TGD AND HYPER-FINITE FACTORS

Matti Pitkänen

Rinnekatu 2-4 A 8, Karkkila, 03620, Finland

April 29, 2024

0.1 PREFACE

Brief summary of TGD

Towards the end of the year 2023 I became convinced that it would be appropriate to prepare collections about books related to TGD and its applications. The finiteness of human lifetime was my first motivation. My second motivation was the deep conviction that TGD will mean a revolution of the scientific world view and I must do my best to make it easier.

The first collection would relate to the TGD proper and its applications to physics. Second collection would relate to TGD inspired theory of consciousness and the third collection to TGD based quantum biology. The books in these collections would focus on much more precise topics than the earlier books and would be shorter. This would make it much easier for the reader to understand what TGD is, when the time is finally mature for the TGD to be taken seriously. This particular book belongs to a collection of books about TGD proper.

The basic ideas of TGD

TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students in the seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 45 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of the embedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, the mainstream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to the same multiplet of the gauge group implying instability of the proton.

Instead of trying to describe in detail the path, which led to TGD as it is now with all its side tracks, it is better to summarize the recent view which of course need not be final.

TGD can be said to be a fusion of special and general relativities. The Relativity Principle (Poincare Invariance) of Special Relativity is combined with the General Coordinate Invariance and Equivalence Principle of General Relativity. TGD involves 3 views of physics: physics geometry, physics as number theory and physics as topological physics in some sense.

Physics as geometry

"Geometro-" in TGD refers to the idea about the geometrization of physics. The geometrization program of Einstein is extended to gauge fields allowing realization in terms of the geometry of surfaces so that Einsteinian space-time as abstract Riemann geometry is replaced with sub-manifold geometry. The basic motivation is the loss of classical conservation laws in General Relativity Theory (GRT)(see **Fig. 1**). Also the interpretation as a generalization of string models by replacing string with 3-D surface is natural.

- Standard model symmetries uniquely fix the choice of 8-D space in which space-time surfaces live to $H = M^4 \times CP_2$ [L77]. Also the notion of twistor is geometrized in terms of surface geometry and the existence of twistor lift fixes the choice of H completely so that TGD is unique [L26, L30](see **Fig. 2**). The geometrization applies even to the quantum theory itself and the space of space-time surfaces - "world of classical worlds" (WCW) - becomes the basic object endowed with Kähler geometry (see **Fig. 3**). The mere mathematical existence of WCW geometry requires that it has maximal isometries, which together twistor lift and number theoretic vision fixes it uniquely [L78].

- General Coordinate Invariance (GCI) for space-time surfaces has dramatic implications. A given 3-surface fixes the space-time surface almost completely as analog of Bohr orbit (preferred extremal). This implies holography and leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K89, L38].

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields in all scales. It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to the phases of ordinary matter predicted by the number theoretic vision and behaving like dark matter but identifiable as matter explaining the missing baryon problem whereas the galactic dark matter would correspond to the dark energy assignable monopole flux tubes as deformations of cosmic strings. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem and p-adic physics solved this problem in terms of p-adic thermodynamics [K17, K41] [L73].

- One of the most recent discoveries of classical TGD is exact general solution of the field equations. Holography can be realized as a generalized holomorphy realized in terms of what I call Hamilton-Jacobi structure [L74]. Space-time surfaces correspond to holomorphic imbeddings of the space-time surface to H with a generalized complex structure defined by the vanishing of 2 analytic functions of 4 generalized complex coordinates of H. These surfaces are automatically minimal surfaces. This is true for any geneneral coordinate invariant action constructed in terms of the induced geometric structures so that the dynamics is universal. Different actions differ only in the sense that singularities at which the minimal surface property fails depend on the action. This affects the scattering amplitudes, which can be constructed in terms of the data related to the singularities [L80].

- Generalized conformal symmetries define an extension of conformal symmetries and one can assign to them Noether charges. Besides this the so called super-symplectic symmetries associated with $\delta M^4_+ \times CP_2$ define isometries of the "world of classical worlds" (WCW), which by holography is essentially the space of Bohr orbits of 3-surfaces as particles so that quantum TGD is expected to reduce to a generalization of wave mechanics.

Physics as number theory

During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

Adelic physics [L23, L24] fusing real and various p-adic physics is part of the number theoretic vision, which provides a kind of dual description for the description based on space-time geometry and the geometry of "world of classical words". Adelic physics predicts two fractal length scale hierarchies: p-adic length scale hierarchy and the hierarchy of dark length scales labelled by $h_{eff} = nh_0$, where n is the dimension of extension of rational. The interpretation of the latter hierarchy is as phases of ordinary matter behaving like dark matter. Quantum coherence is possible in arbitarily long scales. These two hierarchies are closely related. p-Adic primes correspond to ramified primes for a polynomial, whose roots define the extension of rationals: for a given extension this polynomial is not unique.

$M^8 - H$ duality

The concrete realization of the number theoretic vision is based on $M^8 - H$ duality (see **Fig. 4**). What the precise form is this duality is, has been far from clear but the recent form is the simplest one and corresponds to the original view [L79]. M^8 corresponds to octonions O but with the number theoretic metric defined by $Re(o^2)$ rather than the standard norm and giving Minkowskian signature.

The physics in M^8 can be said to be algebraic whereas in H field equations are partial differential equations. The dark matter hierarchy corresponds to a hierarchy of algebraic extensions of rationals inducing that for adeles and has interpretation as an evolutionary hierarchy (see **Fig. 5**). p-Adic physics is an essential part of number theoretic vision and the space-time surfaces are such that at least their M^8 counterparts exists also in p-adic sense. This requires that the analytic function defining the space-time surfaces are polynomials with rational coefficients.

$M^8 - H$ duality relates two complementary visions about physics (see **Fig. 6**), and can be seen as a generalization of the momentum-position duality of wave mechanics, which fails to generalize to quantum field theories (QFTs). $M^8 - H$ duality applies to particles which are 3-surfaces instead of point-like particles.

p-Adic physics

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Principle with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

Hierarchy of Planck constants labelling phases ordinary matter dark matter behaving like dark matter

One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

TGD as an analog of topological QFT

Consider next the attribute "Topological". In condensed matter physical topological physics has become a standard topic. Typically one has fields having values in compact spaces, which are topologically non-trivial. In the TGD framework space-time topology itself is non-trivial as also the topology of $H = M^4 \times CP_2$. Since induced metric is involved with TGD, it is too much to say that TGD is topological QFT but one can for instance say, that space-time surfaces as preferred extremals define representatives for 4-D homological equivalence classes.

The space-time as 4-surface $X^4 \subset H$ has a non-trivial topology in all scales and this together with the notion of many-sheeted space-time brings in something completely new. Topologically trivial Einsteinian space-time emerges only at the QFT limit in which all information about topology is lost (see **Fig. 7**).

Any GCI action satisfying holography=holomorphy principle has the same universal basic extremals: CP_2 type extremals serving basic building bricks of elementary particles, cosmic strings and their thickenings to flux tubes defining a fractal hierarchy of structure extending from CP_2 scale to cosmic scales, and massless extremals (MEs) define space-time correletes for massless particles. World as a set or particles is replaced with a network having particles as nodes and flux tubes as bonds between them serving as correlates of quantum entanglement.

"Topological" could refer also to p-adic number fields obeying p-adic local topology differing radically from the real topology (see **Fig. 8**).

Zero energy ontology

TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

General coordinate invariance leads to the identification of space-time surfaces are analogous to Bohr orbits inside causal diamond (CD). CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). By the already described hologamphy, 3-dimensional data replaces the boundary conditions at single 3-surface involving also normal derivatives with conditions involving no derivates.

In zero energy ontology (ZEO), the superpositions of space-time surfaces inside causal diamond (CD) having their ends at the opposite light-like boundaries of CD, define quantum states. CDs form a scale hierarchy (see **Fig. 9** and **Fig. 10**). Quantum states are modes of WCW spinor fields, essentially wave functions in the space WCW consisting of Bohr orbit-like 4-surfaces.

Quantum jumps occur between these and the basic problem of standard quantum measurement theory disappears. Ordinary state function reductions (SFRs) correspond to "big" SFRs (BSFRs) in which the arrow of time changes (see **Fig. 11**). This has profound thermodynamic implications and the question about the scale in which the transition from classical to quantum takes place becomes obsolete. BSFRs can occur in all scales but from the point of view of an observer with an opposite arrow of time they look like smooth time evolutions.

In "small" SFRs (SSFRs) as counterparts of "weak measurements" the arrow of time does not change and the passive boundary of CD and states at it remain unchanged (Zeno effect).

Equivalence Principle in TGD framework

There have been also longstanding problems related to the relationship between inertial mass and gravitational mass, whose identification has been far from obvious.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of CDs defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle in the form expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

  At quantum level, the Equivalence Principle has a surprisingly strong content. In linear Minkowski coordinates, space-time projection of the M^4 spinor connection representing gravitational gauge potentials the coupling to induced spinor fields vanishes. Also the modified Dirac action for the solutions of the modified Dirac equation seems to vanish identically and in TGD perturbative approach separating interaction terms is not possible.

  The modified Dirac equation however fails at the singularities of the minimal surface representing space-time surface and Dirac action reduces to an integral over singularities for the trace of the second fundamental form slashed between the induced spinor field and its conjugate. Also the M^4 part of the trace is non-vanishing and gives rise to the gravitational coupling. The trace gives both standard model vertices and graviton emission vertices. One could say that at the quantum level gravitational and gauge interactions are eliminated everywhere except at the singularities identifiable as defects of the ordinary smooth structure. The exotic smooth structures [L67], possible only in dimension 4, are ordinary smooth structures apart from these defects serving as vertex representing a creation of a fermion-antifermion pair in the induced gauge potentials. The vertex is universal and essentially the trace of the second fundamental form as an analog of the Higgs field and the gravitational constant is proportional to the square of CP_2 radius.

- There is a delicate difference between inertial and gravitational masses. One can assume that the modes of the imbedding space spinor fields are solutions of massles Dirac equation in either $M^4 \times CP_2$ and therefore eigenstates of inertial momentum or in $CD = cd \times CP_2$: in this case they are only mass eigenstates. The mass spectra are identical for these options. Inertial momenta correspond naturally to the Poincare charges in the space of CDs. For the CD option the spinor modes correspond to mass squared eigenstates for which the mode for H^3 with a given value of light-proper time is a unitary irreducible $SO(1,3)$ representation rather than a representation of translation group. These two eigenmode basis correspond to gravitational basis for spinor modes.

Quantum TGD as a generalization of Einstein's geometrization program

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but it turned that this approach fails due to the extreme non-linearity of the theory.

It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as the space of 3-dimensional surfaces. Later 3-surfaces where replaced with 4-surfaces satifying holography and therefore as analogs of Bohr orbits.

- If one assumes Bohr orbitology, then strong correlations between the 3-surfaces at the ends of CD follow and mean holography. It is natural to identify the quantum states of the Universe (and sub-Univeverses) as modes of a formally classical spinor field in WCW. WCW gamma matrices are expressible in terms of oscillator operators of free second quantized spinor fields of H. The induced spinor fields identified projections of H spinor fields to the space-time surfaces satisfy modified Dirac equation for the modified Dirac equation. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- Quantum TGD can be seen as a theory for free spinor fields in WCW having maximal isometries and the generalization of the Super Virasoro conditions gives rise to the analog massless Dirac equation at the level of WCW.

The world of classical worlds and its symmetries

The notion of "World of Classical Worlds" (WCW) emerged around 1985 but found its basic form around 1990. Holography forced by the realization of General Coordinate Invariance forced/allowed to give up the attempts to make sense of the path integral.

A more concrete way to express this view is that WCW does not consist of 3-surfaces as particle-like entities but almost deterministic Bohr orbits assignable to them as preferred extremals of Kähler action so that quantum TGD becomes wave mechanics in WCW combined with Bohr orbitology. This view has profound implications, which can be formulated in terms of zero energy ontology (ZEO), solving among other things the basic paradox of quantum measurement theory. ZEO forms also the backbone of TGD inspired theory of consciousness and quantum biology.

WCW geometry exists only if it has maximal isometries: this statement is a generalization of the discovery of Freed for loop space geometries [A54]. I have proposed [K35, K20, K88, K63, L78] that WCW could be regarded as a union of generalized symmetric spaces labelled by zero modes which do not contribute to the metric. The induced Kähler field is invariant under symplectic transformations of CP_2 and would therefore define zero mode degrees of freedom if one assumes that WCW metric has symplectic transformations as isometries. In particular, Kähler magnetic fluxes would define zero modes and are quantized closed 2-surfaces. The induced metric appearing in Kähler action is however not zero mode degree of freedom. If the action contains volume term, the assumption about union of symmetric spaces is not well-motivated.

Symplectic transformations are not the only candidates for the isometries of WCW. The basic picture about what these maximal isometries could be, is partially inspired by string models.

- A weaker proposal is that the symplectomorphisms of H define only symplectomorphisms of WCW. Extended conformal symmetries define also a candidate for isometry group. Remarkably, light-like boundary has an infinite-dimensional group of isometries which are in 1-1 correspondence with conformal symmetries of $S^2 \subset S^2 \times R_+ = \delta M^4_+$.

- Extended Kac Moody symmetries induced by isometries of δM^4_+ are also natural candidates for isometries. The motivation for the proposal comes from physical intuition deriving from string models. Note they do not include Poincare symmetries, which act naturally as isometries in the moduli space of causal diamonds (CDs) forming the "spine" of WCW.

- The light-like orbits of partonic 2-surfaces might allow separate symmetry algebras. One must however notice that there is exchange of charges between interior degrees of freedom and partonic 2-surfaces. The essential point is that one can assign to these surface conserved charges when the dual light-like coordinate defines time coordinate. This picture also assumes a slicing of space-time surface by by the partonic orbits for which partonic orbits associated with wormrhole throats and boundaries of the space-time surface would be special. This slicing would correspond to Hamilton-Jacobi structure.

- Fractal hierarchy of symmetry algebras with conformal weights, which are non-negative integer multiples of fundamental conformal weights, is essential and distinguishes TGD from string models. Gauge conditions are true only the isomorphic subalgebra and its commutator with the entire algebra and the maximal gauge symmetry to a dynamical symmetry with generators having conformal weights below maximal value. This view also conforms with p-adic mass calculations.

- The realization of the symmetries for 3-surfaces at the boundaries of CD and for light-like orbits of partonic 2-surfaces is known. The problem is how to extend the symmetries to the interior of the space-time surface. It is natural to expect that the symmetries at partonic orbits and light-cone boundary extend to the same symmetries.

After the developments towards the end of 2023, it seems that the extension of conformal and Kac-Moody symmetries of string models to the TGD framework is understood. What about symplectic symmetries, which were originally proposed as isometries of WCW? In this article this question is discussed in detail and it will be found that these symmetries act naturally on 3-D holographic data and one can identify conserved charges. By holography this is in principle enough and might imply that the actions of holomorphic and symplectic symmetry algebras are dual. Holography=holomorphy hypothesis is discussed also in the case of the modified Dirac equation.

About the construction of scattering amplitudes

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far-reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. After having made several guesses for what the counterpart of S-matrix could be, it became clear that the dream about explicit formulas is unrealistic before one has understood what happens in quantum jump.

- In ZEO [K89, L38] one must distinguish between "small" state function reductions (SSFRs) and "big" SFRs (BSFRs). BSFR is the TGD counterpart of the ordinary SFRs and the arrow of the geometric time changes in it. SSFR follows the counterpart of a unitary time evolution and the arrow of the geometric time is preserved in SSFR. The sequence of SSFRs is the TGD counterpart for the sequence of repeated quantum measurements of the same observables in which nothing happens to the state. In TGD something happens in SSFRs and this gives rise to the flow of consciousness. When the set of the observables measured in SSFR does not commute with the previous set of measured observables, BSFR occurs.

The evolution by SSFRs means that also the causal diamond changes. At quantum level one has a wave function in the finite-dimensional moduli space of CDs which can be said to form a spine of WCW [L76]. CDs form a scale hierarchy. SSFRs are preceded by a dispersion in the moduli space of CDs and SSFR means localization in this space.

- There are several S-matrix like entities. One can assign an analog of the S-matrix to each analog of unitary time evolution preceding a given SSFR. One can also assign an analog S-matrix between the eigenstate basis of the previous set of observables and the eigenstate basis of new observers: this S-matrix characterizes BSFR. One can also assign to zero energy states an S-matrix like entity between the states assignable to the two boundaries of CD. These S-matrix like objects can be interpreted as a complex square root of the density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally.

In standard QFTs Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so-called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. The QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

In the TGD framework this generalization of Feynman diagrams indeed emerges unavoidably.

- The counterparts of elementary particles can be identified as closed monopole flux tubes connecting two parallel Minkowskian space-time sheets and have effective ends which are Euclidean wormhole contacts. The 3-D light-like boundaries of wormhole contacts as orbits of partonic 2-surfaces.

  The intuitive picture is that the 3-D light-like partonic orbits replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. A stronger condition is that fermion number is carried by light-like fermion lines at the partonic orbits, which can be identified as boundaries string world sheets.

- The localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In the TGD framework, the fermionic variant of twistor Grassmann formalism combined with the number theoretic vision [L62, L63] led to a stringy variant of the twistor diagrammatics.

- Fundamental fermions are off-mass-shell in the sense that their momentum components are real algebraic integers in an extension of rationals associated with the space-time surfaces inside CD with a momentum unit determined by the CD size scale. Galois confinement states that the momentum components are integer valued for the physical states.

- The twistorial approach suggests also the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras, which would determine the vertices and scattering amplitudes in terms of poly-local symmetries.

The twistorial approach is however extremely abstract and lacks a concrete physical interpretation. The holography=holomorphy vision led to a breakthough in the construction of the scattering amplitudes by solving the problem of identifying interaction vertices [L80].

1. The basic prediction is that space-time surfaces as analogs of Bohr orbits are holomorphic in a generalized sense and are therefore minimal surfaces. The minimal surface property fails at lower-dimensional singularities and the trace of the second fundamental form (SFF) analogous to acceleration associated with the Bohr orbit of the particle as 3-surface has a delta function like singularity but vanishes elsewhere.

2. The minimal surface property expressess masslessness for both fields and particles as 3-surfaces. At singularities masslessness property fails and singularities can be said to serve as sources which also in QFT define scattering amplitudes.

3. The singularities are analogs of poles and cuts for the 4-D generalization of the ordinary holomorphic functions. Also for the ordinary holomorphic functions the Laplace equation as analog massless field equation and expressing analyticity fails. Complex analysis generalizes to dimension 4.

4. The conditions at the singularity give a generalization of Newton's "F=ma"! I ended up where I started more than 50 years ago!

5. In dimension 4, and only there, there is an infinite number of exotic diff structures [**?**], which differ from ordinary ones at singularities of measure zero analogous to defects. These defects correspond naturally to the singularities of minimal surfaces. One can say that for the exotic diff structure there is no singularity.

6. Group theoretically the trace of the SFF can be regarded as a generalization of the Higgs field, which is non-vanishing only at the vertices and this is enough. Singularities take the role of generalized particle vertices and determine the scattering amplitudes. The second fundamental form contracted with the embedding space gamma matrices and slashed between the second quantized induced spinor field and its conjugate gives the universal vertex involving only fermions (bosons are bound states of fermions in TGD). It contains both gauge and gravitational contributions to the scattering amplitudes and there is a complete symmetry between gravitational and gauge interactions. Gravitational couplings come out correctly as the radius squared of CP_2 as also in the classical picture.

7. The study of the modified Dirac equation leads to the conclusion that vertices as singularities and defects contain the standard electroweak gauge contribution coming from the induced spinor connection and a contribution from the M^4 spinor connection. M^4 part of the generalized Higgs can give rise to a graviton as an $L = 1$ rotational state of the flux tube representing the graviton. It is not clear whether M^4 Kähler gauge potential can give rise to a spin 1 particle. The vielbein part of M^4 spinor connection is pure gauge and could give rise to gravitational topological field theory.

Figures

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 45 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks. The books provide a view of how TGD evolved rather than the final theory and there are archeological layers containing mammoth bones, which reflect earlier views not necessarily consistent with the recent view.

Karkkila, April 21, 2024, Finland

Matti Pitkänen

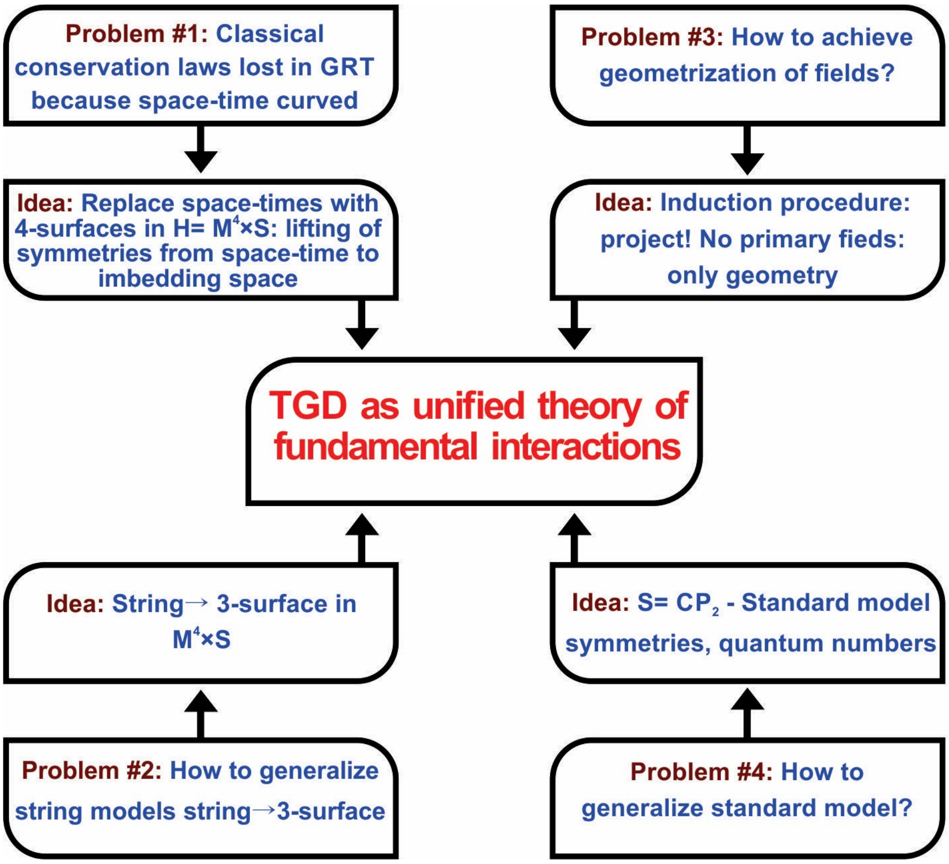

Figure 1: The problems leading to TGD as their solution.

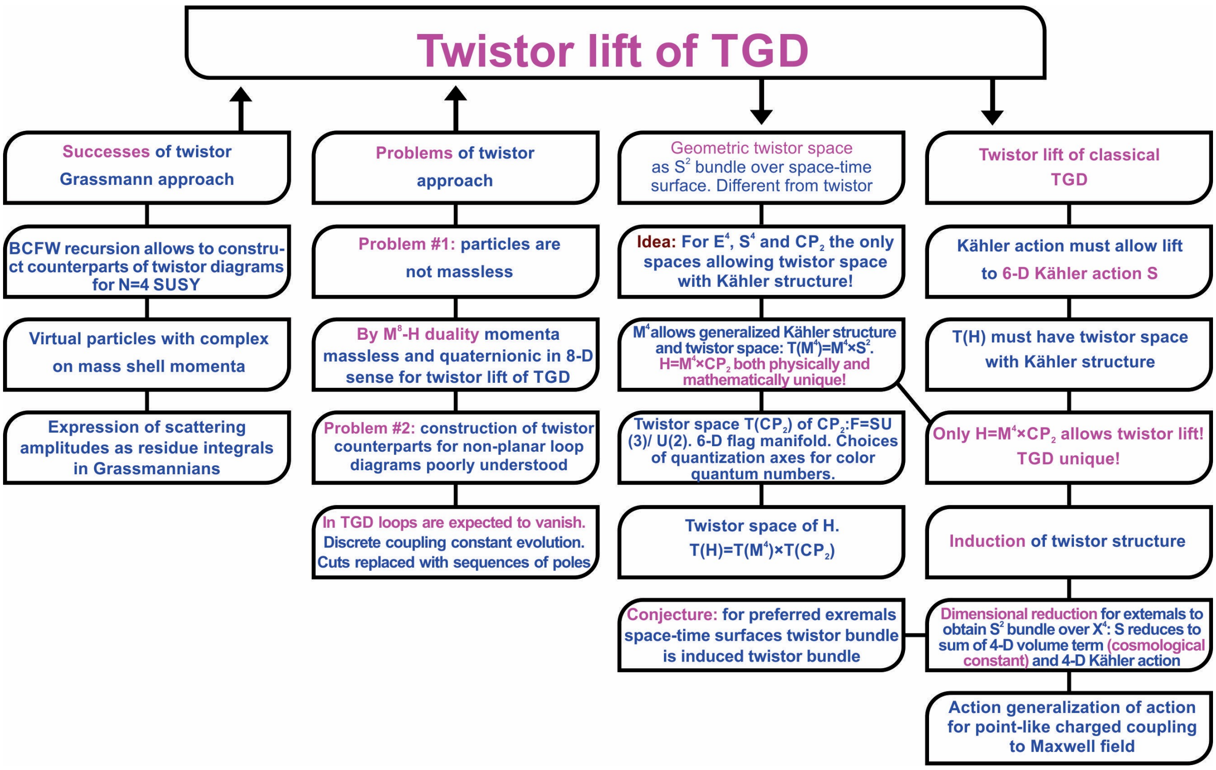

Figure 2: Twistor lift

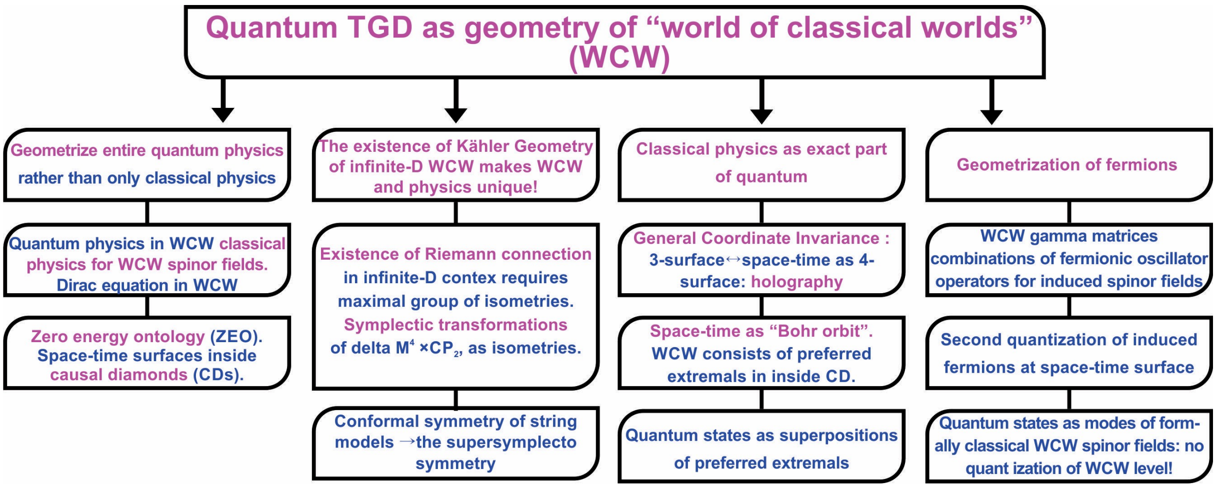

Figure 3: Geometrization of quantum physics in terms of WCW

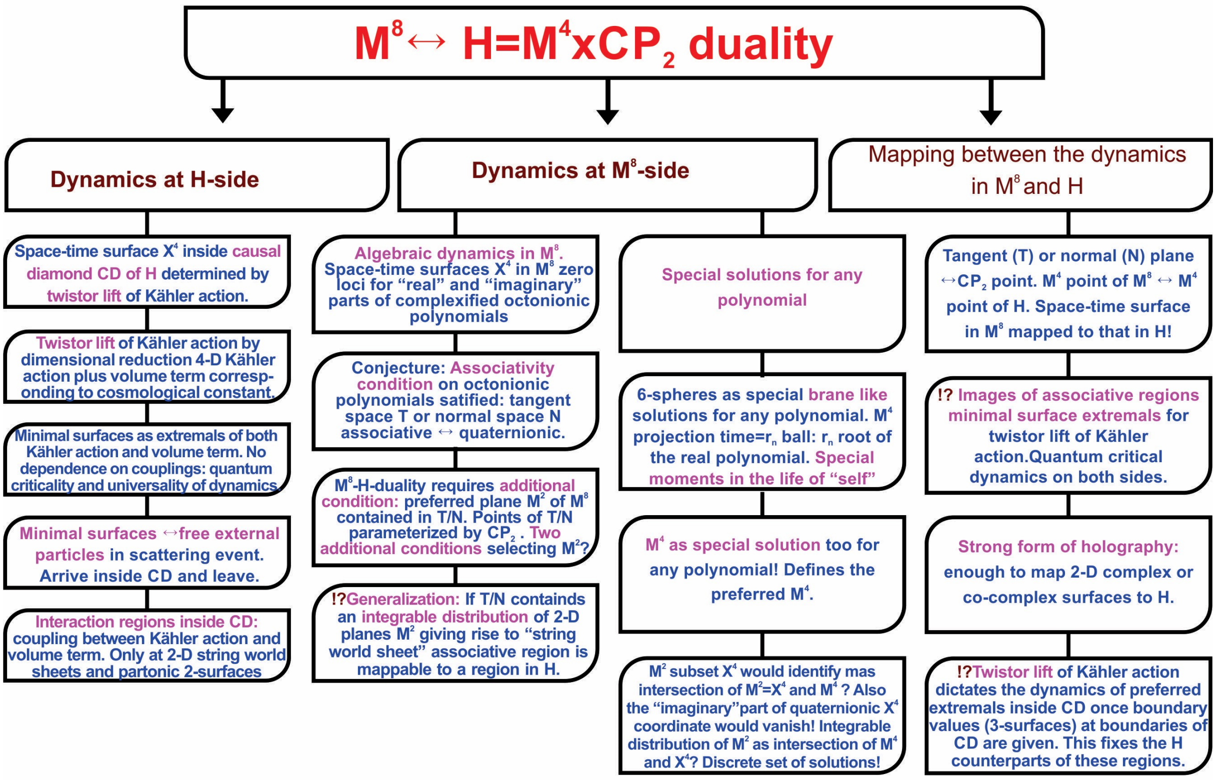

Figure 4: $M^8 - H$ duality

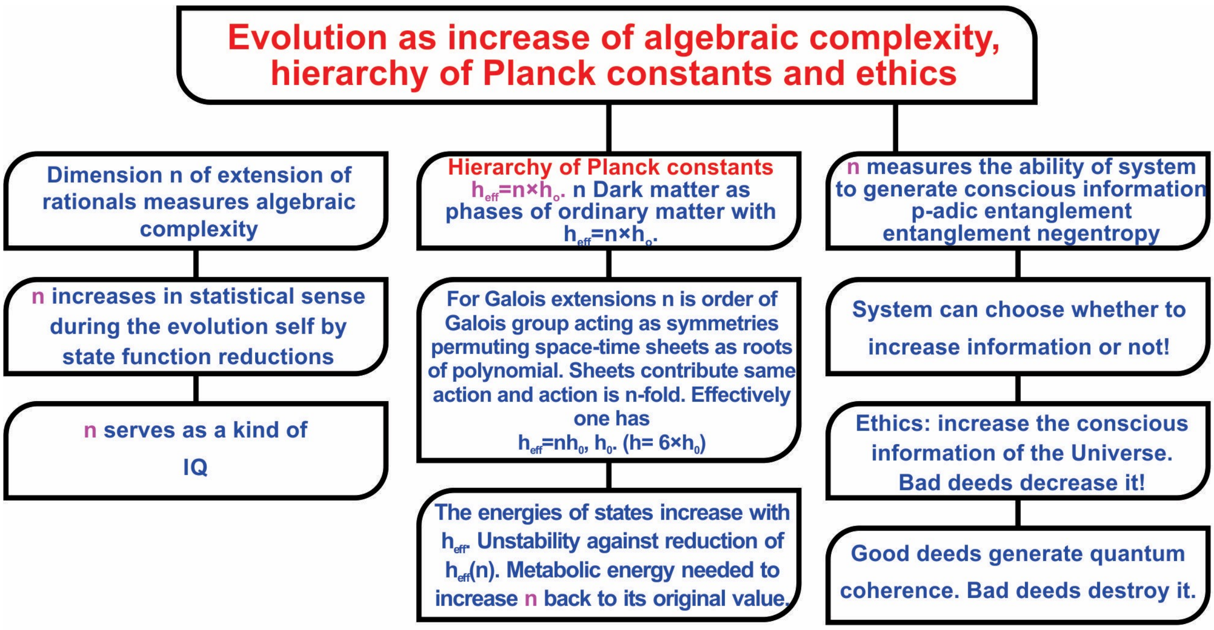

Figure 5: Number theoretic view of evolution

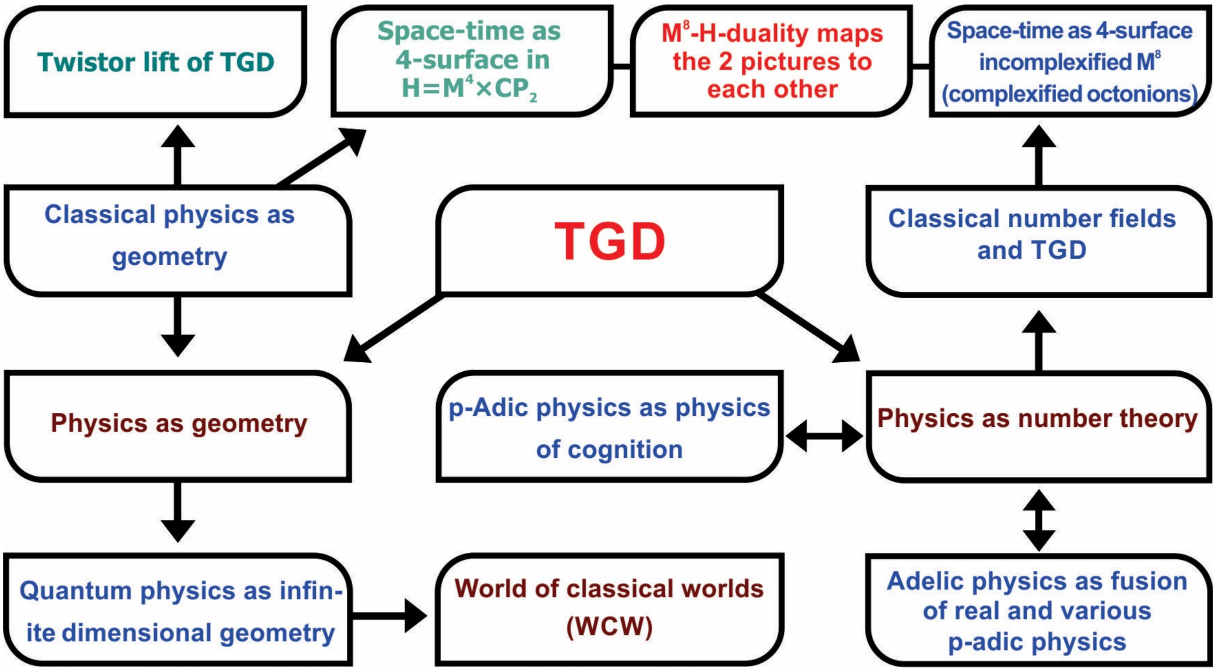

Figure 6: TGD is based on two complementary visions: physics as geometry and physics as number theory.

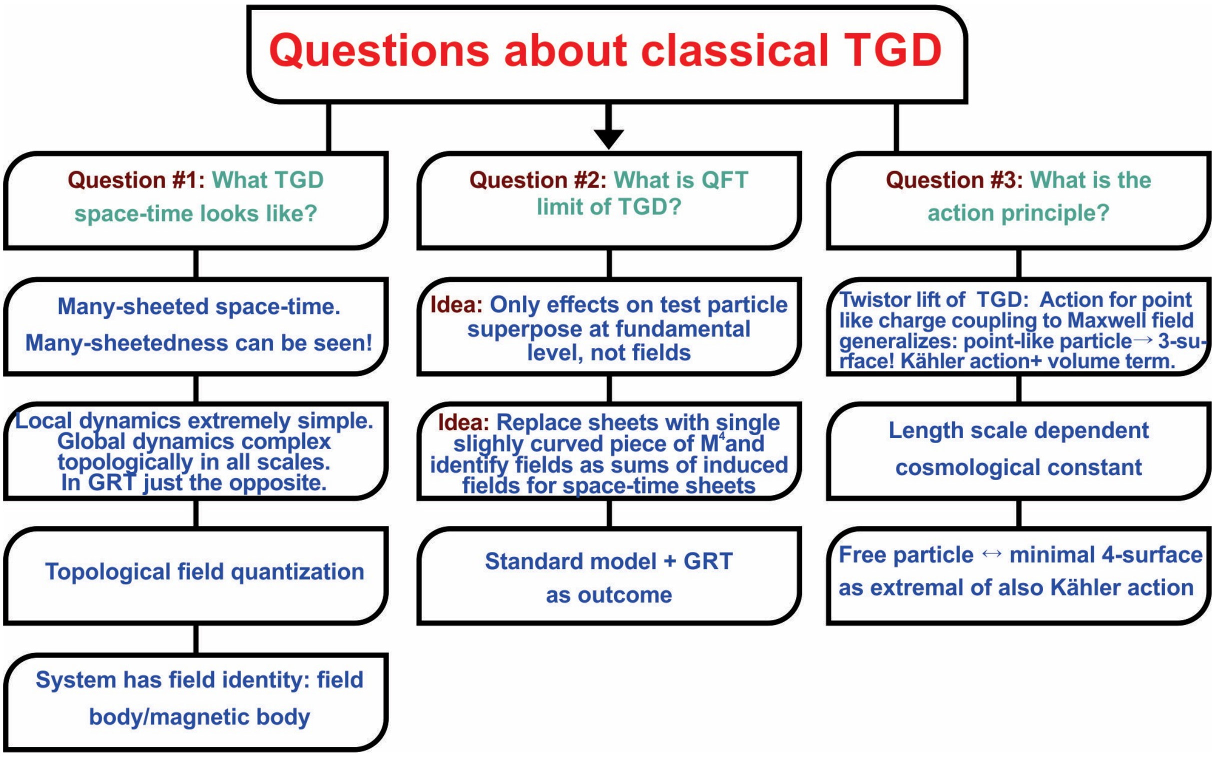

Figure 7: Questions about classical TGD.

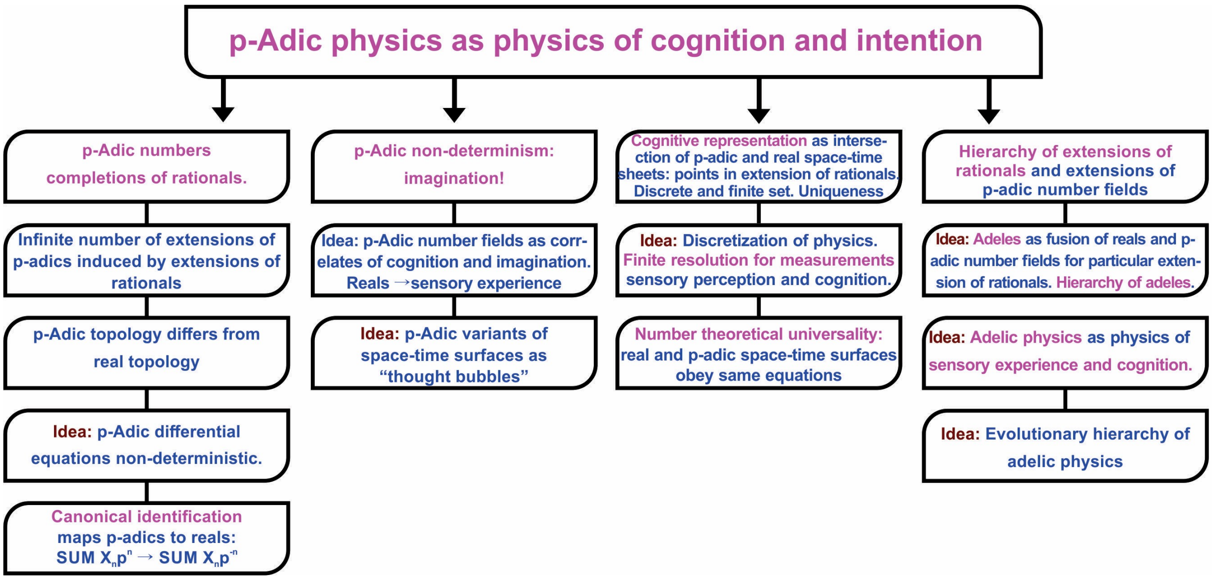

Figure 8: p-Adic physics as physics of cognition and imagination.

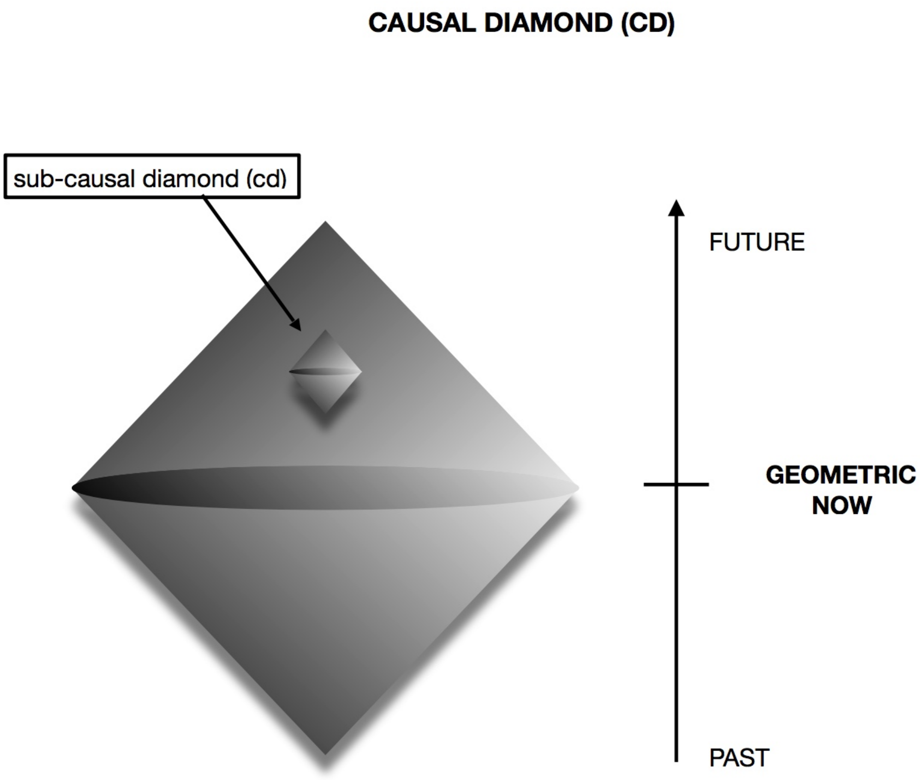

Figure 9: Causal diamond

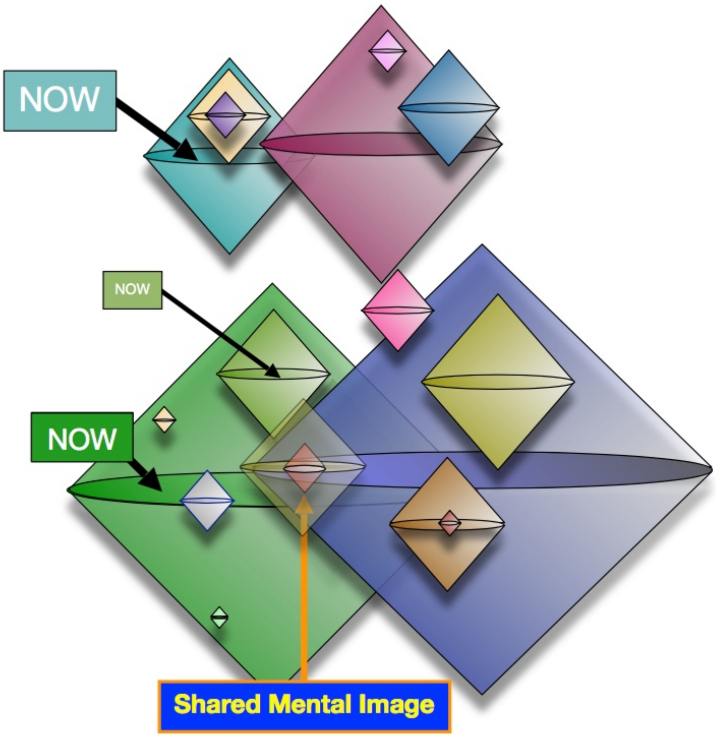

Figure 10: CDs define a fractal "conscious atlas"

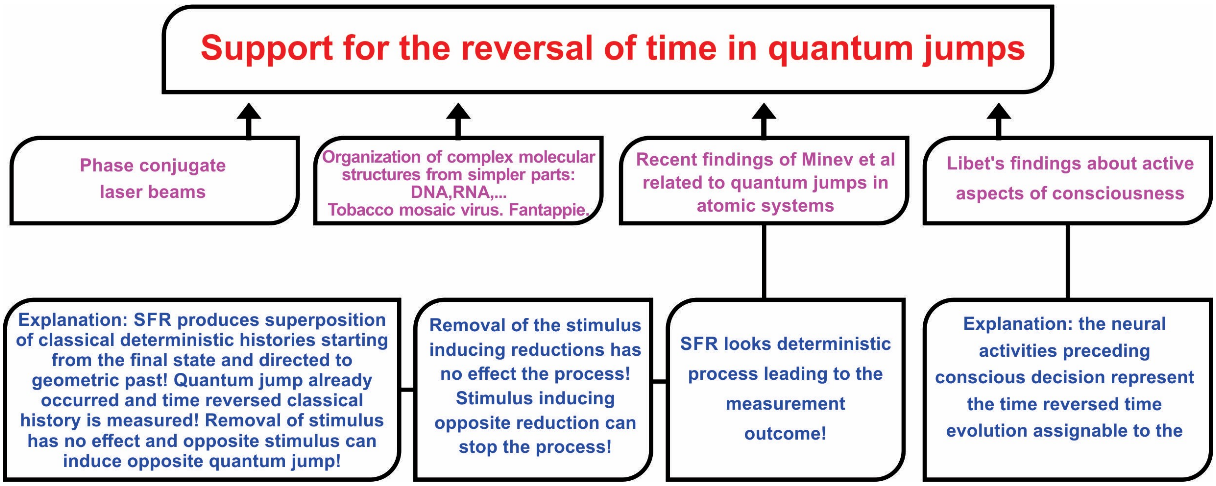

Figure 11: Time reversal occurs in BSFR

ACKNOWLEDGEMENTS

Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family and Kalevi and Ritva Tikkanen and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 45 lonely years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During the last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss my work. Pertti Kärkkäinen is my old physicist friend and has provided continued economic support for a long time. I have also had stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Tommi Ullgren has provided both economic support and encouragement during years. Pekka Rapinoja has offered his help in this respect and I am especially grateful to him for my Python skills.

During the last five years I have had inspiring discussions with many people in Finland interested in TGD. We have had video discussions with Sini Kunnas and had podcast discussions with Marko Manninen related to the TGD based view of physics and consciousness. Marko has also helped in the practical issues related to computers and quite recently he has done a lot of testing of chatGPT helping me to get an overall view of what it is. The discussions in a Zoom group involving Marko Manninen, Tuomas Sorakivi and Rode Majakka have given me the valuable opportunity to clarify my thoughts.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation in CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. I am grateful to Mark McWilliams, Paul Kirsch, Gary Ehlenberg, and Ulla Matfolk and many others for providing links to possibly interesting websites and articles. We have collaborated with Peter Gariaev and Reza Rastmanesh. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the public through the iron wall of academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as an individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without direct support from power holders- even in archives like arXiv.org.

Situation changed as Andrew Adamatsky proposed the writing of a book about TGD when I had already gotten used to the thought that my work would not be published during my lifetime. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loophole. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy.

And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his 73th birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During the period when the right wing held political power this trend was steadily strengthening and the situation is the same as I am writing this. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

Karkkila, August 30, 2023, Finland

Matti Pitkänen

Contents

0.1 PREFACE iii

Acknowledgements **xxiii**

1 Introduction **1**
1.1 Basic Ideas of Topological Geometrodynamics (TGD) 1
1.1.1 Geometric Vision Very Briefly 1
1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion . 4
1.1.3 Basic Objections 6
1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds 7
1.1.5 Construction of scattering amplitudes 10
1.1.6 TGD as a generalized number theory 11
1.1.7 An explicit formula for $M^8 - H$ duality 15
1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy 19
1.1.9 Twistors in TGD and connection with Veneziano duality 20
1.2 Bird's Eye of View about the Topics of "TGD and Hyper-Finite Factors" 24
1.2.1 Hyper-Finite Factors And The Notion Of Measurement Resolution 24
1.2.2 Organization of "TGD and Hyper-Finite Factors" 26
1.3 Sources 26
1.3.1 PART I: HYPER-FINITE FACTORS 27
1.3.2 PART II: CATEGORY THEORY AND QUANTUM TGD 33

I HYPER-FINITE FACTORS **37**

2 Was von Neumann Right After All? **39**
2.1 Introduction 39
2.1.1 Philosophical Ideas Behind Von Neumann Algebras 39
2.1.2 Von Neumann, Dirac, And Feynman 40
2.2 Von Neumann Algebras 40
2.2.1 Basic Definitions 40
2.2.2 Basic Classification Of Von Neumann Algebras 41
2.2.3 Non-Commutative Measure Theory And Non-Commutative Topologies And Geometries 42
2.2.4 Modular Automorphisms 43
2.2.5 Joint Modular Structure And Sectors 43
2.2.6 Basic Facts About Hyper-Finite Factors Of Type III 43
2.3 Braid Group, Von Neumann Algebras, Quantum TGD, And Formation Of Bound States 46
2.3.1 Factors Of Von Neumann Algebras 46
2.3.2 Sub-Factors 46
2.3.3 Ii_1 Factors And The Spinor Structure Of WCW 47
2.3.4 About Possible Space-Time Correlates For The Hierarchy Of II_1 Sub-Factors 48
2.3.5 Could Binding Energy Spectra Reflect The Hierarchy Of Effective Tensor Factor Dimensions? 50
2.3.6 Four-Color Problem, II_1 Factors, And Anyons 50

2.4 Inclusions Of II_1 And III_1 Factors . . . 51
2.4.1 Basic Findings About Inclusions . . . 52
2.4.2 The Fundamental Construction And Temperley-Lieb Algebras . . . 53
2.4.3 Connection With Dynkin Diagrams . . . 53
2.4.4 Indices For The Inclusions Of Type III_1 Factors . . . 54
2.5 TGD And Hyper-Finite Factors Of Type II_1 . . . 55
2.5.1 What Kind Of Hyper-Finite Factors One Can Imagine In TGD? . . . 55
2.5.2 Direct Sum Of HFFs Of Type II_1 As A Minimal Option . . . 57
2.5.3 Bott Periodicity, Its Generalization, And Dimension $D = 8$As An Inherent Property Of The Hyper-Finite II_1 Factor . . . 58
2.5.4 The Interpretation Of Jones Inclusions In TGD Framework . . . 58
2.5.5 WCW, Space-Time, Embedding Space AndHyper-Finite Type II_1 Factors . 60
2.5.6 Quaternions, Octonions, And Hyper-Finite Type II_1Factors . . . 63
2.5.7 Does The Hierarchy Of Infinite Primes Relate To The Hierarchy Of II_1 Factors? 66
2.6 HFFs Of Type III And TGD . . . 67
2.6.1 Problems Associated With The Physical Interpretation Of III_1 Factors . . 67
2.6.2 Quantum Measurement Theory And HFFs Of Type III . . . 68
2.6.3 What Could One Say About II_1 Automorphism Associated With The II_∞ Automorphism Defining Factor Of Type III? . . . 70
2.6.4 What Could Be The Physical Interpretation Of Two Kinds Of Invariants Associated With HFFs Type III? . . . 70
2.6.5 Does The Time Parameter T Represent Time Translation Or Scaling? . . . 71
2.6.6 HFFs Of Type III And The Dynamics In $M^4_\pm$ Degrees Of Freedom? . . . 72
2.6.7 Could The Continuation Of Braidings To Homotopies Involve Δ^{It} Automorphisms . . . 73
2.6.8 HFFs Of Type III As Super-Structures Providing Additional Uniqueness? 73
2.7 Appendix: Inclusions Of Hyper-Finite Factors Of Type II_1 . . . 73
2.7.1 Jones Inclusions . . . 74
2.7.2 Wassermann's Inclusion . . . 74
2.7.3 Generalization From $Su(2)$ To Arbitrary Compact Group . . . 74

3 Evolution of Ideas about Hyper-finite Factors in TGD 76
3.1 Introduction . . . 76
3.1.1 Hyper-Finite Factors In Quantum TGD . . . 76
3.1.2 Hyper-Finite Factors And M-Matrix . . . 77
3.1.3 Connes Tensor Product As A Realization Of Finite Measurement Resolution 78
3.1.4 Concrete Realization Of The Inclusion Hierarchies . . . 78
3.1.5 Analogs of quantum matrix groups from finite measurement resolution? . . 79
3.1.6 Quantum Spinors And Fuzzy Quantum Mechanics . . . 79
3.2 A Vision About The Role Of HFFs In TGD . . . 79
3.2.1 Basic facts about factors . . . 80
3.2.2 TGD and factors . . . 86
3.2.3 Can one identify M-matrix from physical arguments? . . . 91
3.2.4 Finite measurement resolution and HFFs . . . 95
3.2.5 Questions about quantum measurement theory in Zero Energy Ontology . . 100
3.2.6 Planar Algebras And Generalized Feynman Diagrams . . . 105
3.2.7 Miscellaneous . . . 106
3.3 Fresh View About Hyper-Finite Factors In TGD Framework . . . 108
3.3.1 Crystals, Quasicrystals, Non-Commutativity And Inclusions Of Hyperfinite Factors Of Type II_1 . . . 109
3.3.2 HFFs And Their Inclusions In TGD Framework . . . 110
3.3.3 Little Appendix: Comparison Of WCW Spinor Fields With Ordinary Second Quantized Spinor Fields . . . 113
3.4 The idea of Connes about inherent time evolution of certain algebraic structures from TGD point of view . . . 113
3.4.1 Connes proposal and TGD . . . 114
3.5 MIP*= RE: What could this mean physically? . . . 124

3.5.1 Two physically interesting applications . . . 125
3.5.2 The connection with TGD . . . 127
3.6 Analogs Of Quantum Matrix Groups From Finite Measurement Resolution? . . . 132
3.6.1 Well-definedness Of The Eigenvalue Problem As A Constraint To Quantum Matrices . . . 133
3.6.2 The Relationship To Quantum Groups And Quantum Lie Algebras . . . 135
3.6.3 About Possible Applications . . . 138
3.7 Jones Inclusions And Cognitive Consciousness . . . 139
3.7.1 Does One Have A Hierarchy Of U- And M-Matrices? . . . 140
3.7.2 Feynman Diagrams As Higher Level Particles And Their Scattering As Dynamics Of Self Consciousness . . . 140
3.7.3 Logic, Beliefs, And Spinor Fields In The World Of Classical Worlds . . . 143
3.7.4 Jones Inclusions For Hyperfinite Factors Of Type II_1 As A Model For Symbolic And Cognitive Representations . . . 145
3.7.5 Intentional Comparison Of Beliefs By Topological Quantum Computation? 148
3.7.6 The Stability Of Fuzzy Qbits And Quantum Computation . . . 148
3.7.7 Fuzzy Quantum Logic And Possible Anomalies In The Experimental Data For The EPR-Bohm Experiment . . . 148
3.7.8 Category Theoretic Formulation For Quantum Measurement Theory With Finite Measurement Resolution? . . . 150

4 TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, M^8-H Duality, SUSY, and Twistors 153
4.1 Introduction . . . 153
4.1.1 McKay correspondence in TGD framework . . . 154
4.1.2 HFFs and TGD . . . 154
4.1.3 New aspects of M^8-H duality . . . 155
4.1.4 What twistors are in TGD framework? . . . 156
4.2 McKay correspondence . . . 158
4.2.1 McKay graphs . . . 158
4.2.2 Number theoretic view about McKay correspondence . . . 158
4.3 ADE diagrams and principal graphs of inclusions of hyperfinite factors of type II_1 159
4.3.1 Principal graphs and Dynkin diagrams for ADE groups . . . 160
4.3.2 Number theoretic view about inclusions of HFFs and preferred role of $SU(2)$ 160
4.3.3 How could ADE type quantum groups and affine algebras be concretely realized? . . . 161
4.4 M^8-H duality . . . 162
4.4.1 M^8-H duality at the level of space-time surfaces . . . 162
4.4.2 M^8-H duality at the level of momentum space . . . 165
4.4.3 M^8-H duality and the two ways to describe particles . . . 167
4.4.4 M^8-H duality and consciousness . . . 170
4.5 Could standard view about twistors work at space-time level after all? . . . 173
4.5.1 Getting critical . . . 173
4.5.2 The nice results of the earlier approach to M^4 twistorialization . . . 177
4.5.3 ZEO and twistorialization as ways to introduce scales in M^8 physics . . . 178
4.5.4 Hierarchy of length scale dependent cosmological constants in twistorial description . . . 181
4.6 How to generalize twistor Grassmannian approach in TGD framework? . . . 181
4.6.1 Twistor lift of TGD at classical level . . . 182
4.6.2 Octonionic twistors or quantum twistors as twistor description of massive particles . . . 182
4.6.3 Basic facts about twistors and bi-spinors . . . 183
4.6.4 The description for M^4_T option using octo-twistors? . . . 185
4.6.5 Do super-twistors make sense at the level of M^8? . . . 187
4.7 Could one describe massive particles using 4-D quantum twistors? . . . 190
4.7.1 How to define quantum Grassmannian? . . . 190
4.7.2 Two views about quantum determinant . . . 192

4.7.3 How to understand the Grassmannian integrals defining the scattering amplitudes? . . . 193

5 McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product? 195

5.1 Introduction . . . 195
5.1.1 Could one generalize arithmetics by replacing sum and product with direct sum and tensor product? . . . 195
5.1.2 McKay graphs and McKay correspondence . . . 196
5.2 Could the arithmetics based on direct sum and tensor product for the irreps of the Galois group make sense and have physical meaning? . . . 197
5.2.1 Questions . . . 197
5.2.2 Could the notion of quantum arithmetics be useful in the TGD framework? 198
5.3 What could lurk behind McKay correspondence? . . . 199
5.3.1 McKay correspondence . . . 200
5.3.2 Questions . . . 200
5.3.3 TGD view about McKay correspondence . . . 202
5.3.4 Could the inclusion hierarchies of extensions of rationals correspond to inclusion hierarchies of hyperfinite factors? . . . 209
5.4 Appendix: Isometries and holonomies of WCW as counterparts of exact and broken gauge symmetries . . . 210
5.4.1 Isometries of WCW . . . 210
5.4.2 Holonomies of WCW . . . 211

6 Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole 212

6.1 Introduction . . . 212
6.1.1 Basic notions of HFFs from TGD perspective . . . 212
6.1.2 Bird's eye view of HFFs in TGD . . . 213
6.1.3 M^8-H duality and HFFS . . . 214
6.1.4 Infinite primes . . . 215
6.2 Basic notions related to hyperfinite factors of type II_1 from TGD point of view . . 215
6.2.1 Basic concepts related to von Neumann algebras . . . 215
6.2.2 Standard construction for the hierarchy of HFFs . . . 218
6.2.3 Classification of inclusions of HFFs using extended ADE diagrams . . . 219
6.3 TGD and hyperfinite factors of type II_1: a bird's eye of view . . . 220
6.3.1 Identification of HFFs in the TGD framework . . . 220
6.3.2 Could the notion of free probability be relevant in TGD? . . . 222
6.3.3 Some objections against HFFs . . . 224
6.4 M^8-H duality and HFFs . . . 228
6.4.1 Number theoretical level: M^8 picture . . . 229
6.4.2 Geometric level: H picture . . . 232
6.4.3 Wild speculations about McKay correspondence . . . 232
6.5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW) . . . 235
6.5.1 Could twistor lift fix the choice of the action uniquely? . . . 236
6.5.2 Two paradoxes . . . 237
6.6 About the TGD based notions of mass, of twistors and hyperbolic counterpart of Fermi torus . . . 240
6.6.1 Conformal confinement . . . 241
6.6.2 About the notion of twistor space . . . 243
6.6.3 About the analogies of Fermi torus and Fermi surface in H^3 . . . 246
6.7 The notion of generalized integer . . . 248
6.7.1 The first reactions to the abstract . . . 249
6.7.2 Fundamental discretization as a cognitive representation? . . . 250
6.8 Infinite primes as a basic mathematical building block . . . 254
6.8.1 Construction of infinite primes . . . 254

6.8.2 Questions about infinite primes . . . 255
6.8.3 $P = Q$ hypothesis . . . 256
6.9 Summary of the proposed big picture . . . 257
6.9.1 The relation between $M^8 - H$ and $M - M'$ dualities . . . 257
6.9.2 Basic mathematical building blocks . . . 257
6.9.3 Basic algebraic structures at number theoretic side . . . 257
6.9.4 Basic algebraic structures at the geometric side . . . 258
6.10 Appendix: The reduction of quantum TGD to WCW geometry and spinor structure 258
6.10.1 The problems . . . 259
6.10.2 3-D surfaces or 4-surfaces associated to them by holography replace point-like particles . . . 259
6.10.3 WCW Kähler geometry as s geometrization of the entire quantum physics . 259
6.10.4 Quantum physics as physics of free, classical spinor fields in WCW . . . 260
6.10.5 Dirac equation for WCW spinor fields . . . 261
6.10.6 $M^8 - H$ duality at the level of WCW . . . 261

II CATEGORY THEORY AND QUANTUM TGD 263

7 Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness 265
7.1 Introduction . . . 265
7.1.1 Category Theory As A Purely Technical Tool . . . 265
7.1.2 Category Theory Based Formulation Of The Ontology Of TGD Universe . 265
7.1.3 Other Applications . . . 266
7.2 What Categories Are? . . . 267
7.2.1 Basic Concepts . . . 267
7.2.2 Presheaf As A Generalization For The Notion Of Set . . . 267
7.2.3 Generalized Logic Defined By Category . . . 268
7.3 More Precise Characterization Of The Basic Categories And Possible Applications 269
7.3.1 Intuitive Picture About The Category Formed By The Geometric Correlates Of Selves . . . 269
7.3.2 Categories Related To Self And Quantum Jump . . . 269
7.3.3 Communications In TGD Framework . . . 270
7.3.4 Cognizing About Cognition . . . 272
7.4 Logic And Category Theory . . . 273
7.4.1 Is The Logic Of Conscious Experience Based On Set Theoretic Inclusion Or Topological Condensation? . . . 273
7.4.2 Do WCW Spinor Fields Define Quantum Logic And Quantum Topos . . . 274
7.4.3 Category Theory And The Modelling Of Aesthetic And Ethical Judgements 276
7.5 Platonism, Constructivism, And Quantum Platonism . . . 277
7.5.1 Platonism And Structuralism . . . 277
7.5.2 Structuralism . . . 278
7.5.3 The View About Mathematics Inspired By TGD And TGD Inspired Theory Of Consciousness . . . 279
7.5.4 Farey Sequences, Riemann Hypothesis, Tangles, And TGD . . . 283
7.6 Quantum Quandaries . . . 287
7.6.1 The *-Category Of Hilbert Spaces . . . 287
7.6.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob . . . 288
7.6.3 Tqft As A Functor . . . 288
7.6.4 The Situation Is In TGD Framework . . . 289
7.7 How To Represent Algebraic Numbers As Geometric Objects? . . . 291
7.7.1 Can One Define Complex Numbers As Cardinalities Of Sets? . . . 291
7.7.2 In What Sense A Set Can Have Cardinality -1? . . . 292
7.7.3 Generalization Of The Notion Of Rig By Replacing Naturals With P-Adic Integers . . . 294
7.8 Gerbes And TGD . . . 296

7.8.1 What Gerbes Roughly Are? . . . 297
7.8.2 How Do 2-Gerbes Emerge In TGD? . . . 297
7.8.3 How To Understand The Replacement Of 3-Cycles With N-Cycles? . . . 298
7.8.4 Gerbes As Graded-Commutative Algebra: Can One Express All Gerbes As Products Of −1 And 0-Gerbes? . . . 299
7.8.5 The Physical Interpretation Of 2-Gerbes In TGD Framework . . . 299
7.9 Appendix: Category Theory And Construction Of S-Matrix . . . 300

8 Category Theory and Quantum TGD 302
8.1 Introduction . . . 302
8.2 S-Matrix As A Functor . . . 303
8.2.1 The *-Category Of Hilbert Spaces . . . 303
8.2.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob . . . 303
8.2.3 TSFT As A Functor . . . 304
8.2.4 The Situation Is In TGD Framework . . . 304
8.3 Further Ideas . . . 307
8.3.1 Operads, Number Theoretical Braids, And Inclusions Of HFFs . . . 308
8.3.2 Generalized Feynman Diagram As Category? . . . 308
8.4 Planar Operads, The Notion Of Finite Measurement Resolution, And Arrow Of Geometric Time . . . 309
8.4.1 Zeroth Order Heuristics About Zero Energy States . . . 309
8.4.2 Planar Operads . . . 310
8.4.3 Planar Operads And Zero Energy States . . . 311
8.4.4 Relationship To Ordinary Feynman Diagrammatics . . . 313
8.5 Category Theory And Symplectic QFT . . . 313
8.5.1 Fusion Rules . . . 313
8.5.2 What Conditions Could Fix The Symplectic Triangles? . . . 314
8.5.3 Associativity Conditions And Braiding . . . 317
8.5.4 Finite-Dimensional Version Of The Fusion Algebra . . . 318
8.6 Could Operads Allow The Formulation Of The Generalized Feynman Rules? . . . 323
8.6.1 How To Combine Conformal Fields With Symplectic Fields? . . . 323
8.6.2 Symplecto-Conformal Fields In Super-Kac-Moody Sector . . . 324
8.6.3 The Treatment Of Four-Momentum . . . 325
8.6.4 What Does The Improvement Of Measurement Resolution Really Mean? . 328
8.6.5 How Do The Operads Formed By Generalized Feynman Diagrams And Symplecto-Conformal Fields Relate? . . . 329
8.7 Possible Other Applications Of Category Theory . . . 330
8.7.1 Categorification And Finite Measurement Resolution . . . 331
8.7.2 Inclusions Of HFFs And Planar Tangles . . . 333
8.7.3 2-Plectic Structures And TGD . . . 333
8.7.4 TGD Variant For The Category Ncob . . . 334
8.7.5 Number Theoretical Universality And Category Theory . . . 335
8.7.6 Category Theory And Fermionic Parts Of Zero Energy States As Logical Deductions . . . 335
8.7.7 Category Theory And Hierarchy Of Planck Constants . . . 335

9 Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? 336
9.1 Introduction . . . 336
9.2 Basic vision . . . 337
9.2.1 Very concise summary about basic notions and ideas of TGD . . . 337
9.2.2 Tensor networks as categories . . . 339
9.2.3 Yangian as a generalization of symmetries to multilocal symmetries . . . 340
9.3 Some mathematical background about Yangians . . . 341
9.3.1 Yang-Baxter equation (YBE) . . . 341
9.3.2 Yangian . . . 342

9.4 Yangianization in TGD framework . . . 345
9.4.1 Geometrization of super algebras in TGD framework . . . 345
9.4.2 Questions . . . 346
9.4.3 Yangianization of four-momentum . . . 348
9.4.4 Yangianization for mass squared operator . . . 351
9.5 Category theory as a basic tool of TGD . . . 352
9.5.1 Fusion categories . . . 352
9.5.2 Braided categories . . . 353
9.5.3 Categories with reconnections . . . 354
9.6 Trying to imagine the great vision about categorification of TGD . . . 355
9.6.1 Different kind of categories . . . 355
9.6.2 Geometric categories . . . 357

10 Are higher structures needed in the categorification of TGD? 359
10.1 Introduction . . . 359
10.1.1 Higher structures and categorification of physics . . . 359
10.1.2 Evolution of Schreiber's ideas . . . 359
10.1.3 What higher structures are? . . . 361
10.1.4 Possible applications of higher structures to TGD . . . 361
10.2 TGD very briefly . . . 365
10.2.1 World of classical worlds (WCW) . . . 365
10.2.2 Strong form of holography (SH) . . . 367
10.3 The notion of finite measurement resolution . . . 369
10.3.1 Inclusions of HFFs, finite measurement resolution and quantum dimensions 369
10.3.2 Three options for the identification of quantum dimension . . . 370
10.3.3 n-structures and adelic physics . . . 372
10.3.4 Could normal sub-groups of symplectic group and of Galois groups correspond to each other? . . . 373
10.3.5 A possible connection with number theoretic Langlands correspondence . . 375
10.3.6 A formulation of adelic TGD in terms of cognitive representations? . . . 377
10.4 Could McKay correspondence generalize in TGD framework? . . . 379
10.4.1 McKay graphs in mathematics and physics . . . 381
10.4.2 Do McKay graphs of Galois groups give overall view about classical and quantum dynamics of quantum TGD? . . . 386
10.5 Appendix . . . 390
10.5.1 What could be the counterpart of the fake flatness in TGD framework? . . 390
10.5.2 A little glossary . . . 391

11 Is Non-associative Physics and Language Possible only in Many-Sheeted Space-time? 394
11.1 Introduction . . . 394
11.2 Is Non-associative Physics Possible In Many-sheeted Space-time? . . . 395
11.2.1 What Does Non-associativity Mean? . . . 395
11.2.2 Language And Many-sheeted Physics? . . . 396
11.2.3 What About The Hierarchy Of Planck Constants? . . . 397
11.3 Braiding Hierarchy Mathematically . . . 397
11.3.1 How To Represent The Hierarchy Of Braids? . . . 398
11.3.2 Braid Groups As Coverings Of Permutation Groups . . . 398
11.3.3 Braid Having Braids As Strands . . . 399
11.4 General Formulation For The Breaking Of Associativity In The Case Of Operads . 399
11.4.1 How Associativity Could Be Broken? . . . 400
11.4.2 Construction Of Quantum Braid Algebra In TGD Framework . . . 403
11.4.3 Should One Quantize Complex Numbers? . . . 407

i Appendix **410**
A-1 Introduction . . . 410
A-2 Embedding space $M^4 \times CP_2$ . . . 410
A-2.1 Basic facts about CP_2 . . . 411
A-2.2 CP_2 geometry and Standard Model symmetries . . . 415
A-3 Induction procedure and many-sheeted space-time . . . 422
A-3.1 Induction procedure for gauge fields and spinor connection . . . 422
A-3.2 Induced gauge fields for space-times for which CP_2 projection is a geodesic sphere . . . 422
A-3.3 Many-sheeted space-time . . . 423
A-3.4 Embedding space spinors and induced spinors . . . 424
A-3.5 About induced gauge fields . . . 425
A-4 The relationship of TGD to QFT and string models . . . 428
A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces . . . 428
A-4.2 Extension of superconformal invariance . . . 428
A-4.3 String-like objects and strings . . . 428
A-4.4 TGD view of elementary particles . . . 428
A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW) . . . 429
A-5.1 Could twistor lift fix the choice of the action uniquely? . . . 429
A-5.2 Two paradoxes . . . 431
A-6 Number theoretic vision of TGD . . . 434
A-6.1 p-Adic numbers and TGD . . . 434
A-6.2 Hierarchy of Planck constants and dark matter hierarchy . . . 438
A-6.3 $M^8 - H$ duality as it is towards the end of 2021 . . . 439
A-7 Zero energy ontology (ZEO) . . . 440
A-7.1 Basic motivations and ideas of ZEO . . . 440
A-7.2 Some implications of ZEO . . . 441
A-8 Some notions relevant to TGD inspired consciousness and quantum biology . . . 441
A-8.1 The notion of magnetic body . . . 442
A-8.2 Number theoretic entropy and negentropic entanglement . . . 442
A-8.3 Life as something residing in the intersection of reality and p-adicities . . . 442
A-8.4 Sharing of mental images . . . 443
A-8.5 Time mirror mechanism . . . 443

List of Figures

1 The problems leading to TGD as their solution. xii
2 Twistor lift xiii
3 Geometrization of quantum physics in terms of WCW xiv
4 $M^8 - H$ duality xv
5 Number theoretic view of evolution xvi
6 TGD is based on two complementary visions: physics as geometry and physics as number theory. xvii
7 Questions about classical TGD. xviii
8 p-Adic physics as physics of cognition and imagination. xix
9 Causal diamond xx
10 CDs define a fractal "conscious atlas" xxi
11 Time reversal occurs in BSFR xxii

7.1 Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".) 267

Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K2].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A79]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

   M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of

electromagnetic fields are nonvanishing. The correlations functions for weak fields are non-vanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question: super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A26] [B20, B14, B15]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B13]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4 \times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n \times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A61, A78, A45, A70].

The identification of the space-time as a sub-manifold [A62, A110] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. ??** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

[1]There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.1.5 Construction of scattering amplitudes

Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A92, A120, A135]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \to B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K48]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, M^8-H duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and M^8-H duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and M^8-H duality would map the algebraic physics in M^8 to the ordinary physics in $M^4 \times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their M^8-H duals in M^8_c are defined for both reals and various p-adic number fields. This is true if they are defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p=3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired "Universe as Computer" vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.

2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since $M^8 - H$ duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K45].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to “mind stuff”, the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M_c^8

$Y^4 \subset M_c^8$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M_c^4 point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M_c^4 \subset M_c^8$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{(Re(E)^2 - Im(E)^2)}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{(Re(E)^2 - Im(E)^2)}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} \ . \tag{1.1.1}$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M_c^8$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M_c^4 \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1)\times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of M^8-H duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4\times CP_2$?

1. The selection of $SU(3)\subset G_2$ for ordinary M^8-H duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3\subset M^8_c$ and its image $H^3\subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3)\subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of M^8-H duality as something more fundamental than the ordinary M^8-H duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3\times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1)\times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1\oplus 3\oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3\oplus\overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8\oplus 3\oplus\overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3)\times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x)\times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X^3_i.

1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K68].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K59, K60, K57]) support the view that dark matter might be a key player in living matter.

Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [K77]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A79]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   $M^8 - H$ duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L29].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L24]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t-m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

1.2 Bird's Eye of View about the Topics of "TGD and Hyper-Finite Factors"

1.2.1 Hyper-Finite Factors And The Notion Of Measurement Resolution

The work with TGD inspired model [K4, K3] for topological quantum computation [B24] led to the realization that von Neumann algebras [A88], in particular so called hyper-finite factors of type II_1 [A66], could provide the mathematics needed to develop a more explicit view about the construction of scattering amplitudes. Later came the realization that the Clifford algebra of WCW defines a canonical representation of hyper-finite factors of type II_1 and that WCW spinor fields give rise to HFFs of type III_1 encountered also in relativistically invariant quantum field theories [K87].

Philosophical ideas behind von Neumann algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation * and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator A belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type II_1 [A66].

The definitions of adopted by von Neumann allow however more general algebras. Type I_n algebras correspond to finite-dimensional matrix algebras with finite traces whereas I_∞ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type III non-trivial traces are always infinite and the notion of trace becomes useless being replaced by the notion of state which is generalization of the notion of thermodynamical state. The fascinating feature of this notion of state is that it defines a unique modular automorphism of the factor defined apart from unitary inner automorphism and the question is whether this notion or its generalization might be relevant for the construction of M-matrix in TGD.

Von Neumann, Dirac, and Feynman

The association of algebras of type I with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type II_1 as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac [K88] based on the notion of delta function, plus the emergence of generalized Feynman graphs [K29], the possibility to formulate the notion of delta function rigorously in terms of distributions [A100, A69], and the emergence of path integral approach [A122] meant that von Neumann approach was forgotten by particle physicists.

Algebras of type II_1 have emerged only much later in conformal and topological quantum field theories [A59, A127] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [A42, A131] relate closely to type II_1 factors. In topological quantum computation [B24] based on braid groups [A137] modular S-matrices they play an especially important role.

In algebraic quantum field theory [B26] defined in Minkowski space the algebras of observables associated with bounded space-time regions correspond quite generally to the type III_1 hyper-finite factor [B8, B28].

Hyper-finite factors in quantum TGD

The following argument suggests that von Neumann algebras known as hyper-finite factors (HFFs) of type II_1 and even of type III_1- the latter appearing in relativistic quantum field theories could provide the proper mathematical framework for quantum TGD.

1. The Clifford algebra of the infinite-dimensional Hilbert space is a von Neumann algebra known as HFF of type II_1. There also the Clifford algebra at a given point (light-like 3-surface) of WCW is therefore HFF of type II_1. If the fermionic Fock algebra defined by the

fermionic oscillator operators assignable to the induced spinor fields (this is actually not obvious!) is infinite-dimensional it defines a representation for HFF of type II_1. Super-conformal symmetry suggests that the extension of the Clifford algebra defining the fermionic part of a super-conformal algebra by adding bosonic super-generators representing symmetries of WCW respects the HFF property. It could however occur that HFF of type II_∞ results.

2. WCW is a union of sub-WCWs associated with causal diamonds (CD) defined as intersections of future and past directed light-cones. One can allow also unions of CDs and the proposal is that CDs within CDs are possible. Whether CDs can intersect is not clear. One can actually construct the space of CDs as a modulispace.

3. The assumption that the M^4 proper distance a between the tips of CD is quantized in powers of 2 reproduces p-adic length scale hypothesis but one must also consider the possibility that a can have all possible values. Since $SO(3)$ is the isotropy group of CD, the CDs associated with a given value of a and with fixed lower tip are parameterized by the Lobatchevski space $L(a) = SO(3,1)/SO(3)$. Therefore the CDs with a free position of lower tip are parameterized by $M^4 \times L(a)$. a analogous to cosmic time [K69]. Since Lorentz boosts define a non-compact group, the generalization of so called crossed product construction leads to ask whether the local Clifford algebra of WCW is HFF of type III_1.

1.2.2 Organization of "TGD and Hyper-Finite Factors"

The book is organized into two parts. The first part of the book is devoted to hyper-finite factors of type II_1 (HFF) and hierarchy of Planck constants. The notion of HFF is extremely abstract and I must confess that here I must transcend the boundaries of my technical skills and must trust on my physical and mathematical intuition.

1. The spinors of the "world of classical worlds" (WCW) referred to as "configuration space" in the earlier writings) form a fermionic Fock space define a canonical example about hyper-finite factor of type II_1.

   The work with TGD inspired model for quantum computation led to the realization that von Neumann algebras, in particular hyper-finite factors of type II_1 could provide the mathematics needed to develop a more explicit view about the construction of the counterpart of S-matrix in terms of zero energy states in the zero energy ontology (ZEO). This led to a proposal of a general master formula for S-matrix (or rather M-matrix as it was called) with interactions described as a deformation of ordinary tensor product to Connes tensor products.

2. In the second part of the book a category theoretical formulation of quantum TGD is considered. Finite measurement resolution realized in terms of a fractal hierarchy of causal diamonds inside causal diamonds leads to a stringy formulation of quantum TGD involving effective replacement of the 3-D light-like surface with a collection of braid strands representing the ends of strings. A formulation in terms of category theoretic concepts is proposed and leads to a hierarchy of algebras forming what is known as operads.

1.3 Sources

The eight online books about TGD [K83, K78, K62, K53, K16, K49, K32, K71] and nine online books about TGD inspired theory of consciousness and quantum biology [K76, K13, K56, K12, K30, K40, K42, K70, K75] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://tinyurl.com/ybv8dt4n`) contains a lot of material about TGD. In particular, a TGD glossary at `http://tinyurl.com/yd6jf3o7`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://tinyurl.com/ycyrxj4o` founded by Lian Sidorov and in *Prespacetime Journal* (`http://tinyurl.com/ycvktjhn`), *Journal of Consciousness Research and Exploration* (`http://tinyurl.com/yba4f672`), and *DNA Decipher Journal* (`http://tinyurl.

com/y9z52khg), all of them founded by Huping Hu. One can find the list about the articles published at http://tinyurl.com/ybv8dt4n. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.3.1 PART I: HYPER-FINITE FACTORS

Was von Neumann Right After All?

The work with TGD inspired model for topological quantum computation led to the realization that von Neumann algebras, in particular so called hyper-finite factors of type II_1, seem to provide the mathematics needed to develop a more explicit view about the construction of S-matrix. The original discussion has transformed during years from free speculation reflecting in many aspects my ignorance about the mathematics involved to a more realistic view about the role of these algebras in quantum TGD. The discussions of this chapter have been restricted to the basic notions are discussed and only short mention is made to TGD applications discussed in second chapter.

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation * and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator A belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type II_1.

The definitions of adopted by von Neumann allow however more general algebras. Type I_n algebras correspond to finite-dimensional matrix algebras with finite traces whereas I_∞ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type III non-trivial traces are always infinite and the notion of trace becomes useless being replaced by the notion of state which is generalization of the notion of thermodynamical state. The fascinating feature of this notion of state is that it defines a unique modular automorphism of the factor defined apart from unitary inner automorphism and the question is whether this notion or its generalization might be relevant for the construction of M-matrix in TGD. It however seems that in TGD framework based on Zero Energy Ontology identifiable as "square root" of thermodynamics a square root of thermodynamical state is needed.

The inclusions of hyper-finite factors define an excellent candidate for the description of finite measurement resolution with included factor representing the degrees of freedom below measurement resolution. The would also give connection to the notion of quantum group whose physical interpretation has remained unclear. This idea is central to the proposed applications to quantum TGD discussed in separate chapter.

Evolution of Ideas about Hyper-finite Factors in TGD

The work with TGD inspired model for quantum computation led to the realization that von Neumann algebras, in particular hyper-finite factors, could provide the mathematics needed to develop a more explicit view about the construction of M-matrix generalizing the notion of S-matrix in zero energy ontology (ZEO). In this chapter I will discuss various aspects of hyper-finite factors and their possible physical interpretation in TGD framework.

