of Nima Arkani-Hamed et al [B5] about twistorial scattering amplitudes suggest a more detailed physical interpretation of the possible SUSY associated with the right-handed neutrinos.

The fact that right handed neutrinos have only gravitational interaction suggests a radical re-interpretation of SUSY: no SUSY breaking is needed since it is very difficult to distinguish between mass degenerate spartners of ordinary particles. In order to distinguish between different spartners one must be able to compare the gravitomagnetic energies of spartners in slowly varying external gravimagnetic field: this effect is extremely small.

6.6.1 Scattering amplitudes and the positive Grassmannian

The work of Nima Arkani-Hamed and others represents something which makes me very optimistic and I would be happy if I could understand the horrible technicalities of their work. The article Scattering Amplitudes and the Positive Grassmannian by Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, and Trnka [B5] summarizes the recent situation in a form, which should be accessible to ordinary physicist. Lubos Motl has already discussed the article. The following considerations do not relate much to the main message of the article (positive Grassmannians) but more to the question how this approach could be applied in TGD framework.

1. All scattering amplitudes have on shell amplitudes for massless particles as building bricks

The key idea is that all planar amplitudes can be constructed from on shell amplitudes: all virtual particles are actually real. In zero energy ontology I ended up with the representation of TGD analogs of Feynman diagrams using only mass shell massless states with both positive and negative energies. The enormous number of kinematic constraints eliminates UV and IR divergences and also the description of massive particles as bound states of massless ones becomes possible.

In TGD framework quantum classical correspondence requires a space-time correlate for the on mass shell property and it indeed exists. The mathematically ill-defined path integral over all 4-surfaces is replaced with a superposition of preferred extremals of Kähler action analogous to Bohr orbits, and one has only a functional integral over the 3-D ends at the light-like boundaries of causal diamond (Euclidian/Minkowskian space-time regions give real/imaginary Chern-Simons exponent to the vacuum functional). This would be obviously the deeper principle behind on mass shell representation of scattering amplitudes that Nima and others are certainly trying to identify. This principle in turn reduces to general coordinate invariance at the level of the world of classical worlds.

Quantum classical correspondence and quantum ergodicity would imply even stronger condition: the quantal correlation functions should be identical with classical correlation functions for any preferred extremal in the superposition: all preferred extremals in the superposition would be statistically equivalent [K16]. 4-D spin glass degeneracy of Kähler action however suggests that this is is probably too strong a condition applying only to building bricks of the superposition.

Minimal surface property is the geometric counterpart for masslessness and the preferred extremals are also minimal surfaces: this property reduces to the generalization of complex structure at space-time surfaces, which I call Hamilton-Jacobi structure for the Minkowskian signature of the induced metric. Einstein Maxwell equations with cosmological term are also satisfied.

2. Massless extremals and twistor approach

The decomposition $M^4 = M^2 \times E^2$ is fundamental in the formulation of quantum TGD, in the number theoretical vision about TGD, in the construction of preferred extremals, and for the vision about generalized Feynman diagrams. It is also fundamental in the decomposition of the degrees of string to longitudinal and transversal ones. An additional item to the list is that also the states appearing in thermodynamical ensemble in p-adic thermodynamics correspond to four-momenta in M^2 fixed by the direction of the Lorentz boost. In twistor approach to TGD the possibility to decompose also internal lines to massless states at parallel space-time sheets is crucial.

Can one find a concrete identification for $M^2 \times E^2$ decomposition at the level of preferred extremals? Could these preferred extremals be interpreted as the internal lines of generalized Feynman diagrams carrying massless momenta? Could one identify the mass of particle predicted by p-adic thermodynamics with the sum of massless classical momenta assignable to two preferred extremals of this kind connected by wormhole contacts defining the elementary particle?

Candidates for this kind of preferred extremals indeed exist. Local $M^2 \times E^2$ decomposition and light-like longitudinal massless momentum assignable to M^2 characterizes "massless extremals" (MEs, "topological light rays"). The simplest MEs correspond to single space-time sheet carrying a conserved light-like M^2 momentum. For several MEs connected by wormhole contacts the longitudinal massless momenta are not conserved anymore but their sum defines a time-like conserved four-momentum: one has a bound states of massless MEs. The stable wormhole contacts binding MEs together possess Kähler magnetic charge and serve as building bricks of elementary particles. Particles are necessary closed magnetic flux tubes having two wormhole contacts at their ends and connecting the two MEs.

The sum of the classical massless momenta assignable to the pair of MEs is conserved even when they exchange momentum. Quantum classical correspondence requires that the conserved classical rest energy of the particle equals to the prediction of p-adic mass calculations. The massless momenta assignable to MEs would naturally correspond to the massless momenta propagating along the internal lines of generalized Feynman diagrams assumed in zero energy ontology. Masslessness of virtual particles makes also possible twistor approach. This supports the view that MEs are fundamental for the twistor approach in TGD framework.

3. Scattering amplitudes as representations for braids whose threads can fuse at 3-vertices

Just a little comment about the content of the article. The main message of the article is that non-equivalent contributions to a given scattering amplitude in $\mathcal{N}=4$ SYM represent elements of the group of permutations of external lines - or to be more precise - decorated permutations which replace permutation group S_n with $n!$ elements with its decorated version containing $2^n n!$ elements. Besides 3-vertex the basic dynamical process is permutation having the exchange of neighboring lines as a generating permutation completely analogous to fundamental braiding. BCFW bridge has interpretation as a representations for the basic braiding operation.

This supports the TGD inspired proposal (TGD as almost topological QFT) that generalized Feynman diagrams are in some sense also knot or braid diagrams allowing besides braiding operation also two 3-vertices [K6]. The first 3-vertex generalizes the standard stringy 3-vertex but with totally different interpretation having nothing to do with particle decay: rather particle travels along two paths simultaneously after $1 \rightarrow 2$ decay. Second 3-vertex generalizes the 3-vertex of ordinary Feynman diagram (three 4-D lines of generalized Feynman diagram identified as Euclidian space-time regions meet at this vertex). The main idea is that in TGD framework knotting and braiding emerges at two levels.