1. Hyper-finite factors in quantum TGD

The following argument suggests that von Neumann algebras known as hyper-finite factors (HFFs) of type III_1 appearing in relativistic quantum field theories provide also the proper mathematical framework for quantum TGD.

1. The Clifford algebra of the infinite-dimensional Hilbert space is a von Neumann algebra known as HFF of type II_1. Therefore also the Clifford algebra at a given point (light-like 3-surface) of world of classical worlds (WCW) is HFF of type II_1. If the fermionic Fock algebra defined by the fermionic oscillator operators assignable to the induced spinor fields (this is actually not obvious!) is infinite-dimensional it defines a representation for HFF of type II_1. Super-conformal symmetry suggests that the extension of the Clifford algebra defining the fermionic part of a super-conformal algebra by adding bosonic super-generators representing symmetries of WCW respects the HFF property. It could however occur that HFF of type II_∞ results.

2. WCW is a union of sub-WCWs associated with causal diamonds (CD) defined as intersections of future and past directed light-cones. One can allow also unions of CDs and the proposal is that CDs within CDs are possible. Whether CDs can intersect is not clear.

3. The assumption that the M^4 proper distance a between the tips of CD is quantized in powers of 2 reproduces p-adic length scale hypothesis but one must also consider the possibility that a can have all possible values. Since $SO(3)$ is the isotropy group of CD, the CDs associated with a given value of a and with fixed lower tip are parameterized by the Lobatchevski space $L(a) = SO(3,1)/SO(3)$. Therefore the CDs with a free position of lower tip are parameterized by $M^4 \times L(a)$. A possible interpretation is in terms of quantum cosmology with a identified as cosmic time. Since Lorentz boosts define a non-compact group, the generalization of so called crossed product construction strongly suggests that the local Clifford algebra of WCW is HFF of type III_1. If one allows all values of a, one ends up with $M^4 \times M^4_+$ as the space of moduli for WCW.

4. An interesting special aspect of 8-dimensional Clifford algebra with Minkowski signature is that it allows an octonionic representation of gamma matrices obtained as tensor products of unit matrix 1 and 7-D gamma matrices γ_k and Pauli sigma matrices by replacing 1 and γ_k by octonions. This inspires the idea that it might be possible to end up with quantum TGD from purely number theoretical arguments. One can start from a local octonionic Clifford algebra in M^8. Associativity (co-associativity) condition is satisfied if one restricts the octonionic algebra to a subalgebra associated with any hyper-quaternionic and thus 4-D sub-manifold of M^8. This means that the induced gamma matrices associated with the Kähler action span a complex quaternionic (complex co-quaternionic) sub-space at each point of the sub-manifold. This associative (co-associative) sub-algebra can be mapped a matrix algebra. Together with $M^8 - H$ duality this leads automatically to quantum TGD and therefore also to the notion of WCW and its Clifford algebra which is however only mappable to an associative (co-associative(algebra and thus to HFF of type II_1.

2. Hyper-finite factors and M-matrix

HFFs of type III_1 provide a general vision about M-matrix.

1. The factors of type III allow unique modular automorphism Δ^{it} (fixed apart from unitary inner automorphism). This raises the question whether the modular automorphism could be used to define the M-matrix of quantum TGD. This is not the case as is obvious already from the fact that unitary time evolution is not a sensible concept in zero energy ontology.

2. Concerning the identification of M-matrix the notion of state as it is used in theory of factors is a more appropriate starting point than the notion modular automorphism but as a generalization of thermodynamical state is certainly not enough for the purposes of quantum

TGD and quantum field theories (algebraic quantum field theorists might disagree!). Zero energy ontology requires that the notion of thermodynamical state should be replaced with its "complex square root" abstracting the idea about M-matrix as a product of positive square root of a diagonal density matrix and a unitary S-matrix. This generalization of thermodynamical state -if it exists- would provide a firm mathematical basis for the notion of M-matrix and for the fuzzy notion of path integral.

3. The existence of the modular automorphisms relies on Tomita-Takesaki theorem, which assumes that the Hilbert space in which HFF acts allows cyclic and separable vector serving as ground state for both HFF and its commutant. The translation to the language of physicists states that the vacuum is a tensor product of two vacua annihilated by annihilation oscillator type algebra elements of HFF and creation operator type algebra elements of its commutant isomorphic to it. Note however that these algebras commute so that the two algebras are not hermitian conjugates of each other. This kind of situation is exactly what emerges in zero energy ontology (ZEO): the two vacua can be assigned with the positive and negative energy parts of the zero energy states entangled by M-matrix.

4. There exists infinite number of thermodynamical states related by modular automorphisms. This must be true also for their possibly existing "complex square roots". Physically they would correspond to different measurement interactions meaning the analog of state function collapse in zero modes fixing the classical conserved charges equal to the quantal counterparts. Classical charges would be parameters characterizing zero modes.

A concrete construction of M-matrix motivated the recent rather precise view about basic variational principles is proposed. Fundamental fermions localized to string world sheets can be said to propagate as massless particles along their boundaries. The fundamental interaction vertices correspond to two fermion scattering for fermions at opposite throats of wormhole contact and the inverse of the conformal scaling generator L_0 would define the stringy propagator characterizing this interaction. Fundamental bosons correspond to pairs of fermion and antifermion at opposite throats of wormhole contact. Physical particles correspond to pairs of wormhole contacts with monopole Kähler magnetic flux flowing around a loop going through wormhole contacts.

3. Connes tensor product as a realization of finite measurement resolution

The inclusions $\mathcal{N} \subset \mathcal{M}$ of factors allow an attractive mathematical description of finite measurement resolution in terms of Connes tensor product but do not fix M-matrix as was the original optimistic belief.

1. In ZEO $\mathcal{N}$ would create states experimentally indistinguishable from the original one. Therefore $\mathcal{N}$ takes the role of complex numbers in non-commutative quantum theory. The space $\mathcal{M}/\mathcal{N}$ would correspond to the operators creating physical states modulo measurement resolution and has typically fractal dimension given as the index of the inclusion. The corresponding spinor spaces have an identification as quantum spaces with non-commutative $\mathcal{N}$-valued coordinates.

2. This leads to an elegant description of finite measurement resolution. Suppose that a universal M-matrix describing the situation for an ideal measurement resolution exists as the idea about square root of state encourages to think. Finite measurement resolution forces to replace the probabilities defined by the M-matrix with their $\mathcal{N}$ "averaged" counterparts. The "averaging" would be in terms of the complex square root of $\mathcal{N}$-state and a direct analog of functionally or path integral over the degrees of freedom below measurement resolution defined by (say) length scale cutoff.

3. One can construct also directly M-matrices satisfying the measurement resolution constraint. The condition that $\mathcal{N}$ acts like complex numbers on M-matrix elements as far as $\mathcal{N}$-"averaged" probabilities are considered is satisfied if M-matrix is a tensor product of M-matrix in $\mathcal{M}(\mathcal{N}$ interpreted as finite-dimensional space with a projection operator to $\mathcal{N}$. The condition that $\mathcal{N}$ averaging in terms of a complex square root of $\mathcal{N}$ state produces this kind of M-matrix poses a very strong constraint on M-matrix if it is assumed to be universal (apart from variants corresponding to different measurement interactions).

4. Analogs of quantum matrix groups from finite measurement resolution?

The notion of quantum group replaces ordinary matrices with matrices with non-commutative elements. In TGD framework I have proposed that the notion should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution.

In this article I will consider the notion of quantum matrix inspired by recent view about quantum TGD and it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution. The basic idea is to replace complex matrix elements with operators expressible as products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers.

The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. The weak definition of determinant meaning its development with respect to a fixed row or column does not pose additional conditions. Strong definition of determinant requires its invariance under permutations of rows and columns. The permutation of rows/columns turns out to have interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

The resulting quantum matrices define a more general structure than quantum group but provide a concrete representation and interpretation for quantum group in terms of finite measurement resolution if q is a root of unity. For $q = \pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by sign factor invariant under the permutations of both rows and columns. One could also understand the fractal structure of inclusion sequences of hyper-finite factors resulting by recursively replacing operators appearing as matrix elements with quantum matrices.

5. Quantum spinors and fuzzy quantum mechanics

The notion of quantum spinor leads to a quantum mechanical description of fuzzy probabilities. For quantum spinors state function reduction cannot be performed unless quantum deformation parameter equals to $q = 1$. The reason is that the components of quantum spinor do not commute: it is however possible to measure the commuting operators representing moduli squared of the components giving the probabilities associated with "true" and "false". The universal eigenvalue spectrum for probabilities does not in general contain (1,0) so that quantum qbits are inherently fuzzy. State function reduction would occur only after a transition to q=1 phase and decoherence is not a problem as long as it does not induce this transition.

TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors

In this chapter 4 topics are discussed. McKay correspondence, SUSY, and twistors are discussed from TGD point of view, and new aspects of $M^8 - H$ duality are considered.

1. McKay correspondence in TGD framework

There are two mysterious looking correspondences involving ADE groups. McKay correspondence between McKay graphs characterizing tensor products for finite subgroups of $SU(2)$ and Dynkin diagrams of affine ADE groups is the first one. The correspondence between principal diagrams characterizing inclusions of hyper-finite factors of type II_1 (HFFs) with Dynkin diagrams for a subset of ADE groups and Dynkin diagrams for affine ADE groups is the second one.

These correspondences are discussed from number theoretic point of view suggested by TGD and based on the interpretation of discrete subgroups of $SU(2)$ as subgroups of the covering group of quaternionic automorphisms $SO(3)$ (analog of Galois group) and generalization of these groups to semi-direct products $Gal(K) \triangleleft SU(2)_K$ of Galois group for extension K of rationals with the discrete subgroup $SU(2)_K$ of $SU(2)$ with representation matrix elements in K. The identification of the inclusion hierarchy of HFFs with the hierarchy of extensions of rationals and their Galois groups is proposed.

A further mystery whether $Gal(K) \triangleleft SU(2)_K$ could give rise to quantum groups or affine algebras. In TGD framework the infinite-D group of isometries of "world of classical worlds" (WCW) is identified as an infinite-D symplectic group for which the discrete subgroups characterized by

K have infinite-D representations so that hyper-finite factors are natural for their representations. Symplectic algebra SSA allows hierarchy of isomorphic sub-algebras SSA_n. The gauge conditions for SSA_n and $[SSA_n, SSA]$ would define measurement resolution giving rise to hierarchies of inclusions and ADE type Kac-Moody type algebras or quantum algebras representing symmetries modulo measurement resolution.

A concrete realization of ADE type Kac-Moody algebras is proposed. It relies on the group algebra of $Gal(K) \triangleleft SU(2)_K$ and free field representation of ADE type Kac-Moody algebra identifying the free scalar fields in Kac-Moody Cartan algebra as group algebra elements defined by the traces of representation matrices (characters).

2. New aspects of $M^8 - H$ duality

$M^8 - H$ duality is now a central part of TGD and leads to new findings. $M^8 - H$ duality can be formulated both at the level of space-time surfaces and light-like 8-momenta. Since the choice of M^4 in the decomposition of momentum space $M^8 = M^4 \times E^4$ is rather free, it is always possible to find a choice for which light-like 8-momentum reduces to light-like 4-momentum in M^4 - the notion of 4-D mass is relative. This leads to what might be called $SO(4) - SU(3)$ duality corresponding to the hadronic and partonic views about hadron physics. Particles, which are eigenstates of mass squared are massless in $M^4 \times CP_2$ picture and massive in M^8 picture. The massivation in this picture is a universal mechanism having nothing to do with dynamics and results in zero energy ontology automatically if the zero energy states are superpositions of states with different masses. p-Adic thermodynamics describes this massivation. Also a proposal for the realization of ADE hierarchy emerges.

4-D space-time surfaces correspond to roots of octonionic polynomials $P(o)$ with real coefficients corresponding to the vanishing of the real or imaginary part of $P(o)$. These polynomials however allow universal roots, which are not 4-D but analogs of 6-D branes and having topology of S^6. Their M^4 projections are time =constant snapshots $t = r_n, r_M \leq r_n$ 3-balls of M^4 light-cone (r_n is root of $P(x)$). At each point the ball there is a sphere S^3 shrinking to a point about boundaries of the 3-ball. These special values of M^4 time lead to a deeper understanding of ZEO based quantum measurement theory and consciousness theory.

3. Is the identification of twistor space of M^4 really correct?

The critical questions concerning the identification of twistor space of M^4 as $M^4 \times S^2$ led to consider a more conservative identification as CP_3 with hyperbolic signature (3,-3) and replacement of H with $H = cd_{conf} \times CP_2$, where cd_{conf} is CP_2 with hyperbolic signature (1,-3). This approach reproduces the nice results of the earlier picture but means that the hierarchy of CDs in M^8 is mapped to a hierarchy of spaces cd_{conf} with sizes of CDs. This conforms with Poincare symmetry from which everything started since Poincare group acts in the moduli space of octonionic structures of M^8. Note that also the original form of $M^8 - H$ duality continues to make sense and results from the modification by projection from $CP_{3,h}$ to M^4 rather than $CP_{2,h}$.

The outcome of octo-twistor approach applied at level of M^8 together with modified $M^8 - H$ duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor (super-)Grassmannian approach. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. This part of article is not a mere side track since by $M^8 - H$ duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

This article deals with two questions.

1. The ideas related to topological quantum computation suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space

of state space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum.

Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Or could one replace the roots labelling states with representations? Or could even the coefficient field for state space be replaced with the representations? Could one speak about quantum variants of state spaces?

Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums in quantum-classical correspondence, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k, C)$, $k = 2, 3, 4$ to those of $SL(n, C)$ at least. Is there a deep connection between finite subgroups of $SL(n, C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

In the TGD framework $M^8 - H$ duality relates number theoretic and differential geometric views about physics: could it provide some understanding of this mystery? The proposal is that for cognitive representations associated with extended Dynkin diagrams (EEDs), Galois group Gal acts as Weyl group on McKay diagrams defined by irreps of the isotropy group Gal_I of given root of a polynomial which is monic polynomial but with roots replaced with direct sums of irreps of Gal_I. This could work for p-adic number fields and finite fields. One also ends up with a more detailed view about the connection between the hierarchies of inclusion of Galois groups associated with functional composites of polynomials and hierarchies of inclusions of hyperfinite factors of type II_1 assignable to the representation of super-symplectic algebra.

Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole

The theoretical framework behind TGD involves several different strands and the goal is to unify them to a single coherent whole. TGD involves number theoretic and geometric visions about physics and $M^8 - H$ duality, analogous to Langlands duality, is proposed to unify them. Also quantum classical correspondence (QCC) is a central aspect of TGD. One should understand both the $M^8 - H$ duality and QCC at the level of detail.

The following mathematical notions are expected to be of relevance for this goal.

1. Von Neumann algebras, call them M, in particular hyperfinite factors of type II_1 (HFFs), are in a central role. Both the geometric and number theoretic side, QCC could mathematically correspond to the relationship between M and its commutant M'.

   For instance, symplectic transformations leave induced Kähler form invariant and various fluxes of Kähler form are symplectic invariants and correspond to classical physics commuting with quantum physics coded by the super symplectic algebra (SSA). On the number theoretic side, the Galois invariants assignable to the polynomials determining space-time surfaces are analogous classical invariants.

2. The generalization of ordinary arithmetics to quantum arithmetics obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ allows us to replace the notions of finite and p-adic number fields with their quantum variants. The same applies to various algebras.

3. Number theoretic vision leads to adelic physics involving a fusion of various p-adic physics and real physics and to hierarchies of extensions of rationals involving hierarchies of Galois groups involving inclusions of normal subgroups. The notion of adele can be generalized by replacing various p-adic number fields with the p-adic representations of various algebras.

4. The physical interpretation of the notion of infinite prime has remained elusive although a formal interpretation in terms of a repeated quantization of a supersymmetric arithmetic QFT is highly suggestive. One can also generalize infinite primes to their quantum variants. The proposal is that the hierarchy of infinite primes generalizes the notion of adele.

The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves f 3 kinds of algebras A; supersymplectic isometries SSA acting on $\delta M^4_+ \times CP_2$, affine algebras Aff acting on light-like partonic orbits, and isometries I of light-cone boundary δM^4_+, allowing hierarchies A_n.

The braided Galois group algebras at the number theory side and algebras $\{A_n\}$ at the geometric side define excellent candidates for inclusion hierarchies of HFFs. $M^8 - H$ duality suggests that n corresponds to the degree $n$of the polynomial P defining space-time surface and that the n roots of P correspond to n braid strands at H side. Braided Galois group would act in A_n and hierarchies of Galois groups would induce hierarchies of inclusions of HFFs. The ramified primes of P would correspond to physically preferred p-adic primes in the adelic structure formed by p-adic variants of A_n with $+$ and $\times$ replaced with $\oplus$ and $\otimes$.

1.3.2 PART II: CATEGORY THEORY AND QUANTUM TGD

Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. Category theory might provide the desired systematic approach to fuse together the bundles of general ideas related to the construction of quantum TGD proper. Category theory might also have natural applications in the general theory of consciousness and the theory of cognitive representations.

1. The ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences could be expressed elegantly using the language of the category theory. Quantum classical correspondence might allow a mathematical formulation in terms of structure respecting functors mapping the categories associated with the three kinds of existences to each other. Basic vision is following.

   (a) Self hierarchy would have a functorial map to the hierarchy of space-time sheets and also WCW spinor fields reflect it. Thus the self referentiality of conscious experience would have a functorial formulation (it is possible to be conscious about what one *was* conscious).

   (b) The inherent logic for category defined by Heyting algebra must be modified in TGD context. Set theoretic inclusion would be replaced with the topological condensation, which can occur simultaneously to several space-time sheets.

   (c) The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the WCW geometry and realizes the cosmologies within cosmologies scenario.

   (d) In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.

   (e) The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity). The duality would allow to construct new preferred extremals of Kähler action.

2. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience.

3. Categories possess inherent generalized logic based on set theoretic inclusion which in TGD framework is naturally replaced with topological condensation: the outcome is quantum variants for the notions of sieve, topos, and logic. This suggests the possibility of geometrizing the logic of both geometric, objective and subjective existences and perhaps understand why

ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through three-valued logic. Also the right-wrong logic of moral rules and beautiful-ugly logic of aesthetics seem to be too naive and might be replaced with a more general quantum logic.

Category Theory and Quantum TGD

Possible applications of category theory to quantum TGD are discussed. The so called 2-plectic structure generalizing the ordinary symplectic structure by replacing symplectic 2-form with 3-form and Hamiltonians with Hamiltonian 1-forms has a natural place in TGD since the dynamics of the light-like 3-surfaces is characterized by Chern-Simons type action. The notion of planar operad was developed for the classification of hyper-finite factors of type II_1 and its mild generalization allows to understand the combinatorics of the generalized Feynman diagrams obtained by gluing 3-D light-like surfaces representing the lines of Feynman diagrams along their 2-D ends representing the vertices.

The fusion rules for the symplectic variant of conformal field theory, whose existence is strongly suggested by quantum TGD, allow rather precise description using the basic notions of category theory and one can identify a series of finite-dimensional nilpotent algebras as discretized versions of field algebras defined by the fusion rules. These primitive fusion algebras can be used to construct more complex algebras by replacing any algebra element by a primitive fusion algebra. Trees with arbitrary numbers of branches in any node characterize the resulting collection of fusion algebras forming an operad. One can say that an exact solution of symplectic scalar field theory is obtained.

Conformal fields and symplectic scalar field can be combined to form symplecto-formal fields. The combination of symplectic operad and Feynman graph operad leads to a construction of Feynman diagrams in terms of n-point functions of conformal field theory. M-matrix elements with a finite measurement resolution are expressed in terms of a hierarchy of symplecto-conformal n-point functions such that the improvement of measurement resolution corresponds to an algebra homomorphism mapping conformal fields in given resolution to composite conformal fields in improved resolution. This expresses the idea that composites behave as independent conformal fields. Also other applications are briefly discussed.

Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals of Kähler action would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity). The duality would allow to construct new preferred extremals of Kähler action.

Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD?

TGD Universe is extremely simple locally but the presence of various hierarchies make it to look extremely complex globally. Category theory and quantum groups, in particular Yangian or its TGD generalization are most promising tools to handle this complexity. The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.

2. TGD view about generalized Feynman diagrams relies on two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to tree diagrams with braiding. This picture fits very nicely with the picture suggested by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines

between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction but vertices expressed in terms of twistors depend on momenta of fermions.

3. Braiding transformations take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time regions with Minkowskian and Euclidian signature of induced metric respectively permuting two braid strands. R-matrix satisfying Yang-Baxter equation characterizes this operation algebraically.

4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. Reconnection induces an exchange of braid strands defined by the boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries of string world sheets. A generalization of quantum algebras to include also algebraic representation for reconnection is needed. Also reconnection might reduce to a braiding type operation.

Yangians look especially natural quantum algebras from TGD point of view. They are bi-algebras with co-product Δ. This makes the algebra multi-local raising hopes about the understanding of bound states. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Kac-Moody algebras (SKMAs) involved and even with super-conformal algebra (SSA), which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some sub-algebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question.

Are higher structures needed in the categorification of TGD?

The notion of higher structures promoted by John Baez looks very promising notion in the attempts to understand various structures like quantum algebras and Yangians in TGD framework. The stimulus for this article came from the nice explanations of the notion of higher structure by Urs Screiber. The basic idea is simple: replace "=" as a blackbox with an operational definition with a proof for $A = B$. This proof is called homotopy generalizing homotopy in topological sense. n-structure emerges when one realizes that also the homotopy is defined only up to homotopy in turn defined only up...

In TGD framework the notion of measurement resolution defines in a natural manner various kinds of "="s and this gives rise to resolution hierarchies. Hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adeles in adelic TGD and corresponding hierarchy of Galois groups represented geometrically, hierarchy of infinite primes, self hierarchy, etc...

In this article the idea of n-structure is studied in more detail. A rather radical idea is a formulation of quantum TGD using only cognitive representations consisting of points of space-time surface with embedding space coordinates in extension of rationals defining the level of adelic hierarchy. One would use only these discrete points sets and Galois groups. Everything would reduce to number theoretic discretization at space-time level perhaps reducing to that at partonic 2-surfaces with points of cognitive representation carrying fermion quantum numbers.

Even the "world of classical worlds " (WCW) would discretize: cognitive representation would define the coordinates of WCW point. One would obtain cognitive representations of scattering amplitudes using a fusion category assignable to the representations of Galois groups: something diametrically opposite to the immense complexity of the WCW but perhaps consistent with it. Also a generalization of McKay's correspondence suggests itself: only those irreps of the Lie group associated with Kac-Moody algebra that remain irreps when reduced to a subgroup defined by a Galois group of Lie type are allowed as ground states. Also the relation to number theoretic Langlands correspondence is very interesting.

Is Non-associative Physics and Language Possible only in Many-Sheeted Space-time?

Language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D embedding space $M^4 \times CP_2$ is one of the key conjectures of TGD. But what about many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes- that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA. The mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity.

Part I

HYPER-FINITE FACTORS

Chapter 2

Was von Neumann Right After All?

2.1 Introduction

The work with TGD inspired model [K4] for topological quantum computation [B24] led to the realization that von Neumann algebras [A113, A134, A121, A88] , in particular so called hyper-finite factors of type II_1 [A66] , seem to provide the mathematics needed to develop a more explicit view about the construction of S-matrix. The lecture notes of R. Longo [A118] give a concise and readable summary about the basic definitions and results related to von Neumann algebras and I have used this material freely in this chapter.

The original discussion has transformed during years from a free speculation reflecting in many aspects my ignorance about the mathematics involved to a more realistic view about the role of these algebras in quantum TGD. In this chapter I will discuss various aspects of hyperfinite factors with only a brief digression to TGD inspired applications whose evolution discussed in separate chapter [K28].

2.1.1 Philosophical Ideas Behind Von Neumann Algebras

The goal of von Neumann was to generalize the algebra of quantum mechanical observables. The basic ideas behind the von Neumann algebra are dictated by physics. The algebra elements allow Hermitian conjugation * and observables correspond to Hermitian operators. Any measurable function $f(A)$ of operator A belongs to the algebra and one can say that non-commutative measure theory is in question.

The predictions of quantum theory are expressible in terms of traces of observables. Density matrix defining expectations of observables in ensemble is the basic example. The highly non-trivial requirement of von Neumann was that identical a priori probabilities for a detection of states of infinite state system must make sense. Since quantum mechanical expectation values are expressible in terms of operator traces, this requires that unit operator has unit trace: $tr(Id) = 1$.

In the finite-dimensional case it is easy to build observables out of minimal projections to 1-dimensional eigen spaces of observables. For infinite-dimensional case the probably of projection to 1-dimensional sub-space vanishes if each state is equally probable. The notion of observable must thus be modified by excluding 1-dimensional minimal projections, and allow only projections for which the trace would be infinite using the straightforward generalization of the matrix algebra trace as the dimension of the projection.

The non-trivial implication of the fact that traces of projections are never larger than one is that the eigen spaces of the density matrix must be infinite-dimensional for non-vanishing projection probabilities. Quantum measurements can lead with a finite probability only to mixed states with a density matrix which is projection operator to infinite-dimensional subspace. The simple von Neumann algebras for which unit operator has unit trace are known as factors of type II_1 [A66] .

The definitions of adopted by von Neumann allow however more general algebras. Type I_n

algebras correspond to finite-dimensional matrix algebras with finite traces whereas I_∞ associated with a separable infinite-dimensional Hilbert space does not allow bounded traces. For algebras of type III non-trivial traces are always infinite and the notion of trace becomes useless being replaced by the notion of state which is generalization of the notion of thermodynamical state. The fascinating feature of this notion of state is that it defines a unique modular automorphism of the factor defined apart from unitary inner automorphism and the question is whether this notion or its generalization might be relevant for the construction of M-matrix in TGD. It however seems that in TGD framework based on Zero Energy Ontology identifiable as "square root" of thermodynamics a square root of thermodynamical state is needed.

The inclusions of hyper-finite factors define an excellent candidate for the description of finite measurement resolution with included factor representing the degrees of freedom below measurement resolution. The would also give connection to the notion of quantum group whose physical interpretation has remained unclear. This idea is central to the proposed applications to quantum TGD discussed in separate chapter.

2.1.2 Von Neumann, Dirac, And Feynman

The association of algebras of type I with the standard quantum mechanics allowed to unify matrix mechanism with wave mechanics. Note however that the assumption about continuous momentum state basis is in conflict with separability but the particle-in-box idealization allows to circumvent this problem (the notion of space-time sheet brings the box in physics as something completely real).

Because of the finiteness of traces von Neumann regarded the factors of type II_1 as fundamental and factors of type III as pathological. The highly pragmatic and successful approach of Dirac [A115] based on the notion of delta function, plus the emergence of s [A124] , the possibility to formulate the notion of delta function rigorously in terms of distributions [A69, A100] , and the emergence of path integral approach [A122] meant that von Neumann approach was forgotten by particle physicists.

Algebras of type II_1 have emerged only much later in conformal and topological quantum field theories [A127, A59] allowing to deduce invariants of knots, links and 3-manifolds. Also algebraic structures known as bi-algebras, Hopf algebras, and ribbon algebras [A42] relate closely to type II_1 factors. In topological quantum computation [B24] based on braid groups [A137] modular S-matrices they play an especially important role.

In algebraic quantum field theory [B26] defined in Minkowski space the algebras of observables associated with bounded space-time regions correspond quite generally to the type III_1 hyper-finite factor [B8, B28].

I have restricted the considerations of this chapter mostly to the technical aspects and Appendix includes sections about inclusions of HFFs. The evolution of ideas about possible applications to quantum TGD is summarized in chapter, which was originally part of this chapter [K28].

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

2.2 Von Neumann Algebras

In this section basic facts about von Neumann algebras are summarized using as a background material the concise summary given in the lecture notes of Longo [A118] .

2.2.1 Basic Definitions

A formal definition of von Neumann algebra [A134, A121, A88] is as a *-subalgebra of the set of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ closed under weak operator topology, stable under the conjugation $J =^*: x \to x^*$, and containing identity operator Id. This definition allows also von Neumann algebras for which the trace of the unit operator is not finite.

Identity operator is the only operator commuting with a simple von Neumann algebra. A general von Neumann algebra allows a decomposition as a direct integral of simple algebras, which von Neumann called factors. Classification of von Neumann algebras reduces to that for factors.

$\mathcal{B}(\mathcal{H})$ has involution * and is thus a *-algebra. $\mathcal{B}(\mathcal{H})$ has order order structure $A \geq 0$: $(Ax, x) \geq 0$. This is equivalent to $A = BB^*$ so that order structure is determined by algebraic structure. $\mathcal{B}(\mathcal{H})$ has metric structure in the sense that norm defined as supremum of $||Ax||$, $||x|| \leq 1$ defines the notion of continuity. $||A||^2 = inf\{\lambda > 0 : AA^* \leq \lambda I\}$ so that algebraic structure determines metric structure.

There are also other topologies for $\mathcal{B}(\mathcal{H})$ besides norm topology.

1. $A_i \rightarrow A$ strongly if $||Ax - A_ix|| \rightarrow 0$ for all x. This topology defines the topology of C^* algebra. $\mathcal{B}(\mathcal{H})$ is a Banach algebra that is $||AB|| \leq ||A|| \times ||B||$ (inner product is not necessary) and also C^* algebra that is $||AA^*|| = ||A||^2$.

2. $A_i \rightarrow A$ weakly if $(A_ix, y) \rightarrow (Ax, y)$ for all pairs (x, y) (inner product is necessary). This topology defines the topology of von Neumann algebra as a sub-algebra of $\mathcal{B}(\mathcal{H})$.

Denote by M' the commutant of $\mathcal{M}$ which is also algebra. Von Neumann's bicommutant theorem says that $\mathcal{M}$ equals to its own bi-commutant. Depending on whether the identity operator has a finite trace or not, one distinguishes between algebras of type II_1 and type II_∞. II_1 factor allow trace with properties $tr(Id) = 1$, $tr(xy) = tr(yx)$, and $tr(x^*x) > 0$, for all $x \neq 0$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained by completing $\mathcal{M}$ respect to the inner product defined $\langle x|y\rangle = tr(x^*y)$ defines inner product in $\mathcal{M}$ interpreted as Hilbert space. The normalized trace induces a trace in M', natural trace $Tr_{M'}$, which is however not necessarily normalized. JxJ defines an element of M': if $\mathcal{H} = L^2(\mathcal{M})$, the natural trace is given by $Tr_{M'}(JxJ) = tr_M(x)$ for all $x \in M$ and bounded.

2.2.2 Basic Classification Of Von Neumann Algebras

Consider first some definitions. First of all, Hermitian operators with positive trace expressible as products xx^* are of special interest since their sums with positive coefficients are also positive.

In quantum mechanics Hermitian operators can be expressed in terms of projectors to the eigen states. There is a natural partial order in the set of isomorphism classes of projectors by inclusion: $E < F$ if the image of $\mathcal{H}$ by E is contained to the image of $\mathcal{H}$ by a suitable isomorph of F. Projectors are said to be metrically equivalent if there exist a partial isometry which maps the images $\mathcal{H}$ by them to each other. In the finite-dimensional case metric equivalence means that isomorphism classes are identical $E = F$.

The algebras possessing a minimal projection E_0 satisfying $E_0 \leq F$ for any F are called type I algebras. Bounded operators of n-dimensional Hilbert space define algebras I_n whereas the bounded operators of infinite-dimensional separable Hilbert space define the algebra I_∞. I_n and I_∞ correspond to the operator algebras of quantum mechanics. The states of harmonic oscillator correspond to a factor of type I.

The projection F is said to be finite if $F < E$ and $F \equiv E$ implies $F = E$. Hence metric equivalence means identity. Simple von Neumann algebras possessing finite projections but no minimal projections so that any projection E can be further decomposed as $E = F + G$, are called factors of type II.

Hyper-finiteness means that any finite set of elements can be approximated arbitrary well with the elements of a finite-dimensional sub-algebra. The hyper-finite II_∞ algebra can be regarded as a tensor product of hyper-finite II_1 and I_∞ algebras. Hyper-finite II_1 algebra can be regarded as a Clifford algebra of an infinite-dimensional separable Hilbert space sub-algebra of I_∞.

Hyper-finite II_1 algebra can be constructed using Clifford algebras $C(2n)$ of $2n$-dimensional spaces and identifying the element x of $2^n \times 2^n$ dimensional $C(n)$ as the element $diag(x, x)/2$ of $2^{n+1} \times 2^{n+1}$-dimensional $C(n+1)$. The union of algebras $C(n)$ is formed and completed in the weak operator topology to give a hyper-finite II_1 factor. This algebra defines the Clifford algebra of infinite-dimensional separable Hilbert space and is thus a sub-algebra of I_∞ so that hyper-finite II_1 algebra is more regular than I_∞.

von Neumann algebras possessing no finite projections (all traces are infinite or zero) are called algebras of type III. It was later shown by [A46] [A28] that these algebras are labeled by a parameter varying in the range $[0, 1]$, and referred to as algebras of type III_x. III_1 category contains a unique hyper-finite algebra. It has been found that the algebras of observables associated with bounded regions of 4-dimensional Minkowski space in quantum field theories correspond to hyper-finite factors of type III_1 [A118] . Also statistical systems at finite temperature correspond to factors of type III and temperature parameterizes one-parameter set of automorphisms of this algebra [B8] . Zero temperature limit correspond to I_∞ factor and infinite temperature limit to II_1 factor.

2.2.3 Non-Commutative Measure Theory And Non-Commutative Topologies And Geometries

von Neumann algebras and C^* algebras give rise to non-commutative generalizations of ordinary measure theory (integration), topology, and geometry. It must be emphasized that these structures are completely natural aspects of quantum theory. In particular, for the hyper-finite type II_1 factors quantum groups and Kac Moody algebras [B29] emerge quite naturally without any need for ad hoc modifications such as making space-time coordinates non-commutative. The effective 2-dimensionality of quantum TGD (partonic or stringy 2-surfaces code for states) means that these structures appear completely naturally in TGD framework.

Non-commutative measure theory

von Neumann algebras define what might be a non-commutative generalization of measure theory and probability theory [A118] .

1. Consider first the commutative case. Measure theory is something more general than topology since the existence of measure (integral) does not necessitate topology. Any measurable function f in the space $L^\infty(X,\mu)$ in measure space (X,μ) defines a bounded operator M_f in the space $\mathcal{B}(L^2(X,\mu))$ of bounded operators in the space $L^2(X,\mu)$ of square integrable functions with action of M_f defined as $M_f g = fg$.

2. Integral over $\mathcal{M}$ is very much like trace of an operator $f_{x,y} = f(x)\delta(x,y)$. Thus trace is a natural non-commutative generalization of integral (measure) to the non-commutative case and defined for von Neumann algebras. In particular, generalization of probability measure results if the case $tr(Id) = 1$ and algebras of type I_n and II_1 are thus very natural from the point of view of non-commutative probability theory.

The trace can be expressed in terms of a cyclic vector Ω or vacuum/ground state in physicist's terminology. Ω is said to be cyclic if the completion $\overline{M\Omega} = H$ and separating if $x\Omega$ vanishes only for $x = 0$. Ω is cyclic for $\mathcal{M}$ if and only if it is separating for M'. The expression for the trace given by

$$Tr(ab) \quad = \quad \left(\frac{(ab+ba)}{2},\Omega\right) \tag{2.2.1}$$

is symmetric and allows to defined also inner product as $(a,b) = Tr(a^*b)$ in $\mathcal{M}$. If Ω has unit norm $(\Omega,\Omega) = 1$, unit operator has unit norm and the algebra is of type II_1. Fermionic oscillator operator algebra with discrete index labeling the oscillators defines II_1 factor. Group algebra is second example of II_1 factor.

The notion of probability measure can be abstracted using the notion of state. State ω on a C^* algebra with unit is a positive linear functional on $\mathcal{U}$, $\omega(1) = 1$. By so called KMS construction [A118] any state ω in C^* algebra $\mathcal{U}$ can be expressed as $\omega(x) = (\pi(x)\Omega,\Omega)$ for some cyclic vector Ω and π is a homomorphism $\mathcal{U} \to \mathcal{B}(\mathcal{H})$.

Non-commutative topology and geometry

C^* algebras generalize in a well-defined sense ordinary topology to non-commutative topology.

1. In the Abelian case Gelfand Naimark theorem [A118] states that there exists a contravariant functor F from the category of unital abelian C^* algebras and category of compact topological spaces. The inverse of this functor assigns to space X the continuous functions f on X with norm defined by the maximum of f. The functor assigns to these functions having

interpretation as eigen states of mutually commuting observables defined by the function algebra. These eigen states are delta functions localized at single point of X. The points of X label the eigenfunctions and thus define the spectrum and obviously span X. The connection with topology comes from the fact that continuous map $Y \to X$ corresponds to homomorphism $C(X) \to C(Y)$.

2. In non-commutative topology the function algebra $C(X)$ is replaced with a general C^* algebra. Spectrum is identified as labels of simultaneous eigen states of the Cartan algebra of C^* and defines what can be observed about non-commutative space X.

3. Non-commutative geometry can be very roughly said to correspond to *-subalgebras of C^* algebras plus additional structure such as symmetries. The non-commutative geometry of Connes [A29] is a basic example here.

2.2.4 Modular Automorphisms

von Neumann algebras allow a canonical unitary evolution associated with any state ω fixed by the selection of the vacuum state Ω [A118] . This unitary evolution is an automorphism fixed apart form unitary automorphisms $A \to UAU^*$ related with the choice of Ω.

Let ω be a normal faithful state: $\omega(x^*x) > 0$ for any x. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \to x\Omega$. The conjugation * in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \to x^*\Omega$. The closure of S_0 is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. Δ is positive self-adjoint operator and J anti-unitary involution. The following conditions are satisfied

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\ J\mathcal{M}J &= \mathcal{M}' \ . \end{aligned} \tag{2.2.2}$$

Δ^{it} is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation ω as $\pi \to \Delta^{it}\pi\Delta^{-it}$.

2.2.5 Joint Modular Structure And Sectors

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion. The unitary operator $\gamma = J_N J_M$ defines a canonical endomorphisms $M \to N$ in the sense that it depends only up to inner automorphism on $\mathcal{N}$, γ defines a sector of $\mathcal{M}$. The sectors of $\mathcal{M}$ are defined as $Sect(\mathcal{M}) = End(\mathcal{M})/Inn(\mathcal{M})$ and form a semi-ring with respected to direct sum and composition by the usual operator product. It allows also conjugation.

$L^2(\mathcal{M})$ is a normal bi-module in the sense that it allows commuting left and right multiplications. For $a, b \in M$ and $x \in L^2(\mathcal{M})$ these multiplications are defined as $axb = aJb^*Jx$ and it is easy to verify the commutativity using the factor $Jy^*J \in \mathcal{M}'$. [A46] [A29] has shown that all normal bi-modules arise in this way up to unitary equivalence so that representation concepts make sense. It is possible to assign to any endomorphism ρ index $Ind(\rho) \equiv M : \rho(\mathcal{M})$. This means that the sectors are in 1-1 correspondence with inclusions. For instance, in the case of hyper-finite II_1 they are labeled by Jones index. Furthermore, the objects with non-integral dimension $\sqrt{[\mathcal{M} : \rho(\mathcal{M})]}$ can be identified as quantum groups, loop groups, infinite-dimensional Lie algebras, etc...

2.2.6 Basic Facts About Hyper-Finite Factors Of Type III

Hyper-finite factors of type II_1, II_∞ and III_1, III_0, III_λ, $\lambda \in (0, 1)$, allow by definition hierarchy of finite approximations and are unique as von Neumann algebras. Also hyper-finite factors of type II_∞ and type III could be relevant for the formulation of TGD. HFFs of type II_∞ and III could appear at the level operator algebra but that at the level of quantum states one would obtain HFFs of type II_1. These extended factors inspire highly non-trivial conjectures about quantum TGD. The book of Connes [A29] provides a detailed view about von Neumann algebras in general.

Basic definitions and facts

A highly non-trivial result is that HFFs of type II_∞ are expressible as tensor products $II_\infty = II_1 \otimes I_\infty$, where II_1 is hyper-finite [A29] .

1. The existence of one-parameter family of outer automorphisms

The unique feature of factors of type III is the existence of one-parameter unitary group of outer automorphisms. The automorphism group originates in the following manner.

1. Introduce the notion of linear functional in the algebra as a map $\omega : \mathcal{M} \rightarrow C$. ω is said to be hermitian it respects conjugation in $\mathcal{M}$; positive if it is consistent with the notion of positivity for elements of $\mathcal{M}$ in which case it is called weight; state if it is positive and normalized meaning that $\omega(1) = 1$, faithful if $\omega(A) > 0$ for all positive A; a trace if $\omega(AB) = \omega(BA)$, a vector state if $\omega(A)$ is "vacuum expectation" $\omega_\Omega(A) = (\Omega, \omega(A)\Omega)$ for a non-degenerate representation $(\mathcal{H}, \pi)$ of $\mathcal{M}$ and some vector $\Omega \in \mathcal{H}$ with $||\Omega|| = 1$.

2. The existence of trace is essential for hyper-finite factors of type II_1. Trace does not exist for factors of type III and is replaced with the weaker notion of state. State defines inner product via the formula $(x, y) = \phi(y^*x)$ and $*$ is isometry of the inner product. *-operator has property known as pre-closedness implying polar decomposition $S = J\Delta^{1/2}$ of its closure. Δ is positive definite unbounded operator and J is isometry which restores the symmetry between $\mathcal{M}$ and its commutant $\mathcal{M}'$ in the Hilbert space $\mathcal{H}_\phi$, where $\mathcal{M}$ acts via left multiplication: $\mathcal{M}' = J\mathcal{M}J$.

3. The basic result of Tomita-Takesaki theory is that Δ defines a one-parameter group $\sigma^t_\phi(x) = \Delta^{it}x\Delta^{-it}$ of automorphisms of $\mathcal{M}$ since one has $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$. This unitary evolution is an automorphism fixed apart from unitary automorphism $A \rightarrow UAU^*$ related with the choice of ϕ. For factors of type I and II this automorphism reduces to inner automorphism so that the group of outer automorphisms is trivial as is also the outer automorphism associated with ω. For factors of type III the group of these automorphisms divided by inner automorphisms gives a one-parameter group of $Out(\mathcal{M})$ of outer automorphisms, which does not depend at all on the choice of the state ϕ.

More precisely, let ω be a normal faithful state: $\omega(x^*x) > 0$ for any x. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \rightarrow x\Omega$. The conjugation * in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \rightarrow x^*\Omega$. The closure of S_0 is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. Δ is positive self-adjoint operator and J anti-unitary involution. The following conditions are satisfied

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\ J\mathcal{M}J &= \mathcal{M}' \ . \end{aligned} \tag{2.2.3}$$

Δ^{it} is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation ω as $\pi \rightarrow \Delta^{it}\pi\Delta^{-it}$. What makes this result thought provoking is that it might mean a universal quantum dynamics apart from inner automorphisms and thus a realization of general coordinate invariance and gauge invariance at the level of Hilbert space.

2. Classification of HFFs of type III

Connes achieved an almost complete classification of hyper-finite factors of type III completed later by others. He demonstrated that they are labeled by single parameter $0 \leq \lambda \leq 1]$ and that factors of type III_λ, $0 \leq \lambda < 1$ are unique. Haagerup showed the uniqueness for $\lambda = 1$. The idea was that the group has an invariant, the kernel $T(M)$ of the map from time like R to $Out(M)$, consisting of those values of the parameter t for which σ^t_ϕ reduces to an inner automorphism and to unity as outer automorphism. Connes also discovered also an invariant, which he called spectrum $S(\mathcal{M})$ of $\mathcal{M}$ identified as the intersection of spectra of $\Delta_\phi \backslash \{0\}$, which is closed multiplicative subgroup of R^+.

Connes showed that there are three cases according to whether $S(\mathcal{M})$ is

1. R^+, type III_1
2. $\{\lambda^n, n \in Z\}$, type III_λ.
3. $\{1\}$, type III_0.

   The value range of λ is this by convention. For the reversal of the automorphism it would be that associated with $1/\lambda$.

Connes constructed also an explicit representation of the factors $0 < \lambda < 1$ as crossed product II_∞ factor $\mathcal{N}$ and group Z represented as powers of automorphism of II_∞ factor inducing the scaling of trace by λ. The classification of HFFs of type III reduced thus to the classification of automorphisms of $\mathcal{N} \otimes \mathcal{B}(\mathcal{H}$. In this sense the theory of HFFs of type III was reduced to that for HFFs of type II_∞ or even II_1. The representation of Connes might be also physically interesting.

Probabilistic view about factors of type III

Second very concise representation of HFFs relies on thermodynamical thinking and realizes factors as infinite tensor product of finite-dimensional matrix algebras acting on state spaces of finite state systems with a varying and finite dimension n such that one assigns to each factor a density matrix characterized by its eigen values. Intuitively one can think the finite matrix factors as associated with n-state system characterized by its energies with density matrix ρ defining a thermodynamics. The logarithm of the ρ defines the single particle quantum Hamiltonian as $H = log(\rho)$ and $\Delta = \rho = exp(H)$ defines the automorphism σ_ϕ for each finite tensor factor as $exp(iHt)$. Obviously free field representation is in question.

Depending on the asymptotic behavior of the eigenvalue spectrum one obtains different factors [A29] .

1. Factor of type I corresponds to ordinary thermodynamics for which the density matrix as a function of matrix factor approaches sufficiently fast that for a system for which only ground state has non-vanishing Boltzmann weight.
2. Factor of type II_1 results if the density matrix approaches to identity matrix sufficiently fast. This means that the states are completely degenerate which for ordinary thermodynamics results only at the limit of infinite temperature. Spin glass could be a counterpart for this kind of situation.
3. Factor of type III results if one of the eigenvalues is above some lower bound for all tensor factors in such a way that neither factor of type I or II_1 results but thermodynamics for systems having infinite number of degrees of freedom could yield this kind of situation.

This construction demonstrates how varied representations factors can have, a fact which might look frustrating for a novice in the field. In particular, the infinite tensor power of $M(2, C)$ with state defined as an infinite tensor power of $M(2, C)$ state assigning to the matrix A the complex number $(\lambda^{1/2} A_{11} + \lambda^{-1/2}\ \phi(A) = A_{22})/(\lambda^{1/2} + \lambda^{-1/2})$ defines HFF III_λ [A29] , [C1] . Formally the same algebra which for $\lambda = 1$ gives ordinary trace and HFF of type II_1, gives III factor only by replacing trace with state. This simple model was discovered by Powers in 1967.

It is indeed the notion of state or thermodynamics is what distinguishes between factors. This looks somewhat weird unless one realizes that the Hilbert space inner product is defined by the "thermodynamical" state ϕ and thus probability distribution for operators and for their thermal expectation values. Inner product in turn defines the notion of norm and thus of continuity and it is this notion which differs dramatically for $\lambda = 1$ and $\lambda < 1$ so that the completions of the algebra differ dramatically.

In particular, there is no sign about I_∞ tensor factor or crossed product with Z represented as automorphisms inducing the scaling of trace by λ. By taking tensor product of I_∞ factor represented as tensor power with induces running from $-\infty$ to 0 and II_1 HFF with indices running from 1 to ∞ one can make explicit the representation of the automorphism of II_∞ factor inducing scaling of trace by λ and transforming matrix factors possessing trace given by square root of index $\mathcal{M} : \mathcal{N}$ to those with trace 2.

2.3 Braid Group, Von Neumann Algebras, Quantum TGD, And Formation Of Bound States

The article of Vaughan Jones in [A137] discusses the relation between knot theory, statistical physics, and von Neumann algebras. The intriguing results represented stimulate concrete ideas about how to understand the formation of bound states quantitatively using the notion of join along boundaries bond. All mathematical results represented in the following discussion can be found in [A137] and in the references cited therein so that I will not bother to refer repeatedly to this article in the sequel.

2.3.1 Factors Of Von Neumann Algebras

Von Neumann algebras M are algebras of bounded linear operators acting in Hilbert space. These algebras contain identity, are closed with respect to Hermitian conjugation, and are topologically complete. Finite-dimensional von Neuman algebras decompose into a direct sum of algebras M_n, which act essentially as matrix algebras in Hilbert spaces $\mathcal{H}_{nm}$, which are tensor products $C^n \otimes \mathcal{H}_m$. Here $\mathcal{H}_m$ is an m-dimensional Hilbert space in which M_n acts trivially. m is called the multiplicity of M_n.

A factor of von Neumann algebra is a von Neumann algebra whose center is just the scalar multiples of identity. The algebra of bounded operators in an infinite-dimensional Hilbert space is certainly a factor. This algebra decomposes into "atoms" represented by one-dimensional projection operators. This kind of von Neumann algebras are called type I factors.

The so called type II_1 factors and type III factors came as a surprise even for Murray and von Neumann. II_1 factors are infinite-dimensional and analogs of the matrix algebra factors M_n. They allow a trace making possible to define an inner product in the algebra. The trace defines a generalized dimension for any subspace as the trace of the corresponding projection operator. This dimension is however continuous and in the range $[0, 1]$: the finite-dimensional analog would be the dimension of the sub-space divided by the dimension of $\mathcal{H}_n$ and having values $(0, 1/n, 2/n, ..., 1)$. II_1 factors are isomorphic and there exists a minimal "hyper-finite" II_1 factor is contained by every other II_1 factor.

Just as in the finite-dimensional case, one can to assign a multiplicity to the Hilbert spaces where II_1 factors act on. This multiplicity, call it $dim_M(\mathcal{H})$ is analogous to the dimension of the Hilbert space tensor factor $\mathcal{H}_m$, in which II_1 factor acts trivially. This multiplicity can have all positive real values. Quite generally, von Neumann factors of type I and II_1 are in many respects analogous to the coefficient field of a vector space.

2.3.2 Sub-Factors

Sub-factors $N \subset M$, where N and M are of type II_1 and have same identity, can be also defined. The observation that M is analogous to an algebraic extension of N motivates the introduction of index $|M : N|$, which is essentially the dimension of M with respect to N. This dimension is an analog for the complex dimension of CP_2 equal to 2 or for the algebraic dimension of the extension of p-adic numbers.

The following highly non-trivial results about the dimensions of the tensor factors hold true.

1. If $N \subset M$ are II_1 factors and $|M : N| < 4$, there is an integer $n \geq 3$ such $|M : N| = r = 4cos^2(\pi/n)$, $n \geq 3$.

2. For each number $r = 4cos^2(\pi/n)$ and for all $r \geq 4$ there is a sub-factor $R_r \subset R$ with $|R : R_r| = r$.

   One can say that M effectively decomposes to a tensor product of N with a space, whose dimension is quantized to a certain algebraic number r. The values of r corresponding to $n = 3, 4, 5, 6...$ are $r = 1, 2, 1 + \Phi \simeq 2.61, 3, ...$ and approach to the limiting value $r = 4$. For $r \geq 4$ the dimension becomes continuous.

An even more intriguing result is that by starting from $N \subset M$ with a projection e_N: $M \to N$ one can extend M to a larger II_1 algebra $\langle M, e_N \rangle$ such that one has

$$\begin{aligned} |\langle M,e_N\rangle : M| &= |M:N| \ , \\ tr(xe_N) &= |M:N|^{-1}tr(x) \ , \ \ x\in M \ . \end{aligned} \qquad (2.3.1)$$

One can continue this process and the outcome is a tower of II_1 factors $M_i \subset M_{i+1}$ defined by $M_1 = N$, $M_2 = M$, $M_{i+1} = \langle M_i, e_{M_{i-1}}\rangle$. Furthermore, the projection operators $e_{M_i} \equiv e_i$ define a Temperley-Lieb representation of the braid algebra via the formulas

$$\begin{aligned} e_i^2 &= e_i \ , \\ e_ie_{i\pm1}e_i &= \tau e_i \ , \ \ \tau = 1/|M:N| \\ e_ie_j &= e_je_i \ , \ \ |i-j|\geq 2 \ . \end{aligned} \qquad (2.3.2)$$

Temperley Lieb algebra will be discussed in more detail later. Obviously the addition of a tensor factor of dimension r is analogous with the addition of a strand to a braid.

The hyper-finite algebra R is generated by the set of braid generators $\{e_1, e_2, ...\}$ in the braid representation corresponding to r. Sub-factor R_1 is obtained simply by dropping the lowest generator e_1, R_2 by dropping e_1 and e_2, etc..

2.3.3 Ii$_1$ Factors And The Spinor Structure Of WCW

The following observations serve as very suggestive guidelines for how one could interpret the above described results in TGD framework.

1. The discrete spectrum of dimensions $1, 2, 1 + \Phi, 3, ..$ below $r < 4$ brings in mind the discrete energy spectrum for bound states whereas the for $r \geq 4$ the spectrum of dimensions is analogous to a continuum of unbound states. The fact that r is an algebraic number for $r < 4$ conforms with the vision that bound state entanglement corresponds to entanglement probabilities in an extension of rationals defining a finite-dimensional extension of p-adic numbers for every prime p.

2. The discrete values of r correspond precisely to the angles ϕ allowed by the unitarity of Temperley-Lieb representations of the braid algebra with $d = -\sqrt{r}$. For $r \geq 4$ Temperley-Lieb representation is not unitary since $cos^2(\pi/n)$ becomes formally larger than one (n would become imaginary and continuous). This could mean that $r \geq 4$, which in the generic case is a transcendental number, represents unbound entanglement, which in TGD Universe is not stable against state preparation and state function reduction processes.

3. The formula $tr(xe_N) = |M : N|^{-1}tr(x)$ is completely analogous to the formula characterizing the normalization of the link invariant induced by the second Markov move in which a new strand is added to a braid such that it braids only with the leftmost strand and therefore does not change the knot resulting as a link closure. Hence the addition of a single strand seems to correspond to an introduction of an r-dimensional sub-factor to II_1 factor.

In TGD framework the generation of bound state has the formation of (possibly braided join along boundaries bonds as a space-time correlate and this encourages a rather concrete interpretation of these findings. Also the I_1 factors themselves have a nice interpretation in terms of the WCW spinor structure.

1. The interpretation of II_1 factors in terms of Clifford algebra of WCW

The Clifford algebra of an infinite-dimensional Hilbert space defines a II_1 factor. The counterparts for e_i would naturally correspond to the analogs of projection operators $(1 + \sigma_i)/2$ and thus to operators of form $(1+\Sigma_{ij})/2$, defined by a subset of sigma matrices. The first guess is that the index pairs are $(i, j) = (1, 2), (2, 3), (3, 4), ...$ The dimension of the Clifford algebra is 2^N for N-dimensional space so that $\Delta N = 1$ would correspond to $r = 2$ in the classical case and to one qubit. The problem with this interpretation is $r > 2$ has no physical interpretation: the formation of bound states is expected to reduce the value of r from its classical value rather than increase it.

One can however consider also the sequence $(i,j) = (1, 1{+}k), (1{+}k, 1{+}2k), (1{+}2k, 1{+}3k), ...$ For $k = 2$ the reduction of r from $r = 4$ would be due to the loss of degrees of freedom due to the formation of a bound state and $(r = 4, \Delta N = 2)$ would correspond to the classical limit resulting at the limit of weak binding. The effective elimination of the projection operators from the braid algebra would reflect this loss of degrees of freedom. This interpretation could at least be an appropriate starting point in TGD framework.

In TGD Universe physical states correspond to WCW spinor fields, whose gamma matrix algebra is constructed in terms of second quantized free induced spinor fields defined at space-time sheets. The original motivation was the idea that the quantum states of the Universe correspond to the modes of purely classical free spinor fields in the infinite-dimensional configuration space of 3-surfaces (the "world of classical worlds", WCW) possessing general coordinate invariant (in 4-dimensional sense!) Kähler geometry. Quantum information-theoretical motivation could have come from the requirement that these fields must be able to code information about the properties of the point (3-surface, and corresponding space-time sheet). Scalar fields would treat the 3-surfaces as points and are thus not enough. Induced spinor fields allow however an infinite number of modes: according to the naïve Fourier analyst's intuition these modes are in one-one correspondence with the points of the 3-surface. Second quantization gives much more. Also non-local information about the induced geometry and topology must be coded, and here quantum entanglement for states generated by the fermionic oscillator operators coding information about the geometry of 3-surface provides enormous information storage capacity.

In algebraic geometry also the algebra of the embedding space of algebraic variety divided by the ideal formed by functions vanishing on the surface codes information about the surface: for instance, the maximal ideals of this algebra code for the points of the surface (functions of embedding space vanishing at a particular point). The function algebra of the embedding space indeed plays a key role in the construction of WCW-geometry besides second quantized fermions.

The Clifford algebra generated by the WCW gamma matrices at a given point (3-surface) of WCW of 3-surfaces could be regarded as a II_1-factor associated with the local tangent space endowed with Hilbert space structure (WCW Kähler metric). The counterparts for e_i would naturally correspond to the analogs of projection operators $(1 + \sigma_i)/2$ and thus operators of form $(G_{\overline{A}B} \times 1 + \Sigma_{\overline{A}B})$ formed as linear combinations of components of the Kähler metric and of the sigma matrices defined by gamma matrices and contracted with the generators of the isometries of WCW. The addition of single complex degree of freedom corresponds to $\Delta N = 2$ and $r = 4$ and the classical limit and would correspond to the addition of single braid. $(r < 4, \Delta N < 2)$ would be due to the binding effects.

$r = 1$ corresponds to $\Delta N = 0$. The first interpretation is in terms of strong binding so that the addition of particle does not increase the number of degrees of freedom. In TGD framework $r = 1$ might also correspond to the addition of zero modes which do not contribute to the WCW metric and spinor structure but have a deep physical significance. $(r = 2, \Delta N = 1)$ would correspond to strong binding reducing the spinor and space-time degrees of freedom by a factor of half. $r = \Phi^2$ $(n = 5)$ *resp.* $r = 3$ $(n = 6)$ corresponds to $\Delta N_r \simeq 1.3885$ *resp.* $\Delta N_r = 1.585$. Using the terminology of quantum field theories, one might say that in the infinite-dimensional context a given complex bound state degree of freedom possesses anomalous real dimension $r < 2$. $r \geq 4$ would correspond to a unbound entanglement and increasingly classical behavior.

2.3.4 About Possible Space-Time Correlates For The Hierarchy Of II_1 Sub-Factors

By quantum classical correspondence the infinite-dimensional physics at WCW level should have definite space-time correlates. In particular, the dimension r should have some fractal dimension as a space-time correlate.

1. Quantum classical correspondence

Join along boundaries bonds serve as correlates for bound state formation. The presence of join along boundaries bonds would lead to a generation of bound states just by reducing the degrees of freedom to those of connected 3-surface. The bonds would constrain the two 3-surfaces to single space-like section of embedding space.

This picture would allow to understand the difficulties related to Bethe-Salpeter equations for bound states based on the assumption that particles are points moving in M^4. The restriction of particles to time=constant section leads to a successful theory which is however non-relativistic. The basic binding energy would relate to the entanglement of the states associated with the bonded 3-surfaces. Since the classical energy associated with the bonds is positive, the binding energy tends to be reduced as r increases.

By spin glass degeneracy join along boundaries bonds have an infinite number of degrees of freedom in the ordinary sense. Since the system is infinite-dimensional and quantum critical, one expects that the number r of degrees freedom associated with a single join along boundaries bond is universal. Since join along boundaries bonds correspond to the strands of a braid and are correlates for the bound state formation, the natural guess is that $r = 4cos^2(\pi/n)$, $n = 3, 4, 5, ...$ holds true. $r < 4$ should characterize both binding energy and the dimension of the effective tensor factor introduced by a new join along boundaries bond.

The assignment of 2 "bare" and $\Delta N \leq 2$ renormalized real dimensions to single join along boundaries bond is consistent with the effective two-dimensionality of anyon systems and with the very notion of the braid group. The picture conforms also with the fact that the degrees of freedom in question are associated with metrically 2-dimensional light-like boundaries (of say magnetic flux tubes) acting as causal determinants. Also vibrational degrees of freedom described by Kac-Moody algebra are present and the effective 2-dimensionality means that these degrees of freedom are not excited and only topological degrees of freedom coded by the position of the puncture remain.

$(r \geq 4, \Delta N \geq 2)$, if possible at all, would mean that the tensor factor associated with the join along boundaries bond is effectively more than 4-dimensional due to the excitation of the vibrational Kac-Moody degrees of freedom. The finite value of r would mean that most of theme are eliminated also now but that their number is so large that bound state entanglement is not possible anymore.

The introduction of non-integer dimension could be seen as an effective description of an infinite-dimensional system as a finite-dimensional system in the spirit of renormalization group philosophy. The non-unitarity of $r \geq 4$ Temperley-Lieb representations could mean that they correspond to unbound entanglement unstable against state function reduction and preparation processes. Since this kind of entanglement does not survive in quantum jump it is not representable in terms of braid groups.

2. Does r define a fractal dimension of CP_2 projection of partonic 2-surface?

On basis of the quantum classical correspondence one expects that r should define some fractal dimension at the space-time level. Since r varies in the range $1, .., 4$ and corresponds to the fractal dimension of 2-D Clifford algebra the corresponding spinors would have dimension $d = \sqrt{r}$. There are two options.

1. $D = r/2$ is suggested on basis of the construction of quantum version of M^d.

2. $D = log_2(r)$ is natural on basis of the dimension $d = 2^{D/2}$ of spinors in D-dimensional space.

r can be assigned with CP_2 degrees of freedom in the model for the quantization of Planck constant based on the explicit identification of Josephson inclusions in terms of finite subgroups of $SU(2) \subset SU(3)$. Hence D should relate to the CP_2 projection of the partonic 2-surface and one could have $D = D(X^2)$, the latter being the average dimension of the CP_2 projection of the partonic 2-surface for the preferred extremals of Kähler action.

Since a strongly interacting non-perturbative phase should be in question, the dimension for the CP_2 projection of the space-time surface must be at least $D(X^4) = 2$ to guarantee that non-vacuum extremals are in question. This is true for $D(X^2) = r/2 \geq 1$. The logarithmic formula $D(X^2) = log_2(r) \geq 0$ gives $D(X^2) = 0$ for $n = 3$ meaning that partonic 2-surfaces are vacua: space-time surface can still be a non-vacuum extremal.

As n increases, the number of CP_2 points covering a given M^4 point and related by the finite subgroup of $G \subset SU(2) \subset SU(3)$ defining the inclusion increases so that the fractal dimension of the CP_2 projection is expected to increase also. $D(X^2) = 2$ would correspond to the space-time surfaces for which partons have topological magnetic charge forcing them to have a 2-dimensional CP_2 projection. There are reasons to believe that the projection must be homologically non-trivial geodesic sphere of CP_2.

2.3.5 Could Binding Energy Spectra Reflect The Hierarchy Of Effective Tensor Factor Dimensions?

If one takes completely seriously the idea that join along boundaries bonds are a correlate of binding then the spectrum of binding energies might reveal the hierarchy of the fractal dimensions $r(n)$. Hydrogen atom and harmonic oscillator have become symbols for bound state systems. Hence it is of interest to find whether the binding energy spectrum of these systems might be expressed in terms of the "binding dimension" $x(n) = 4 - r(n)$ characterizing the deviation of dimension from that at the limit of a vanishing binding energy. The binding energies of hydrogen atom are in a good approximation given by $E(n)/E(1) = 1/n^2$ whereas in the case of harmonic oscillator one has $E(n)/E_0 = 2n + 1$. The constraint $n \geq 3$ implies that the principal quantum number must correspond $n - 2$ in the case of hydrogen atom and to $n - 3$ in the case of harmonic oscillator.

Before continuing one must face an obvious objection. By previous arguments different values of r correspond to different values of $\hbar$. The value of $\hbar$ cannot however differ for the states of hydrogen atom. This is certainly true. The objection however leaves open the possibility that the states of the light-like boundaries of join along boundaries bonds correspond to reflective level and represent some aspects of the physics of, say, hydrogen atom.

In the general case the energy spectrum satisfies the condition

$$\frac{E_B(n)}{E_B(3)} = \frac{f(4 - r(n))}{f(3)} \ , \tag{2.3.3}$$

where f is some function. The simplest assumption is that the spectrum of binding energies $E_B(n) = E(n) - E(\infty)$ is a linear function of $r(n) - 4$:

$$\frac{E_B(n)}{E_B(3)} = \frac{4 - r(n)}{3} = \frac{4}{3}sin^2(\frac{\pi}{n}) \rightarrow \frac{4\pi^2}{3} \times \frac{1}{n^2} \ . \tag{2.3.4}$$

In the linear approximation the ratio $E(n+1)/E(n)$ approaches $(n/n+1)^2$ as in the case of hydrogen atom but for small values the linear approximation fails badly. An exact correspondence results for

$$\frac{E(n)}{E(1)} = \frac{1}{n^2} \ ,$$
$$n = \frac{1}{\pi \arcsin\left(\sqrt{1 - r(n+2)/4}\right)} - 2 \ .$$

Also the ionized states with $r \geq 4$ would correspond to bound states in the sense that two particle would be constrained to move in the same space-like section of space-time surface and should be distinguished from genuinely free states when particles correspond to disjoint space-time sheets.

For the harmonic oscillator one express $E(n) - E(0)$ instead of $E(n) - E(\infty)$ as a function of $x = 4 - r$ and one would have

$$\frac{E(n)}{E(0)} = 2n + 1 \ ,$$
$$n = \frac{1}{\pi \arcsin\left(\sqrt{1 - r(n+3)/4}\right)} - 3 \ .$$

In this case ionized states would not be possible due to the infinite depth of the harmonic oscillator potential well.

2.3.6 Four-Color Problem, II$_1$ Factors, And Anyons

The so called four-color problem can be phrased as a question whether it is possible to color the regions of a plane map using only four colors in such a way that no adjacent regions have the same color (for an enjoyable discussion of the problem see [A94]). One might call this kind of coloring complete. There is no loss of generality in assuming that the map can be represented as a graph with regions represented as triangle shaped faces of the graph. For the dual graph the coloring of faces becomes coloring of vertices and the question becomes whether the coloring is possible in such a way that no vertices at the ends of the same edge have same color. The problem can be generalized by replacing planar maps with maps defined on any two-dimensional surface with or without boundary and arbitrary topology. The four-color problem has been solved with an extensive use of computer [A34] but it would be nice to understand why the complete coloring with four colors is indeed possible.

There is a mysterious looking connection between four-color problem and the dimensions $r(n) = 4cos^2(\pi/n)$, which are in fact known as Beraha numbers in honor of the discoverer of this connection [A73] . Consider a more general problem of coloring two-dimensional map using m colors. One can construct a polynomial $P(m)$, so called chromatic polynomial, which tells the number of colorings satisfying the condition that no neighboring vertices have the same color. The vanishing of the chromatic polynomial for an integer value of m tells that the complete coloring using m colors is not possible.

$P(m)$ has also other than integer valued real roots. The strange discovery due to Beraha is that the numbers $B(n)$ appear as approximate roots of the chromatic polynomial in many situations. For instance, the four non-integral real roots of the chromatic polynomial of the truncated icosahedron are very close to $B(5)$, $B(7)$, $B(8)$ and $B(9)$. These findings led Beraha to formulate the following conjecture. Let P_i be a sequence of chromatic polynomials for a graph for which the number of vertices approaches infinity. If r_i is a root of the polynomial approaching a well-defined value at the limit $i \to \infty$, then the limiting value of $r(i)$ is Beraha number.

A physicist's proof for Beraha's conjecture based on quantum groups and conformal theory has been proposed [A73] . It is interesting to look for the a possible physical interpretation of 4-color problem and Beraha's conjecture in TGD framework.

1. In TGD framework $B(n)$ corresponds to a renormalized dimension for a 2-spin system consisting of two qubits, which corresponds to 4 different colors. For $B(n) = 4$ two spin 1/2 fermions obeying Fermi statistics are in question. Since the system is 2-dimensional, the general case corresponds to two anyons with fractional spin $B(n)/4$ giving rise to $B(n) < 4$ colors and obeying fractional statistics instead of Fermi statistics. One can replace coloring problem with the problem whether an ideal antiferro-magnetic lattice using anyons with fractional spin $B(n)/4$ is possible energetically. In other words, does this system form a quantum mechanical bound state even at the limit when the lengths of the edges approach to zero.

2. The failure of coloring means that there are at least two neighboring vertices in the lattice with the property that the spins at the ends of the same edge are in the same direction. Lattice defect would be in question. At the limit of an infinitesimally short edge length the failure of coloring is certainly not an energetically favored option for fermionic spins ($m = 4$) but is allowed by anyonic statistics for $m = B(n) < 4$. Thus one has reasons to expect that when anyonic spin is $B(n)/4$ the formation of a purely 2-anyon bound states becomes possible and they form at the limit of an infinitesimal edge length a kind of topological macroscopic quantum phase with a non-vanishing binding energy. That $B(n)$ are roots of the chromatic polynomial at the continuum limit would have a clear physical interpretation.

3. Only $B(n) < 4$ defines a sub-factor of von Neumann algebra allowing unitary Temperley-Lieb representations. This is consistent with the fact that for $m = 4$ complete coloring must exists. The physical argument is that otherwise a macroscopic quantum phase with non-vanishing binding energy could result at the continuum limit and the upper bound for r from unitarity would be larger than 4. For $m = 4$ the completely anti-ferromagnetic state would represent the ground state and the absence of anyon-pair condensate would mean a vanishing binding energy.

2.4 Inclusions Of II_1 And III_1 Factors

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. For type I algebras the inclusions are trivial and tensor product description applies as such. For factors of II_1 and III the inclusions are highly non-trivial. The inclusion of type II_1 factors were understood by Vaughan Jones [A2] and those of factors of type III by Alain Connes [A28] .

Sub-factor $\mathcal{N}$ of $\mathcal{M}$ is defined as a closed *-stable C-subalgebra of $\mathcal{M}$. Let $\mathcal{N}$ be a sub-factor of type II_1 factor $\mathcal{M}$. Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = dim_N(L^2(\mathcal{M})) = Tr_{N'}(id_{L^2(\mathcal{M})})$. One can say that the dimension of completion of $\mathcal{M}$ as $\mathcal{N}$ module is in question.

2.4.1 Basic Findings About Inclusions

What makes the inclusions non-trivial is that the position of $\mathcal{N}$ in $\mathcal{M}$ matters. This position is characterized in case of hyper-finite II_1 factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of $\mathcal{M}$ as $\mathcal{N}$ module and also as the inverse of the dimension defined by the trace of the projector from $\mathcal{M}$ to $\mathcal{N}$. It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either $\mathcal{M}$ or $\mathcal{M}$, only the embedding.

The basic facts proved by Jones are following [A2] .

1. For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$\begin{aligned} &a)\ \ \mathcal{M} : \mathcal{N} = 4cos^2(\pi/h) \ \ , \quad h \geq 3 \ \ , \\ &b)\ \ \mathcal{M} : \mathcal{N} \geq 4 \ \ . \end{aligned} \tag{2.4.1}$$

   the numbers at right hand side are known as Beraha numbers [A73] . The comments below give a rough idea about what finiteness of principal graph means.

2. As explained in [B29] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (dimg(g) - r)/r$. The Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of SU(2) and the interpretation proposed in [A131] is following. The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: SU(2) itself, circle group U(1), and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection. The diagrams corresponding to finite subgroups are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with n=6,7,8 for tetrahedron, cube, dodecahedron. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed.

The interpretation of [A131] is that the subfactors correspond to inclusions $\mathcal{N} \subset \mathcal{M}$ defined in the following manner.

1. Let G be a finite subgroup of SU(2). Denote by R the infinite-dimensional Clifford algebras resulting from infinite-dimensional tensor power of $M_2(C)$ and by R_0 its subalgebra obtained by restricting $M_2(C)$ element of the first factor to be unit matrix. Let G act by automorphisms in each tensor factor. G leaves R_0 invariant. Denote by R_0^G and R^G the sub-algebras which remain element wise invariant under the action of G. The resulting Jones inclusions $R_0^G \subset R^G$ are consistent with the ADE correspondence.

2. The argument suggests the existence of quantum versions of subgroups of SU(2) for which representations are truncations of those for ordinary subgroups. The results have been generalized to other Lie groups.

3. Also $SL(2, C)$ acts as automorphisms of $M_2(C)$. An interesting question is what happens if one allows G to be any discrete subgroups of SL(2,C). Could this give inclusions with $\mathcal{M} : \mathcal{N} > 4$?. The strong analogy of the spectrum of indices with spectrum of energies with hydrogen atom would encourage this interpretation: the subgroup SL(2,C) not reducing to those of SU(2) would correspond to the possibility for the particle to move with respect to each other with constant velocity.

2.4.2 The Fundamental Construction And Temperley-Lieb Algebras

It was shown by Jones [A71] that for a given Jones inclusion with $\beta = \mathcal{M} : \mathcal{N} < \infty$ there exists a tower of finite II_1 factors $\mathcal{M}_k$ for $k = 0, 1, 2, ...$ such that

1. $\mathcal{M}_0 = \mathcal{N}$, $\mathcal{M}_1 = \mathcal{M}$,
2. $\mathcal{M}_{k+1} = End_{\mathcal{M}_{k-1}}\mathcal{M}_k$ is the von Neumann algebra of operators on $L^2(\mathcal{M}_k)$ generated by $\mathcal{M}_k$ and an orthogonal projection $e_k : L^2(\mathcal{M}_k) \to L^2(\mathcal{M}_{k-1})$ for $k \geq 1$, where $\mathcal{M}_k$ is regarded as a subalgebra of $\mathcal{M}_{k+1}$ under right multiplication.

It can be shown that $\mathcal{M}_{k+1}$ is a finite factor. The sequence of projections on $\mathcal{M}_\infty = \cup_{k\geq 0}\mathcal{M}_k$ satisfies the relations

$$\begin{array}{ll} e_i^2 = e_i \ , & e_i^{=} e_i \ , \\ e_i = \beta e_i e_j e_i & \text{for } \ |i-j| = 1 \ , \\ e_i e_j = e_j e_i & \text{for } \ |i-j| \geq 2 \ . \end{array} \tag{2.4.2}$$

The construction of hyper-finite II_1 factor using Clifford algebra $C(2)$ represented by 2×2 matrices allows to understand the theorem in $\beta = 4$ case in a straightforward manner. In particular, the second formula involving β follows from the identification of x at $(k-1)^{th}$ level with $(1/\beta)diag(x, x)$ at k^{th} level.

By replacing 2×2 matrices with $\sqrt{\beta} \times \sqrt{\beta}$ matrices one can understand heuristically what is involved in the more general case. $\mathcal{M}_k$ is $\mathcal{M}_{k-1}$ module with dimension $\sqrt{\beta}$ and $\mathcal{M}_{k+1}$ is the space of $\sqrt{\beta} \times \sqrt{\beta}$ matrices $\mathcal{M}_{k-1}$ valued entries acting in $\mathcal{M}_k$. The transition from $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$ linear maps of $\mathcal{M}_k$ happens in the transition to the next level. x at $(k-1)^{th}$ level is identified as $(x/\beta) \times Id_{\sqrt{\beta}\times\sqrt{\beta}}$ at the next level. The projection e_k picks up the projection of the matrix with $\mathcal{M}_{k-1}$ valued entries in the direction of the $Id_{\sqrt{\beta}\times\sqrt{\beta}}$.

The union of algebras $A_{\beta,k}$ generated by $1, e_1, ..., e_k$ defines Temperley-Lieb algebra A_β [A129] . This algebra is naturally associated with braids. Addition of one strand to a braid adds one generator to this algebra and the representations of the Temperley Lieb algebra provide link, knot, and 3-manifold invariants [A137] . There is also a connection with systems of statistical physics and with Yang-Baxter algebras [A41] .

A further interesting fact about the inclusion hierarchy is that the elements in $\mathcal{M}_i$ belonging to the commutator $\mathcal{N}'$ of $\mathcal{N}$ form finite-dimensional spaces. Presumably the dimension approaches infinity for $n \to \infty$.

2.4.3 Connection With Dynkin Diagrams

The possibility to assign Dynkin diagrams ($\beta < 4$) and extended Dynkin diagrams ($\beta = 4$ to Jones inclusions can be understood heuristically by considering a characterization of so called bipartite graphs [A133] , [B29] by the norm of the adjacency matrix of the graph.

Bipartite graphs Γ is a finite, connected graph with multiple edges and black and white vertices such that any edge connects white and black vertex and starts from a white one. Denote by $w(\Gamma)$ ($b(\Gamma)$) the number of white (black) vertices. Define the adjacency matrix $\Lambda = \Lambda(\Gamma)$ of size $b(\Gamma) \times w(\Gamma)$ by

$$w_{b,w} = \left\{ \begin{array}{ll} m(e) & \text{if there exists } e \text{ such that } \ \delta e = b - w \ , \\ 0 & \text{otherwise } . \end{array} \right. \tag{2.4.3}$$

Here $m(e)$ is the multiplicity of the edge e.

Define norm $||\Gamma||$ as

$$\begin{array}{rcl} ||X|| & = & max\{||X||; \ ||x|| \leq 1\} \ , \\ ||\Gamma|| & = & ||\Lambda(\Gamma)|| = \left|\left| \begin{array}{ll} 0 & \Lambda(\Gamma) \\ \Lambda(\Gamma)^t & 0 \end{array} \right|\right| \ . \end{array} \tag{2.4.4}$$

Note that the matrix appearing in the formula is $(m+n) \times (m+n)$ symmetric square matrix so that the norm is the eigenvalue with largest absolute value.

Suppose that Γ is a connected finite graph with multiple edges (sequences of edges are regarded as edges). Then

1. If $||\Gamma|| \leq 2$ and if Γ has a multiple edge, $||\Gamma|| = 2$ and $\Gamma = \tilde{A}_1$, the extended Dynkin diagram for $SU(2)$ Kac Moody algebra.

2. $||\Gamma|| < 2$ if and only Γ is one of the Dynkin diagrams of A,D,E. In this case $||\Gamma|| = 2cos(\pi/h)$, where h is the Coxeter number of Γ.

3. $||\Gamma|| = 2$ if and only if Γ is one of the extended Dynkin diagrams $\tilde{A}, \tilde{D}, \tilde{E}$.

This result suggests that one can indeed assign to the Jones inclusions Dynkin diagrams. To really understand how the inclusions can be characterized in terms bipartite diagrams would require a deeper understanding of von Neumann algebras. The following argument only demonstrates that bipartite graphs naturally describe inclusions of algebras.

1. Consider a bipartite graph. Assign to each white vertex linear space $W(w)$ and to each edge of a linear space $W(b,w)$. Assign to a given black vertex the vector space $\oplus_{\delta e=b-w} W(b,w) \otimes W(w)$ where (b,w) corresponds to an edge ending to b.

2. Define $\mathcal{N}$ as the direct sum of algebras $End(W(w))$ associated with white vertices and $\mathcal{M}$ as direct sum of algebras $\oplus_{\delta e=b-w} End(W(b,w)) \otimes End(W(w))$ associated with black vertices.

3. There is homomorphism $N \rightarrow M$ defined by embedding direct sum of white endomorphisms x to direct sum of tensor products x with the identity endomorphisms associated with the edges starting from x.

It is possible to show that Jones inclusions correspond to the Dynkin diagrams of A_n, D_{2n}, and E_6, E_8 and extended Dynkin diagrams of ADE type. In particular, the dual of the bi-partite graph associated with $\mathcal{M}_{n-1} \subset \mathcal{M}_n$ obtained by exchanging the roles of white and black vertices describes the inclusion $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ so that two subsequent Jones inclusions might define something fundamental (the corresponding space-time dimension is $2 \times log_2(\mathcal{M} : \mathcal{N}) \leq 4$.

2.4.4 Indices For The Inclusions Of Type III_1 Factors

Type III_1 factors appear in relativistic quantum field theory defined in 4-dimensional Minkowski space [B8] . An overall summary of basic results discovered in algebraic quantum field theory is described in the lectures of Longo [A118] . In this case the inclusions for algebras of observables are induced by the inclusions for bounded regions of M^4 in axiomatic quantum field theory. Tomita's theory of modular Hilbert algebras [A111] , [B28] forms the mathematical corner stone of the theory.

The basic notion is Haag-Kastler net [A106] consisting of bounded regions of M^4. Double cone serves as a representative example. The von Neumann algebra $\mathcal{A}(O)$ is generated by observables localized in bounded region O. The net satisfies the conditions implied by local causality:

1. Isotony: $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

2. Locality: $O_1 \subset O_2'$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$ with O' defined as $\{x : \langle x, y\rangle < 0 \text{ for all } y \in O\}$.

3. Haag duality $\mathcal{A}(O')' = \mathcal{A}(O)$.

   Besides this Poincare covariance, positive energy condition, and the existence of vacuum state is assumed.

DHR (Doplicher-Haag-Roberts) [A52] theory allows to deduce the values of Jones index and they are squares of integers in dimensions $D > 2$ so that the situation is rather trivial. The 2-dimensional case is distinguished from higher dimensional situations in that braid group replaces permutation group since the paths representing the flows permuting identical particles can be linked in $X^2 \times T$ and anyonic statistics [D1, D2] becomes possible. In the case of 2-D Minkowski space M^2 Jones inclusions with $\mathcal{M} : \mathcal{N} < 4$ plus a set of discrete values of $\mathcal{M} : \mathcal{N}$ in the range $(4,6)$ are possible. In [A118] some values are given ($\mathcal{M} : \mathcal{N} = 5, 5.5049..., 5.236...., 5.828...$).

At least intersections of future and past light cones seem to appear naturally in TGD framework such that the boundaries of future/past directed light cones serve as seats for incoming/outgoing states defined as intersections of space-time surface with these light cones. III_1 sectors cannot thus be excluded as factors in TGD framework. On the other hand, the construction of S-matrix at space-time level is reduced to II_1 case by effective 2-dimensionality.

2.5 TGD And Hyper-Finite Factors Of Type II_1

By effective 2-dimensionality of the construction of quantum states the hyper-finite factors of type II_1 fit naturally to TGD framework. In particular, infinite dimensional spinors define a canonical representations of this kind of factor. The basic question is whether only hyper-finite factors of type II_1 appear in TGD framework. Affirmative answer would allow to interpret physical M-matrix as time like entanglement coefficients.

2.5.1 What Kind Of Hyper-Finite Factors One Can Imagine In TGD?

The working hypothesis has been that only hyper-finite factors of type II_1 appear in TGD. The basic motivation has been that they allow a new view about M-matrix as an operator representable as time-like entanglement coefficients of zero energy states so that physical states would represent laws of physics in their structure. They allow also the introduction of the notion of measurement resolution directly to the definition of reaction probabilities by using Jones inclusion and the replacement of state space with a finite-dimensional state space defined by quantum spinors. This hypothesis is of course just an attractive working hypothesis and deserves to be challenged.

WCW spinors

For WCW spinor s the HFF II_1 property is very natural because of the properties of infinite-dimensional Clifford algebra and the inner product defined by the WCW geometry does not allow other factors than this. A good guess is that the values of conformal weights label the factors appearing in the tensor power defining WCW spinor s. Because of the non-degeneracy and super-symplectic symmetries the density matrix representing metric must be essentially unit matrix for each conformal weight which would be the defining characteristic of hyper-finite factor of type II_1.

Bosonic degrees of freedom

The bosonic part of the super-symplectic algebra consists of Hamiltonians of CH in one-one correspondence with those of $\delta M^4_\pm \times CP_2$. Also the Kac-Moody algebra acting leaving the light-likeness of the partonic 3-surfaces intact contributes to the bosonic degrees of freedom. The commutator of these algebras annihilates physical states and there are also Virasoro conditions associated with ordinary conformal symmetries of partonic 2-surface [K19] . The labels of Hamiltonians of WCW and spin indices contribute to bosonic degrees of freedom.

Hyper-finite factors of type II_1 result naturally if the system is an infinite tensor product finite-dimensional matrix algebra associated with finite dimensional systems [A29] . Unfortunately, neither Virasoro, symplectic nor Kac-Moody algebras do have decomposition into this kind of infinite tensor product. If bosonic degrees for super-symplectic and super-Kac Moody algebra indeed give I_∞ factor one has HFF if type II_∞. This looks the most natural option but threatens to spoil the beautiful idea about M-matrix as time-like entanglement coefficients between positive and negative energy parts of zero energy state.

The resolution of the problem is surprisingly simple and trivial after one has discovered it. The requirement that state is normalizable forces to project M-matrix to a finite-dimensional sub-space in bosonic degrees of freedom so that the reduction $I_\infty \to I_n$ occurs and one has the reduction $II_\infty \to II_1 \times I_n = II_1$ to the desired HFF.

One can consider also the possibility of taking the limit $n \to \infty$. One could indeed say that since I_∞ factor can be mapped to an infinite tensor power of $M(2, C)$ characterized by a state which is not trace, it is possible to map this representation to HFF by replacing state with trace [A29] . The question is whether the forcing the bosonic foot to fermionic shoe is physically natural. One could also regard the II_1 type notion of probability as fundamental and also argue that it is required by full super-symmetry realized also at the level of many-particle states rather than mere single particle states.

How the bosonic cutoff is realized?

Normalizability of state requires that projection to a finite-dimensional bosonic sub-space is carried out for the bosonic part of the M-matrix. This requires a cutoff in quantum numbers of super-conformal algebras. The cutoff for the values of conformal weight could be formulated by replacing integers with Z_n or with some finite field $G(p,1)$. The cutoff for the labels associated with Hamiltonians defined as an upper bound for the dimension of the representation looks also natural.

Number theoretical braids which are discrete and finite structures would define space-time correlate for this cutoff. p-Adic length scale $p \simeq 2^k$ hypothesis could be interpreted as stating the fact that only powers of p up to p^k are significant in p-adic thermodynamics which would correspond to finite field $G(k,1)$ if k is prime. This has no consequences for p-adic mass calculations since already the first two terms give practically exact results for the large primes associated with elementary particles [K49] .

Finite number of strands for the theoretical braids would serve as a correlate for the reduction of the representation of Galois group S_∞ of rationals to an infinite produce of diagonal copies of finite-dimensional Galois group so that same braid would repeat itself like a unit cell of lattice i condensed matter [K38] .

HFF of type III for field operators and HFF of type II_1 for states?

One could also argue that the Hamiltonians with fixed conformal weight are included in fermionic II_1 factor and bosonic factor I_∞ factor, and that the inclusion of conformal weights leads to a factor of type III. Conformal weight could relate to the integer appearing in the crossed product representation $III = Z \times_{cr} II_\infty$ of HFF of type III [A29] .

The value of conformal weight is non-negative for physical states which suggests that Z reduces to semigroup N so that a factor of type III would reduce to a factor of type II_∞ since trace would become finite. If unitary process corresponds to an automorphism for II_∞ factor, the action of automorphisms affecting scaling must be uni-directional. Also thermodynamical irreversibility suggests the same. The assumption that state function reduction for positive energy part of state implies unitary process for negative energy state and vice versa would only mean that the shifts for positive and negative energy parts of state are opposite so that $Z \rightarrow N$ reduction would still hold true.

HFF of type II_1 for the maxima of Kähler function?

Probabilistic interpretation allows to gain heuristic insights about whether and how hyper-finite factors of type type II_1 might be associated with WCW degrees of freedom. They can appear both in quantum fluctuating degrees of freedom associated with a given maximum of Kähler function and in the discrete space of maxima of Kähler function.

Spin glass degeneracy is the basic prediction of classical TGD and means that instead of a single maximum of Kähler function analogous to single free energy minimum of a thermodynamical system there is a fractal spin glass energy landscape with valleys inside valleys. The discretization of WCW in terms of the maxima of Kähler function crucial for the p-adicization problem, leads to the analog of spin glass energy landscape and hyper-finite factor of type II_1 might be the appropriate description of the situation.

The presence of the tensor product structure is a powerful additional constraint and something analogous to this should emerge in WCW degrees of freedom. Fractality of the many-sheeted space-time is a natural candidate here since the decomposition of the original geometric structure to parts and replacing them with the scaled down variant of original structure is the geometric analog of forming a tensor power of the original structure.

2.5.2 Direct Sum Of HFFs Of Type II_1 As A Minimal Option

HFF II_1 property for the Clifford algebra of WCW means a definite distinction from the ordinary Clifford algebra defined by the fermionic oscillator operators since the trace of the unit matrix of the Clifford algebra is normalized to one. This does not affect the anti-commutation relations at the basic level and delta functions can appear in them at space-time level. At the level of momentum space I_∞ property requires discrete basis and anti-commutators involve only Kronecker deltas. This conforms with the fact that HFF of type II_1 can be identified as the Clifford algebra associated with a separable Hilbert space.

II_∞ factor or direct sum of HFFs of type II_1?

The expectation is that super-symplectic algebra is a direct sum over HFFs of type II$_1$ labeled by the radial conformal weight. In the same manner the algebra defined by fermionic anti-commutation relations at partonic 2-surface would decompose to a direct sum of algebras labeled by the conformal weight associated with the light-like coordinate of X^3_l. Super-conformal symmetry suggests that also the configuration space degrees of freedom correspond to a direct sum of HFFs of type II_1.

One can of course ask why not $II_\infty = I_\infty \times II_1$ structures so that one would have single factor rather than a direct sum of factors.

1. The physical motivation is that the direct sum property allow to decompose M-matrix to direct summands associated with various sectors with weights whose moduli squared have an interpretation in terms of the density matrix. This is also consistent with p-adic thermodynamics where conformal weights take the place of energy eigen values.

2. II_∞ property would predict automorphisms scaling the trace by an arbitrary positive real number $\lambda \in R_+$. These automorphisms would require the scaling of the trace of the projectors of Clifford algebra having values in the range $[0, 1]$ and it is difficult to imagine how these automorphisms could be realized geometrically.

How HFF property reflects itself in the construction of geometry of WCW?

The interesting question is what HFF property and finite measurement resolution realizing itself as the use of projection operators means concretely at the level of WCW geometry.

Super-Hamiltonians define the Clifford algebra of the configuration space. Super-conformal symmetry suggests that the unavoidable restriction to projection operators instead of complex rays is realized also WCW degrees of freedom. Of course, infinite precision in the determination of the shape of 3-surface would be physically a completely unrealistic idea.

In the fermionic situation the anti-commutators for the gamma matrices associated with WCW individual Hamiltonians in 3-D sense are replaced with anti-commutators where Hamiltonians are replaced with projectors to subspaces of the space spanned by Hamiltonians. This projection is realized by restricting the anti-commutator to partonic 2-surfaces so that the anti-commutator depends only the restriction of the Hamiltonian to those surfaces.

What is interesting that the measurement resolution has a concrete particle physical meaning since the parton content of the system characterizes the projection. The larger the number of partons, the better the resolution about WCW degrees of freedom is. The degeneracy of WCW metric would be interpreted in terms of finite measurement resolution inherent to HFFs of type II_1, which is not due to Jones inclusions but due to the fact that one can project only to infinite-D subspaces rather than complex rays.

Effective 2-dimensionality in the sense that WCW Hamiltonians reduce to functionals of the partonic 2-surfaces of X^3_l rather than functionals of X^3_l could be interpreted in this manner. For a wide class of Hamiltonians actually effective 1-dimensionality holds true in accordance with conformal invariance.

The generalization of WCW Hamiltonians and super-Hamiltonians by allowing integrals over the 2-D boundaries of the patches of X^3_l would be natural and is suggested by the requirement of discretized 3-dimensionality at the level of WCW.

By quantum classical correspondence the inclusions of HFFs related to the measurement resolution should also have a geometric description. Measurement resolution corresponds to braids in given time scale and as already explained there is a hierarchy of braids in time scales coming as negative powers of two corresponding to the addition of zero energy components to positive/negative energy state. Note however that particle reactions understood as decays and fusions of braid strands could also lead to a notion of measurement resolution.

2.5.3 Bott Periodicity, Its Generalization, And Dimension $D = 8$As An Inherent Property Of The Hyper-Finite II_1 Factor

Hyper-finite II_1 factor can be constructed as infinite-dimensional tensor power of the Clifford algebra $M_2(C) = C(2)$ in dimension $D = 2$. More precisely, one forms the union of the Clifford algebras $C(2n) = C(2)^{\otimes n}$ of $2n$-dimensional spaces by identifying the element $x \in C(2n)$ as block diagonal elements $diag(x, x)$ of $C(2(n + 1))$. The union of these algebras is completed in weak operator topology and can be regarded as a Clifford algebra of real infinite-dimensional separable Hilbert space and thus as sub-algebra of I_∞. Also generalizations obtained by replacing complex numbers by quaternions and octions are possible.

1. The dimension 8 is an inherent property of the hyper-finite II_1 factor since Bott periodicity theorem states $C(n+8) = C_n(16)$. In other words, the Clifford algebra $C(n+8)$ is equivalent with the algebra of 16×16 matrices with entries in $C(n)$. Or articulating it still differently: $C(n+8)$ can be regarded as 16×16 dimensional module with $C(n)$ valued coefficients. Hence the elements in the union defining the canonical representation of hyper-finite II_1 factor are $16^n \times 16^n$ matrices having $C(0)$, $C(2)$, $C(4)$ or $C(6)$ valued valued elements.

2. The idea about a local variant of the infinite-dimensional Clifford algebra defined by power series of space-time coordinate with Taylor coefficients which are Clifford algebra elements fixes the interpretation. The representation as a linear combination of the generators of Clifford algebra of the finite-dimensional space allows quantum generalization only in the case of Minkowski spaces. However, if Clifford algebra generators are representable as gamma *matrices*, the powers of coordinate can be absorbed to the Clifford algebra and the local algebra is lost. Only if the generators are represented as quantum versions of octonions allowing no matrix representation because of their non-associativity, the local algebra makes sense. From this it is easy to deduce both quantum and classical TGD.

2.5.4 The Interpretation Of Jones Inclusions In TGD Framework

By the basic self-referential property of von Neumann algebras one can consider several interpretations of Jones inclusions consistent with sub-system-system relationship, and it is better to start by considering the options that one can imagine.

How Jones inclusions relate to the new view about sub-system?

Jones inclusion characterizes the embedding of sub-system $\mathcal{N}$ to $\mathcal{M}$and $\mathcal{M}$ as a finite-dimensional $\mathcal{N}$-module is the counterpart for the tensor product in finite-dimensional context. The possibility to express $\mathcal{M}$ as $\mathcal{N}$ module $\mathcal{M}/\mathcal{N}$ states fractality and can be regarded as a kind of self-referential "Brahman=Atman identity" at the level of infinite-dimensional systems.

Also the mysterious looking almost identity $CH^2 = CH$ for the WCW would fit nicely with the identity $M \oplus M = M$. $M \otimes M \subset M$ in WCW Clifford algebra degrees of freedom is also implied and the construction of $\mathcal{M}$ as a union of tensor powers of $C(2)$ suggests that $M \otimes M$ allows $\mathcal{M} : \mathcal{N} = 4$ inclusion to $\mathcal{M}$. This paradoxical result conforms with the strange self-referential property of factors of II_1.

The notion of many-sheeted space-time forces a considerable generalization of the notion of sub-system and simple tensor product description is not enough. Topological picture based on the length scale resolution suggests even the possibility of entanglement between sub-systems of un-entangled sub-systems. The possibility that hyper-finite II_1-factors describe the physics of TGD also in bosonic degrees of freedom is suggested by WCW super-symmetry. On the other hand, bosonic degrees could naturally correspond to I_∞ factor so that hyper-finite II_∞ would be the net result.

The most general view is that Jones inclusion describes all kinds of sub-system-system inclusions. The possibility to assign conformal field theory to the inclusion gives hopes of rather detailed view about dynamics of inclusion.

1. The topological condensation of space-time sheet to a larger space-time sheet mediated by wormhole contacts could be regarded as Jones inclusion. $\mathcal{N}$ would correspond to the condensing space-time sheet, $\mathcal{M}$ to the system consisting of both space-time sheets, and $\sqrt{\mathcal{M}:\mathcal{N}}$ would characterize the number of quantum spinorial degrees of freedom associated with the interaction between space-time sheets. Note that by general results $\mathcal{M}:\mathcal{N}$ characterizes the fractal dimension of quantum group ($\mathcal{M}:\mathcal{N}<4$) or Kac-Moody algebra ($\mathcal{M}:\mathcal{N}=4$) [B29] .

2. The branchings of space-time sheets (space-time surface is thus homologically like branching like of Feynman diagram) correspond naturally to n-particle vertices in TGD framework. What is nice is that vertices are nice 2-dimensional surfaces rather than surfaces having typically pinch singularities. Jones inclusion would naturally appear as inclusion of operator spaces $\mathcal{N}_i$ (essentially Fock spaces for fermionic oscillator operators) creating states at various lines as sub-spaces $N_i \subset M$ of operators creating states in common von Neumann factor $\mathcal{M}$. This would allow to construct vertices and vertices in natural manner using quantum groups or Kac-Moody algebras.

   The fundamental $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \otimes_N \mathcal{M}$ inclusion suggests a concrete representation based on the identification $N_i = M$, where M is the universal Clifford algebra associated with incoming line and $\mathcal{N}$ is defined by the condition that $\mathcal{M}/\mathcal{N}$ is the quantum variant of Clifford algebra of H. N-particle vertices could be defined as traces of Connes products of the operators creating incoming and outgoing states. It will be found that this leads to a master formula for S-matrix if the generalization of the old-fashioned string model duality implying that all generalized Feynman diagrams reduce to diagrams involving only single vertex is accepted.

3. If 4-surfaces can branch as the construction of vertices requires, it is difficult to argue that 3-surfaces and partonic/stringy 2-surfaces could not do the same. As a matter fact, the master formula for S-matrix to be discussed later explains the branching of 4-surfaces as an apparent effect. Despite this one can consider the possibility that this kind of joins are possible so that a new kind of mechanism of topological condensation would become possible. 3-space-sheets and partonic 2-surfaces whose p-adic fractality is characterized by different p-adic primes could be connected by "joins" representing branchings of 2-surfaces. The structures formed by soap film foam provide a very concrete illustration about what would happen. In the TGD based model of hadrons [K51] it has been assumed that join along boundaries bonds (JABs) connect quark space-time space-time sheets to the hadronic space-time sheet. The problem is that, at least for identical primes, the formation of join along boundaries bond fuses two systems to single bound state. If JABs are replaced joins, this objection is circumvented.

4. The space-time correlate for the formation of bound states is the formation of JABs. Standard intuition tells that the number of degrees of freedom associated with the bound state is smaller than the number of degrees of freedom associated with the pair of free systems. Hence the inclusion of the bound state to the tensor product could be regarded as Jones inclusion. On the other hand, one could argue that the JABs carry additional vibrational degrees of freedom so that the idea about reduction of degrees of freedom might be wrong: free system could be regarded as sub-system of bound state by Jones inclusion. The self-referential holographic properties of von Neumann algebras allow both interpretations: any system can be regarded as sub-system of any system in accordance with the bootstrap idea.

5. Maximal deterministic regions inside given space-time sheet bounded by light-like causal determinants define also sub-systems in a natural manner and also their inclusions would naturally correspond to Jones inclusions.

6. The TGD inspired model for topological quantum computation involves the magnetic flux tubes defined by join along boundaries bonds connecting space-time sheets having light-like boundaries. These tubes condensed to background 3-space can become linked and knotted

and code for quantum computations in this manner. In this case the addition of new strand to the system corresponds to Jones inclusion in the hierarchy associated with inclusion $\mathcal{N} \subset \mathcal{M}$. The anyon states associated with strands would be represented by a finite tensor product of quantum spinors assignable to $\mathcal{M}/\mathcal{N}$ and representing quantum counterpart of H-spinors.

One can regard $\mathcal{M} : \mathcal{N}$ degrees of freedom correspond to quantum group or Kac-Moody degrees of freedom. Quantum group degrees of freedom relate closely to the conformal and topological degrees of freedom as the connection of II_1 factors with topological quantum field theories and braid matrices suggests itself. For the canonical inclusion this factorization would correspond to factorization of quantum H-spinor from WCW spinor .

A more detailed study of canonical inclusions to be carried out later demonstrates what this factorization corresponds at the space-time level to a formation of space-time sheets which can be regarded as multiple coverings of M^4 and CP_2 with invariance group $G = G_a \times G_b \subset SL(2,C)\times SU(2)$, $SU(2) \subset SU(3)$. The unexpected outcome is that Planck constants assignable to M^4 and CP_2 degrees of freedom depend on the canonical inclusions. The existence of macroscopic quantum phases with arbitrarily large Planck constants is predicted.

It would seem possible to assign the $\mathcal{M} : \mathcal{N}$ degrees quantum spinorial degrees of freedom to the interface between subsystems represented by $\mathcal{N}$ and $\mathcal{M}$. The interface could correspond to the wormhole contacts, joins, JABs, or light-like causal determinants serving as boundary between maximal deterministic regions, etc... In terms of the bipartite diagrams representing the inclusions, joins (say) would correspond to the edges connecting white vertices representing sub-system (the entire system without the joins) to black vertices (entire system).

About the interpretation of $\mathcal{M} : \mathcal{N}$ degrees of freedom

The Clifford algebra $\mathcal{N}$ associated with a system formed by two space-time sheet can be regarded as $1 \leq \mathcal{M} : \mathcal{N} \leq 4$-dimensional module having $\mathcal{N}$ as its coefficients. It is possible to imagine several interpretations the degrees of freedom labeled by β.

1. The $\beta = \mathcal{M} : \mathcal{N}$ degrees of freedom could relate to the interaction of the space-time sheets. Beraha numbers appear in the construction of S-matrices of topological quantum field theories and an interpretation in terms of braids is possible. This would suggest that the interaction between space-time sheets can be described in terms of conformal quantum field theory and the S-matrices associated with braids describe this interaction. Jones inclusions would characterize the effective number of active conformal degrees of freedom. At $n = 3$ limit these degrees of freedom disappear completely since the conformal field theory defined by the Chern-Simons action describing this interaction would become trivial ($c = 0$ as will be found).

2. The interpretation in terms of embedding space Clifford algebra would suggest that β-dimensional Clifford algebra of $\sqrt{\beta}$-dimensional spinor space is in question. For $\beta = 4$ the algebra would be the Clifford algebra of 2-dimensional space. $\mathcal{M}/\mathcal{N}$ would have interpretation as complex quantum spinors with components satisfying $z_1z_2 = qz_2z_1$ and its conjugate and having fractal complex dimension $\sqrt{\beta}$. This would conform with the effective 2-dimensionality of TGD. For $\beta < 4$ the fractal dimension of partonic quantum spinors defining the basic conformal fields would be reduced and become $d = 1$ for $n = 3$: the interpretation is in terms of strong correlations caused by the non-commutativity of the components of quantum spinor. For number theoretical generalizations of infinite-dimensional Clifford algebras $Cl(C)$ obtained by replacing C with Abelian complexification of quaternions or octonions one would obtain higher-dimensional spinors.

2.5.5 WCW, Space-Time, Embedding Space AndHyper-Finite Type II_1 Factors

The preceding considerations have by-passed the question about the relationship of WCW tangent space to its Clifford algebra. Also the relationship between space-time and embedding space and their quantum variants could be better. In particular, one should understand how effective 2-dimensionality can be consistent with the 4-dimensionality of space-time.

Super-conformal symmetry and WCW Poisson algebra as hyper-finite type II_1 factor

It would be highly desirable to achieve also a description of the WCW degrees of freedom using von Neumann algebras. Super-conformal symmetry relating fermionic degrees of freedom and WCW degrees of freedom suggests that this might be the case. Super-symplectic algebra has as its generators configuration space Hamiltonians and their super-counterparts identifiable as CH gamma matrices. Super-symmetry requires that the Clifford algebra of CH and the Hamiltonian vector fields of CH with symplectic central extension both define hyper-finite II_1 factors. By super-symmetry Poisson bracket corresponds to an anti-commutator for gamma matrices. The ordinary quantized version of Poisson bracket is obtained as $\{P_i, Q_j\} \to [P_i, Q_j] = J_{ij} Id$. Finite trace version results by assuming that Id corresponds to the projector CH Clifford algebra having unit norm. The presence of zero modes means direct integral over these factors.

WCW gamma matrices anti-commuting to identity operator with unit norm corresponds to the tangent space $T(CH)$ of CH. Thus it would be not be surprising if $T(CH)$ could be imbedded in the sigma matrix algebra as a sub-space of operators defined by the gamma matrices generating this algebra. At least for $\beta = 4$ construction of hyper-finite II_1 factor this definitely makes sense.

The dimension of WCW defined as the trace of the projection operator to the sub-space spanned by gamma matrices is obviously zero. Thus WCW has in this sense the dimensionality of single space-time point. This sounds perhaps absurd but the generalization of the number concept implied by infinite primes indeed leads to the view that single space-time point is infinitely structured in the number theoretical sense although in the real sense all states of the point are equivalen. The reason is that there is infinitely many numbers expressible as ratios of infinite integers having unit real norm in the real sense but having different p-adic norms.

How to understand the dimensions of space-time and embedding space?

One should be able to understand the dimensions of 3-space, space-time and embedding space in a convincing matter in the proposed framework. There is also the question whether space-time and embedding space emerge uniquely from the mathematics of von Neumann algebras alone.

1. The dimensions of space-time and embedding space

Two sub-sequent inclusions dual to each other define a special kind of inclusion giving rise to a quantum counterpart of $D = 4$ naturally. This would mean that space-time is something which emerges at the level of cognitive states.

The special role of classical division algebras in the construction of quantum TGD [K74] , $D = 8$ Bott periodicity generalized to quantum context, plus self-referential property of type II_1 factors might explain why 8-dimensional embedding space is the only possibility.

State space has naturally quantum dimension $D \leq 8$ as the following simple argument shows. The space of quantum states has quark and lepton sectors which both are super-symmetric implying $D \leq 4$ for each. Since these sectors correspond to different Hamiltonian algebras (triality one for quarks and triality zero for leptonic sector), the state space has quantum dimension $D \leq 8$.

2. How the lacking two space-time dimensions emerge?

3-surface is the basic dynamical unit in TGD framework. This seems to be in conflict with the effective 2-dimensionality [K74] meaning that partonic 2-surface code for quantum states, and with the fact that hyper-finite II_1 factors have intrinsic quantum dimension 2.

A possible resolution of the problem is that the foliation of 3-surface by partonic two-surfaces defines a one-dimensional direct integral of isomorphic hyper-finite type II_1 factors, and the zero mode labeling the 2-surfaces in the foliation serves as the third spatial coordinate. For a given 3-surface the contribution to the WCW metric can come only from 2-D partonic surfaces defined as intersections of 3-D light-like CDs with $X^7_\pm$ [K20] . Hence the direct integral should somehow relate to the classical non-determinism of Kähler action.

1. The one-parameter family of intersections of light-like CD with $X^7_\pm$ inside $X^4 \cap X^7_\pm$ could indeed be basically due to the classical non-determinism of Kähler action. The contribution to the metric from the normal light-like direction to $X^3 = X^4 \cap X^7_\pm$ can cause the vanishing of the metric determinant $\sqrt{g_4}$ of the space-time metric at $X^2 \subset X^3$ under some conditions on X^2. This would mean that the space-time surface $X^4(X^3)$ is not uniquely determined by the

minimization principle defining the value of the Kähler action, and the complete dynamical specification of X^3 requires the specification of partonic 2-surfaces X_i^2 with $\sqrt{g_4} = 0$.

2. The known solutions of field equations [K10] define a double foliation of the space-time surface defined by Hamilton-Jacobi coordinates consisting of complex transversal coordinate and two light-like coordinates for M^4 (rather than space-time surface). Number theoretical considerations inspire the hypothesis that this foliation exists always [K74] . Hence a natural hypothesis is that the allowed partonic 2-surfaces correspond to the 2-surfaces in the restriction of the double foliation of the space-time surface by partonic 2-surfaces to X^3, and are thus locally parameterized by single parameter defining the third spatial coordinate.

3. There is however also a second light-like coordinate involved and one might ask whether both light-like coordinates appear in the direct sum decomposition of II_1 factors defining $T(CH)$. The presence of two kinds of light-like CDs would provide the lacking two space-time coordinates and quantum dimension $D = 4$ would emerge at the limit of full non-determinism. Note that the duality of space-like partonic and light-like stringy 2-surfaces conforms with this interpretation since it corresponds to a selection of partonic/stringy 2-surface inside given 3-D CD whereas the dual pairs correspond to different CDs.

4. That the quantum dimension would be $2D_q = \beta < 4$ above CP_2 length scale conforms with the fact that non-determinism is only partial and time direction is dynamically frozen to a high degree. For vacuum extremals there is strong non-determinism but in this case there is no real dynamics. For CP_2 type extremals, which are not vacuum extremals as far action and small perturbations are considered, and which correspond to $\beta = 4$ there is a complete non-determinism in time direction since the M^4 projection of the extremal is a light-like random curve and there is full 4-D dynamics. Light-likeness gives rise to conformal symmetry consistent with the emergence of Kac Moody algebra [K10] .

3. Time and cognition

In a completely deterministic physics time dimension is strictly speaking redundant since the information about physical states is coded by the initial values at 3-dimensional slice of space-time. Hence the notion of time should emerge at the level of cognitive representations possible by to the non-determinism of the classical dynamics of TGD.

Since Jones inclusion means the emergence of cognitive representation, the space-time view about physics should correspond to cognitive representations provided by Feynman diagram states with zero energy with entanglement defined by a two-sided projection of the lowest level S-matrix. These states would represent the “laws of quantum physics” cognitively. Also space-time surface serves as a classical correlate for the evolution by quantum jumps with maximal deterministic regions serving as correlates of quantum states. Thus the classical non-determinism making possible cognitive representations would bring in time. The fact that quantum dimension of space-time is smaller than $D = 4$ would reflect the fact that the loss of determinism is not complete.

4. Do space-time and embedding space emerge from the theory of von Neumann algebras and number theory?

The considerations above force to ask whether the notions of space-time and embedding space emerge from von Neumann algebras as predictions rather than input. The fact that it seems possible to formulate the S-matrix and its generalization in terms of inherent properties of infinite-dimensional Clifford algebras suggest that this might be the case.

Inner automorphisms as universal gauge symmetries?

The continuous outer automorphisms Δ^{it} of HFFs of type III are not completely unique and one can worry about the interpretation of the inner automorphisms. A possible resolution of the worries is that inner automorphisms act as universal gauge symmetries containing various super-conformal symmetries as a special case. For hyper-finite factors of type II_1 in the representation as an infinite tensor power of $M_2(C)$ this would mean that the transformations non-trivial in a finite number of tensor factors only act as analogs of local gauge symmetries. In the representation as a group algebra of S_∞ all unitary transformations acting on a finite number of braid strands act as gauge transformations whereas the infinite powers $P\times P\times ...$, $P\in S_n$, would act as counterparts of global gauge transformations. In particular, the Galois group of the closure of rationals would act as local gauge transformations but diagonally represented finite Galois groups would act like global gauge transformations and periodicity would make possible to have finite braids as space-time correlates without a loss of information.

Do unitary isomorphisms between tensor powers of II_1 define vertices?

What would be left would be the construction of unitary isomorphisms between the tensor products of the HFFs of type $II_1\otimes I_n = II_1$ at the partonic 2-surfaces defining the vertices. This would be the only new element added to the construction of braiding M-matrices.

As a matter fact, this element is actually not completely new since it generalizes the fusion rules of conformal field theories, about which standard example is the fusion rule $\phi_i = {c_i}^{jk}\phi_j\phi_k$ for primary fields. These fusion rules would tell how a state of incoming HFF decomposes to the states of tensor product of two outgoing HFFs.

These rules indeed have interpretation in terms of Connes tensor products $\mathcal{M}\otimes_{\mathcal{N}}...\otimes_{\mathcal{N}}\mathcal{M}$ for which the sub-factor $\mathcal{N}$ takes the role of complex numbers [A63] so that one has $\mathcal{M}$ becomes $\mathcal{N}$ bimodule and "quantum quantum states" have $\mathcal{N}$ as coefficients instead of complex numbers. In TGD framework this has interpretation as quantum measurement resolution characterized by $\mathcal{N}$ (the group G characterizing leaving the elements of $\mathcal{N}$ invariant defines the measured quantum numbers).

2.5.6 Quaternions, Octonions, And Hyper-Finite Type II_1Factors

Quaternions and octonions as well as their hyper counterparts obtained by multiplying imaginary units by commuting $\sqrt{-1}$ and forming a sub-space of complexified division algebra, are in in a central role in the number theoretical vision about quantum TGD [K74] . Therefore the question arises whether complexified quaternions and perhaps even octonions could be somehow inherent properties of von Neumann algebras. One can also wonder whether the quantum counterparts of quaternions and octonions could emerge naturally from von Neumann algebras. The following considerations allow to get grasp of the problem.

Quantum quaternions and quantum octonions

Quantum quaternions have been constructed as deformation of quaternions [A126] . The key observation that the Glebsch Gordan coefficients for the tensor product $3\otimes 3 = 5\oplus\oplus 3\oplus 1$ of spin 1 representation of $SU(2)$ with itself gives the anti-commutative part of quaternionic product as spin 1 part in the decomposition whereas the commutative part giving spin 0 representation is identifiable as the scalar product of the imaginary parts. By combining spin 0 and spin 1 representations, quaternionic product can be expressed in terms of Glebsh-Gordan coefficients. By replacing GGC:s by their quantum group versions for group $sl(2)_q$, one obtains quantum quaternions.

There are two different proposals for the construction of quantum octonions [A103, A1] . Also now the idea is to express quaternionic and octonionic multiplication in terms of Glebsch-Gordan coefficients and replace them with their quantum versions.

1. The first proposal [A103] relies on the observation that for the tensor product of $j = 3$ representations of $SU(2)$ the Glebsch-Gordan coefficients for $7\otimes 7\rightarrow 7$ in $7\otimes 7 = 9\oplus 7\oplus 5\oplus 3\oplus 1$ defines a product, which is equivalent with the antisymmetric part of the product of octonionic imaginary units. As a matter fact, the antisymmetry defines 7-dimensional Malcev algebra defined by the anti-commutator of octonion units and satisfying b definition the identity

$$[[x,y,z]\,,x] = [x,y,[x,z]] \quad , \quad [x,y,z]\equiv [x,[y,z]]+[y,[z,x]]+[z,[x,y]] \quad . \tag{2.5.1}$$

7-element Malcev algebra defining derivations of octonionic algebra is the only complex Malcev algebra not reducing to a Lie algebra. The $j = 0$ part of the product corresponds also now to scalar product for imaginary units. Octonions are constructed as sums of $j = 0$ and $j = 3$ parts and quantum Glebsch-Gordan coefficients define the octonionic product.

2. In the second proposal [A1] the quantum group associated with $SO(8)$ is used. This representation does not allow unit but produces a quantum version of octonionic triality assigning to three octonions a real number.

Quaternionic or octonionic quantum mechanics?

There have been numerous attempts to introduce quaternions and octonions to quantum theory. Quaternionic or octonionic quantum mechanics, which means the replacement of the complex numbers as coefficient field of Hilbert space with quaternions or octonions, is the most obvious approach (for example and references to the literature see for instance [A101] .

In both cases non-commutativity poses serious interpretational problems. In the octonionic case the non-associativity causes even more serious obstacles [B32, A101] , [B32] .

1. Assuming that an orthonormalized state basis with respect to an octonion valued inner product has been found, the multiplication of any basis with octonion spoils the orthonormality. The proposal to circumvent this difficulty discussed in [B32] , [B32] eliminates non-associativity by assuming that octonions multiply states one by one (rather than multiplying each other before multiplying the state). Effectively this means that octonions are replaced with 8×8-matrices.

2. The definition of the tensor product leads also to difficulties since associativity is lost (recall that Yang-Baxter equation codes for associativity in case of braid statistics [A42]).

3. The notion of hermitian conjugation is problematic and forces a selection of a preferred imaginary unit, which does not look nice. Note however that the local selection of a preferred imaginary unit is in a key role in the proposed construction of space-time surfaces as

   hyper-quaternionic or co-hyper-quaternionic surfaces and allows to interpret space-time surfaces either as surfaces in 8-D Minkowski space M^8 of hyper-octonions or in $M^4 \times CP_2$. This selection turns out to have quite different interpretation in the proposed framework.

Hyper-finite factor II_1 has a natural Hyper-Kähler structure

In the case of hyper-finite factors of type II_1 quaternions a more natural approach is based on the generalization of the Hyper-Kähler structure rather than quaternionic quantum mechanics. The reason is that also WCW tangent space should and is expected to have this structure [K20] . The Hilbert space remains a complex Hilbert space but the quaternionic units are represented as operators in Hilbert space. The selection of the preferred unit is necessary and natural. The identity operator representing quaternionic real unit has trace equal to one, is expected to give rise to the series of quantum quaternion algebras in terms of inclusions $\mathcal{N} \subset \mathcal{M}$ having interpretation as N-modules.

The representation of the quaternion units is rather explicit in the structure of hyper-finite II_1 factor. The $\mathcal{M} : \mathcal{N} \equiv \beta = 4$ hierarchical construction can be regarded as Connes tensor product of infinite number of 4-D Clifford algebras of Euclidian plane with Euclidian signature of metric ($diag(-1,-1)$). This algebra is nothing but the quaternionic algebra in the representation of quaternionic imaginary units by Pauli spin matrices multiplied by i.

The imaginary unit of the underlying complex Hilbert space must be chosen and there is whole sphere S^2 of choices and in every point of WCW the choice can be made differently. The space-time correlate for this local choice of preferred hyper-octonionic unit [K74] . At the level of WCW geometry the quaternion structure of the tangent space means the existence of Hyper-Kähler structure guaranteeing that WCW has a vanishing Einstein tensor. It it would not vanish, curvature scalar would be infinite by symmetric space property (as in case of loop spaces) and induce a divergence in the functional integral over 3-surfaces from the expansion of $\sqrt{g}$ [K20] .

The quaternionic units for the II_1 factor, are simply limiting case for the direct sums of 2×2 units normalized to one. Generalizing from $\beta = 4$ to $\beta < 4$, the natural expectation is that the representation of the algebra as $\beta = \mathcal{M} : \mathcal{N}$-dimensional $\mathcal{N}$-module gives rise to quantum quaternions with quaternion units defined as infinite sums of $\sqrt{\beta} \times \sqrt{\beta}$ matrices.

At Hilbert space level one has an infinite Connes tensor product of 2-component spinor spaces on which quaternionic matrices have a natural action. The tensor product of Clifford algebras gives the algebra of 2×2 quaternionic matrices acting on 2-component quaternionic spinors (complex 4-component spinors). Thus double inclusion could correspond to (hyper-)quaternionic structure at space-time level. Note however that the correspondence is not complete since hyper-quaternions appear at space-time level and quaternions at Hilbert space level.

Von Neumann algebras and octonions

The octonionic generalization of the Hyper-Kähler manifold does not make sense as such since octonionic units are not representable as linear operators. The allowance of anti-linear operators inherently present in von Neumann algebras could however save the situation. Indeed, the Cayley-Dickson construction for the division algebras (for a nice explanation see [A94]), which allows to extend any * algebra, and thus also any von Neumann algebra, by adding an imaginary unit it and identified as *, comes in rescue.

The basic idea of the Cayley-Dickson construction is following. The * operator, call it J, representing a conjugation defines an *anti-linear* operator in the original algebra A. One can extend A by adding this operator as a new element to the algebra. The conditions satisfied by J are

$$a(Jb) = J(a^*b) \ \ , \quad (aJ)b = (ab^*)J \ \ , \quad (Ja)(bJ^{-1}) = (ab)^* \ \ . \tag{2.5.2}$$

In the associative case the conditions are equivalent to the first condition.

It is intuitively clear that this addition extends the hyper-Kähler structure to an octonionic structure at the level of the operator algebra. The quantum version of the octonionic algebra is fixed by the quantum quaternion algebra uniquely and is consistent with the Cayley-Dickson construction. It is not clear whether the construction is equivalent with either of the earlier proposals [A103, A1] . It would however seem that the proposal is simpler.

Physical interpretation of quantum octonion structure

Without further restrictions the extension by J would mean that vertices contain operators, which are superpositions of linear and anti-linear operators. This would give superpositions of states and their time-reversals and mean that state could be a superposition of states with opposite values of say fermion numbers. The problem disappears if either the linear operators A or anti-linear operators JA can be used to construct physical states from vacuum. The fact, that space-time surfaces are either hyper-quaternionic or co-hyper-quaternionic, is a space-time correlate for this restriction.

The $HQ - coHQ$ duality discussed in [K74] states that the descriptions based on hyper-quaternionic and co-hyper-quaternionic surfaces are dual to each other. The duality can have two meanings.

1. The vacuum is invariant under J so that one can use either complexified quaternionic operators A or their co-counterparts of form JA to create physical states from vacuum.

2. The vacuum is not invariant under J. This could relate to the breaking of CP and T invariance known to occur in meson-antimeson systems. In TGD framework two kinds of vacua are predicted corresponding intuitively to vacua in which either the product of all positive or negative energy fermionic oscillator operators defines the vacuum state, and these two vacua could correspond to a vacuum and its J conjugate, and thus to positive and negative energy states. In this case the two state spaces would not be equivalent although the physics associated with them would be equivalent.

The considerations of [K74] related to the detailed dynamics of $HQ - coHQ$ duality demonstrate that the variational principles defining the dynamics of hyper-quaternionic and co-hyper-quaternionic space-time surfaces are antagonistic and correspond to world as seen by a conscientous book-keeper on one hand and an imaginative artist on the other hand. HQ case is conservative: differences measured by the magnitude of Kähler action tend to be minimized, the dynamics is highly predictive, and minimizes the classical energy of the initial state. $coHQ$ case is radical: differences are maximized (this is what the construction of sensory representations would require). The interpretation proposed in [K74] was that the two space-time dynamics are just different predictions for what would happen (has happened) if no quantum jumps would occur (had occurred). A stronger assumption is that these two views are associated with systems related by time reversal symmetry.

What comes in mind first is that this antagonism follows from the assumption that these dynamics are actually time-reversals of each other with respect to M^4 time (the rapid elimination of differences in the first dynamics would correspond to their rapid enhancement in the second dynamics). This is not the case so that T and CP symmetries are predicted to be broken in accordance with the CP breaking in meson-antimeson systems [K46] and cosmological matter-antimatter asymmetry [K69] .

2.5.7 Does The Hierarchy Of Infinite Primes Relate To The Hierarchy Of II_1 Factors?

The hierarchy of Feynman diagrams accompanying the hierarchy defined by Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset ...$ gives a concrete representation for the hierarchy of cognitive dynamics providing a representation for the material world at the lowest level of the hierarchy. This hierarchy seems to relate directly to the hierarchy of space-time sheets.

Also the construction of infinite primes [K72] leads to an infinite hierarchy. Infinite primes at the lowest level correspond to polynomials of single variable x_1 with rational coefficients, next level to polynomials x_1 for which coefficients are rational functions of variable x_2, etc... so that a natural ordering of the variables is involved.

If the variables x_i are hyper-octonions (subs-space of complexified octonions for which elements are of form $x + \sqrt{-1}y$, where x is real number and y imaginary octonion and $\sqrt{-1}$ is commuting imaginary unit, this hierarchy of states could provide a realistic representation of physical states as far as quantum numbers related to embedding space degrees of freedom are considered in M^8 picture dual to $M^4 \times CP_2$ picture [K74] . Infinite primes are mapped to space-time surfaces in a way analogous to the mapping of polynomials to the loci of their zeros so that infinite primes, integers, and rationals become concrete geometrical objects.

Infinite primes are also obtained by a repeated second quantization of a super-symmetric arithmetic quantum field theory. Infinite rational numbers correspond in this description to pairs of positive energy and negative energy states of opposite energies having interpretation as pairs of initial and final states so that higher level states indeed represent transitions between the states. For these reasons this hierarchy has been interpreted as a correlate for a cognitive hierarchy coding information about quantum dynamics at lower levels. This hierarchy has also been assigned with the hierarchy of space-time sheets. Just as the hierarchy of generalized Feynman diagrams provides self representations of the lowest matter level and is coded by it, finite primes code the hierarchy of infinite primes.

Infinite primes, integers, and rationals have finite p-adic norms equal to 1, and one can wonder whether a Hilbert space like structure with dimension given by an infinite prime or integer makes sense, and whether it has anything to do with the Hilbert space for which dimension is infinite in the sense of the limiting value for a dimension of sub-space. The Hilbert spaces with dimension equal to infinite prime would define primes for the tensor product of these spaces. The dimension of this kind of space defined as any p-adic norm would be equal to one.

One cannot exclude the possibility that infinite primes could express the infinite dimensions of hyper-finite III_1 factors, which cannot be excluded and correspond to that part of quantum TGD which relates to the embedding space rather than space-time surface. Indeed, infinite primes code naturally for the quantum numbers associated with the embedding space. Secondly, the appearance of 7-D light-like causal determinants $X^7_\pm = M^4_\pm \times CP_2$ forming nested structures in the construction of S-matrix brings in mind similar nested structures of algebraic quantum field theory [B8] . If this is were the case, the hierarchy of Beraha numbers possibly associated with the phase resolution could correspond to hyper-finite factors of type II_1, and the decomposition of space-time surface to regions labeled by p-adic primes and characterized by infinite primes could correspond to hyper-finite factors of type III_1 and represent embedding space degrees of freedom.

The state space would in this picture correspond to the tensor products of hyper-finite factors of type II_1 and III_1 (of course, also factors I_n and I_∞ are also possible). III_1 factors could be assigned to the sub-WCWs defined by 3-surfaces in regions of M^4 expressible in terms of unions and intersections of $X^7_\pm = M^4_\pm \times CP_2$. By conservation of four-momentum, bounded regions of this kind are possible only for the states of zero net energy appearing at the higher levels of hierarchy. These sub-WCWs would be characterized by the positions of the tips of light cones $M^4_\pm \subset M^4$ involved. This indeed brings in continuous spectrum of four-momenta forcing to introduce non-separable Hilbert spaces for momentum eigen states and necessitating III_1 factors. Infinities would be avoided since the dynamics proper would occur at the level of space-time surfaces and involve only II_1 factors.

2.6 HFFs Of Type III And TGD

One can imagine several ways for how HFFs of type III could emerge in TGD although the proposed view about M-matrix in zero energy ontology suggests that HFFs of type III_1 should be only an auxiliary tool at best. Same is suggested with interpretational problems associated with them. Both TGD inspired quantum measurement theory, the idea about a variant of HFF of type II_1 analogous to a local gauge algebra, and some other arguments, suggest that HFFs of type III could be seen as a useful idealization allowing to make non-trivial conjectures both about quantum TGD and about HFFs of type III. Quantum fields would correspond to HFFs of type III and II_∞ whereas physical states (M-matrix) would correspond to HFF of type II_1. I have summarized first the problems of III_1 factors so that reader can decide whether the further reading is worth of it.

2.6.1 Problems Associated With The Physical Interpretation Of III_1 Factors

Algebraic quantum field theory approach [B26, B8] has led to a considerable understanding of relativistic quantum field theories in terms of hyper-finite III_1 factors. There are however several reasons to suspect that the resulting picture is in conflict with physical intuition. Also the infinities of non-trivial relativistic QFTs suggest that something goes wrong.

Are the infinities of quantum field theories due the wrong type of von Neumann algebra?

The infinities of quantum field theories involve basically infinite traces and it is now known that the algebras of observables for relativistic quantum field theories for bounded regions of Minkowski space correspond to hyper-finite III_1 algebras, for which non-trivial traces are always infinite. This might be the basic cause of the divergence problems of relativistic quantum field theory.

On basis of this observations there is some temptation to think that the finite traces of hyper-finite II_1 algebras might provide a resolution to the problems but not necessarily in QFT context. One can play with the thought that the subtraction of infinities might be actually a process in which III_1 algebra is transformed to II_1 algebra. A more plausible idea suggested by dimensional regularization is that the elimination of infinities actually gives rise to II_1 inclusion at the limit $\mathcal{M} : \mathcal{N} \rightarrow 4$. It is indeed known that the dimensional regularization procedure of quantum field theories can be formulated in terms of bi-algebras assignable to Feynman diagrams and [A30] and the emergence of bi-algebras suggests that a connection with II_1 factors and critical role of dimension $D = 4$ might exist.

Continuum of inequivalent representations of commutation relations

There is also a second difficulty related to type III algebras. There is a continuum of inequivalent representations for canonical commutation relations [A117] . In thermodynamics this is blessing since temperature parameterizes these representations. In quantum field theory context situation is however different and this problem has been usually put under the rug.

Entanglement and von Neumann algebras

In quantum field theories where 4-D regions of space-time are assigned to observables. In this case hyper-finite type III_1 von Neumann factors appear. Also now inclusions make sense and has been studiedin fact, the parameters characterizing Jones inclusions appear also now and this due to the very general properties of the inclusions.

The algebras of type III_1 have rather counter-intuitive properties from the point of view of entanglement. For instance, product states between systems having space-like separation are not possible at all so that one can speak of intrinsic entanglement [A81] . What looks worse is that the decomposition of entangled state to product states is highly non-unique.

Mimicking the steps of von Neumann one could ask what the notion of observables could mean in TGD framework. Effective 2-dimensionality states that quantum states can be constructed using the data given at partonic or stringy 2-surfaces. This data includes also information about normal derivatives so that 3-dimensionality actually lurks in. In any case this would mean that observables are assignable to 2-D surfaces. This would suggest that hyper-finite II_1 factors appear in quantum TGD at least as the contribution of single space-time surface to S-matrix is considered. The contributions for WCW degrees of freedom meaning functional (not path-) integral over 3-surfaces could of course change the situation.

Also in case of II_1 factors, entanglement shows completely new features which need not however be in conflict with TGD inspired view about entanglement. The eigen values of density matrices are infinitely degenerate and quantum measurement can remove this degeneracy only partially. TGD inspired theory of consciousness has led to the identification of rational (more generally algebraic entanglement) as bound state entanglement stable in state function reduction. When an infinite number of states are entangled, the entanglement would correspond to rational (algebraic number) valued traces for the projections to the eigen states of the density matrix. The symplectic transformations of CP_2 are almost $U(1)$ gauge symmetries broken only by classical gravitation. They imply a gigantic spin glass degeneracy which could be behind the infinite degeneracies of eigen states of density matrices in case of II_1 factors.

2.6.2 Quantum Measurement Theory And HFFs Of Type III

The attempt to interpret the HFFs of type III in terms of quantum measurement theory based on Jones inclusions leads to highly non-trivial conjectures about these factors.

Could the scalings of trace relate to quantum measurements?

What should be understood is the physical meaning of the automorphism inducing the scaling of trace. In the representation based of factors based on infinite tensor powers the action of g should transform single $n \times n$ matrix factor with density matrix Id/n to a density matrix e_{11} of a pure state.

Obviously the number of degrees of freedom is affected and this can be interpreted in terms of appearance or disappearance of correlations. Quantization and emergence of non-commutativity indeed implies the emergence of correlations and effective reduction of degrees of freedom. In particular, the fundamental quantum Clifford algebra has reduced dimension $\mathcal{M} : \mathcal{N} = r \leq 4$ instead of $r = 4$ since the replacement of complex valued matrix elements with $\mathcal{N}$ valued ones implies non-commutativity and correlations.

The transformation would be induced by the shift of finite-dimensional state to right or left so that the number of matrix factors overlapping with I_∞ part increases or is reduced. Could it have interpretation in terms of quantum measurement for a quantum Clifford factor? Could quantum measurement for $\mathcal{M}/\mathcal{N}$ degrees of freedom reducing the state in these degrees of freedom to a pure state be interpreted as a transformation of single finite-dimensional matrix factor to a type I factor inducing the scaling of the trace and could the scalings associated with automorphisms of HFFs of type III also be interpreted in terms of quantum measurement?

This interpretation does not as such say anything about HFF factors of type III since only a decomposition of II_1 factor to I_2^k factor and II_1 factor with a reduced trace of projector to the latter. However, one can ask whether the scaling of trace for HFFs of type III could correspond to a situation in which infinite number of finite-dimensional factors have been quantum measured. This would correspond to the inclusion $\mathcal{N} \subset \mathcal{M}_\infty = \cup_n \mathcal{M}_n$ where $\mathcal{N} \subset \mathcal{M} \subset ...\mathcal{M}_n...$ defines the canonical inclusion sequence. Physicist can of course ask whether the presence of infinite number of I_2-, or more generally, I_n-factors is at all relevant to quantum measurement and it has already become clear that situation at the level of M-matrix reduces to I_n.

Could the theory of HHFs of type III relate to the theory of Jones inclusions?

The idea about a connection of HFFs of type III and quantum measurement theory seems to be consistent with the basic facts about inclusions and HFFs of type III_1.

1. Quantum measurement would scale the trace by a factor $2^k/\sqrt{\mathcal{M}:\mathcal{N}}$ since the trace would become a product for the trace of the projector to the newly born $M(2,C)^{\otimes k}$ factor and the trace for the projection to $\mathcal{N}$ given by $1/\sqrt{\mathcal{M}:\mathcal{N}}$. The continuous range of values $\mathcal{M}:\mathcal{N}\geq 4$ gives good hopes that all values of λ are realized. The prediction would be that $2^k\sqrt{\mathcal{M}:\mathcal{N}}\geq 1$ holds always true.

2. The values $\mathcal{M}:\mathcal{N}\in\{r_n=4cos^2(\pi/n)\}$ for which the single $M(2,C)$ factor emerges in state function reduction would define preferred values of the inverse of $\lambda=\sqrt{\mathcal{M}:\mathcal{N}/4}$ parameterizing factors III_λ. These preferred values vary in the range $[1/2,1]$.

3. $\lambda=1$ at the end of continuum would correspond to HFF III_1 and to Jones inclusions defined by infinite cyclic subgroups dense in $U(1)\subset SU(2)$ and this group combined with reflection. These groups correspond to the Dynkin diagrams A_∞ and D_∞. Also the classical values of $\mathcal{M}:\mathcal{N}=n^2$ characterizing the dimension of the quantum Clifford $\mathcal{M}:\mathcal{N}$ are possible. In this case the scaling of trace would be trivial since the factor n to the trace would be compensated by the factor $1/n$ due to the disappearance of $\mathcal{M}/\mathcal{N}$ factor III_1 factor.

4. Inclusions with $\mathcal{M}:\mathcal{N}=\infty$ are also possible and they would correspond to $\lambda=0$ so that also III_0 factor would also have a natural identification in this framework. These factors correspond to ergodic systems and one might perhaps argue that quantum measurement in this case would give infinite amount of information.

5. This picture makes sense also physically. p-Adic thermodynamics for the representations of super-conformal algebra could be formulated in terms of factors of type I_∞ and in excellent approximation using factors I_n. The generation of arbitrary number of type II_1 factors in quantum measurement allow this possibility.

The end points of spectrum of preferred values of λ are physically special

The fact that the end points of the spectrum of preferred values of λ are physically special, supports the hopes that this picture might have something to do with reality.

1. The Jones inclusion with $q=exp(i\pi/n)$, $n=3$ (with principal diagram reducing to a Dynkin diagram of group SU(3)) corresponds to $\lambda=1/2$, which corresponds to HFF III_1 differing in essential manner from factors III_λ, $\lambda<1$. On the other hand, $SU(3)$ corresponds to color group which appears as an isometry group and important subgroup of automorphisms of octonions thus differs physically from the ADE gauge groups predicted to be realized dynamically by the TGD based view about McKay correspondence [K38] .

2. For $r=4$ SU(2) inclusion parameterized by extended ADE diagrams $M(2,C)^{\otimes 2}$ would be created in the state function reduction and also this would give $\lambda=1/2$ and scaling by a factor of 2. Hence the end points of the range of discrete spectrum would correspond to the same scaling factor and same HFF of type III. SU(2) could be interpreted either as electro-weak gauge group, group of rotations of th geodesic sphere of $\delta M^4_\pm$, or a subgroup of SU(3). In TGD interpretation for McKay correspondence a phase transition replacing gauge symmetry with Kac-Moody symmetry.

3. The scalings of trace by factor 2 seem to be preferred physically which should be contrasted with the fact that primes near prime powers of 2 and with the fact that quantum phases $q=exp(i\pi/n)$ with n equal to Fermat integer proportional to power of 2 and product of the Fermat primes (the known ones are 5, 17, 257, and $2^{16}+1$) are in a special role in TGD Universe.

2.6.3 What Could One Say About II_1 Automorphism Associated With The II_∞ Automorphism Defining Factor Of Type III?

An interesting question relates to the interpretation of the automorphisms of II_∞ factor inducing the scaling of trace.

1. If the automorphism for Jones inclusion involves the generator of cyclic automorphism subgroup Z_n of II_1 factor then it would seem that for other values of λ this group cannot be cyclic. SU(2) has discrete subgroups generated by arbitrary phase q and these are dense in $U(1) \subset SU(2)$ sub-group. If the interpretation in terms of Jones inclusion makes sense then the identification $\lambda = \sqrt{\mathcal{M} : \mathcal{N}}/2^k$ makes sense.

2. If HFF of type II_1 is realized as group algebra of infinite symmetric group [K38] , the outer automorphism induced by the diagonally imbedded finite Galois groups can induce only integer values of n and Z_n would correspond to cyclic subgroups. This interpretation conforms with the fact that the automorphisms in the completion of inner automorphisms of HFF of type II_1 induce trivial scalings. Therefore only automorphisms which do not belong to this completion can define HFFs of type III.

2.6.4 What Could Be The Physical Interpretation Of Two Kinds Of Invariants Associated With HFFs Type III?

TGD predicts two kinds of counterparts for S-matrix: M-matrix and U-matrix. Both are expected to be more or less universal.

There are also *two* kinds of invariants and automorphisms associated with HFFs of type III.

1. The first invariant corresponds to the scaling $\lambda \in]0,1[$ of the trace associated with the automorphism of factor of II_∞. Also the end points of the interval make sense. The inverse of this scaling accompanies the inverse of this automorphism.

2. Second invariant corresponds to the time scales $t = T_0$ for which the outer automorphism σ_t reduces to inner automorphism. It turns out that T_0 and λ are related by the formula $\lambda^{iT_0} = 1$, which gives the allowed values of T_0 as $T_0 = n2\pi/log(\lambda)$ [A29] . This formula can be understood intuitively by realizing that λ corresponds to the eigenvalue of the density matrix $\Delta = e^H$ in the simplest possible realization of the state ϕ.

The presence of two automorphisms and invariants brings in mind U matrix characterizing the unitary process occurring in quantum jump and M-matrix characterizing time like entanglement.

1. If one accepts the vision based on quantum measurement theory then λ corresponds to the scaling of the trace resulting when quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ reduces to a tensor power of $M(2,C)$ factor in the state function reduction. The proposed interpretation for U process would be as the inverse of state function reduction transforming this factor back to $\mathcal{M}/\mathcal{N}$. Thus U process and state function reduction would correspond naturally to the scaling and its inverse. This picture might apply not only in single particle case but also for zero energy states which can be seen as states associated the a tensor power of HFFs of type II_1 associated with partons.

2. The implication is that U process can occur only in the direction in which trace is reduced. This would suggest that the full III_1 factor is not a physical notion and that one must restrict the group Z in the crossed product $Z \times_{cr} II_\infty$ to the group N of non-negative integers. In this kind of situation the trace is well defined since the traces for the terms in the crossed product comes as powers λ^{-n} so that the net result is finite. This would mean a reduction to II_∞ factor.

3. Since time t is a natural parameter in elementary particle physics experiment, one could argue that σ_t could define naturally M-matrix. Time parameter would most naturally correspond to a parameter of scaling affecting all $M^4_\pm$ coordinates rather than linear time. This conforms also with the fundamental role of conformal transformations and scalings in TGD framework.

The identification of the full M-matrix in terms of σ does not seem to make sense generally. It would however make sense for incoming and outgoing number theoretic braids so that σ could define universal braiding M-matrices. Inner automorphisms would bring in the dependence on experimental situation. The reduction of the braiding matrix to an inner automorphism for critical values of t which could be interpreted in terms of scaling by power of p. This trivialization would be a counterpart for the elimination of propagator legs from M-matrix element. Vertex itself could be interpreted as unitary isomorphism between tensor product of incoming and outgoing HFFs of type II_1 would code all what is relevant about the particle reaction.

2.6.5 Does The Time Parameter T Represent Time Translation Or Scaling?

The connection $T_n = n2\pi/log(\lambda)$ would give a relationship between the scaling of trace and value of time parameter for which the outer automorphism represented by σ reduces to inner automorphism. It must be emphasized that the time parameter t appearing in σ need not have anything to do with time translation. The alternative interpretation is in terms of $M^4_\pm$ scaling (implying also time scaling) but one cannot exclude even preferred Lorentz boosts in the direction of quantization axis of angular momentum.

Could the time parameter correspond to scaling?

The central role of conformal invariance in quantum TGD suggests that t parameterizes scaling rather than translation. In this case scalings would correspond to powers of $(K\lambda)^n$. The numerical factor K which cannot be excluded a priori, seems to reduce to $K = 1$.

1. The scalings by powers of p have a simple realization in terms of the representation of HFF of type II_∞ as infinite tensor power of $M(p, C)$ with suitably chosen densities matrices in factors to get product of I_∞ and II_1 factor. These matrix algebras have the remarkable property of defining prime tensor power factors of finite matrix algebras. Thus p-adic fractality would reflect directly basic properties of matrix algebras as suggested already earlier. That scalings by powers of p would correspond to automorphism reducing to inner automorphisms would conform with p-adic fractality.

2. Also scalings by powers $[\sqrt{\mathcal{M} : \mathcal{N}}/2^k]^n$ would be physically preferred if one takes previous arguments about Jones inclusions seriously and if also in this case scalings are involved. For $q = exp(i\pi/n)$, $n = 5$ the minimal value of n allowing universal topological quantum computation would correspond to a scaling by Golden Mean and these fractal scalings indeed play a key role in living matter. In particular, Golden Mean makes it visible in the geometry of DNA.

Could the time parameter correspond to time translation?

One can consider also the interpretation of σ_t as time translation. TGD predicts a hierarchy of Planck constants parameterized by rational numbers such that integer multiples are favored. In particular, integers defining ruler and compass polygons are predicted to be in a very special role physically. Since the geometric time span associated with zero energy state should scale as Planck constant one expects that preferred values of time t associated with σ are quantized as rational multiples of some fundamental time scales, say the basic time scale defined by CP_2 length or p-adic time scales.

1. For $\lambda = 1/p$, p prime, the time scale would be $T_n = nT_1$, $T_1 = T_0 = 2\pi/log(p)$ which is not what p-adic length scale hypothesis would suggest.

2. For Jones inclusions one would have $T_n/T_0 = n2\pi/log(2^{2k}/\mathcal{M} : \mathcal{N})$. In the limit when λ becomes very small (the number k of reduced $M(2, C)$ factors is large one obtains $T_n = (n/k)t_1$, $T_1 = T_0\pi/log(2)$. Approximate rational multiples of the basic length scale would be obtained as also predicted by the general quantization of Planck constant.

p-Adic thermodynamics from first principles

Quantum field theory at non-zero temperature can be formulated in the functional integral formalism by replacing the time parameter associated with the unitary time evolution operator $U(t)$ with a complexified time containing as imaginary part the inverse of the temperature: $t \rightarrow t + i\hbar/T$. In the framework of standard quantum field theory this is a mere computational trick but the time parameter associated with the automorphisms σ_t of HFF of type III is a temperature like parameter from the beginning, and its complexification would naturally lead to the analog of thermal QFT.

Thus thermal equilibrium state would be a genuine quantum state rather than fictive but useful auxiliary notion. Thermal equilibrium is defined separately for each incoming parton braid and perhaps even braid (partons can have arbitrarily large size). At elementary particle level p-adic thermodynamics could be in question so that particle massivation would have first principle description. p-Adic thermodynamics is under relatively mild conditions equivalent with its real counterpart obtained by the replacement of p^{L_0} interpreted as a p-adic number with p^{-L_0} interpreted as a real number.

2.6.6 HFFs Of Type III And The Dynamics In $M^4_\pm$ Degrees Of Freedom?

HFFs of type III could be also assigned with the poorly understood dynamics in $M^4_\pm$ degrees of freedom which should have a lot of to do with four-dimensional quantum field theory. Hyper-finite factors of type III_1 might emerge when one extends II_1 to a local algebra by multiplying it with hyper-octonions replaced as analog of matrix factor and considers hyper-quaternionic subalgebra. The resulting algebra would be the analog of local gauge algebra and the elements of algebra would be analogous to conformal fields with complex argument replaced with hyper-octonionic, -quaternionic, or -complex one. Since quantum field theory in M^4 gives rise to hyper-finite III_1 factors one might guess that the hyper-quaternionic restriction indeed gives these factors.

The expansion of the local HFF II_∞ element as $O(m) = \sum_n m^n O_n$, where M^4 coordinate m is interpreted as hyper-quaternion, could have interpretation as expansion in which O_n belongs to $\mathcal{N} g^n$ in the crossed product $\mathcal{N} \times_{cr} \{g^n, n \in Z\}$. The analogy with conformal fields suggests that the power g^n inducing λ^n fold scaling of trace increases the conformal weight by n.

One can ask whether the scaling of trace by powers of λ defines an inclusion hierarchy of sub-algebras of conformal sub-algebras as suggested by previous arguments. One such hierarchy would be the hierarchy of sub-algebras containing only the generators O_m with conformal weight $m \geq n$, $n \in Z$.

It has been suggested that the automorphism Δ could correspond to scaling inside light-cone. This interpretation would fit nicely with Lorentz invariance and TGD in general. The factors III_λ with λ generating semi-subgroups of integers (in particular powers of primes) could be of special physical importance in TGD framework. The values of t for which automorphism reduces to inner automorphism should be of special physical importance in TGD framework. These automorphisms correspond to scalings identifiable in terms of powers of p-adic prime p so that p-adic fractality would find an explanation at the fundamental level.

If the above mentioned expansion in powers of m^n of $M^4_\pm$ coordinate makes sense then the action of σ^t representing a scaling by p^n would leave the elements O invariant or induce a mere inner automorphism. Conformal weight n corresponds naturally to n-ary p-adic length scale by uncertainty principle in p-adic mass calculations.

The basic question is the physical interpretation of the automorphism inducing the scaling of trace by λ and its detailed action in HFF. This scaling could relate to a scaling in M^4 and to the appearance in the trace of an integral over M^4 or subspace of it defining the trace. Fractal structures suggests itself strongly here. At the level of construction of physical states one always selects some minimum non-positive conformal weight defining the tachyonic ground state and physical states have non-negative conformal weights. The interpretation would be as a reduction to HHF of type II_∞ or even II_1.

2.6.7 Could The Continuation Of Braidings To Homotopies Involve Δ^{It} Automorphisms

The representation of braidings as special case of homotopies might lead from discrete automorphisms for HFFs type II_1 to continuous outer automorphisms for HFFs of type III_1. The question is whether the periodic automorphism of II_1 represented as a discrete sub-group of $U(1)$ would be continued to $U(1)$ in the transition.

The automorphism of II_∞ HFF associated with a given value of the scaling factor λ is unique. If Jones inclusions defined by the preferred values of λ as $\lambda = \sqrt{\mathcal{M}:\mathcal{N}}/2^k$ (see the previous considerations), then this automorphism could involve a periodic automorphism of II_1 factor defined by the generator of cyclic subgroup Z_n for $\mathcal{M}:\mathcal{N} < 4$ besides additional shift transforming II_1 factor to I_∞ factor and inducing the scaling.

2.6.8 HFFs Of Type III As Super-Structures Providing Additional Uniqueness?

If the braiding M-matrices are as such highly unique. One could however consider the possibility that they are induced from the automorphisms σ_t for the HFFs of type III restricted to HFFs of type II_∞. If a reduction to inner automorphism in HFF of type III implies same with respect to HFF of type II_∞ and even II_1, they could be trivial for special values of time scaling t assignable to the partons and identifiable as a power of prime p characterizing the parton. This would allow to eliminate incoming and outgoing legs. This elimination would be the counterpart of the division of propagator legs in quantum field theories. Particle masses would however play no role in this process now although the power of padic prime would fix the mass scale of the particle.

2.7 Appendix: Inclusions Of Hyper-Finite Factors Of Type II_1

Many names have been assigned to inclusions: Jones, Wenzl, Ocneacnu, Pimsner-Popa, Wasserman [A104] . It would seem to me that the notion Jones inclusion includes them all so that various names would correspond to different concrete realizations of the inclusions conjugate under outer automorphisms.

1. According to [A104] for inclusions with $\mathcal{M}:\mathcal{N} \leq 4$ (with $A_1^{(1)}$ excluded) there exists a countable infinity of sub-factors with are pairwise non inner conjugate but conjugate to $\mathcal{N}$.

2. Also for any finite group G and its outer action there exists uncountably many sub-factors which are pairwise non inner conjugate but conjugate to the fixed point algebra of G [A104] . For any amenable group G the inclusion is also unique apart from outer automorphism [A63] .

Thus it seems that not only Jones inclusions but also more general inclusions are unique apart from outer automorphism.

Any *-endomorphism σ, which is unit preserving, faithful, and weakly continuous, defines a sub-factor of type II_1 factor [A104] . The construction of Jones leads to a standard inclusion sequence $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}^1 \subset ....$ This sequence means addition of projectors e_i, $i < 0$, having visualization as an addition of braid strand in braid picture. This hierarchy exists for all factors of type II. At the limit $\mathcal{M}^\infty = \cup_i \mathcal{M}^i$ the braid sequence extends from $-\infty$ to ∞. Inclusion hierarchy can be understood as a hierarchy of Connes tensor powers $\mathcal{M} \otimes_\mathcal{N} \mathcal{M} .... \otimes_\mathcal{N} \mathcal{M}$. Also the ordinary tensor powers of hyper-finite factors of type II_1 (HFF) as well as their tensor products with finite-dimensional matrix algebras are isomorphic to the original HFF so that these objects share the magic of fractals.

Under certain assumptions the hierarchy can be continued also in opposite direction. For a finite index an infinite inclusion hierarchy of factors results with the same value of index. σ is said to be basic if it can be extended to *-endomorphisms from $\mathcal{M}^1$ to $\mathcal{M}$. This means that the hierarchy of inclusions can be continued in the opposite direction: this means elimination of strands in the braid picture. For finite factors (as opposed to hyper-finite ones) there are no basic *-endomorphisms of $\mathcal{M}$ having fixed point algebra of non-abelian G as a sub-factor [A104] .

2.7.1 Jones Inclusions

For hyper-finite factors of type II_1 Jones inclusions allow basic *-endomorphism. They exist for all values of $\mathcal{M} : \mathcal{N} = r$ with $r \in \{4cos^2(\pi/n)|n \geq 3\} \cap [4, \infty)$ [A104] . They are defined for an algebra defined by projectors e_i, $i \geq 1$. All but nearest neighbor projectors commute. $\lambda = 1/r$ appears in the relations for the generators of the algebra given by $e_i e_j e_i = \lambda e_i$, $|i-j| = 1$. $\mathcal{N} \subset \mathcal{M}$ is identified as the double commutator of algebra generated by e_i, $i \geq 2$.

This means that principal graph and its dual are equivalent and the braid defined by projectors can be continued not only to $-\infty$ but that also the dropping of arbitrary number of strands is possible [A104] . It would seem that ADE property of the principal graph meaning single root length codes for the duality in the case of $r \leq 4$ inclusions.

Irreducibility holds true for $r < 4$ in the sense that the intersection of $Q' \cap P = P' \cap P = C$. For $r \geq 4$ one has $dim(Q' \cap P) = 2$. The operators commuting with Q contain besides identify operator of Q also the identify operator of P. Q would contain a single finite-dimensional matrix factor less than P in this case. Basic *-endomorphisms with $\sigma(P) = Q$ is $\sigma(e_i) = e_{i+1}$. The difference between genuine symmetries of quantum TGD and symmetries which can be mimicked by TGD could relate to the irreducibility for $r < 4$ and raise these inclusions in a unique position. This difference could partially justify the hypothesis [K27] that only the groups $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2, C) \times SU(3)$ define orbifold coverings of $H_\pm = M^4_\pm \times CP_2 \to H_\pm/G_a \times G_b$.

2.7.2 Wassermann's Inclusion

Wasserman's construction of $r = 4$ factors clarifies the role of the subgroup of $G \subset SU(2)$ for these inclusions. Also now $r = 4$ inclusion is characterized by a discrete subgroup $G \subset SU(2)$ and is given by $(1 \otimes \mathcal{M})^G \subset (M_2(C) \times \mathcal{M})^G$. According to [A104] Jones inclusions are irreducible also for $r = 4$. The definition of Wasserman inclusion for $r = 4$ seems however to imply that the identity matrices of both $\mathcal{M}^G$ and $(M(2, C) \otimes \mathcal{M})^G$ commute with $\mathcal{M}^G$ so that the inclusion should be reducible for $r = 4$.

Note that G leaves both the elements of $\mathcal{N}$ and $\mathcal{M}$ invariant whereas $SU(2)$ leaves the elements of $\mathcal{N}$ invariant. $M(2, C)$ is effectively replaced with the orbifold $M(2, C)/G$, with G acting as automorphisms. The space of these orbits has complex dimension $d = 4$ for finite G.

For $r < 4$ inclusion is defined as $M^G \subset M$. The representation of G as outer automorphism must change step by step in the inclusion sequence ... $\subset \mathcal{N} \subset \mathcal{M} \subset$ ... since otherwise G would act trivially as one proceeds in the inclusion sequence. This is true since each step brings in additional finite-dimensional tensor factor in which G acts as automorphisms so that although $\mathcal{M}$ can be invariant under $G_{\mathcal{M}}$ it is not invariant under $G_{\mathcal{N}}$.

These two inclusions might accompany each other in TGD based physics. One could consider $r < 4$ inclusion $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ with G acting non-trivially in $\mathcal{M}/\mathcal{N}$ quantum Clifford algebra. $\mathcal{N}$ would decompose by $r = 4$ inclusion to $\mathcal{N}_1 \subset \mathcal{N}$ with $SU(2)$ taking the role of G. $\mathcal{N}/\mathcal{N}_1$ quantum Clifford algebra would transform non-trivially under $SU(2)$ but would be G singlet.

In TGD framework the G-invariance for $SU(2)$ representations means a reduction of S^2 to the orbifold S^2/G. The coverings $H_\pm \to H_\pm/G_a \times G_b$ should relate to these double inclusions and $SU(2)$ inclusion could mean Kac-Moody type gauge symmetry for $\mathcal{N}$. Note that the presence of the factor containing only unit matrix should relate directly to the generator d in the generator set of affine algebra in the McKay construction [K38] . The physical interpretation of the fact that almost all ADE type extended diagrams ($D^{(1)}_n$ must have $n \geq 4$) are allowed for $r = 4$ inclusions whereas D_{2n+1} and E_6 are not allowed for $r < 4$, remains open.

2.7.3 Generalization From $Su(2)$ To Arbitrary Compact Group

The inclusions with index $\mathcal{M} : \mathcal{N} < 4$ have one-dimensional relative commutant $\mathcal{N}' \cup \mathcal{M}$. The most obvious conjecture that $\mathcal{M} : \mathcal{N} \geq 4$ corresponds to a non-trivial relative commutant is wrong. The index for Jones inclusion is identifiable as the square of quantum dimension of the fundamental representation of $SU(2)$. This identification generalizes to an arbitrary representation of arbitrary compact Lie group.

In his thesis Wenzl [A76] studied the representations of Hecke algebras $H_n(q)$ of type A_n obtained from the defining relations of symmetric group by the replacement $e_i^2 = (q-1)e_i+q$. H_n is isomorphic to complex group algebra of S_n if q is not a root of unity and for $q = 1$ the irreducible representations of $H_n(q)$ reduce trivially to Young's representations of symmetric groups. For primitive roots of unity $q = exp(i2\pi/l)$, $l = 4, 5...$, the representations of $H_n(\infty)$ give rise to inclusions for which index corresponds to a quantum dimension of any irreducible representation of $SU(k)$, $k \geq 2$. For $SU(2)$ also the value $l = 3$ is allowed for spin 1/2 representation.

The inclusions are obtained by dropping the first m generators e_k from $H_\infty(q)$ and taking double commutant of both H_∞ and the resulting algebra. The relative commutant corresponds to $H_m(q)$. By reducing by the minimal projection to relative commutant one obtains an inclusion with a trivial relative commutant. These inclusions are analogous to a discrete states superposed in continuum. Thus the results of Jones generalize from the fundamental representation of $SU(2)$ to all representations of all groups $SU(k)$, and in fact to those of general compact groups as it turns out.

The generalization of the formula for index to square of quantum dimension of an irreducible representation of $SU(k)$ reads as

$$\mathcal{M} : \mathcal{N} \;=\; \prod_{1\leq r<s\leq k} \frac{sin^2\left((\lambda_r - \lambda_s + s - r)\pi/l\right)}{sin^2\left((s-r)n/l\right)} \quad . \tag{2.7.1}$$

Here λ_r is the number of boxes in the r^{th} row of the Yang diagram with n boxes characterizing the representations and the condition $1 \leq k \leq l-1$ holds true. Only Young diagrams satisfying the condition $l - k = \lambda_1 - \lambda_{r_{max}}$ are allowed.

The result would allow to restrict the generalization of the embedding space in such a way that only cyclic group Z_n appears in the covering of $M^4 \to M^4/G_a$ or $CP_2 \to CP_2/G_b$ factor. Be as it may, it seems that quantum representations of any compact Lie group can be realized using the generalization of the embedding space. In the case of SU(2) the interpretation of higher-dimensional quantum representations in terms of Connes tensor products of 2-dimensional fundamental representations is highly suggestive.

The groups $SO(3,1) \times SU(3)$ and $SL(2,C) \times U(2)_{ew}$ have a distinguished position both in physics and quantum TGD and the vision about physics as a generalized number theory implies them. Also the general pattern for inclusions selects these groups, and one can say that the condition that all possible statistics are realized is guaranteed by the choice $M^4 \times CP_2$.

1. $n > 2$ for the quantum counterparts of the fundamental representation of $SU(2)$ means that braid statistics for Jones inclusions cannot give the usual fermionic statistics. That Fermi statistics cannot "emerge" conforms with the role of infinite-D Clifford algebra as a canonical representation of HFF of type II_1. $SO(3,1)$ as isometries of H gives Z_2 statistics via the action on spinors of M^4 and $U(2)$ holonomies for CP_2 realize Z_2 statistics in CP_2 degrees of freedom.

2. $n > 3$ for more general inclusions in turn excludes Z_3 statistics as braid statistics in the general case. $SU(3)$ as isometries induces a non-trivial Z_3 action on quark spinors but trivial action at the embedding space level so that Z_3 statistics would be in question.

Chapter 3

Evolution of Ideas about Hyper-finite Factors in TGD

3.1 Introduction

This chapter has emerged from a splitting of a chapter devote to the possible role of von Neumann algebras known as hyper-finite factors in quantum TGD. Second chapter emerging from the splitting is a representation of basic notions to chapter "Was von Neumann right after all?" [K87] representing only very briefly ideas about application to quantum TGD only briefly.

In the sequel the ideas about TGD applications are reviewed more or less chronologically. A summary about evolution of ideas is in question, not a coherent final structure, and as always the first speculations - in this case roughly for a decade ago - might look rather weird. The vision has however gradually become more realistic looking as deeper physical understanding of factors has evolved slowly.

The mathematics involved is extremely difficult for a physicist like me, and to really learn it at the level of proofs one should reincarnate as a mathematician. Therefore the only practical approach relies on the use of physical intuition to see whether HFFs might the correct structure and what HFFs do mean. What is needed is a concretization of the extremely abstract mathematics involved: mathematics represents only the bones to which physics should add flesh.

3.1.1 Hyper-Finite Factors In Quantum TGD

The following argument suggests that von Neumann algebras known as hyper-finite factors (HFFs) of type III_1 appearing in relativistic quantum field theories provide also the proper mathematical framework for quantum TGD.

1. The Clifford algebra of the infinite-dimensional Hilbert space is a von Neumann algebra known as HFF of type II_1. There also the Clifford algebra at a given point (light-like 3-surface) of world of classical worlds (WCW) is therefore HFF of type II_1. If the fermionic Fock algebra defined by the fermionic oscillator operators assignable to the induced spinor fields (this is actually not obvious!) is infinite-dimensional it defines a representation for HFF of type II_1. Super-conformal symmetry suggests that the extension of the Clifford algebra defining the fermionic part of a super-conformal algebra by adding bosonic super-generators representing symmetries of WCW respects the HFF property. It could however occur that HFF of type II_∞ results.

2. WCW is a union of sub-WCWs associated with causal diamonds (CD) defined as intersections of future and past directed light-cones. One can allow also unions of CDs and the proposal is that CDs within CDs are possible. Whether CDs can intersect is not clear.

3. The assumption that the M^4 proper distance a between the tips of CD is quantized in powers of 2 reproduces p-adic length scale hypothesis but one must also consider the possibility that a can have all possible values. Since $SO(3)$ is the isotropy group of CD, the CDs associated

with a given value of a and with fixed lower tip are parameterized by the Lobatchevski space $L(a) = SO(3,1)/SO(3)$. Therefore the CDs with a free position of lower tip are parameterized by $M^4 \times L(a)$. A possible interpretation is in terms of quantum cosmology with a identified as cosmic time [K69] . Since Lorentz boosts define a non-compact group, the generalization of so called crossed product construction strongly suggests that the local Clifford algebra of WCW is HFF of type III_1. If one allows all values of a, one ends up with $M^4 \times M^4_+$ as the space of moduli for WCW.

4. An interesting special aspect of 8-dimensional Clifford algebra with Minkowski signature is that it allows an octonionic representation of gamma matrices obtained as tensor products of unit matrix 1 and 7-D gamma matrices γ_k and Pauli sigma matrices by replacing 1 and γ_k by octonions. This inspires the idea that it might be possible to end up with quantum TGD from purely number theoretical arguments. This seems to be the case. One can start from a local octonionic Clifford algebra in M^8. Associativity condition is satisfied if one restricts the octonionic algebra to a subalgebra associated with any hyper-quaternionic and thus 4-D sub-manifold of M^8. This means that the Kähler-Dirac gamma matrices associated with the Kähler action span a complex quaternionic sub-space at each point of the sub-manifold. This associative sub-algebra can be mapped a matrix algebra. Together with $M^8 - H$ duality [K88, K19] this leads automatically to quantum TGD and therefore also to the notion of WCW and its Clifford algebra which is however only mappable to an associative algebra and thus to HFF of type II_1.

3.1.2 Hyper-Finite Factors And M-Matrix

HFFs of type III_1 provide a general vision about M-matrix.

1. The factors of type III allow unique modular automorphism Δ^{it} (fixed apart from unitary inner automorphism). This raises the question whether the modular automorphism could be used to define the M-matrix of quantum TGD. This is not the case as is obvious already from the fact that unitary time evolution is not a sensible concept in zero energy ontology.

2. Concerning the identification of M-matrix the notion of state as it is used in theory of factors is a more appropriate starting point than the notion modular automorphism but as a generalization of thermodynamical state is certainly not enough for the purposes of quantum TGD and quantum field theories (algebraic quantum field theorists might disagree!). Zero energy ontology requires that the notion of thermodynamical state should be replaced with its "complex square root" abstracting the idea about M-matrix as a product of positive square root of a diagonal density matrix and a unitary S-matrix. This generalization of thermodynamical state -if it exists- would provide a firm mathematical basis for the notion of M-matrix and for the fuzzy notion of path integral.

3. The existence of the modular automorphisms relies on Tomita-Takesaki theorem, which assumes that the Hilbert space in which HFF acts allows cyclic and separable vector serving as ground state for both HFF and its commutant. The translation to the language of physicists states that the vacuum is a tensor product of two vacua annihilated by annihilation oscillator type algebra elements of HFF and creation operator type algebra elements of its commutant isomorphic to it. Note however that these algebras commute so that the two algebras are not hermitian conjugates of each other. This kind of situation is exactly what emerges in zero energy ontology: the two vacua can be assigned with the positive and negative energy parts of the zero energy states entangled by M-matrix.

4. There exists infinite number of thermodynamical states related by modular automorphisms. This must be true also for their possibly existing "complex square roots". Physically they would correspond to different measurement interactions giving rise to Kähler functions of WCW differing only by a real part of holomorphic function of complex coordinates of WCW and arbitrary function of zero mode coordinates and giving rise to the same Kähler metric of WCW.

3.1.3 Connes Tensor Product As A Realization Of Finite Measurement Resolution

The inclusions $\mathcal{N} \subset \mathcal{M}$ of factors allow an attractive mathematical description of finite measurement resolution in terms of Connes tensor product but do not fix M-matrix as was the original optimistic belief.