1. At the level of space-time surface string world sheets at which the induced spinor fields (except right-handed neutrino [K16]) are localized due to the conservation of electric charge can form 2-knots and can intersect at discrete points in the generic case. The boundaries of strings world sheets at light-like wormhole throat orbits and at space-like 3-surfaces defining the ends of the space-time at light-like boundaries of causal diamonds can form ordinary 1-knots, and get linked and braided. Elementary particles themselves correspond to closed loops at the ends of space-time surface and can also get knotted (possible effects are discussed in [K6]).

2. One can assign to the lines of generalized Feynman diagrams lines in M^2 characterizing given causal diamond. Therefore the 2-D representation of Feynman diagrams has concrete physical interpretation in TGD. These lines can intersect and what suggests itself is a description of non-planar diagrams (having this kind of intersections) in terms of an algebraic knot theory. A natural guess is that it is this knot theoretic operation which allows to describe also non-planar diagrams by reducing them to planar ones as one does when one constructs knot invariant by reducing the knot to a trivial one. Scattering amplitudes would be basically knot invariants.

"Almost topological" has also a meaning usually not assigned with it. Thurston's geometrization conjecture stating that geometric invariants of canonical representation of manifold as Riemann geometry, defined topological invariants, could generalize somehow. For instance, the geometric invariants of preferred extremals could be seen as topological or more refined invariants (symplectic, conformal in the sense of 4-D generalization of conformal structure). If quantum ergodicity holds true, the statistical geometric invariants defined by the classical correlation functions of various induced classical gauge fields for preferred extremals could be regarded as this kind of invariants for sub-manifolds. What would distinguish TGD from standard topological QFT would be that the invariants in question would involve length scale and thus have a physical content in the usual sense of the word!

6.6.2 Could $\mathcal{N}$ =2 or $\mathcal{N} = 4$ SUSY have something to do with TGD?

$\mathcal{N} = 4$ SYM has been the theoretical laboratory of Nima and others. $\mathcal{N} = 4$ SYM is definitely a completely exceptional theory, and one cannot avoid the question whether it could in some sense be part of fundamental physics. In TGD framework right handed neutrinos have remained a mystery: whether one should assign space-time SUSY to them or not. Could they give rise to something resembling $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SUSY with fermion number conservation?

1. Earlier results

My latest view is that *fully* covariantly constant right-handed neutrinos decouple from the dynamics completely. I will repeat first the earlier arguments which consider only fully covariantly constant right-handed neutrinos.

1. $\mathcal{N} = 1$ SUSY is certainly excluded since it would require Majorana property not possible in TGD framework since it would require superposition of left and right handed neutrinos and lead to a breaking of lepton number conservation. Could one imagine SUSY in which both MEs between which particle wormhole contacts reside have $\mathcal{N} = 2$ SUSY which combine to form an $\mathcal{N} = 4$ SUSY?

2. Right-handed neutrinos which are covariantly constant right-handed neutrinos in both M^4 degrees of freedom cannot define a non-trivial theory as shown already earlier. They have no electroweak nor gravitational couplings and carry no momentum, only spin.

   The fully covariantly constant right-handed neutrinos with two possible helicities at given ME would define representation of SUSY at the limit of vanishing light-like momentum. At this limit the creation and annihilation operators creating the states would have vanishing anti-commutator so that the oscillator operators would generate Grassmann algebra. Since creation and annihilation operators are hermitian conjugates, the states would have zero norm and the states generated by oscillator operators would be pure gauge and decouple from physics. This is the core of the earlier argument demonstrating that $\mathcal{N} = 1$ SUSY is not possible in TGD framework: LHC has given convincing experimental support for this belief.

2. Could massless right-handed neutrinos covariantly constant in CP_2 degrees of freedom define $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SUSY?

Consider next right-handed neutrinos, which are covariantly constant in CP_2 degrees of freedom but have a light-like four-momentum. In this case fermion number is conserved but this is consistent with $\mathcal{N} = 2$ SUSY at both MEs with fermion number conservation. $\mathcal{N} = 2$ SUSYs could emerge from $\mathcal{N} = 4$ SUSY when one half of SUSY generators annihilate the states, which is a basic phenomenon in supersymmetric theories.

1. At space-time level right-handed neutrinos couple to the space-time geometry - gravitation - although weak and color interactions are absent. One can say that this coupling forces them to move with light-like momentum parallel to that of ME. At the level of space-time surface right-handed neutrinos have a spectrum of excitations of four-dimensional analogs of conformal spinors at string world sheet (Hamilton-Jacobi structure).

   For MEs one indeed obtains massless solutions depending on longitudinal M^2 coordinates only since the induced metric in M^2 differs from the light-like metric only by a contribution which is light-like and contracts to zero with light-like momentum in the same direction. These solutions are analogs of (say) left movers of string theory. The dependence on E^2 degrees of freedom is holomorphic. That left movers are only possible would suggest that one has only single helicity and conservation of fermion number at given space-time sheet rather than 2 helicities and non-conserved fermion number: two real Majorana spinors combine to single complex Weyl spinor.

2. At imbedding space level one obtains a tensor product of ordinary representations of $\mathcal{N} = 2$ SUSY consisting of Weyl spinors with opposite helicities assigned with the ME. The state content is same as for a reduced $\mathcal{N} = 4$ SUSY with four $\mathcal{N} = 1$ Majorana spinors replaced by two complex $\mathcal{N} = 2$ spinors with fermion number conservation. This gives 4 states at both space-time sheets constructed from ν_R and its antiparticle. Altogether the two MEs give 8 states, which is one half of the 16 states of $\mathcal{N} = 4$ SUSY so that a degeneration of this symmetry forced by non-Majorana property is in question.

3. Is the dynamics of $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM possible in right-handed neutrino sector?

Could $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM be a part of quantum TGD? Could TGD be seen a fusion of a degenerate $\mathcal{N} = 4$ SYM describing the right-handed neutrino sector and string theory like theory describing the contribution of string world sheets carrying other leptonic and quark spinors? Or could one imagine even something simpler?

What is interesting that the net momenta assigned to the right handed neutrinos associated with a pair of MEs would correspond to the momenta assignable to the particles and obtained by p-adic mass calculations. It would seem that right-handed neutrinos provide a representation of the momenta of the elementary particles represented by wormhole contact structures. Does this mimicry generalize to a full duality so that all quantum numbers and even microscopic dynamics of defined by generalized Feynman diagrams (Euclidian space-time regions) would be represented by right-handed neutrinos and MEs? Could a generalization of $\mathcal{N} = 4$ SYM with non-trivial gauge group with proper choices of the ground states helicities allow to represent the entire microscopic dynamics?

Irrespective of the answer to this question one can compare the TGD based view about supersymmetric dynamics with what I have understood about $\mathcal{N} = 4$ SYM.

1. In the scattering of MEs induced by the dynamics of Kähler action the right-handed neutrinos play a passive role. Kähler-Dirac equation forces them to adopt the same direction of four-momentum as the MEs so that the scattering reduces to the geometric scattering for MEs as one indeed expects on basic of quantum classical correspondence. In ν_R sector the basic scattering vertex involves four MEs and could be a re-sharing of the right-handed neutrino content of the incoming two MEs between outgoing two MEs respecting fermion number conservation. Therefore $\mathcal{N} = 4$ SYM with fermion number conservation would represent the scattering of MEs at quantum level.

2. $\mathcal{N} = 4$ SUSY would suggest that also in the degenerate case one obtains the full scattering amplitude as a sum of permutations of external particles followed by projections to the directions of light-like momenta and that BCFW bridge represents the analog of fundamental braiding operation. The decoration of permutations means that each external line is effectively doubled. Could the scattering of MEs can be interpreted in terms of these decorated permutations? Could the doubling of permutations by decoration relate to the occurrence of pairs of MEs?

   One can also revert these questions. Could one construct massive states in $\mathcal{N} = 4$ SYM using pairs of momenta associated with particle with integer label k and its decorated copy with label $k + n$? Massive external particles obtained in this manner as bound states of massless ones could solve the IR divergence problem of $\mathcal{N} = 4$ SYM.

3. The description of amplitudes in terms of leading singularities means picking up of the singular contribution by putting the fermionic propagators on mass shell. In the recent case it would give the inverse of massless Dirac propagator acting on the spinor at the end of the internal line annihilating it if it is a solution of Dirac equation.

   The only way out is a kind of cohomology theory in which solutions of Dirac equation represent exact forms. Dirac operator defines the exterior derivative d and virtual lines correspond to non-physical helicities with $d\Psi \neq 0$. Virtual fermions would be on mass-shell fermions with non-physical polarization satisfying $d^2\Psi = 0$. External particles would be those with physical polarization satisfying $d\Psi = 0$, and one can say that the Feynman diagrams containing physical helicities split into products of Feynman diagrams containing only non-physical helicities in internal lines.

4. The fermionic states at wormhole contacts should define the ground states of SUSY representation with helicity +1/2 and -1/2 rather than spin 1 or -1 as in standard realization of $\mathcal{N}=4$ SYM used in the article. This would modify the theory but the twistorial and Grassmannian description would remain more or less as such since it depends on light-likeness and momentum conservation only.

4. 3-vertices for sparticles are replaced with 4-vertices for MEs

In $\mathcal{N}=4$ SYM the basic vertex is on mass-shell 3-vertex which requires that for real light-like momenta all 3 states are parallel. One must allow complex momenta in order to satisfy energy conservation and light-likeness conditions. This is strange from the point of view of physics although number theoretically oriented person might argue that the extensions of rationals involving also imaginary unit are rather natural.

The complex momenta can be expressed in terms of two light-like momenta in 3-vertex with one real momentum. For instance, the three light-like momenta can be taken to be p, k, and $p-ka$ with $k = ap_R$. Here p (incoming momentum) and p_R are real light-like momenta satisfying $p \cdot p_R = 0$ but with opposite sign of energy, and a is complex number. What is remarkable that also the negative sign of energy is necessary also now.

Should one allow complex light-like momenta in TGD framework? One can imagine two options.

1. Option I: no complex momenta. In zero energy ontology the situation is different due to the presence of a pair of MEs meaning replaced of 3-vertices with 4-vertices or 6-vertices, the allowance of negative energies in internal lines, and the fact that scattering is of sparticles is induced by that of MEs. In the simplest vertex a massive external particle with non-parallel MEs carrying non-parallel light-like momenta can decay to a pair of MEs with light-like momenta. This can be interpreted as 4-ME-vertex rather than 3-vertex (say) BFF so that complex momenta are not needed. For an incoming boson identified as wormhole contact the vertex can be seen as BFF vertex.

   To obtain space-like momentum exchanges one must allow negative sign of energy and one has strong conditions coming from momentum conservation and light-likeness which allow non-trivial solutions (real momenta in the vertex are not parallel) since basically the vertices are 4-vertices. This reduces dramatically the number of graphs. Note that one can also consider vertices in which three pairs of MEs join along their ends so that 6 MEs (analog of 3-boson vertex) would be involved.

2. Option II: complex momenta are allowed. Proceeding just formally, the $\sqrt{g_4}$ factor in Kähler action density is imaginary in Minkowskian and real in Euclidian regions. It is now clear that the formal approach is correct: Euclidian regions give rise to Kähler function and Minkowskian regions to the analog of Morse function. TGD as almost topological QFT inspires the conjecture about the reduction of Kähler action to boundary terms proportional to Chern-Simons term. This is guaranteed if the condition $j^\mu_K A_\mu = 0$ holds true: for the known extremals this is the case since Kähler current j_K is light-like or vanishing for them. This would seem that Minkowskian and Euclidian regions provide dual descriptions of physics. If so, it would not be surprising if the real and complex parts of the four-momentum were parallel and in constant proportion to each other.

   This argument suggests that also the conserved quantities implied by the Noether theorem have the same structure so that charges would receive an imaginary contribution from Minkowskian regions and a real contribution from Euclidian regions (or vice versa). Four-momentum would be complex number of form $P = P_M + iP_E$. Generalized light-likeness condition would give $P_M^2 = P_E^2$ and $P_M \cdot P_E = 0$. Complexified momentum would have 6 free components. A stronger condition would be $P_M^2 = 0 = P_E^2$ so that one would have two light-like momenta "orthogonal" to each other. For both relative signs energy P_M and P_E would be actually parallel: parameterization would be in terms of light-like momentum and scaling factor. This would suggest that complex momenta do not bring in anything new and Option II reduces effectively to Option I. If one wants a complete analogy with the usual twistor approach then $P_M^2 = P_E^2 \neq 0$ must be allowed.