1. In zero energy ontology $\mathcal{N}$ would create states experimentally indistinguishable from the original one. Therefore $\mathcal{N}$ takes the role of complex numbers in non-commutative quantum theory. The space $\mathcal{M}/\mathcal{N}$ would correspond to the operators creating physical states modulo measurement resolution and has typically fractal dimension given as the index of the inclusion. The corresponding spinor spaces have an identification as quantum spaces with non-commutative $\mathcal{N}$-valued coordinates.

2. This leads to an elegant description of finite measurement resolution. Suppose that a universal M-matrix describing the situation for an ideal measurement resolution exists as the idea about square root of state encourages to think. Finite measurement resolution forces to replace the probabilities defined by the M-matrix with their $\mathcal{N}$ "averaged" counterparts. The "averaging" would be in terms of the complex square root of $\mathcal{N}$-state and a direct analog of functionally or path integral over the degrees of freedom below measurement resolution defined by (say) length scale cutoff.

3. One can construct also directly M-matrices satisfying the measurement resolution constraint. The condition that $\mathcal{N}$ acts like complex numbers on M-matrix elements as far as $\mathcal{N}$ "averaged" probabilities are considered is satisfied if M-matrix is a tensor product of M-matrix in $\mathcal{M}(\mathcal{N}$ interpreted as finite-dimensional space with a projection operator to $\mathcal{N}$. The condition that $\mathcal{N}$ averaging in terms of a complex square root of $\mathcal{N}$ state produces this kind of M-matrix poses a very strong constraint on M-matrix if it is assumed to be universal (apart from variants corresponding to different measurement interactions).

3.1.4 Concrete Realization Of The Inclusion Hierarchies

A concrete construction of M-matrix motivated by the recent rather precise view about basic variational principles of TGD allows to identify rather concretely the inclusions of HFFs in TGD framework and relate them to the hierarchies of broken conformal symmetries accompanying quantum criticalities.

1. Fundamental fermions localized to string world sheets can be said to propagate as massless particles along their boundaries. The fundamental interaction vertices correspond to two fermion scattering for fermions at opposite throats of wormhole contact and the inverse of the conformal scaling generator L_0 would define the stringy propagator characterizing this interaction. Fundamental bosons correspond to pairs of fermion and antifermion at opposite throats of wormhole contact. Physical particles correspond to pairs of wormhole contacts with monopole Kähler magnetic flux flowing around a loop going through wormhole contacts.

2. The formulation of scattering amplitudes in terms of Yangian of the super-symplectic algebra leads to a rather detailed view about scattering amplitudes [K77]. In this formulation scattering amplitudes are representations for sequences of algebraic operations connecting collections of elements of Yangian and sequences produce the same result. A huge generalization of the duality symmetry of the hadronic string models is in question.

3. The reduction of the hierarchy of Planck constants $h_{eff}/h = n$ to a hierarchy of quantum criticalities accompanied by a hierarchy of sub-algebras of super-symplectic algebra acting as conformal gauge symmetries leads to the identification of inclusions of HFFs as inclusions of WCW Clifford algebras characterizing by $n(i)$ and $n(i+1) = m(i) \times n(i)$ so that hierarchies of von Neuman algebras, of Planck constants, and of quantum criticalities would be very closely related. In the transition $n(i) \to n(i+1) = m(i) \times n(i)$ the measurement accuracy indeed increases since some conformal gauge degrees of freedom are transformed to physical ones. An open question is whether one could interpret $m(i)$ as the integer characterizing

inclusion: the problem is that also $m(i) = 2$ with $\mathcal{M} : \mathcal{N} = 4$ seems to be allowed whereas Jones inclusions allow only $m \geq 3$.

Even more, number theoretic universality and strong form of holography leads to a detailed vision about the construction of scattering amplitudes suggesting that the hierarchy of algebraic extensions of rationals relates to the above mentioned hierarchies.

3.1.5 Analogs of quantum matrix groups from finite measurement resolution?

The notion of quantum group replaces ordinary matrices with matrices with non-commutative elements. In TGD framework I have proposed that the notion should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution.

In this article I will consider the notion of quantum matrix inspired by recent view about quantum TGD and it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution. The basic idea is to replace complex matrix elements with operators expressible as products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers.

The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. The weak definition of determinant meaning its development with respect to a fixed row or column does not pose additional conditions. Strong definition of determinant requires its invariance under permutations of rows and columns. The permutation of rows/columns turns out to have interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

The resulting quantum matrices define a more general structure than quantum group but provide a concrete representation and interpretation for quantum group in terms of finite measurement resolution if q is a root of unity. For $q = \pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by sign factor invariant under the permutations of both rows and columns. One could also understand the fractal structure of inclusion sequences of hyper-finite factors resulting by recursively replacing operators appearing as matrix elements with quantum matrices.

3.1.6 Quantum Spinors And Fuzzy Quantum Mechanics

The notion of quantum spinor leads to a quantum mechanical description of fuzzy probabilities. For quantum spinors state function reduction cannot be performed unless quantum deformation parameter equals to $q = 1$. The reason is that the components of quantum spinor do not commute: it is however possible to measure the commuting operators representing moduli squared of the components giving the probabilities associated with "true" and "false". The universal eigenvalue spectrum for probabilities does not in general contain (1,0) so that quantum qbits are inherently fuzzy. State function reduction would occur only after a transition to q=1 phase and de-coherence is not a problem as long as it does not induce this transition.

This chapter represents a summary about the development of the ideas with last sections representing the recent latest about the realization and role of HFFs in TGD. I have saved the reader from those speculations that have turned out to reflect my own ignorance or are inconsistent with what I regarded established parts of quantum TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

3.2 A Vision About The Role Of HFFs In TGD

It is clear that at least the hyper-finite factors of type II_1 assignable to WCW spinors must have a profound role in TGD. Whether also HFFs of type III_1 appearing also in relativistic quantum field theories emerge when WCW spinors are replaced with spinor fields is not completely clear. I have proposed several ideas about the role of hyper-finite factors in TGD framework. In particular, Connes tensor product is an excellent candidate for defining the notion of measurement resolution.

In the following this topic is discussed from the perspective made possible by ZEO and the recent advances in the understanding of M-matrix using the notion of bosonic emergence. The conclusion is that the notion of state as it appears in the theory of factors is not enough for the purposes of quantum TGD. The reason is that state in this sense is essentially the counterpart of thermodynamical state. The construction of M-matrix might be understood in the framework of factors if one replaces state with its "complex square root" natural if quantum theory is regarded as a "complex square root" of thermodynamics. It is also found that the idea that Connes tensor product could fix M-matrix is too optimistic but an elegant formulation in terms of partial trace for the notion of M-matrix modulo measurement resolution exists and Connes tensor product allows interpretation as entanglement between sub-spaces consisting of states not distinguishable in the measurement resolution used. The partial trace also gives rise to non-pure states naturally.

The newest element in the vision is the proposal that quantum criticality of TGD Universe is realized as hierarchies of inclusions of super-conformal algebras with conformal weights coming as multiples of integer n, where n varies. If n_1 divides n_2 then various super-conformal algebras C_{n_2} are contained in C_{n_1}. This would define naturally the inclusion.

3.2.1 Basic facts about factors

In this section basic facts about factors are discussed. My hope that the discussion is more mature than or at least complementary to the summary that I could afford when I started the work with factors for more than half decade ago. I of course admit that this just a humble attempt of a physicist to express physical vision in terms of only superficially understood mathematical notions.

Basic notions

First some standard notations. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators of Hilbert space $\mathcal{H}$ bounded in the norm topology with norm defined by the supremum for the length of the image of a point of unit sphere $\mathcal{H}$. This algebra has a lot of common with complex numbers in that the counterparts of complex conjugation, order structure and metric structure determined by the algebraic structure exist. This means the existence involution -that is *- algebra property. The order structure determined by algebraic structure means following: $A \geq 0$ defined as the condition $(A\xi, \xi) \geq 0$ is equivalent with $A = B^*B$. The algebra has also metric structure $||AB|| \leq ||A||||B|$ (Banach algebra property) determined by the algebraic structure. The algebra is also C^* algebra: $||A^*A|| = ||A||^2$ meaning that the norm is algebraically like that for complex numbers.

A von Neumann algebra $\mathcal{M}$ [A24] is defined as a weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$ and has therefore all the above mentioned properties. From the point of view of physicist it is important that a sub-algebra is in question.

In order to define factors one must introduce additional structure.

1. Let $\mathcal{M}$ be subalgebra of $\mathcal{B}(\mathcal{H})$ and denote by $\mathcal{M}'$ its commutant $(\mathcal{H})$ commuting with it and allowing to express $\mathcal{B}(\mathcal{H})$ as $\mathcal{B}(\mathcal{H}) = \mathcal{M} \vee \mathcal{M}'$.

2. A factor is defined as a von Neumann algebra satisfying $\mathcal{M}'' = \mathcal{M}$ $\mathcal{M}$ is called factor. The equality of double commutant with the original algebra is thus the defining condition so that also the commutant is a factor. An equivalent definition for factor is as the condition that the intersection of the algebra and its commutant reduces to a complex line spanned by a unit operator. The condition that the only operator commuting with all operators of the factor is unit operator corresponds to irreducibility in representation theory.

3. Some further basic definitions are needed. $\Omega \in \mathcal{H}$ is cyclic if the closure of $\mathcal{M}\Omega$ is $\mathcal{H}$ and separating if the only element of $\mathcal{M}$ annihilating Ω is zero. Ω is cyclic for $\mathcal{M}$ if and only if it is separating for its commutant. In so called standard representation Ω is both cyclic and separating.

4. For hyperfinite factors an inclusion hierarchy of finite-dimensional algebras whose union is dense in the factor exists. This roughly means that one can approximate the algebra in arbitrary accuracy with a finite-dimensional sub-algebra.

The definition of the factor might look somewhat artificial unless one is aware of the underlying physical motivations. The motivating question is what the decomposition of a physical system to non-interacting sub-systems could mean. The decomposition of $\mathcal{B}(\mathcal{H})$ to $\vee$ product realizes this decomposition.

1. Tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is the decomposition according to the standard quantum measurement theory and means the decomposition of operators in $\mathcal{B}(\mathcal{H})$ to tensor products of mutually commuting operators in $\mathcal{M} = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{M}' = \mathcal{B}(\mathcal{H}_2)$. The information about $\mathcal{M}$ can be coded in terms of projection operators. In this case projection operators projecting to a complex ray of Hilbert space exist and arbitrary compact operator can be expressed as a sum of these projectors. For factors of type I minimal projectors exist. Factors of type I_n correspond to sub-algebras of $\mathcal{B}(\mathcal{H})$ associated with infinite-dimensional Hilbert space and I_∞ to $\mathcal{B}(\mathcal{H})$ itself. These factors appear in the standard quantum measurement theory where state function reduction can lead to a ray of Hilbert space.

2. For factors of type II no minimal projectors exists whereas finite projectors exist. For factors of type II_1 all projectors have trace not larger than one and the trace varies in the range $(0, 1]$. In this case cyclic vectors Ω exist. State function reduction can lead only to an infinite-dimensional subspace characterized by a projector with trace smaller than 1 but larger than zero. The natural interpretation would be in terms of finite measurement resolution. The tensor product of II_1 factor and I_∞ is II_∞ factor for which the trace for a projector can have arbitrarily large values. II_1 factor has a unique finite tracial state and the set of traces of projections spans unit interval. There is uncountable number of factors of type II but hyper-finite factors of type II_1 are the exceptional ones and physically most interesting.

3. Factors of type III correspond to an extreme situation. In this case the projection operators E spanning the factor have either infinite or vanishing trace and there exists an isometry mapping $E\mathcal{H}$ to $\mathcal{H}$ meaning that the projection operator spans almost all of $\mathcal{H}$. All projectors are also related to each other by isometry. Factors of type III are smallest if the factors are regarded as sub-algebras of a fixed $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ corresponds to isomorphism class of Hilbert spaces. Situation changes when one speaks about concrete representations. Also now hyper-finite factors are exceptional.

4. Von Neumann algebras define a non-commutative measure theory. Commutative von Neumann algebras indeed reduce to $L^\infty(X)$ for some measure space (X, μ) and vice versa.

Weights, states and traces

The notions of weight, state, and trace are standard notions in the theory of von Neumann algebras.

1. A weight of von Neumann algebra is a linear map from the set of positive elements (those of form a^*a) to non-negative reals.

2. A positive linear functional is weight with $\omega(1)$ finite.

3. A state is a weight with $\omega(1) = 1$.

4. A trace is a weight with $\omega(aa^*) = \omega(a^*a)$ for all a.

5. A tracial state is a weight with $\omega(1) = 1$.

A factor has a trace such that the trace of a non-zero projector is non-zero and the trace of projection is infinite only if the projection is infinite. The trace is unique up to a rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. Factors of type I_n the values of trace are equal to multiples of $1/n$. For a factor of type I_∞ the value of trace are $0, 1, 2, ...$ For factors of type II_1 the values span the range $[0, 1]$ and for factors of type II_∞ n the range $[0, \infty)$. For factors of type III the values of the trace are 0, and ∞.

Tomita-Takesaki theory

Tomita-Takesaki theory is a vital part of the theory of factors. First some definitions.

1. Let $\omega(x)$ be a faithful state of von Neumann algebra so that one has $\omega(xx^*) > 0$ for $x > 0$. Assume by Riesz lemma the representation of ω as a vacuum expectation value: $\omega = (\cdot\Omega, \Omega)$, where Ω is cyclic and separating state.

2. Let

$$L^{\infty}(\mathcal{M}) \equiv \mathcal{M} \ , \quad L^2(\mathcal{M}) = \mathcal{H} \ , \qquad L^1(\mathcal{M}) = \mathcal{M}_* \ , \tag{3.2.1}$$

   where $\mathcal{M}_*$ is the pre-dual of $\mathcal{M}$ defined by linear functionals in $\mathcal{M}$. One has ${\mathcal{M}_*}^* = \mathcal{M}$.

3. The conjugation $x \rightarrow x^*$ is isometric in $\mathcal{M}$ and defines a map $\mathcal{M} \rightarrow L^2(\mathcal{M})$ via $x \rightarrow x\Omega$. The map $S_0; x\Omega \rightarrow x^*\Omega$ is however non-isometric.

4. Denote by S the closure of the anti-linear operator S_0 and by $S = J\Delta^{1/2}$ its polar decomposition analogous that for complex number and generalizing polar decomposition of linear operators by replacing (almost) unitary operator with anti-unitary J. Therefore $\Delta = S^*S > 0$ is positive self-adjoint and J an anti-unitary involution. The non-triviality of Δ reflects the fact that the state is not trace so that hermitian conjugation represented by S in the state space brings in additional factor $\Delta^{1/2}$.

5. What x can be is puzzling to physicists. The restriction fermionic Fock space and thus to creation operators would imply that Δ would act non-trivially only vacuum state so that $\Delta > 0$ condition would not hold true. The resolution of puzzle is the allowance of tensor product of Fock spaces for which vacua are conjugates: only this gives cyclic and separating state. This is natural in ZEO.

The basic results of Tomita-Takesaki theory are following.

1. The basic result can be summarized through the following formulas

$$\Delta^{it} M \Delta^{-it} = \mathcal{M} \ , J\mathcal{M}J = \mathcal{M}' \ .$$

2. The latter formula implies that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic algebras. The first formula implies that a one parameter group of modular automorphisms characterizes partially the factor. The physical meaning of modular automorphisms is discussed in [A47, A93] Δ is Hermitian and positive definite so that the eigenvalues of $log(\Delta)$ are real but can be negative. Δ^{it} is however not unitary for factors of type II and III. Physically the non-unitarity must relate to the fact that the flow is contracting so that hermiticity as a local condition is not enough to guarantee unitarity.

3. $\omega \rightarrow \sigma_t^{\omega} = Ad\Delta^{it}$ defines a canonical evolution -modular automorphism- associated with ω and depending on it. The Δ:s associated with different ω:s are related by a unitary inner automorphism so that their equivalence classes define an invariant of the factor.

Tomita-Takesaki theory gives rise to a non-commutative measure theory which is highly non-trivial. In particular the spectrum of Δ can be used to classify the factors of type II and III.

Modular automorphisms

Modular automorphisms of factors are central for their classification.

1. One can divide the automorphisms to inner and outer ones. Inner automorphisms correspond to unitary operators obtained by exponentiating Hermitian Hamiltonian belonging to the factor and connected to identity by a flow. Outer automorphisms do not allow a representation as a unitary transformations although $log(\Delta)$ is formally a Hermitian operator.

2. The fundamental group of the type II_1 factor defined as fundamental group group of corresponding II_∞ factor characterizes partially a factor of type II_1. This group consists real numbers λ such that there is an automorphism scaling the trace by λ. Fundamental group typically contains all reals but it can be also discrete and even trivial.

3. Factors of type III allow a one-parameter group of modular automorphisms, which can be used to achieve a partial classification of these factors. These automorphisms define a flow in the center of the factor known as flow of weights. The set of parameter values λ for which ω is mapped to itself and the center of the factor defined by the identity operator (projector to the factor as a sub-algebra of $\mathcal{B}(\mathcal{H})$) is mapped to itself in the modular automorphism defines the Connes spectrum of the factor. For factors of type III_λ this set consists of powers of $\lambda < 1$. For factors of type III_0 this set contains only identity automorphism so that there is no periodicity. For factors of type III_1 Connes spectrum contains all real numbers so that the automorphisms do not affect the identity operator of the factor at all.

The modules over a factor correspond to separable Hilbert spaces that the factor acts on. These modules can be characterized by M-dimension. The idea is roughly that complex rays are replaced by the sub-spaces defined by the action of $\mathcal{M}$ as basic units. M-dimension is not integer valued in general. The so called standard module has a cyclic separating vector and each factor has a standard representation possessing antilinear involution J such that $\mathcal{M}' = J\mathcal{M}J$ holds true (note that J changes the order of the operators in conjugation). The inclusions of factors define modules having interpretation in terms of a finite measurement resolution defined by $\mathcal{M}$.

Crossed product as a way to construct factors of type III

By using so called crossed product crossedproduct for a group G acting in algebra A one can obtain new von Neumann algebras. One ends up with crossed product by a two-step generalization by starting from the semidirect product $G \triangleleft H$ for groups defined as $(g_1, h_1)(g_2, h_2) = (g_1 h_1(g_2), h_1 h_2)$ (note that Poincare group has interpretation as a semidirect product $M^4 \triangleleft SO(3,1)$ of Lorentz and translation groups). At the first step one replaces the group H with its group algebra. At the second step the the group algebra is replaced with a more general algebra. What is formed is the semidirect product $A \triangleleft G$ which is sum of algebras Ag. The product is given by $(a_1, g_1)(a_2, g_2) = (a_1 g_1(a_2), g_1 g_2)$. This construction works for both locally compact groups and quantum groups. A not too highly educated guess is that the construction in the case of quantum groups gives the factor $\mathcal{M}$ as a crossed product of the included factor $\mathcal{N}$ and quantum group defined by the factor space $\mathcal{M}/\mathcal{N}$.

The construction allows to express factors of type III as crossed products of factors of type II_∞ and the 1-parameter group G of modular automorphisms assignable to any vector which is cyclic for both factor and its commutant. The ergodic flow θ_λ scales the trace of projector in II_∞ factor by $\lambda > 0$. The dual flow defined by G restricted to the center of II_∞ factor does not depend on the choice of cyclic vector.

The Connes spectrum - a closed subgroup of positive reals - is obtained as the exponent of the kernel of the dual flow defined as set of values of flow parameter λ for which the flow in the center is trivial. Kernel equals to $\{0\}$ for III_0, contains numbers of form $log(\lambda)Z$ for factors of type III_λ and contains all real numbers for factors of type III_1 meaning that the flow does not affect the center.

Inclusions and Connes tensor product

Inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras have physical interpretation as a mathematical description for sub-system-system relation. In [K87] there is more extensive TGD colored description of inclusions and their role in TGD. Here only basic facts are listed and the Connes tensor product is explained.

For type I algebras the inclusions are trivial and tensor product description applies as such. For factors of II_1 and III the inclusions are highly non-trivial. The inclusion of type II_1 factors were understood by Vaughan Jones [A2] and those of factors of type III by Alain Connes [A28] .

Formally sub-factor $\mathcal{N}$ of $\mathcal{M}$ is defined as a closed *-stable C-subalgebra of $\mathcal{M}$. Let $\mathcal{N}$ be a sub-factor of type II_1 factor $\mathcal{M}$. Jones index $\mathcal{M} : \mathcal{N}$ for the inclusion $\mathcal{N} \subset \mathcal{M}$ can be defined as $\mathcal{M} : \mathcal{N} = dim_N(L^2(\mathcal{M})) = Tr_{N'}(id_{L^2(\mathcal{M})})$. One can say that the dimension of completion of $\mathcal{M}$ as $\mathcal{N}$ module is in question.

Basic findings about inclusions

What makes the inclusions non-trivial is that the position of $\mathcal{N}$ in $\mathcal{M}$ matters. This position is characterized in case of hyper-finite II_1 factors by index $\mathcal{M} : \mathcal{N}$ which can be said to the dimension of $\mathcal{M}$ as $\mathcal{N}$ module and also as the inverse of the dimension defined by the trace of the projector from $\mathcal{M}$ to $\mathcal{N}$. It is important to notice that $\mathcal{M} : \mathcal{N}$ does not characterize either $\mathcal{M}$ or $\mathcal{M}$, only the embedding.

The basic facts proved by Jones are following [A2] .

1. For pairs $\mathcal{N} \subset \mathcal{M}$ with a finite principal graph the values of $\mathcal{M} : \mathcal{N}$ are given by

$$
\begin{aligned}
& a) \;\; \mathcal{M} : \mathcal{N} = 4cos^2(\pi/h) \;\; , \quad h \geq 3 \;\; , \\
& b) \;\; \mathcal{M} : \mathcal{N} \geq 4 \;\; .
\end{aligned}
\tag{3.2.2}
$$

   the numbers at right hand side are known as Beraha numbers [A73] . The comments below give a rough idea about what finiteness of principal graph means.

2. As explained in [B29] , for $\mathcal{M} : \mathcal{N} < 4$ one can assign to the inclusion Dynkin graph of ADE type Lie-algebra g with h equal to the Coxeter number h of the Lie algebra given in terms of its dimension and dimension r of Cartan algebra r as $h = (dimg(g) - r)/r$. For $\mathcal{M} : \mathcal{N} < 4$ ordinary Dynkin graphs of D_{2n} and E_6, E_8 are allowed. The Dynkin graphs of Lie algebras of $SU(n)$, E_7 and D_{2n+1} are however not allowed. $E_6, E_7, andE_8$ correspond to symmetry groups of tetrahedron, octahedron/cube, and icosahedron/dodecahedron. The group for octahedron/cube is missing: what could this mean?

   For $\mathcal{M} : \mathcal{N} = 4$ one can assign to the inclusion an extended Dynkin graph of type ADE characterizing Kac Moody algebra. Extended ADE diagrams characterize also the subgroups of $SU(2)$ and the interpretation proposed in [A131] is following-

   The ADE diagrams are associated with the $n = \infty$ case having $\mathcal{M} : \mathcal{N} \geq 4$. There are diagrams corresponding to infinite subgroups: A_∞ corresponding to $SU(2)$ itself, $A_{-\infty,\infty}$ corresponding to circle group $U(1)$, and infinite dihedral groups (generated by a rotation by a non-rational angle and reflection.

   One can construct also inclusions for which the diagrams corresponding to finite subgroups $G \subset SU(2)$ are extension of A_n for cyclic groups, of D_n dihedral groups, and of E_n with $n = 6, 7, 8$ for tetrahedron, cube, dodecahedron. These extensions correspond to ADE type Kac-Moody algebras.

   The extension is constructed by constructing first factor R as infinite tensor power of $M_2(C)$ (complexified quaternions). Sub-factor R_0 consists elements of of R of form $Id \otimes x$. $SU(2)$ preserves R_0 and for any subgroup G of $SU(2)$ one can identify the inclusion $N \subset M$ in terms of $N = R_0^G$ and $M = R^G$, where $N = R_0^G$ and $M = R^G$ consists of fixed points of R_0 and R under the action of G. The principal graph for $N \subset M$ is the extended Coxeter-Dynk graph for the subgroup G.

   Physicist might try to interpret this by saying that one considers only sub-algebras R_0^G and R^G of observables invariant under G and obtains extended Dynkin diagram of G defining an

ADE type Kac-Moody algebra. Could the condition that Kac-Moody algebra elements with non-vanishing conformal weight annihilate the physical states state that the state is invariant under R_0 defining measurement resolution. Besides this the states are also invariant under finite group G? Could R_0^G and R^G correspond just to states which are also invariant under finite group G.

Connes tensor product

The basic idea of Connes tensor product is that a sub-space generated sub-factor $\mathcal{N}$ takes the role of the complex ray of Hilbert space. The physical interpretation is in terms of finite measurement resolution: it is not possible to distinguish between states obtained by applying elements of $\mathcal{N}$.

Intuitively it is clear that it should be possible to decompose $\mathcal{M}$ to a tensor product of factor space $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$:

$$\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N} \ . \tag{3.2.3}$$

One could regard the factor space $\mathcal{M}/\mathcal{N}$ as a non-commutative space in which each point corresponds to a particular representative in the equivalence class of points defined by $\mathcal{N}$. The connections between quantum groups and Jones inclusions suggest that this space closely relates to quantum groups. An alternative interpretation is as an ordinary linear space obtained by mapping $\mathcal{N}$ rays to ordinary complex rays. These spaces appear in the representations of quantum groups. Similar procedure makes sense also for the Hilbert spaces in which $\mathcal{M}$ acts.

Connes tensor product can be defined in the space $\mathcal{M}\otimes\mathcal{M}$ as entanglement which effectively reduces to entanglement between $\mathcal{N}$ sub-spaces. This is achieved if $\mathcal{N}$ multiplication from right is equivalent with $\mathcal{N}$ multiplication from left so that $\mathcal{N}$ acts like complex numbers on states. One can imagine variants of the Connes tensor product and in TGD framework one particular variant appears naturally as will be found.

In the finite-dimensional case Connes tensor product of Hilbert spaces has a rather simple representation. If the matrix algebra N of $n \times n$ matrices acts on V from right, V can be regarded as a space formed by $m \times n$ matrices for some value of m. If N acts from left on W, W can be regarded as space of $n \times r$ matrices.

1. In the first representation the Connes tensor product of spaces V and W consists of $m \times r$ matrices and Connes tensor product is represented as the product VW of matrices as $(VW)_{mr}e^{mr}$. In this representation the information about N disappears completely as the interpretation in terms of measurement resolution suggests. The sum over intermediate states defined by N brings in mind path integral.

2. An alternative and more physical representation is as a state

$$\sum_n V_{mn}W_{nr}e^{mn} \otimes e^{nr}$$

in the tensor product $V \otimes W$.

3. One can also consider two spaces V and W in which N acts from right and define Connes tensor product for $A^\dagger \otimes_N B$ or its tensor product counterpart. This case corresponds to the modification of the Connes tensor product of positive and negative energy states. Since Hermitian conjugation is involved, matrix product does not define the Connes tensor product now. For $m = r$ case entanglement coefficients should define a unitary matrix commuting with the action of the Hermitian matrices of N and interpretation would be in terms of symmetry. HFF property would encourage to think that this representation has an analog in the case of HFFs of type II_1.

4. Also type I_n factors are possible and for them Connes tensor product makes sense if one can assign the inclusion of finite-D matrix algebras to a measurement resolution.

Factors in quantum field theory and thermodynamics

Factors arise in thermodynamics and in quantum field theories [A118, A47, A93] . There are good arguments showing that in HFFs of III_1 appear are relativistic quantum field theories. In non-relativistic QFTs the factors of type I appear so that the non-compactness of Lorentz group is essential. Factors of type III_1 and III_λ appear also in relativistic thermodynamics.

The geometric picture about factors is based on open subsets of Minkowski space. The basic intuitive view is that for two subsets of M^4, which cannot be connected by a classical signal moving with at most light velocity, the von Neumann algebras commute with each other so that $\vee$ product should make sense.

Some basic mathematical results of algebraic quantum field theory [A93] deserve to be listed since they are suggestive also from the point of view of TGD.

1. Let $\mathcal{O}$ be a bounded region of R^4 and define the region of M^4 as a union $\cup_{|x|<\epsilon}(\mathcal{O}+x)$ where $(\mathcal{O}+x)$ is the translate of O and $|x|$ denotes Minkowski norm. Then every projection $E \in \mathcal{M}(\mathcal{O})$ can be written as WW^* with $W \in \mathcal{M}(\mathcal{O}_\epsilon)$ and $W^*W = 1$. Note that the union is not a bounded set of M^4. This almost establishes the type III property.

2. Both the complement of light-cone and double light-cone define HFF of type III_1. Lorentz boosts induce modular automorphisms.

3. The so called split property suggested by the description of two systems of this kind as a tensor product in relativistic QFTs is believed to hold true. This means that the HFFs of type III_1 associated with causally disjoint regions are sub-factors of factor of type I_∞. This means

$$\mathcal{M}_1 \subset \mathcal{B}(\mathcal{H}_1) \times 1 \ , \ \mathcal{M}_2 \subset 1 \otimes \mathcal{B}(\mathcal{H}_2) \ .$$

   An infinite hierarchy of inclusions of HFFs of type III_1s is induced by set theoretic inclusions.

3.2.2 TGD and factors

The following vision about TGD and factors relies heavily on zero energy ontology, TGD inspired quantum measurement theory, basic vision about quantum TGD, and bosonic emergence.

The problems

Concerning the role of factors in TGD framework there are several problems of both conceptual and technical character.

1. Conceptual problems

It is safest to start from the conceptual problems and take a role of skeptic.

1. Under what conditions the assumptions of Tomita-Takesaki formula stating the existence of modular automorphism and isomorphy of the factor and its commutant hold true? What is the physical interpretation of the formula $\mathcal{M}' = J\mathcal{M}J$ relating factor and its commutant in TGD framework?

2. Is the identification $M = \Delta^{it}$ sensible is quantum TGD and ZEO, where M-matrix is "complex square root" of exponent of Hamiltonian defining thermodynamical state and the notion of unitary time evolution is given up? The notion of state ω leading to Δ is essentially thermodynamical and one can wonder whether one should take also a "complex square root" of ω to get M-matrix giving rise to a genuine quantum theory.

3. TGD based quantum measurement theory involves both quantum fluctuating degrees of freedom assignable to light-like 3-surfaces and zero modes identifiable as classical degrees of freedom assignable to interior of the space-time sheet. Zero modes have also fermionic counterparts. State preparation should generate entanglement between the quantal and classical states. What this means at the level of von Neumann algebras?

4. What is the TGD counterpart for causal disjointness. At space-time level different space-time sheets could correspond to such regions whereas at embedding space level causally disjoint CDs would represent such regions.

2. Technical problems

There are also more technical questions.

1. What is the von Neumann algebra needed in TGD framework? Does one have a a direct integral over factors? Which factors appear in it? Can one construct the factor as a crossed product of some group G with direct physical interpretation and of naturally appearing factor A? Is A a HFF of type II_∞? assignable to a fixed CD? What is the natural Hilbert space $\mathcal{H}$ in which A acts?

2. What are the geometric transformations inducing modular automorphisms of II_∞ inducing the scaling down of the trace? Is the action of G induced by the boosts in Lorentz group. Could also translations and scalings induce the action? What is the factor associated with the union of Poincare transforms of CD? $log(\Delta)$ is Hermitian algebraically: what does the non-unitarity of $exp(log(\Delta)it)$ mean physically?

3. Could Ω correspond to a vacuum which in conformal degrees of freedom depends on the choice of the sphere S^2 defining the radial coordinate playing the role of complex variable in the case of the radial conformal algebra. Does *-operation in $\mathcal{M}$ correspond to Hermitian conjugation for fermionic oscillator operators and change of sign of super conformal weights?

The exponent of the Kähler-Dirac action gives rise to the exponent of Kähler function as Dirac determinant and fermionic inner product defined by fermionic Feynman rules. It is implausible that this exponent could as such correspond to ω or Δ^{it} having conceptual roots in thermodynamics rather than QFT. If one assumes that the exponent of the Kähler-Dirac action defines a "complex square root" of ω the situation changes. This raises technical questions relating to the notion of square root of ω.

1. Does the complex square root of ω have a polar decomposition to a product of positive definite matrix (square root of the density matrix) and unitary matrix and does $\omega^{1/2}$ correspond to the modulus in the decomposition? Does the square root of Δ have similar decomposition with modulus equal equal to $\Delta^{1/2}$ in standard picture so that modular automorphism, which is inherent property of von Neumann algebra, would not be affected?

2. Δ^{it} or rather its generalization is defined modulo a unitary operator defined by some Hamiltonian and is therefore highly non-unique as such. This non-uniqueness applies also to $|\Delta|$. Could this non-uniqueness correspond to the thermodynamical degrees of freedom?

ZEO and factors

The first question concerns the identification of the Hilbert space associated with the factors in ZEO. As the positive or negative energy part of the zero energy state space or as the entire space of zero energy states? The latter option would look more natural physically and is forced by the condition that the vacuum state is cyclic and separating.

1. The commutant of HFF given as $\mathcal{M}' = J\mathcal{M}J$, where J is involution transforming fermionic oscillator operators and bosonic vector fields to their Hermitian conjugates. Also conformal weights would change sign in the map which conforms with the view that the light-like boundaries of CD are analogous to upper and lower hemispheres of S^2 in conformal field theory. The presence of J representing essentially Hermitian conjugation would suggest that positive and zero energy parts of zero energy states are related by this formula so that state space decomposes to a tensor product of positive and negative energy states and M-matrix can be regarded as a map between these two sub-spaces.

2. The fact that HFF of type II_1 has the algebra of fermionic oscillator operators as a canonical representation makes the situation puzzling for a novice. The assumption that the vacuum is cyclic and separating means that neither creation nor annihilation operators can annihilate it. Therefore Fermionic Fock space cannot appear as the Hilbert space in the Tomita-Takesaki theorem. The paradox is circumvented if the action of * transforms creation operators acting on the positive energy part of the state to annihilation operators acting on negative energy part of the state. If J permutes the two Fock vacuums in their tensor product, the action of S indeed maps permutes the tensor factors associated with $\mathcal{M}$ and $\mathcal{M}'$.

It is far from obvious whether the identification $M = \Delta^{it}$ makes sense in ZEO.

1. In ZEO M-matrix defines time-like entanglement coefficients between positive and negative energy parts of the state. M-matrix is essentially "complex square root" of the density matrix and quantum theory similar square root of thermodynamics. The notion of state as it appears in the theory of HFFs is however essentially thermodynamical. Therefore it is good to ask whether the "complex square root of state" could make sense in the theory of factors.

2. Quantum field theory suggests an obvious proposal concerning the meaning of the square root: one replaces exponent of Hamiltonian with imaginary exponential of action at $T \to 0$ limit. In quantum TGD the exponent of Kähler-Dirac action giving exponent of Kähler function as real exponent could be the manner to take this complex square root. Kähler-Dirac action can therefore be regarded as a "square root" of Kähler action.

3. The identification $M = \Delta^{it}$ relies on the idea of unitary time evolution which is given up in ZEO based on CDs? Is the reduction of the quantum dynamics to a flow a realistic idea? As will be found this automorphism could correspond to a time translation or scaling for either upper or lower light-cone defining CD and can ask whether Δ^{it} corresponds to the exponent of scaling operator L_0 defining single particle propagator as one integrates over t. Its complex square root would correspond to fermionic propagator.

4. In this framework $J\Delta^{it}$ would map the positive energy and negative energy sectors to each other. If the positive and negative energy state spaces can identified by isometry then $M = J\Delta^{it}$ identification can be considered but seems unrealistic. $S = J\Delta^{1/2}$ maps positive and negative energy states to each other: could S or its generalization appear in M-matrix as a part which gives thermodynamics? The exponent of the Kähler-Dirac action does not seem to provide thermodynamical aspect and p-adic thermodynamics suggests strongly the presence exponent of $exp(-L_0/T_p)$ with T_p chose in such manner that consistency with p-adic thermodynamics is obtained. Could the generalization of $J\Delta^{n/2}$ with Δ replaced with its "square root" give rise to padic thermodynamics and also ordinary thermodynamics at the level of density matrix? The minimal option would be that power of Δ^{it} which imaginary value of t is responsible for thermodynamical degrees of freedom whereas everything else is dictated by the unitary S-matrix appearing as phase of the "square root" of ω.

Zero modes and factors

The presence of zero modes justifies quantum measurement theory in TGD framework and the relationship between zero modes and HFFs involves further conceptual problems.

1. The presence of zero modes means that one has a direct integral over HFFs labeled by zero modes which by definition do not contribute to WCW line element. The realization of quantum criticality in terms of Kähler-Dirac action [K88] suggests that also fermionic zero mode degrees of freedom are present and correspond to conserved charges assignable to the critical deformations of the pace-time sheets. Induced Kähler form characterizes the values of zero modes for a given space-time sheet and the symplectic group of light-cone boundary characterizes the quantum fluctuating degrees of freedom. The entanglement between zero modes and quantum fluctuating degrees of freedom is essential for quantum measurement theory. One should understand this entanglement.

2. Physical intuition suggests that classical observables should correspond to longer length scale than quantal ones. Hence it would seem that the interior degrees of freedom outside CD should correspond to classical degrees of freedom correlating with quantum fluctuating degrees of freedom of CD.

3. Quantum criticality means that Kähler-Dirac action allows an infinite number of conserved charges which correspond to deformations leaving metric invariant and therefore act on zero modes. Does this super-conformal algebra commute with the super-conformal algebra associated with quantum fluctuating degrees of freedom? Could the restriction of elements of quantum fluctuating currents to 3-D light-like 3-surfaces actually imply this commutativity. Quantum holography would suggest a duality between these algebras. Quantum measurement theory suggests even 1-1 correspondence between the elements of the two super-conformal algebras. The entanglement between classical and quantum degrees of freedom would mean that prepared quantum states are created by operators for which the operators in the two algebras are entangled in diagonal manner.

4. The notion of finite measurement resolution has become key element of quantum TGD and one should understand how finite measurement resolution is realized in terms of inclusions of hyper-finite factors for which sub-factor defines the resolution in the sense that its action creates states not distinguishable from each other in the resolution used. The notion of finite measurement resolution suggests that one should speak about entanglement between sub-factors and corresponding sub-spaces rather than between states. Connes tensor product would code for the idea that the action of sub-factors is analogous to that of complex numbers and tracing over sub-factor realizes this idea.

5. Just for fun one can ask whether the duality between zero modes and quantum fluctuating degrees of freedom representing quantum holography could correspond to $\mathcal{M}' = J\mathcal{M}J$? This interpretation must be consistent with the interpretation forced by zero energy ontology. If this crazy guess is correct (very probably not!), both positive and negative energy states would be observed in quantum measurement but in totally different manner. Since this identity would simplify enormously the structure of the theory, it deserves therefore to be shown wrong.

Crossed product construction in TGD framework

The identification of the von Neumann algebra by crossed product construction is the basic challenge. Consider first the question how HFFs of type II_∞ emerge, how modular automorphisms act on them, and how one can understand the non-unitary character of the Δ^{it} in an apparent conflict with the hermiticity and positivity of Δ.

1. The Clifford algebra at a given point of WCW(CD) (light-like 3-surfaces with ends at the boundaries of CD) defines HFF of type II_1 or possibly a direct integral of them. For a given CD having compact isotropy group $SO(3)$ leaving the rest frame defined by the tips of CD invariant the factor defined by Clifford algebra valued fields in WCW(CD) is most naturally HFF of type II_∞. The Hilbert space in which this Clifford algebra acts, consists of spinor fields in WCW(CD). Also the symplectic transformations of light-cone boundary leaving light-like 3-surfaces inside CD can be included to G. In fact all conformal algebras leaving CD invariant could be included in CD.

2. The downwards scalings of the radial coordinate r_M of the light-cone boundary applied to the basis of WCW (CD) spinor fields could induce modular automorphism. These scalings reduce the size of the portion of light-cone in which the WCW spinor fields are non-vanishing and effectively scale down the size of CD. $exp(iL_0)$ as algebraic operator acts as a phase multiplication on eigen states of conformal weight and therefore as apparently unitary operator. The geometric flow however contracts the CD so that the interpretation of $exp(itL_0)$ as a unitary modular automorphism is not possible. The scaling down of CD reduces the value of the trace if it involves integral over the boundary of CD. A similar reduction is implied by the downward shift of the upper boundary of CD so that also time translations would induce

modular automorphism. These shifts seem to be necessary to define rest energies of positive and negative energy parts of the zero energy state.

3. The non-triviality of the modular automorphisms of II_∞ factor reflects different choices of ω. The degeneracy of ω could be due to the non-uniqueness of conformal vacuum which is part of the definition of ω. The radial Virasoro algebra of light-cone boundary is generated by $L_n = L_{-n}^*$, $n \neq 0$ and $L_0 = L_0^*$ and negative and positive frequencies are in asymmetric position. The conformal gauge is fixed by the choice of $SO(3)$ subgroup of Lorentz group defining the slicing of light-cone boundary by spheres and the tips of CD fix $SO(3)$ uniquely. One can however consider also alternative choices of $SO(3)$ and each corresponds to a slicing of the light-cone boundary by spheres but in general the sphere defining the intersection of the two light-cone does not belong to the slicing. Hence the action of Lorentz transformation inducing different choice of $SO(3)$ can lead out from the preferred state space so that its representation must be non-unitary unless Virasoro generators annihilate the physical states. The non-vanishing of the conformal central charge c and vacuum weight h seems to be necessary and indeed can take place for super-symplectic algebra and Super Kac-Moody algebra since only the differences of the algebra elements are assumed to annihilate physical states.

Modular automorphism of HFFs type III_1 can be induced by several geometric transformations for HFFs of type III_1 obtained using the crossed product construction from II_∞ factor by extending CD to a union of its Lorentz transforms.

1. The crossed product would correspond to an extension of II_∞ by allowing a union of some geometric transforms of CD. If one assumes that only CDs for which the distance between tips is quantized in powers of 2, then scalings of either upper or lower boundary of CD cannot correspond to these transformations. Same applies to time translations acting on either boundary but not to ordinary translations. As found, the modular automorphisms reducing the size of CD could act in HFF of type II_∞.

2. The geometric counterparts of the modular transformations would most naturally correspond to any non-compact one parameter sub-group of Lorentz group as also QFT suggests. The Lorentz boosts would replace the radial coordinate r_M of the light-cone boundary associated with the radial Virasoro algebra with a new one so that the slicing of light-cone boundary with spheres would be affected and one could speak of a new conformal gauge. The temporal distance between tips of CD in the rest frame would not be affected. The effect would seem to be however unitary because the transformation does not only modify the states but also transforms CD.

3. Since Lorentz boosts affect the isotropy group $SO(3)$ of CD and thus also the conformal gauge defining the radial coordinate of the light-cone boundary, they affect also the definition of the conformal vacuum so that also ω is affected so that the interpretation as a modular automorphism makes sense. The simplistic intuition of the novice suggests that if one allows wave functions in the space of Lorentz transforms of CD, unitarity of Δ^{it} is possible. Note that the hierarchy of Planck constants assigns to CD preferred M^2 and thus direction of quantization axes of angular momentum and boosts in this direction would be in preferred role.

4. One can also consider the HFF of type III_λ if the radial scalings by negative powers of 2 correspond to the automorphism group of II_∞ factor as the vision about allowed CDs suggests. $\lambda = 1/2$ would naturally hold true for the factor obtained by allowing only the radial scalings. Lorentz boosts would expand the factor to HFF of type III_1. Why scalings by powers of 2 would give rise to periodicity should be understood.

The identification of M-matrix as modular automorphism Δ^{it}, where t is complex number having as its real part the temporal distance between tips of CD quantized as 2^n and temperature as imaginary part, looks at first highly attractive, since it would mean that M-matrix indeed exists mathematically. The proposed interpretations of modular automorphisms do not support the idea that they could define the S-matrix of the theory. In any case, the identification as modular automorphism would not lead to a magic universal formula since arbitrary unitary transformation is involved.

Quantum criticality and inclusions of factors

Quantum criticality fixes the value of Kähler coupling strength but is expected to have also an interpretation in terms of a hierarchies of broken conformal gauge symmetries suggesting hierarchies of inclusions.

1. In ZEO 3-surfaces are unions of space-like 3-surfaces at the ends of causal diamond (CD). Space-time surfaces connect 3-surfaces at the boundaries of CD. The non-determinism of Kähler action allows the possibility of having several space-time sheets connecting the ends of space-time surface but the conditions that classical charges are same for them reduces this number so that it could be finite. Quantum criticality in this sense implies non-determinism analogous to that of critical systems since preferred extremals can co-incide and suffer this kind of bifurcation in the interior of CD. This quantum criticality can be assigned to the hierarchy of Planck constants and the integer n in $h_{eff} = n \times h$ [K27] corresponds to the number of degenerate space-time sheets with same Kähler action and conserved classical charges.

2. Also now one expects a hierarchy of criticalities and since criticality and conformal invariance are closely related, a natural conjecture is that the fractal hierarchy of sub-algebras of conformal algebra isomorphic to conformal algebra itself and having conformal weights coming as multiples of n corresponds to the hierarchy of Planck constants. This hierarchy would define a hierarchy of symmetry breakings in the sense that only the sub-algebra would act as gauge symmetries.

3. The assignment of this hierarchy with super-symplectic algebra having conformal structure with respect to the light-like radial coordinate of light-cone boundary looks very attractive. An interesting question is what is the role of the super-conformal algebra associated with the isometries of light-cone boundary $R_+ \times S^2$ which are conformal transformations of sphere S^2 with a scaling of radial coordinate compensating the scaling induced by the conformal transformation. Does it act as dynamical or gauge symmetries?

4. The natural proposal is that the inclusions of various superconformal algebras in the hierarchy define inclusions of hyper-finite factors which would be thus labelled by integers. Any sequences of integers for which n_i divides n_{i+1} would define a hierarchy of inclusions proceeding in reverse direction. Physically inclusion hierarchy would correspond to an infinite hierarchy of criticalities within criticalities.

3.2.3 Can one identify M-matrix from physical arguments?

Consider next the identification of M-matrix from physical arguments from the point of view of factors.

A proposal for M-matrix

The proposed general picture reduces the core of U-matrix to the construction of S-matrix possibly having the real square roots of density matrices as symmetry algebra. This structure can be taken as a template as one tries to to imagine how the construction of M-matrix could proceed in quantum TGD proper.

1. At the bosonic sector one would have converging functional integral over WCW . This is analogous to the path integral over bosonic fields in QFTs. The presence of Kähler function would make this integral well-defined and would not encounter the difficulties met in the case of path integrals.

2. In fermionic sector 1-D Dirac action and its bosonic counterpart imply that spinors modes localized at string world sheets are eigenstates of induced Dirac operator with generalized

eigenvalue $p^k\gamma_k$ defining light-like 8-D momentum so that one would obtain fermionic propagators massless in 8-D sense at light-light geodesics of embedding space. The 8-D generalization of twistor Grassmann approach is suggestive and would mean that the residue integral over fermionic virtual momenta gives only integral over massless momenta and virtual fermions differ from real fermions only in that they have non-physical polarizations so that massless Dirac operator replacing the propagator does not annihilate the spinors at the other end of the line.

3. Fundamental bosons (not elementary particles) correspond to wormhole contacts having fermion and antifermion at opposite throats and bosonic propagators are composite of massless fermion propagators. The directions of virtual momenta are obviously strongly correlated so that the approximation as a gauge theory with gauge symmetry breaking in almost massless sector is natural. Massivation follows necessary from the fact that also elementary particles are bound states of two wormhole contacts.

4. Physical fermions and bosons correspond to pairs of wormhole contacts with throats carrying Kähler magnetic charge equal to Kähler electric charge (dyon). The absence of Dirac monopoles (as opposed to homological magnetic monopoles due to CP_2 topology) implies that wormhole contacts must appear as pairs (also large numbers of them are possible and 3 valence quarks inside baryons could form Kähler magnetic tripole). Hence elementary particles would correspond to pairs of monopoles and are accompanied by Kähler magnetic flux loop running along the two space-time sheets involved as well as fermionic strings connecting the monopole throats.

   There seems to be no specific need to assign string to the wormhole contact and if is a piece of deformed CP_2 type vacuum extremal this might not be even possible: the Kähler-Dirac gamma matrices would not span 2-D space in this case since the CP_2 projection is 4-D. Hence massless fermion propagators would be assigned only with the boundaries of string world sheets at Minkowskian regions of space-time surface. One could say that physical particles are bound states of massless fundamental fermions and the non-collinearity of their four-momenta can make them massive. Therefore the breaking of conformal invariance would be due to the bound state formation and this would also resolve the infrared divergence problems plaguing Grassmann twistor approach by introducing natural length scale assignable to the size of particles defined by the string like flux tube connecting the wormhole contacts. This point is discussed in more detail in [K77].

   The bound states would form representations of super-conformal algebras so that stringy mass formula would emerge naturally. p-Adic mass calculations indeed assume conformal invariance in $CP2$ length scale assignable to wormhole contacts. Also the long flux tube strings contribute to the particle masses and would explain gauge boson masses.

5. The interaction vertices would correspond topologically to decays of 3-surface by splitting in complete analogy with ordinary Feynman diagrams. At the level of orbits of partonic 2-surface the vertices would be represented by partonic 2-surfaces. In [K77] the interpretation of scattering ampiltudes as sequences of algebraic operations for the Yangian of super-symplectic algebra is proposed: product and co-product would define time 3-vertex and its time reversal. At the level of fermions the diagrams reduce to braid diagrams since fermions are "free". At vertices fermions can however reflect in time direction so that fermion-antifermion annihilations in classical fields can be said to appear in the vertices.

   The Yangian is generated by super-symplectic fermionic Noether charges assignable to the strings connecting partonic 2-surfaces. The interpretation of vertices as algebraic operations implies that all sequences of operations connecting given collections of elements of Yangian at the opposite boundaries of CD give rise to the same amplitude. This means a huge generalization of the duality symmetry of hadronic string models that I have proposed already earlier: the chapter [K9] is a remnant of an "idea that came too early". The propagators are associated with the fermionic lines identifiable as boundaries of string world sheets. These lines are light-like geodesics of H and fermion lines correspond topartial wave in the space S^3 of light like 8-momenta with fixed M^4 momentum. For external lines M^8 momentum corresponds to the $M^4 \times CP_2$ quantum numbers of a spinor harmonic.

The amplitudes can be formulated using only partonic 2-surfaces and string world sheets and the algebraic continuation to achieve number theoretic Universality should be rather straightforward: the parameters characterizing 2-surfaces - by conformal invariance various conformal moduli - in the algebraic extension of rationals are replaced with real and various p-adic numbers.

6. Wormhole contacts represent fundamental interaction vertex pairs and propagators between them and one has stringy super-conformal invariance. Therefore there are excellent reasons to expect that the perturbation theory is free of divergences. Without stringy contributions for massive conformal excitations of wormhole contacts one would obtain the usual logarithmic UV divergences of massless gauge theories. The fact that physical particles are bound states of massless particles, gives good hopes of avoiding IR divergences of massless theories.

The figures **??**, **??** (`http://tgdtheory.fi/appfigures/elparticletgd.jpg` `http://tgdtheory.fi/appfigures/tgdgrpahs.jpg`) in the appendix of this book illustrate the relationship between TGD diagrammatics, QFT diagrammatics and stringy diagrammatics. In [K77] a more detailed construction based on the generalization of twistor approach and the idea that scattering amplitudes represent sequences of algebraic operation in the Yangian of super-symplectic algebra, is considered.

Quantum TGD as square root of thermodynamics

ZEO (ZEO) suggests strongly that quantum TGD corresponds to what might be called square root of thermodynamics. Since fermionic sector of TGD corresponds naturally to a hyper-finite factor of type II_1, and super-conformal sector relates fermionic and bosonic sectors (WCW degrees of freedom), there is a temptation to suggest that the mathematics of von Neumann algebras generalizes: in other worlds it is possible to speak about the complex square root of ω defining a state of von Neumann algebra [A118] [K87]. This square root would bring in also the fermionic sector and realized super-conformal symmetry. The reduction of determinant with WCW vacuum functional would be one manifestation of this supersymmetry.

The exponent of Kähler function identified as real part of Kähler action for preferred extremals coming from Euclidian space-time regions defines the modulus of the bosonic vacuum functional appearing in the functional integral over WCW. The imaginary part of Kähler action coming from the Minkowskian regions is analogous to action of quantum field theories and would give rise to interference effects distinguishing thermodynamics from quantum theory. This would be something new from the point of view of the canonical theory of von Neumann algebra. The saddle points of the imaginary part appear in stationary phase approximation and the imaginary part serves the role of Morse function for WCW.

The exponent of Kähler function depends on the real part of t identified as Minkowski distance between the tips of CD. This dependence is not consistent with the dependence of the canonical unitary automorphism Δ^{it} of von Neumann algebra on t [A118], [K87] and the natural interpretation is that the vacuum functional can be included in the definition of the inner product for spinors fields of WCW. More formally, the exponent of Kähler function would define ω in bosonic degrees of freedom.

Note that the imaginary exponent is more natural for the imaginary part of Kähler action coming from Minkowskian region. In any case, one has combination of thermodynamics and QFT and the presence of thermodynamics makes the functional integral mathematically well-defined.

Number theoretic vision requiring number theoretical universality suggests that the value of CD size scales as defined by the distance between the tips is expected to come as integer multiples of CP_2 length scale - at least in the intersection of real and p-adic worlds. If this is the case the continuous faimily of modular automorphisms would be replaced with a discretize family.

Quantum criticality and hierarchy of inclusions

Quantum criticality and related fractal hierarchies of breakings of conformal symmetry could allow to understand the inclusion hierarchies for hyper-finite factors. Quantum criticality - implied by the condition that the Kähler-Dirac action gives rise to conserved currents assignable to the deformations of the space-time surface - means the vanishing of the second variation of Kähler action for these deformations. Preferred extremals correspond to these 4-surfaces and $M^8 - M^4 \times CP_2$ duality would allow to identify them also as associative (co-associative) space-time surfaces.

Quantum criticality is basically due to the failure of strict determinism for Kähler action and leads to the hierarchy of dark matter phases labelled by the effective value of Planck constant $h_{eff} = n \times h$. These phases correspond to space-time surfaces connecting 3-surfaces at the ends of CD which are multi-sheeted having n conformal equivalence classes.

Conformal invariance indeed relates naturally to quantum criticality. This brings in n discrete degrees of freedom and one can technically describe the situation by using n-fold singular covering of the embedding space [K27]. One can say that there is hierarchy of broken conformal symmetries in the sense that for $h_{eff} = n \times h$ the sub-algebra of conformal algebras with conformal weights coming as multiples of n act as gauge symmetries. This implies that classical symplectic Noether charges vanish for this sub-algebra. The quantal conformal charges associated with induced spinor fields annihilate the physical states. Therefore it seems that the measured quantities are the symplectic charges and there is not need to introduce any measurement interaction term and the formalism simplifies dramatically.

The resolution increases with $h_{eff}/h = n$. Also the number of of strings connecting partonic 2-surfaces (in practice elementary particles and their dark counterparts plus bound states generated by connecting dark strings) characterizes physically the finite measurement resolution. Their presence is also visible in the geometry of the space-time surfaces through the conditions that induced W fields vanish at them (well-definedness of em charge), and by the condition that the canonical momentum currents for Kähler action define an integrable distribution of planes parallel to the string world sheet. In spirit with holography, preferred extremal is constructed by fixing string world sheets and partonic 2-surfaces and possibly also their light-like orbits (should one fix wormhole contacts is not quite clear). If the analog of AdS/CFT correspondence holds true, the value of Kähler function is expressible as the energy of string defined by area in the effective metric defined by the anti-commutators of K-D gamma matrices.

Super-symplectic algebra, whose charges are represented by Noether charges associated with strings connecting partonic 2-surfaces extends to a Yangian algebra with multi-stringy generators [K77]. The better the measurement resolution, the larger the maximal number of strings associated with the multilocal generator.

Kac-Moody type transformations preserving light-likeness of partonic orbits and possibly also the light-like character of the boundaries of string world sheets carrying modes of induced spinor field underlie the conformal gauge symmetry. The minimal option is that only the light-likeness of the string end world line is preserved by the conformal symmetries. In fact, conformal symmetries was originally deduced from the light-likeness condition for the M^4 projection of CP_2 type vacuum extremals.

The inclusions of super-symplectic Yangians form a hierarchy and would naturally correspond to inclusions of hyperfinite factors of type II_1. Conformal symmetries acting as gauge transformations would naturally correspond to degrees of freedom below measurement resolution and would correspond to included subalgebra. As h_{eff} increases, infinite number of these gauge degrees of freedom become dynamical and measurement resolution is increased. This picture is definitely in conflict with the original view but the reduction of criticality in the increase of h_{eff} forces it.

Summary

On basis of above considerations it seems that the idea about "complex square root" of the state ω of von Neumann algebras might make sense in quantum TGD. Also the discretized versions of modular automorphism assignable to the hierarchy of CDs would make sense and because of its non-uniqueness the generator Δ of the canonical automorphism could bring in the flexibility needed one wants thermodynamics. Stringy picture forces to ask whether Δ could in some situation be proportional $exp(L_0)$, where L_0 represents as the infinitesimal scaling generator of either super-symplectic algebra or super Kac-Moody algebra (the choice does not matter since the differences of the generators annihilate physical states in coset construction). This would allow to reproduce real thermodynamics consistent with p-adic thermodynamics. Note that also p-adic thermodynamics would be replaced by its square root in ZEO.

3.2.4 Finite measurement resolution and HFFs

The finite resolution of quantum measurement leads in TGD framework naturally to the notion of quantum M-matrix for which elements have values in sub-factor $\mathcal{N}$ of HFF rather than being complex numbers. M-matrix in the factor space $\mathcal{M}/\mathcal{N}$ is obtained by tracing over $\mathcal{N}$. The condition that $\mathcal{N}$ acts like complex numbers in the tracing implies that M-matrix elements are proportional to maximal projectors to $\mathcal{N}$ so that M-matrix is effectively a matrix in $\mathcal{M}/\mathcal{N}$ and situation becomes finite-dimensional. It is still possible to satisfy generalized unitarity conditions but in general case tracing gives a weighted sum of unitary M-matrices defining what can be regarded as a square root of density matrix.

About the notion of observable in ZEO

Some clarifications concerning the notion of observable in zero energy ontology are in order.

1. As in standard quantum theory observables correspond to hermitian operators acting on either positive or negative energy part of the state. One can indeed define hermitian conjugation for positive and negative energy parts of the states in standard manner.

2. Also the conjugation $A \rightarrow JAJ$ is analogous to hermitian conjugation. It exchanges the positive and negative energy parts of the states also maps the light-like 3-surfaces at the upper boundary of CD to the lower boundary and vice versa. The map is induced by time reflection in the rest frame of CD with respect to the origin at the center of CD and has a well defined action on light-like 3-surfaces and space-time surfaces. This operation cannot correspond to the sought for hermitian conjugation since JAJ and A commute.

3. In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute to the Kähler action. Chern-Simons Dirac terms to which Kähler action reduces could be responsible for the breaking of CP and T symmetries as they appear in CKM matrix.

4. ZEO gives Cartan sub-algebra of the Lie algebra of symmetries a special status. Only Cartan algebra acting on either positive or negative states respects the zero energy property but this is enough to define quantum numbers of the state. In absence of symmetry breaking positive and negative energy parts of the state combine to form a state in a singlet representation of group. Since only the net quantum numbers must vanish ZEO allows a symmetry breaking respecting a chosen Cartan algebra.

5. In order to speak about four-momenta for positive and negative energy parts of the states one must be able to define how the translations act on CDs. The most natural action is a shift of the upper (lower) tip of CD. In the scale of entire CD this transformation induced Lorentz boost fixing the other tip. The value of mass squared is identified as proportional to the average of conformal weight in p-adic thermodynamics for the scaling generator L_0 for either super-symplectic or Super Kac-Moody algebra.

Inclusion of HFFs as characterizer of finite measurement resolution at the level of S-matrix

The inclusion $\mathcal{N} \subset \mathcal{M}$ of factors characterizes naturally finite measurement resolution. This means following things.

1. Complex rays of state space resulting usually in an ideal state function reduction are replaced by $\mathcal{N}$-rays since $\mathcal{N}$ defines the measurement resolution and takes the role of complex numbers in ordinary quantum theory so that non-commutative quantum theory results. Non-commutativity corresponds to a finite measurement resolution rather than something exotic occurring in Planck length scales. The quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ creates physical states

modulo resolution. The fact that $\mathcal{N}$ takes the role of gauge algebra suggests that it might be necessary to fix a gauge by assigning to each element of $\mathcal{M}/\mathcal{N}$ a unique element of $\mathcal{M}$. Quantum Clifford algebra with fractal dimension $\beta = \mathcal{M} : \mathcal{N}$ creates physical states having interpretation as quantum spinors of fractal dimension $d = \sqrt{\beta}$. Hence direct connection with quantum groups emerges.

2. The notions of unitarity, hermiticity, and eigenvalue generalize. The elements of unitary and hermitian matrices and $\mathcal{N}$-valued. Eigenvalues are Hermitian elements of $\mathcal{N}$ and thus correspond entire spectra of Hermitian operators. The mutual non-commutativity of eigenvalues guarantees that it is possible to speak about state function reduction for quantum spinors. In the simplest case of a 2-component quantum spinor this means that second component of quantum spinor vanishes in the sense that second component of spinor annihilates physical state and second acts as element of $\mathcal{N}$ on it. The non-commutativity of spinor components implies correlations between then and thus fractal dimension is smaller than 2.

3. The intuition about ordinary tensor products suggests that one can decompose Tr in $\mathcal{M}$ as

$$Tr_{\mathcal{M}}(X) = Tr_{\mathcal{M}/\mathcal{N}} \times Tr_{\mathcal{N}}(X) \ . \tag{3.2.4}$$

Suppose one has fixed gauge by selecting basis $|r_k\rangle$ for $\mathcal{M}/\mathcal{N}$. In this case one expects that operator in $\mathcal{M}$ defines an operator in $\mathcal{M}/\mathcal{N}$ by a projection to the preferred elements of $\mathcal{M}$.

$$\langle r_1|X|r_2\rangle = \langle r_1|Tr_{\mathcal{N}}(X)|r_2\rangle \ . \tag{3.2.5}$$

4. Scattering probabilities in the resolution defined by $\mathcal{N}$ are obtained in the following manner. The scattering probability between states $|r_1\rangle$ and $|r_2\rangle$ is obtained by summing over the final states obtained by the action of $\mathcal{N}$ from $|r_2\rangle$ and taking the analog of spin average over the states created in the similar from $|r_1\rangle$. $\mathcal{N}$ average requires a division by $Tr(P_{\mathcal{N}}) = 1/\mathcal{M} : \mathcal{N}$ defining fractal dimension of $\mathcal{N}$. This gives

$$p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times \langle r_1|Tr_{\mathcal{N}}(SP_{\mathcal{N}}S^{\dagger})|r_2\rangle \ . \tag{3.2.6}$$

This formula is consistent with probability conservation since one has

$$\sum_{r_2} p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times Tr_N(SS^{\dagger}) = \mathcal{M} : \mathcal{N} \times Tr(P_N) = 1 \ . \tag{3.2.7}$$

5. Unitarity at the level of $\mathcal{M}/\mathcal{N}$ can be achieved if the unit operator Id for $\mathcal{M}$ can be decomposed into an analog of tensor product for the unit operators of $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$ and M decomposes to a tensor product of unitary M-matrices in $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$. For HFFs of type II projection operators of $\mathcal{N}$ with varying traces are present and one expects a weighted sum of unitary M-matrices to result from the tracing having interpretation in terms of square root of thermodynamics.

6. This argument assumes that $\mathcal{N}$ is HFF of type II_1 with finite trace. For HFFs of type III_1 this assumption must be given up. This might be possible if one compensates the trace over $\mathcal{N}$ by dividing with the trace of the infinite trace of the projection operator to $\mathcal{N}$. This probably requires a limiting procedure which indeed makes sense for HFFs.

Quantum M-matrix

The description of finite measurement resolution in terms of inclusion $\mathcal{N} \subset \mathcal{M}$ seems to boil down to a simple rule. Replace ordinary quantum mechanics in complex number field C with that in $\mathcal{N}$. This means that the notions of unitarity, hermiticity, Hilbert space ray, etc.. are replaced with their $\mathcal{N}$ counterparts.

The full M-matrix in $\mathcal{M}$ should be reducible to a finite-dimensional quantum M-matrix in the state space generated by quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ which can be regarded as a finite-dimensional matrix algebra with non-commuting $\mathcal{N}$-valued matrix elements. This suggests that full M-matrix can be expressed as M-matrix with $\mathcal{N}$-valued elements satisfying $\mathcal{N}$-unitarity conditions.

Physical intuition also suggests that the transition probabilities defined by quantum S-matrix must be commuting hermitian $\mathcal{N}$-valued operators inside every row and column. The traces of these operators give $\mathcal{N}$-averaged transition probabilities. The eigenvalue spectrum of these Hermitian matrices gives more detailed information about details below experimental resolution. $\mathcal{N}$-hermicity and commutativity pose powerful additional restrictions on the M-matrix.

Quantum M-matrix defines $\mathcal{N}$-valued entanglement coefficients between quantum states with $\mathcal{N}$-valued coefficients. How this affects the situation? The non-commutativity of quantum spinors has a natural interpretation in terms of fuzzy state function reduction meaning that quantum spinor corresponds effectively to a statistical ensemble which cannot correspond to pure state. Does this mean that predictions for transition probabilities must be averaged over the ensemble defined by "quantum quantum states"?

Quantum fluctuations and inclusions

Inclusions $\mathcal{N} \subset \mathcal{M}$ of factors provide also a first principle description of quantum fluctuations since quantum fluctuations are by definition quantum dynamics below the measurement resolution. This gives hopes for articulating precisely what the important phrase "long range quantum fluctuations around quantum criticality" really means mathematically.

1. Phase transitions involve a change of symmetry. One might hope that the change of the symmetry group $G_a \times G_b$ could universally code this aspect of phase transitions. This need not always mean a change of Planck constant but it means always a leakage between sectors of embedding space. At quantum criticality 3-surfaces would have regions belonging to at least two sectors of H.

2. The long range of quantum fluctuations would naturally relate to a partial or total leakage of the 3-surface to a sector of embedding space with larger Planck constant meaning zooming up of various quantal lengths.

3. For M-matrix in $\mathcal{M}/\mathcal{N}$ regarded as $calN$ module quantum criticality would mean a special kind of eigen state for the transition probability operator defined by the M-matrix. The properties of the number theoretic braids contributing to the M-matrix should characterize this state. The strands of the critical braids would correspond to fixed points for $G_a \times G_b$ or its subgroup.

M-matrix in finite measurement resolution

The following arguments relying on the proposed identification of the space of zero energy states give a precise formulation for M-matrix in finite measurement resolution and the Connes tensor product involved. The original expectation that Connes tensor product could lead to a unique M-matrix is wrong. The replacement of ω with its complex square root could lead to a unique hierarchy of M-matrices with finite measurement resolution and allow completely finite theory despite the fact that projectors have infinite trace for HFFs of type III_1.

1. In ZEO the counterpart of Hermitian conjugation for operator is replaced with $\mathcal{M} \to J\mathcal{M}J$ permuting the factors. Therefore $N \in \mathcal{N}$ acting to positive (negative) energy part of state corresponds to $N \to N' = JNJ$ acting on negative (positive) energy part of the state.

2. The allowed elements of N much be such that zero energy state remains zero energy state. The superposition of zero energy states involved can however change. Hence one must have that the counterparts of complex numbers are of form $N = JN_1J \vee N_2$, where N_1 and N_2 have same quantum numbers. A superposition of terms of this kind with varying quantum numbers for positive energy part of the state is possible.

3. The condition that N_{1i} and N_{2i} act like complex numbers in $\mathcal{N}$-trace means that the effect of $JN_{1i}J \vee N_{2i}$ and $JN_{2i}Ji \vee N_{1i}$ to the trace are identical and correspond to a multiplication by a constant. If $\mathcal{N}$ is HFF of type II_1 this follows from the decomposition $\mathcal{M} = \mathcal{M}/\mathcal{N} \otimes \mathcal{N}$ and from $Tr(AB) = Tr(BA)$ assuming that M is of form $M = M_{\mathcal{M}/\mathcal{N}} \times P_{\mathcal{N}}$. Contrary to the original hopes that Connes tensor product could fix the M-matrix there are no conditions on $M_{\mathcal{M}/\mathcal{N}}$ which would give rise to a finite-dimensional M-matrix for Jones inclusions. One can replaced the projector $P_{\mathcal{N}}$ with a more general state if one takes this into account in * operation.

4. In the case of HFFs of type III_1 the trace is infinite so that the replacement of Tr_N with a state ω_N in the sense of factors looks more natural. This means that the counterpart of * operation exchanging N_1 and N_2 represented as $SA\Omega = A^*\Omega$ involves Δ via $S = J\Delta^{1/2}$. The exchange of N_1 and N_2 gives altogether Δ. In this case the KMS condition $\omega_{\mathcal{N}}(AB) = \omega_{\mathcal{N}}\Delta A)$ guarantees the effective complex number property [A8] .

5. Quantum TGD more or less requires the replacement of ω with its "complex square root" so that also a unitary matrix U multiplying Δ is expected to appear in the formula for S and guarantee the symmetry. One could speak of a square root of KMS condition [A8] in this case. The QFT counterpart would be a cutoff involving path integral over the degrees of freedom below the measurement resolution. In TGD framework it would mean a cutoff in the functional integral over WCW and for the modes of the second quantized induced spinor fields and also cutoff in sizes of causal diamonds. Discretization in terms of braids replacing light-like 3-surfaces should be the counterpart for the cutoff.

6. If one has M-matrix in $\mathcal{M}$ expressible as a sum of M-matrices of form $M_{\mathcal{M}/\mathcal{N}} \times M_{\mathcal{N}}$ with coefficients which correspond to the square roots of probabilities defining density matrix the tracing operation gives rise to square root of density matrix in M.

Is universal M-matrix possible?

The realization of the finite measurement resolution could apply only to transition probabilities in which $\mathcal{N}$-trace or its generalization in terms of state ω_N is needed. One might however dream of something more.

1. Maybe there exists a universal M-matrix in the sense that the same M-matrix gives the M-matrices in finite measurement resolution for all inclusions $\mathcal{N} \subset \mathcal{M}$. This would mean that one can write

$$M = M_{\mathcal{M}/\mathcal{N}} \otimes M_{\mathcal{N}} \tag{3.2.8}$$

for any physically reasonable choice of $\mathcal{N}$. This would formally express the idea that M is as near as possible to M-matrix of free theory. Also fractality suggests itself in the sense that $M_{\mathcal{N}}$ is essentially the same as $M_{\mathcal{M}}$ in the same sense as $\mathcal{N}$ is same as $\mathcal{M}$. It might be that the trivial solution $M = 1$ is the only possible solution to the condition.

2. $M_{\mathcal{M}/\mathcal{N}}$ would be obtained by the analog of $Tr_{\mathcal{N}}$ or ω_N operation involving the "complex square root" of the state ω in case of HFFs of type III_1. The QFT counterpart would be path integration over the degrees of freedom below cutoff to get effective action.

3. Universality probably requires assumptions about the thermodynamical part of the universal M-matrix. A possible alternative form of the condition is that it holds true only for canonical choice of "complex square root" of ω or for the S-matrix part of M:

$$S = S_{\mathcal{M}/\mathcal{N}} \otimes S_{\mathcal{N}} \tag{3.2.9}$$

for any physically reasonable choice $\mathcal{N}$.

4. In TGD framework the condition would say that the M-matrix defined by the Kähler-Dirac action gives M-matrices in finite measurement resolution via the counterpart of integration over the degrees of freedom below the measurement resolution.

An obvious counter argument against the universality is that if the M-matrix is "complex square root of state" cannot be unique and there are infinitely many choices related by a unitary transformation induced by the flows representing modular automorphism giving rise to new choices. This would actually be a well-come result and make possible quantum measurement theory.

In the section "Handful of problems with a common resolution" it was found that one can add to both Kähler action and Kähler-Dirac action a measurement interaction term characterizing the values of measured observables. The measurement interaction term in Kähler action is Lagrange multiplier term at the space-like ends of space-time surface fixing the value of classical charges for the space-time sheets in the quantum superposition to be equal with corresponding quantum charges. The term in Kähler-Dirac action is obtained from this by assigning to this term canonical momentum densities and contracting them with gamma matrices to obtain Kähler-Dirac gamma matrices appearing in 3-D analog of Dirac action. The constraint terms would leave Kähler function and Kähler metric invariant but would restrict the vacuum functional to the subset of 3-surfaces with fixed classical conserved charges (in Cartan algebra) equal to their quantum counterparts.

Connes tensor product and space-like entanglement

Ordinary linear Connes tensor product makes sense also in positive/negative energy sector and also now it makes sense to speak about measurement resolution. Hence one can ask whether Connes tensor product should be posed as a constraint on space-like entanglement. The interpretation could be in terms of the formation of bound states. The reducibility of HFFs and inclusions means that the tensor product is not uniquely fixed and ordinary entanglement could correspond to this kind of entanglement.

Also the counterpart of p-adic coupling constant evolution would makes sense. The interpretation of Connes tensor product would be as the variance of the states with respect to some subgroup of $U(n)$ associated with the measurement resolution: the analog of color confinement would be in question.

2-vector spaces and entanglement modulo measurement resolution

John Baez and collaborators [A85] are playing with very formal looking formal structures obtained by replacing vectors with vector spaces. Direct sum and tensor product serve as the basic arithmetic operations for the vector spaces and one can define category of n-tuples of vectors spaces with morphisms defined by linear maps between vectors spaces of the tuple. n-tuples allow also element-wise product and sum. They obtain results which make them happy. For instance, the category of linear representations of a given group forms 2-vector spaces since direct sums and tensor products of representations as well as n-tuples make sense. The 2-vector space however looks more or less trivial from the point of physics.

The situation could become more interesting in quantum measurement theory with finite measurement resolution described in terms of inclusions of hyper-finite factors of type II_1. The reason is that Connes tensor product replaces ordinary tensor product and brings in interactions via irreducible entanglement as a representation of finite measurement resolution. The category in question could give Connes tensor products of quantum state spaces and describing interactions. For instance, one could multiply M-matrices via Connes tensor product to obtain category of M-matrices having also the structure of 2-operator algebra.

1. The included algebra represents measurement resolution and this means that the infinite-D sub-Hilbert spaces obtained by the action of this algebra replace the rays. Sub-factor

takes the role of complex numbers in generalized QM so that one obtains non-commutative quantum mechanics. For instance, quantum entanglement for two systems of this kind would not be between rays but between infinite-D subspaces corresponding to sub-factors. One could build a generalization of QM by replacing rays with sub-spaces and it would seem that quantum group concept does more or less this: the states in representations of quantum groups could be seen as infinite-dimensional Hilbert spaces.

2. One could speak about both operator algebras and corresponding state spaces modulo finite measurement resolution as quantum operator algebras and quantum state spaces with fractal dimension defined as factor space like entities obtained from HFF by dividing with the action of included HFF. Possible values of the fractal dimension are fixed completely for Jones inclusions. Maybe these quantum state spaces could define the notions of quantum 2-Hilbert space and 2-operator algebra via direct sum and tensor production operations. Fractal dimensions would make the situation interesting both mathematically and physically.

Suppose one takes the fractal factor spaces as the basic structures and keeps the information about inclusion.

1. Direct sums for quantum vectors spaces would be just ordinary direct sums with HFF containing included algebras replaced with direct sum of included HFFs.

2. The tensor products for quantum state spaces and quantum operator algebras are not anymore trivial. The condition that measurement algebras act effectively like complex numbers would require Connes tensor product involving irreducible entanglement between elements belonging to the two HFFs. This would have direct physical relevance since this entanglement cannot be reduced in state function reduction. The category would defined interactions in terms of Connes tensor product and finite measurement resolution.

3. The sequences of super-conformal symmetry breakings identifiable in terms of inclusions of super-conformal algebras and corresponding HFFs could have a natural description using the 2-Hilbert spaces and quantum 2-operator algebras.

3.2.5 Questions about quantum measurement theory in Zero Energy Ontology

The following summary about quantum measurement theory in ZEO is somewhat out-of-date and somewhat sketchy. For more detailed view see [K45, K82, K6].

Fractal hierarchy of state function reductions

In accordance with fractality, the conditions for the Connes tensor product at a given time scale imply the conditions at shorter time scales. On the other hand, in shorter time scales the inclusion would be deeper and would give rise to a larger reducibility of the representation of $\mathcal{N}$ in $\mathcal{M}$. Formally, as $\mathcal{N}$ approaches to a trivial algebra, one would have a square root of density matrix and trivial S-matrix in accordance with the idea about asymptotic freedom.

M-matrix would give rise to a matrix of probabilities via the expression $P(P_+ \rightarrow P_-) = Tr[P_+ M^\dagger P_- M]$, where P_+ and P_- are projectors to positive and negative energy energy $\mathcal{N}$-rays. The projectors give rise to the averaging over the initial and final states inside $\mathcal{N}$ ray. The reduction could continue step by step to shorter length scales so that one would obtain a sequence of inclusions. If the U-process of the next quantum jump can return the M-matrix associated with $\mathcal{M}$ or some larger HFF, U process would be kind of reversal for state function reduction.

Analytic thinking proceeding from vision to details; human life cycle proceeding from dreams and wild actions to the age when most decisions relate to the routine daily activities; the progress of science from macroscopic to microscopic scales; even biological decay processes: all these have an intriguing resemblance to the fractal state function reduction process proceeding to shorter and shorter time scales. Since this means increasing thermality of M-matrix, U process as a reversal of state function reduction might break the second law of thermodynamics.

The conservative option would be that only the transformation of intentions to action by U process giving rise to new zero energy states can bring in something new and is responsible for evolution. The non-conservative option is that the biological death is the U-process of the next quantum jump leading to a new life cycle. Breathing would become a universal metaphor for what happens in quantum Universe. The 4-D body would be lived again and again.

quantum classical correspondence is realized at parton level?

Quantum classical correspondence must assign to a given quantum state the most probable space-time sheet depending on its quantum numbers. The space-time sheet $X^4(X^3)$ defined by the Kähler function depends however only on the partonic 3-surface X^3, and one must be able to assign to a given quantum state the most probable X^3 - call it X^3_{max} - depending on its quantum numbers.

$X^4(X^3_{max})$ should carry the gauge fields created by classical gauge charges associated with the Cartan algebra of the gauge group (color isospin and hypercharge and electromagnetic and Z^0 charge) as well as classical gravitational fields created by the partons. This picture is very similar to that of quantum field theories relying on path integral except that the path integral is restricted to 3-surfaces X^3 with exponent of Kähler function bringing in genuine convergence and that 4-D dynamics is deterministic apart from the delicacies due to the 4-D spin glass type vacuum degeneracy of Kähler action.

Stationary phase approximation selects X^3_{max} if the quantum state contains a phase factor depending not only on X^3 but also on the quantum numbers of the state. A good guess is that the needed phase factor corresponds to either Chern-Simons type action or an action describing the interaction of the induced gauge field with the charges associated with the braid strand. This action would be defined for the induced gauge fields. YM action seems to be excluded since it is singular for light-like 3-surfaces associated with the light-like wormhole throats (not only $\sqrt{det(g_3)}$ but also $\sqrt{det(g_4)}$ vanishes).

The challenge is to show that this is enough to guarantee that $X^4(X^3_{max})$ carries correct gauge charges. Kind of electric-magnetic duality should relate the normal components F_{ni} of the gauge fields in $X^4(X^3_{max})$ to the gauge fields F_{ij} induced at X^3. An alternative interpretation is in terms of quantum gravitational holography.

One is forced to introduce gauge couplings and also electro-weak symmetry breaking via the phase factor. This is in apparent conflict with the idea that all couplings are predictable. The essential uniqueness of M-matrix in the case of HFFs of type II_1 (at least) however means that their values as a function of measurement resolution time scale are fixed by internal consistency. Also quantum criticality leads to the same conclusion. Obviously a kind of bootstrap approach suggests itself.

Quantum measurements in ZEO

ZEO based quantum measurement theory leads directly to a theory of conscious entities. The basic idea is that state function reduction localizes the second boundary of CD so that it becomes a piece of light-cone boundary (more precisely $\delta M^4_\pm \times CP_2$).

Repeated reductions are possible as in standard quantum measurement theory and leave the passive boundary of CD. Repeated reduction begins with U process generating a superposition of CDs with the active boundary of CD being de-localized in the moduli space of CDs, and is followed by a localization in this moduli space so that single CD is the outcome. This process tends to increase the distance between the ends of the CD and has interpretation as a space-time correlate for the flow of subjective time.

Self as a conscious entity corresponds to this sequence of repeated reductions on passive boundary of CD. The first reduction at opposite boundary means death of self and its re-incarnation at the opposite boundary of CD. Also the increase of Planck constant and generation of negentropic entanglement is expected to be associated with this state function reduction.

Weak form of NMP is the most plausible variational principle to characterize the state function reduction. It does not require maximal negentropy gain for state function reductions but allows it. In other words, the outcome of reduction is n-dimensional eigen space of density matrix space but this space need not have maximum possible dimension and even 1-D ray is possible in which case the entanglement negentropy vanishes for the final state and system becomes isolated from the rest of the world. Weak form of NMP brings in free will and can allow also larger negentropy gain than the strong form if n is a product of primes. The beauty of this option is that one can understand how the generalization of p-adic length scale hypothesis emerges.

Hyper-finite factors of type II_1 and quantum measurement theory with a finite measurement resolution

The realization that the von Neumann algebra known as hyper-finite factor of type II_1 is tailor made for quantum TGD has led to a considerable progress in the understanding of the mathematical structure of the theory and these algebras provide a justification for several ideas introduced earlier on basis of physical intuition.

Hyper-finite factor of type II_1 has a canonical realization as an infinite-dimensional Clifford algebra and the obvious guess is that it corresponds to the algebra spanned by the gamma matrices of WCW. Also the local Clifford algebra of the embedding space $H = M^4 \times CP_2$ in octonionic representation of gamma matrices of H is important and the entire quantum TGD emerges from the associativity or co-associativity conditions for the sub-algebras of this algebra which are local algebras localized to maximal associative or co-associate sub-manifolds of the embedding space identifiable as space-time surfaces.

The notion of inclusion for hyper-finite factors provides an elegant description for the notion of measurement resolution absent from the standard quantum measurement theory.