5. *Is SUSY breaking possible or needed?*

It is difficult to imagine the breaking of the proposed kind of SUSY in TGD framework, and the first guess is that all these 4 super-partners of particle have identical masses. p-Adic thermodynamics does not distinguish between these states and the only possibility is that the p-adic primes differ for the spartners. But is the breaking of SUSY really necessary? Can one really distinguish between the 8 different states of a given elementary particle using the recent day experimental methods?

1. In electroweak and color interactions the spartners behave in an identical manner classically. The coupling of right-handed neutrinos to space-time geometry however forces the right-handed neutrinos to adopt the same direction of four-momentum as MEs has. Could some gravitational effect allow to distinguish between spartners? This would be trivially the case if the p-adic mass scales of spartners would be different. Why this should be the case remains however an open question.

2. In the case of unbroken SUSY only spin distinguishes between spartners. Spin determines statistics and the first naive guess would be that bosonic spartners obey totally different atomic physics allowing condensation of selectrons to the ground state. Very probably this is not true: the right-handed neutrinos are de-localized to 4-D MEs and other fermions correspond to wormhole contact structures and 2-D string world sheets.

   The coupling of the spin to the space-time geometry seems to provide the only possible manner to distinguish between spartners. Could one imagine a gravimagnetic effect with energy splitting proportional to the product of gravimagnetic moment and external gravimagnetic field B? If gravimagnetic moment is proportional to spin projection in the direction of B, a non-trivial effect would be possible. Needless to say this kind of effect is extremely small so that the unbroken SUSY might remain undetected.

3. If the spin of sparticle be seen in the classical angular momentum of ME as quantum classical correspondence would suggest then the value of the angular momentum might allow to distinguish between spartners. Also now the effect is extremely small.

6. *What can one say about scattering amplitudes?*

One expect that scattering amplitudes factorize with the only correlation between right-handed neutrino scattering and ordinary particle scattering coming from the condition that the four-momentum of the right-handed neutrino is parallel to that of massless extremal of more general preferred extremal having interpretation as a geometric counterpart of radiation quantum. This momentum is in turn equal to the massless four-momentum associated with the space-time sheet in question such that the sum of classical four-momenta associated with the space-time sheets equals to that for all wormhole throats involved. The right-handed neutrino amplitude itself would be simply constant. This certainly satisfies the SUSY constraint and it is actually difficult to find other candidates for the amplitude. The dynamics of right-handed neutrinos would be therefore that of spectator following the leader.

6.6.3 Right-handed neutrino as inert neutrino?

6.7 Right-handed neutrino as inert neutrino?

There is a very interesting posting by Jester in Resonaances with title How many neutrinos in the sky? [?]. Jester tells about the recent 9 years WMAP data [?] and compares it with earlier 7 years data. In the earlier data the effective number of neutrino types was $N_{eff} = 4.34 \pm 0.87$ and in the recent data it is $N_{eff} = 3.26 \pm 0.35$. WMAP alone would give $N_{eff} = 3.89 \pm 0.67$ also in the recent data but also other data are used to pose constraints on N_{eff}.

To be precise, N_{eff} could include instead of fourth neutrino species also some other weakly interacting particle. The only criterion for contributing to N_{eff} is that the particle is in thermal equilibrium with other massless particles and thus contributes to the density of matter considerably during the radiation dominated epoch.

Jester also refers to the constraints on N_{eff} from nucleosynthesis, which show that $N_{eff} \sim 4$ us slightly favored although the entire range $[3,5]$ is consistent with data.

It seems that the effective number of neutrinos could be 4 instead of 3 although latest WMAP data combined with some other measurements favor 3. Later a corrected version of the eprint appeared [**?**] telling that the original estimate of N_{eff} contained a mistake and the correct estimate is $N_{eff} = 3.84 \pm 0.40$.

An interesting question is what $N_{eff} = 4$ could mean in TGD framework?

1. One poses to the modes of the Kähler-Dirac equation the following condition: electric charge is conserved in the sense that the time evolution by Kähler-Dirac equation does not mix a mode with a well-defined em charge with those with different em charge. The implication is that all modes except pure right handed neutrino are restricted at string world sheets. The first guess is that string world sheets are minimal surfaces of space-time surface (rather than those of imbedding space). One can also consider minimal surfaces of imbedding space but with effective metric defined by the anti-commutators of the Kähler-Dirac gamma matrices. This would give a direct physical meaning for this somewhat mysterious effective metric.

   For the neutrino modes localized at string world sheets mixing of left and right handed modes takes place and they become massive. If only 3 lowest genera for partonic 2-surfaces are light, one has 3 neutrinos of this kind. The same applies to all other fermion species. The argument for why this could be the case relies on simple observation [K2]: the genera g=0, 1, 2 have the property that they allow for all values of conformal moduli Z_2 as a conformal symmetry (hyper-ellipticity). For $g > 2$ this is not the case. The guess is that this additional conformal symmetry is the reason for lightness of the three lowest genera.

2. Only purely right-handed neutrino is completely de-localized in 4-volume so that one cannot assign to it genus of the partonic 2-surfaces as a topological quantum number and it effectively gives rise to a fourth neutrino very much analogous to what is called sterile neutrino. De-localized right-handed neutrinos couple only to gravitation and in case of massless extremals this forces them to have four-momentum parallel to that of ME: only massless modes are possible. Very probably this holds true for all preferred extremals to which one can assign massless longitudinal momentum direction which can vary with spatial position.

3. The coupling of ν_R is to gravitation alone and all electroweak and color couplings are absent. According to standard wisdom de-localized right-handed neutrinos cannot be in thermal equilibrium with other particles. This according to standard wisdom. But what about TGD?

   One should be very careful here: de-localized right-handed neutrinos is proposed to give rise to SUSY (not $\mathcal{N} = 1$ requiring Majorana fermions) and their dynamics is that of passive spectator who follows the leader. The simplest guess is that the dynamics of right handed neutrinos at the level of amplitudes is completely trivial and thus trivially supersymmetric. There are however correlations between four-momenta.

   (a) The four-momentum of ν_R is parallel to the light-like momentum direction assignable to the massless extremal (or more general preferred extremal). This direct coupling to the geometry is a special feature of the Kähler-Dirac operator and thus of sub-manifold gravity.

   (b) On the other hand, the sum of massless four-momenta of two parallel pieces of preferred extremals is the - in general massive - four-momentum of the elementary particle defined by the wormhole contact structure connecting the space-time sheets (which are glued along their boundaries together since this is seems to be the only manner to get rid of boundary conditions requiring vacuum extremal property near the boundary). Could this direct coupling of the four-momentum direction of right-handed neutrino to geometry and four-momentum directions of other fermions be enough for the right handed neutrinos to be counted as a fourth neutrino species in thermal equilibrium? This might be the case!

One cannot of course exclude the coupling of 2-D neutrino at string world sheets to 4-D purely right handed neutrinos analogous to the coupling inducing a mixing of sterile neutrino with ordinary neutrinos. Also this could help to achieve the thermal equilibrium with 2-D neutrino species.

6.8 Experimental evidence for sterile neutrino?

Many physicists are somewhat disappointed to the results from LHC: the expected discovery of Higgs has been seen as the main achievement of LHC hitherto. Much more was expected. To my opinion there is no reason for disappointment. The exclusion of the standard SUSY at expected energy scale is very far reaching negative result. Also the fact that Higgs mass is too small to be stable without fine tuning is of great theoretical importance. The negative results concerning heavy dark matter candidates are precious guidelines for theoreticians. The non-QCD like behavior in heavy ion collisions and proton-ion collisions is bypassed my mentioning something about AdS/CFT correspondence and non-perturbative QCD effects. I tend to see these effects as direct evidence for M_{89} hadron physics [K8].

In any case, something interesting has emerged quite recently. Resonaances tells that the recent analysis [**?**] of X-ray spectrum of galactic clusters claims the presence of monochromatic 3.5 keV photon line. The proposed interpretation is as a decay product of sterile 7 keV neutrino transforming first to a left-handed neutrino and then decaying to photon and neutrino via a loop involving W boson and electron. This is of course only one of the many interpretations. Even the existence of line is highly questionable.

One of the poorly understood aspects of TGD is right-handed neutrino, which is obviously the TGD counterpart of the inert neutrino.

1. The old idea is that covariantly constant right handed neutrino could generate $\mathcal{N}=2$ supersymmetry in TGD Universe. In fact, all modes of induced spinor field would generate superconformal symmetries but electroweak interactions would break these symmetries for the modes carrying non-vanishing electroweak quantum numbers: they vanish for ν_R. This picture is now well-established at the level of WCW geometry [K21]: super-conformal generators are labelled angular momentum and color representations plus two conformal weights: the conformal weight assignable to the light-like radial coordinate of light-cone boundary and the conformal weight assignable to string coordinate. It seems that these conformal weights are independent. The third integer labelling the states would label genuinely Yangian generators: it would tell the poly-locality of the generator with locus defined by partonic 2-surface: generators acting on single partonic 2-surface, 2 partonic 2-surfaces, ...

2. It would seem that even the SUSY generated by ν_R must be badly broken unless one is able to invent dramatically different interpretation of SUSY. The scale of SUSY breaking and thus the value of the mass of right-handed neutrino remains open also in TGD. In lack of better one could of course argue that the mass scale must be CP_2 mass scale because right-handed neutrino mixes considerably with the left-handed neutrino (and thus becomes massive) only in this scale. But why this argument does not apply also to left handed neutrino which must also mix with the right-handed one!

3. One can of course criticize the proposed notion of SUSY: wonder whether fermion + extremely weakly interacting ν_R at same wormhole throat (or interior of 3-surface) can behave as single coherent entity as far spin is considered [K19] ?

4. The condition that the modes of induced spinor field have a well-defined electromagnetic charge eigenvalue [K16] requires that they are localized at 2-D string world sheets or partonic 2-surfaces: without this condition classical W boson fields would mix the em charged and neutral modes with each other. Right-handed neutrino is an exception since it has no

electroweak couplings. Unless right-handed neutrino is covariantly constant, the Kähler-Dirac gamma matrices can however mix the right-handed neutrino with the left handed one and this can induce transformation to charged mode. This does not happen if each Kähler-Dirac gamma matrix can be written as a linear combination of either M^4 or CP_2 gamma matrices and Kähler-Dirac equation is satisfied separately by M^4 and CP_2 parts of the Kähler-Dirac equation.

5. Is the localization of the modes other than covariantly constant neutrino to string world sheets a consequence of dynamics or should one assume this as a separate condition? If one wants similar localization in space-time regions of Euclidian signature - for which CP_2 type vacuum extremal is a good representative - one must assume it as a separate condition. In number theoretic formulation string world sheets/partonic 2-surfaces would be commutative/co-commutative sub-manifolds of space-time surfaces which in turn would be associative or co-associative sub-manifolds of imbedding space possessing (hyper-)octonionic tangent space structure. For this option also right-handed neutrino would be localized to string world sheets. Right-handed neutrino would be covariantly constant only in 2-D sense.

   One can consider the possibility that ν_R is de-localized to the entire 4-D space-time sheet. This would certainly modify the interpretation of SUSY since the number of degrees of freedom would be reduced for ν_R.

6. Non-covariantly constant right-handed neutrinos could mix with left-handed neutrinos but not with charged leptons if the localization to string world sheets is assumed for modes carrying non-vanishing electroweak quantum numbers. This would make possible the decay of right-handed to neutrino plus photon, and one cannot exclude the possibility that ν_R has mass 7 keV.

   Could this imply that particles and their spartners differ by this mass only? Could it be possible that practically unbroken SUSY could be there and we would not have observed it? Could one imagine that sfermions have annihilated leaving only states consisting of fundamental fermions? But shouldn't the total rate for the annihilation of photons to hadrons be two times the observed one? This option does not sound plausible.

   What if one assumes that given sparticle is charactrized by the same p-adic prime as corresponding particle but is dark in the sense that it corresponds to non-standard value of Planck constant. In this case sfermions would not appear in the same vertex with fermions and one could escape the most obvious contradictions with experimental facts. This leads to the notion of shadron: shadrons would be [K19] obtained by replacing quarks with dark squarks with nearly identical masses. I have asked whether so called X and Y bosons having no natural place in standard model of hadron could be this kind of creatures.