1. The included sub-factor creates in ZEO states not distinguishable from the original one and the formally the coset space of factors defining quantum spinor space defines the space of physical states modulo finite measurement resolution.

2. The quantum measurement theory for hyperfinite factors differs from that for factors of type I since it is not possible to localize the state into single ray of state space. Rather, the ray is replaced with the sub-space obtained by the action of the included algebra defining the measurement resolution. The role of complex numbers in standard quantum measurement theory is taken by the non-commutative included algebra so that a non-commutative quantum theory is the outcome.

3. This leads also to the notion of quantum group. For instance, the finite measurement resolution means that the components of spinor do not commute anymore and it is not possible to reduce the state to a precise eigenstate of spin. It is however perform a reduction to an eigenstate of an observable which corresponds to the probability for either spin state.

4. For HFFs the dimension of infinite-dimensional state space is finite and 1 by convention. For included HFF $\mathcal{N} \subset \mathcal{M}$ the dimension of the tensor factor space containing only the degrees of freedom which are above measurement resolution is given by the index of inclusion $d = \mathcal{M} : \mathcal{N}$. One can say that the dimension associated with degrees of freedom below measurement resolution is $D = 1/d$. This number is never large than 1 for the inclusions and contains a set of discrete values $d = 4cos^2(2\pi/n)$, $n \geq 3$, plus the continuum above it. The fractal generalization of the formula for entanglement entropy gives $S = -log(1/D) = -log(d) \leq 0$ so that one can say that the entanglement negentropy assignable to the projection operators to the sub-factor is positive except for $n = 3$ for which it vanishes. The non-measured degrees of freedom carry information rather than entropy.

5. Clearly both HFFs of type I and II allow entanglement negentropy and allow to assign it with finite measurement resolution. In the case of factors its is not clear whether the weak form of NMP allows makes sense.

As already explained, the topology of the many-sheeted space-time encourages the generalization of the notion of quantum entanglement in such a way that unentangled systems can possess entangled sub-systems. One can say that the entanglement between sub-selves is not visible in the resolution characterizing selves. This makes possible sharing and fusion of mental images central for TGD inspired theory of consciousness. These concepts find a deeper justification from the quantum measurement theory for hyper-finite factors of type II_1 for which the finite measurement resolution is basic notion.

Hierarchies of conformal symmetry breakings, Planck constants, and inclusions of HFFs

The basic almost prediction of TGD is a fractal hierarchy of breakings of symplectic symmetry as a gauge symmetry.

It is good to briefly summarize the basic facts about the symplectic algebra assigned with $\delta M^4_\pm \times CP_2$ first.

1. Symplectic algebra has the structure of Virasoro algebra with respect to the light-like radial coordinate r_M of the light-cone boundary taking the role of complex coordinate for ordinary conformal symmetry. The Hamiltonians generating symplectic symmetries can be chosen to be proportional to functions $f_n(r_M)$. What is the natural choice for $f_n(r_M)$ is not quite clear. Ordinary conformal invariance would suggests $f_n(r_M) = r_M^n$. A more adventurous possibility is that the algebra is generated by Hamiltonians with $f_n(r_M) = r^{-s}$, where s is a root of Riemann Zeta so that one has either $s = 1/2 + iy$ (roots at critical line) or $s = -2n$, $n > 0$ (roots at negative real axis).

2. The set of conformal weights would be linear space spanned by combinations of all roots with integer coefficients $s = n - iy$, $s = \sum n_i y_i$, $n > -n_0$, where $-n_0 \geq 0$ is negative conformal weight. Mass squared is proportional to the total conformal weight and must be real demanding $y = \sum y_i = 0$ for physical states: I call this conformal confinement analogous to color confinement. One could even consider introducing the analog of binding energy as "binding conformal weight".

   Mass squared must be also non-negative (no tachyons) giving $n_0 \geq 0$. The generating conformal weights however have negative real part -1/2 and are thus tachyonic. Rather remarkably, p-adic mass calculations force to assume negative half-integer valued ground state conformal weight. This plus the fact that the zeros of Riemann Zeta has been indeed assigned with critical systems forces to take the Riemannian variant of conformal weight spectrum with seriousness. The algebra allows also now infinite hierarchy of conformal sub-algebras with weights coming as n-ples of the conformal weights of the entire algebra.

3. The outcome would be an infinite number of hierarchies of symplectic conformal symmetry breakings. Only the generators of the sub-algebra of the symplectic algebra with radial conformal weight proportional to n would act as gauge symmetries at given level of the hierarchy. In the hierarchy n_i divides n_{i+1}. In the symmetry breaking $n_i \rightarrow n_{i+1}$ the conformal charges, which vanished earlier, would become non-vanishing. Gauge degrees of freedom would transform to physical degrees of freedom.

4. What about the conformal Kac-Moody algebras associated with spinor modes. It seems that in this case one can assume that the conformal gauge symmetry is exact just as in string models.

The natural interpretation of the conformal hierarchies $n_i \rightarrow n_{i+1}$ would be in terms of increasing measurement resolution.

1. Conformal degrees of freedom below measurement resolution would be gauge degrees of freedom and correspond to generators with conformal weight proportional to n_i. Conformal hierarchies and associated hierarchies of Planck constants and n-fold coverings of space-time surface connecting the 3-surfaces at the ends of causal diamond would give a concrete realization of the inclusion hierarchies for hyper-finite factors of type II_1 [K87].

   n_i could correspond to the integer labelling Jones inclusions and associating with them the quantum group phase factor $U_n = exp(i2\pi/n)$, $n \geq 3$ and the index of inclusion given by $|M : N| = 4cos^2(2\pi/n)$ defining the fractal dimension assignable to the degrees of freedom above the measurement resolution. The sub-algebra with weights coming as n-multiples of the basic conformal weights would act as gauge symmetries realizing the idea that these degrees of freedom are below measurement resolution.

2. If $h_{eff} = n \times h$ defines the conformal gauge sub-algebra, the improvement of the resolution would scale up the Compton scales and would quite concretely correspond to a zoom analogous to that done for Mandelbrot fractal to get new details visible. From the point of view

of cognition the improving resolution would fit nicely with the recent view about h_{eff}/h as a kind of intelligence quotient.

This interpretation might make sense for the symplectic algebra of $\delta M^4_{\pm} \times CP_2$ for which the light-like radial coordinate r_M of light-cone boundary takes the role of complex coordinate. The reason is that symplectic algebra acts as isometries.

3. If Kähler action has vanishing total variation under deformations defined by the broken conformal symmetries, the corresponding conformal charges are conserved. The components of WCW Kähler metric expressible in terms of second derivatives of Kähler function can be however non-vanishing and have also components, which correspond to WCW coordinates associated with different partonic 2-surfaces. This conforms with the idea that conformal algebras extend to Yangian algebras generalizing the Yangian symmetry of $\mathcal{N} = 4$ symmetric gauge theories. The deformations defined by symplectic transformations acting gauge symmetries the second variation vanishes and there is not contribution to WCW Kähler metric.

4. One can interpret the situation also in terms of consciousness theory. The larger the value of h_{eff}, the lower the criticality, the more sensitive the measurement instrument since new degrees of freedom become physical, the better the resolution. In p-adic context large n means better resolution in angle degrees of freedom by introducing the phase $exp(i2\pi/n)$ to the algebraic extension and better cognitive resolution. Also the emergence of negentropic entanglement characterized by $n \times n$ unitary matrix with density matrix proportional to unit matrix means higher level conceptualization with more abstract concepts.

The extension of the super-conformal algebra to a larger Yangian algebra is highly suggestive and gives and additional aspect to the notion of measurement resolution.

1. Yangian would be generated from the algebra of super-conformal charges assigned with the points pairs belonging to two partonic 2-surfaces as stringy Noether charges assignable to strings connecting them. For super-conformal algebra associated with pair of partonic surface only single string associated with the partonic 2-surface. This measurement resolution is the almost the poorest possible (no strings at all would be no measurement resolution at all!).

2. Situation improves if one has a collection of strings connecting set of points of partonic 2-surface to other partonic 2-surface(s). This requires generalization of the super-conformal algebra in order to get the appropriate mathematics. Tensor powers of single string super-conformal charges spaces are obviously involved and the extended super-conformal generators must be multi-local and carry multi-stringy information about physics.

3. The generalization at the first step is simple and based on the idea that co-product is the "time inverse" of product assigning to single generator sum of tensor products of generators giving via commutator rise to the generator. The outcome would be expressible using the structure constants of the super-conformal algebra schematically a $Q^1_A = f_A^{BC} Q_B \otimes Q_C$. Here Q_B and Q_C are super-conformal charges associated with separate strings so that 2-local generators are obtained. One can iterate this construction and get a hierarchy of n-local generators involving products of n stringy super-conformal charges. The larger the value of n, the better the resolution, the more information is coded to the fermionic state about the partonic 2-surface and 3-surface. This affects the space-time surface and hence WCW metric but not the 3-surface so that the interpretation in terms of improved measurement resolution makes sense. This super-symplectic Yangian would be behind the quantum groups and Jones inclusions in TGD Universe.

4. n gives also the number of space-time sheets in the singular covering. One possible interpretation is in terms measurement resolution for counting the number of space-time sheets. Our recent quantum physics would only see single space-time sheet representing visible manner and dark matter would become visible only for $n > 1$.

It is not an accident that quantum phases are assignable to Yangian algebras, to quantum groups, and to inclusions of HFFs. The new deep notion added to this existing complex of high level mathematical concepts are hierarchy of Planck constants, dark matter hierarchy, hierarchy of criticalities, and negentropic entanglement representing physical notions. All these aspects represent new physics.

3.2.6 Planar Algebras And Generalized Feynman Diagrams

Planar algebras [A14] are a very general notion due to Vaughan Jones and a special class of them is known to characterize inclusion sequences of hyper-finite factors of type II_1 [A49] . In the following an argument is developed that planar algebras might have interpretation in terms of planar projections of generalized Feynman diagrams (these structures are metrically 2-D by presence of one light-like direction so that 2-D representation is especially natural). In [K14] the role of planar algebras and their generalizations is also discussed.

Planar algebra very briefly

First a brief definition of planar algebra.

1. One starts from planar k-tangles obtained by putting disks inside a big disk. Inner disks are empty. Big disk contains $2k$ braid strands starting from its boundary and returning back or ending to the boundaries of small empty disks in the interior containing also even number of incoming lines. It is possible to have also loops. Disk boundaries and braid strands connecting them are different objects. A black-white coloring of the disjoint regions of k-tangle is assumed and there are two possible options (photo and its negative). Equivalence of planar tangles under diffeomorphisms is assumed.

2. One can define a product of k-tangles by identifying k-tangle along its outer boundary with some inner disk of another k-tangle. Obviously the product is not unique when the number of inner disks is larger than one. In the product one deletes the inner disk boundary but if one interprets this disk as a vertex-parton, it would be better to keep the boundary.

3. One assigns to the planar k-tangle a vector space V_k and a linear map from the tensor product of spaces V_{k_i} associated with the inner disks such that this map is consistent with the decomposition k-tangles. Under certain additional conditions the resulting algebra gives rise to an algebra characterizing multi-step inclusion of HFFs of type II_1.

4. It is possible to bring in additional structure and in TGD framework it seems necessary to assign to each line of tangle an arrow telling whether it corresponds to a strand of a braid associated with positive or negative energy parton. One can also wonder whether disks could be replaced with closed 2-D surfaces characterized by genus if braids are defined on partonic surfaces of genus g. In this case there is no topological distinction between big disk and small disks. One can also ask why not allow the strands to get linked (as suggested by the interpretation as planar projections of generalized Feynman diagrams) in which case one would not have a planar tangle anymore.

General arguments favoring the assignment of a planar algebra to a generalized Feynman diagram

There are some general arguments in favor of the assignment of planar algebra to generalized Feynman diagrams.

1. Planar diagrams describe sequences of inclusions of HFF:s and assign to them a multi-parameter algebra corresponding indices of inclusions. They describe also Connes tensor powers in the simplest situation corresponding to Jones inclusion sequence. Suppose that also general Connes tensor product has a description in terms of planar diagrams. This might be trivial.

2. Generalized vertices identified geometrically as partonic 2-surfaces indeed contain Connes tensor products. The smallest sub-factor N would play the role of complex numbers meaning that due to a finite measurement resolution one can speak only about N-rays of state space and the situation becomes effectively finite-dimensional but non-commutative.

3. The product of planar diagrams could be seen as a projection of 3-D Feynman diagram to plane or to one of the partonic vertices. It would contain a set of 2-D partonic 2-surfaces. Some of them would correspond vertices and the rest to partonic 2-surfaces at future and past directed light-cones corresponding to the incoming and outgoing particles.

4. The question is how to distinguish between vertex-partons and incoming and outgoing partons. If one does not delete the disk boundary of inner disk in the product, the fact that lines arrive at it from both sides could distinguish it as a vertex-parton whereas outgoing partons would correspond to empty disks. The direction of the arrows associated with the lines of planar diagram would allow to distinguish between positive and negative energy partons (note however line returning back).

5. One could worry about preferred role of the big disk identifiable as incoming or outgoing parton but this role is only apparent since by compactifying to say S^2 the big disk exterior becomes an interior of a small disk.

A more detailed view

The basic fact about planar algebras is that in the product of planar diagrams one glues two disks with identical boundary data together. One should understand the counterpart of this in more detail.

1. The boundaries of disks would correspond to 1-D closed space-like stringy curves at partonic 2-surfaces along which fermionic anti-commutators vanish.

2. The lines connecting the boundaries of disks to each other would correspond to the strands of number theoretic braids and thus to braidy time evolutions. The intersection points of lines with disk boundaries would correspond to the intersection points of strands of number theoretic braids meeting at the generalized vertex.

   [Number theoretic braid belongs to an algebraic intersection of a real parton 3-surface and its p-adic counterpart obeying same algebraic equations: of course, in time direction algebraicity allows only a sequence of snapshots about braid evolution].

3. Planar diagrams contain lines, which begin and return to the same disk boundary. Also “vacuum bubbles” are possible. Braid strands would disappear or appear in pairwise manner since they correspond to zeros of a polynomial and can transform from complex to real and vice versa under rather stringent algebraic conditions.

4. Planar diagrams contain also lines connecting any pair of disk boundaries. Stringy decay of partonic 2-surfaces with some strands of braid taken by the first and some strands by the second parton might bring in the lines connecting boundaries of any given pair of disks (if really possible!).

5. There is also something to worry about. The number of lines associated with disks is even in the case of k-tangles. In TGD framework incoming and outgoing tangles could have odd number of strands whereas partonic vertices would contain even number of k-tangles from fermion number conservation. One can wonder whether the replacement of boson lines with fermion lines could imply naturally the notion of half-k-tangle or whether one could assign half-k-tangles to the spinors of WCW (“world of classical worlds”) whereas corresponding Clifford algebra defining HFF of type II_1 would correspond to k-tangles.

3.2.7 Miscellaneous

The following considerations are somewhat out-of-date: hence the title “Miscellaneous”.

Connes tensor product and fusion rules

One should demonstrate that Connes tensor product indeed produces an M-matrix with physically acceptable properties.

The reduction of the construction of vertices to that for n-point functions of a conformal field theory suggest that Connes tensor product is essentially equivalent with the fusion rules for conformal fields defined by the Clifford algebra elements of $CH(CD)$ (4-surfaces associated with 3-surfaces at the boundary of causal diamond CD in M^4), extended to local fields in M^4 with gamma matrices acting on WCW spinor s assignable to the partonic boundary components.

Jones speculates that the fusion rules of conformal field theories can be understood in terms of Connes tensor product [A131] and refers to the work of Wassermann about the fusion of loop group representations as a demonstration of the possibility to formula the fusion rules in terms of Connes tensor product [A32] .

Fusion rules are indeed something more intricate that the naïve product of free fields expanded using oscillator operators. By its very definition Connes tensor product means a dramatic reduction of degrees of freedom and this indeed happens also in conformal field theories.

1. For non-vanishing n-point functions the tensor product of representations of Kac Moody group associated with the conformal fields must give singlet representation.

2. The ordinary tensor product of Kac Moody representations characterized by given value of central extension parameter k is not possible since k would be additive.

3. A much stronger restriction comes from the fact that the allowed representations must define integrable representations of Kac-Moody group [A44] . For instance, in case of $SU(2)_k$ Kac Moody algebra only spins $j \leq k/2$ are allowed. In this case the quantum phase corresponds to $n = k + 2$. $SU(2)$ is indeed very natural in TGD framework since it corresponds to both electro-weak $SU(2)_L$ and isotropy group of particle at rest.

Fusion rules for localized Clifford algebra elements representing operators creating physical states would replace naïve tensor product with something more intricate. The naïvest approach would start from M^4 local variants of gamma matrices since gamma matrices generate the Clifford algebra Cl associated with $CH(CD)$. This is certainly too naïve an approach. The next step would be the localization of more general products of Clifford algebra elements elements of Kac Moody algebras creating physical states and defining free on mass shell quantum fields. In standard quantum field theory the next step would be the introduction of purely local interaction vertices leading to divergence difficulties. In the recent case one transfers the partonic states assignable to the light-cone boundaries $\delta M^4_\pm(m_i) \times CP_2$ to the common partonic 2-surfaces X^2_V along $X^3_{L,i}$ so that the products of field operators at the same space-time point do not appear and one avoids infinities.

The remaining problem would be the construction an explicit realization of Connes tensor product. The formal definition states that left and right $\mathcal{N}$ actions in the Connes tensor product $\mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$ are identical so that the elements $nm_1 \otimes m_2$ and $m_1 \otimes m_2 n$ are identified. This implies a reduction of degrees of freedom so that free tensor product is not in question. One might hope that at least in the simplest choices for $\mathcal{N}$ characterizing the limitations of quantum measurement this reduction is equivalent with the reduction of degrees of freedom caused by the integrability constraints for Kac-Moody representations and dropping away of higher spins from the ordinary tensor product for the representations of quantum groups. If fusion rules are equivalent with Connes tensor product, each type of quantum measurement would be characterized by its own conformal field theory.

In practice it seems safest to utilize as much as possible the physical intuition provided by quantum field theories. In [K18] a rather precise vision about generalized Feynman diagrams is developed and the challenge is to relate this vision to Connes tensor product.

Connection with topological quantum field theories defined by Chern-Simons action

There is also connection with topological quantum field theories (TQFTs) defined by Chern- Simons action [A59] .

1. The light-like 3-surfaces X_l^3 defining propagators can contain unitary matrix characterizing the braiding of the lines connecting fermions at the ends of the propagator line. Therefore the modular S-matrix representing the braiding would become part of propagator line. Also incoming particle lines can contain similar S-matrices but they should not be visible in the M-matrix. Also entanglement between different partonic boundary components of a given incoming 3-surface by a modular S-matrix is possible.

2. Besides CP_2 type extremals MEs with light-like momenta can appear as brehmstrahlung like exchanges always accompanied by exchanges of CP_2 type extremals making possible momentum conservation. Also light-like boundaries of magnetic flux tubes having macroscopic size could carry light-like momenta and represent similar brehmstrahlung like exchanges. In this case the modular S-matrix could make possible topological quantum computations in $q \neq 1$ phase [K4] . Notice the somewhat counter intuitive implication that magnetic flux tubes of macroscopic size would represent change in quantum jump rather than quantum state. These quantum jumps can have an arbitrary long geometric duration in macroscopic quantum phases with large Planck constant [K25] .

There is also a connection with topological QFT defined by Chern-Simons action allowing to assign topological invariants to the 3-manifolds [A59] . If the light-like CDs $X_{L,i}^3$ are boundary components, the 3-surfaces associated with particles are glued together somewhat like they are glued in the process allowing to construct 3-manifold by gluing them together along boundaries. All 3-manifold topologies can be constructed by using only torus like boundary components.

This would suggest a connection with 2+1-dimensional topological quantum field theory defined by Chern-Simons action allowing to define invariants for knots, links, and braids and 3-manifolds using surgery along links in terms of Wilson lines. In these theories one consider gluing of two 3-manifolds, say 3-spheres S^3 along a link to obtain a topologically non-trivial 3-manifold. The replacement of link with Wilson lines in $S^3\#S^3 = S^3$ reduces the calculation of link invariants defined in this manner to Chern-Simons theory in S^3.

In the recent situation more general structures are possible since arbitrary number of 3-manifolds are glued together along link so that a singular 3-manifolds with a book like structure are possible. The allowance of CDs which are not boundaries, typically 3-D light-like throats of wormhole contacts at which induced metric transforms from Minkowskian to Euclidian, brings in additional richness of structure. If the scaling factor of CP_2 metric can be arbitrary large as the quantization of Planck constant predicts, this kind of structure could be macroscopic and could be also linked and knotted. In fact, topological condensation could be seen as a process in which two 4-manifolds are glued together by drilling light-like CDs and connected by a piece of CP_2 type extremal.

3.3 Fresh View About Hyper-Finite Factors In TGD Framework

In the following I will discuss the basic ideas about the role of hyper-finite factors in TGD with the background given by a work of more than half decade. First I summarize the input ideas which I combine with the TGD inspired intuitive wisdom about HFFs of type II_1 and their inclusions allowing to represent finite measurement resolution and leading to notion of quantum spaces with algebraic number valued dimension defined by the index of the inclusion.

Also an argument suggesting that the inclusions define “skewed” inclusions of lattices to larger lattices giving rise to quasicrystals is proposed. The core of the argument is that the included HFF of type II_1 algebra is a projection of the including algebra to a subspace with dimension $D \leq 1$. The projection operator defines the analog of a projection of a bigger lattice to the included lattice. Also the fact that the dimension of the tensor product is product of dimensions of factors just like the number of elements in finite group is product of numbers of elements of coset space and subgroup, supports this interpretation.

One also ends up with a detailed identification of the hyper-finite factors in orbital degrees of freedom in terms of symplectic group associated with $\delta M^4_\pm \times CP_2$ and the group algebras of their discrete subgroups define what could be called “orbital degrees of freedom” for WCW spinor fields. By very general argument this group algebra is HFF of type II, maybe even II_1.

3.3.1 Crystals, Quasicrystals, Non-Commutativity And Inclusions Of Hyperfinite Factors Of Type II_1

I list first the basic ideas about non-commutative geometries and give simple argument suggesting that inclusions of HFFs correspond to "skewed" inclusions of lattices as quasicrystals.

1. Quasicrystals (see `http://tinyurl.com/67kz3qo`) (say Penrose tilings) [A19] can be regarded as subsets of real crystals and one can speak about "skewed" inclusion of real lattice to larger lattice as quasicrystal. What this means that included lattice is obtained by projecting the larger lattice to some lower-dimensional subspace of lattice.

2. The argument of Connes concerning definition of non-commutative geometry can be found in the book of Michel Lapidus at page 200. Quantum space is identified as a space of equivalence classes. One assigns to pairs of elements inside equivalence class matrix elements having the element pair as indices (one assumes numerable equivalence class). One considers irreducible representations of the algebra defined by the matrices and identifies the equivalent irreducible representations. If I have understood correctly, the equivalence classes of irreps define a discrete point set representing the equivalence class and it can often happen that there is just single point as one might expect. This I do not quite understand since it requires that all irreps are equivalent.

3. It seems that in the case of linear spaces - von Neumann algebras and accompanying Hilbert spaces - one obtains a connection with the inclusions of HFFs and corresponding quantum factor spaces that should exist as analogs of quantum plane. One replaces matrices with elements labelled by element pairs with linear operators in HFF of type II_1. Index pairs correspond to pairs in linear basis for the HFF or corresponding Hilbert space.

4. Discrete infinite enumerable basis for these operators as a linear space generates a lattice in summation. Inclusion $N \subset M$ defines inclusion of the lattice/crystal for N to the corresponding lattice of M. Physical intuition suggests that if this inclusion is "skewed" one obtains quasicrystal. The fact the index of the inclusion is algebraic number suggests that the coset space M/N is indeed analogous to quasicrystal.

   More precisely, the index of inclusion is defined for hyper-finite factors of type II_1 using the fact that quantum trace of unit matrix equals to unity $Tr(Id(M)) = 1$, and from the tensor product composition $M = (M/N) \times N$ given $Tr(Id(M)) = 1 = Ind(M/N)Tr(P(M \to N))$, where $P(M \to N$ is projection operator from M to N. Clearly, $Ind(M/N) = 1/Tr(P(M \to N))$ defines index as a dimension of quantum space M/N.

   For Jones inclusions characterized by quantum phases $q = exp(i2\pi/n)$, $n = 3, 4, ...$ the values of index are given by $Ind(M/N) = 4cos^2(\pi/n)$, $n = 3, 4, ...$ There is also another range inclusions $Ind(M/N) \geq 4$: note that $Tr(P(M \to N))$ defining the dimension of N as included sub-space is never larger than one for HFFs of type II_1. The projection operator $P(M \to N)$ is obviously the counterpart of the projector projecting lattice to some lower-dimensional sub-space of the lattice.

5. Jones inclusions are between linear spaces but there is a strong analogy with non-linear coset spaces since for the tensor product the dimension is product of dimensions and for discrete coset spaces G/H one has also the product formula $n(G) = n(H) \times n(G/H)$ for the numbers of elements. Noticing that space of quantum amplitudes in discrete space has dimension equal to the number of elements of the space, one could say that Jones inclusion represents quantized variant for classical inclusion raised from the level of discrete space to the level of space of quantum states with the number of elements of set replaced by dimension. In fact, group algebras of infinite and enumerable groups defined HFFs of type II under rather general conditions (see below).

   Could one generalize Jones inclusions so that they would apply to non-linear coset spaces analogs of the linear spaces involved ? For instance, could one think of infinite-dimensional groups G and H for which Lie-algebras defining their tangent spaces can be regarded as HFFs of type II_1? The dimension of the tangent space is dimension of the non-linear manifold: could this mean that the non-linear infinite-dimensional inclusions reduce to tangent space level and thus to the inclusions for Lie-algebras regarded hyper-finite factors of type II_1 or more generally, type II? This would would rise to quantum spaces which have finite but algebraic valued quantum dimension and in TGD framework take into account the finite measurement resolution.

6. To concretize this analogy one can check what is the number of points map from 5-D space containing aperiodic lattice as a projection to a 2-D irrational plane containing only origin as common point with the 5-D lattice. It is easy to get convinced that the projection is 1-to-1 so that the number of points projected to a given point is 1. By the analogy with Jones inclusions this would mean that the included space has same von Neumann dimension 1 - just like the including one. In this case quantum phase equals $q = exp(i2\pi/n)$, $n = 3$ - the lowest possible value of n. Could one imagine the analogs of $n > 3$ inclusions for which the number of points projected to a given point would be larger than 1? In 1-D case the rational lines $y = (k/l)x$ define 1-D rational analogs of quasi crystals. The points $(x, y) = (m, n)$, $m\ mod\ l = 0$ are projected to the same point. The number of points is now infinite and the ratio of points of 2-D lattice and 1-D crystal like structure equals to l and serves as the analog for the quantum dimension $d_q = 4cos^2(\pi/n)$.

To sum up, this this is just physicist's intuition: it could be wrong or something totally trivial from the point of view of mathematician. The main message is that the inclusions of HFFs might define also inclusions of lattices as quasicrystals.

3.3.2 HFFs And Their Inclusions In TGD Framework

In TGD framework the inclusions of HFFs have interpretation in terms of finite measurement resolution. If the inclusions define quasicrystals then finite measurement resolution would lead to quasicrystals.

1. The automorphic action of N in $M \supset N$ and in associated Hilbert space H_M where N acts generates physical operators and accompanying stas (operator rays and rays) not distinguishable from the original one. States in finite measurement resolution correspond to N-rays rather than complex rays. It might be natural to restrict to unitary elements of N.

   This leads to the need to construct the counterpart of coset space M/N and corresponding linear space H_M/H_N. Physical intuition tells that the indices of inclusions defining the "dimension" of M/N are algebraic numbers given by Jones index formula.

2. Here the above argument would assign to the inclusions also inclusions of lattices as quasicrystals.

Degrees of freedom for WCW spinor field

Consider first the identification of various kinds of degrees of freedom in TGD Universe.

1. Very roughly, WCW ("world of classical worlds") spinor is a state generated by fermionic creation operators from vacuum at given 3-surface. WCW spinor field assigns this kind of spinor to each 3-surface. WCW spinor fields decompose to tensor product of spin part (Fock state) and orbital part ("wave" in WCW) just as ordinary spinor fields.

2. The conjecture motivated by super-symmetry has been that both WCW spinors and their orbital parts (analogs of scalar field) define HFFs of type II_1 in quantum fluctuating degrees of freedom.

3. Besides these there are zero modes, which by definition do not contribute to WCW Kähler metric.

   (a) If the zero zero modes are symplectic invariants, they appear only in conformal factor of WCW metric. Symplectically invariant zero modes represent purely classical degrees of freedom - direction of a pointer of measurement apparatus in quantum measurement - and in given experimental arrangement they entangle with quantum fluctuating degrees of freedom in one-one manner so that state function reduction assigns to the outcome of state function reduction position of pointer. I forget symplectically invariant zero modes and other analogous variables in the following and concentrate to the degrees of freedom contributing WCW line-element.

(b) There are also zero modes which are not symplectic invariants and are analogous to degrees of freedom generated by the generators of Kac-Moody algebra having vanishing conformal weight. They represent "center of mass degrees of freedom" and this part of symmetric algebra creates the representations representing the ground states of Kac-Moody representations. Restriction to these degrees of freedom gives QFT limit in string theory. In the following I will speak about "cm degrees of freedom".

The general vision about symplectic degrees of freedom (the analog of "orbital degrees of freedom" for ordinary spinor field) is following.

1. WCW (assignable to given CD) is a union over the sub-WCWs labeled by zero modes and each sub-WCW representing quantum fluctuating degrees of freedom and "cm degrees of freedom" is infinite-D symmetric space. If symplectic group assignable to $\delta M^4_+ \times CP_2$ acts as as isometries of WCW then "orbital degrees of freedom" are parametrized by the symplectic group or its coset space (note that light-cone boundary is 3-D but radial dimension is light-like so that symplectic - or rather contact structure - exists).

   Let S^2 be $r_M = constant$ sphere at light-cone boundary (r_M is the radial light-like coordinate fixed apart from Lorentz transformation). The full symplectic group would act as isometries of WCW but does not - nor cannot do so - act as symmetries of Kähler action except in the huge vacuum sector of the theory correspond to vacuum extremals.

2. WCW Hamiltonians can be deduced as "fluxes" of the Hamiltonians of $\delta M^4_+ \times CP_2$ taken over partonic 2-surfaces. These Hamiltonians expressed as products of Hamiltonians of S^2 and CP_2 multiplied by powers r^n_M. Note that r_M plays the role of the complex coordinate z for Kac-Moody algebras and the group G defining KM is replaced with symplectic group of $S^2 \times CP_2$. Hamiltonians can be assumed to have well-defined spin (SO(3)) and color (SU(3)) quantum numbers.

3. The generators with vanishing radial conformal weight ($n = 0$) correspond to the symplectic group of $S^2 \times CP_2$. They are not symplectic invariants but are zero modes. They would correspond to "cm degrees of freedom" characterizing the ground states of representations of the full symplectic group.

Discretization at the level of WCW

The general vision about finite measurement resolution implies discretization at the level of WCW.

1. Finite measurement resolution at the level of WCW means discretization. Therefore the symplectic groups of $\delta M^4_+ \times CP_2$ *resp.* $S^2 \times CP_2$ are replaced by an enumerable discrete subgroup. WCW is discretized in both quantum fluctuating degrees of freedom and "center of mass" degrees of freedom.

2. The elements of the group algebras of these discrete groups define the "orbitals parts" of WCW spinor fields in discretization. I will later develop an argument stating that they are HFFs of type II - maybe even II_1. Note that also function spaces associated with the coset spaces of these discrete subgroups could be considered.

3. Discretization applies also in the spin degrees of freedom. Since fermionic Fock basis generates quantum counterpart of Boolean algebra the interpretation in terms of the physical correlates of Boolean cognition is motivated (fermion number 1/0 and various spins in decomposition to a tensor product of lower-dimensional spinors represent bits). Note that in ZEO fermion number conservation does not pose problems and zero states actually define what might be regarded as quantum counterparts of Boolean rules $A \rightarrow B$.

4. Note that 3-surfaces correspond by the strong form of GCI/holography to collections of partonic 2-surfaces and string world sheets of space-time surface intersecting at discrete set of points carrying fermionic quantum numbers. WCW spinors are constructed from second quantized induced spinor fields and fermionic Fock algebra generates HFF of type II_1.

Does WCW spinor field decompose to a tensor product of two HFFs of type II_1?

The group algebras associated with infinite discrete subgroups of the symplectic group define the discretized analogs of waves in WCW having quantum fluctuating part and cm part. The proposal is that these group algebras are HFFs of type II_1. The spinorial degrees of freedom correspond to fermionic Fock space and this is known to be HFF. Therefore WCW spinor fields would defined tensor product of HFFs of type II_1. The interpretation would be in terms of supersymmetry at the level of WCW. Super-conformal symmetry is indeed the basic symmetry of TGD so that this result is a physical "must". The argument goes as follows.

1. In non-zero modes WCW is symplectic group of $\delta M^4_+ \times CP_2$ (call this group just *Sympl*) reduces to the analog of Kac-Moody group associated with $S^2 \times CP_2$, where S^2 is $r_M =$ constant sphere of light-cone boundary and z is replaced with radial coordinate. The Hamiltonians, which do not depend on r_M would correspond to zero modes and one could not assign metric to them although symplectic structure is possible. In "cm degrees of freedom" one has symplectic group associated with $S^2 \times CP_2$.

2. Finite measurement resolution, which seems to be coded already in the structure of the preferred extremals and of the solutions of the Kähler-Dirac equation, suggests strongly that this symplectic group is replaced by its discrete subgroup or symmetric coset space. What this group is, depends on measurement resolution defined by the cutoffs inherent to the solutions. These subgroups and coset spaces would define the analogs of Platonic solids in WCW!

3. Why the discrete infinite subgroups of *Sympl* would lead naturally to HFFs of type II? There is a very general result stating that group algebra of an enumerable discrete group, which has infinite conjugacy classes, and is amenable so that its regular representation in group algebra decomposes to all unitary irreducibles is HFF of type II. See for examples about HFFs of type II listed in Wikipedia article (see `http://tinyurl.com/y8445w8q`) [A7].

4. Suppose that the group algebras associated the discrete subgroups *Sympl* are indeed HFFs of type II or even type II_1. Their inclusions would define finite measurement resolution the orbital degrees of freedom for WCW spinor fields. Included algebra would create rays of state space not distinguishable experimentally. The inclusion would be characterized by the inclusion of the lattice defined by the generators of included algebra by linearity. One would have inclusion of this lattice to a lattice associated with a larger discrete group. Inclusions of lattices are however known to give rise to quasicrystals (Penrose tilings are basic example), which define basic non-commutative structures. This is indeed what one expects since the dimension of the coset space defined by inclusion is algebraic number rather than integer.

5. Also in fermionic degrees of freedom finite measurement resolution would be realized in terms of inclusions of HFFs- now certainly of type II_1. Therefore one could obtain hierarchies of lattices included as quasicrystals.

What about zero modes which are symplectic invariants and define classical variables? They are certainly discretized too. One might hope that one-one correlation between zero modes (classical variables) and quantum fluctuating degrees of freedom suggested by quantum measurement theory allows to effectively eliminate them. Besides zero modes there are also modular degrees of freedom associated with partonic 2-surfaces defining together with their 4-D tangent space data basis objects by strong form of holography. Also these degrees of freedom are automatically discretized. But could one consider finite measurement resolution also in these degrees of freedom. If the symplectic group of $S^2 \times CP_2$ defines zero modes then one could apply similar argument also in these degrees of freedom to discrete subgroups of $S^2 \times CP_2$.

3.3.3 Little Appendix: Comparison Of WCW Spinor Fields With Ordinary Second Quantized Spinor Fields

In TGD one identifies states of Hilbert space as WCW spinor fields. The analogy with ordinary spinor field helps to understand what they are. I try to explain by comparison with QFT.

Ordinary second quantized spinor fields

Consider first ordinary fermionic QFT in fixed space-time. Ordinary spinor is attached to an space-time point and there is $2^{D/2}$ dimensional space of spin degrees of freedom. Spinor field attaches spinor to every point of space-time in a continuous/smooth manner. Spinor fields satisfying Dirac equation define in Euclidian metric a Hilbert space with a unitary inner product. In Minkowskian case this does not work and one must introduce second quantization and Fock space to get a unitary inner product. This brings in what is essentially a basic realization of HFF of type II_1 as allowed operators acting in this Fock space. It is operator algebra rather than state space which is HFF of type II_1 but they are of course closely related.

Classical WCW spinor fields as quantum states

What happens TGD where one has quantum superpositions of 4-surface/3-surfaces by GCI/partonic 2-surfaces with 4-D tangent space data by strong form of GCI.

1. First guess: space-time point is replaced with 3-surface. Point like particle becomes 3-surface representing particle. WCW spinors are fermionic Fock states at this surface. WCW spinor fields are Fock state as a functional of 3-surface. Inner product decomposes to Fock space inner product plus functional integral over 3-surfaces (no path integral!). One could speak of quantum multiverse. Not single space-time but quantum superposition of them. This quantum multiverse character is something new as compared to QFT.

2. Second guess: forced by ZEO, by geometrization of Feynman diagrams, etc.

    (a) 3-surfaces are actually not connected 3-surfaces. They are collections of components at both ends of CD and connected to single connected structure by 4-surface. Components of 3-surface are like incoming and outgoing particles in connected Feynman diagrams. Lines are identified as regions of Euclidian signature or equivalently as the 3-D light-like boundaries between Minkowskian and Euclidian signature of the induced metric.

    (b) Spinors(!!) are defined now by the fermionic Fock space of second quantized induced spinor fields at these 3-surfaced and by holography at 4-surface. This fermionic Fock space is assigned to all multicomponent 3-surfaces defined in this manner and WCW spinor fields are defined as in the first guess. This brings integration over WCW to the inner product.

3. Third, even more improved guess: motivated by the solution ansatz for preferred extremals and for Kähler-Dirac equation [K88] giving a connection with string models.

    The general solution ansatz restricts all spinor components but right-handed neutrino to string world sheets and partonic 2-surfaces: this means effective 2-dimensionality. String world sheets and partonic 2-surfaces intersect at the common ends of light-like and space-like braids at ends of CD and at along wormhole throat orbits so that effectively discretization occurs. This fermionic Fock space replaces the Fock space of ordinary second quantization.

3.4 The idea of Connes about inherent time evolution of certain algebraic structures from TGD point of view

Jonathan Disckau asked me about what I think about the proposal of Connes represented in the summary of progress of noncommutative geometry in "Noncommutative Geometry Year 2000" [A31] (see `https://arxiv.org/abs/math/0011193`) that certain mathematical structures have inherent time evolution coded into their structure.

I have written years ago about Connes's proposal. At that time I was trying to figure out how to understand the construction of scattering amplitudes in the TGD framework and the proposal of Connes looked attractive. Later I had to give up this idea. However, the basic idea is beautiful. One should only replace the notion of time evolution from a one-parameter group of automorphisms to something more interesting. Also time evolution as increasing algebraic complexity is a more attractive interpretation.

The inclusion hierarchies of hyperfinite factors (HFFs) - closely related to the work of Connes - are a key element of TGD and crucial for understanding evolutionary hierarchies in TGD. Is it possible that mathematical structure evolves in time in some sense? The TGD based answer is that quantum jump as a fundamental evolutionary step - moment of subjective time evolution - is a necessary new element. The sequence of moments of consciousness as quantum jumps would have an interpretation as hopping around in the space of mathematical structures leading to increasingly complex structures.

The generalization of the idea of Connes is discussed in this framework. In particular, the inclusion hierarchies of hyper-finite factors, the extension hierarchies of rationals, and fractal inclusion hierarchies of subalgebras of supersymplectic algebra isomorphic with the entire algebra are proposed to be more or less one and the same thing in TGD framework.

The time evolution operator of Connes could corresponds to super-symplectic algebra (SSA) to the time evolution generated by $exp(iL_0\tau)$ so that the operator Δ of Connes would be identified as $\Delta = exp(L_0)$. This identification allows number theoretical universality if τ is quantized. Furthermore, one ends up with a model for the subjective time evolution by small state function reductions (SSFRs) for SSA with SSA$_n$ gauge conditions: the unitary time evolution for given SSFR would be generated by a linear combination of Virasoro generators not annihilating the states. This model would generalize the model for harmonic oscillator in external force allowing exact S-matrix.

3.4.1 Connes proposal and TGD

In this section I develop in more detail the analog of Connes proposal in TGD framework.

What does Connes suggest?

One must first make clear what the automorphism of HFFs discovered by Connes is.

1. Tomita-Takesaki theory

Tomita-Takesaki theory is a vital part of the theory of factors. I have described the theory earlier [K48, K28].

First some definitions.

1. Let $\omega(x)$ be a faithful state of von Neumann algebra so that one has $\omega(xx^*) > 0$ for $x > 0$. Assume by Riesz lemma the representation of ω as a vacuum expectation value: $\omega = (\cdot\Omega, \Omega)$, where Ω is cyclic and separating state.
2. Let

$$L^{\infty}(\mathcal{M}) \equiv \mathcal{M} \ , \quad L^2(\mathcal{M}) = \mathcal{H} \ , \quad L^1(\mathcal{M}) = \mathcal{M}_* \ , \qquad (3.4.1)$$

   where $\mathcal{M}_*$ is the pre-dual of $\mathcal{M}$ defined by linear functionals in $\mathcal{M}$. One has ${\mathcal{M}_*}^* = \mathcal{M}$.
3. The conjugation $x \rightarrow x^*$ is isometric in $\mathcal{M}$ and defines a map $\mathcal{M} \rightarrow L^2(\mathcal{M})$ via $x \rightarrow x\Omega$. The map $S_0; x\Omega \rightarrow x^*\Omega$ is however non-isometric.
4. Denote by S the closure of the anti-linear operator S_0 and by $S = J\Delta^{1/2}$ its polar decomposition analogous that for complex number and generalizing polar decomposition of linear operators by replacing (almost) unitary operator with anti-unitary J. Therefore $\Delta = S^*S > 0$ is positive self-adjoint and J an anti-unitary involution. The non-triviality of Δ reflects the fact that the state is not trace so that hermitian conjugation represented by S in the state space brings in additional factor $\Delta^{1/2}$.

5. What x can be is puzzling to physicists. The restriction fermionic Fock space and thus to creation operators would imply that Δ would act non-trivially only vacuum state so that $\Delta > 0$ condition would not hold true. The resolution of puzzle is the allowance of tensor product of Fock spaces for which vacua are conjugates: only this gives cyclic and separating state. This is natural in ZEO.

The basic results of Tomita-Takesaki theory are following.

1. The basic result can be summarized through the following formulas

$$\Delta^{it} M \Delta^{-it} = \mathcal{M} \ \ , J\mathcal{M}J = \mathcal{M}' \ \ .$$

2. The latter formula implies that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic algebras. The first formula implies that a one parameter group of modular automorphisms characterizes partially the factor. The physical meaning of modular automorphisms is discussed in [A47, A93] Δ is Hermitian and positive definite so that the eigenvalues of $log(\Delta)$ are real but can be negative. Δ^{it} is however not unitary for factors of type II and III. Physically the non-unitarity must relate to the fact that the flow is contracting so that hermiticity as a local condition is not enough to guarantee unitarity.

3. $\omega \rightarrow \sigma_t^{\omega} = Ad\Delta^{it}$ defines a canonical evolution -modular automorphism- associated with ω and depending on it. The Δ:s associated with different ω:s are related by a unitary inner automorphism so that their equivalence classes define an invariant of the factor.

Tomita-Takesaki theory gives rise to a non-commutative measure theory which is highly non-trivial. In particular the spectrum of Δ can be used to classify the factors of type II and III.

The definition of Δ^{it} reduces in eigenstate basis of Δ to the definition of complex function d^{it}. Note that is positive so that the logarithm of d is real.

In TGD framework number theoretic universality poses additional conditions. In diagonal basis $e^{log(d)it}$ must exist. A simply manner to solve the conditions is $e = exp(m/r)$ existing p-adically for an extension of rational allowing r:th root of e. This requires also quantization of as a root of unity so that the exponent reduces to a root of unity.

2. Modular automorphisms

Modular automorphisms of factors are central for their classification.

1. One can divide the automorphisms to inner and outer ones. Inner automorphisms correspond to unitary operators obtained by exponentiating Hermitian Hamiltonian belonging to the factor and connected to identity by a flow. Outer automorphisms do not allow a representation as a unitary transformations although $log(\Delta)$ is formally a Hermitian operator.

2. The fundamental group of the type II_1 factor defined as fundamental group group of corresponding II_∞ factor characterizes partially a factor of type II_1. This group consists of real numbers λ such that there is an automorphism scaling the trace by λ. Fundamental group typically contains all reals but it can be also discrete and even trivial.

3. Factors of type III allow a one-parameter group of modular automorphisms, which can be used to achieve a partial classification of these factors. These automorphisms define a flow in the center of the factor known as flow of weights. The set of parameter values λ for which ω is mapped to itself and the center of the factor defined by the identity operator (projector to the factor as a sub-algebra of $\mathcal{B}(\mathcal{H})$) is mapped to itself in the modular automorphism defines the Connes spectrum of the factor. For factors of type III_λ this set consists of powers of $\lambda < 1$. For factors of type III_0 this set contains only identity automorphism so that there is no periodicity. For factors of type III_1 Connes spectrum contains all real numbers so that the automorphisms do not affect the identity operator of the factor at all.

The modules over a factor correspond to separable Hilbert spaces that the factor acts on. These modules can be characterized by M-dimension. The idea is roughly that complex rays are replaced by the sub-spaces defined by the action of $\mathcal{M}$ as basic units. M-dimension is not integer valued in general. The so called standard module has a cyclic separating vector and each factor has a standard representation possessing antilinear involution J such that $\mathcal{M}' = J\mathcal{M}J$ holds true (note that J changes the order of the operators in conjugation). The inclusions of factors define modules having interpretation in terms of a finite measurement resolution defined by $\mathcal{M}$.

3. Objections against the idea of Connes

One can represent objections against this idea.

1. Ordinary time evolution in wave mechanics is a unitary automorphism, so that in this framework they would not have physical meaning but act as gauge transformations. If outer automorphisms define time evolutions, they must act as gauge transformations. One would have an analog of gauge field theory in HFF. This would be of course highly interesting: when I gave up the idea of Connes, I did not consider this possibility. Super-symplectic algebras having fractal structure are however extremely natural candidate for defining HFF and there is infinite number of gauge conditions.

2. An automorphism is indeed in question so that the algebraic system would not be actually affected. Therefore one cannot say that HFF has inherent time evolution and time. However, one can represent in HFF dynamical systems obeying this inherent time evolution. This possibility is highly interesting as a kind of universal gauge theory.

   On the other hand, outer automorphisms affect the trace of the projector defining the identity matrix for a given factor. Does the scaling factor Λ represent some kind of renormalization operation? Could it relate to the action of scalings in the TGD framework where scalings replace time translations at the fundamental level? What the number theoretic vision of TGD could mean? Could this quantize the continuous spectrum of the scalings Λ for HFFs so that they belong to the extension? Could one have a spectrum of Λ for each extension of rationals? Are different extensions related by inclusions of HFFs?

3. The notion of time evolution itself is an essentially Newtonian concept: selecting a preferred time coordinate breaks Lorentz invariance. In TGD however time coordinate is replace by scaling parameter and the situation changes.

4. The proposal of Connes is not general enough if evolution is interpreted as an increase of complexity.

For these reasons I gave up the automorphism proposed by Connes as a candidate for defining time evolution giving rise to scattering amplitudes in TGD framework.

Two views about TGD

The two dual views about what TGD is described briefly in [L49].

1. Physics as geometry of the world of "world of classical worlds" (WCW) identified as the space of space-time surfaces in $M^4 \times CP_2$ [K63]. Twistor lift of TGD [L26] implies that the space-time surfaces are minimal surfaces which can be also regarded as extermals of the Kähler action. This implies holography required by the general coordinate invariance in TGD framework.

2. TGD as generalized number theory forcing to generalize physics to adelic physics [L24] fusing real physics as correlate of sensory experience and various p-adic physics as correlates of cognition. Now space-times are naturally co-associative surfaces in complexified M^8 (complexified octonions) defined as "roots" of octonionic polynomials determined by polynomials with rational coefficients [L45, L46, L55]. Now holography extends dramatically: finite number of rational numbers/roots of rational polynomial/points of space-time region dictate it.

$M^8 - H$ duality relates these two views and is actually a generalization of Fourier transform and realizes generalization of momentum-position duality.

The notion of time evolution in TGD

Concerning various time evolutions in TGD, the general situation is now rather well understood.

There are two quantal time evolutions: geometric one assignable to single CD and and subjective time evolution which reflects the generalization of point-like particle to a 3-surface and the introduction of CD as 4-D perceptive field of particle in ZEO [L38].

1. Geometric time evolution corresponds to the standard scattering amplitudes for which I have a general formula now in terms of zero energy ontology (ZEO) [L51, L45, L46, L55]. The analog of S-matrix corresponds to entanglement coefficients between members of zero energy state at opposite boundaries of causal diamond (CD).

2. Subjective time evolution of conscious entity corresponds to a sequence of "small" state function reductions (SSFRs) as moments of consciousness: each SSFR is preceded by an analog of unitary time evolution, call it U. SSFRs are the TGD counterparts of "weak" measurements.

   $U(t)$ is generated by the scaling generator L_0 scaling light-like radial coordinate of light-cone boundary and is a generalization of corresponding operator in superconformal and string theories and defined for super-symplectic algebras acting as isometries of the world of classical worlds (WCW) [L55]. $U(t)$ is not the exponential of energy as a generator of time translation as in QFTs but an exponential of the mass squared operator and corresponds to the scaling of radial light-like coordinate r of the light-like boundary of CD: r is analogous to the complex coordinate z in conformal field theories.

   Also "big" SFRs (BSFRs) are possible and correspond to "ordinary" SFRs and in TGD framework mean death of self in the universal sense and followed by reincarnation as time reversed subjective time evolution [L32].

3. There is also classical time evolution at the level of space-time surfaces. Here the assumption that X^4 belongs to $H = M^4 \times CP_2$ defines Minkowski coordinates of M^4 as almost unique space-time coordinates of X^4 is the M^4 projection of X^4 is 4-D. This generalizes also to the case of M^8. Symmetries make it possible to identify an essentially a unique time coordinate.

   This means enormous simplification. General coordinate invariance is a marvellous symmetry but it leads to the problem of specifying space-time coordinates that is finding preferred coordinates. This seems impossible since 3-metric is dynamical. M^4 provides a fixed reference system and the problem disappears. M^4 is dynamical by its Minkowskian signature and one can speak about classical signals.

4. There is also classical time evolution for the induced spinor fields. At the level of H the spinor field is a superposition of modes of the massless Dirac operator (massless in 8-D sense). This spinor field is free and second quantized. Second quantization of induced spinor trivializes and this is absolutely crucial for obtaining scattering amplitudes for fermions and avoiding the usual problems for quantization of fermions in curved background.

   The induced spinor field is a restriction of this spinor field to the space-time surface and satisfies modified Dirac equation automatically. There is no need for second quantization at the level of space-time surface and propagators etc.... are directly calculable. This is an enormous simplification.

There are therefore as many as 4 time evolutions and subjective time evolution by BSFRs and possibly also by SSFRs is a natural candidate for time evolution as genuine evolution as emergence of more complex algebraic structures.

Could the inherent time evolution of HFF have a physical meaning in TGD after all?

The idea about inherent time evolution defined by HFF itself as one parameter group of outer automorphisms is very attractive by its universality: physics would become part of mathematics.

1. Thermodynamic interpretation, with inverse temperature identified as an analog of time coordinate, comes first in mind but need not be the correct interpretation.

2. Outer automorphisms should act at a very fundamental level analogous to the state space of topological field theories. Fundamental group is after all in question! The assignment of the S-matrix of particle physics to the outer automorphism does not look reasonable since the time evolution would be with respect to the linear Minkowski coordinate, which is not Lorentz invariant.

For these reasons I gave up the idea of Connes when considering it for the first time. However, TGD inspired theory of consciousness as a generalization of quantum measurement theory has evolved since then and the situation is different now.

The sequence of SSFRs defines subjective time evolution having no counterpart in QFTs. Each SSFR is preceded by a unitary time evolution, which however corresponds to the scaling of the light-like radial coordinate of the light-cone boundary [L55] rather than time translation. Hamiltonian is replaced with the scaling generator L_0 acting as Lorentz invariant mass squared operator so that Lorentz invariance is not lost.

Could the time evolution assignable to L_0 correspond to the outer automorphism of Connes when one poses an infinite number of gauge conditions making inner automorphisms gauge transformations? The connection of Connes proposal with conformal field theories and with TGD is indeed suggestive.

1. Conformally invariant systems obey infinite number of gauge conditions stating that the conformal generators L_n, $n > 0$, annihilate physical states and carry vanishing Noether charges.

   These gauge conditions bring in mind the condition that infinitesimal inner automorphisms do not change the system physically. Does this mean that Connes outer automorphism generates the time evolution and inner automorphisms act as gauge symmetries? One would have an analog of gauge field theory in HFF.

2. In TGD framework one has an infinite hierarchy of systems satisfying conditions analogous to the conformal gauge conditions. The generators of the super-symplectic algebra (SCA) acting as isometries of the "world of classical worlds" (WCW) are labelled by non-negative conformal weight n and it has infinite hierarchy of algebras SCA$_k$ isomorphic to it with conformal weights given by k-multiple of those of the entire algebra, $k = 1, 2, ...$

   Gauge conditions state for SCA_k that the generators of SCA_k and its commutator with SCA annihilate physical states. The interpretation is in terms of a hierarchy of improving measurement resolutions with degrees of freedom below measurement resolution acting like gauge transformations.

   The Connes automorphism would "see" only the time evolution in the degrees of freedom above measurement resolution and as k increases, their number would increase.

In the case of hyperfinite factors of type II_1 (HFFs) the fundamental group of corresponding factor II_∞ consists of all reals: I hope I am right here.

1. The hyperfinite factors of type II_1 and corresponding factors II_∞ are natural in the TGD context. Therefore the spectrum would consist of reals unless one poses additional conditions.

2. Could the automorphisms correspond to the scalings of the lightcone proper time, which replace time translations as fundamental dynamics. Also in string models scalings take the role of time translations.

3. In zero energy ontology (ZEO) the scalings would act in the moduli space of causal diamonds which is finite-dimensional. This moduli space defines the backbone of the "world of classical worlds". WCW itself consists of a union of sub-WCs as bundle structures over CDs [**?**]. The fiber consists of space-time surfaces inside a given CD analogous to Bohr orbits and satisfying holography reducing to generalized holomorphy. The scalings as automorphisms scale the causal diamonds. The space of CDs is a finite-dimensional coset space and has also other symmetry transformations.

4. The number theoretic vision suggests a quantization of the spectrum of Λ so that for a given extension of rationals the spectrum would belong to the extension. HFFs would be labelled at least partially by the extensions of rationals. The recent view of M^8-H duality [L79] is dramatically simpler than the earlier view [L45, L46, **?**] and predicts that the space-time regions are determined by a pair of analytic functions with rational coefficients forced by number theoretical universality meaning that the space-time surfaces have interpretation also as p-adic surfaces.

   The simplest analytic functions are polynomials with integer coefficients and if one requires that the coefficients are smaller than the degree of the polynomial, the number of polynomials is finite for a given degree. This would give very precise meaning for the concept of number theoretic evolution.

   There would be an evolutionary hierarchy of pairs of polynomials characterized by increasing complexity and one can assign to these polynomials extension of rationals characterized by ramified primes depending on the polynomials. The ramified primes would have interpretation as p-adic primes characterizing the space-time region considered. Extensions of rationals and ramified primes could also characterize HFFs. This is a rather dramatic conjecture at the level of pure mathematics.

5. Scalings define renormalization group in standard physics. Now they scale the size of the CD. Could the scalings as automorphisms of HFFs correspond to discrete renormalization operations?

Three views about finite measurement resolution

Evolution could be seen physically as improving finite measurement resolution: this applies to both sensory experience and cognition. There are 3 views about finite measurement resolution (FMR) in TGD.

1. Hyper finite factors (HFFs) and FMR

HFFs are an essential part of Connes's work and I encountered them about 15 years ago or so [K87, K28].

The inclusions of hyper-finite factors HFFs provide one of the three - as it seems equivalent - ways to describe finite measurement resolution (FMR) in TGD framework: the included factor defines an analog for gauge degrees of freedom which correspond to those below measurement resolution.

2. Cognitive representations and FMR

Another description for FMR in the framework of adelic physics would be in terms of cognitive representations [L34]. First some background about M^8-H duality.

1. There are number theoretic and geometric views about dynamics. In algebraic dynamics at the level of M^8, the space-time surfaces are roots of polynomials. There are no partial differential equations like in the geometric dynamics at the level of H.

2. The algebraic "dynamics" of space-time surfaces in M^8 is dictated by co-associativity, which means that the normal space of the space-time surface is associative and thus quaternionic. That normal space rather than tangent space must be associative became clear last year [L45, L46].

3. M^8-H duality maps these algebraic surfaces in M^8 to $H=M^4\times CP_2$ and the one obtains the usual dynamics based on variational principle giving minimal surfaces which are non-linear analogs for the solutions of massless field equations. Instead of polynomials the natural functions at the level of H are periodic functions used in Fourier analysis [L55].

At level of complexified M^8 cognitive representation would consist of points of co-associative space-time surface X^4 in complexified M^8 (complexified octonions), whose coordinates belong to extension of rationals and therefore make sense also p-adically for extension of p-adic numbers induced by extension of rationals. M^8-H duality maps the cognitive representations to H.

Cognitive representations form a hierarchy: the larger the extension of rationals, the larger the number of points in the extension and in the unique discretization of space-time surface. Therefore also the measurement resolution improves.

The surprise was that the cognitive representations which are typically finite, are for the "roots" of octonionic polynomials infinite [L45, L46]. Also in this case the density of points of cognitive representation increases as the dimension of extensions increases.

The understanding of the physical interpretation of M^8-H duality increased dramatically during the last half year.

1. X^4 in M^8 is highly analogous to momentum space (4-D analog of Fermi ball one might say) and H to position space. Physical states correspond to discrete sets of points - 4-momenta - in X^4. This is just the description used in particle physics for physical states. Time and space in this description are replaced by energy and 4-momentum. At the level of H one space-time and classical fields and one talks about frequencies and wavelengths instead of momenta.

2. M^8-H duality is a generalization of Fourier transform. Hitherto I have assumed that the space-time surface in M^8 is mapped to H. The momentum space interpretation at the level of M^8 however requires that the image must be a superposition of translates of the image in plane wave with some momentum: only the translates inside some bigger CD are allowed - this means infrared cutoff.

   The total momentum as sum of momenta for two half-cones of CD in M^8 is indeed well-defined. One has a generalization of a plane wave over translational degrees of freedom of CD and restricted to a bigger CD.

   At the limit of infinitely large size for bigger CD, the result is non-vanishing only when the sum of the momenta for two half-cones of CD vanishes: this corresponds to conservation of 4-momentum as a consequence of Poincare invariance rather than assumption as in the earlier approach [L55].

   This generalizes the position-momentum duality of wave mechanics lost in quantum field theory. Point-like particle becomes a quantum superposition of space-time surfaces inside the causal diamond (CD). Plane wave is a plane wave for the superposition of space-time surfaces inside CD having the cm coordinates of CD as argument.

3. Inclusion hierarchy of supersymplectic algebras and FMR

The third inclusion hierarchy allowing to describe finite measurement resolution is defined by supersymplectic algebras acting as the isometries of the "world of classical worlds" (WCW) consisting of space-time surfaces are preferred extremals ("roots" of polynomials in M^8 and minimal surfaces satisfying infinite-D set of additional "gauge conditions" in H).

At a given level of hierarchy generators with conformal weight larger than n act like gauge generators as also their commutators with generators with conformal weight smaller than n correspond to vanishing Noether charges. This defines "gauge conditions".

To sum up, there are therefore 3 hierarchies allowing to describe finite measurement resolution and they must be essentially equivalent in TGD framework.

Three evolutionary hierarchies

There are three evolutionary hierarchies: hierarchies of extensions of extensions of... ofrationals...; inclusions of inclusions of .... of HFFs, and inclusions of isomorphic super symplectic algebras.

1. Extensions of rationals

The extensions of rationals become algebraically increasingly complex as their dimension increases. The co-associative space-time surfaces in M^8 are "roots" of real polynomials with rational coefficients to guarantee number theoretical universality and this means space-time surfaces are characterized by extension of rationals.

Each extension of rationals defines extensions for p-adic number fields and entire adele. The interpretation is as a cognitive leap: the system's intelligence/algebraic complexity increases when the extension is extended further.

The extensions of extensions of .... define hierarchies with Galois groups in certain sense products of extensions involved. Exceptional extensions are those which do not allow this decomposition. In this case Galois group is a simple group. Simple groups are primes of finite groups and correspond to elementary particles of cognition. Kind of fundamental, non-decomposable ideas. Mystic might speak of pure states of consciousnesswith no thoughts.

In the evolution by quantum jumps the dimension of extension increases in statistical sense and evolution is unavoidable. This evolution is due to subjective time evolution by quantum jumps, something which is in spirit with Connes proposal but replaces time evolution by a sequence of evolutionary leaps.

2. Inclusions of HFFs as a hierarchy

HFFs are fractals. They have infinite inclusion hierarchies in which sub-HFF isomorphicto entire HFFs is included to HFF.

Also the hierarchies of inclusions define evolutionary hierarchies: HFF which is isomorphic with original becomes larger and in some sense more complex than the included factor. Also now one has sequences of inclusions of inclusions of.... These sequences would correspond to sequences for extensions of extensions... of rationals. Note that the inclusion hierarchy would be the basic object: not only single HFF in the hierarchy.

3. Inclusions of supersymplectic algebras as an evolutionary hierarchy

The third hierarchy is defined by the fractal hierarchy of sub-algebras of supersymplectic algebra isomorphic to the algebra itself. At a given level of hierarchy generators with conformal weight larger than n correspond to gauge degrees of freedom. As n increases the number of physical degrees of freedom above measurement resolution increases which means evolution. This hierarchy should correspond rather concretely to that for the extensions of rationals. These hierarchies would be essentially one and the same thing in the TGD Universe.

TGD based model for subjective time development

The understanding of subjective time development as sequences of SSFRs preceded by unitary "time" evolution has improved quite considerably recently [L55]. The idea is that the subjective time development as a sequence of scalings at the light-cone boundary generated by the vibrational part $\hat{L}_0$ of the scaling generator $L_0 = p^2 - \hat{L_0}$ (L_0 annihilates the physical states). Also p-adic mass calculations use $\hat{L}_0$.

For more than 10 years ago [K48, K28], I considered the possibility that Connes time evolution operator that he assigned with thermo-dynamical time could have a significant role in the definition of S-matrix in standard sense but had to give up the idea.

It however seems that for super-symplectic algebra $\hat{L}_0$ generates an outer automorphism since the algebra has only generators with conformal with $n > 0$ and its extension to included also generators with $n \leq 0$ is required to introduce L_0: since L_0 contains annihilation operators, it indeed generates outer automorphism in SCA. The two views could be equivalent! Whereas Connes considered thermo-dynamical time evolution, in TGD framework the time evolution would be subjective time evolution by SSFRs.

1. The guess would be that the exponential of the scaling operator L_0 gives the time evolution. The problem is that L_0 annihilates the physical states. The solution of the problem would be the same as in p-adic thermodynamics. L_0 decomposes as $L_0 = p^2 - \hat{L_0}$ and the vibrational part $\hat{L}_0$ this gives mass spectrum as eigenvalues of p^2. The thermo-dynamical state in p-adic thermodynamics is $p^{\hat{L}_0\beta}$. This operator exists p-adically in the p-adic number field defined by prime p.

2. Could unitary subjective time development involve the operator $exp(i2\pi L_0\tau)$ $\tau = log(T/T_0)$? This requires $T/T_0 = exp(n/m)$ guaranteeing that exponential is a root of unity for an eigenstate of L_0. The scalings are discretized and scalings come as powers of $e^{1/m}$. This is possible using extensions of rationals generated by a root of e. The unique feature of p-adics is that e^p is ordinary p-adic number. This alone would give periodic time evolution for eigenstates of L_0 with integer eigenvalues n.

SSA and SSA_n

Supersymplectic algebra SSA has fractal hierarchies of subalgebras SSA_n. The integers in a given hierarchy are of forn $n_1, n_1n_2, n_1n_2n_3, ...$ and correspond naturally to hierarchies of inclusions of HFFs. Conformal weights are positive: $n > 0$. For ordinary conformal algebras also negative weights are allowed. Yangians have only non-negative weights. This is of utmost importance.

SSA_n with generators have radial light-like conformal weights coming as multiples of n. SSA_n annihilates physical states and $[SSA_n, SSA]$ does the same. Hence the generators with conformal weight larger than n annihilate the physical states.

What about generators with conformal weights smaller than n? At least a subset of them need not annihilate the physical states. Since L_n are superpositions of creation operators, the idea that analogs of coherent states could be in question.

It would be nice to have a situation in which L_n, $n < m$ commute. $[L_k, L_l] = 0$ effectively for $k + l \geq m$.

The simplest way to obtain a set of effectively commuting operators is to take the generators L_k, $[m/2] < k < m$, where $[m/2]$ is nearest integer larger than $m/2$.

This raises interesting questions.

1. Could the Virasoro generators $O(\{c_k\}) = \sum_{k\in[m/2],m]} c_kL_k$ as linear combinations of creation operators generate a set of coherent states as eigenstates of their Hermitian conjugates.

2. Some facts about coherent states are in order.

   (a) When one adds to quantum harmonic oscillator Hamiltonian oscillator a time dependent perturbation which lasts for a finite the vacuum state evolves to an oscillator vacuum whose position is displacemented. The displacement is complex and is a Fourier component of the external force $f(t)$ corresponding to the harmonic oscillator frequency ω. Time evolution picks up only this component.

   (b) Coherent state property means that the state is eigenstate of the annihilation creation operator with eivengeu $\alpha = -ig(\omega)$ where $g(omega) = \int f(u)exp(-i\omega u)du$ is Fourier transform of $f(t)$.

   (c) Coherent states are not orthogonal and form an overcomplete set. The overlaps of coherent states are proportional to a Gaussian depending on the complex parameters characterizing them. One can however develop any state in terms of coherent states as a unique expansion since one can represent unitary in terms of coherent states.

   (d) Coherent state obtained from the vacuum state by time evolution in presence of $f(t)$ by a unitary displacement operator $D(\alpha) = exp(\alpha a^\dagger - \overline{\alpha}a)$. (`https://en.wikipedia.org/wiki/Displacement_operator`).

   The displacement operator is a unitary operator and in the general case the displacement is complex. The product of two displacement operators would be apart from a phase factor a displacement operator associated with the sum of displacements.

   (e) Harmonic oscillator coherent states are indeed maximally classical since wave packets have minimal width in both q and p space. Furthermore, the classical expectation values for q and p obey classical equations of motion.

These observations raise interesting questions about how the evolution by SSFRs could be modelled.

1. Instead of harmonic oscillator in q-space, one would have time evolution in the space of scalings of causal diamond parameterized by the scaling parameter $\tau = log(T/T_0)$, where T can be identified as the radial light-like coordinate of light-cone boundary.

   The analogs of harmonic oscillator states would be defined in this space and would be essentially wave packets with ground state minimizing the width of the wave packet.

2. The role of harmonic oscillator Hamiltonian in absence of external force would be taken by the generator $\hat{L}_0$ ($L_0 = p^2 - \hat{L}_0$ acts trivially) and gives rise to mass squared quantization. The situation would be highly analogous to that in p-adic thermodynamics. The role of ω

would be taken by the minimal conformal weight h_{min} such that the eigenvalues of L_0 are its multiples. It seems that this weight must be equal to $h_{min} = 1$.

The commutations of $\hbar L_0$ with L_k, $k > 0$ would be as for L_0 so what the replacement should not affect the situation.

3. The scaling parameter τ is analogous to the spatial coordinate q for the harmonic oscillator. Can one identify the analog of the external force $f(t)$ acting during unitary evolution between two SSFRs? Or is it enough to use only the analog of $g(\omega \rightarrow h_{min} = 1)$ - that is the coefficients C_k.

   To identify $f(t)$, one needs a time coordinate t. This was already identified as τ. This one would have $q = t$, which looks strange. The space in which time evolution is the space of scalings and the time evolutions are scalings and thus time evolution means translation in this space. The analog for this would be Hamiltonian $H = i\hbar d/dq$.

   Number theoretical universality allows only the values of $\tau = r/s$ whose exponents give roots of unity. Also $exp(n\tau)$ makes sense p-adically for these values. This would mean that the Fourier transform defining g would become discrete and be sum over the values $f(\tau = r/s)$.

4. What happens if one replaces $\hat{L}_0$ with L_0. In this case one would have the replacement of ω with $h_{vac} = 0$. Also the analog of Fourier transform with zero frequency makes sense. $\hat{L_0} = p^2 - L_0$ is the most natural choice for the Hamiltonian defining the time evolution operator but is trivial. Could $\Delta^{i\tau}$ describe the inherent time evolution. It would be outer automorphism since it is not defined solely in terms of SCA. So: could one have $\Delta = exp(\hat{L_0})$ so that $\Delta^{i\tau}$ coincide with $exp(i\hat{L_0}\tau)$? This would mean the identification

$$\Delta = exp(\hat{L_0}) \ ,$$

   which is a positive definite operator. The exponents coming from $exp(iL_0\tau)$ can be number theoretically universal if $\tau = log(T/T_0)$ is a rational number implying $T/T_0 = exp(r/s)$, which is possible number theoretically) and the extension of rationals contains some roots of e.

5. Could one have $\Delta = L_0$? Also now that positivity condition would be satisfied if SSA conformal weights satisfy $n > 0$.

   The problem with this operation is that it is not number theoretically universal since the exponents $exp(ilog(n)\tau)$ do not exist p-adically without introducing infinite-D extension of p-adic number making $log(n)$ well-defined.

   What is however intriguing is that the "time" evolution operator $\Delta^{i\tau}$ in the eigenstate basis would have trace equal to $Tr(\Delta^{i\tau}) \sum d(n)n^{i\tau}$, where $d(n)$ is the degeneracy of the state. This is a typical zeta function: for Riemann Zeta one has $d(n) = 1$.

   For $\Delta = exp(L_0)$ option $Tr(\Delta^{i\tau}) = \sum d(n)exp(in\tau)$ exists for $\tau = r/s$ if r:th root of e belongs to the extension of p-adics.

To sum up, one would have Gaussian wave packet as harmonic oscillator vacuum in the space of scaled variants of CD. The unitary time evolution associated with SSFR would displace the peak of the wave packet to a larger scalings. The Gaussian wave function in the space of scaled CDs has been proposed earlier.

Could this time evolution make sense and be even realistic?

1. The analogs of harmonic oscillator states are defined in the space of scalings as Gaussians and states obtained from them using oscillator operators. There would be a wave function in the moduli space of CDs analogous to a state of harmonic oscillator.

2. SSFR following the time evolutions would project to an eigenstate of harmonic oscillator having in general displaced argument. The unitary displacement operator D should commute with the operators having the members of zero energy states at the passive boundary of CD as eigenstates. This poses strong conditions. At least number theoretic measurements could satisfy these conditions.

3. SSFRs are identified as weak measurements as near as possible to classical measurements. Time evolution by the displacement would be indeed highly analogous to classical time evolution for theeharmonic oscillator.

4. The unitary displacement operator corresponds to the arbitrary external force on the harmonic oscillator and it seems that it would be selected in SSFR for the unitary evolution after SSFR. This means fixing the coefficients C_k in the operator $\sum C_k L_k$.

What is the subjective "time" evolution operator when in the case of SSA_n?

1. The scaling analog of the unitary displacement operator D as $D = \sum exp(\sum C_k L_k - \overline{C}_k L_{-k})$ is highly suggestive and would take the oscillator vacuum to a coherent state. Coefficients C_k would be proportional to τ. There would be a large number of choices for the unitary displacement operator. One can also consider complex values of τ since one has complexified M^8.

2. There should be a normalization for the coefficients: without this it is not possible to talk about a special value of τ does not make sense. For instance, the sum of their moduli squared could be equal to 1. This would give interpretation as a quantum state in the degrees of freedom considered. The width of the Gaussian would increase slowly during the unitary time evolution and be proportional to $log(T/T_0)$.

   The width of the Gaussian would increase slowly as a function of T during the unitary time evolution and be proportional to $log(T/T_0)$. The condition that c_k are proportional the same complex number times τ is too strong.

3. The arbitrariness in the choice of C_k would bring in a kind of non-determinism as a selection of this superposition. The ability to engineer physical systems is in conflict with the determinism of classical physics and also difficult to understand in standard quantum physics. Could one interpret this choice as an analog for engineering a Hamiltonian as in say quantum computation or build-up of an electric circuit for some purpose? Could goal directed action correspond to this choice?

   If so engineerable degrees of freedom would correspond to intermediate degrees of freedom associated with L_k, $[m/2] \leq k \leq m$. They would be totally absent for $k = 1$ and this would correspond to a situation analogous to the standard physics without any intentional action.

3.5 MIP*= RE: What could this mean physically?

I received a very interesting link to a popular article (`https://cutt.ly/sfd5UQF`) explaining a recently discovered deep result in mathematics having implications also in physics. The article [A97] (`https://cutt.ly/rffiYdc`) by Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen has a rather concise title "**MIP*=RE**". In the following I try to express the impressions of a (non-mainstream) physicist about the result.

The following is the result expressed using the concepts of computer science about which I know very little at the hard technical level. The results are however told to state something highly non-trivial about physics.

1. RE (recursively enumerable languages) denotes all problems solvable by computer. P denotes the problems solvable in a polynomial time. NP does not refer to a non-polynomial time but to "non-deterministic polynomial acceptable problems" - I hope this helps the reader- I am a little bit confused! It is not known whether P = NP is true.

2. IP problems (P is now for "prover" that can be solved by a collaboration of an interrogator and prover who tries to convince the interrogator that her proof is convincing with high enough probability. MIP involves multiple l provers treated as criminals trying to prove that they are innocent and being not allowed to communicate. MIP* is the class of solvable problems in which the provers are allowed to entangle.

The finding, which is characterized as shocking, is that *all* problems solvable by a Turing computer belong to this class: MIP*=RE. All problems solvable by computer would reduce to problems in the class MIP*! Quantum computation would indeed add something genuinely new to the classical computation.

Quantum entanglement would play an essential role in quantum computation. Also the implications for physics are highly non-trivial.

1. Connes embedding problem asking whether all infinite-D matrices can always be approximated by finite-D matrices has a negative solution.Therefore MIP*= RE does not hold true for hyperfinite factors of type II_1 (HFFs) central in quantum TGD. Also the Tirelson problem finds a solution. The measurements of commuting observers performed by two observers are equivalent to the measurements of tensor products of observables only in finite-D case and for HFFs. That quantum entanglement would not have any role in HFFs is in conflict with intuition.

2. In the TGD framework finite measurement resolution is realized in terms of HFFs at Hilbert space level and in terms of cognitive representations at space-time level defined purely number-theoretically. This leads to a hierarchy of adeles defined by extensions of rationals and the Hilbert spaces must have algebraic extensions of rationals as a coefficient field. Therefore one cannot in general case find a unitary transformation mapping the entangled situation to an unentangled one, and quantum entanglement plays a key role. It seems that computationalism formulated in terms of recursive functions of natural numbers must be formulated for the hierarchy of extensions of rationals in terms of algebraic integers.

3. In TGD inspired theory of consciousness entanglement between observers could be seen as a kind of telepathy helping to perform conscious quantum computations. Zero energy ontology also suggests a modification of the views about quantum computation. TGD can be formulated also for real and p-adic continua identified as correlates of sensory experience and cognition, and it seems that computational paradigm need not apply in the continuum cases.

3.5.1 Two physically interesting applications

There are two physically interesting applications of the theorem interesting also from the TGD point of view and force to make explicit the assumptions involved.

About the quantum physical interpretation of MP*

To proceed one must clarify the quantum physical interpretation of MIP*.

Quantum measurement requires entanglement of the observer O with the measured system M. What is basically measured is the density matrix of M (or equivalently that of O). State function reduction gives as an outcome a state, which corresponds to an eigenvalue of the density matrix. Note that this state can be an entangled state if the density matrix has degenerate eigenvalues. Quantum measurement can be regarded as a question to the measured system: "What are the values of given commuting observables?". The final state of quantum measurement provides an eigenstate of the observables as the answer to this question. M would be in the role of the prover and O_i would serve as interrogators.

In the first case multiple interrogators measurements would entangle M with unentangled states of the tensor product $H_1 \otimes H_2$ for O followed by a state function reduction splitting the state of M to un-entangled state in the tensor product $M_1 \otimes M_2$.

In the second case the entire M would be interrogated using entanglement of M with entangled states of $H_1 \otimes H_2$ using measurements of several commuting observables. The theorem would state that interrogation in this manner is more efficient in infinite-D case unless HFFs are involved.

3. This interpretation differs from the interpretation in terms of computational problem solving in which one would have several provers and one interrogator. Could these interpretations be dual as the complete symmetry of the quantum measurement with respect to O and M suggests? In the case of multiple provers (analogous to accused criminals) it is advantageous to isolate them. In the case of multiple interrogators the best result is obtained if the interrogator does not effectively split itself into several ones.

Connes embedding problem and the notion of finite measurement/cognitive resolution

Alain Connes formulated what has become known as Connes embedding problem. The question is whether infinite matrices forming factor of type II_1 can be *always* approximated by finite-D matrices that is imbedded in a *hyperfinite* factor of type II_1 (HFF). Factors of type II and their HFFs are special classes of von Neumann algebras possibly relevant for quantum theory.

This result means that if one has measured of a complete set of for a product of commuting observables acting in the full space, one can find in the finite-dimensional case a unitary transformation transforming the observables to tensor products of observables associated with the factors of a tensor product. In the infinite-D case this is not true.

What seems to put alarms ringing is that in TGD there are excellent arguments suggesting that the state space has HFFs as building bricks. Does the result mean that entanglement cannot help in quantum computation in TGD Universe? I do not want to live in this kind of Universe!

Tsirelson problem

Tsirelson problem (see this) is another problem mentioned in the popular article as a physically interesting application. The problem relates to the mathematical description of quantum measurement.

Three systems are considered. There are two systems O_1 and O_2 representing observers and the third representing the measured system M. The measurement reducing the entanglement between M and O_1 or O_2 can regarded as producing correspondence between state of M and O_1 or O_2, and one can think that O_1 or O_2 measures only obserservables in its own state space as a kind of image of M. There are two ways to see the situation. The provers correspond now to the observers and the two situations correspond to provers without and with entanglement.

Consider first a situation in which one has single Hilbert space H and single observer O. This situation is analogous to IP.

1. The state of the system is described statistically by a density matrix - not necessarily pure state -, whose diagonal elements have interpretation as reduction probabilities of states in this bases. The measurement situation fixes the state basis used. Assume an ensemble of identical copies of the system in this state. Assume that one has a complete set of commuting observables.

2. By measuring all observables for the members of the ensemble one obtains the probabilities as diagonal elements of the density matrix. If the observable is the density matrix having no- degenerate eigenvalues, the situation is simplified dramatically. It is enough to use the density matrix as an observable. TGD based quantum measurement theory assumes that the density matrix describing the entanglement between two subsystems is in a universal observable measure in state function reductions reducing their entanglement.

3. Can one deduce also the state of M as a superposition of states in the basic chosen by the observer? This basis need not be the same as the basis defined by - say density matrix if the system has interacted with some system and this ineracton has led to an eigenstate of the density matrix. Assume that one can prepare the latter basis by a physical process such as this kind of interaction.

   The coefficients of the state form a set of N^2 complex numbers defining a unitary $N \times N$ matrix. Unitarity conditions give N conditions telling that the complex rows and unit vectors: these numbers are given by the measurement of all observables. There are also $N(N-1)$ conditions telling that the rows are orthogonal. Together these $N+N(N-1) = N^2$ numbers fix the elements of the unitary matrix and therefore the complex coefficients of the state basis of the system can be deduced from a complete set of measurements for all elements of the basis.

Consider now the analog of the MIS* involving more than one observer. For simplicity consider two observers.

1. Assume that the state space H of M decomposes to a tensor product $H = H_1 \otimes H_2$ of state spaces H_1 and H_2 such that O_1 measures observables X_1 in H_1 and O_2 measuresobservables X_2 in H_2. The observables X_1 and X_2 commute since they act in different tensor factors. The absence of interaction between the factors corresponds to the inability of the provers to communicate. As in the previous case, one can deduce the probabilities for the various outcomes of the joint measurements interpreted as measurements of a complete set of observables $X_1 \otimes X_2$.

2. One can also think that the two systems form a single system O so that O_1 and O_2 can entangle. This corresponds to a situation in which entanglement between the provers is allowed. Now X_1 and X_2 are not in general independent but also now they must commute. One can deduce the probabilities for various outcomes as eigenstates of observables X_1X_2 and deduce the diagonal elements of the density matrix as probabilities.

Are these ways to see the situation equivalent? Tsirelson demonstrated that this is the case for finite-dimensional Hilbert spaces, which can indeed be decomposed to a tensor product of factors associated with O_1 and O_2. This means that one finds a unitary transformation transforming the entangled situation to an unentangled one and to tensor product observables.

For the infinite-dimensional case the situation remained open. According to the article, the new result implies that this is not the case. For hyperfinite factors the situation can be approximated with a finite-D Hilbert space so that the situations are equivalent in arbitrary precise approximation.

3.5.2 The connection with TGD

The result looks at first a bad news from the TGD point of view, where HFFs are highly suggestive. One must be however very careful with the basic definitions.

Measurement resolution

Measurement resolution is the basic notion.

1. There are intuitive physicist's arguments demonstrating that in TGD the operator algebras involved with TGD are HFFs provides a description of finite measurement resolution. The inclusion of HFFs defines the notion of resolution: included factor represents the degrees of freedom not seen in the resolution used [K87, K28] (`http://tgdtheoryd.fi/pfpool/vNeumann.pdf`) and `http://tgdtheoryd.fi/pfpool/vNeumannnew.pdf`).