The interpretation of 3.5 keV photons as decay products of right-handed neutrinos is of course totally ad hoc. Another TGD inspired interpretation would be as photons resulting from the decays of excited nuclei to their ground state.

1. Nuclear string model [K9] predicts that nuclei are string like objects formed from nucleons connected by color magnetic flux tubes having quark and antiquark at their ends. These flux tubes are long and define the "magnetic body" of nucleus. Quark and antiquark have opposite em charges for ordinary nuclei. When they have different charges one obtains exotic state: this predicts entire spectrum of exotic nuclei for which statistic is different from what proton and neutron numbers deduced from em charge and atomic weight would suggest. Exotic nuclei and large values of Planck constant could make also possible cold fusion [K5].

2. What the mass difference between these states is, is not of course obvious. There is however an experimental finding [**?**] (see *Analysis of Gamma Radiation from a Radon Source: Indications of a Solar Influence*) that nuclear decay rates oscillate with a period of year and the rates correlate with the distance from Sun. A possible explanation is that the gamma rays from Sun in few keV range excite the exotic nuclear states with different decay rate so that the average decay rate oscillates [K9]. Note that nuclear excitation energies in keV range would also make possible interaction of nuclei with atoms and molecules.

3. This allows to consider the possibility that the decays of exotic nuclei in galactic clusters generates 3.5 keV photons. The obvious question is why the spectrum would be concentrated at 3.5 keV in this case (second question is whether the energy is really concentrated at 3.5 keV: a lot of theory is involved with the analysis of the experiments). Do the energies of excited states depend on the color bond only so that they would be essentially same for all nuclei? Or does single excitation dominate in the spectrum? Or is this due to the fact that the thermal radiation leaking from the core of stars excites predominantly single state? Could $E = 3.5$ keV correspond to the maximum intensity for thermal radiation in stellar core? If so, the temperature of the exciting radiation would be about $T \simeq E/3 \simeq 1.2 \times 10^7$ K. This in the temperature around which formation of Helium by nuclear fusion has begun: the temperature at solar core is around 1.57×10^7 K.

6.9 Still one attempt understand generalized Feynman diagrams

The only manner to develop the understanding about generalized Feynman diagrams is to articulate the basic questions again and again in the hope that something new might emerge. There are many questions to be answered.

What Grasmannian twistorialization means when imbedding space spinor fields are the fundamental objects. How does ZEO make twistorialization possible? How twistorialization emerges from the functional integral in WCW from the proposed stringy construction of spinor modes.

One must also understand in detail the realization of super-conformal symmetries and how n-point functions of conformal field theory are associated with scattering amplitudes, and how cm degrees of freedom described using imbedding space spinor harmonics are treated in the scattering amplitudes. Also the braiding and knotting should be understood. The challenge is to find a universal form for the vertices and to identify the propagators. Also the modular degrees of freedom of partonic 2-surfaces explaining family replication phenomenon should be taken into account.

6.9.1 Zero energy ontology, twistors, and Grassmannian description?

In ZEO also virtual wormhole throats are massless particles and four-momentum conservation at vertices identifiable as partonic 2-surfaces at which wormhole throats meet expressed in terms of twistors leads to Grassmannian formulation automatically. This feature is thus not specific to $\mathcal{N} = 4$ SYM.

Momentum conservation and massless on mass-shell conditions at vertices defined as partonic 2-surfaces at which the orbits of wormhole contacts meet, are extremely restrictive, and one has good hopes that huge reduction in the number of twistorial diagrams takes place and could even lead to finite number of diagrams (number theoretic arguments favor this).

6.9.2 Realization of super-conformal algebra

Thanks to the advances in the construction of preferred extremals and solutions of the Kähler-Dirac equation there has been considerable progress in the understanding of super-conformal invariance and its 4-D generalization [K16].

1. In ordinary SYM ground states correspond to both maximal helicites or only second maximal helicity of super multiplet ($\mathcal{N} = 4$ case). Now these ground states are replaced by the modes of imbedding space spinor fields assignable to center of mass degrees of freedom for partonic 2-surfaces. The light-like four-momenta of these modes can be expressed in terms of twistor variables. Spin-statistics connection seems to require that the total number of fermions and anti-fermions associated with given wormhole throat is always odd.

2. Super-algebra consists of oscillator operators with non-vanishing quark or lepton number. By conformal invariance fermionic oscillator operators obey 1-D anti-commutation relations. The integral over CD boundary defines a bi-linear form analogous to inner product. If a reduction to single particle level takes place, the vertex is expressible as a matrix element between two fermion-anti-fermion states: the first one assignable to the incoming and outgoing wormhole throats one and second to the virtual boson identified as wormhole contact on one hand. The exchange boson entangled fermion-anti-fermion state represented by a bi-local generalization

of the gauge current. This picture applying to gauge boson exchanges generalizes in rather obvious manner.

3. Unitary demands correlation between fermionic oscillator operators and spinor harmonics of imbedding space as following argument suggets. The bilocal generalization of gauge current defines a "norm" for spinor modes as generalization of what in QFT regarded as charge. On basis of experience with Dirac spinors one expect that this norm is not positive definite. This "norm" must be consistent with the unitarity of the scattering amplitude and the experience with QFT suggests a correlation between creation/annihilation operator character of fermionic oscillator operators and the sign of the "norm" in imbedding space degrees of freedom.

4. The modes with negative norm should correspond to negative energy fermions and annihilation operators and modes with positive norm to positive energy fermions and creation operators. Therefore the anti-commutators of fermionic oscillator operators must be linear in four-momentum or its longitudinal projection and thus proportional to $p^k\gamma_k$ or $p^k_L\gamma_k$.

   On the other hand, the primary anti-commutators for the induced spinor fields are proportional to the Kähler-Dirac gamma matrix in a direction normal to the 1-D quantization curve at the boundary of string world sheet or at the partonic 2-surface. These two anti-commutators should be consistent.

   (a) Does the functional integral somehow lead from the primordial anti-commutators to the anti-commutators involving longitudinal momentum and perhaps 1-D delta function in the intersection of M^2 with CD boundary (light-like line)?
   (b) Or does the connection between the two quantizations emerge as boundary conditions stating that the normal component of Kähler-Dirac gamma matrix at the boundary and along string world sheet equals to $p^k_L\gamma_k$? This would also realize quantum classical correspondence in the sense that the longitudinal momentum is reflected in the geometry of the space-time sheet. Quaternionic space-time surfaces indeed contain integrable distribution of $M^2(x) \subset M^4$ at their tangent spaces. The restriction to braid strands would mean that the condition indeed makes sense. Note that braid strands should correspond to same $M^2(x)$.

6.9.3 How conformal time evolution corresponds to physical time evolution?

The only internally consistent option is conformally invariant meaning that induced spinor fields anti-commute only along as set of 1-D curves belonging to partonic 2-surfaces. This means that one can speak about conformal time evolution at partonic 2-surface.

This suggests a huge simplification of the conformal dynamics.

1. Conformal time evolution can be translated to time evolution along light-like orbit of wormhole throat by projecting the intersections of this surface with shifted light-cone boundary to the upper or lower light-like boundary of CD: whether it is upper or lower boundary of CD depends on the arrow of imbedding space time associated with the zero energy state. All partonic 2-surfaces would be mapped to same light-cone boundary. The orbits of braid strands at wormhole throat project to orbits at light-cone boundary in question and can be further projected to the sphere $r_M = constant$ at light-boundary. 3-D dynamics would project to simplest possible stringy 2-D dynamics (spherical string orbit) and dictated by conformal invariance.

2. The conformal field theory in question is for conformal fermionic fields realized in terms of fermionic oscillator operators and n-point functions correspond to fermionic n-point functions. The non-triviality of dynamics in these degrees of freedom follows from the non-triviality of the conformal field theory. The entire collection of partonic 2-surfaces at the ends of CD would reduce to its projection to S^2.

3. One can try to build a geometric view about the situation using as a guideline conformal Hamiltonian quantum evolution. Time=constant slices would correspond to 1-D curve or

collection of them. At these slices fermionic oscillator operators would satisfy the conformal anti-commutation relations. This kind of slice would be associated with both ends of CD. Braid strands would connect these 1-D slices as kind of hairs. One can however ask whether there is any need to restrict the end points of braid strands to line on a curve at which fermionic oscillator operators satisfy stringy anti-commutation relations.

6.9.4 What happens in 3-vertices?

The vision is that only 3-vertices are needed. Idealize particles as wormhole contacts (in reality pair of wormhole contacts connected by a flux tube would describe elementary particles). A very convenient visualization of wormhole contact is as a very short string like object with throats at its ends so that stringy diagrammatics allows to identify the vertices as the analogs of open string vertices. One can even consider the possibility that string theory amplitudes define the vertices. This would conform with the p-adic mass calculations applying conformal invariance in CP_2 scale. Note also that partonic 2-surfaces are effectively replaced by braids so that very stringy picture results.

1. Consider a three vertex representing the emission of boson by incoming fermion (FFB) or by incoming boson (BBB) described as wormhole contact such that throats carry fermion and anti-fermion number in the bosonic case. In the fermionic the first throat carries fermion and second one represents vacuum state. The exchanged boson can be regarded as fermion anti-fermion pair such that second fermion travels to future and second one to the past in the vertex. 3-vertex would reduce to two 2-vertices representing the transformation of fermion line from incoming line to exchanged line or from latter to outgoing line.

2. The minimal option is that the same vertex describes the situation if both cases. Essentially a combination of incoming free fermions to boson like state is in question and corresponds in string picture an exchange of open string between open strings. If so, second wormhole throat is passive and suffers forward scattering in the vertex. The fermion and anti-fermion of the exchanged virtual boson (the light-like momenta of wormhole throats need not be collinear for virtual bosons and also the sign of energy can be different form them) would suffer scattering before the transformation to fermions belonging to incoming and outgoing wormhole contact.

One expects the vertices to factorize into products of two kinds of factors: the inner products of fermionic Fock states defined by conformal n-point functions at sphere of light-cone boundary, and the bi-linear forms for the modes of imbedding space spinor fields involving integral over cm degrees of freedom and allowing twistorialization by previous arguments. Let us continue with the simple example in which wormhole throats carry fermion number 0 or 1.

1. If second wormhole throat is passive, it is enough to construct only FFB vertex, with B identified as a wormhole contact carrying fermion and anti-fermion. One has 4 fermions altogether, and one expects that in cm degrees of freedom incoming and outgoing fermion are un-correlated whereas the fermions of the boson exchange are correlated and the correlation is expressible as the analog of gauge current.

2. This suggests a sum over bi-local counterparts of electro-weak and color gauge currents at opposite ends of the exchanged line. Bi-local gauge currents would contain a spinor mode from both wormhole throats, and the strict locality of M^4 gauge currents would be replaced with a bi-locality in CP_2 scale.

3. The current assignable to a particular boson exchange must involve the matrix element of corresponding charge matrix between spinor modes besides the quantity. Is it possible to find a general expression for the sum over current - current interaction terms? If this is the case, there would be no need to perform the summation over bosonic exchanges explicitly. One would have the analog for the $\sum_n |n\rangle\langle n|$ in propagator line but summation allowing the momenta of fermion and anti-fermion to be arbitrary massless momenta rather than summing up to the on mass shell momentum of boson. The counterpart of gauge coupling should be universal and naturally given by Kähler coupling.

4. The TGD counterparts of scalar and pseudo-scalar bosons would be vector bosons with polarization in CP_2 direction and they could be also seen both as Higgs like states and Euclidian pions assignable to wormhole contacts. Genuine H-scalars are excluded implied by 8-D chiral symmetry implying also separate conservation of B and L.

In the general case the wormhole throats carry arbitrary odd fermion number but for fermion numbers $n > 1$ at any wormhole throat exotic super-partner with propagator decaying faster than $1/p^2$ is in question. Furthermore, wormhole contact is accompanied by second wormhole contact since the flux lines of monopole flux must closed. Therefore one has a pair of "long" string like flux tubes connected by short flux tubes at their ends. Its length is given by weak length scale quite generally or possibly by Compton length. The other end of the long flux tube can also contain fermions at both flux tubes.