   Hyperfinite factors involve new structures like quantum groups and quantum algebras reflecting the presence of additional symmetries: actually the "world of classical worlds" (WCW) as the space of space-time surfaces as maximal group of isometries and this group has a fractal hierarchy of isomorphic groups imply inclusion hierarchies of HFFs. By the analogs of gauge conditions this infinite-D group reduces to a hierarchy of effectively finite-D groups. For quantum groups the infinite number of irreps of the corresponding compact group effectively reduces to a finite number of them, which conforms with the notion of hyper-finiteness.

   It looks that the reduction of the most general quantum theory to TGD-like theory relying on HFFs is not possible. This would not be surprising taking into account gigantic symmetries responsible for the cancellation of infinities in TGD framework and the very existence of WCW geometry.

2. Second TGD based approach to finite resolution is purely number theoretic [L23] and involves adelic physics as a fusion of the real physics with various p-adic physics as correlates of

cognition. Cognitive representations are purely number theoretic and unique discretizations of space-time surfaces defined by a given extension of rationals forming an evolutionary hierarchy: the coordinates for the points of space-time as a 4-surface of the embedding space $H = M^4 \times CP_2$ or of its dual M^8 are in the extension of rationals defining the adele. In the case of M^8 the preferred coordinates are unique apart from time translation. These two views would define descriptions of the finite resolution at the level of space-time and Hilbert space. In particular, the hierarchies of extensions of rationals should define hierarchies of inclusions of HFFs.

For hyperfinite factors the analog of MIP*=RE cannot hold true. Doesn't the TGD Universe allow a solution of all the problems solvable by Turing Computer? There is a loophole in this argument.

1. The point is that for the hierarchy of extensions of rationals also Hilbert spaces have as a coefficient field the extension of rationals! Unitary transformations are restricted to matrices with elements in the extension. In general it is not possible to realize the unitary transformation mapping the entangled situation to an un-entangled one! The weakening of the theorem would hold true for the hierarchy of adeles and entanglement would give something genuinely new for quantum computation!

2. A second deep implication is that the density matrix characterizing the entanglement between two systems cannot in general be diagonalized such that all diagonal elements identifiable as probabilities would be in the extension considered. One would have stable or partially stable entanglement (could the projection make sense for the states or subspaces with entanglement probability in the extension). For these bound states the binding mechanism is purely number theoretical. For a given extension of p-adic numbers one can assign to algebraic entanglement also information measure as a generalization of Shannon entropy as a p-adic entanglement entropy (real valued). This entropy can be negative and the possible interpretation is that the entanglement carries conscious information.

What about transcendental extensions?

During the writing of this article an interesting question popped up.

1. Also transcendental extensions of rationals are possible, and one can consider the generalization of the computationalism by also allowing functions in transcendental extensions. Could the hierarchy of algebraic extensions could continue with transcendental extensions? Could one even play with the idea that the discovery of transcendentals meant a quantum leap leading to an extension involving for instance e and π as basic transcendentals? Could one generalize the notion of polynomial root to a root of a function allowing Taylor expansion $f(x) = \sum q_n x^n$ with rational coefficients so that the roots of $f(x) = 0$ could be used define transcendental extensions of rationals?

2. Powers of e or its root define and infinite-D extensions having the special property that they are finite-D for p-adic number fields because e^p is ordinary p-adic number. In the p-adic context e can be defined as a root of the equation $x^p - \sum p^n/n! = 0$ making sense also for rationals. The numbers $log(p_i)$ such that p_i appears a factor for integers smaller than p define infinite-D extension of both rationals and p-adic numbers. They are obtained as roots of $e^x - p_i = 0$.

3. The numbers $(2n+1)\pi$ $(2n\pi)$ can be defined as roots of $sin(x) = 0$ $(cos(x) = 0$. The extension by π is infinite-dimensional and the conditions defining it would serve as consistency conditions when the extension contains roots of unity and effectively replaces them.

4. What about other transcendentals appearing in mathematical physics? The values $\zeta(n)$ of Riemann Zeta appearing in scattering amplitudes are for even values of n given by $\zeta(2n) = (-1)^{n+1} B_{2n} (2\pi)^{2n}/2(2n+1)!$. This follows from the functional identity for Riemann zeta and from the expression $\zeta(-n) = (-1)^n B_{n+1}/(n+1)$ ($(B(-1/2) = -1/2)$ (`https:`

//cutt.ly/dfgtgmw). The Bernoulli numbers B_n are rational and vanish for odd values of n. An open question is whether also the odd values are proportional to π^n with a rational coefficient or whether they represent "new" transcendentals.

What about the situation for the continuum version of TGD?

At least the cognitively finitely representable physics would have the HFF property with coefficient field of Hilbert spaces replaced by an extension of rationals. Number theoretical universality would suggest that HFF property characterizes also the physics of continuum TGD. This leads to a series of questions.

1. Does the new theorem imply that in the continuum version of TGD all quantum computations allowed by the Turing paradigm for real coefficients field for quantum states are not possible: $MIP* \subset RE$? The hierarchy of extensions of rationals allows utilization of entanglement, and one can even wonder whether one could have $MIP* = RE$ at the limit of algebraic numbers.

2. Could the number theoretic vision force change also the view about quantum computation? What does RE actually mean in this framework? Can one really assume complex entanglement coefficients in computation. Does the computational paradigm makes sense at all in the continuum picture?

   Are both real and p-adic continuum theories unreachable by computation giving rise to cognitive representations in the algebraic intersubsection of the sensory and cognitive worlds? I have indeed identified real continuum physics as a correlate for sensory experience and various p-adic physics as correlates of cognition in TGD: They would represent the computionally unreachable parts of existence.

   Continuum physics involves transcendentals and in mathematics this brings in analytic formulas and partial differential equations. At least at the level of mathematical consciousness the emergence of the notion of continuum means a gigantic step. Also this suggests that transcendentality is something very real and that computation cannot catch all of it.

3. Adelic theorem allows to express the norm of a rational number as a product of inverses of its p-adic norms. Very probably this representation holds true also for the analogs of rationals formed from algebraic integeres. Reals can be approximated by rationals. Could extensions of all p-adic numbers fields restricted to the extension of rationals say about real physics only what can be expressed using language?

Also fermions are highly interesting in the recent context. In TGD spinor structure can be seen as a square root of Kähler geometry, in particular for the "world of classical worlds" (WCW). Fermions are identified as correlates of Boolean cognition. The continuum case for fermions does not follow as a naïve limit of algebraic picture.

1. The quantization of the induced spinors in TGD looks different in discrete and continuum cases. Discrete case is very simple since equal-time anticommutators give discrete Kronecker deltas. In the continuum case one has delta functions possibly causing infinite vacuum energy like divergences in conserved Noether charges (Dirac sea).

2. In [L47] (https://cutt.ly/zfftoK6) I have proposed how these problems could be avoided by avoiding anticommutators giving delta-function. The proposed solution is based on zero energy ontology and TGD based view about space-time. One also obtains a long-sought-for concrete realization for the idea that second quantized induce spinor fields are obtained as restrictions of second quantized free spinor fields in $H = M^4 \times CP_2$ to space-time surface. The fermionic variant of $M^8 - H$-duality [L48] provides further insights and gives a very concrete picture about the dynamics of fermions in TGD.

These considerations relate in an interesting manner to consciousness. Quantum entanglement makes in the TGD framework possible telepathic sharing of mental images represented by sub-selves of self. For the series of discretizations of physics by HFFs and cognitive representations associated with extensions of rationals, the result indeed means something new.

What does one mean with quantum computation in TGD Universe?

The TGD approach raises some questions about computation.

1. The ordinary computational paradigm is formulated for Turing machines manipulating natural numbers by recursive algorithms. Programs would essentially represent a recursive function $n \to f(n)$. What happens to this paradigm when extensions of rationals define cognitive representations as unique space-time discretizations with algebraic numbers as the limit giving rise to a dense in the set of reals.

   The usual picture would be that since reals can be approximated by rationals, the situation is not changed. TGD however suggests that one should replace at least the quantum version of the Turing paradigm by considering functions mapping algebraic integers (algebraic rational) to algebraic integers.

   Quite concretely, one can manipulate algebraic numbers without approximation as a rational and only at the end perform this approximation and computations would construct recursive functions in this manner. This would raise entanglement to an active role even if one has HFFs and even if classical computations could still look very much like ordinary computation using integers.

   This suggests that computationalism usually formulated in terms of recursive functions of natural or rational numbers could be replaced with a hierarchy of computationalisms for the hierarchy of extensions of rationals. One would have recursively definable functions defined and having values in the extensions of rationals. These functions would be analogs of analytic functions (or polynomials) with the complex variable replaced with an integer or a rational of the extension. This poses very powerful constraints and there are good reasons to expect an increase of computational effectiveness. One can hope that at the limit of algebraic numbers of these functions allow arbitrarily precise approximations to real functions. If the real world phenomena can be indeed approximated by cognitive representations in the TGD sense, one can imagine a highly interesting approach to AI.

2. ZEO brings in also time reversal occurring in "big" (ordinary) quantum jumps and this modifies the views about quantum computation. In ZEO based conscious quantum computation halting means "death" and "reincarnation" of conscious entity, self? How the processes involving series of haltings in this sense differs from ordinary quantum computation: could one shorten the computation time by going forth and back in time.

There are many interesting questions to be considered. $M^8 - H$ duality gives justifications for the vision about algebraic physics. TGD leads also to the notion of infinite prime and I have considered the possibility that infinite primes could give a precise meaning for the dimension of infinite-D Hilbert space. Could the number-theoretic view about infinite be considerably richer than the idea about infinity as limit would suggest [K72].

The construction of infinite primes is analogous to a repeated second quantization of arithmetic supersymmetric quantum field theory allowing also bound states at each level and a concrete correspondence with the hierarchy of space-time sheets is suggestive. For the infinite primes at the lowest level of the hierarchy single particle states correspond to rationals and bound states to polynomials and therefore to the sets of their roots. This strongly suggests a connection with M^8 picture.

Could the number field of computable reals (p-adics) be enough for physics?

For some reason I have managed to not encounter the notion of computable number (see `https://cutt.ly/pTeSSfR`) as opposed to that of non-computable number (see `https://cutt.ly/gTeD9vF`). The reason is perhaps that I have been too lazy to take computationalism seriously enough.

Computable real number is a number, which can be produced to an arbitrary accuracy by a Turing computer, which by definition has a finite number of internal states, has input which is natural number and produces output which is natural numbers. Turing computer computes values of a function from natural numbers to itself by applying a recursive algorithm.

The following three formal definitions of the notion are equivalent.

1. The real number a is computable, if it can be expressed in terms of a computable function $n \rightarrow f(n)$ from natural numbers to natural numbers characterized by the property

$$\frac{f(n)-1}{n} \leq a \leq (\frac{f(n)+1}{n} .$$

For rational $a = q$, $f(n) = nq$ satisfies the conditions. Note that this definition does not work for p-adic numbers since they are not well-ordered.

2. The number a is computable if for an arbitrarily small rational number ϵ there exists a computable function producing a rational number r satisfying $|r - x \leq \epsilon$. This definition works also for p-adic numbers since it involves only the p-adic norm which has values which are powers of p and is therefore real valued.

3. a is computable if there exists a computable sequence of rational numbers r_i converging to a such that $|a - r_i| \leq 2^{-i}$ holds true. This definition works also for 2-adic numbers and its variant obtained by replacing 2 with the p-adic prime p makes sense for p-adic numbers.

The set R_c of computable real numbers and the p-adic counterparts $Q_{p,c}$ of R_c, have highly interesting properties.

1. R_c is enumerable and therefore can be mapped to a subset of rationals: even the ordering can be preserved. Also $Q_{p,c}$ is enumerable but now one cannot speak of ordering. As a consequence, most real (p-adic) numbers are non-computable. Note that the pinary expansion of a rational is periodic after some pinary digit. For a p-adic transcendental this is not the case.

2. Algebraic numbers are computable so that one can regard R_c as a kind of completion of algebraic numbers obtained by adding computable reals. For instance, π and e are computable. 2π can be computed by replacing the unit circle with a regular polygon with n sides and estimating the length as nL_n. L_n the length of the side. e can be computed from the standard formula. Interestingly, e^p is an ordinary p-adic number. An interesting question is whether there are other similar numbers. Certainly many algebraic numbers correspond to ordinary p-adic numbers.

3. R_c ($Q_{p,c}$) is a number field since the arithmetic binary operations $+, -\times, /$ are computable. Also differential and integral calculus can be constructed. The calculation of a derivative as a limit can be carried out by restricting the consideration to computable reals and there is always a computable real between two computable reals. Also Riemann sum can be evaluated as a limit involving only computable reals.

4. An interesting distinction between real and p-adic numbers is that in the sum of real numbers the sum of arbitrarily high digits can affect even all lower digits so that it requires computational work to predict the outcome. For p-adic numbers memory digits affect only the higher digits. This is why p-adic numbers are tailor made for computational purposes.Canonical identification $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ used in p-adic mass calculations to map p-adic mass squared to its real counterpart [K41] maps p-adics to reals in a continuous manner. For integers this corresponds is 2-to-1 due to the fact that the p-adic numbers $-1 = (p-1)/(1-p)$ and $1/p$ are mapped to p.

5. For computable numbers, one cannot define the relation =. One can only define equality in some resolution ϵ. The category theoretical view about equality is also effective and conforms with the physical view.

   Also the relations $\leq$ and $\geq$ fail to have computable counterparts since only the absolute value $|x - y|$ can appear in the definition and one loses the information about the well-ordered nature of reals. For p-adic numbers there is no well-ordering so that nothing is lost. A restriction to non-equal pairs however makes order relation computable. For p-adic numbers the same is true.

6. Computable number is obviously definable but there are also definanable numbers, which are not computable. Examples are Gödel numbers in a given coding scheme for statements, which are true but not provable. More generally, the Gödel numbers coding for undecidable problems such as the halting problem are uncomputable natural numbers in a given coding scheme. Chaitin's constant, which gives the probability that random Turing computation halts, is a non-computable but definable real number.

7. Computable numbers are arithmetic numbers, which are numbers definable in terms of first order logic using Peano's axioms. First order logic does not allow statements about statements and one has an entire hierarchy of statements about... about statements. The hierarchy of infinite primes defines an analogous hierarchy in the TGD framework and is formally similar to a hierarchy of second quantizations [K72].

3.6 Analogs Of Quantum Matrix Groups From Finite Measurement Resolution?

The notion of quantum group [**?**]eplaces ordinary matrices with matrices with non-commutative elements. This notion is physically very interesting, and in TGD framework I have proposed that it should relate to the inclusions of von Neumann algebras allowing to describe mathematically the notion of finite measurement resolution [**?**] These ideas have developed slowly through various side tracks.

In the sequel I will consider the notion of quantum matrix inspired by the recent view about quantum TGD relying on the notion of finite measurement resolution and show that under some additional conditions it provides a concrete representation and physical interpretation of quantum groups in terms of finite measurement resolution.

1. The basic idea is to replace complex matrix elements with operators, which are products of non-negative hermitian operators and unitary operators analogous to the products of modulus and phase as a representation for complex numbers. Modulus and phase would be non-commuting and have commutation relation analogous to that between momentum and plane-wave in accordance with the idea about quantization of complex numbers.

2. The condition that determinant and sub-determinants exist is crucial for the well-definedness of eigenvalue problem in the generalized sense. Strong/weak permutation symmetry of determinant requires its invariance apart from sign change under permutations of rows and/or columns. Weak permutation symmetry means development of determinant with respect to a fixed row or column and does not pose additional conditions. For weak permutation symmetry the permutation of rows/columns would however have a natural interpretation as braiding for the hermitian operators defined by the moduli of operator valued matrix elements and here quantum group structure emerges.

3. The commutativity of all sub-determinants is essential for the replacement of eigenvalues with eigenvalue spectra of hermitian operators and sub-determinants define mutually commuting set of operators.

Quantum matrices define a more general structure than quantum group but provide a concrete representation for them in terms of finite measurement resolution, in particular when q is a root of unity. For $q = \pm 1$ (Bose-Einstein or Fermi-Dirac statistics) one obtains quantum matrices for which the determinant is apart from possible change by a sign factor invariant under the permutations of both rows and columns. One can also understand the recursive fractal structure of inclusion sequences of hyper-finite factors resulting by replacing operators appearing as matrix elements with quantum matrices and a concrete connection with quantum groups emerges.

In Zero Energy Ontology (ZEO) M-matrix serving as the basic building brick of unitary U-matrix and identified as a hermitian square root of density matrix provides a possible application for this vision. Especially fascinating is the possibility of hierarchies of measurement resolutions represented as inclusion sequences realized as recursive construction of M-matrices. Quantization would emerge already at the level of complex numbers appearing as M-matrix elements.

This approach might allow to unify various ideas behind TGD. For instance, Yangian algebras emerging naturally in twistor approach are examples of quantum algebras. The hierarchy of Planck constants should have close relationship with inclusions and fractal hierarchy of sub-algebras of super-symplectic and other conformal algebras.

3.6.1 Well-definedness Of The Eigenvalue Problem As A Constraint To Quantum Matrices

Intuition suggests that the presence of degrees of freedom below measurement resolution implies that one must use density matrix description obtained by taking trace over the unobserved degrees of freedom. One could argue that in state function reduction with finite measurement resolution the outcome is not a pure state, or not even negentropically entangled state (possible in TGD framework) but a state described by a density matrix. The challenge is to describe the situation mathematically in an elegant manner.

1. There is present an infinite number of degrees of freedom below measurement resolution with which measured degrees of freedom entangle so that their presence affects the situation. One has a system with finite number degrees of freedom such as two-state system described by a quantum spinor. In this case observables as hermitian operators described by 2×2 matrices would be replaced by quantum matrices with elements, which in general do not commute.

   An attractive generalization of complex numbers appearing as elements of matrices is obtained by replacing them with products $H_{ij} = h_{ij}u_{ij}$ of hermitian operators h_{ij} with non-negative spectrum (modulus of complex number) and unitary operators u_{ij} (phase of complex number) suggests itself. The commutativity of h_{ij} and u_{ij} would look nice but is not necessary and is in conflict with the idea that modulus and phase of an amplitudes do not commute in quantum mechanics.

   Very probably this generalization is trivial for mathematician. One could indeed interpret the generalization in terms of a tensor product of finite-dimensional matrices with possibly infinite-dimensional space of operators of Hilbert space. For the physicist the situation might be different as the following proposal for what hermitian quantum matrices could be suggests.

2. The modulus of complex number is replaced with a hermitian operator having non-negative eigenvalues. The representation as $h = AA^\dagger + A^\dagger A$ is would guarantee this. The phase of complex number would be replaced by a unitary operator U possibly allowing the representation $U = exp(iT)$, T hermitian. The commutativity condition

   $$[h_{ij}, u_{ij}] = 0 \tag{3.6.1}$$

   for a given matrix element is also suggestive but as already noticed, Uncertainty Principle suggests that modulus and phase do not commute as operators. The commutator of modulus and phase would naturally be equal to that between momentum operator and plane wave:

   $$[h_{ij}, u_{ij}] = i\hbar \times u_{ij} \quad , \tag{3.6.2}$$

   Here $\hbar = h/2\pi$ can be chosen to be unity in standard quantum theory. In TGD it can be generalized to a hermitian operator H_{eff}/h with an integer valued spectrum of eigenvalues given by $h_{eff}/h = n$ so that ordinary and dark matter sectors would be unified to single structure mathematically.

3. The notions of eigenvalues and eigenvectors for a hermitian operator should generalize. Now hermitian operator H would be a matrix with formally the same structure as $N \times N$ hermitian matrix in commutative number field - say complex numbers - possibly satisfying additional conditions.

Hermitian matrix can be written as

$$H_{ij} = h_{ij}u_{ij} \quad \text{for i>j} \quad H_{ij} = u_{ij}h_{ij} \quad \text{for i<j} \ , \quad H_{ii} = h_i \ . \tag{3.6.3}$$

Hermiticity conditions $H_{ij} = H^{\dagger}_{ji}$ give

$$h_{ij} = h_{ji} \ , \quad u_{ij} = u^{\dagger}_{ji} \ . \tag{3.6.4}$$

Here it has been assumed that one has quantum SU(2). For quantum U(2) one would have $U_{11} = U^{\dagger}_{22} = h_a u_a$ with u_a commuting with other operators. The form of the conditions is same as for ordinary hermitian matrices and it is not necessary to assume commutativity $[h_{ij}, u_{ij}] = 0$. Generalization of Pauli spin matrices provides a simple illustration.

4. The well-definedness of eigenvalue problem gives a strong constraint on the notion of hermitian quantum matrix. Eigenvalues of hermitian operator are determined by the vanishing of determinant $det(H - \lambda I)$. Its expression involves sub-determinants and one must decide whether to demand that the definition of determinant is independent of which column or row one chooses to develop the determinant.

   For ordinary matrix the determinant is expressible as sum of symmetric functions:

$$det(H - \lambda I) = \sum \lambda^n S_n(H) \ . \tag{3.6.5}$$

   Elementary symmetric functions S_n - n-functions in following - have the property that they are sums of contributions from to n-element paths along the matrix with the property that path contains no vertical or horizontal steps. One has a discrete analog of path integral in which time increases in each step by unit. The analogy with fermionic path integral is also obvious. In the non-commutative case non-commutativity poses problems since different orderings of rows (or columns) along the same n-path give different results.

   (a) For the first option one gives up the condition that determinant can be developed with respect to any row or column and defines determinant by developing it with respect to say first row or first column. If one developing with respect to the column (row) the permutations of rows (columns) do not affect the value of determinant or sub-determinants but permutations of columns (rows) do so unless one poses additional conditions stating that the permutations do not affect given contribution to the determinant or sub-determinant. It turns out that this option must be applied in the case of ordinary quantum group. For quantum phase $q = \pm 1$ the determinant is invariant under permutations of both rows and columns.

   (b) Second manner to get rid of difficulty would be that n-path does not depend on the ordering of the rows (columns) differ only by the usual sign factor. For 2×2 case this would give

$$ad - bc = da - cb \ , \quad \text{(Option 2)} \tag{3.6.6}$$

   These conditions state the invariance of the n-path under permutation group S_n permuting rows or columns.

   (c) For the third option the elements along n-paths commute: paths could be said to be "classical". The invariance of N-path in this sense guarantees the invariance of all n-paths. In 2-D case this gives

$$[a, d] = 0 \ , \quad [b, c] = 0 \ . \quad \text{(Option 3)} \tag{3.6.7}$$

5. One should have a well-defined eigenvalue problem. If the n-functions commute, one can diagonalize the corresponding operators simultaneously and the eigenvalues problem reduces to possibly infinite number of ordinary eigenvalue problems corresponding to restrictions to given set of eigenvalues associated with $N-1$ symmetric functions. This gives an additional constraint on quantum matrices.

   In 2-dimensional case one would have the condition

$$[ad-bc, a+d] = 0 \ . \tag{3.6.8}$$

   Depending on how strong S_2 invariance one requires, one obtains 0, 1, 2 nontrivial conditions for 2×2 quantum matrices and 1 condition from the commutativity of n-functions besides hermiticity conditions.

   For $N \times N$-matrices one would have $N!-1$ non-trivial conditions from the strong form of permutation invariance guaranteeing the permutation symmetry of n-functions and $N(N-1)/2$ conditions from the commutativity of n-functions.

6. The eigenvectors of the density matrix are obtained in the usual manner for each eigenvalue contributing to quantum eigenvalue. Also the diagonalization can be carried out by a unitary transformation for each eigenvalue separately. Hence the standard approach seems to generalize almost trivially.

   What makes the proposal non-trivial and possibly physically interesting is that the hermitian operators are not assumed to be just tensor products of $N \times N$ hermitian matrices with hermitian operators in Hilbert space.

The notion of unitary quantum matrix should also make sense. The naïve guess is that the exponentiation of a linear combination of ordinary hermitian matrices with coefficients, which are hermitian matrices gives quantum unitary matrices. In the case of U(1) the replacement of exponentiation parameter t in $exp(itX)$ with a hermitian operator gives standard expression for the exponent and it is trivial to see that unitary conditions are satisfied also in this case. Also in the case of SU(2) it is easy to verify that the guess is correct. One must also check that one indeed obtains a group: it could also happen that only semi-group is obtained.

In any case, one could speak of quantum matrix groups with coordinates replaced by hermitian matrices. These quantum matrix group need not be identical with quantum groups in the standard sense of the word. Maybe this could provide one possible meaning for quantization in the case of groups and perhaps also in the case of coset spaces G/H.

3.6.2 The Relationship To Quantum Groups And Quantum Lie Algebras

It is interesting to find out whether quantum matrices give rise to quantum groups under suitable additional conditions. The child's guess for these conditions is that the permutation of rows and columns correspond to braiding for the hermitian moduli h_{ij} defined by unitary operators U_{ij}.

Quantum groups and quantum matrices

The conditions for hermiticity and unitary do not involve quantum parameter q, which suggests that the naïve generalization of the notion of unitary matrix gives unitary group obtained by replacing complex number field with operator algebra gives group with coordinates defined by hermitian operators rather than standard quantum group. This turns out to be the case and it seems that quantum matrices provide a concrete representation for quantum group. The notion of braiding as that for operators h_{ij} can be said to emerge from the notion of quantum matrix.

1. Exponential of quantum hermitian matrix is excellent candidate for quantum unitary matrix. One should check the exponentiation indeed gives rise to a quantum unitary matrix. For $q = \pm 1$ this seems obvious but one should check this separately for other roots of unity. Instead of considering the general case, we consider explicit ansatz for unitary U(2) quantum

matrix as $U = [a, b; -b^\dagger, a^\dagger]$. The conditions for unitary quantum group in the proposed sense would state the orthonormality and unit norm property of rows/columns.

The explicit form of the conditions reads as

$$\begin{array}{ll} ab - ba = 0 \ , & ab^\dagger = b^\dagger a \ , \\ aa^\dagger + bb^\dagger = 1 \ , & a^\dagger a + b^\dagger b = 1 \ . \end{array} \tag{3.6.9}$$

The orthogonality conditions are unique and reduce to the vanishing of commutators.

Normalization conditions involve a choice of ordering. One possible manner to avoid the problem is to assume that both orderings give same unit length for row or column (as done above). If only the other option is assumed then only third or fourth equations is needed. The invariance of determinant under permutation of rows would imply $[a, a^\dagger] = [b, b^\dagger] = 0$ and the ordering problem would disappear.

2. One can look what conditions the explicit representation $U_{ij} = h_{ij}u_{ij}$ or equivalently $[h_a u_a, h_b u_b; -u_b^\dagger h_b, u_a^\dagger h_a]$ gives. The intuitive expectation is that U(2) matrix decomposes to a product of commutating SU(2) matrix and U(1) matrices. This implies that u_a commutes with the other matrices involved. One obtains the conditions

$$h_a h_b = h_b(u_b h_a u_b^\dagger) \ , \quad h_b h_a = (u_b h_a u_b^\dagger) h_b \ . \tag{3.6.10}$$

These conditions state that the permutation of h_a and h_b analogous to braiding operation is a unitary operation.

For the purposes of comparison consider now the corresponding conditions for $SU(2)_q$ matrix.

1. The $SU(2)_q$ matrix $[a, b; b^\dagger, a^\dagger]$ with *real* value of q (see `http://tinyurl.com/yb8tycag`) satisfies the conditions

$$\begin{array}{lll} ba = qab \ , & b^\dagger a = qab^\dagger, & bb^\dagger = b^\dagger b \ , \\ a^\dagger a + q^2 b^\dagger b = 1 \ , & aa^\dagger + bb^\dagger = 1 \ . & \end{array} \tag{3.6.11}$$

This gives $[a^\dagger, a] = (1 - q^2) b^\dagger b$. The above conditions would correspond to $q = \pm 1$ but with complex numbers replaced with operator algebra. q-commutativity obviously replaces ordinary commutativity in the conditions and one can speak of q-orthonormality.

For complex values of q - in particular roots of unity - the condition $a^\dagger a + q^2 b^\dagger b = 1$ is in general not self-consistent since hermitian conjugation transforms q^2 to its complex conjugate. Hence this condition must be dropped for complex roots of unity.

2. Only for $q = \pm 1$ corresponding to Bose-Einstein and Fermi-Dirac statistics the conditions are consistent with the invariance of n-functions (determinant) under permutations of both rows and columns. Indeed, if 2×2 q-determinant is developed with respect to column, the permutation of rows does not affect its value. This is trivially true also in $N \times N$ dimensional case since the permutation of rows does not affect the n-paths at all.

If the symmetry under permutations is weakened, nothing prevents from posing quantum orthogonality conditions also now and the decomposition to a product of positive and hermitian matrices give a concrete meaning to the notion of quantum group.

Do various $n-$functions commute with each other for $SU(2)_q$? The only commutator of this kind is that for the trace and determinant and should vanish:

$$\left[b + b^\dagger, aa^\dagger + bb^\dagger\right] = 0 \ . \tag{3.6.12}$$

Since $a^\dagger a$ and $aa^\dagger$ are linear combinations of $b^\dagger b = b^\dagger b$, they vanish. Hence it seems that TGD based view about quantum groups is consistent with the standard view.

3. One can look these conditions in TGD framework by restricting the consideration to the case of SU(2) ($u_a = 1$) and using the ansatz $U = [h_a, h_b u_b; -u_b^\dagger h_b, h_a]$. Orthogonality conditions read as

$$h_a h_b = q h_b (u_b h_a u_b^\dagger) \ , \quad h_b h_a = q (u_b h_a u_b^\dagger) h_b \ .$$

If q is root of unity, these conditions state that the permutation of h_a and h_b analogous to a unitary braiding operation apart from a multiplication with quantum phase q. For $q = \pm 1$ the sign-factor is that in standard statistics. Braiding picture could help guess the commutators of h_{ij} in the case of $N \times N$ quantum matrices. The permutations of rows and columns would have interpretation as braidings and one could say that braided commutators of matrix elements vanish.

The conditions from the normalization give

$$h_a^2 + h_b^2 = 1 \ , \quad h_a^2 + q^2 (u_b^\dagger h_b^2 u_b) = 1 \ . \tag{3.6.13}$$

For complex q the latter condition does not make sense since $h_a^2 - 1$ and $u_b^\dagger h_b^2 u_b$ are hermitian matrices with real eigenvalues. Also for real values of $q \neq \pm 1$ one obtains contradicion since the spectra of unitarily related hermitian operators would differ by scaling factor q^2. Hence one must give up the condition involving q^2 unless one has $q = \pm 1$. Note that the term proportional to q^2 does not allow interpretation in terms of braiding.

4. Roots of unity are natural number theoretically as values of q but number theoretical universality allows the generic value of q would be a complex number existing simultaneously in all p-adic number properly extended. This would suggest the spectrum of q to come as

$$q(m,n) = e^{1/m} exp(\frac{i 2\pi}{n}) \ . \tag{3.6.14}$$

The motivation comes from the fact that e^p is ordinary p-adic number for all p-adic number fields so e and also any root of e defines a finite-dimensional extension of p-adic numbers [K86] [L8]. The roots of unity would be associated to the discretization of the ordinary angles in case of compact matrix groups. Roots of e would be associated with the discretization of hyperbolic angles needed in the case of non-compact matrix groups such as SL(2,C).

Also now unification of various values of q to single single operator Q, which is product of *commuting* hermitian and unitary operators and commuting with the hermitian operator H representing the spectrum of Planck constant would code the spectrum. Skeptic can of course wonder, whether the modulus and phase of Q can be assumed to commute. The relationship between integers associated with H and Q is interesting.

Quantum Lie algebras and quantum matrices

What about quantum Lie algebras? There are many notions of quantum Lie algebra and quantum group. General formulas for the commutation relations are well-known for Drinfeld-Jimbo type quantum groups (see `http://tinyurl.com/yb8tycag`). The simplest guess is that one just poses the defining conditions for quantum group, replaces complex numbers as coefficient module with operator algebra, and poses the above described conditions making possible to speak about eigen-values and eigen vectors. One might however hope that this representation allows to realize the non-commutativity of matrix elements of quantum Lie algebra in a concrete manner.

1. For SU(2) the commutation relations for the elements X_+, X_-, h read as

   $$[h, X_\pm] = \pm X_\pm \ \ , \quad [X_+, X_-] = h \ \ . \tag{3.6.15}$$

   Here one can use the 2×2 matrix representations for the ladder operators $X^\pm$ and diagonal angular momentum generator h.

2. For $\mathrm{SU}(2)_q$ one has

   $$[h, X_\pm] = \pm X_\pm \ \ , \quad [X_+, X_-] = \tfrac{q^h - q^{-h}}{q - q^{-1}} \ \ . \tag{3.6.16}$$

3. Using the ansatz for the generators but allowing hermitian operator coefficients in non-diagonal generators $X_\pm$, one obtains the condition

   For $\mathrm{SU}(2)_q$ one would have

   $$[X_+, X_-] = h_+^2 = h_-^2 = \frac{q^h - q^{-h}}{q - q^{-1}} \ \ . \tag{3.6.17}$$

   Clearly, the proposal might make possible to have concrete representations for the quantum Lie algebras making the decomposition to measurable and directly non-measurable degrees of freedom explicit.

The conclusion is that finite measurement resolution does not lead automatically to standard quantum groups although the proposed realization is consistent with them. Also the quantum phases $q = \pm 1$ $n = 1, 2$ are realized and correspond to strong permutation symmetry and Bose-Einstein and Fermi statistics.

3.6.3 About Possible Applications

The realization for the notion of finite measurement resolution is certainly the basic application but one can imagine also other applications where hermitian and unitary matrices appear.

Density matrix description of degrees of freedom below measurement resolution

Density matrix ρ obtained by tracing over non-observable degrees of freedom is a fundamental example about a hermitian matrix satisfying the additional condition $Tr(\rho) = 1$.

1. A state function reduction with a finite measurement resolution would lead to a non-pure state. This state would be describable using $N \times N$-dimensional quantum hermitian quantum density matrix satisfying the condition $Tr(\rho) = 1$ (or more generally $Tr_q(\rho) = 1$), and satisfying the additional conditions allowing to reduce its diagonalization to that for a collection of ordinary density matrices so that the eigenvalues of ordinary density matrix would be replaced by N quantum eigenvalues defined by infinite-dimensional diagonalized density matrices.

2. One would have N quantum eigenvalues - quantum probabilities - each decomposing to possibly infinite set of ordinary probabilities assignable to the degrees of freedom below measurement resolution and defining density matrix for non-pure states resulting in state function reduction.

Some questions

Some further questions pop up naturally.

1. One might hope that the quantum counterparts of hermitian operators are in some sense universal, at least in TGD framework (by quantum criticality). Could the condition that the commutator of hermitian generators is proportional to $i\hbar$ times hermitian generator pose additional constraints? In 2-D case this condition is satisfied for quantum SU(2) generators and very probably the same is true also in the general case. The possible problems result from the non-commutativity but $(XY)^\dagger = Y^\dagger X^\dagger$ identity takes care that there are no problems.

2. One can also raise physics related questions. What one can say about most general quantum Hamiltonians and their energy spectra, say quantum hydrogen atom? What about quantum angular momentum? If the proposed construction is only a concretization of abstract quantum group construction, then nothing new is expected at the level of representations of quantum groups.

3. Could the spectrum of h_{eff} define a quantum h as a hermitian positive definite operator? Could this allow a description for the presence of dark matter, which is not directly observable? Same question applies to the quantum parameter q.

4. M-matrices are basic building bricks of scattering amplitudes in ZEO. M-matrix is produce of hermitian "complex" square root H of density matrix satisfying $H^2 = \rho$ and unitary S-matrix S. It has been proposed that these matrices commute. The previous consideration relying on basic quantum thinking suggests that they relate like translation generator in radial direction and phase defined by angle and thus satisfy $[H, S] = i(H_{eff}/h) \times S$. This would give enormously powerful additional condition to S-matrix. One can also ask whether M-matrices in presence of degrees of freedom below measurement resolution is quantum version of M-matrix in the proposed sense.

5. Fractality is of of the key notions of TGD and characterizes also hyperfinite factors. I have proposed some realizations of fractality such as infinite primes and finite-dimensional Hilbert spaces taking the role of natural numbers and ordinary sum and product replaced with direct sum and tensor product. One could also imagine a fractal hierarchy of quantum matrices obtained by replacing the operators appearing as matrix elements of quantum matrix element by quantum matrices. This hierarchy could relate to the sequence of inclusions of HFFs.

3.7 Jones Inclusions And Cognitive Consciousness

WCW spinors have a natural interpretation in terms of a quantum version of Boolean algebra. Beliefs of various kinds are the basic element of cognition and obviously involve a representation of the external world or part of it as states of the system defining the believer. Jones inclusions mediating unitary mappings between the spaces of WCWs spinors of two systems are excellent candidates for these maps, and it is interesting to find what one kind of model for beliefs this picture leads to.

The resulting quantum model for beliefs provides a cognitive interpretation for quantum groups and predicts a universal spectrum for the probabilities that a given belief is true. This spectrum depends only on the integer n characterizing the quantum phase $q = exp(i2\pi/n)$ characterizing the Jones inclusion. For $n \neq \infty$ the logic is inherently fuzzy so that absolute knowledge is impossible. $q = 1$ gives ordinary quantum logic with qbits having precise truth values after state function reduction.

3.7.1 Does One Have A Hierarchy Of U- And M-Matrices?

U-matrix describes scattering of zero energy states and since zero energy states can be illustrated in terms of Feynman diagrams one can say that scattering of Feynman diagrams is in question. The initial and final states of the scattering are superpositions of Feynman diagrams characterizing the corresponding M-matrices which contain also the positive square root of density matrix as a factor.

The hypothesis that U-matrix is the tensor product of S-matrix part of M-matrix and its Hermitian conjugate would make U-matrix an object deducible by physical measurements. One cannot of course exclude that something totally new emerges. For instance, the description of quantum jumps creating zero energy state from vacuum might require that U-matrix does not reduce in this manner. One can assign to the U-matrix a square like structure with S-matrix and its Hermitian conjugate assigned with the opposite sides of a square.

One can imagine of constructing higher level physical states as composites of zero energy states by replacing the S-matrix with M-matrix in the square like structure. These states would provide a physical representation of U-matrix. One could define U-matrix for these states in a similar manner. This kind of hierarchy could be continued indefinitely and the hierarchy of higher level U and M-matrices would be labeled by a hierarchy of n-cubes, $n = 1, 2,...$ TGD inspired theory of consciousness suggests that this hierarchy can be interpreted as a hierarchy of abstractions represented in terms of physical states. This hierarchy brings strongly in mind also the hierarchies of n-algebras and n-groups and this forces to consider the possibility that something genuinely new emerges at each step of the hierarchy. A connection with the hierarchies of infinite primes [K72] and Jones inclusions are suggestive.

3.7.2 Feynman Diagrams As Higher Level Particles And Their Scattering As Dynamics Of Self Consciousness

The hierarchy of inclusions of hyper-finite factors of II_1 as counterpart for many-sheeted space-time lead inevitably to the idea that this hierarchy corresponds to a hierarchy of generalized Feynman diagrams for which Feynman diagrams at a given level become particles at the next level. Accepting this idea, one is led to ask what kind of quantum states these Feynman diagrams correspond, how one could describe interactions of these higher level particles, what is the interpretation for these higher level states, and whether they can be detected.

Jones inclusions as analogs of space-time surfaces

The idea about space-time as a 4-surface replicates itself at the level of operator algebra and state space in the sense that Jones inclusion can be seen as a representation of the operator algebra $\mathcal{N}$ as infinite-dimensional linear sub-space (surface) of the operator algebra $\mathcal{M}$. This encourages to think that generalized Feynman diagrams could correspond to image surfaces in II_1 factor having identification as kind of quantum space-time surfaces.

Suppose that the modular S-matrices are representable as the inner automorphisms $\Delta(\mathcal{M}_k^{it}$ assigned to the external lines of Feynman diagrams. This would mean that $\mathcal{N} \subset \mathcal{M}_k$ moves inside $calM_k$ along a geodesic line determined by the inner automorphism. At the vertex the factors $calM_k$ to fuse along $\mathcal{N}$ to form a Connes tensor product. Hence the copies of $\mathcal{N}$ move inside $\mathcal{M}_k$ like incoming 3-surfaces in H and fuse together at the vertex. Since all $\mathcal{M}_k$ are isomorphic to a universal factor $\mathcal{M}$, many-sheeted space-time would have a kind of quantum image inside II_1 factor consisting of pieces which are $d = \mathcal{M} : \mathcal{N}/2$-dimensional quantum spaces according to the identification of the quantum space as subspace of quantum group to be discussed later. In the case of partonic Clifford algebras the dimension would be indeed $d \leq 2$.

The hierarchy of Jones inclusions defines a hierarchy of S-matrices

It is possible to assign to a given Jones inclusion $\mathcal{N} \subset \mathcal{M}$ an entire hierarchy of Jones inclusions $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 ...$, $\mathcal{M}_0 = N$, $\mathcal{M}_1 = M$. A possible interpretation for these inclusions would be as a sequence of topological condensations.

This sequence also defines a hierarchy of Feynman diagrams inside Feynman diagrams. The factor $\mathcal{M}$ containing the Feynman diagram having as its lines the unitary orbits of $\mathcal{N}$ under $\Delta_{\mathcal{M}}$

becomes a parton in $\mathcal{M}_1$ and its unitary orbits under $\Delta_{\mathcal{M}_1}$ define lines of Feynman diagrams in M_1. The concrete representation for M-matrix or projection of it to some subspace as entanglement coefficients of partons at the ends of a braid assignable to the space-like 3-surface representing a vertex of a higher level Feynman diagram. In this manner quantum dynamics would be coded and simulated by quantum states.

The outcome can be said to be a hierarchy of Feynman diagrams within Feynman diagrams, a fractal structure for which many particle scattering events at a given level become particles at the next level. The particles at the next level represent dynamics at the lower level: they have the property of "being about" representing perhaps the most crucial element of conscious experience. Since net conserved quantum numbers can vanish for a system in TGD Universe, this kind of hierarchy indeed allows a realization as zero energy states. Crossing symmetry can be understood in terms of this picture and has been applied to construct a model for M-matrix at high energy limit [K18] .

One might perhaps say that quantum space-time corresponds to a double inclusion and that further inclusions bring in N-parameter families of space-time surfaces.

Higher level Feynman diagrams

The lines of Feynman diagram in $\mathcal{M}_{n+1}$ are geodesic lines representing orbits of $\mathcal{M}_n$ and this kind of lines meet at vertex and scatter. The evolution along lines is determined by $\Delta_{\mathcal{M}_{n+1}}$. These lines contain within themselves $\mathcal{M}_n$ Feynman diagrams with similar structure and the hierarchy continues down to the lowest level at which ordinary elementary particles are encountered.

For instance, the generalized Feynman diagrams at the second level are ribbon diagrams obtained by thickening the ordinary diagrams in the new time direction. The interpretation as ribbon diagrams crucial for topological quantum computation and suggested to be realizable in terms of zero energy states in [K4] is natural. At each level a new time parameter is introduced so that the dimension of the diagram can be arbitrarily high. The dynamics is not that of ordinary surfaces but the dynamics induced by the $\Delta_{\mathcal{M}_n}$.

Quantum states defined by higher level Feynman diagrams

The intuitive picture is that higher level quantum states corresponds to the self reflective aspect of existence and must provide representations for the quantum dynamics of lower levels in their own structure. This dynamics is characterized by M-matrix whose elements have representation in terms of Feynman diagrams.

1. These states correspond to zero energy states in which initial states have "positive energies" and final states have "negative energies". The net conserved quantum numbers of initial and final state partons compensate each other. Gravitational energies, and more generally gravitational quantum numbers defined as absolute values of the net quantum numbers of initial and final states do not vanish. One can say that thoughts have gravitational mass but no inertial mass.

2. States in sub-spaces of positive and negative energy states are entangled with entanglement coefficients given by M-matrix at the level below.

To make this more concrete, consider first the simplest non-trivial case. In this case the particles can be characterized as ordinary Feynman diagrams, or more precisely as scattering events so that the state is characterized by $\hat{S} = P_{in}SP_{out}$, where S is S-matrix and P_{in} *resp.* P_{out} is the projection to a subspace of initial *resp.* final states. An entangled state with the projection of S-matrix giving the entanglement coefficients is in question.

The larger the domains of projectors P_{in} and P_{out}, the higher the representative capacity of the state. The norm of the non-normalized state $\hat{S}$ is $Tr(\hat{S}\hat{S}^\dagger) \leq 1$ for II_1 factors, and at the limit $\hat{S} = S$ the norm equals to 1. Hence, by II_1 property, the state always entangles infinite number of states, and can in principle code the entire S-matrix to entanglement coefficients.

The states in which positive and negative energy states are entangled by a projection of S-matrix might define only a particular instance of states for which conserved quantum numbers vanish. The model for the interaction of Feynman diagrams discussed below applies also to these more general states.

The interaction of $\mathcal{M}_n$ Feynman diagrams at the second level of hierarchy

What constraints can one pose to the higher level reactions? How Feynman diagrams interact? Consider first the scattering at the second level of hierarchy ($\mathcal{M}_1$), the first level $\mathcal{M}_0$ being assigned to the interactions of the ordinary matter.

1. Conservation laws pose constraints on the scattering at level $\mathcal{M}_1$. The Feynman diagrams can transform to new Feynman diagrams only in such a way that the net quantum numbers are conserved separately for the initial positive energy states and final negative energy states of the diagram. The simplest assumption is that positive energy matter and negative energy matter know nothing about each other and effectively live in separate worlds. The scattering matrix form Feynman diagram like states would thus be simply the tensor product $S \otimes S^\dagger$, where S is the S-matrix characterizing the lowest level interactions and identifiable as unitary factor of M-matrix for zero energy states. Reductionism would be realized in the sense that, apart from the new elements brought in by $\Delta_{\mathcal{M}_n}$ defining single particle free dynamics, the lowest level would determine in principle everything occurring at the higher level providing representations about representations about... for what occurs at the basic level. The lowest level would represent the physical world and higher levels the theory about it.

2. The description of hadronic reactions in terms of partons serves as a guide line when one tries to understand higher level Feynman diagrams. The fusion of hadronic space-time sheets corresponds to the vertices $\mathcal{M}_1$. In the vertex the analog of parton plasma is formed by a process known as parton fragmentation. This means that the partonic Feynman diagrams belonging to disjoint copies of $\mathcal{M}_0$ find themselves inside the same copy of $\mathcal{M}_0$. The standard description would apply to the scattering of the initial *resp.* final state partons.

3. After the scattering of partons hadronization takes place. The analog of hadronization in the recent case is the organization of the initial and final state partons to groups I_i and F_i such that the net conserved quantum numbers are same for I_i and F_i. These conditions can be satisfied if the interactions in the plasma phase occur only between particles belonging to the clusters labeled by the index i. Otherwise only single particle states in $\mathcal{M}_1$ would be produced in the reactions in the generic case. The cluster decomposition of S-matrix to a direct sum of terms corresponding to partitions of the initial state particles to clusters which do not interact with each other obviously corresponds to the "hadronization". Therefore no new dynamics need to be introduced.

4. One cannot avoid the question whether the parton picture about hadrons indeed corresponds to a higher level physics of this kind. This would require that hadronic space-time sheets carry the net quantum numbers of hadrons. The net quantum numbers associated with the initial state partons would be naturally identical with the net quantum numbers of hadron. Partons and they negative energy conjugates would provide in this picture a representation of hadron about hadron. This kind of interpretation of partons would make understandable why they cannot be observed directly. A possible objection is that the net gravitational mass of hadron would be three times the gravitational mass deduced from the inertial mass of hadron if partons feed their gravitational fluxes to the space-time sheet carrying Earth's gravitational field.

5. This picture could also relate to the suggested duality between string and parton pictures [K74] . In parton picture hadron is formed from partons represented by space-like 2-surfaces X^2_i connected by join along boundaries bonds. In string picture partonic 2-surfaces are replaced with string orbits. If one puts positive and negative energy particles at the ends of string diagram one indeed obtains a higher level representation of hadron. If these pictures are dual then also in parton picture positive and negative energies should compensate each other. Interestingly, light-like 3-D causal determinants identified as orbits of partons could be interpreted as orbits of light like string word sheets with "time" coordinate varying in space-like direction.

Scattering of Feynman diagrams at the higher levels of hierarchy

This picture generalizes to the description of higher level Feynman diagrams.

1. Assume that higher level vertices have recursive structure allowing to reduce the Feynman diagrams to ordinary Feynman diagrams by a procedure consisting of finite steps.

2. The lines of diagrams are classified as incoming or outgoing lines according to whether the time orientation of the line is positive or negative. The time orientation is associated with the time parameter t_n characterizing the automorphism $\Delta^{it_n}_{\mathcal{M}_\backslash}$. The incoming and outgoing net quantum numbers compensate each other. These quantum numbers are basically the quantum numbers of the state at the lowest level of the hierarchy.

3. In the vertices the $\mathcal{M}_{n+1}$ particles fuse and $\mathcal{M}_n$ particles form the analog of quark gluon plasma. The initial and final state particles of $\mathcal{M}_n$ Feynman diagram scatter independently and the S-matrix S_{n+1} describing the process is tensor product $S_n \otimes S_n^\dagger$. By the clustering property of S-matrix, this scattering occurs only for groups formed by partons formed by the incoming and outgoing particles $\mathcal{M}_n$ particles and each outgoing $\mathcal{M}_{n+1}$ line contains and irreducible $\mathcal{M}_n$ diagram. By continuing the recursion one finally ends down with ordinary Feynman diagrams.

3.7.3 Logic, Beliefs, And Spinor Fields In The World Of Classical Worlds

Beliefs can be characterized as Boolean value maps $\beta_i(p)$ telling whether i believes in proposition p or not. Additional structure is brought in by introducing the map $\lambda_i(p)$ telling whether p is true or not in the environment of i. The task is to find quantum counterpart for this model.

The spectrum of probabilities for outcomes in state function reduction with finite measurement resolution is universal

Consider quantum two-spinor as a model of a system with finite measurement resolution implying that state function reduction do not anymore lead to a spin state with a precise value but that one can only predict the probability distribution for the outcome of measurement. These probabilities can be also interpreted as truth values of a belief in finite cognitive resolution.

It is actually possible to calculate the spectrum of the probabilities of truth values with rather mild additional assumptions.

1. Since the Hermitian operators $X_1 = (z^1\overline{z^1} + \overline{z^1}z^1)/2$ and $X_2 = (z^2\overline{z^2} + \overline{z^2}z^2)/2$ commute, physical states can be chosen to be eigen states of these operators and it is possible to assign to the truth values probabilities given by $p_1 = X_1/R^2$ and $p_2 = X_2/R^2$, $R^2 = X_1 + X_2$.

2. By introducing the analog of the harmonic oscillator vacuum as a state $|0\rangle$ satisfying $z^1|0\rangle = z^2|0\rangle = 0$, one obtains eigen states of X_1 and X_2 as states $|n_1, n_2\rangle = \overline{z^1}^{n_1}\overline{z^2}^{n_2}|0\rangle$, $n_1 \geq 0, n_2 \geq 0$. The eigenvalues of X_1 and X_2 are given by a modified harmonic oscillator spectrum as

$$X_1 = (1/2 + n_1q^{n_2})r \ \ , \ \ X_2 = (1/2 + n_2q^{n_1})r \ \ .$$

   The reality of eigenvalues (hermiticity) is guaranteed if one has $n_1 = N_1n$ and $n_1 = N_2n$ and implies that the spectrum of eigen states gets increasingly thinner for $n \to \infty$. This must somehow reflect the fractal dimension. The fact that large values of oscillator quantum numbers n_1 and n_2 correspond to the classical limit suggests that modulo condition guarantees approximate classicality of the logic for $n \to \infty$.

3. The probabilities p_1 and p_2 for the truth values given by $(p_1, p_2) = (1/2+N_1n, 1/2+N_2n)/[1+(N_1 + N_2)n]$ are rational and allow an interpretation as both real and p-adic numbers. This also conforms with the frequency interpretation for probabilities. All states are are inherently fuzzy and only at the limits $N_1 \gg N_2$ and $N_2 \gg N_1$ non-fuzzy states result. As noticed, $n = \infty$ must be treated separately and corresponds to an ordinary non-fuzzy qbit logic. At $n \to \infty$ limit one has $(p_1, p_2) = (N_1, N_2)/(N_1, N_2)$: at this limit $N_1 = 0$ or $N_2 = 0$ states are non-fuzzy.

4. A possible interpretation for the fuzziness is in terms of finite measurement resolution. The quantized probabilities could be assigned with diagonalized density matrix regarded as matrix with elements which are commuting hermitian operators. The generalized eigenvalues would be eigenvalues spectra. States would not be pure expect at the limits$N_1 \gg N_2$ and $N_2 \gg N_1$. The non-purity of the state could be understood in terms of entanglement with the degrees of freedom below measurement resolution describable in terms of inclusion of von Neumann algebras. One could perhaps say that in finite measurement resolution the outcome of state function reduction is always non-pure state characterized by a universal density matrix obtained by tracing over non-visible degrees of freedom.

WCW spinors as logic statements

In TGD framework the infinite-dimensional WCW (CH) spinor fields defined in CH, the "world of classical worlds", describe quantum states of the Universe [K88] . WCW spinor field can be regarded as a state in infinite-dimensional Fock space and are labeled by a collection of various two valued indices like spin and weak isospin. The interpretation is as a collection of truth values of logic statements one for each fermionic oscillator operator in the state. For instance, spin up and down would correspond to two possible truth values of a proposition characterized by other quantum numbers of the mode.

The hierarchy of space-time sheet could define a physical correlate for the hierarchy of higher order logics (statements about statements about...). The space-time sheet containing N fermions topologically condensed at a larger space-time sheet behaves as a fermion or boson depending on whether N is odd or even. This hierarchy has also a number theoretic counterpart: the construction of infinite primes [K72] corresponds to a repeated second quantization of a super-symmetric quantum field theory.

Quantal description of beliefs

The question is whether TGD inspired theory of consciousness allows a fundamental description of beliefs.

1. Beliefs define a model about some subsystem of universe constructed by the believer. This model can be understood as some kind of representation of real word in the state space representing the beliefs.

2. One can wonder what is the difference between real and p-adic variants of WCW spinor fields and whether they could represent reality and beliefs about reality. WCW spinors (as opposed to spinor fields) are constructible in terms of fermionic oscillator operators and seem to be universal in the sense that one cannot speak about p-adic and real WCW spinors as different objects. Real/p-adic spinor fields however have real/p-adic space-time sheets as arguments. This would suggest that there is no fundamental difference between the logic statements represented by p-adic and real WCW spinors.

3. This vision is realized if the intersection of reality and various p-adicities corresponds to an algebraically universal set of consisting of partonic 2-surfaces and string world sheets for which defining parameters are WCW coordinates in an algebraic extension of rationals defining that for p-adic number fields. Induced spinor fields would be localized at string world sheets and their intersections with partonic 2-surfaces and would be number theoretically universal. If second quantized induced spinor fields are correlates of Boolean cognition, which is behind the entire mathematics, their number theoretical universality is indeed a highly natural condition. Also fermionic anticommutation relations are number theoretically universal. By conformal invariance the conformal moduli of string world sheets and partonic 2-surface would be the natural WCW coordinates for the 2-surfaces in question and I proposed their p-adicization already in p-adic mass calculations for two decades ago.

   This picture would provide an elegant realization for the p-adicization. There would be ne need to map real space-time surfaces directly to p-adic ones and vice versa and one would avoid problems related to general coordinate invariance (GCI) completely. Strong form of holography would assign to partonic surfaces the real and p-adic variants. Already p-adic mass calculations support the presence of cognition in all length scales.

These observations suggest a more concrete view about how beliefs emerge physically.

The idea that p-adic WCW spinor fields could serve as representations of beliefs and real WCW spinor fields as representations of reality looks very nice and conforms with the adelic vision that space-time is adele - a book-like structure contains space-time sheets in various number fields as pages glued together along back for which the parameters characterizing space-time surface are numbers in an algebraic extension of rationals. Real space-time surfaces would be correlates for sensory experience and p-adic space-time sheets for cognition.

3.7.4 Jones Inclusions For Hyperfinite Factors Of Type II_1 As A Model For Symbolic And Cognitive Representations

Consider next a more detailed model for how cognitive representations and beliefs are realized at quantum level. This model generalizes trivially to symbolic representations.

The Clifford algebra of gamma matrices associated with WCW spinor fields corresponds to a von Neumann algebra known as hyper-finite factor of type II_1. The mathematics of these algebras is extremely beautiful and reproduces basic mathematical structures of modern physics (conformal field theories, quantum groups, knot and braid groups,....) from the mere assumption that the world of classical worlds possesses infinite-dimensional Kähler geometry and allows spinor structure.

The almost defining feature is that the infinite-dimensional unit matrix of the Clifford algebra in question has by definition unit trace. Type II_1 factors allow also what are known as Jones inclusions of Clifford algebras $\mathcal{N} \subset \mathcal{M}$. What is special to II_1 factors is that the induced unitary mappings between spinor spaces are genuine inclusions rather than 1-1 maps.

The S-matrix associated with the real-to-p-adic quantum transition inducing belief from reality would naturally define Jones inclusion of CH Clifford algebra $\mathcal{N}$ associated with the real space-time sheet to the Clifford algebra $\mathcal{M}$ associated with the p-adic space-time sheet. The moduli squared of S-matrix elements would define probabilities for pairs or real and belief states.

In Jones inclusion $\mathcal{N} \subset \mathcal{M}$ the factor $\mathcal{N}$ is included in factor $\mathcal{M}$ such that $\mathcal{M}$ can be expressed as $\mathcal{N}$-module over quantum space $\mathcal{M}/\mathcal{N}$ which has fractal dimension given by Jones index $\mathcal{M} : \mathcal{N} = 4cos^2(\pi/n) \leq 4$, $n = 3, 4, ...$ varying in the range $[1, 4]$. The interpretation is as the fractal dimension corresponding to a dimension of Clifford algebra acting in $d = \sqrt{\mathcal{M} : \mathcal{N}}$-dimensional spinor space: d varies in the range $[1, 2]$. The interpretation in terms of a quantal variant of logic is natural.

Probabilistic beliefs

For $\mathcal{M} : \mathcal{N} = 4$ ($n = \infty$) the dimension of spinor space is $d = 2$ and one can speak about ordinary 2-component spinors with $\mathcal{N}$-valued coefficients representing generalizations of qubits. Hence the inclusion of a given $\mathcal{N}$-spinor as M-spinor can be regarded as a belief on the proposition and for the decomposition to a spinor in N-module $\mathcal{M}/\mathcal{N}$ involves for each index a choice $\mathcal{M}/\mathcal{N}$ spinor component selecting super-position of up and down spins. Hence one has a superposition of truth values in general and one can speak only about probabilistic beliefs. It is not clear whether one can choose the basis in such a way that $\mathcal{M}/\mathcal{N}$ spinor corresponds always to truth value 1. Since WCW spinor field is in question and even if this choice might be possible for a single 3-surface, it need not be possible for deformations of it so that at quantum level one can only speak about probabilistic beliefs.

Fractal probabilistic beliefs

For $d < 2$ the spinor space associated with $\mathcal{M}/\mathcal{N}$ can be regarded as quantum plane having complex quantum dimension d with two non-commuting complex coordinates z^1 and z^2 satisfying $z^1z^2 = qz^2z^1$ and $\overline{z^1z^2} = \overline{q}\overline{z^2z^1}$. These relations are consistent with hermiticity of the real and imaginary parts of z^1 and z^2 which define ordinary quantum planes. Hermiticity also implies that one can identify the complex conjugates of z^i as Hermitian conjugates.

The further commutation relations $[z^1,\overline{z^2}] = [z^2,\overline{z^1}] = 0$ and $[z^1,\overline{z^1}] = [z^2,\overline{z^2}] = r$ give a closed algebra satisfying Jacobi identities. One could argue that $r \geq 0$ should be a function $r(n)$ of the quantum phase $q = exp(i2\pi/n)$ vanishing at the limit $n \to \infty$ to guarantee that the algebra becomes commutative at this limit and truth values can be chosen to be non-fuzzy. $r = sin(\pi/n)$ would be the simplest choice. As will be found, the choice of $r(n)$ does not however affect at all the spectrum for the probabilities of the truth values. $n = \infty$ case corresponding to non-fuzzy quantum logic is also possible and must be treated separately: it corresponds to Kac Moody algebra instead of quantum groups.

The non-commutativity of complex spinor components means that z^1 and z^2 are not independent coordinates: this explains the reduction of the number of the effective number of truth values to $d < 2$. The maximal reduction occurs to $d = 1$ for $n = 3$ so that there is effectively only single truth value and one could perhaps speak about taboo or dogma or complete disappearance of the notions of truth and false (this brings in mind reports about meditative states: in fact $n = 3$ corresponds to a phase in which Planck constant becomes infinite so that the system is maximally quantal).

As non-commuting operators the components of d-spinor are not simultaneously measurable for $d < 2$. It is however possible to measure simultaneously the operators describing the probabilities $z^1\overline{z^1}$ and $z^2\overline{z^2}$ for truth values since these operators commute. An inherently fuzzy Boolean logic would be in question with the additional feature that the spinorial counterparts of statement and its negation cannot be regarded as independent observables although the corresponding probabilities satisfy the defining conditions for commuting observables.

If one can speak of a measurement of probabilities for $d < 2$, it differs from the ordinary quantum measurement in the sense that it cannot involve a state function reduction to a pure qubit meaning irreducible quantal fuzziness. One could speak of fuzzy qbits or fqbits (or quantum qbits) instead of qbits. This picture would provide the long sought interpretation for quantum groups.

The previous picture applies to all representations $M_1 \subset M_2$, where M_1 and M_2 denote either real or p-adic Clifford algebras for some prime p. For instance, real-real Jones inclusion could be interpreted as symbolic representations assignable to a unitary mapping of the states of a subsystem M_1 of the external world to the state space M_2 of another real subsystem. $p_1 \to p_2$ unitary inclusions would in turn map cognitive representations to cognitive representations. There is a strong temptation to assume that these Jones inclusions define unitary maps realizing universe as a universal quantum computer mimicking itself at all levels utilizing cognitive and symbolic representations. Subsystem-system inclusion would naturally define one example of Jones inclusion.

The spectrum of probabilities of truth values is universal

It is actually possible to calculate the spectrum of the probabilities of truth values with rather mild additional assumptions.

1. Since the Hermitian operators $X_1 = (z^1\overline{z^1} + \overline{z^1}z^1)/2$ and $X_2 = (z^2\overline{z^2} + \overline{z^2}z^2)/2$ commute, physical states can be chosen to be eigen states of these operators and it is possible to assign to the truth values probabilities given by $p_1 = X_1/R^2$ and $p_2 = X_2/R^2$, $R^2 = X_1 + X_2$.

2. By introducing the analog of the harmonic oscillator vacuum as a state $|0\rangle$ satisfying $z^1|0\rangle = z^2|0\rangle = 0$, one obtains eigen states of X_1 and X_2 as states $|n_1,n_2\rangle = \overline{z^1}^{n_1}\overline{z^2}^{n_2}|0\rangle$, $n_1 \geq 0, n_2 \geq 0$. The eigenvalues of X_1 and X_2 are given by a modified harmonic oscillator spectrum as

$$X_1 = (1/2 + n_1 q^{n_2})r \ \ , \ \ X_2 = (1/2 + n_2 q^{n_1})r \ \ .$$

   The reality of eigenvalues (hermiticity) is guaranteed if one has $n_1 = N_1 n$ and $n_1 = N_2 n$ and implies that the spectrum of eigen states gets increasingly thinner for $n \to \infty$. This must somehow reflect the fractal dimension. The fact that large values of oscillator quantum numbers n_1 and n_2 correspond to the classical limit suggests that modulo condition guarantees approximate classicality of the logic for $n \to \infty$.

3. The probabilities p_1 and p_2 for the truth values given by $(p_1,p_2) = (1/2+N_1 n, 1/2+N_2 n)/[1+(N_1+N_2)n]$ are rational and allow an interpretation as both real and p-adic numbers. This

also conforms with the frequency interpretation for probabilities. All states are are inherently fuzzy and only at the limits $N_1 \gg N_2$ and $N_2 \gg N_1$ non-fuzzy states result. As noticed, $n = \infty$ must be treated separately and corresponds to an ordinary non-fuzzy qbit logic. At $n \to \infty$ limit one has $(p_1, p_2) = (N_1, N_2)/(N_1, N_2)$: at this limit $N_1 = 0$ or $N_2 = 0$ states are non-fuzzy.

4. A possible interpretation for the fuzziness is in terms of finite measurement resolution. The quantized probabilities could be assigned with diagonalized density matrix regarded as matrix with elements which are commuting hermitian operators. The generalized eigenvalues would be eigenvalues spectra. States would not be pure expect at the limits$N_1 \gg N_2$ and $N_2 \gg N_1$. The non-purity of the state could be understood in terms of entanglement with the degrees of freedom below measurement resolution describable in terms of inclusion of von Neumann algebras. One could perhaps say that in finite measurement resolution the outcome of state function reduction is always non-pure state characterized by a universal density matrix obtained by tracing over non-visible degrees of freedom.

How to define variants of belief quantum mechanically?

Probabilities of true and false for Jones inclusion characterize the plausibility of the belief and one can ask whether this description is enough to characterize states such as knowledge, misbelief, doubt, delusion, and ignorance. The truth value of $\beta_i(p)$ is determined by the measurement of probability assignable to Jones inclusion on the p-adic side. The truth value of $\lambda_i(p)$ is determined by a similar measurement on the real side. β and λ appear completely symmetrically and one can consider all kinds of triplets $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ assuming that there exist unitary S-matrix like maps mediating a sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3$ of Jones inclusions. Interestingly, the hierarchies of Jones inclusions are a key concept in the theory of hyper-finite factors of type II_1 and pair of inclusions plays a fundamental role.

Let us restrict the consideration to the situation when $\mathcal{M}_1$ corresponds to a real subsystem of the external world, $\mathcal{M}_2$ its real representation by a real subsystem, and $\mathcal{M}_3$ to p-adic cognitive representation of M_3. Assume that both real and p-adic sides involve a preferred state basis for qubits representing truth and false.

Assume first that both $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\mathcal{M}_2 \subset \mathcal{M}_3$ correspond to $d = 2$ case for which ordinary quantum measurement or truth value is possible giving outcome true or false. Assume further that the truth values have been measured in both M_2 and M_3.

1. Knowledge corresponds to the proposition $\beta_i(p) \wedge \lambda_i(p)$.

2. Misbelief to the proposition $\beta_i(p) \wedge \neq \lambda_i(p)$.
Knowledge and misbelief would involve both the measurement of real and p-adic probabilities
.

3. Assume next that one has $d < 2$ form $\mathcal{M}_2 \subset \mathcal{M}_3$. Doubt can be regarded neither belief or disbelief: $\beta_i(p) \wedge \neq \beta_i(\neq p)$: belief is inherently fuzzy although proposition can be non-fuzzy.

   Assume next that truth values in $\mathcal{M}_1 \subset \mathcal{M}_2$ inclusion corresponds to $d < 2$ so that the basic propositions are inherently fuzzy.

4. Delusion is a belief which cannot be justified: $\beta_i(p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. This case is possible if $d = 2$ holds true for $\mathcal{M}_2 \subset \mathcal{M}_3$. Note that also misbelief that cannot be shown wrong is possible.
In this case truth values cannot be quantum measured for $\mathcal{M}_1 \subset \mathcal{M}_2$ but can be measured for $\mathcal{M}_2 \subset \mathcal{M}_3$. Hence the states are products of pure $\mathcal{M}_3$ states with fuzzy $\mathcal{M}_2$ states.

5. Ignorance corresponds to the proposition $\beta_i(p) \wedge \neq \beta_i(\neq p) \wedge \lambda_i(p) \wedge \neq \lambda(\neq p))$. Both real representational states and belief states are inherently fuzzy.

Quite generally, only for $d_1 = d_2 = 2$ ideal knowledge and ideal misbelief are possible. Fuzzy beliefs and logics approach to ordinary one at the limit $n \to \infty$, which according to the proposal of [K68] corresponds to the ordinary value of Planck constant. For other cases these notions are only approximate and quantal approach allows to characterize the goodness of the approximation.

A new kind of inherent quantum uncertainty of knowledge is in question and one could speak about a Uncertainty Principle for cognition and symbolic representations. Also the unification of symbolic and various kinds of cognitive representations deserves to be mentioned.

3.7.5 Intentional Comparison Of Beliefs By Topological Quantum Computation?

Intentional comparison would mean that for a given initial state also the final state of the quantum jump is fixed. This requires the ability to engineer S-matrix so that it leads from a given state to single state only. Any S-matrix representing permutation of the initial states fulfills these conditions. This condition is perhaps unnecessarily strong.

Quantum computation is basically the engineering of S-matrix so that it represents a superposition of parallel computations. In TGD framework topological quantum computation based on the braiding of magnetic flux tubes would be represented as an evolution characterized by braid [K4] . The dynamical evolution would be associated with light-like boundaries of braids. This evolution has dual interpretations either as a limit of time evolution of quantum state (program running) or a quantum state satisfying conformal invariance constraints (program code).

The dual interpretation would mean that conformally invariant states are equivalent with engineered time evolutions and topological computation realized as braiding connecting the quantum states to be compared (beliefs represented as many-fermion states at the boundaries of magnetic flux tubes) could give rise to conscious computational comparison of beliefs. The complexity of braiding would give a measure for how much the states to be compared differ.

Note that quantum computation is defined by a unitary map which could also be interpreted as symbolic representation of states of system M_1 as states of system M_2 mediated by the braid of join along boundaries bonds connecting the two space-time sheets in question and having light-like boundaries. These considerations suggest that the idea about S-matrix of the Universe should be generalized so that the dynamics of the Universe is dynamics of mimicry described by an infinite collection of fermionic S-matrices representable in terms of Jones inclusions.

3.7.6 The Stability Of Fuzzy Qbits And Quantum Computation

The stability of fqbits against state function reduction might have deep implications for quantum computation since quantum spinors would be stable against state function reduction induced by the perturbations inducing de-coherence in the normal situation. If this is really true, and if the only dangerous perturbations are those inducing the phase transition to qbits, the implications for quantum computation could be dramatic. Of course, the rigidity of qbits could be just another way to say that topological quantum computations are stable against thermal perturbations not destroying anyons [K4] .

The stability of fqbits could also be another manner to state the stability of rational, or more generally algebraic, bound state entanglement against state function reduction, which is one of the basic hypothesis of TGD inspired theory of consciousness [K43] . For sequences of Jones inclusions or equivalently, for multiple Connes tensor products, one would obtain tensor products of quantum spinors making possible arbitrary complex configurations of fqbits. Anyonic braids in topological quantum computation would have interpretation as representations for this kind of tensor products.

3.7.7 Fuzzy Quantum Logic And Possible Anomalies In The Experimental Data For The EPR-Bohm Experiment

The experimental data for EPR-Bohm experiment [J4] excluding hidden variable interpretations of quantum theory. What is less known that the experimental data indicates about possibility of an anomaly challenging quantum mechanics [J1] . The obvious question is whether this anomaly might provide a test for the notion of fuzzy quantum logic inspired by the TGD based quantum measurement theory with finite measurement resolution.

The anomaly

The experimental situation involves emission of two photons from spin zero system so that photons have opposite spins. What is measured are polarizations of the two photons with respect to polarization axes which differ from standard choice of this axis by rotations around the axis of photon momentum characterized by angles α and β. The probabilities for observing polarizations (i,j), where i,j is taken Z_2 valued variable for a convenience of notation are $P_{ij}(\alpha,\beta)$, are predicted to be $P_{00} = P_{11} = cos^2(\alpha-\beta)/2$ and $P_{01} = P_{10} = sin^2(\alpha-\beta)/2$.

Consider now the discrepancies.

1. One has four identities $P_{i,i} + P_{i,i+1} = P_{ii} + P_{i+1,i} = 1/2$ having interpretation in terms of probability conservation. Experimental data of [J4] are not consistent with this prediction [J2] and this is identified as the anomaly.

2. The QM prediction $E(\alpha,\beta) = \sum_i (P_{i,i} - P_{i,i+1}) = cos(2(\alpha-\beta)$ is not satisfied neither: the maxima for the magnitude of E are scaled down by a factor $\simeq .9$. This deviation is not discussed in [J2] .

Both these findings raise the possibility that QM might not be consistent with the data. It turns out that fuzzy quantum logic predicted by TGD and implying that the predictions for the probabilities and correlation must be replaced by ensemble averages, can explain anomaly b) but not anomaly a). A "mundane" explanation for anomaly a) is proposed.

Predictions of fuzzy quantum logic for the probabilities and correlations

1. The description of fuzzy quantum logic in terms statistical ensemble

The fuzzy quantum logic implies that the predictions $P_{i,j}$ for the probabilities should be replaced with ensemble averages over the ensembles defined by fuzzy quantum logic. In practice this means that following replacements should be carried out:

$$\begin{aligned} P_{i,j} &\rightarrow P^2 P_{i,j} + (1-P)^2 P_{i+1,j+1} \\ &+ P(1-P)\left[P_{i,j+1} + P_{i+1,j}\right] \ . \end{aligned} \tag{3.7.1}$$

Here P is one of the state dependent universal probabilities/fuzzy truth values for some value of n characterizing the measurement situation. The concrete predictions would be following

$$\begin{aligned} P_{0,0} = P_{1,1} &\rightarrow A\frac{cos^2(\alpha-\beta)}{2} + B\frac{sin^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{cos^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ P_{0,1} = P_{1,0} &\rightarrow A\frac{sin^2(\alpha-\beta)}{2} + B\frac{cos^2(\alpha-\beta)}{2} \\ &= (A-B)\frac{sin^2(\alpha-\beta)}{2} + \frac{B}{2} \ , \\ A &= P^2 + (1-P)^2 \ , \ B = 2P(1-P) \ . \end{aligned} \tag{3.7.2}$$

The prediction is that the graphs of probabilities as a function as function of the angle $\alpha-\beta$ are scaled by a factor $1-4P(1-P)$ and shifted upwards by $P(1-P)$. The value of P, and one might hope even the value of n labeling Jones inclusion and the integer m labeling the quantum state might be deducible from the experimental data as the upward shift. The basic prediction is that the maxima of curves measuring probabilities $P_(i,j)$ have minimum at $B/2 = P(1-P)$ and maximum is scaled down to $(A-B)/2 = 1/2 - 2P(1-P)$.

If the P is same for all pairs i,j, the correlation $E = \sum_i (P_{ii} - P_{i,i+1})$ transforms as

$$E(\alpha,\beta) \rightarrow \left[1-4P(1-P)\right] E(\alpha,\beta) \ . \tag{3.7.3}$$

Only the normalization of $E(\alpha,\beta)$ as a function of $\alpha-\beta$ reducing the magnitude of E occurs. In particular the maximum/minimum of E are scaled down from $E=\pm 1$ to $E=\pm(1-4P(1-P))$.

From the figure 1b) of [J2] the scaling down indeed occurs for magnitudes of E with same amount for minimum and maximum. Writing $P=1-\epsilon$ one has $A-B\simeq 1-4\epsilon$ and $B\simeq 2\epsilon$ so that the maximum is in the first approximation predicted to be at $1-4\epsilon$. The graph would give $1-P\simeq\epsilon\simeq .025$. Thus the model explains the reduction of the magnitude for the maximum and minimum of E which was not however considered to be an anomaly in [J1, J2] .

A further prediction is that the identities $P(i,i)+P(i+1,i)=1/2$ should still hold true since one has $P_{i,i}+P_{i,i+1}=(A-B)/2+B=1$. This is implied also by probability conservation. The four curves corresponding to these identities do not however co-incide as the figure 6 of [J2] demonstrates. This is regarded as the basic anomaly in [J1, J2] . From the same figure it is also clear that below $\alpha-\beta<10$ degrees $P_{++}=P_{--}$ $\Delta P_{+-}=-\Delta P_{-+}$ holds true in a reasonable approximation. After that one has also non-vanishing ΔP_{ii} satisfying $\Delta P_{++}=-\Delta P_{--}$. This kind of splittings guarantee the identity $\sum_{ij}P_{ij}=1$. These splittings are not visible in E.

Since probability conservation requires $P_{ii}+P_{ii+1}=1$, a mundane explanation for the discrepancy could be that the failure of the conditions $P_{i,i}+P_{ii+1}=1$ means that the measurement efficiency is too low for P_{+-} and yields too low values of $P_{+-}+P_{--}$ and $P_{+-}+P_{++}$. The constraint $\sum_{ij}P_{ij}=1$ would then yield too high value for P_{-+}. Similar reduction of measurement efficiency for P_{++} could explain the splitting for $\alpha-\beta>10$ degrees.

Clearly asymmetry with respect to exchange of photons or of detectors is in question.

1. The asymmetry of two photon state with respect to the exchange of photons could be considered as a source of asymmetry. This would mean that the photons are not maximally entangled. This could be seen as an alternative "mundane" explanation.

2. The assumption that the parameter P is different for the detectors does not change the situation as is easy to check.

3. One manner to achieve splittings which resemble observed splittings is to assume that the value of the probability parameter P depends on the *polarization pair*: $P=P(i,j)$ so that one has $(P(-,+),P(+,-))=(P+\Delta,P-\Delta)$ and $(P(-,-),P(+,+))=(P+\Delta_1,P-\Delta_1)$. $\Delta\simeq .025$ and $\Delta_1\simeq\Delta/2$ could produce the observed splittings qualitatively. One would however always have $P(i,i)+P(i,i+1)\geq 1/2$. Only if the procedure extracting the correlations uses the constraint $\sum_{i,j}P_{ij}=1$ effectively inducing a constant shift of P_{ij} downwards an asymmetry of observed kind can result. A further objection is that there are no special reason for the values of $P(i,j)$ to satisfy the constraints.

2. Is it possible to say anything about the value of n in the case of EPR-Bohm experiment?

To explain the reduction of the maximum magnitudes of the correlation E from 1 to $\sim .9$ in the experiment discussed above one should have $p_1\simeq .9$. It is interesting to look whether this allows to deduce any information about the valued of n. At the limit of large values of N_in one would have $(N_1-N_2)/(N_1+N_2)\simeq .4$ so that one cannot say anything about n in this case. $(N_1,N_2)=(3,1)$ satisfies the condition exactly. For $n=3$, the smallest possible value of n, this would give $p_1\simeq .88$ and for $n=4$ $p_1=.41$. With high enough precision it might be possible to select between $n=3$ and $n=4$ options if small values of N_i are accepted.

3.7.8 Category Theoretic Formulation For Quantum Measurement Theory With Finite Measurement Resolution?

I have been trying to understand whether category theory might provide some deeper understanding about quantum TGD, not just as a powerful organizer of fuzzy thoughts but also as a tool providing genuine physical insights. Marni Dee Sheppeard (or Kea in her blog Arcadian Functor at `http://tinyurl.com/yb3lsbjq`) is also interested in categories but in much more technical sense. Her dream is to find a category theoretical formulation of M-theory as something, which is not the 11-D something making me rather unhappy as a physicist with second foot still deep in the muds of low energy phenomenology.

Locales, frames, Sierpinski topologies and Sierpinski space

The ideas below popped up when Kea mentioned in M-theory lesson 51 the notions of locale and frame [A5] . In Wikipedia I learned that complete Heyting algebras, which are fundamental to category theory, are objects of three categories with differing arrows. CHey, Loc and its opposite category Frm (arrows reversed). Complete Heyting algebras are partially ordered sets which are complete lattices. Besides the basic logical operations there is also algebra multiplication (I have considered the possible role of categories and Heyting algebras in TGD in [K15]). From Wikipedia I also learned that locales and the dual notion of frames form the foundation of pointless topology [A17] . These topologies are important in topos theory which does not assume axiom of choice.

The so called particular point topology [A13] assumes a selection of single point but I have the physicist's feeling that it is otherwise rather near to pointless topology. Sierpinski topology [A21] is this kind of topology. Sierpinski topology is defined in a simple manner: the set is open only if it contains a given preferred point p. The dual of this topology defined in the obvious sense exists also. Sierpinski space consisting of just two points 0 and 1 is the universal building block of these topologies in the sense that a map of an arbitrary space to Sierpinski space provides it with Sierpinski topology as the induced topology. In category theoretical terms Sierpinski space is the initial object in the category of frames and terminal object in the dual category of locales. This category theoretic reductionism looks highly attractive.

Particular point topologies, their generalization, and number theoretical braids

Pointless, or rather particular point topologies might be very interesting from physicist's point of view. After all, every classical physical measurement has a finite space-time resolution. In TGD framework discretization by number theoretic braids replaces partonic 2-surface with a discrete set consisting of algebraic points in some extension of rationals: this brings in mind something which might be called a topology with a set of particular algebraic points. Could this preferred set belongs to any open set in the particular point topology appropriate in this situation?

Perhaps the physical variant for the axiom of choice could be restricted so that only sets of algebraic points in some extension of rationals can be chosen freely and the choices is defined by the intersection of p-adic and real partonic 2-surfaces and in the framework of TGD inspired theory of consciousness would thus involve the interaction of cognition with the material world. The extension would depend on the position of the physical system in the algebraic evolutionary hierarchy defining also a cognitive hierarchy. Certainly this would fit very nicely to the formulation of quantum TGD unifying real and p-adic physics by gluing real and p-adic number fields to single super-structure via common algebraic points.

Analogs of particular point topologies at the level of state space: finite measurement resolution

There is also a finite measurement resolution in Hilbert space sense not taken into account in the standard quantum measurement theory based on factors of type I. In TGD framework one indeed introduces quantum measurement theory with a finite measurement resolution so that complex rays become included hyper-finite factors of type II_1 (HFFs).

1. Could topology with particular algebraic points have a generalization allowing a category theoretic formulation of the quantum measurement theory without states identified as complex rays?

2. How to achieve this? In the transition of ordinary Boolean logic to quantum logic in the old fashioned sense (von Neuman again!) the set of subsets is replaced with the set of subspaces of Hilbert space. Perhaps this transition has a counterpart as a transition from Sierpinski topology to a structure in which sub-spaces of Hilbert space are quantum subspaces with complex rays replaced with the orbits of subalgebra defining the measurement resolution. Sierpinski space $\{0,1\}$ would in this generalization be replaced with the quantum counterpart of the space of 2-spinors. Perhaps one should also introduce q-category theory with Heyting algebra being replaced with q-quantum logic.

Fuzzy quantum logic as counterpart for Sierpinksi space

The program formulated above might indeed make sense. The lucky association induced by Kea's blog was to the ideas about fuzzy quantum logic realized in terms of quantum 2-spinor that I had developed a couple of years ago. Fuzzy quantum logic would reflect the finite measurement resolution. I just list the pieces of the argument.

Spinors and qbits: Spinors define a quantal variant of Boolean statements, qbits. One can however go further and define the notion of quantum qbit, qqbit. I indeed did this for couple of years ago (the last section of this chapter).

Q-spinors and qqbits: For q-spinors the two components a and b are not commuting numbers but non-Hermitian operators: $ab = qba$, q a root of unity. This means that one cannot measure both a and b simultaneously, only either of them. $aa^\dagger$ and $bb^\dagger$ however commute so that probabilities for bits 1 and 0 can be measured simultaneously. State function reduction is not possible to a state in which a or b gives zero. The interpretation is that one has q-logic is inherently fuzzy: there are no absolute truths or falsehoods. One can actually predict the spectrum of eigenvalues of probabilities for say 1. Obviously quantum spinors would be state space counterparts of Sierpinski space and for $q \neq 1$ the choice of preferred spinor component is very natural. Perhaps this fuzzy quantum logic replaces the logic defined by the Heyting algebra.

Q-locale: Could one think of generalizing the notion of locale to quantum locale by using the idea that sets are replaced by sub-spaces of Hilbert space in the conventional quantum logic. Q-openness would be defined by identifying quantum spinors as the initial object, q-Sierpinski space. a (resp. b for the dual category) would define q-open set in this space. Q-open sets for other quantum spaces would be defined as inverse images of a (resp. b) for morphisms to this space. Only for q=1 one could have the q-counterpart of rather uninteresting topology in which all sets are open and every map is continuous.

Q-locale and HFFs: The q-Sierpinski character of q-spinors would conform with the very special role of Clifford algebra in the theory of HFFs, in particular, the special role of Jones inclusions to which one can assign spinor representations of $SU(2)$. The Clifford algebra and spinors of the world of classical worlds identifiable as Fock space of quark and lepton spinors is the fundamental example in which 2-spinors and corresponding Clifford algebra serves as basic building brick although tensor powers of any matrix algebra provides a representation of HFF.

Q-measurement theory: Finite measurement resolution (q-quantum measurement theory) means that complex rays are replaced by sub-algebra rays. This would force the Jones inclusions associated with $SU(2)$ spinor representation and would be characterized by quantum phase q and bring in the q-topology and q-spinors. Fuzzyness of qqbits of course correlates with the finite measurement resolution.

Q-n-logos: For other q-representations of SU(2) and for representations of compact groups (Appendix) one would obtain something which might have something to do with quantum n-logos, quantum generalization of n-valued logic. All of these would be however less fundamental and induced by q-morphisms to the fundamental representation in terms of spinors of the world of classical worlds. What would be however very nice that if these q-morphisms are constructible explicitly it would become possible to build up q-representations of various groups using the fundamental physical realization - and as I have conjectured [K62] - McKay correspondence and huge variety of its generalizations would emerge in this manner.

The analogs of Sierpinski spaces: The discrete subgroups of $SU(2)$, and quite generally, the groups Z_n associated with Jones inclusions and leaving the choice of quantization axes invariant, bring in mind the n-point analogs of Sierpinski space with unit element defining the particular point. Note however that $n \geq 3$ holds true always so that one does not obtain Sierpinski space itself. If all these n preferred points belong to any open set it would not be possible to decompose this preferred set to two subsets belonging to disjoint open sets. Recall that the generalized embedding space related to the quantization of Planck constant is obtained by gluing together coverings $M^4 \times CP_2 \rightarrow M^4 \times CP_2/G_a \times G_b$ along their common points of base spaces. The topology in question would mean that if some point in the covering belongs to an open set, all of them do so. The interpretation would be that the points of fiber form a single inseparable quantal unit.

Number theoretical braids identified as as subsets of the intersection of real and p-adic variants of algebraic partonic 2-surface define a second candidate for the generalized Sierpinski space with a set of preferred points.

Chapter 4

TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors

4.1 Introduction

There are two mysterious looking correspondences involving ADE groups. McKay correspondence between McKay graphs characterizing tensor products for finite subgroups of $SU(2)$ and Dynkin diagrams of affine ADE groups is the first one. The correspondence between principal diagrams characterizing inclusions of hyper-finite factors of type II_1 (HFFs) with Dynkin diagrams for a subset of ADE groups and Dynkin diagrams for affine ADE groups is the second one.

I have considered the interpretation of McKay correspondence in TGD framework already earlier [K87, K28] but the decision to look it again led to a discovery of a bundle of new ideas allowing to answer several key questions of TGD.

1. Asking questions about $M^8 - H$ duality at the level of 8-D momentum space [L18] led to a realization that the notion of mass is relative as already the existence of alternative QFT descriptions in terms of massless and massive fields suggests (electric-magnetic duality). Depending on choice $M^4 \subset M^8$, one can describe particles as massless states in $M^4 \times CP_2$ picture (the choice is M^4_L depending on state) and as massive states (the choice is fixed M^4_T) in M^8 picture. p-Adic thermal massivation of massless states in M^4_L picture can be seen as a universal dynamics independent mechanism implied by ZEO. Also a revised view about zero energy ontology (ZEO) based quantum measurement theory as theory of consciousness suggests itself.

2. Hyperfinite factors of type II_1 (HFFs) [K87, K28] and number theoretic discretization in terms of what I call cognitive representations [L29] provide two alternative approaches to the notion of finite measurement resolution in TGD framework. One obtains rather concrete view about how these descriptions relate to each other at the level of 8-D space of light-like momenta. Also ADE hierarchy can be understood concretely.

3. The description of 8-D twistors at momentum space-level is also a challenge of TGD. 8-D twistorializations in terms of octo-twistors (M^4_T description) and $M^4 \times CP_2$ twistors (M^4_L description) emerge at embedding space level. Quantum twistors could serve as a twistor description at the level of space-time surfaces.

4.1.1 McKay correspondence in TGD framework

Consider first McKay correspondence in more detail.

1. McKay correspondence states that the McKay graphs characterizing the tensor product decomposition rules for representations of discrete and finite sub-groups of $SU(2)$ are Dynkin diagrams for the affine ADE groups obtained by adding one node to the Dynkin diagram of ADE group. Could this correspondence make sense for any finite group G rather than only discrete subgroups of $SU(2)$? In TGD Galois group of extensions K of rationals can be any finite group G. Could Galois group take the role of G?

2. Why the subgroups of $SU(2)$ should be in so special role? In TGD framework quaternions and octonions play a fundamental role at M^8 side of $M^8 - H$ duality [L18]. Complexified M^8 represents complexified octonions and space-time surfaces X^4 have quaternionic tangent or normal spaces. $SO(3)$ is the automorphism group of quaternions and for number theoretical discretizations induced by extension K of rationals it reduces to its discrete subgroup $SO(3)_K$ having $SU(2)_K$ as a covering. In certain special cases corresponding to McKay correspondence this group is finite discrete group acting as symmetries of Platonic solids. Could this make the Platonic groups so special? Could the semi-direct products $Gal(K) \triangleleft SU(2)_K$ take the role of discrete subgroups of $SU(2)$?

4.1.2 HFFs and TGD

The notion of measurement resolution is definable in terms of inclusions of HFFs and using number theoretic discretization of X^4. These definitions should be closely related.

1. The inclusions $\mathcal{N} \subset \mathcal{M}$ of HFFs with index $\mathcal{M} : \mathcal{N} < 4$ are characterized by Dynkin diagrams for a subset of ADE groups. The TGD inspired conjecture is that the inclusion hierarchies of extensions of rationals and of corresponding Galois groups could correspond to the hierarchies for the inclusions of HFFs. The natural realization would be in terms of HFFs with coefficient field of Hilbert space in extension K of rationals involved.

   Could the physical triviality of the action of unitary operators $\mathcal{N}$ define measurement resolution? If so, quantum groups assignable to the inclusion would act in quantum spaces associated with the coset spaces $\mathcal{M}/\mathcal{N}$ of operators with quantum dimension $d = \mathcal{M} : \mathcal{N}$. The degrees of freedom below measurement resolution would correspond to gauge symmetries assignable to $\mathcal{N}$.

2. Adelic approach [L24] provides an alternative approach to the notion of finite measurement resolution. The cognitive representation identified as a discretization of X^4 defined by the set of points with points having H (or at least M^8 coordinates) in K would be common to all number fields (reals and extensions of various p-adic number fields induced by K). This approach should be equivalent with that based on inclusions. Therefore the Galois groups of extensions should play a key role in the understanding of the inclusions.

How HFFs could emerge from TGD?

1. The huge symmetries of "world of classical words" (WCW) could explain why the ADE diagrams appearing as McKay graphs and principal diagrams of inclusions correspond to affine ADE algebras or quantum groups. WCW consists of space-time surfaces X^4, which are preferred extremals of the action principle of the theory defining classical TGD connecting the 3-surfaces at the opposite light-like boundaries of causal diamond $CD = cd \times CP_2$, where cd is the intersection of future and past directed light-cones of M^4 and contain part of $\delta M^4_\pm \times CP_2$. The symplectic transformations of $\delta M^4_+ \times CP_2$ are assumed to act as isometries of WCW. A natural guess is that physical states correspond to the representations of the super-symplectic algebra SSA.

2. The sub-algebras SSA_n of SSA isomorphic to SSA form a fractal hierarchy with conformal weights in sub-algebra being n-multiples of those in SSA. SSA_n and the commutator $[SSA_n, SSA]$ would act as gauge transformations. Therefore the classical Noether charges for

these sub-algebras would vanish. Also the action of these two sub-algebras would annihilate the quantum states. Could the inclusion hierarchies labelled by integers $.. < n_1 < n_2 < n_3 ....$ with n_{i+1} divisible by n_i would correspond hierarchies of HFFs and to the hierarchies of extensions of rationals and corresponding Galois groups? Could n correspond to the dimension of Galois group of K.

3. Finite measurement resolution defined in terms of cognitive representations suggests a reduction of the symplectic group SG to a discrete subgroup SG_K, whose linear action is characterized by matrix elements in the extension K of rationals defining the extension. The representations of discrete subgroup are infinite-D and the infinite value of the trace of unit operator is problematic concerning the definition of characters in terms of traces. One can however replace normal trace with quantum trace equal to one for unit operator. This implies HFFs and the hierarchies of inclusions of HFFs [K87, K28]. Could inclusion hierarchies for extensions of rationals correspond to inclusion hierarchies of HFFs and of isomorphic sub-algebras of SSA?

Quantum spinors are central for HFFs. A possible alternative interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group. This has also application in TGD inspired theory of consciousness [K28]: the idea is that the truth value of Boolean statement is fuzzy. At the level of quantum measurement theory this would mean that the outcome of quantum measurement is not anymore precise eigenstate but that one obtains only probabilities for the appearance of different eigenstate. One might say that probability of eigenstates becomes a fundamental observable and measures the strength of belief.

4.1.3 New aspects of $M^8 - H$ duality

$M^8 - H$ duality ($H = M^4 \times CP_2$) [L18] has become one of central elements of TGD. $M^8 - H$ duality implies two descriptons for the states.

1. $M^8 - H$ duality assumes that space-time surfaces in M^8 have associative tangent- or normal space M^4 and that these spaces share a common sub-space $M^2 \subset M^4$, which corresponds to complex subspace of octonions (also integrable distribution of $M^2(x)$ can be considered). This makes possible the mapping of space-time surfaces $X^4 \subset M^8$ to $X^4 \subset H = M^4 \times CP_2$) giving rise to $M^8 - H$ duality.

2. $M^8 - H$ duality makes sense also at the level of 8-D momentum space in one-one correspondence with light-like octonions. In $M^8 = M^4 \times E^4$ picture light-like 8-momenta are projected to a fixed quaternionic $M^4_T \subset M^8$. The projections to $M^4_T \supset M^2$ momenta are in general massive. The group of symmetries is for E^4 parts of momenta is $Spin(SO(4)) = SU(2)_L \times SU(2)_R$ and identified as the symmetries of low energy hadron physics.

   $M^4 \supset M^2$ can be also chosen so that the light-like 8-momentum is parallel to $M^4_L \subset M^8$. Now CP_2 codes for the E^4 parts of 8-momenta and the choice of M^4_L and color group $SU(3)$ as a subgroup of automorphism group of octonions acts as symmetries. This correspond to the usual description of quarks and other elementary particles. This leads to an improved understanding of $SO(4) - SU(3)$ duality. A weaker form of this duality $S^3 - CP_2$ duality: the 3-spheres S^3 with various radii parameterizing the E^4 parts of 8-momenta with various lengths correspond to discrete set of 3-spheres S^3 of CP_2 having discrete subgroup of $U(2)$ isometries.

3. The key challenge is to understand why the MacKay graphs in McKay correspondence and principal diagrams for the inclusions of HFFs correspond to ADE Lie groups or their affine variants. It turns out that a possible concrete interpretation for the hierarchy of finite subgroups of $SU(2)$ appears as discretizations of 3-sphere S^3 appearing naturally at M^8 side of $M^8 - H$ duality. Second interpretation is as covering of quaternionic Galois group. Also the coordinate patches of CP_2 can be regarded as piles of 3-spheres and finite measurement resolution. The discrete groups of $SU(2)$ define in a natural way a hierarchy of measurement

resolutions realized as the set of light-like M^8 momenta. Also a concrete interpretation for Jones inclusions as inclusions for these discretizations emerges.

4. A radically new view is that descriptions in terms of massive and massless states are alternative options leads to the interpretation of p-adic thermodynamics as a completely universal massivation mechanism having nothing to do with dynamics. The problem is the paradoxical looking fact that particles are massive in H picture although they should be massless by definition. The massivation is unavoidable if zero energy states are superposition of massive states with varying masses. The M^4_L in this case most naturally corresponds to that associated with the dominating part of the state so that higher mass contributions can be described by using p-adic thermodynamics and mass squared can be regarded as thermal mass squared calculable by p-adic thermodynamics.

5. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory. 4-D space-time surfaces correspond to roots of octonionic polynomials $P(o)$ with real coefficients corresponding to the vanishing of the real or imaginary part of $P(o)$.

   These polynomials however allow universal roots, which are not 4-D but analogs of 6-D branes and having topology of S^6. Their M^4 projections are time =constant snapshots $t = r_n, r_M \leq r_n$ 3-balls of M^4 light-cone (r_n is root of $P(x)$). At each point the ball there is a sphere S^3 shrinking to a point about boundaries of the 3-ball.

   What suggests itself is following "braney" picture. 4-D space-time surfaces intersect the 6-spheres at 2-D surfaces identifiable as partonic 2-surfaces serving as generalized vertices at which 4-D space-time surfaces representing particle orbits meet along their ends. Partonic 2-surfacew would define the space-time regions at which one can pose analogs of boundary values fixing the space-time surface by preferred extremal property. This would realize strong form of holography (SH): 3-D holography is implied already by ZEO.

   This picture forces to consider a modification of the recent view about ZEO based theory of consciousness. Should one replace causal diamond (CD) with light-cone, which can be however either future or past directed. "Big" state function reductions (BSR) meaning the death and re-incarnation of self with opposite arrow of time could be still present. An attractive interpretation for the moments $t = r_n$ would be as moments assignable to "small" state function reductions (SSR) identifiable as "weak" measurements giving rise to sensory input of conscious entity in ZEO based theory of consciousness. One might say that conscious entity becomes gradually conscious about its roots in increasing order. The famous question "What it feels to be a bat" would reduce to "What it feels to be a polynomial?"! One must be however very cautious here.

4.1.4 What twistors are in TGD framework?

The basic problem of the ordinary twistor approach is that the states must be massless in 4-D sense. In TGD framework particles would be massless in 8-D sense. The meaning of 8-D twistorialization at space-time level is relatively well understood but at the level of momentum space the situation is not at all so clear.

1. In TGD particles are massless in 8-D sense. For M^4_L description particles are massless in 4-D sense and the description at momentum space level would be in terms of products of ordinary M^4 twistors and CP_2 twistors. For M^4_T description particles are massive in 4-D sense. How to generalize the twistor description to 8-D case?

   The incidence relation for twistors and the need to have index raising and lowering operation in 8-D situation suggest the replacement of the ordinary l twistors with either with octo-twistors or non-commutative quantum twistors.

2. I have assumed that what I call geometric twistor space of M^4 is simply $M^4 \times S^2$. It however turned out that one can consider standard twistor space CP_3 with metric signature (3,-3) as an alternative. This option reproduces the nice results of the earlier approach but the philosophy is different: there is no fundamental length scale but the hierarchy of causal

diamonds (CDs) predicted by zero energy ontology (ZEO) gives rise to the breaking of the exact scaling invariance of M^8 picture. This forces to modify $M^8 - H$ correspondence so that it involves map from M^4 to CP_3 followed by a projection to hyperbolic variant $CP_{2,h}$ of CP_2. Note that also the original form of $M^8 - H$ duality continues to make sense and results from the modification by projection from $CP_{3,h}$ to M^4 rather than $CP_{2,h}$.

M^4 in H would not be be replaced with conformally compactified M^4 (M^4_{conf}) but conformally compactified cd (cd_{conf}) for which a natural identification is as CP_2 with second complex coordinate replaced with hypercomplex coordinate. The sizes of twistor spaces of cd_{conf} using CP_2 size as unit would reflect the hierarchy of size scales for CDs. The consideration on the twistor space of M^8 in similar picture leads to the identification of corresponding twistor space as HP_3 - quaternionic variant of CP_3: the counterpart of CD_8 would be HP_2.

3. Octotwistors can be expressed as pairs of quaternionic twistors. Octotwistor approach suggests a generalization of twistor Grassmannian approach obtained by replacing the bi-spinors with complexified quaternions and complex Grassmannians with their quaternionic counterparts. Although TGD is not a quantum field theory, this proposal makes sense for cognitive representations identified as discrete sets of spacetime points with coordinates in the extension of rationals defining the adele [L24] implying effective reduction of particles to point-like particles.

4. The outcome of octo-twistor approach together with $M^8 - H$ duality leads to a nice picture view about twistorial description of massive states based on quaternionic generalization of twistor Grassmannian approach. A radically new view is that descriptions in terms of massive and massless states are alternative options, and correspond to two different alternative twistorial descriptions and leads to the interpretation of p-adic thermodynamics as completely universal massivation mechanism having nothing to do with dynamics. As a side product emerges a deeper understanding of ZEO based quantum measurement theory and consciousness theory relying on the universal roots of octonionic polynomials of M^8, which are not 4-D but analogs of 6-D branes. By $M^8 - H$ duality the finite sub-groups of $SU(2)$ of McKay correspondence appear quite concretely in the description of the measurement resolution of 8-momentum.

What about super-twistors in TGD framework?

1. The parallel progress in the understanding SUSY in TGD framework [L39] in turn led to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.

2. The great surprise from physics point of view is that in fermionic sector only quarks are allowed by $SO(1,7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of embedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.

3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

What about the interpretation of quantum twistors? They could make sense as 4-D spacetime description analogous to description at space-time level. Now one can consider generalization of the twistor Grassmannian approach in terms of quantum Grassmannians.

4.2 McKay correspondence

Consider first McKay correspondence from TGD point of view.

4.2.1 McKay graphs

McKay graps are defined in the following manner. Consider group G which is discrete but not necessarily finite. If the group is finite there is a finite number of irreducible representations χ_I. Select preferred representation V - usually V is taken to be the fundamental representation of G and form tensor products $\chi_I \otimes V$. Suppose irrep χ_J appears n_{ij} times in the tensor product $\chi_I \otimes \chi_0$. Assign to each representation χ_I dot and connect the dots of χ_I and χ_J by n_{ij} arrows. This gives rise to MacKay graph.

Consider now the situation for finite-D groups of $SU(2)$. 2-D $SU(2)$ spinor representation as a fundamental representation is essential for obtaining the identification of McKay graphs as Dynkin diagrams of simply laced affine algebras having only single line connecting the roots (the angle between positive roots is 120 degrees) (see `http://tinyurl.com/z48d92t`).

1. For $SU(2)$ representations one has the basic rule $j_1 - 1/2 \leq j \leq j_1 + 1/2$ for the tensor product $j_1 \otimes 1/2$. This rule must be broken for finite subgroups of $SU(2)$ since the number of representations if finite so that branching point appears in McKay graph. In branching point the decomposition of $j_1 \otimes 1/2$ decomposes to 3 lower-dimensional representations of the finite subgroup takes place.

2. Simply lacedness means that given representation appears only once in $chi_I \otimes V$, when V is 2-D representation as it can be for a subgroup of $SU(2)$. Additional exceptional properties is the absence of loops ($n_{ii} = 0$) and connectedness of McKay graph.

3. One can consider binary icosahedral group (double covering of icosahedral group A_5 with order 60) as an example (for the McKay graph see `http://tinyurl.com/y2h55jwp`). The representations of A_5 are $1_A, 3_A, 3'_B, 4_A, 5_A$, where integer tells the dimension. Note that $SO(3)$ does not allow 4-D representation. For binary icosahedral group one obtains also the representations $2_A, 2'_B, 4_B, 6_A$. The McKay graph of E_8 contains one branching point in which one has the tensor product of 6-D and 2-D representations 6_A and 2_A giving rise to $5_A \oplus 3_C \oplus 4_B$.

McKay graphs can be defined for any finite group and they are not even unions of simply laced diagrams unless one has subgroups of $SU(2)$. Still one can wonder whether McKay correspondence generalizes from subgroups of $SU(2)$ to all finite groups. At first glance this does not seem possible but there might be some clever manner to bring in all finite groups.

The proposal has been that this McKay correspondence has a deeper meaning. Could simply laced affine ADE algebras, ADE type quantum algebras, and/or corresponding finite groups act as symmetry algebras in TGD framework?

4.2.2 Number theoretic view about McKay correspondence

Could the physical picture provided by TGD help to answer the above posed questions?

1. Could one identify discrete subgroups of $SU(2)$ with those of the covering group $SU(2)$ of $SO(3)$ of quaternionic automorphisms defining the continuous analog of Galois group and reducing to a discrete subgroup for a finite resolution characterized by extension K of rationals. The tensor products of 2-D spinor representation of these discrete subgroups $SU(2)_K$ would give rise to irreps appearing in the McKay graph.

2. In adelic physics [L24] extensions K of rationals define an evolutionary hierarchy with effective Planck constant $h_{eff}/h_0 = n$ identified as the dimension of K. The action of discrete and finite subgroups of various symmetry groups can be represented as Galois group action making sense at the level of X^4 [L18] for what I have called cognitive representations. By $M^8 - H$ duality also the Galois group of quaternions and its discrete subgroups appear naturally.

This suggests a possible generalization of McKay correspondence so that it would apply to all finite groups G. Any finite group G can appear as Galois group. The Galois group $Gal(K)$ characterizing the extension of rationals induces in turn extensions of p-adic number fields appearing in the adele. Could the representation of G as Galois group of extension of rationals allow to generalize McKay correspondence?

Could the following argument inspired by these observations make sense?

1. $SU(2)$ is identified as spin covering of the quaternionic automorphism group. One can define the subgroups in matrix representation of $SU(2)$ based on complex numbers. One can replace complex numbers with the extension of rationals and speak of group $SU(2)_K$ identified as a discrete subgroup of $SU(2)$ having in general infinite order.

   The discrete finite subgroups $G \subset SU(2)$ appearing in the standard McKay correspondence correspond to extensions K of rationals for which one has $G = SU(2)_K$. These special extensions can be identified by studying the matrix elements of the representation of G and include the discrete groups Z_n acting as rotation symmetries of the Platonic solids. For instance, for icosahedral group Z_2,Z_3 and Z_5 are involved and correspond to roots of unity.

2. The semi-direct product $Gal \triangleleft SU(2)_K$ with group action

$$(gal_1, g_1)(gal_2, g_2) = (gal_1 \circ gal_2, g_1(gal_1 g_2))$$

   makes sense. The action of $Gal \triangleleft SU(2)_K$ in the representation is therefore well-defined. Since all finite groups G can appear as Galois groups, it seems that one obtains extension of the McKay correspondence to semi-direct products involving all finite groups G representable as Galois groups.

3. A good guess is that the number of representations of $SU(2)_K$ involved is infinite if $SU(2)_K$ has infinite order. For $\tilde{A}_n$ and $\tilde{D}_n$ the branching occurs only at the ends of the McKay graph. For E_k, $k = 6, 7, 8$ the branching occurs in middle of the graph (see `http://tinyurl.com/y2h55jwp`). What happens for arbitrary G. Does the branching occur at all? One could argue that if the discrete subgroup has infinite order, the representation can be completed to a representation of $SU(2)$ in terms of real numbers so that the McKay graphs must be identical.

4. A concrete realization of ADE type Kac-Moody algebras is proposed. It relies on the group algebra of $Gal(K) \triangleleft SU(2)_K$ and free field representation of ADE type Kac-Moody algebra identifying the free scalar fields in Kac-Moody Cartan algebra as group algebra elements defined by the traces of representation matrices (characters).

5. A possible interpretation of quantum spinors is in terms of quantum measurement theory with finite measurement resolution in which precise eigenstates as measurement outcomes are replaced with universal probability distributions defined by quantum group [K28]. TGD inspired theory of consciousness is a possible application.

   Also the notion of quantum twistor [L43] can be considered. In TGD particles are massless in 8-D sense and in general massive in 4-D sense but 4-D twistors are needed also now so that a modification of twistor approach is needed. The incidence relation for twistors suggests the replacement of the usual twistors with non-commutative quantum twistors.

4.3 ADE diagrams and principal graphs of inclusions of hyperfinite factors of type II_1

Dynkin diagrams for ADE groups and corresponding affine groups characterize also the inclusions of hyperfinite factors of type II_1 (HFFs) [K28].

4.3.1 Principal graphs and Dynkin diagrams for ADE groups

1. If the index $\beta = \mathcal{M} : \mathcal{N}$ of the Jones inclusion satisfies $\beta < 4$, the affine Dynkin diagrams of $SU(n)$ (discrete symmetry groups of n-polygons) and E_7 (symmetry group of octahedron and cube) and $D(2n+1)$ (symmetries of double 2n+1-polygons) are not allowed.

2. Vaughan Jones [A131] (see `http://tinyurl.com/y8jzvogn`) has speculated that these finite subgroups could correspond to quantum groups as kind of degenerations of Kac-Moody groups. Modulo arithmetics defined by the integer n defining the quantum phase suggests itself strongly. For $\beta = 4$ one can construct inclusions characterized by extended Dynkin diagram and any finite sub-group of $SU(2)$. In this case affine ADE hierarchy appear as principal graphs characterizing the inclusions. For $\beta < 4$ the finite measurement resolution could reduce affine algebra to quantum algebra.

3. The rule is that for odd values of n defining the quantum phase the Dynkin diagram does not appear. If Dynkin diagrams correspond to quantum groups, one can ask whether they allow only quantum group representations with quantum phase $q = exp(i\pi/n)$ equal to even root of unity.

4.3.2 Number theoretic view about inclusions of HFFs and preferred role of $SU(2)$

Consider next the TGD inspired interpretation.

1. TGD suggests the interpretation in terms of representations of $Gal(K(G)) \triangleleft G$ for finite subgroups G of $SU(2)$, where $K(G)$ would be an extension associated with G. This would generalize to subgroups of $SU(2)$ with infinite order in the case of arbitrary Galois group. Quantum groups have finite number of representations in 1-1-correspondence with terms of finite-D representations of G. Could the representations of $Gal(K(G)) \triangleleft G$ correspond to the representations of quantum group defined by G?

   This would conform with the vision that there are two ways to realize finite measurement resolution. The first one would be in terms of inclusions of hyper-finite factors. Second would be in terms cognitive representations defining a number theoretic discretization of X^4 with embedding space coordinates in the extension of rationals in which Galois group acts.

   In fact, also the discrete subgroup of infinite-D group of symplectic transformations of $\Delta M^4_+ \times CP_2$ would act in the cognitive representations and this suggests a far reaching implications concerning the understanding of the cognitive representations, which pose a formidable looking challenge of finding the set of points of X^4 in given extension of rationals [L37].

2. This brings in mind also the model for bio-harmony in which genetic code is defined in terms of Hamiltonian cycles associated with icosahedral and tetrahedral geometries [L7, L31]. One can wonder why the Hamiltonian cycles for cubic/octahedral geometry do not appear. On the other hand, according to Vaughan the Dynkin diagram for E_7 is missing from the hierarchy of so principal graphs characterizing the inclusions of HFFs for $\beta < 4$ (a fact that I failed to understand). Could the genetic code directly reflect the properties of the inclusion hierarchy?

How would the hierarchies of inclusions of HFFs and extensions of rationals relate to each other?

1. I have proposed that the inclusion hierarchies of extensions K of rationals accompanied by similar hierarchies of Galois groups $Gal(K)$ could correspond to a fractal hierarchy of sub-algebras of hyperfinite factor of type II_1. Quantum group representations in ADE hierarchy would somehow correspond to these inclusions. The analogs of coset spaces for two algebras in the hierarchy define would quantum group representations with quantum dimension characterizing the inclusion.

2. The quantum group in question would correspond to a quantum analog of finite-D group of $SU(2)$ which would be in completely unique role mathematically and physically. The infinite-D group in question could be the symplectic group of $\delta M^4_+ \times CP_2$ assumed to act as isometries of WCW. In the hierarchy of Galois groups the quantum group of finite group $G \subset SU(2)$ would correspond to Galois group of an extension K.

3. The trace of unit matrix defining the character associated with unit element is infinite for these representations for factors of type I. Therefore it is natural to assume that hyper-finite factor of type II_1 for which the trace of unit matrix can be normalized to 1. Sub-factors would have trace of projector with trace smaller than 1.

4. Do the ADE diagrams for groups $Gal(K(G)) \triangleleft G$ indeed correspond to quantum groups and affine algebras? Why the finite groups should give rise to affine/Kac-Moody algebras? In number theoretic vision a possible answer would be that depending on the value of the index β of inclusion the symplectic algebra reduces in the number theoretic discretization by gauge conditions specifying the inclusion either to Kac-Moody group ($\beta = 4$) or to quantum group ($\beta < 4$).

What about subgroups of groups other than $SU(2)$? According to Vaughan there has been work about inclusion hierarchies of $SU(3)$ and other groups and it seems that the results generalize and finite subgroups of say $SU(3)$ appear. In this case the tensor products of finite sub-groups McKay graphs do not however correspond to the principal graphs for inclusions. Could the number theoretic vision come in rescue with the replacement of discrete subgroup with Galois group and the identification of $SU(2)$ as the covering for the Galois group of quaternions?

4.3.3 How could ADE type quantum groups and affine algebras be concretely realized?

The questions discussed are following. How to understand the correspondence between the McKay graph for finite group $G \subset SU(2)$ and ADE (affine) group Dynkin diagram for $\beta < 4$ ($\beta = 4$)? How the nodes of McKay grap representing the irreps of finite group can correspond to the positive roots of a Dynkin diagram, which are essentially vectors defined by eigenvalues of Cartan algebra generators for complexified Lie-algebra basis.

The first thing that comes in mind is the construction of representation of Kac-Moody algebra using scalar fields labelled by Cartan algebra generators (see `http://tinyurl.com/y9lkeelk`): these representations are discussed by Edward Frenkel [A55]. The charged generators of Kac-Moody algebra in the complement of Cartan algebra are obtained by exponentiating the contractions of the vector formed by these scalar fields with roots to get $\alpha \cdot \Phi = \alpha_i \Phi^i$. The charged field is represented as a normal ordered product $: exp(i\alpha \cdot \Phi) :$. If one can assign to each irrep of G a scalar field in a natural manner one could achieve this.

Neglect first the presence of the group algebra of $Gal(K(G)) \triangleleft G$. The standard rule for the dimensions of the representations of finite groups reads as $\sum_i d_I^2 = n(G)$. For double covering of G one obtains this rule separately for integer spin representations and half-odd integers spin representations. An interesting possibility is that this could be interpreted in terms of supersymmetry at the level of group algebra in which representation of dimension d_I appears d_I times.

The group algebra of G and its covering provide a universal manner to realize these representations in terms of a basis for complex valued functions in group (for extensions of rationals also the values of the functions must belong to the extension).

1. Representation with dimension d_I occurs d_I times and corresponds to $d_I \times d_I$ representation matrices D^I_{mn} of representation χ_I, whose columns and rows provide representations for left- and right-sided action of G. The tensor product rules for the representations χ_I can be formulated as double tensor products. For basis states $|J, n\rangle$ in χ_I and $|J, n\rangle$ in χ_J one has

$$|I, m\rangle_{\otimes} |J, n\rangle = c^{K,p}_{I,m|J,n} |K, p\rangle \ ,$$

where $c^{K,p}_{J,n|J,n}$ are Glebch-Gordan coefficients.

2. For the tensor product of matrices D^I_{mn} and D^J_{mn} one must apply this rule to both indices. The orthogonality properties of Glebsch-Gordan coefficients guarantee that the tensor product contains only terms in which one has same representation at left- and right-hand side. The orthogonality rule is

$$\sum_{m,n} c^{K,p}_{I,m|J,n} c^{K,q}_{I,r|J,s} \propto \delta_{K,L} \quad .$$

3. The number of states is $n(G)$ whereas the number $I(G)$ of irreps corresponds to the dimension of Cartan algebra of Kac-Moody algebra or of quantum group is smaller. One should be able to pick only one state from each representation D^I.

   The condition that the state X of group algebra is invariant under automorphism gXg^{-1} implies that the allowed states as function in group algebra are traces $Tr(D^I)(g)$ of the representation matrices. The traces of representation matrices indeed play fundamental role as automorphism invariants. This suggests that the scalar fields Φ_I in Kac-Moody algebra correspond to Hilbert space coefficients of $Tr(D^I)(g)$ as elements of group algebra labelled by the representation. The exponentiation of $\alpha \cdot \Phi$ would give the charged Kac-Moody algebra generators as free field representation.

4. For infinite sub-groups $G \subset SU(2)$ $d(G)$ is infinite. The traces are finite also in this case if the dimensions of the representations involved are finite. If one interprets the unit matrix as a function having value 1 in entire group $Tr(Id)$ diverges. Unit dimension for HFFs provide a more natural notion of dimension $d = n(G)$ of group algebra $n(G)$ as $d = n(G) = 1$. Therefore HFFs would emerge naturally.

It is easy to take into account $Gal(K(G))$. One can represent the elements of semi-direct product $Gal(K(G)) \triangleleft G$ as functions in $Gal(K(G)) \times G$ and the proposed construction brings in also the tensor products in the group algebra of $Gal(K(G))$. It is however essential that group algebra elements have values in K. This brings in tensor products of representations Gal and G and the number of representations is $n(Gal) \times n(G)$. The number of fields Φ_I as also the number of Cartan algebra generators of ADE Lie algebra increases from $I(G)$ to $I(Gal) \times I(G)$. The reduction of the extension of coefficient field for the Kac-Moody algebra from complex numbers to K splits the Hilbert space to sectors with smaller number of states.

4.4 $M^8 - H$ duality

The generalization of the standard twistor Grassmannian approach to TGD remains a challenge for TGD and one can imagine several approaches. $M^8 - H$ duality is essential for these approaches and will be discussed in the sequel.

The original form of $M^8 - H$ duality assumed $H = M^4 \times CP_2$ but quite recently it turned out that one could replace the twistor space of M^4 identified as $M^4 \times S^2$ with $CP_{3,h}$, which is hyperbolic variant of CP_3. This option forces to replace H with $H = CP_{2,h} \times CP_2$. $M^8 - H$ duality would consist of a map of M^4 point to corresponding twistor sphere in $CP_{3,h}$ and its projection to $CP_{2,h}$. This option will be discussed in the section about twistor lift of TGD.

4.4.1 $M^8 - H$ duality at the level of space-time surfaces

$M^8 - H$ duality [L18] relates two views about space-time surfaces X^4: as algebraic surfaces in complexified octonionic M^8 and as minimal surfaces with singularities in $H = M^4 \times CP_2$.

1. Octonion structure at the level of M^8 means a selection of a suitable decomposition $M^8 = M^4 \times E^4$ in turn determining $H = M^4 \times CP_2$. Choices of M^4 share a preferred 2-plane $M^2 \subset M^4$ belonging to the tangent space of allowed space-time surfaces $X^4 \subset M^8$ at various points. One can parameterize the tangent space of $X^4 \subset M^8$ with this property by a point of CP_2. Therefore X^4 can be mapped to a surface in $H = M^4 \times CP_2$: one M^8-duality. One can consider also the possibility that the choice of M^2 is local but that the distribution of $M^2(x)$ is integrable and defines string world sheet in M^4. In this case $M^2(x)$ is mapped to same $M^2 \subset H$.

2. Since 8-momenta p_8 are light-like one can always find a choice of $M^4_L \subset M^8$ such that p_8 belongs to M^4_L and is thus light-like. The momentum has in the general case a component orthogonal to M^2 so that M^4_L is unique by quaternionicity: quaternionic cross product for tangent space quaternions gives the third imaginary quaternionic unit. For a fixed M^4, call it M^4_T, the M^4 projections of momenta are time-like. When momentum belongs to M^2 the choices is non-unique and any $M^4 \subset M^2$ is allowed.

3. Space-time surfaces $X^4 \subset M^8$ have either quaternionic tangent- or normal spaces. Quantum classical correspondence (QCC) requires that charges in Cartan algebra co-incide with their classical counters parts determined as Noether charges by the action principle determining X^4 as preferred extremal. Parallelity of 8-momentum currents with tangent space of X^4 would conform with the naïve view about QCC. It does not however hold true for the contributions to four-momentum coming from string world sheet singularities (string world sheet boundaries are identified as carriers of quantum numbers), where minimal surface property fails.

An important aspect of $M^8 - H$ duality is the description of space-time surfaces $X^4_c \subset M^8_c$ as roots for the "real" or "imaginary" part in quaternionic sense of complexified-octonionic polynomial with real coefficients: these options correspond to complexified-quaternionic tangent - or normal spaces. The real space-time surfaces would be naturally obtained as "real" parts with respect to i of their complexified counterparts by projection from M^8_c to M^4_c. One could drop the subscripts "$_c$" but in the sequel they are kept.

Remark:O_c,O_c,C_c,R_c will be used in the sequel for complexifications of octonions, quaternions, etc.. number fields using commuting imaginary unit i appearing naturally via the roots of real polynomials.

$M^8 - H$ duality makes sense under 2 additional assumptions to be considered in the following more explicitly than in earlier discussions.

1. Space-time surface is identified as a 4-D root for a H_c-valued "imaginary" or "real" part of O_c valued polynomial obtained as an O_c continuation of a real polynomial P with rational coefficients, which can be chosen to be integers. For $P(x) = x^n + ..$ ordinary roots are algebraic integers. The 4-D space-time surface is projection of this surface from M^8_c to M^8.

   The tangent space of space-time surface and thus space-time surface itself contains a preferred $M^2_c \subset M^4_c$ or more generally, an integrable distribution of tangent spaces $M^2_c(x)$. The string world sheet like entity defined by this distribution is 2-D surface $X^2_c \subset X^4_c$ in R_c sense.

   X^2c can be fixed by posing to the non-vanishing Q_c-valued part of octonionic polynomial condition that the C_c valued "real" or "imaginary" part in C_c sense for this polynomial vanishes. M^2_c would be the simplest solution but also more general complex sub-manifolds $X^2_c \subset M^4_c$ are possible. In general one would obtain book like structures as collections of several string world sheets having real axis as back.

   By assuming that R_c-valued "real" or "imaginary" part of the polynomial at this 2-surface vanishes. one obtains preferred M^1_c or E^1_c containing octonionic real and preferred imaginary unit or distribution of the imaginary unit having interpretation as complexified string. Together these kind 1-D surfaces in R_c sense would define local quantization axis of energy and spin. The outcome would be a realization of the hierarchy $R_{\rightarrow}C_c \rightarrow H_c \rightarrow O_c$ realized as surfaces.

   Remark: Also M^4_c appears as a special solution for any polynomial P. M^4_c seems to be like a universal reference solution with which to compare other solutions. M^4_c would intersect all other solutions along string world sheets X^2_c. Also this would give rise to a book like structures with 2-D string world sheet representing the back of given book. The physical interpretation of these book like structures remains open in both cases.

   I have proposed that string world sheets as singularities correspond to 2-D folds of space-time surfaces at which the dimension of the quaternionic tangent space degenerates from 4 to 2 [L35] [K7]. This interpretation is consistent with the identification as a book like structure with 2-pages. Also 1-D real and imaginary manifols could be interpreted as folds or equivalently books with 2 pages.

2. Associativity condition for tangent-/normal space is second essential condition and means that tangent - or normal space is quaternionic. The conjecture is that the identification in terms of roots of polynomials guarantees this and one can formulate this as rather convincing argument [L19, L20, L21].

One cannot exclude rational functions and or even real analytic functions in the sense that Taylor coefficients are octonionically real (propotional to octonionic real unit). Number theoretical vision - adelic physics [L24], suggests that polynomial coefficients are rational or perhaps in extensions of rationals. The real coefficients could in principle be replaced with complex numbers $a + ib$, where i commutes with the octonionic units and defines complexifiation of octonions. i appears also in the roots defining complex extensions of rationals.

1. In general the zero loci for imaginary or real part are 4-D but the 7-D light-cone δM^8_+ of M^8 with tip at the origin of coordinates is an exception [L18]. At δM^8_+ the octonionic coordinate o is light-like and one can write $o = re$, where 8-D time coordinate and radial coordinate are related by $t = r$ and one has $e = (1 + e_r)/\sqrt{2}$ such that one as $e^2 = e$.

   Polynomial $P(o)$ can be written at δM^8_+ as $P(o) = P(r)e$ and its roots correspond to 6-spheres S^6 represented as surfaces $t_M = t = r_N$, $r_M = \sqrt{r_N^2 - r_E^2} \leq r_N$, $r_E \leq r_N$, where the value of Minkowski time $t = r = r_N$ is a root of $P(r)$ and r_M denotes radial Minkowski coordinate. The points with distance r_M from origin of $t = r_N$ ball of M^4 has as fiber 3-sphere with radius $r = \sqrt{r_N^2 - r_E^2}$. At the boundary of S^3 contracts to a point.

2. These 6-spheres are analogous to 6-D branes in that the 4-D solutions would intersect them in the generic case along 2-D surfaces X^2. The boundaries $r_M = r_N$ of balls belong to the boundary of M^4 light-cone. In this case the intersection would be that of 4-D and 3-D surface, and empty in the generic case (it is however quite not clear whether topological notion of "genericity" applies to octonionic polynomials with very special symmetry properties).

3. The 6-spheres $t_M = r_N$ would be very special. At these 6-spheres the 4-D space-time surfaces X^4 as usual roots of $P(o)$ could meet. Brane picture suggests that the 4-D solutions connect the 6-D branes with different values of r_n.

   The basic assumption has been that particle vertices are 2-D partonic 2-surfaces and light-like 3-D surfaces - partonic orbits identified as boundaries between Minkowskian and Euclidian regions of space-time surface in the induced metric (at least at H level) - meet along their 2-D ends X^2 at these partonic 2-surfaces. This would generalize the vertices of ordinary Feynman diagrams. Obviously this would make the definition of the generalized vertices mathematically elegant and simple.

   Note that this does not require that space-time surfaces X^4 meet along 3-D surfaces at S^6. The interpretation of the times t_n as moments of phase transition like phenomena is suggestive. ZEO based theory of consciousness suggests interpretation as moments for state function reductions analogous to weak measurements ad giving rise to the flow of experienced time.

4. One could perhaps interpret the free selection of 2-D partonic surfaces at the 6-D roots as initial data fixing the 4-D roots of polynomials. This would give precise content to strong form of holography (SH), which is one of the central ideas of TGD and strengthens the 3-D holography coded by ZEO alone in the sense that pairs of 3-surfaces at boundaries of CD define unique preferred extremals. The reduction to 2-D holography would be due to preferred extremal property realizing the huge symplectic symmetries and making $M^8 - H$ duality possible as also classical twistor lift.

   I have also considered the possibility that 2-D string world sheets in M^8 could correspond to intersections $X^4 \cap S^6$? This is not possible since time coordinate t_M constant at the roots and varies at string world sheets.

   Note that the compexification of M^8 (or equivalently octonionic E^8) allows to consider also different variants for the signature of the 6-D roots and hyperbolic spaces would appear for $(\epsilon_1, \epsilon_i, .., \epsilon_8)$, $epsilon_i = \pm 1$ signatures. Their physical interpretation - if any - remains open at this moment.

5. The universal 6-D brane-like solutions S^6_c have also lower-D counterparts. The condition determining X^2 states that the C_c-valued "real" or "imaginary" for the non-vanishing Q_c-valued "real" or "imaginary" for P vanishes. This condition allows universal brane-like solution as a restriction of O_c to M^4_c (that is CD_c) and corresponds to the complexified time=constant hyperplanes defined by the roots $t = r_n$ of P defining "special moments in the life of self" assignable to CD. The condition for reality in R_c sense in turn gives roots of $t = r_n$ a hyper-surfaces in M^2_c.

4.4.2 $M^8 - H$ duality at the level of momentum space

$M^8 - H$ duality occurs also at the level of momentum space and has different meaning now.

1. At M^8 level 8-momenta are quaternionic and light-like. The choices of $M^4_L \supset M^2$ for which 8-momentum in M^4_L, are parameterized by CP_2 parameterizing also the choices of tangent or normal spaces of $X^4 \subset M^8$ at space-time level. This maps M^8 light-like momenta to light-like M^4_L momenta and to CP_2 point characterizing the M^4 and depending on 8-momentum. One can introduce CP_2 wave functions expressible in terms of spinor harmonics and generators of of a tensor product of Super-Virasoro algebras.

2. For a fixed choice M^4_T momenta in general time-like and the E^4 component of 8-momentum has value equal to mass squared. E^4 momenta are points of 3-sphere so that $SO(3)$ harmonics with $SO(4)$ symmetry could parametrize the states. The quantum numbers are $M^4_T \supset M^2$ momenta with fixed mass and the two angular momenta with identical values for S^3 harmonics, which correspond to the quantum states of a spherical quantum mechanical rigid body, and are given by the matrix elements $D^j_{m,n}$ $SU(2)$ group elements ($SO(4)$ decomposes to $SU(2)_L) \times SU(2)_R$ acting from left and right).

   This picture suggests what one might call $SO(4) - SU(3)$ duality at the level of momentum space. There would be two descriptions of states: as massless states with $SU(3)$ symmetry and massive states with $SO(4)$ symmetry.

3. What about the space formed by the choices of the space of the light-like 8-momenta? This space is the space for the choices of preferred M^2 and parameterized by the 6-D (symmetric space $G_2/SU(3)$, where $SU(3) \subset G_2$ leaving complex plane M^2 invariant is subgroup of quaternionic automorphic group $G(2)$ leaving octonionic real unit defining the rest system invariant. This space is moduli space for octonionic structures each of which defines family of space-time surfaces. 8-D Lorent transformations produce even more general octonionic structures. The space for the choices of color quantization axes is $SU(3)/U(1) \times U(1)$, the twistor space of CP_2.

Do M^4_L and M^4_T have analogs at the space-time level?

As found, the solutions of octonionic polynomials consisting of 4-D roots and special 6-D roots coming as 6-sphere S^6 s at 7-D light-cone of M^8. The roots at $t = r$ light-cone boundary are given by the roots $r = r_N$ of the polynomial $P(t)$ and correspond to M^4 slices $t_M = r_N, r_M \leq r_N$. At point r_M S^3 fiber as radius $r(S^3) = \sqrt{r_N^2 - r_M^2}$ and contracts to a point at its boundaries.

Could M^4_L and M_T have analogies at the space-time level?

1. The sphere S^3 associated M^4_T could have counterpart at the level of space-time description. The momenta in M^4_T would naturally be mapped to momenta in the section $t = r_n$ in this case the S^3:s of different mass squared values would naturally correspond to S^3:s assignable to the points of the balls $t = r_n$ and code for mass squared value.

   The counterpart of M^4_L should correspond to light-cone boundary but what does CP_2 correspond? Could the pile of S^3 associated with $t = r_n$ correspond also to CP_2. Could this be the case if there is wormhole contact carrying monopole flux at the origin so that the analog for the replacement of 3-sphere at $r_{CP_2} = \infty$ with homologically non-trivial 2-sphere would be realized?

2. Does the 6-sphere as a root polynomial have counterpart in H? The image should be consistent with M^8-H duality and correspond to a fixed structure depending on the root r_n only. Since S^3 associated with the E^4 momenta reduces to a point for M^4_L, the natural guess is that S^6 reduces to $t=r_n, 0\leq r_M\leq r_n$ surface in H.

S^3-CP_2 duality

S^3-CP_2 duality at the level of quantum numbers suggest strongly itself. What does this require? One can approach the problem from two different perspectives.

1. The first approach would be that the representations of $SU(3)$ and $SO(4)$ groups somehow correspond to each other: one could speak of $SU(3)-SO(4)$ duality [K74, K86]. The original form of this duality was this. The color symmetries of quark physics at high energies would be dual to the $SO(4)=SU(2)_L\times SU(2)_R$ symmetries of the low energy hadron physics. Since the physical objects are partons and hadrons formed from the one cannot expect the duality to hold true at the level of details for the representations, and the comparison of the representations makes this clear.

2. The second approach relies on the notion of cognitive representation meaning discretization of CP_2 and S^3 and counting of points of cognitive representations providing discretization in terms of M^8 or H points belonging to the extension of rationals considered. In this case it is more natural to talk about S^3-CP_2 duality.

   The basic observation is that the open region $0\leq r<\infty$ of CP_2 in Eguchi-Hanson coordinates with r labeling 3-spheres $S^3(r)$ with finite radius can be regarded as pile of $S^3(r)$. In discretization one would have discrete pile of these 3-spheres with finite number of points in the extension of rationals. They points of given S^3 could be related by isometries in special cases.

How S^3-CP_2 duality at the level of light-like M^8 momenta could emerge?

1. Consider first the situation in which one chooses $M^4\supset M^2$ sub-spaces so that momentum projection to it is light-like. For cognitive representation the choices of $M^4\supset M^2$ correspond to ad discrete set of points of CP_2 and thus points in the pile of S^3 with discrete radii since all E^4 parts of momenta with fixed length squared to zero in this choice and each E^4 momentum with fixed lengthand thus identifiable as discrete point of S^3 would correspond to one choice.

   All these choices would give rise to a pile of S^3:s identifiable as set $0\leq r<\infty$ of CP_2. The number of CP_2 points would be same as total number of points in the pile of discrete S^3s. This is what S^3-CP_2 duality would say.

   Remark: The volumes of CP_2 and S^3 with unit radius are $8\pi^2$ and $2\pi^2$ so that ration is rational number.

2. Consider now the situation for M^4_T so that one has non-vanishing M^4 mass squared equal to E^4 mass squared, having discretized values. The E^4 would momenta correspond to points for a pile of discretized S^3 and thus to the points of CP_2 by above argument. One would have S^3-CP_2 correspondence also now as one indeed expects since the two ways to see the situation should be equivalent.

3. In the space of light-like M^8 momenta E^8 momenta could naturally organize into representations of finite discrete subgroups of $SU(2)$ appearing in McKay correspondence with ADE groups: the groups are cyclic groups, dihedral groups, and the isometry groups associated with tetrahedron, octahedron (cube) and icosahedron (dodecahedron) (see `http://tinyurl.com/yyyn9p95`).

4. Could a concrete connection with the inclusion hierarchy of HFFs be based on increasing momentum resolution realized in terms of these groups at sphere S^3. Notice however that for cyclic and dihedral groups the orbits are circles and pairs of circles for dihedral groups so that the discretization looks too simple and is rotationally asymmetric. Discretization should improve as n increases.

One can of course ask why C_n and D_n with single direction of rotation axes would appear? Could it be that the directions of rotation axis correspond to the directions defined by the vertices of the 5 Platonic solids. Or could the orbits of fixed axis under the 5 Platonic orbits be allowed. Also this looks still too simple.

Could the discretization labelled by n_{max} be determined by the product of the groups up to n_{max} and define a group with infinite order. One can consider also groups defined by subsets $\{n_1, n_2...n_3\}$ and these a pair of sequences with larger sequence containing the smaller one could perhaps define an inclusion. The groups C_n and D_n allow prime decomposition in obvious manner and it seems enough to include to the product only the groups C_p and D_p, where p is prime as generators so that one would have set $\{p_1, ...p_n\}$ of primes labelling these groups besides the Platonic groups. The extension of rationals used poses a cutoff on the number of groups involved and on the group elements representable since since too high roots of unity resulting in the multiplication of C_{p_i} and D_{p_j} do not belong to the extension.

At the level of momentum space the hierarchy of finite discrete groups of $SU(2)$ would define the notion measurement resolution. The discrete orbits of $SU(2) \times U(1)$ at S^3 would be analogous to tessellations of sphere S^2 known as Platonic solids at sphere S^2 and appearing in the ADE correspondence assignable to Jones inclusions as description of measurement resolution. This would also explain also why Z_2 coverings of the subgroups of $SO(3)$ appear in ADE sequence.

This picture is probably not enough for the needs of adelic physics [L24] allowing all extensions of rationals. Besides roots of unity only the roots of small integers $2, 3, 5$ associated with the geometry of Platonic solids would be included in these discretizations. One could interpret these discretizations in terms of subgroups of discrete automorphism groups of quaternions. Also the extensions of rationals are probably needed.

Could $S^3 - CP_2$ duality make sense at space-time level? Consider the space-time analog for the projection of M^8 momenta to fixed M^4_T.

1. Suppose that the 3-surfaces determining the space-time surfaces as algebraic surfaces in $X^4 \subset M^8$ are given at the surfaces $t = r_N, r_M \leq r_N$ and have a 3-D fiber which should be surface in CP_2. On can assign to each point of this ball $S^3(r_M)$ with radius going to zero at $r_M = r_N$. One has pile of $S^3(r_M)$ which corresponds to the region $0 \leq r < \infty$ of CP_2. This set is discretized. Suppose that the discretization is like momentum discretization. If so, the points would correspond to points of CP_2. It is not however clear why the discretization should be so symmetric as in momentum discretization.

2. The initial values could be chosen by choosing discrete set of points in this pile of S^3:s and this would give rise to a discrete set of points of CP_2 fixing tangent or normal plane of X^4 at these points. One should show that the selection of a point of S^6 at each point indeed determines quaternionic tangent or normal plane plane for a given polynomial $P(o)$ in M^8.

It would seem that this correspondence need not hold true.

4.4.3 $M^8 - H$ duality and the two ways to describe particles

The isometry groups for $M^4 \times CP_2$ is $P \times SU(3)$ (P for Poincare group). The isometry group for $M^8 = M^4 \times E^4$ with a fixed choice of M^4 breaks down to $P \times SO(4)$. A further breaking by selection $M^4 \subset M^2$ of preferred octonionic complex plane M^2 necessary in the algebraic approach to space-time surfaces $X^4 \subset M^8$ brings in preferred rest system and reduces the Poincare symmetry further. At the space-time level the assumption that the tangent space of X^4 contains fixed M^2 or at least integral distribution of $M^2(x) \subset M^4$ is necessary for $M^8 - H$ duality [L18].

The representations $SO(4)$ and $SU(3)$ could provide alternative description of physics so that one would have what I have called $SO(4)-SU(3)$ duality [K74]. This duality could manifest in the description of strong interaction physics in terms of hadrons and quarks respectively (conserved vector current hypothesis and partially conserved axial current hypothesis based on $Spin(SO(4)) = SU(2) \times SU(2)_R$. The challenge is to understand in more detail this duality. This could allow also to understand better how the two twistor descriptions might relate.

$SO(4) - SU(3)$ duality implies two descriptions for the states and scattering amplitudes.

Option I: One uses projection of 8-momenta to a fixed $M^4_T \supset M^2$.

Option II: One assumes that $M^4_L \supset M^2$ is defines the frame in which quaternionic octonion momentum is parallel to M^4_L: this M^4_L depends on particle state and describes this dependence in terms of wave function in CP_2.

Option I: fixed $M^4_T \supset M^2$

For Option I the description would be in terms of a *fixed* $M^4_T \subset M^8 = M^4_T \times E^4$ and $M^2 \subset M^4_T$ fixed for both options. For given quaternionic light-like M^8 momentum one would have projection to M^4_T, which is in general massive. E^4 momentum would have same the length squared by light-likeness.

De-localization M^4_T mass squared equal to $p^2(M^4_T) = m^2$ in E^4 can be described in terms of $SO(4)$ harmonics at sphere having $p^2(E^4) = m^2$. This would be the description applied to hadrons and leptons and particles treated as massive particles. Particle mass would be due to the fixed choice of M^4_T. What dictates this choice is an interesting question. An interesting question is how these descriptions relate to QFT Higgs mechanism as (in principle) alternative descriptions: the choice of fixed M^4_T could be seen as analog for the generation of vacuum expectation of Higgs selecting preferred direction in the space of Higgs fields.

Option II: varying $M^4_L \supset M^2$

For Option II the description would use $M^4_L \supset M^2$, which is *not fixed* but chosen so that it contains light-like M^8 momentum. This would give light-like momentum in M^4_L identifiable as quaternionic sub-space of complexified octonions.

1. One could assign to the state wave function function for the choices of M^4 and by quaternionicity of 8-momenta this would correspond to a state in super-conformal representation with vanishing M^4_L mass: CP_2 point would code the information about E^4 component light-like 8-momentum. This description would apply to the partonic description of hadrons in terms of massless quarks and gluons.

2. For this option one could use the product of ordinary M^4 twistors and CP_2 twistors. One challenge would be the generalization of the twistor description to the case of CP_2 twistors.

p-Adic particle massivation and ZEO

The two pictures about description of light-like M^8 momenta do not seem to be quite consistent with the recent view about TGD in which H-harmonics describe massivation of massless particles. What looks like a problem is following.

1. The resulting particles are massive in M^4. But they should be massless in $M^4 \times CP_2$ description. The non-vanishing mass would suggest the correct description in terms of Option I for which the description in terms of E^4 momenta with length equal to mass and thus identifiable as points of S^3. Momentum space wave functions at S^3 - essentially rigid body wave functions given by representation matrices of $SU(2)$ could characterize the states rather than CP_2 harmonic.

2. The description based on CP_2 color partial waves however works and this would favor Option II with vanishing M^4 mass. What goes wrong?

To understand what might be involved, consider p-adic mass calculations.

1. The massivation of physical fermion states includes also the action of super-conformal generators changing the mass. The particles are originally massless and p-adic mass squared is generated thermally and mapped to real mass squared by canonical identification map.

   For CP_2 spinor harmonics mass squared is of order CP_2 mass squared and thus gigantic. Therefore the mass squared is assumed to contain negative tachyonic ground state contribution due to the negative half-odd integer valued conformal weight $h_{vac} < 0$ of vacuum. The origin of this contribution has remained a mystery in p-adic thermodynamics but it makes possible to construct massless states. h_{vac} cancels the spinorial contributions and the contribution from positive conformal weights of super-conformal generators so that the particle states are massless before thermalization. This would conform with the idea of using varying M^4_L and thus CP_2 description.

2. What does the choice of M^4 mean in terms of super-conformal representations? Could the mysterious vacuum conformal weight h_{vac} provide a description for the effect of the needed $SU(3)$ rotation of M^4 from standard orientation on super-conformal representation. The effect would be very simple and in certain sense reversal to the effect of Higgs vacuum expectation value in that it would cancel mass rather than generate it.

   An important prediction would be that heavy states should be absent from the spectrum in the sense that mass squared would be p-adically of order $O(p)$ or $O(p^2)$ (in real sense of order $O(1/p)$ or $O(1/p^2)$). The trick would be that the generation of h_0 as a representation of SU(3) rotation of M^4 makes always the dominating contribution to the mass of the state vanishing.

   Remark: If the canonical identification I mapping the p-adic mass integers to their real numbers is of the simplest form $m = \sum_n x_n p^n \to I(m) = \sum_n x_n p^{-n}$, it can happen that the image of rational m/n with p-adic norm not larger than 1 represented as p-adic integer by expanding it in powers of p, can be near to the maximal value of p and the mass of the state can be of order CP_2 mass - about 10^{-4} Planck masses. If the canonical identification is defined as $m/n \to I/(m)/I(n)$ the image of the mass is small for small values of m and n.

3. Unfortunately, it is easy to get convinced that this explanation of h_{vac} is not physically attractive. Identical mass spectra at the level of M^8 and H looks like a natural implication of $M^8 - H$-duality. $SU(3)$ rotation of M^4 in M^8 cannot however preserve the general form of $M^4 \times CP_2$ mass squared spectrum for the M^4 projections of M^8 momenta at level of M^8.

   Remark: For $H = M^4 \times CP_2$ the mass squared in given representation of Super-conformal symmetries is given as a sum of CP_2 mass squared for the spinor harmonic determining the ground state and of a Virasoro contribution proportional to a non-negative integer. The masses are required to independent of electroweak quantum numbers.

One can imagine two further identifications for the origin of h_{vac}.

1. Take seriously the possibility of complex momenta allowed by the complexification of M^8 by commuting imagine unit i and also suggested by the generalization of the twistorialization. The real and imaginary parts of light-like complex 8-momenta $p_8 = p_{8,Re} + ip_{8,Im}$ are orthogonal to each other: $p_{8,Re} \cdot p_{8,Im} = 0$ so that both real and imaginary parts of p_8 are light-like: $p^2_{8,Re} = p^2_{8,Im} = 0$. The M^4 mass squared can be written has $m^2 = m^2_{Re} - m^2_{Im}$ with $h_{vac} \propto -m^2_{Im}$. The representations of Super-conformal algebra would be labelled by $h_{vac} \propto m^2_{Im}$.

   Could the needed wrong sign contribution to CP_2 mass squared mass make sense? CP_2 type extremals having light-like geodesic as M^4 projection are locally identical with CP_2 but because of light-like projection they can be regarded as CP_2 with a hole and thus non-compact. Boundary conditions allow analogs of CP_2 harmonics for which spinor d'Alembertian would have complex eigenvalues.

   Does quantum-classical correspondence allow complex momenta: can the classical four-momenta assignable to 6-D Kähler action be complex? The value of Kähler coupling strength allows the action to have complex phase but parts with different phases are not allowed. Could the imaginary part to 4-momentum could come from the CP_2 type extremal with Euclidian signature of metric?

2. Second - not necessarily independent - idea relies on the observation that in TGD one has besides the usual conformal algebra acting on complex coordinate z also its analog acting on the light-like radial coordinate r of light-cone boundary. At light-cone boundary one has both super-symplectic symmetries of $\Delta M^4_+ \times CP_2$ and extension of super-conformal symmetries of sphere S^2 to analogs of conformal symmetries depending on z and r and it seems that one

must chose between these two options. Also the extension of ordinary Kac-Moody algebra acts at the light-like orbits of partonic 2-surfaces.

There are two scaling generators: the usual $L_0 = zd/dz$ and the second generator $L_{0,1} = ird/dr$. For $L_{0,1}$ at light-cone boundary powers of z^n are replaced with $(r/r_0)^{ik} = exp(iku)$, $u = log(r/r_0))$. Could it be that mass squared operator is proportional to $L_0 + L_{0,1}$ having eigenvalues $h = n - k$? The absence of tachyons requires $h \geq 0$. Could k characterize given Super-Virasoro representation? Could $k \geq 0$ serve as an analog of positive energy condition allowing to analytically continue $exp(iku)$ to upper u-plane? How to interpret this continuation?

The 3-D generalization of super-symplectic symmetries at light-cone boundary and extended Ka-Moody symmetries at partonic 2-surfaces should be possible in some sense. Could the continuation to the upper u-plane correspond to the continuation of the extended conformal symmetries from light-cone boundary to future light-one and from light-partonic 2-surfaces to space-time interior?

Why p-adic massivation should occur at all? Here ZEO comes in rescue.

1. In ZEO one can have superposition of states with different 4-momenta, mass values and also other charges: this does not break conservation laws. How to fix M^4 in this case? One cannot do it separately for the states in superposition since they have different masses. The most natural choices is as the M^4 associated with the dominating contribution to the zero energy state. The outcome would be thermal massivation described excellently by p-adic thermodynamics [K41]. Recently a considerable increase in the understanding of hadron and weak boson masses took place [L44].

2. In ZEO quantum theory is square root of thermodynamics in a well-defined formal sense, and one can indeed assign to p-adic partition function a complex square root as a genuine zero energy state. Since mass varies, one must describe the presence of higher mass excitations in zero energy state as particles in M^4 assigned with the dominating part of the state so that the observed particle mass squared is essentially p-adic thermal expectation value over thermal excitations. p-Adic thermodynamics would thus describe the fact that the choice of M^4_L cannot not ideal in ZEO and massivation would be possible only in ZEO.

3. Current quarks and constituent quarks are basic notions of hadron physics. Constituent quarks with rather large masses appear in the low energy description of hadrons and current quarks in high energy description of hadronic reactions. That both notions work looks rather paradoxical. Could massive quarks correspond to M_T picture and current quarks to M^4_L picture but with p-adic thermodynamics forced by the superposition of mass eigenstates with different masses.

   The massivation of ordinary massless fermion involves mixing of fermion chiralities. This means that the $SU(3)$ rotation determined by the dominating component in zero energy state must induce this mixing. This should be understood.

4.4.4 $M^8 - H$ duality and consciousness

$M^8 - H$ duality is one of the key ideas of TGD and one can ask whether it has implications for TGD inspired theory of consciousness and it indeed forces to challenge the recent ZEO based view about consciousness [L27] .

Objections against ZEO based theory of consciousness

Consider first objections against ZEO based view about consciousness.

1. ZEO (zero energy ontology) based view about conscious entity can be regarded as a sequence of "small" state function reductions (SSRs) identifiable as analogs of so called weak measurements at the active boundary of causal diamond (CD) receding reduction by reduction farther away from the passive boundary, which is unchanged as also the members of state pairs at it. One can say that weak measurements commute with the observables, whose

eigenstates the states at passive boundary are. This asymmetry assigns arrow of time to the self having CD as embedding space correlate. "Big" state function reductions (BSRs) would change the roles of boundaries of CD and the arrow of time. The interpretation is as death and re-incarnation of the conscious entity with opposite arrow of time.

The question is whether quantum classical correspondence (QCC) could allow to say something about the time intervals between subsequent values of temporal distance between weak state function reductions.

2. The questionable aspect of this view is that $t_M = constant$ sections look intuitively more natural as seats of quantum states than light-cone boundaries forming part of CD boundaries. The boundaries of CD are however favoured by the huge symplectic symmetries assignable to the boundary of M^4 light-cone with points replaced with CP_2 at level of H. These symmetries are crucial or the existence of the geometry of WCW ("world of classical worlds").

3. Second objection is that the size of CD increases steadily: this nice from the point of view of cosmology but the idea that CD as correlate for a conscious entity increases from CP_2 size to cosmological scales looks rather weird. For instance, the average energy of the state assignable to either boundary of CD would increase. Since zero energy state is a superposition of states with different energies classical conservation law for energy does not prevent this [L40]: essentially quantal effect due to the fact that the zero energy states are not exact eigenstates of energy could be in question. In BSRs the energy would gradually increase. Admittedly this looks strange and one must be keen for finding more conventional options.

4. Third objection is that re-incarnated self would not have any "childhood" since CD would increase all the time.

One can ask whether $M^8 - H$ duality and this braney picture has implications for ZEO based theory of consciousness. Certain aspects of $M^8 - H$ duality indeed challenge the recent view about consciousness based on ZEO (zero energy ontology) and ZEO itself.

1. The moments $t = r_n$ defining the 6-branes correspond classically to special moments for which phase transition like phenomena occur. Could $t = r_n$ have a special role in consciousness theory?

   (a) For some SSRs the increase of the size of CD reveals new $t = r_n$ plane inside CD. One can argue that these SSRS define very special events in the life of self. This would not modify the original ZEO considerably but could give a classical signature for how many ver special moments of consciousness have occurred: the number of the roots of P would be a measure for the lifetime of self and there would be the largest root after which BSR would occur.

   (b) Second possibility is more radical. One could one think of replacing CD with single truncated future- or past-directed light-cone containing the 6-D universal roots of P up to some r_n defining the upper boundary of the truncated cone? Could $t = r_n$ define a sequence of moments of consciousness? To me it looks more natural to assume that they are associated with very special moments of consciousness.

2. For both options SSRs increase the number of roots r_n inside CD/truncated light-one gradually and thus its size? When all roots of $P(o)$ would have been measured - meaning that the largest value r_{max} of r_n is reached -, BSR would be unavoidable.

   BSR could replace $P(o)$ with $P_1(r_1 - o)$: r_1 must be real and one should have $r_1 > r_{max}$. The new CD/truncated light-cone would be in opposite direction and time evolution would be reversed. Note that the new CD could have much smaller size size if it contains only the smallest root r_0. One important modification of ZEO becomes indeed possible. The size of CD after BSR could be much smaller than before it. This would mean that the re-incarnated self would have "childhood" rather than beginning its life at the age of previous self - kind of fresh start wiping the slate clean.

   One can consider also a less radical BSR preserving the arrow of time and replacing the polynomial with a new one, say a polynomial having higher degree (certainly in statistical sense so that algebraic complexity would increase).

Could one give up the notion of CD?

A possible alternative view could be that one the boundaries of CD are replaced by a pair of two $t = r_N$ snapshots $t = r_0$ and $t = r_N$. Or at least that these surfaces somehow serve as correlates for mental images. The theory might allow reformulation also in this case, and I have actually used this formulation in popular lectures since it is easier to understand by laymen.

1. Single truncated light-cone, whose size would increase in each SSR would be present now since the spheres correspond to balls of radius r_n at times r_n. If $r_0 = 0$, which is the case for $P(o) \propto o$, the tip of the light-cone boundary is one root. One cannot avoid association with big bang cosmology. For $P(0) \neq r_0$ the first conscious moment of the cosmology corresponds to $t = r_0$. One can wonder whether the emergence of consciousness in various scales could be described in terms of the varying value of the smallest root r_0 of $P(o)$.

   If one allows BSR:s this picture differs from the earlier one in that CDs are replaced with alternation of light-cones with opposite directions and their intersections would define CD.

2. For this option the preferred values of t for SSRs would naturally correspond to the roots of the polynomial defining $X^4 \subset M^8$. Moments of consciousness as state function reductions would be due to collisions of 4-D space-time surfaces X^4 with 6-D branes! They would replace the sequence of scaled CD sizes. CD could be replaced with light-one and with the increasing sequence $(r_0, ...r_n)$ of roots defining the ticks of clock and having positive and negative energy states at the boundaries r_0 and r_n.

3. What could be the interpretation for BSRs representing death of a conscious entity in the new variant of ZEO? Why the arrow of time would change? Could it be because there are no further roots of $P(o)$? The number of roots of $P(o)$ would give the number of small state function reductions?

   What would happen to $P(o)$ in BSR? The vision about algebraic evolution as increase of the dimension for the extension of rationals would suggest that the degree of $P(o)$ increases as also the number of roots if all complex roots are allowed. Could the evolution continue in the same direction or would it start to shift the part of boundary corresponding to the lowest root in opposite direction of time. Now one would have more roots and more algebraic complexity so that evolutionary step would occur.

   In the time reversal one would have naturally $t_{max} \geq r_{n_{max}}$ for the new polynomial $P(t-t_{max})$ having $r_{n_{max}}$ as its smallest root. The light-cone in M^8 with tip at $t = t_{max}$ would be in opposite direction now and also the slices $t - t_{max} = r'_n$ would increase in opposite direction! One would have two light-cones with opposite directions and the $t = r_n$ sections would replace boundaries of CDs. The reborn conscious entity would start from the lowest root so that also it would experience childhood.

This option could solve the argued problems of the previous scenario and give concrete connection with the classical physics in accordance with QCC. On the other hand, a minimal modification of original scenario combined with $M^8 - H$ duality with moments $t = r_n$ as special moments in the life of conscious entity allows also to solve these problems if the active boundary of CD is interpreted as boundary beyond which classical signals cannot contribute to perceptions.

What could be the minimal modification of ZEO based view about consciousness?

What would be the minimal modification of the earlier picture? Could one *assume* that CDs serve as embedding space correlates for the perceptive field?

1. Zero energy states would be defined as before that is in terms of 3-surfaces at boundaries of CD: this would allow a realization of huge symmetries of WCW and the active boundary A of CD would define the boundary of the region from which self can receive classical information about environment. The passive boundary P of CD would define the boundary of the region providing classical information about the state of self. Also now BSR would mean death and reincarnation with an opposite arrow of time. Now however CD would shrink in BSR before starting to grow in opposite time direction. Conscious entity would have “childhood”.

2. If the geometry of CD were fixed, the size scale of the $t = r_n$ balls of M^4 would first increase and then start to decrease and contract to a point eventually at the tip of CD. One must however remember that the size of $t = r_n$ planes increases all the time as also the size of CD in the sequences of SSRs. Moments $t = r_n$ could represent special moments in the life of conscious entity taking place in SSRs in which $t = r_n$ hyperplane emerges inside CD with increased size. The recent surprising findings challenging the Bohrian view about quantum jumps [L32] can be understood in this picture [L32].

3. $t = r_n$ planes could also serve as correlates for memories. As CD increases at active boundary new events as $t = r_n$ planes would take place and give rise to memories. The states at $t = r_n$ planes are analogous to seats of boundary conditions in strong holography and the states at these planes might change in state function reductions - this would conform with the observations that our memories are not absolute.

To sum up, the original view about ZEO seems to be essentially correct. The introduction of moments $t = r_n$ as special moments in the life of self looks highly attractive as also the possibility of wiping the slate clear by reduction of the size of CD in BSR.

4.5 Could standard view about twistors work at space-time level after all?

While asking what super-twistors in TGD might be, I became critical about the recent view concerning what I have called geometric twistor space of M^4 identified as $M^4 \times S^2$ rather than CP_3 with hyperbolic metric. The basic motivations for the identification come from M^8 picture in which there is number theoretical breaking of Poincare and Lorentz symmetries. Second motivation was that M^4_{conf} - the conformally compactified M^4 - identified as group $U(2)$ [B3] (see `http://tinyurl.com/y35k5wwo`) assigned as base space to the standard twistor space CP_3 of M^4, and having metric signature (3,-3) is compact and is stated to have metric defined only modulo conformal equivalence class.

As found in the previous section, TGD strongly suggests that M^4 in $H = M^4 \times CP_2$ should be replaced with hyperbolic variant of CP_2, and it seems to me that these spaces are not identical. Amusingly, $U(2)$ and CP_2 are fiber and base in the representation of $SU(3)$ as fiber space so that the their identification does not seem plausible.

On can however ask whether the selection of a representative metric from the conformal equivalence class could be seen as breaking of the scaling invariance implied also by ZEO introducing the hierarchy of CDs in M^8. Could it be enough to have M^4 only at the level of M^8 and conformally compactified M^4 at the level of H? Should one have $H = cd_{conf} \times CP_2$? What cd_{conf} would be: is it hyperbolic variant of CP_2?

4.5.1 Getting critical

The only way to make progress is to become very critical now and then. These moments of almost despair usually give rise to a progress. At this time I got very critical about the TGD inspired identification of twistor spaces of M^4 and CP_2 and their properties.

Getting critical about geometric twistor space of M^4

Let us first discuss the recent picture and how to modify it so that it is consistent with the hierarchy of CDs. The key idea is that the twistor space and its base space represents CD so that one obtains scale hierarchy of twistor spaces as a realization of broken scale invariance giving rise to the p-adic length scale hierarchy.

1. I have identified the twistor space of M^4 simply as $T(M^4) = M^4 \times S^2$. The interpretation would be at the level of octonions as a product of M^4 and choices of M^2 as preferred complex sub-space of octonions with S^2 parameterizing the directions of spin quantization axes. Real octonion axis would correspond to time coordinate. One could talk about the space of of light-like directions. Light-like vector indeed defines M^2. This view could be defended by

the breaking of both translation and Lorentz invariance in the octonionic approach due to the choice of M^2 and by the fact that it seems to work.

Remark: $M^8 = M^4 \times E^4$ is complexified to M_c^8 by adding a commuting imaginary unit i appearing in the extensions of rationals and ordinary M^8 represents its particular sub-space. Also in twistor approach one uses often complexified M^4.

2. The objection is that it is ordinary twistor space identifiable as CP_3 with (3,-3) signature of metric is what works in the construction of twistorial amplitudes. CP_3 has metric as compact space and coset space. Could this choice of twistor space make sense after all as geometric twistor space?

   Here one must pause and recall that the original key idea was that Poincare invariance is symmetry of TGD for $X^4 \subset M^4 \times CP_2$. Now Poincare symmetry has been transformed to a symmetry acting at the level of M^8 in the moduli space of octonion structures defined by the choice of the direction of octonionic real axis reducing Poincare group to $T \times SO(3)$ consisting of time translations and rotations. Fixing of M^2 reducrs the group to $T \times SO(2)$ and twistor space can be seen as the space for selections of quantization axis of energy and spin.

3. But what about the space H? The first guess is $H = M^4_{conf} \times CP_2$. According to [B3] (see `http://tinyurl.com/y35k5wwo`) one has $M^4_{conf} = U(2)$ such that $U(1)$ factor is time- like and $SU(2)$ factor is space-like. One could understand $M^4_{conf} = U(2)$ as resulting by addition and identification of metrically 2-D light-cone boundaries at $t = \pm\infty$. This is topologically like compactifying E^3 to S^3 and gluing the ends of cylinder $S^3 \times D^1$ together to the $S^3 \times S^1$.

   The conformally compactified Minkowski space M^4_{conf} should be analogous to base space of CP_3 regarded as bundle with fiber S^2. The problem is that one cannot imagine an analog of fiber bundle structure in CP_3 having $U(2)$ as base. The identification $H = M^4_{conf} \times CP_2$ does not make sense.

4. In ZEO based breaking of scaling symmetry it is CD that should be mapped to the analog of M^4_{conf} - call it cd_{conf}. The only candidate is $cd_{conf} = CP_2$ with one hypercomplex coordinate. To understand why one can start from the following picture. The light-like boundaries of CD are metrically equivalent to spheres. The light-like boundaries at $t = \pm\infty$ are identified as in the case of M^4_{conf}. In the case of CP_2 one has 3 homologically trivial spheres defining coordinate patches. This suggests that cd_{conf} is simply CP_2 with second complex coordinate made hypercomplex. M^4 and E^4 differ only by the signature and so would do cd_{conf} and CP_2.

   The twistor spheres of CP_3 associated with points of M^4 intersect at point if the points differ by light-like vector so that one has and singular bundle structure. This structure should have analog for the compactification of CD. CP_3 has also bundle structure $CP_3 \rightarrow CP_2$. The S^2 fibers and base are homologically non-trivial and complex analogs of mutually orthogonal line and plane and intersect at single point. This defines the desired singular bundle structure via the assignment of S^2 to each point of CP_2.

   The M^4 points must belong to the interior of cd and this poses constraints on the distance of M^4 points from the tips of cd. One expects similar hierarchy of cds at the level of momentum space.

5. In this picture $M^4_{conf} = U(2)$ could be interpreted as a base space for the space of CDs with fixed direction of time axis identified as direction of octonionic real axis associated with various points of M^4 and therefore of M^4_{conf}. For Euclidian signature one would have base and fiber of the automorphism sub-group $SU(3)$ regarded as $U(2)$ bundle over CP_2: now one would have CP_2 bundle over $U(2)$. This is perhaps not an accident, and one can ask whether these spaces could be interpreted as representing local trivialization of $SU(3)$ as $U(2) \times CP_2$. This would give to metric cross terms between $U(2)$ and CP_2.

6. The proposed identification can be tested by looking whether it generalizes. What the twistor space for entire M^8 would be? $cd = CD_4$ is replaced with CD_8 and the discussion of the preceding chapter demonstrated that the only possible identification of the twistor space is

now is as the 12-D hyperbolic variant of HP_3 whereas $CD_{8,conf}$ would correspond to 8-D hyperbolic variant of HP_2 analogous to hyperbolic variant of CP_2.

The outcome of these considerations is surprising.

1. One would have $T(H) = CP_3 \times F$ and $H = CP_{2,H} \times CP_2$ where $CP_{2,H}$ has hyperbolic metric with metric signature $(1, -3)$ having M^4 as tangent space so that the earlier picture can be understood as an approximation. This would reduce the construction of preferred extremals of 6-D Kähler action in $T(H)$ to a construction of polynomial holomorphic surfaces and also the minimal surfaces with singularities at string world sheets should result as bundle projection. Since $M^8 - H$ duality must respect algebraic dynamics the maximal degree of the polynomials involved must be same as the degree of the octonionic polynomial in M^8.

2. The hyperbolic variant Kähler form and also spinor connection of hyperbolic CP_2 brings in new physics beyond standard model. This Kähler form would serve as the analog of Kähler form assigned to M^4 earlier, and suggested to explain the observed CP breaking effects and matter antimatter asymmetry for which there are two explanations [L39].

Some comments about the Minkowskian signature of the hyperbolic counterparts of CP_3 and CP_2 are in order.

1. Why the metric of CP_3 could not be Euclidian just as the metric of F? The basic objection is that propagation of fields is not possible in Euclidian signature and one completely loses the earlier picture provided by $M^4 \times CP_2$. The algebraic dynamics in M^8 picture can hardly replace it.

2. The map assigning to the point M^4 a point of CP_3 involves Minkowskian sigma matrices but it seems that the Minkowskian metric of CP_3 is not explicitly involved in the construction of scattering amplitudes. Note however that the antisymmetric bi-spinor metric for the spin 1/2 representation of Lorentz group and its conjugate bring in the signature. $U(2, 2)$ as representation of conformal symmetries suggests $(2, 2)$ signature for 8-D complex twistor space with 2+2 complex coordinates representing twistors.

   The signature of CP_3 metric is not explicitly visible in the construction of twistor amplitudes but analytic continuations are carried out routinely. One has also complexified M^4 and M^8 and one could argue that the problems disappear. In the geometric situation the signatures of the subspaces differ dramatically. As already found, analytic continuation could allow to define the variants of twistor spaces elegantly by replacing a complex coordinate with a hyperbolic one.

   Remark: For E^4 CP_3 is Euclidian and if one has $E^4_{conf} = U(2)$, one could think of replacing the Cartesian product of twistor spaces with $SU(3)$ group having $M^4_{conf} = U(2)$ as fiber and CP_2 as base. The metric of $SU(3)$ appearing as subgroup of quaternionic automorphisms leaving $M^4 \subset M^8$ invariant would decompose to a sum of M^4_{conf} metric and CP_2 metric plus cross terms representing correlations between the metrics of M^4_{conf} and CP_2. This is probably mere accident.