6.9.5 The identification of propagators

A natural guess is that the propagator for single fermion state is just the longitudinal Dirac propagator D_{p_L} for a massless fermion in $M^4 \supset M^2$. For states, which by statistics constraint always contain an odd number $M = 2N + 1$ of fermions and anti-fermions, the propagator would be M: th power of fermionic longitudinal propagator so that it would reduce to $p_L^{-2N} D_{p_L}$ meaning that only the single fermion states would be behave like ordinary elementary particles. States with higher fermion number would represent radiative corrections reflecting the non-point-like nature of partons. Longitudinal mass squared would be equal to the sum of the contribution from CP_2 degrees of freedom and the integer valued conformal contribution from spinor harmonics. The M^4 momenta associated with wormhole throats would be light-like. In the prescription using fermionic longitudinal propagators assigned to the braid strands, braid strands are analogous to the edges of polygons appearing in twistor Grassmannian approach.

6.9.6 Some open questions

A long list of open questions remains without a final answer. Consider first twistor Grassmannian approach.

1. Does this prescription follow from quantum criticality? Recall that quantum criticality formulated in terms of preferred extremals and Kähler-Dirac equation leads to a stringy perturbation theory involving fermionic propagator defined by the Kähler-Dirac operator and functional integral over WCW for the deformations of space-time surface preserving the preferred extremal property [K16]. This propagator could be called space-time propagator to distinguish it from the imbedding space propagator associated with the longitudinal momentum.

2. One expects that one still has topological Feynman diagrammatic expansion (besides that defined by functional integral over small deformations of space-time surface with given topology) involving in principle an arbitrary number of vertices defined by the intermediate partonic 2-surfaces. Momentum conservation and massless on mass-shell conditions however pose powerful restrictions on the allowed diagrams, and one might hope that the simplicity of the outcome is comparable to Grassmannian twistor approach for $\mathcal{N} = 4$ SYM. One can even hope that the number of contributing diagrams is finite. The important point would be that Grassmannian diagrams would give the outcome of the functional integral over 3-surfaces. Twistorial Grassmann representation is the first guess hitherto for the explicit outcome of the functional integral over WCW.

3. The lines of Feynman graph are replaced with braids. A new element is that braid strands are braided as curves inside light-like 3-surfaces defined by the orbit of the wormhole throat. Twistorial construction applies only to the planar amplitudes of $\mathcal{N} = 4$ SYM. Can one imagine TGD counterparts for non-planar amplitudes in TGD framework or does the stringy picture imply that they are completely absent?

   A possible answer to the question is based on the M^2 projection of the lines of braid strands (or on the projection to the 2-surface defined by an integrable distribution of tangent planes

$M^2(x)$). For non-planar diagrams the projections intersect and the intersection cannot be eliminated by a small deformation. It does not make sense to say that line goes over or below the second line. One can speak only about crossings. In the theory or algebraic knots [A7] algebraic knots with crossings are possible [K6]. Could algebraic knot theory allow to reduce non-planar diagrams to sums of planar diagrams?

4. Does one obtain Yangian symmetry using longitudinal propagators and by integrating over the moduli labeling among other things the choices of the preferred plane $M^2 \subset M^4$ or integrable distribution of preferred planes $M^2(x) \subset M^4$? The integral over the choices $M^2 \subset M^4$ gives formally a Lorentz invariant outcome. Does it also give rise to physically acceptable scattering amplitudes? Are the gauge conditions for the incoming gauge boson states formulated in terms of longitudinal momentum and thus allowing also the third polarization physical? Can one apply this gauge condition also to the virtual boson like exchanges?

5. It is still somewhat unclear whether one should assume single global choice of M^2 or an integrable distribution of $M^2(x)$.

   (a) The choice of $M^2(x)$ must be same for all braid strands of given partonic 2-surface and remain constant along braid strand and therefore be same also at second end of the strand. Otherwise the fermionic propagator would vary along braid strand. A possible additional condition on braids is that braid strands correspond to the same choice of $M^2(x)$. In quantum measurement theory this corresponds to the choice of same spin quantization axes for all fermions inside parton and is physically extremely natural condition. The implication is that one can indeed assign a fixed M^2 with CD and choice of braid strands via boundary conditions. The simplest boundary conditions would require $M^2(x)$ to be constant at light-like 3-surfaces and at the ends of space-time surface at boundaries of CD. This is in spirit with holography stating that quantum measurements can be carried out only at these 3-surfaces (or at least those at the ends of CD).

   (b) One cannot exclude the possibility that $M^2(x)$ does not depend on x for a particular space-time sheet and even entire CD although this looks rather strong a restriction. On the other hand, one can ask whether the preferred M^2 assigned with CD should be generalized to an integrable distribution $M^2(x)$ assigned with CD such that $M^2(x)$ is contained in the tangent space of preferred Minkowskian extremal.

   (c) Is the functional integral over integrable distributions $M^2(x)$ needed? It would be analogous to a functional integral over string world sheets. It is enough to integrate over Lorentz transforms of a given distribution $M^2(x)$ to achieve Lorentz invariance. This because the choice of the integrable distribution of $M^2(x)$ for space-time surface reduces effectively to the choice of M^2 for the disconnected pieces of generalized Feynman diagram. Physical intuition suggests that a particular choice of $M^2(x)$ corresponds to fixing of zero modes of WCW and is essentially fixing of classical variables needed to fix quantization axes. The fixing of value distributions of induced Kähler fields n 4-D sense at partonic 2-surfaces would be similar fixation of zero modes.

   (d) If only M^2 momentum makes it visible in anti-commutators, how the other components of four-momentum can make themselves visible in dynamics? This is possible via momentum conservation at vertices making possible twistor Grassmannian approach. The dynamics in transversal momenta would be dictated completely by the conservation laws.

There are also other challenges.

1. Family replication phenomenon has TGD based explanation in terms of the conformal moduli of partonic 2-surfaces. How conformal moduli should be taken into account in the Feynman diagrammatics? Phenomena like topological mixing inducing in turn the mixing of partonic 2-topologies responsible for CKM mixing in TGD Universe should be understood in this description.

2. Number theoretical universality requires that also the p-adic variants of the amplitudes should make sense. One could even require that the amplitudes decompose to products of parts belonging to different number fields [K18]. If one were able to formulate this vision precisely, it would provide powerful constraints on the amplitudes. For instance, a reduction of the amplitudes to a sum over finite number of generalized Feynman diagrams is plausible since this would guarantee that individual contributions which must give rise to algebraic numbers for algebraic 4-momenta, would sum up to an algebraic number.

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