$M^8 - H$ duality and twistor space counterparts of space-time surfaces

It seems that by identifying $CP_{3,h}$ as the twistor space of M^4, one could develop $M^8 - H$ duality to a surprisingly detailed level from the conditions that the dimensional reduction guaranteed by the identification of the twistor spheres takes place and the extensions of rationals associated with the polynomials defining the space-time surfaces at M^8- and twistor space sides are the same. The reason is that minimal surface conditions reduce to holomorphy meaning algebraic conditions involving first partial derivatives in analogy with algebraic conditions at M^8 side but involving no derivatives.

1. The simplest identification of twistor spheres is by $z_1 = z_2$ for the complex coordinates of the spheres. One can consider replacing z_i by its Möbius transform but by a coordinate change the condition reduces to $z_1 = z_2$.

2. At M^8 side one has either $RE(P) = 0$ or $IM(P) = 0$ for octonionic polynomial obtained as continuation of a real polynomial P with rational coefficients giving 4 conditions (RE/IM denotes real/imaginary part in quaternionic sense). The condition guarantees that tangent/normal space is associative.

   Since quaternion can be decomposed to a sum of two complex numbers: $q = z_1 + Jz_2$ $RE(P) = 0$ correspond to the conditions $Re(RE(P)) = 0$ and $Im(RE(P)) = 0$. $IM(P) = 0$ in turn reduces to the conditions $Re(IM(P)) = 0$ and $Im(IM(P)) = 0$.

3. The extensions of rationals defined by these polynomial conditions must be the same as at the octonionic side. Also algebraic points must be mapped to algebraic points so that cognitive representations are mapped to cognitive representations. The counterparts of both $RE(P) = 0$ and $IM(P) = 0$ should be satisfied for the polynomials at twistor side defining the same extension of rationals.

4. $M^8 - H$ duality must map the complex coordinates $z_{11} = Re(RE)$ and $z_{12} = Im(RE)$ ($z_{21} = Re(IM)$ and $z_{22} = Im(IM)$) at M^8 side to complex coordinates u_{i1} and u_{i2} with $u_{i1}(0) = 0$ and $u_{i2}(0) = 0$ for $i = 1$ or $i = 2$, at twistor side.

   Roots must be mapped to roots in the same extension of rationals, and no new roots are allowed at the twistor side. Hence the map must be linear: $u_{i1} = a_i z_{i1} + b_i z_{i2}$ and $u_{i2} = c_i z_{i1} + d_i z_{i2}$ so that the map for given value of i is characterized by SL(2,Q) matrix $(a_i, b_i; c_i, d_i)$.

5. These conditions do not yet specify the choices of the coordinates (u_{i1}, u_{i2}) at twistor side. At CP_2 side the complex coordinates would naturally correspond to Eguchi-Hanson complex coordinates (w_1, w_2) determined apart from color $SU(3)$ rotation as a counterpart of $SU(3)$ as sub-group of automorphisms of octonions.

   If the base space of the twistor space $CP_{3,h}$ of M^4 is identified as $CP_{2,h}$, the hyper-complex counterpart of CP_2, the analogs of complex coordinates would be (w_3, w_4) with w_3 hypercomplex and w_4 complex. A priori one could select the pair (u_{i1}, u_{i2}) as any pair $(w_{k(i)}, w_{l(i)})$, $k(i) \neq l(i)$. These choices should give different kinds of extremals: such as CP_2 type extremals, string like objects, massless extremals, and their deformations.

String world sheet singularitees and world-line singularities as their light-like boundaries at the light-like orbits of partonic 2-surfaces are conjectured to characterize preferred extremals as surfaces of H at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom so that the extremal is not simultaneously an extremal of both Kähler action and volume term as elsewhere. What could be the counteparts of these surfaces in M^8?

1. The interpretation of the pre-images of these singularities in M^8 should be number theoretic and related to the identification of quaternionic imaginary units. One must specify two non-parallel octonionic imaginary units e^1 and e^2 to determine the third one as their cross product $e^3 = e^1 \times e^2$. If e^1 and e^2 are parallel at a point of octonionic surface, the cross product vanishes and the dimension of the quaternionic tangent/normal space reduces from $D = 4$ to $D = 2$.

2. Could string world sheets/partonic 2-surfaces be images of 2-D surfaces in M^8 at which this takes place? The parallelity of the tangent/normal vectors defining imaginary units e_i, $i = 1, 2$ states that the component of e_2 orthogonal to e_1 vanishes. This indeed gives 2 conditions in the space of quaternionic units. Effectively the 4-D space-time surface would degenerate into 2-D at string world sheets and partonic 2-surfacesa as their duals. Note that this condition makes sense in both Euclidian and Minkowskian regions.

3. Partonic orbits in turn would correspond surfaces at which the dimension reduces to D=3 by light-likeness - this condition involves signature in an essential manner - and string world sheets would have 1-D boundaries at partonic orbits.

Getting critical about implicit assumptions related to the twistor space of CP_2

One can also criticize the earlier picture about implicit assumptions related the twistor spaces of CP_2.

1. The possibly singular decomposition of F to a product of S^2 and CP_2 would has a description similar to that for CP_3. One could assign to each point of CP_2 base homologically non-trivial sphere intersecting it orthogonally.

2. I have assumed that the twistor space $T(CP_2) = F = SU(3)/U(1) \times U(1)$ allows Kaluza-Klein type metric meaning that the metric decomposes to a sum of the metrics assignable to the base CP_2 and fiber S^2 plus cross terms representing interaction between these degrees of freedom. It is easy to check that this assumption holds true for Hopf fibration $S^3 \rightarrow S^2$ having circle $U(1)$ as fiber (see `http://tinyurl.com/qbvktsx`). If Kaluza-Klein picture holds true, the metric of F would decompose to a sum of CP_2 metric and S^2 metric plus cross terms representing correlations between the metrics of CP_2 and S^2.

3. One should demonstrate that $F = SU(3)/U(1) \times U(1)$ has metric with the expected Kaluza-Klein property. One can represent $SU(3)$ matrices as products XYZ of 3 matrices. X represents a point of base space CP_2 as matrix, Y represents the point of the fiber $S^2 = U(2)/U(1) \times U(1)$ of F in similar manner as $U(2)$ matrix, and the Z represents $U(1) \times U(1)$ element as diagonal matrix [B3](see `http://tinyurl.com/y6c3pp2g`).

   By dropping $U(1) \times U(1)$ matrix one obtains a coordinatization of F. To get the line element of F in these coordinates one could put the coordinate differentials of $U(1) \times U(1)$ to zero in an expression of $SU(3)$ line element. This should leave sum of the metrics of CP_2 and S^2 with constant scales plus cross terms. One might guess that the left- and righ-invariance of the $SU(3)$ metric under $SU(3)$ implies KK property.

Also CP_3 should have the KK structure if one wants to realize the breaking of scaling invariance as a selection of the scale of the conformally compactified M^4. In absence of KK structure the space-time surface would depend parametrically on the point of the twistor sphere S^2.

4.5.2 The nice results of the earlier approach to M^4 twistorialization

The basic nice results of the earlier picture should survive in the new picture.

1. Central for the entire approach is twistor lift of TGD replacing space-time surfaces with 6-D surfaces in 12-D $T(M^4) \times T(CP_2)$ having space-time surfaces as base and twistor sphere S^2 as fiber. Dimensional reduction identifying twistor spheres of $T(M^4)$ an $T(CP_2)$ and makes these degrees of freedom non-dynamical.

2. Dimensionally reduced action 6-D Kähler action is sum of 4-D Kähler action and a volume term coming from S^2 contribution to the induced Kähler form. On interpretation is as a generalization of Maxwell action for point like charge by making particle a 3-surface.

   The interpretation of volume term is in terms of cosmological constant. I have proposed that a hierarchy of length scale dependent cosmological constants emerges. The hierarchy of cosmological constants would define the running length scale in coupling constant evolution and would correspond to a hierarchy of preferred p-aic length scales with preferred p-adic primes identified as ramified primes of extension of rationals.

3. The twistor spheres associated $M^4 \times S^2$ and F were assumed to have same radii and most naturally same Euclidian signature: this looks very nice since there would be only single fundamental length equal to CP_2 radius determining the radius of its twistor sphere. The vision to be discussed would be different. There would be no fundamental scale and length scales would emerge through the length scale hierarchy assignable to CDs in M^8 and mapped to length scales for twistor spaces.

   The identification of twistor spheres with same radius would give only single value of cosmological constant and the problem of understanding the huge discrepancy between empirical value and its naïve estimate would remain. I have argued that the Kähler forms and metrics of the two twistor spheres can be rotated with respect to each other so that the induced metric and Kähler form are rotated with respect to each other, and the magnetic energy density assignable to the sum of the induced Kähler forms is not maximal.

The definition of Kähler forms involving preferred coordinate frame would gives rise to symmetry breaking. The essential element is interference of real Kähler forms. If the signatures of twistor spheres were opposite, the Kähler forms differ by imaginary unit and the interference would not be possible.

Interference could give rise to a hierarchy of values of cosmological constant emerging as coefficient of the Kähler magnetic action assignable to $S^2(X^4)$ and predict length scale dependent value of cosmological constant and resolve the basic problem related to the extremely small value of cosmological constant.

4. One could criticize the allowance of relative rotation as adhoc: note that the resulting cosmological constant becomes a function depending on S^2 point. For instance, does the rotation really produce preferred extremals as minimal surfaces extremizing also Kähler action except at string world sheets? Each point of S^2 would correspond to space-time surface X^4 with different value of cosmological constant appearing as a parameter. Moreover, non-trivial relative rotation spoils the covariant constancy and $J^2(S^2) = -g(S^2)$ property for the S^2 part of Kähler form, and that this does not conform with the very idea of twistor space.

5. One nice implication would be that space-time surfaces would be minimal surfaces apart from 2-D string world sheet singularities at which there is a transfer of canonical momentum currents between Kähler and volume degrees of freedom. One can also consider the possibility that the minimal surfaces correspond to surfaces give as roots of 3 polynomials of hypercomplex coordinate of M^2 and remaining complex coordinates.

   Minimal surface property would be direct translation of masslessness and conform with the twistor view. Singular surfaces would represent analogs of Abelian currents. The universal dynamics for minimal surfaces would be a counterpart for the quantum criticality. At M^8 level the preferred complex plane M^2 of complexified octonions would represent the singular string world sheets and would be forced by number theory.

   Masslessness would be realized as generalized holomorphy (allowing hyper-complexity in M^2 plane) as proposed in the original twistor approach but replacing holomorphic fields in twistor space with 6-D twistor spaces realized as holomorphic 6-surfaces.

4.5.3 ZEO and twistorialization as ways to introduce scales in M^8 physics

M^8 physics as such has no scales. One motivation for ZEO is that it brings in the scales as sizes of causal diamonds (CDs).

ZEO generates scales in M^8 physics

Scales are certainly present in physics and must be present also in TGD Universe.

1. In TGD Universe CP_2 scale plays the role of fundamental length scale, there is also the length scale defined by cosmological constant and the geometric mean of these two length scales defining a scale of order 10^{-4} meters emerging in the earlier picture and suggesting a biological interpretation.

   The fact that conformal inversion $m^k \rightarrow R^2 m^k/a^2$, $a^2 = m^k m_k$ is a conformal transformation mapping hyperboloids with $a \geq R$ and $a \leq R$ to each other, suggests that one can relate CP_2 scale and cosmological scale defined by Λ by inversion so that cell length scale would define one possible radius of cd_{conf}.

2. In fact, if one has $R(cd_{conf}) = x \times R(CP_2)$ one obtains by repeated inversions a hierarchy $R(k) = x^k R$ and for $x = \sqrt{p}$ one obtains p-adic length scale hierarchy coming as powers of $\sqrt{p}$, which can be also negative. This suggests a connection with p-adic length scale hypothesis and connections between long length scale and short length scale physics: they could be related by inversion. One could perhaps see Universe as a kind of Leibnizian monadic system in which monads reflect each other with respect to hyperbolic surfaces $a = constant$. This would conform with the holography.

3. Without additional assumptions there is a complete scaling invariance at the level of M^8. The scales could come from the choice of 8-D causal diamond CD_8 as intersection of 8-D future and past directed light-cones inducing choice of cd in M^4. CD serves as a correlate for the perceptive field of a conscious entity in TGD inspired theory of consciousness and is crucial element of zero energy ontology (ZEO) allowing to solve the basic problem of quantum measurement theory.

Twistorial description of CDs

Could the map of the surfaces of 4-surfaces of M^8 to $cd_{conf} \times CP_2$ by a modification of $M^8 - H$ correspondence allow to describe these scales? If so, compactification via twistorialization and $M^8 - H$ correspondence would be the manner to describe these scales as something emergent rather than fundamental.

1. The simplest option is that the scale of cd_{conf} corresponds to that of CD_8 and CD_4. One should also understand what CP_2 scale corresponds. The simplest option is that CP_2 scale defines just length unit since it is difficult to imagine how this scale could appear at M^8 level. cd_{conf} scale squared would be multiple or CP_2 scale squared, say prime multiple of it, and assignable to ramified primes of extension of rationals. Inversions would produce further scales. Inversion would allow kind of hologram like representation of physics in long length scales in arbitrary short length scales and vice versa.

2. The compactness of cd_{conf} corresponds to periodic time assignable to over-critical cosmologies starting with big bang and ending with big crunch. Also CD brings in mind over-critical cosmology, and one can argue that the dynamics at the level of cd_{conf} reflects the dynamics of ZEO at the level of M^8.

Modification of H and $M^8 - H$ correspondence

It is often said that the metric of M^4_{conf} is defined only modulo conformal scaling factor. This would reflect projectivity. One can however endow projective space CP_3 with a metric with isometry group $SU(2,2)$ and the fixing of the metric is like gauge choice by choosing representative in the projective equivalence class. Thus CP_3 with signature (3,-3) might perhaps define geometric twistor space with base cd_{conf} rather than M^4_{conf} very much like the twistor space $T(CP_2) = F = SU(3)/U(1) \times U(1)$ at the level. Second projection would be to M^4 and map twistor sphere to a point of M^4. The latter bundle structure would be singular since for points of M^4 with light-like separation the twistor spheres have a common point: this is an essential feature in the construction of twistor amplitudes.

New picture requires a modification of the view about H and about M^8-H correspondence.

1. H would be replaced with $cd_{conf} \times CP_2$ and the corresponding twistor space with $CP_3 \times F$. $M^8 - H$ duality involves the decomposition $M^2 \subset M^4 \subset M^8 = M^4 \times CP_2$, where M^4 is quaternionic sub-space containing preferred place M^2. The tangent or normal space of X^4 would be characterized by a point of CP_2 and would be mapped to a point of CP_2 and the point of CP_2 - or rather point plus the space S^2 or light-like vectors characterizing the choices of M^2 - would mapped to the twistor sphere S^2 of CP_3 by the standard formulas.

   $S^2(cd_{conf})$ would correspond to the choices of the direction of preferred octonionic imaginary unit fixing M^2 as quantization axis of spin and $S^2(CP_2)$ would correspond to the choice of isospin quantization axis: the quantization axis for color hyperspin would be fixed by the choice of quaternionic $M^4 \subset M^8$. Hence one would have a nice information theoretic interpretation.

2. The M^4 point mapped to twistor sphere $S^2(CP_3)$ would be projected to a point of cd_{conf} and define $M^8 - H$ correspondence at the level of M^4. This would define compactification and associate two scales with it. Only the ratio $R(cd_{conf})/R(CP_2)$ matters by the scaling invariance at M^8 level and one can just fixe the scale assignable to $T(CP_2)$ and call it CP_2 length scale.

One should have a concrete construction for the hyperbolic variants of CP_n.

1. One can represent Minkowski space and its variants with varying signatures as sub-spaces of complexified quaternions, and it would seem that the structure of sub-space must be lifted to the level of the twistor space. One could imagine variants of projective spaces CP_n, $n = 2, 3$ as and HP_n, $n = 2, 3$. They would be obtained by multiplying imaginary quaternionic unit I_k with the imaginary unit i commuting with quaternionic units. If the quaternions λ involved with the projectivization $(q_1, ..., q_n) \equiv \lambda(q_1, ..., q_n)$ are ordinary quaternions, the multiplication respects the signature of the subspace. By non-commutativity of quaternions one can talk about left- and right projective spaces.

2. One would have extremely close correspondence between M^4 and CP_2 degrees of freedom reflecting the $M^8 - H$ correspondence. The projection $CP_3 \rightarrow CP_2$ for E^4 would be replaced with the projection for the hyperbolic analogs of these spaces in the case of M^4. The twistor space of M^4 identified as hyperbolic variant of CP_3 would give hyperbolic variant of CP_2 as conformally compactified *cd*. The flag manifold $F = SU(3)/U(1) \times U(1)$ as twistor space of CP_2 would also give CP_2 as base space.

The general solution of field equations at the level of $T(H)$ would correspond to holomorphy in general sense for the 6-surfaces defined by 3 vanishing conditions for holomorphic functions - 6 real conditions. Effectively this would mean the knowledge of the exact solutions of field equations also at the level of H: TGD would be an integrable theory. Scattering amplitudes would in turn constructible from these solutions using ordinary partial differential equations [L39].

1. The first condition would identify the complex coordinates of $S^2(cd_{conf})$ and $S^2(CP_2)$: here one cannot exclude relative rotation represented as a holomorphic transformation but for $R(cd_{conf}) \gg R(CP_2)$ the effect of the rotation is small.

2. Besides this there would be vanishing conditions for 2 holomorphic polynomials. The coordinate pairs corresponding to $M^2 \subset M^4$ would correspond to hypercomplex behavior with hyper complex coordinate $u = \pm t - z$. t and z could be assigned with $U(1)$ fibers of Hopf fibrations $SU(2) \rightarrow S^2$.

3. The octonionic polynomial $P(o)$ of degree $n = h_{eff}/h_0$ with rational coefficients fixes the extension of rationals and since the algebraic extension should be same at both sides, the polynomials in twistor space should have same degree. This would give enormous boos concerning the understanding of the proposed cancellation of fermionic Wick contractions in SUSY scattering amplitudes forced by number theoretic vision [L39].

Possible problems related to the signatures

The different signatures for the metrics of the twistor spheres of cd_{conf} and CP_2 can pose technical problems.

1. Twistor lift would replace X^4 with 6-D twistor space X^6 represented as a 6-surface in $T(M^4) \times T(CP_2)$. X^6 is defined by dimensional reduction in which the twistor spheres $S^2(cd_{conf})$ and $S^2(CP_2)$ are identified and define the twistor sphere $S^2(X^4)$ of X^6 serving as a fiber whereas space-time surface X^4 serves as a base. The simplest identification is as $(\theta, \phi)_{S^2(M^4)} = (\theta, \phi)_{S^2(CP_2)}$: the same can be done for the complex coordinates $z_{S^2(M^4_{conf})} = z_{S^2(CP_2))}$). An open question is whether a Möbius transformation could relate the complex coordinates. The metrics of the spheres are of opposite sign and differ only by the scaling factors $R^2(cd_{conf})$ and $R^2(CP_2)$.

2. For cd_{conf} option the signatures of the 2 twistor spheres would be opposite (time-like for cd_{conf}). For $R(cd_{conf})/R(CP_2) = 1$. $J^2 = -g$ is the only consistent option unless the signature of space is not totally positive or negative and implies that the Kähler forms of the two twistor spheres differ by i. The magnetic contribution from $S^2(X^4)$ would give rise to an infinite value of cosmological constant proportional to $1/\sqrt{g_2}$, which would diverge $R(cd_{conf})/R(CP_2) = 1$. There is however no need to assume this condition as in the original approach.

4.5.4 Hierarchy of length scale dependent cosmological constants in twistorial description

At the level of M^8 the hierarchy of CDs defines a hierarchy of length scales and must correspond to a hierarchy of length scale dependent cosmological constants. Even fundamental scales would emerge.

1. If one has $R(cd_{conf})/R(CP_2) >> 1$ as the idea about macroscopic cd_{conf} would suggest, the contribution of $S^2(cd_{conf})$ to the cosmological constant dominates and the relative rotation of metrics and Kähler form cannot affect the outcome considerably. Therefore different mechanism producing the hierarchy of cosmological constants is needed and the freedom to choose rather freely the ratio $R(cd_{conf})/R(CP_2)$ would provide the mechanism. What looked like a weakness would become a strength.

2. $S^2(cd_{conf}$ would have time-like metric and could have large scale. Is this really acceptable? Dimensional reduction essential for the twistor induction however makes $S^2(cd_{conf})$ non-dynamical so that time-likeness would not be visible even for large radii of $S^2(cd_{conf})$ expected if the size of cd_{conf} can be even macroscopic. The corresponding contribution to the action as cosmological constant has the sign of magnetic action and also Kähler magnetic energy is positive. If the scales are identical so that twistor spheres have same radius, the contributions to the induced metric cancel each other and the twistor space becomes metrically 4-D.

3. At the limit $R(cd_{conf}) \rightarrow RCP_2)$ cosmological constant coming from magnetic energy density diverges for $J^2 = -G$ option since it is proportional to $1/\sqrt{g_2}$. Hence the scaling factors must be different. The interpretation is that cosmological constant has arbitrarily large values near CP_2 length scale. Note however that time dependence is replaced with scale dependence and space-time sheets with different scales have only wormhole contacts.

It would seem that this approach could produce the nice results of the earlier approach. The view about how the hierarchy of cosmological constants emerges would change but the idea about reducing coupling constant evolution to that for cosmological constant would survive. The interpretation would be in terms of the breaking of scale invariance manifesting as the scales of CDs defining the scales for the twistor spaces involved. New insights about p-adic coupling constant evolution emerge and one finds a new "must" for ZEO. $H = M^4 \times CP_2$ picture would emerge as an approximation when cd_{conf} is replaced with its tangent space M^4. The consideration of the quaternionic generalization of twistor space suggests natural identification of the conformally compactified twistor space as being obtained from CP_2 by making second complex coordinate hyperbolic. This need not conform with the identification as $U(2)$.

4.6 How to generalize twistor Grassmannian approach in TGD framework?

One should be able to generalize twistor Grassmannian approach in TGD framework. The basic modification is replacement of 4-D light-like momenta with their 8-D counterparts. The octonionic interpretation encourages the idea that twistor approach could generalize to 8-D context. Higher-dimensional generalizations of twistors have been proposed but the basic problem is that the index raising and lifting operations for twistors do not generalize (see `http://tinyurl.com/y24lkwce`).

1. For octonionic twistors as pairs of quaternionic twistors index raising would not be lost working for M_T option and light-like M^8 momenta can be regarded sums of M^4_T and E^4 parts as also twistors. Quaternionic twistor components do not commute and this is essential for incidence relation requiring also the possibility to raise or lower the indices of twistors. Ordinary complex twistor Grassmannians would be replaced with their quaternionic countparts. The twistor space as a generalization of CP_3 would be 3-D quaternionic projective space $T(M^8) = HP_3$ with Minkowskian signature (6,6) of metric and having real dimension 12 as one might expect.

Another option realizing non-commutativity could be based on the notion of quantum twistor to be also discussed.

2. Second approach would rely on the identification of $M^4 \times CP_2$ twistor space as a Cartesian product of twistor spaces of M^4 and CP_2. For this symmetries are not broken, $M^4_L \subset M^8$ depends on the state and is chosen so that the projection of M^8 momentum is light-like so that ordinary twistors and CP_2 twistors should be enough. $M^8 - H$ relates varying M^4_L based and M^4_T based descriptions.

3. The identification of the twistor space of M^4 as $T(M^4) = M^4 \times S^2$ can be motivated by octonionic considerations but might be criticized as non-standard one. The fact that quaternionic twistor space HP_3 looks natural for M^8 forces to ask whether $T(M^4) = CP_3$ endowed with metric having signature (3,3) could work in the case of M^4. In the sequel also a vision based on the identification $T(M^4) = CP_3$ endowed with metric having signature (3,3) will be discussed.

4.6.1 Twistor lift of TGD at classical level

In TGD framework twistor structure is generalized [K77, L26, K8, L30]. The inspiration for TGD approach to twistorialization has come from the work of Nima Arkani-Hamed and colleagues [B20, B14, B15, B17, B36, B21, B7]. The new element is the formulation of twistor lift also at the level of classical dynamics rather than for the scattering amplitudes only [K77, K8, L26, L30].

1. The 4-D light-like momenta in ordinary twistor approach are replaced by 8-D light-like momenta so that massive particles in 4-D sense become possible. Twistor lift of TGD takes places also at the space-time level and is geometric counterpart for the Penrose's replacement of massless fields with twistors. Roughly, space-time surfaces are replaced with their 6-D twistor spaces represented as 6-surfaces. Space-time surfaces as preferred extremals are minimal surfaces with 2-D string world sheets as singularities. This is the geometric manner to express masslessness. X^4 is simultaneously also extremal of 4-D Kähler action outside singularities: this realizes preferred extremal property.

2. One can say that twistor structure of X^4 is induced from the twistor structure of $H = M^4 \times CP_2$, whose twistor space $T(H)$ is the Cartesian product of geometric twistor space $T(M^4) = M^4 \times CP_1$ and $T(CP_2) = SU(3)/U(1) \times U(1)$. The twistor space of M^4 assigned to momenta is usually taken as a variant of CP_3 with metric having Minkowski signature and both CP_1 fibrations appear in the more precise definition of $T(M^4)$. Double fibration [B34] (see `http://tinyurl.com/yb4bt74l`) means that one has fibration from $M^4 \times CP_1$ - the trivial CP_1 bundle defining the geometric twistor space to the twistors space identified as complex projective space defining conformal compactification of M^4. Double fibration is essential in the twistorialization of TGD [K29].

3. The basic objects in the twistor lift of classical TGD are 6-D surfaces in $T(H)$ having the structure of twistor space in the sense that they are CP_1 bundles having X^4 as base space. Dimensional reduction to CP_1 bundle effectively eliminates the dynamics in CP_1 degrees of freedom and its only remnant is the value of cosmological constant appearing as coefficient of volume term of the dimensionally reduced action containing also 4-D Kähler action. Cosmological term depends on p-adic length scales and has a discrete spectrum [L30, L29].

CP_1 has also an interpretation as a projective space constructed from 2-D complex spinors. Could the replacement of these 2-spinors with their quantum counterparts defining in turn quantum CP_1 realize finite quantum measurement resolution in M^4 degrees of freedom? Projective invariance for the complex 2-spinors would mean that one indeed has effectively CP_1.

4.6.2 Octonionic twistors or quantum twistors as twistor description of massive particles

For M^4_T option the particles are massive and the one encounters the problem whether and how to generalize the ordinary twistor description.

4.6.3 Basic facts about twistors and bi-spinors

It is convenient to start by summarizing the basic facts about bi-spinors and their conjugates allowing to express massless momenta as $p^{aa'} = \lambda^a \tilde{\lambda}^{a'}$ with $\tilde{\lambda}$ defined as complex conjugate of λ and having opposite chirality (see `http://tinyurl.com/y6bnznyn`).

1. When λ is scaled by a complex number $\tilde{\lambda}$ suffers an opposite scaling. The bi-spinors allow the definition of various inner products

$$\begin{aligned} \langle \lambda, \mu \rangle &= \epsilon_{ab}\lambda^a \mu^b \ , \\ \left[\tilde{\lambda}, \tilde{\mu}\right] &= \epsilon_{a'b'}\tilde{\lambda}^{a'}\tilde{\mu}^{b'} \ , \\ p\cdot q &= \langle \lambda, \mu \rangle \left[\tilde{\lambda}, \tilde{\mu}\right] \ , \ \ (q_{aa'} = \mu_a \tilde{\mu}_{a'}) \ . \end{aligned} \tag{4.6.1}$$

2. Spinor indices are lowered and raised using antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. If the particle has spin one can assign it a positive or negative helicity $h = \pm 1$. Positive helicity can be represented by introducing artitrary negative (positive) helicity bispinor μ_a ($\mu_{a'}$) not parallel to λ_a ($\mu_{a'}$) so that one can write for the polarization vector

$$\begin{aligned} \epsilon_{aa'} &= \frac{\mu_a \tilde{\lambda}_{a'}}{\langle \mu, \lambda \rangle} \ , \ \text{positive helicity} \ , \\ \epsilon_{aa'} &= \frac{\lambda_a \tilde{\mu}_{a'}}{\left[\tilde{\mu}, \tilde{\lambda}\right]} \ , \ \text{negative helicity} \ . \end{aligned} \tag{4.6.2}$$

   In the case of momentum twistors the μ part is determined by different criterion to be discussed later.

3. What makes 4-D twistors unique is the existence of the index raising and lifting operations using ϵ tensors. In higher dimensions they do not exist and this causes difficulties. For octonionic twistors with quaternionic components possibly only in $D = 8$ the situation changes.

To get a very rough idea about twistor Grassmannian approach idea, consider tree amplitudes of $\mathcal{N} = 4$ SUSY as example and it is convenient to drop the group theory factor $Tr(T_1 T_2 \cdots T_n)$. The starting point is the observation that tree amplitude for which more than $n-2$ gluons have the same helicity vanish. MHV amplitudes have exactly $n-2$ gluons of same helicity- taken by a convention to be negative- have extremely simple form in terms of the spinors and reads as

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle} \tag{4.6.3}$$

When the sign of the helicities is changed $\langle .. \rangle$ is replaced with [..].

An essential point in what follows is that the amplitudes are expressible in terms of the antisymmetric bi-linears $\langle \lambda_i, \lambda_j \rangle$ making sense also for octotwistors and identifiable as quaternions rather than octonions.

$M^8 - H$ duality and two alternative twistorializations of TGD

$M^8 - H$ duality suggests two alternative twistorializations of TGD.

1. The first approach would be in terms of M^8 twistors suggested by quaternionic light-lineness of 8-momenta. M^8 twistors would be Cartesian products of M^4 and E^4 twistors. One can imagine a straightforward generalization of twistor scattering amplitudes in terms of generalized Grassmannian approach replacing complex Grassmannian with quaternionic Grassmannian, which is a mathematically well-defined notion.

2. Second approach would rely on $M^4 \times CP_2$ twistors, which are products of M^4 twistors and CP_2 twistors: this description works nicely at classical space-time level but at the level of momentum space the problem is how to describe massivation of M^4 momenta using twistors.

Why the components of twistors must be non-commutative?

How to modify the 4-D twistor description of light-like 4-momenta so that it applies to massive 4-momenta?

1. Twistor consists of a pair $(\mu_{\dot{\alpha}}, \lambda^{\alpha})$ of bi-spinors in conjugate representations of $SU(2)$. One can start from the 4-D incidence relations for twistors

$$\mu_{\dot{\alpha}} = p_{\alpha\dot{\alpha}}\lambda^{\alpha} \quad .$$

Here $p_{\alpha\dot{\alpha}}$ denotes the representation of four-momentum $p^k\sigma_k$. The antisymmetric permutation symbols $\epsilon^{\alpha\beta}$ and its dotted version define antisymmetric "inner product" in twistor space. By taking the inner product of μ with itself, one obtains the commutation relation $\mu_1\mu_2 - \mu_2\mu_1 = 0$, which is consistent with right-hand side for massless particles with $p_kp^k = 0$.

2. In TGD framework particles are massless only in 8-D sense so that the right hand side in the contraction is in general non-vanishing. In massive case one can replace four-momentum with unit vector. This requires

$$\langle \mu 1, \mu_2\rangle = \mu_1\mu_2 - \mu_2\mu_1 \neq 0 \quad .$$

The components of 2-spinor become non-commutative.

This raises two questions.

1. Could the replacement of complex twistors by quaternionic twistors make them non-commutative and allow massive states?

2. Could non-commutative quantum twistors solve the problem caused by the light-likeness of momenta allowing 4-D twistor description?

Octotwistors or quantum twistors?

One should be able to generalize twistor amplitudes and twistor Grassmannian approach to TGD framework, where particles are massless in 8-D sense and massive in 4-D sense. Could twistors be replaced by octonionic or quantum twistors.

1. One can express mass squared as a product of commutators of components of the twistors λ and $\tilde{\lambda}$, which is essentially the conjugate of λ:

$$p \cdot p = \langle \lambda, \lambda\rangle \left[\tilde{\lambda}, \tilde{\lambda}\right] \quad . \tag{4.6.4}$$

This operator should be non-vanishing for non-vanishing mass squared. Both terms in the product vanish unless commutativity fails so that mass vanishes. The commutators should have the quantum state as its eigenstate.

2. Also 4-momentum components should have well-defined values. Four-momentum has expression $p^{aa'} = \lambda^a\tilde{\lambda}^{a'}$ in massless case. This expression should generalized to massive case as such. Eigenvalue condition and reality of the momentum components requires that the components $p^{aa'}$ are commuting Hermitian operators.

In twistor Grassmannian approach complex but light-like momenta are possible as analogs of virtual momenta. Also in TGD framework the complexity of Kähler coupling strength allows to consider complex momenta. For twistor lift they however differ from real momenta only by a phase factor associated with the $1/\alpha_K$ associated with 6-D Kähler action.

Remark: I have considered also the possibility that states are eigenstates only for the longitudinal M^2 projection of 4-momentum with quark model of hadrons serving as a motivation.

(a) Could this equation be obtained in massive case by regarding λ^a and $\tilde{\lambda}^{a'}$ as commuting octo-spinors and their complex conjugates? Octotwistors would naturally emerge in the description at embedding space level. I have already earlier considered the notion of octotwistor [K73] [L18]).

(b) Or could it be obtained for quantum bi-spinors having same states as eigenstates. Could quantum twistors as generalization of the ordinary twistors correspond to the reduction of the description from the level of M^8 or H to at space-time level so that one would have 4-D twistors and massive particles with 4-momentum identifiable as Noether charge for the action principle determining preferred extremals? I have considered also the notion of quantum spinor earlier [K28, K48, K43, K1, K84].

3. In the case of quantum twistors the generalization of the product of the quantities $\langle\lambda_i,\lambda_{i+1}\rangle$ appearing in the formula should give rise to c-number in the case of quantum spinors. Can one require that the quantities $\langle\lambda_i,\lambda_{i+1}\rangle$ or even $\langle\lambda_i,\lambda_j\rangle$ are c-numbers simultaneously? This would also require that $\langle\lambda,\lambda\rangle$ is non-vanishing c-number in massive case: also incidence relation suggest this condition. Could one think λ as an operator such that $\langle\lambda,\lambda\rangle$ has eigenvalue spectrum corresponding to the quantities $\langle\lambda_i,\lambda_{i+1}\rangle$ appearing in the scattering amplitude?

4.6.4 The description for M^4_T option using octo-twistors?

For option I with massive M^4_T projection of 8-momentum one could imagine twistorial description by using M^8 twistors as products of M^4_T and E^4 twistors, and a rather straightforward generalization of standard twistor Grassmann approach can be considered.

Could twistor Grassmannians be replaced with their quaternionic variants?

The first guess would simply replace $Gr(k,n)$ with $Gr(2k,2n)$ 4-D twistors 8-D twistors. From twistor amplitudes with quaternionic M^8-momenta one could construct physical amplitudes by going from 8-momentum basis to the 4-momentum- basis with wave functions in irreps of $SO(3)$. Life is however not so simple.

1. The notion of ordinary twistor involves in an essential manner Pauli matrices σ_i satisfying the well-known anti-commutation relations. They should be generalized. In fact, σ_0 and $\sqrt{-1}\sigma_i$ can be regarded as a matrix representation for quaternionic units. They should have analogs in 8-D case.

   Octonionic units ie_i indeed provide this analog of sigma matrices. Octonionic units for the complexification of octonions allow to define incidence relation and representation of 8-momenta in terms of octo-spinors. They do not however allow matrix representation whereas time-like octonions allow interpretation as quaternion in suitable bases and thus matrix representation. Index raising operation is essential for twistors and makes dimension $D=4$ very special. For naïve generalizations of twistors to higher dimensions this operation is lost (see `http://tinyurl.com/y24lkwce`).

2. Could one avoid multiplication of more than two octo-twistors in Grassmann amplitudes leading to difficulties with associativity. An important observation is that in the expressions for the twistorial scattering amplitudes only products $\langle\lambda_i,\lambda_j\rangle$ or $[\tilde{\lambda}_i,\tilde{\lambda}_{i+1}]$ but not both occur. These products are associative even if the spinors are replaced by quaternionic spinors.

   These operations are antisymmetric in the arguments, which suggests cross product for quaternions giving rise to imaginary quaternion so that the product of objects would give rise to a product of imaginary quaternions. This might be a problem since a large number of terms in the product would approach to zero for random imaginary quaternions.

An ad hoc guess would be that scattering probability is proportional to the product of amplitude as product $\langle\lambda_i, \lambda_j\rangle$ and its "hermitian conjugate" with the conjugates $[\tilde{\lambda}_i, \tilde{\lambda}_{i+1}]$ in the reverse order (this does not affect the outcome) so that the result would be real. Scattering amplitude would be more like quaternion valued operator. Could one have a formulation of quantum theory or at least TGD view about quantum theory allowing this?

3. If ordinary massless 4-momenta correspond to quaternionic sigma matrices, twistors can be regarded as pairs of 2-spinors in matrix representation. Octonionic 8-momenta should correspond to pairs of 4-spinors. As already noticed, octonions do not however allow matrix representation! Octonions for a fixed decomposition $M^8 = M^4 \times E^4$ can be however decomposed to linear combination of two quaternions just like complex numbers to a combination of real numbers. These quaternions would have matrix representation and quaternionic analogs of twistor pair $(\mu, \tilde{\lambda})$. One could perhaps formulate the generalization of twistor Grassmann amplitudes using these pairs. This would suggest replacement of complex bi-spinors with complexified quaternions in the ordinary formalism. This might allow to solve problems with associativity if only $\langle\lambda_i, \lambda_j\rangle$ or $[\tilde{\lambda}_i, \tilde{\lambda}_{i+1}]$ appear in the amplitudes.

4. The argument in the momentum conserving delta function $\delta(\lambda_i\tilde{\lambda}_i)$ should be real so that the conjugation with respect to i would not change the argument and non-commutativity would not be problem. In twistor Grassmann amplitudes the argument $C \cdot Z$ of delta momentum conserving function is linear in the components of complex twistor Z. If complex twistor is replaced with quaternionic twistor, the Grassmannian coordinates C in delta functions $\delta(C \cdot Z)$ must be replaced with quaternionic one.

The replacement of complex Grassmannians $Gr_C(k, n)$ with quaternionic Grassmannians $Gr_H(k, n)$ is therefore highly suggestive. Quaternionic Grassmannians (see `http://tinyurl.com/y23jsffn`) are quotients of symplectic Lie groups $Gr_H(k, n) = U_n(H)/(U_r(H) \times U_{n-r}(H))$ and thus well-defined. In the description using $Gl_H(k, n)$ matrices the matrix elements would be quaternions and $k \times k$ minors would be quaternionic determinants.

Remark: Higher-D projective spaces of octonions do not exist so that in this sense dimension $D = 8$ for embedding space would be maximal.

Twistor space of M^8 as quaternionic projective space HP_3?

The simplest Grassmannian corresponds to twistor space and one can look what one obtains in this case. One can also try to understand how to cope with the problems caused by Minkowskian signature.

1. In previous section it was found that the modification of H to $H = cd_{conf} \times CP_2$ with $cd_{conf} = CP_{2,h}$ identifiable as CP_2 with Minkowskian signature of metric is strongly suggestive.

2. For E^8 quaternionic twistor space as analog of CP_3 would be its quaternionic variant HP_3 with expected dimension $D = 16 - 4 = 12$. Twistor sphere would be replaced with its quaternionic counterpart $SU(2)_H/U(1)_H$ having dimension 4 as expected. $CD_{8,conf}$ as conformally compactified CD_8 must be 8-D. The space HP_2 has dimension 8 and is analog of CP_2 appearing as analog of base space of CP_3 identified as conformally compactified 4-D causal diamond cd_{conf}. The quaternionic analogy of $M^4_{conf} = U(2)$ identified as conformally compactified M^4 would be $U(2)_H$ having dimension $D = 10$ rather than 8.

   HP_3 and HP_2 might work for E^8 but it seems that the 4-D analog of twistor sphere should have signature (2,-2) whereas base space should have signature (1,-7). Some kind of hyperbolic analogs of these spaces obtained by replacing quaternions with their hypercomplex variant seem to be needed. The same receipe in the twistorialization of M^4 would give cd_{conf} as analog of CP_2 with second complex coordinate made hyperbolic. I have already considered the construction of hyperbolic analogs of CP_2 and CP_3 as projective spaces. These results apply to HP_2 and HP_3.

3. What about octonions? Could one define octonionic projective plane OP_2 and its hyperbolic variants corresponding to various sub-spaces of M^8? Euclidian OP_2 known as Cayley plane exists as discovered by Ruth Moufang in 1933. Octonionic higher-D projective spaces and Grassmannians do not however exist so that one cannot assign OP_3 as twistor spaces.

Can one obtain scattering amplitudes as quaternionic analogs of residue integrals?

Can one obtain complex valued scattering amplitudes (i commuting with octonionic units) in this framework?

1. The residue integral over quaternionic C-coordinates should make sense, and pick up the poles as vanishing points of minors. The outcome of repeated residue integrations should give a sum over poles with complex residues.

2. Residue calculus requires analyticity. The problem is that quaternion analyticity based on a generalization of Cauchy-Riemann equations allows only linear functions. One could define quaternion (and octonion) analyticity in restricted sense using powers series with real coefficients (or in extension involving i commuting with octonion units). The quaternion/octonion analytic functions with real coefficients are closed with respect to sum and product. I have used this definition in the proposed construction of algebraic dynamics for in $X^4 \subset M^8$ [L18].

3. Could one define the residue integral purely algebraically? Could complexity of the coefficients (i) force complex outcome: if pole q_0 is not quaternionically real the function would not allow decompose to $f(q)/(q-q_0)$ with f allowing similar Taylor series at pole. If so, then the formulas of Grassmannian formalism could generalize more or less as such at M^8 level and one could map the predictions to predictions of $M^4 \times CP_2$ approach by analog of Fourier transform transforming these quantum state basis to each other.

This option looks rather interesting and involves the key number theoretic aspects of TGD in a crucial manner.

4.6.5 Do super-twistors make sense at the level of M^8?

By $M^8 - H$ duality [L18] there are two levels involved: M^8 and H. These levels are encountered both at the space-time level and momentum space level. Do super-octonions and super-twistors make sense at M^8 level?

1. At the level of M^8 the high uniqueness and linearity of octonion coordinates makes the notion of super-octonion natural. By $SO(8)$ triality octonionic coordinates (bosonic octet 8_0), octonionic spinors (fermionic octet 8_1), and their conjugates (anti-fermionic octet 8_{-1}) would for triplet related by triality. A possible problem is caused by the presence of separately conserved B and L. Together with fermion number conservation this would require $\mathcal{N}=4$ or even $\mathcal{N}=4$ SUSY, which is indeed the simplest and most beautiful SUSY.

2. At the level of the 8-D momentum space octonionic twistors would be pairs of two quaternionic spinors as a generalization of ordinary twistors. Super octo-twistors would be obtained as generalization of these.

The progress in the understanding of the TGD version of SUSY [L39] led to a dramatic progress in the understanding of super-twistors.

1. In non-twistorial description using space-time surfaces and Dirac spinors in H, embedding space coordinates are replaced with super-coordinates and spinors with super-spinors. Theta parameters are replaced with quark creation and annihilation operators. Super-coordinate is a super-polynomial consisting of monomials with vanishing total quark number and appearing in pairs of monomial and its conjugate to guarantee hermiticity.

   Dirac spinor is a polynomial consisting of powers of quark creation operators multiplied by monomials similar to those appearing in the super-coordinate. Anti-leptons are identified as spartners ofquarks identified as local 3-quark states. The multi-spinors appearing in the expansions describe as such local many-quark-antiquark states so that super-symmetrization means also second quantization. Fermionic and bosonic states assignable to H-geometry interact since super-Dirac action contains induced metric and couplings to induced gauge potentials.

2. The same recipe works at the level of twistor space. One introduces twistor super-coordinates analogous to super-coordinates of H and M^8. The super YM field of $\mathcal{N}=4$ SUSY is replaced with super-Dirac spinor in twistor space. The spin degrees of freedom associated with twistor spheres S^2 would bring in 2 additional spin-like degrees of freedom.

   The most plausible option is that the new spin degrees are frozen just like the geometric S^2 degrees of freedom. The freezing of bosonic degrees of freedom is implied by the construction of twistor space of X^4 by dimensional reduction as a 6-D surface in the product of twistor spaces of M^4 and CP_2. Chirality conditions would allow only single spin state for both spheres.

3. Number theoretical vision implies that the number of Wick contractions of quarks and antiquarks cannot be larger than the degree of the octonionic polynomial, which in turn should be same as that of the polynomials of twistor space giving rise to the twistor space of space-time surface as 6-surface. The resulting conditions correspond to conserved currents identifiable as Noether currents assignable to symmetries.

Also Grassmannian is replaced with super-Grassmannian and super-coordinates as matrix elements of super matrices are introduced.

1. The integrand of the Grassmannian integral defining the amplitude can be expanded in Taylor series with respect to theta parameters associated with the super coordinates C as rows of super $G(k,n)$ matrix.

2. The delta function $\delta(C,Z)$ factorizing into a product of delta functions is also expanded in Taylor series to get derivatives of delta function in which only coordinates appear. By partial integration the derivatives acting on delta function are transformed to derivatives acting on integrand already expanded in Taylor series in theta parameters. The integration over the theta parameters using the standard rules gives the amplitudes associated with different powers of theta parameters associated with Z and from this expression one can pick up the scattering amplitudes for various helicities of external particles.

The super-Grassmannian formalism is extremely beautiful but one must remember that one is dealing with quantum field theory. It is not at all clear whether this kind of formalism generalizes to TGD framework, where particle are 3-surfaces [L18]. The notion of cognitive representation effectively reducing 3-surfaces to a set of point-like particles strongly suggests that the generalization exists.

The progress in understanding of M^8-H duality throws also light to the problem whether SUSY is realized in TGD and what SUSY breaking does mean. It seems now clear that sparticles are predicted and SUSY remains in the simplest scenario exact but that p-adic thermodynamics causes thermal massivation: unlike Higgs mechanism, this massivation mechanism is universal and has nothing to do with dynamics. This is due to the fact that zero energy states are superpositions of states with different masses. The selection of p-adic prime characterizing the sparticle causes the mass splitting between members of super-multiplets although the mass formula is same for all of them.

The increased undestanding of what twistorialization leads to an improved understanding of what twistor space in TGD could be. It turns out that the hyperbolic variant $CP_{3,h}$ of the standard twistor space CP_3 is a more natural identification than the earlier $M^4\times S^2$ also in TGD framework but with a scale corresponding to the scale of CD at the level of M^8 so that one obtains a scale hierarchy of twistor spaces. Twistor space has besides the projection to M^4 also a bundle projection to the hyperbolic variant $CP_{2,h}$ of CP_2 so that a remarkable analogy between M^4 and CP_2 emerges. One can formulate super-twistor approach to TGD using the same formalism as will be discussed in this article for the formulation at the level of H. This requires introducing besides 6-D Kähler action and its super-variant also spinors and their super-variants in super-twistor space. The two formulations are equivalent apart from the hierarchy of scales for the twistor space. Also M^8 allows analog of twistor space as quaternionic Grassmannian HP_3 with signature (6,6). What about super- variant of twistor lift of TGD? consider first the situation before the twistorialization.

1. The parallel progress in the understanding SUSY in TGD framework [L39] leads to the identification of the super-counterparts of M^8, H and of twistor spaces modifying dramatically

the physical interpretation of SUSY. Super-spinors in twistor space would provide the description of quantum states. Super-Grassmannians would be involved with the construction of scattering amplitudes. Quaternionic super Grassmannians would be involved with M^8 description.

2. In fermionic sector only quarks are allowed by $SO(1,7)$ triality and that anti-leptons are local 3-quark composites identifiable as spartners of quarks. Gauge bosons, Higgs and graviton would be also spartners and assignable to super-coordinates of embedding space expressible as super-polynomials of quark oscillator operators. Super-symmetrization means also quantization of fermions allowing local many-quark states.

3. SUSY breaking would be caused by the same universal mechanism as ordinary massivation of massless states. The mass formulas would be supersymmetric but the choice of p-adic prime identifiable as ramified prime of extension of rationals would depend on the state of super-multiplet. ZEO would make possible symmetry breaking without symmetry breaking as Wheeler might put it.

Super-counterpart of twistor lift using the proposed formalism

The construction of super-coordinates and super-spinors [L39] suggests a straightforward formulation of the super variant of twistor lift . One should only replace the super-embedding space and super-spinors with super-twistor space and corresponding super-spinors and formulate the theory using 6-D super-Kähler action and super-Dirac equation and the same general prescription for constructing S-matrix. Dimensional reduction should give essentially the 4-D theory apart from the variation of the radius of the twistor space predicting variation of cosmological constant. The size scale of CD would correspond to the size scale of the twistor space for M^4 and for CP_2 the size scale would serve as unit and would not vary.

The first step is the construction of ordinary variant of Kähler action and modified Dirac action for 6-D surfaces in 12-D twistor space.

1. Replace the spinors of H with the spinors of 12-D twistor space and assume only quark chirality. By the bundle property of the twistor space one can express the spinors as tensor products of spinors of the twistor spaces $T(M^4)$ and $T(CP_2)$. One can express the spinors of $T(M^4)$ tensor products of spinors of M^4 - and S^2 spinors locally and spinors of $T(CP_2)$ as tensor products of CP_2 - and S^2 spinors locally. Chirality conditions should reduce the number of 2 spin components for both $T(M^4)$ and $T(CP_2)$ to one so that there are no additional spin degrees of freedom.

   The dimensional reduction can be generalized by identifying the two S^2 fibers for the preferred extremals so that one obtains induced twistor structure. In spinorial sector the dimensional reduction must identify spinorial degrees of freedom of the two S^2s by the proposed chirality conditions also make them non-dynamical. The S^2 spinors covariantly constant in S^2 degrees of freedom.

2. Define the spinor structure of 12-D twistor space, define induced spinor structure at 6-D surfaces defining the twistor space of space-time surface. Define the twistor counterpart of the analog of modified Dirac action using same general formulas as in case of H.

Construct next the super-variant of this structure.

1. Introduce second quark oscillator operators labelled by the points of cognitive representation in 12-D twistor space effectively replacing 6-D surface with its discretization and having quantized quark field q as its continuum counterpart. Replace the coordinates of the 12-D twistor space with super coordinates h_s expressed in terms of quark and anti-quark oscillator operators labelled by points of cognitive representation, and having interpretation as quantized quark field q restricted to the points of representation.

2. Express 6-D Kähler action and Dirac action density in terms of super-coordinates h_s. The local monomials of q appear in h_s and therefore also in the expansion of super-variants of modified gamma matrices defined by 6-D ähler action as contractions of canonical momentum

currents of the action density L_K with the gamma matrices of 12-D twistor space. In super-Kähler action also the local composites of q giving rise to currents formed from the local composites of 3-quarks and antiquarks and having interpretation as leptons and anti-leptons occur - leptons would be therefore spartners of squarks.

3. Perform super-expansion also for the induced spinor field q_s in terms of monomials of q. $q_s(q)$ obeys super-Dirac equation non-linear in q. But also q should satisfy super-Dirac action as an analog of quantized quark field and non-linearity indeed forces also q to have has super-expansion. Thus both quark field q and super-quark field q_s both satisfy super-Dirac equation.

   The only possibility is $q_s = q$ stating fixed point property under $q \rightarrow q_s$ having interpretation in terms of quantum criticality fixing the values of the coefficients of various terms in q_s and in the super-coordinate h_s having interpretation as coupling constants. One has quantum criticality and discrete coupling constant evolution with respect to extension of rationals characterizing adelic physics.

4. Super-Dirac action vanishes for its solutions and the exponent of super-action reduces to exponent of super-Kähler action, whose matrix elements between positive and negative energy parts of zero energy states give S-matrix elements.

   Super-Dirac action has however an important function: the derivatives of quark currents appearing in the super-Kähler action can be transformed to a linear strictly local action of super spinor connection ($\partial_\alpha \rightarrow A_{\alpha,s}$ effectively). Without this lattice discretization would be needed and cognitive representation would not be enough.

To sum up, the super variants of modified gamma matrices of the 6-surface would satisfy the condition $D_{\alpha,s}\Gamma^\alpha_s = 0$ expressing preferred extremal property and guaranteeing super-hermicity of D_s. q_s would obey super-Dirac equation $D_s q_s = 0$. The self-referential identification $q = q_s$ would express quantum criticality of TGD.

4.7 Could one describe massive particles using 4-D quantum twistors?

The quaternionic generalization of twistors looks almost must. But before this I considered also the possibility that ordinary twistors could be generalized to quantum twistors to describe particle massivation. Quantum twistors could provide space-time level description, which requires 4-D twistors, which cannot be ordinary M^4 twistors. Also the classical 4-momenta, which by QCC would be equal to M^8 momenta, are in general massive so that the ordinary twistor approach cannot work. One cannot of course exclude the possibility that octo-twistors are enough or that M^8_L description is equivalent with space-time description using quantum twistors.

4.7.1 How to define quantum Grassmannian?

The approach to twistor amplitude relies on twistor Grassmann approach [B16, B11, B10, B19, B20, B7] (see `http://tinyurl.com/yxllwcsn`). This approach should be replaced by replacing Grassmannian $GR(K,N) = Gl(n,C)/Gl(n-m,C) \times Gl(m,C)$ with quantum Grassmannian.

naïve approach to the definition of quantum Grassmannian

Quantum Grassmannian is a notion studied in mathematics and the approach of [A95] (see `http://tinyurl.com/y5q6kv6b`) looks reasonably comprehensible even for physicist. I have already earlier tried to understand quantum algebras and their possible role in TGD [K9]. It is however better to start as ignorant physicist and proceed by trial and error and find whether mathematicians have ended up with something similar.

1. Twistor Grassmannian scattering amplitudes involving k negative helicity gluons involve product of $k \times k$ minors of an $k \times n$ matrix C taken in cyclic order. C defines $k \times n$ coordinates for Grassmannian $Gr(k,n)$ of which part is redundant by the analogs of gauge

symmetries $Gl(n-m, C) \times Gl(m, C)$. Here n is the number of external gluons and k the number of negative helicity gluons. The $k \times k$ determinants taken in cyclic order appear in the integrand over Grassmannian. Also the quantum variants of these determinants and integral over quantum Grassmannian should be well-defined and residue calculus gives hopes for achieving this.

2. One should define quantum Grassmannian as algebra according to my physicist's understanding algebra can be defined by starting from a free algebra generated by a set of elements - now the matrix elements of quantum matrix. One poses on these elements relations to get the algebra considered. What could these conditions be in the recent case.

3. A natural condition is that the definition allows induction in the sense that its restriction to quantum sub-matrices is consistent with the general definition of $k \times n$ quantum matrices. In particular, one can identify the columns and rows of quantum matrices as instances of quantum vectors.

4. How to generalize from 2×2 case to $k \times n$ case? The commutation relations for neighboring elements of rows and columns are fixed by induction. In 4×4 corresponding to M^4 twistors one would obtain for $(a_1, ..., a_4)$. $a_i a_{i+1} = q a_{i+1} a_i$ cyclically ($k = 1$ follows $k = 4$).

   What about commutations of a_i and a_{i+k}, $k > 1$. Is there need to say anything about these commutators? In twistor Grassmann approach only connected $k \times k$ minors in cyclic order appear. Without additional relations the algebra might be too large. One could argue that the simplest option is that one has $a_i a_{i+k} = q a_{i+k} a_i$ for k odd $a_i a_{i+k} = q^{-1} a_{i+k} a_i$ for k even. This is required from the consistency with cyclicity. These conditions would allow to define also sub-determinants, which do not correspond to connected $k \times k$ squares by moving the elements to a a connected patch by permutations of rows and columns.

5. What about elements along diagonal? The induction from 2×2 would require the commutativity of elements along right-left diagonals. Only commutativity of the elements along left-right diagonal be modified. Or is the commutativity lost only along directions parallel to left-right diagonal? The problem is that the left-right and right-left directions are transformed to each other in odd permutations. This would suggest that only even permutations are allowed in the definition of determinant

6. Could one proceed inductively and require that one obtains the algebra for 2×2 matrices for all 2×2 minors? Does this apply to all 2×2 minors or only to connected 2×2 minors with cyclic ordering of rows and columns so that top and bottom row are nearest neighbors as also right and left column. Also in the definition of 3×3 determinant only the connected developed along the top row or left column only 2×2 determinants involving nearest neighbor matrix elements appear. This generalizes to $k \times k$ case.

It is time to check how wrong the naïve intuition has been. Consider 2×2 matrices as simple example. In this case this gives only 1 condition ($ad - bc = -da + cb$) corresponding to the permutation of rows or columns. Stronger condition suggested by higher-D case would be $ad = da$ and $bc = cb$. The definition of 2×2 in [A95] however gives for quantum 2-matrices $(a, b; c, d)$ the conditions

$$\begin{array}{ll} ac = qca \ , & bd = qda \ , \\ ab = qba \ , cd = qdc \ , & \\ bc = cb \ , & ad - da = (q - q^{-1})bc \ . \end{array} \tag{4.7.1}$$

The commutativity along left-right diagonal is however lost for $q \neq 1$ so that quantum determinant depends on what row or column is used to expand it. The modification of the commutation relations along rows and columns is what one might expect and wants in order to achieve non-commutativity of twistor components making possible massivation in M^4 sense.

The limit $q \to 1$ corresponds to non-trivial algebra in general and would correspond to $\beta = 4$ for inclusions of HFFs expected to give representations of Kac-Moody algebras. At this limit only massless particles in 4-D sense are allowed. This suggests that the reduction of Kac-Moody algebras to quantum groups corresponds to symmetry breaking associated with massivation in 4-D sense.

Mathematical definition of quantum Grassmannian

It would seem that the proposed approach is reasonable. The article [A130] (see http://tinyurl.com/yycflgrd) proposing a definition of quantum determinant explains also the basic interpretation of what the non-commutativity of elements of quantum matrices does mean.

1. The first observation is that the commutation of the elements of quantum matrix corresponds to braiding rather than permutation and this operation is represented by R-matrix. The formula for the action of braiding is

$$R^{ab}_{cd}t^c_e t^d_f = t^a_d t^b_c R^{cd}_{ef} \quad . \tag{4.7.2}$$

   Here R-matrix is a solution of Yang-Baxter equaion and characterizes completely the commutation relations between the elements of quantum matrix. The action of braiding is obtained by applying the inverse of R-matrix from left to the equation. By iterating the braidings of nearest neighbors one can deduce what happens in the braiding exchanging quantum matrix elements which are not nearest neighbors. What is nice that the R-matrix would fix the quantum algebra, in particular quantum Grassmannian completely.

2. In the article the notion of quantum determinant is discussed and usually the definition of quantum determinant involves also the introduction of metric g^{ab} allowing the raising of the indices of the permutation symbol. One obtains formulas relating metric and R-matrix and restricting the choice of the metric. Note however that if ordinary permutation symbol is used there is no need to introduce the metric.

The definition quantum Grassmannian proposed does not involve hermitian conjugates of the matrices involved. One can define the elements of Grassmannian and Grassmannian residue integrals without reference to complex conjugation: could one do without hermitian conjugates? On the other hand, Grassmannians have complex structure and Kähler structure: could this require hermitian conjugates and commutation relations for these?

4.7.2 Two views about quantum determinant

If one wants to define quantum matrices in $Gr(k,n)$ so that quantal twistor-Grassmann amplitudes make sense, the first challenge is to generalize the notion of $k \times k$ determinant.

One can consider two approaches concerning the definition of quantum determinant.

1. The first guess is that determinant should not depend on the ordering of rows or columns apart from the standard sign factor. This option fails unless one modifies the definition of permutation symbol.

2. The alternative view is that permutation symbol is ordinary and there is dependence on the row or column with respect to which one develops. This dependence would however disappear in the scattering amplitudes. If the poles and corresponding residues associated with the $k \times k$-minors of the twistor amplitude remain invariant under the permutation, this is not a problem. In other words, the scattering amplitudes are invariant under braid group. This is what twistor Grassmann approach implies and also TGD predict.

For the first option quantum determinant would be braiding invariant. The standard definition of quantum determinant is discussed in detail in [A130] (see http://tinyurl.com/yycflgrd).

1. The commutation of the elements of quantum matrix corresponds to braiding rather than permutation and as found, this operation is represented by R-matrix.

2. Quantum determinant would change only by sign under the braidings of neighboring rows and columns. The braiding for the elements of quantum matrix would compensate the braiding for quantum permutation symbol. Permutation symbol is assumed to be q-antisymmetric under braiding of any adjacent indices. This requires that permutation $i_k \leftrightarrow i_{k+1}$ regarded

as braiding gives a contraction of quantum permutation symbol $\epsilon_{i_1,...1_k}$ with $R^{ij}_{i_k i_{k+1}}$ plus scaling by some normalization factor λ besides the change of sign.

$$\epsilon_{a_1...a_k a_{k+1}...a_n} = -\lambda \epsilon_{a_1...ij...a_n} R^{ji}_{a_k a_{k+1}} \quad . \tag{4.7.3}$$

The value of λ can be calculated.

3. The calculation however leads to the result that quantum determinant $\mathcal{D}$ satisfies $\mathcal{D}^2 = 1$! If the result generalizes for sub-determinants defined by $k \times k$-minors (, which need not be the case) would have determinants satisfying $\mathcal{D}^2 = 1$, and the idea about vanishing of $k \times k$-minor essential for getting non-trivial twistor scattering amplitude as residue would not make sense.

It seems that the braiding invariant definition of quantum determinant, which of course involves technical assumptions) is too restrictive. Does this mean that the usual definition requiring development with respect to preferred row is the physically acceptable option? This makes sense if only the integral but not integrand is invariant under braidings. Braiding symmetry would be analogous to gauge invariance.

4.7.3 How to understand the Grassmannian integrals defining the scattering amplitudes?

The beauty of the twistor Grassmannian approach is that the residue integrals over quantum $Gr(k,n)$ would reduce to sum over poles (or possibly integrals over higher-D poles). Could residue calculus provide a manner to integrate q-number valued functions of q-numbers? What would be the minimal assumptions allowing to obtain scattering amplitudes as c-numbers?

Consider first what the integrand to be replaced with its quantum version looks like.

1. Twistor scattering amplitudes involve also momentum conserving delta function expressible as $\delta(\lambda_a \tilde{\lambda}^a)$. This sum and - as it seems - also the summands should be c-numbers - in other words one has eigenstates of the operators defining the summands.

2. By introducing Grassmannian space $Gr(k,n)$ with coordinates $C_{\alpha,i}$ (see `http://tinyurl.com/yxllwcsn`), one can linearize $\delta(\lambda_a \tilde{\lambda}^a)$ to a product of delta functions $\delta(C \cdot Z) = \delta(C \cdot \tilde{\lambda}) \times \delta(C^{\perp} \cdot \lambda)$ (I have not written the delta function is Grassmann parameters related to super coordinates). Z is the n-vector formed by the twistors associated with incoming particles.

   The $4 \times k$ components of $C_{\alpha,k} Z^k$ should be c-numbers at least when they vanish. One should define quantum twistors and quantum Grassmannian and pose the constraints on the poles.

How to achieve the goal? Before proceeding it is good to recall the notion of non-commutative geometry (see `http://tinyurl.com/yxrcr8xv`). Ordinary Riemann geometry can be obtained from exterior algebra bundle, call it E. The Hilbert space of square integrable sections in E carries a representation of the space of continuous functions $C(M)$ by multiplication operators. Besides this there is unbounded differential operator D, which so called signature operator and defined in terms of exterior derivative and its dual: $D = d + d^*$. This spectral triple of algebra, Hilbert space, and operator D allows to deduce the Riemann geometry.

The dream is that one could assign to non-commutative algebras non-commutative spaces using this spectral triple. The standard q-p quantization is example of this: one obtains now Lagrange manifolds as ordinary commutative manifolds.

Consider now the situation in the case of quantum Grassmannian.

1. In the recent case the points defining the poles of the function - it might be that the eventual poles are not a set of discrete points but a higher-dimensional object - would form the commutative part of non-commutative quantum space. In this space the product of quantum minors would become ordinary number as also the argument $C \cdot Z$ of the delta function. This commutative sub-space would correspond to a space in which maximum number of minors vanish and residues reduce to c-numbers.

Thus poles of the integrand of twistor amplitude would correspond to eigenstates for some $k \times k$ minors of Grassmannian with a vanishing eigenvalue. The residue at the pole at given step in the recursion pole by pole need not be c-number but the further residue integrals should eventually lead to a c-number or c-number valued integrand.

2. The most general option would be that the conditions hold true only in the sense that some $k \times k$ minors for $k \geq 2$ are c-numbers and have a vanishing eigenvalue but that smaller minors need not have this property. Also $C_{\alpha,k}Z^k$ should be c-numbers and vanish. Residue calculus would give rise to lower-D integrals in step-wise manner.

   The simplest and most general option is that one can speak only about eigenvalues of $k \times k$ minors. At pole it is enough to have one minor for which eigenvalue vanishes whereas other minors could remain quantal. In the final reduction the product of all non-vanishing $k \times k$ minors appearing in cyclic order in the integrand should have a well-defined c-number as eigenvalue. Does this allow the appearance of only cyclic minors.

   A stronger condition would be that all non-vanishing minors reduce to their eigenvalues. Could it be that only the n cyclic minors can commute simultaneously and serve as analogs of q-coordinates in phase space? The complex dimension of $G_C(n,k)$ is $d = (n-k)k$. If the space spaced by minors corresponds to Lagrangian manifold with real dimension not larger than d, one has $k \leq d = (n-k)k$. This gives $k \leq n/2(1+\sqrt{1-2/n})$ For $k = 2$ this gives $k \leq n/2$. For $n \to \infty$ one has $k \leq n/2 + 1$. For $k > n/2$ one can change the roles of positive and negative helicities. It has been found that in certain sense the Grassmannian contributing to the twistor amplitude is positive.

   The notion of positivity found to characterize the part of Grassmannian contributing to the residue integral and also the minors and the argument of delta function [B18](see `http://tinyurl.com/yd9tf2ya`) would suggest that it is also real sub-space in some sense and this finding supports this picture.

   The delta function constraint forcing $C \cdot Z$ to zero must also make sense. $C \cdot Z$ defines $k \times 6$ matrix and also now one must consider eigenvalues of $C \cdot Z$. Positivity suggest reality also now. Z adds $4 \times n$ degrees of freedom and the number $6 \times k$ of additional conditions is smaller than $4 \times n$. $6k \leq 4 \times n$ combined with $k \leq n/2$ gives $k \leq n/2$ so that the conditions seems to be consistent.

3. The c-number property for the cyclic minors could define the analog of Lagrangian manifold for the phase space or Kähler manifold. One can of course ask, whether Kähler structure of $Gr(k,n)$ could generalize to quantum context and give the integration region as a sub-manifold of Lagrangian manifold of $Gr(k,n)$ and whether the twistor amplitudes could reduce to integral over sub-manifold of Lagrangian manifold of ordinary $Gr(k,n)$.

To sum up, I have hitherto thought that TGD allows to get rid of the idea of quantization of coordinates. Now I have encountered this idea from totally unexpected perspective in an attempt to understand how 8-D masslessness and its twistor description could relate to 4-D one. Grassmannians are however very simple and symmetric objects and have natural coordinates as $k \times n$ matrices interpretable as quantum matrices. The notion of quantum group could find very concrete application as a solution to the basic problem of the standard twistor approach. Therefore one can consider the possibility that they have quantum counterparts and at least the residue integrals reducing to c-numbers make sense for quantum Grassmannians in algebraic sense.

Chapter 5

McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product?

5.1 Introduction

This article deals with two questions.

1. The ideas related to topological quantum computation [L69] suggests that it might make sense to replace quantum states with representations of the Galois group or even the coefficient space of Hilbert space with a quantum analog of a number field with tensor product and direct sum replacing the multiplication and sum. I have considered this kind of idea already earli [K54].

   Could one generalize arithmetics by replacing sum and product with direct sum $\oplus$ and tensor product $\otimes$ and consider group representations as analogs of numbers? Could one replace the roots labelling states with group representations? Or could even the coefficient field for the state space be replaced with a ring of representations? Could one speak about quantum variants of state spaces?

   Could this give a kind of quantum arithmetics or even quantum number theory and possibly also a new kind of quantum analog of group theory. If the direct sums are mapped to ordinary sums of algebraic numbers in quantum-classical correspondence interpreted as a kind of category theoretic morphism, this map could make sense under some natural conditions.

2. McKay graphs (quivers) have irreducible representations as nodes and characterize the tensor product rules for the irreps of finite groups. How general is the McKay correspondence relating these graphs to the Dynkin diagrams of ADE type affine algebras? Could it generalize from finite subgroups of $SL(k,C)$, $k = 2,3,4$ [A65, A64] to those of $SL(n,C)$. Is there a deep connection between finite subgroups of $SL(n,C)$, and affine algebras. Could number theory or its quantum counterpart provide insights to the problem?

5.1.1 Could one generalize arithmetics by replacing sum and product with direct sum and tensor product?

In the model for topological quantum computation (TQC) [B5, B4] quantum states in the representations of groups are replaced with entire representations (anyons). One can argue that this helps to guarantee statibility: this generalization could be regarded as error correction code. In TGD, these representations would correspond to irreps of Galois groups or of discrete subgroups of the covering group for automorphisms of quaternions. Also discrete subgroups of $SL(2,C)$ assignable naturally to the tessellations of H^3 can be considered.

Tensor product $\otimes$ and direct sum $\oplus$ are commutative operations and very much like operations of ordinary arithmetics. One can also speak of positive integer multiples of representation. The algebras of irreps of various algebraic structures generated by $\oplus$ and $\otimes$ are applied quite generally in mathematics and especially so in gauge theories and conformal field theories and are known as fusion algebras (`https://cutt.ly/TLU3hvJ`) and quivers (`https://cutt.ly/xLU3zrM`).

Could the replacement of the roots of the EDD of the ADE group with representations of the finite subgroup of $SL(2,C)$ associated with the diagram make sense? The trivial representation would correspond to an additional node and lead to an extended Dynkin diagram (EDD).

Could one regard the irreps as quantum roots of an ordinary monic polynomial so that the ordinary algebraic numbers would have representation as state spaces? Could one obtain the full root diagram by a generalization of the Weyl group operation as reflection of root with respect to root? The first guess is that the isotropy group Gal_I of a root acts as a subgroup of Gal defines the polynomial, which gives the roots replaced by irreps and that Gal itself acts in the same role as the Weyl group.

McKay graph characterizes the rules for the tensor product compositions for the irreps of a finite group G, in particular Galois group. There is an excellent description of McKay graphs on the web (see `https://cutt.ly/zLzoAwF`). The article describes first the special McKay graphs for finite subgroups of $SL(2,C)$ and their geometric interpretation in terms of the geometry of Platonic solids and their denerate versions as regular polygons and shows that they turn out to correspond to EDDs for ADE type Lie algebras. Also general McKay graphs are considered.

5.1.2 McKay graphs and McKay correspondence

The McKay graphs are a special case of quiver diagrams (`https://cutt.ly/xLU3zrM`) and code for the tensor product decomposition rules for the irreps of finite groups [A105, A82].

For a general finite group, McKay graphs can be constructed in the following way. Consider any finite group G and its irreducible representations (irreps) ξ_i and assign to ξ_i vertices. Select one irrep V and assign also to it a vertex. For all tensor products $\xi_i \otimes V$ and decompose them to a direct sum of irreps ξ_j. If ξ_j is contained to $V \otimes \xi_i$ a_{ij} times, draw a_{ij} directed arrows connecting vertex i to vertex j. One obtains a weighted, directed graph with incidence matrix a_{ij}. Adjacency matrix plays a central role in graph theory.

McKay correspondence is only one of the mysteries related to MacKay graphs for finite subgroups of $SL(k,C)$, $k = 2,3,4$ and presumably also $k > 4$ [A65, A64]. The MacKay graphs correspond to EDDs for ADE type Lie groups having interpretations as Dynkin diagrams for ADE type affine algebras.

The classification of singularities of complex surfaces represents another example of McKay correspondence.

1. ADE Dynkin diagrams provide a classification of Kleinian singularities of complex surfaces having real dimension 4 and satisfying a polynomial equation $P(z_1,z_2,z_3) = 0$ with $P(0,0,0) = 0$ so that the singularity is at origin [A82] (`https://cutt.ly/5LQPyhy`). The finite subgroups of $SL(2,C)$ naturally appear as symmetries of the singularities at origin.

2. In the TGD framework, this kind of complex surfaces could correspond to surfaces with an Euclidean signature of induced metric as 4-surfaces in $E^2 \times CP_2 \subset M^4 \times CP_2$. What I call CP_2 type extremals have light-like M^4 projection as deformations of the canonically imbedded CP_2. These surfaces could correspond to deformations of CP_2 type extremals. One can ask whether one could assign ADE type affine algebras as affine algebras with these singularities.

5.2 Could the arithmetics based on direct sum and tensor product for the irreps of the Galois group make sense and have physical meaning?

The idea about the generalization of the mathematical structures based on integer arithmetics with arithmetics replacing $+$ and $\times$ with direct sum $\oplus$ and tensor product $\otimes$ raises a bundle of questions. This idea makes sense also for the finite subgroups of $SU(2)$ defining the covering group of quaternion automorphism having a role similar to that of the Galois group.

What motivates this proposal is that the extensions of rationals and their Galois groups are central in TGD. Polynomials P with integer coefficients are proposed to determine space-time surfaces by M^8-H duality in terms of holography based on the realization of dynamics in M^8 in terms of roots of P having interpretation as mass shells. Holography is realized in terms of the condition that the normal space of the space-time surface going through the mass shells has associative normal space [L45, L46].

5.2.1 Questions

The following questions and considerations are certainly very naive from the point of view of a professional mathematician and the main motivation for the mathematical self ridicule is that there are fascinating physical possibilities involved.

The basic question is whether $\otimes$ and $\oplus$ can give rise to quantum variants of rings of integers and even algebraic integers defined in terms of quantum roots of ordinary polynomial equations and could one even generalize the notion of number field: do quantum variants of extensions of rationals, finite fields, and p-adic number fields make sense?

Recall that also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L23, L24] [K49, K33, K34].

Quantum polynomials

To build extensions of rationals, one must have polynomials. The notion of polynomial playing central role in M^8-H duality [L45, L46], or rather the notion of a root of polynomial, generalizes.

1. Polynomials would look exactly like ordinary monic polynomials, with the real unit replaced with identity representation but their quantum roots would be expressible as direct sums of irreps associated with a given extension of rationals.

2. One would obtain roots as direct sums of the generators of the extension which could correspond to irreps of the isotropy group Gal_I of Galois group Gal. McKay graph would define the multiplication rules for the tensor products appearing in the polynomial whose coefficients would be quantum counterparts of ordinary (positive) integers.

3. Also a generalization of an imaginary unit could make sense for p-adic ring and finite fields as a root of a polynomial. Note that $\sqrt{-1}$ can exist for p-adic number fields. Also p-adic number fields and the adelic physics relying on the fusion of p-adic physics and real physics play a central role in TGD [L23, L24] [K49, K33, K34].

Does one obtain additive and multiplicative group structures, rings, and fields?

Could one give to the space spanned by irreps a structure of ring or even field?

1. Could one replace algebraic integers of the ordinary extension of rationals with direct sums of the n_C irreps of Galois group G, where n_C is the number of classes of G? Note that the dimensions n_i of irreps satisfy the formula $\sum n_i^2 = n_C$.

   If $\oplus$ corresponds to $+$ for ordinary integers, only non-negative integers can appear as coefficients so that one would have semigroups with respect to both $\oplus$ and $\otimes$.

2. The inverse with respect to $\oplus$ requires that negative multiples of quantum integers make sense. This is possible in p-adic topology: the number -1 would correspond to the quantum part of the integer $(p-1)\sum_{\oplus} p^{\oplus n}$. The summands in this expression would have p-adic norms p^{-n}. This allows to define also the negatives of other roots playing the role of generator of the quantum extension of rationals.

3. Is even the quantum analog of a number field possible? If one requires multiplicative inverse, only the finite field option remains under consideration since the quantum variant of $1/p^k$ does not make sense since one has $p \equiv= 0$. If one requires group structure for only $\oplus$, quantum p-adics remain under consideration.

Can one map the numbers of quantum extensions of rationals the numbers of ordinary extensions?

Concerning the physical interpretation, it would be important to map the quantum variants of algebraic integers to their real counterparts. Mathematicians might talk of some kind of category theoretical correspondence.

1. Since the same polynomial would have ordinary roots and quantum roots, the natural question is whether the quantum roots can be mapped to the ordinary roots.

2. If the quantum roots correspond to roots of the Dynkin diagram as quantum numbers in quantum extension of rationals, it should be able to map all quantum roots of the ADE type affine algebra to ordinary roots. This requires that sums with respect to $\oplus$ correspond to sums with respect to $+$: additivity of quantum numbers would hold true at both levels and one would have category theoretic correspondence as algebraic isomorphisms.

   Note that Galois confinement means that 4-momenta and other quantum numbers of states are integer valued, when one uses the momentum scale defined by causal diamond (CD). This means that they would correspond to $\oplus$ multiples of trivial representation of the Galois group acting as Weyl group.

3. What about the tensor products of roots appearing in the McKay graph? Can one require that the products with respect to $\otimes$ correspond to products with respect to $\times$. Only $\otimes$ does appear in the generation of the quantum roots of a given KM algebra representation.

   What about quantum variants of quantum states? If the quantum variants of p-adic integers or finite fields appear also as a coefficient field of quantum states, one can always express the coefficients as direct sums of quantum roots and map these sums to sums of ordinary polynomial roots, that is algebraic numbers. Extensions of rationals can appear as coefficient fields for Hilbert spaces.

   If one assumes that only quantum variants of p-adic numbers with a finite number of the pinary digits and their negatives are possible, they can be mapped to numbers in algebraic extension. One could overcome the problems related to the definition of inner product when finite field or p-adic numbers define the coefficient field for Hilbert state.

4. For generalized finite fields, the notions of vector space and matrix algebra, hermiticity and unitarity, and eigenvalue problem could be generalized. For instance, eigenvalues of a Hermitian operator could be just real numbers. Also a relatively straightforward looking generalization of group theory can be imagined, and would be obtained by replacing the elements of the matrix group with the elements of a generalized finite field.

5.2.2 Could the notion of quantum arithmetics be useful in the TGD framework?

These ideas might find an application in TGD.

1. The quantum generalization of the notion of rationals, p-adic number fields, and finite fields could be defended as something more than a mere algebraic game. In particular, in TGD the ramified primes of extension of rationals correspond to physically important p-adic primes,

especially the largest ramified prime of the extension. Algebraic prime is a generalization of the notion of ordinary prime. Also its generalization could make sense and give rise to the notion of quantum prime.

Unfortunately, the extension of finite field F_p induced by a given extension of rationals does not exist for the ramified primes appearing as divisors in the discriminant determined by the product of root differences.

Could the generalization of the notion of finite field save the situation? Topological quantum computations (TQC) relying on Galois representations as counterparts for anyons would mean an increase of the abstraction level replacing numbers of algebraic extension with representations of Galois group as their cognitive representations.

One can assign also to the possibly unique monic polynomial P_c defining the n_c-dimensional extension, a discriminant, call it D_c. For the primes dividing the discriminant D of P but not D_c, the quantum counterpart of the finite-field extension could make sense.

2. In TGD, the roots of polynomials define 3-D mass and energy shells in M^8 in turn defining holographic data defining 4-D surface in M^8 mapped to space-time surfaces in H by M^8-H duality. Could one consider quantum variants of the polynomial equations defining space-time surfaces by holography in the generalized extensions of rationals based on representations of Galois groups?

   Could monic polynomials define quantum variants of 4-surfaces or at least of discretizations of hyperbolic spaces H^3 as 3-D sections of 4-surface in M^8 defined as roots of polynomial P and containing holographic data as cognitive representation? Mass shells would be mapped by M^8-H duality to light-cone proper time hyperboloids in H.

   The interiors of 4-surfaces in M^8 would contain very few points of cognitive representation as momentum components in the extension of rationals defined by the polynomial P. Mass shells and their H images would be different and represent a kind of cognitive explosion. The presence of fermions (quarks) at the points of cognitive representation of given mass shells would make them active.

3. Could the transition from the classical to a quantum theory, which also describes cognition, replace discrete classical mass shells as roots of a polynomial in M^8 with roots with direct sums of irreps of the Galois group?

   This idea would conform with category theoretic thinking which leaves the internal structure of the basic object, such as point, open. That points of cognitive representations would be actually irreducible representations of the Galois groups would reveal a kind of cognitive hidden variables and quantum cognition.

These ideas are now completely new. I have earlier considered the possibility that points could have an infinite complex internal structure and that the "world of classical worlds" could be actually M^8 or H with points having this structure [K72]. I have also considered the possibility that Hilbert spaces could have arithmetic structure based on $\otimes$ and $\oplus$ with Hilbert spaces with prime dimension defining the primes [K54].

"Do not quantize" has been my motto for all these years but in this framework, it might be possible to talk about quantization of cognition as a deformation of number theory obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary numbers with representations of Galois group. Perhaps this quantization could apply to cognition.

5.3 What could lurk behind McKay correspondence?

The appearance of EDDs in so many contexts having apparently no connection with affine algebras is an almost religious mystery and one cannot avoid the question of whether there is a deep connection between some finite groups G, in particular finite subgroups of $SL(n,C)$, and affine algebras. In the TGD framework M^8-H duality relates number theoretic and differential geometric views about physics and the natural question whether it could provide some understanding of this mystery.

$M^8 - H$ duality also suggests how to understand the Langlands correspondence: during years I have tried to understand Langlands correspondence [A57, A56] from the TGD perspective [K38, L11].

5.3.1 McKay correspondence

There is an excellent article of Khovanov [A105] describing the details of McKay correspondence for the discrete subgroups of $SL(2,C)$ (`https://cutt.ly/1LQDqce`). There is also an article "McKay correspondence" by Nakamura about various aspects of McKay correspondence [A82] (`https://cutt.ly/5LQPyhy`).

1. Consider finite subgroups G of $SL(2,C)$. The McKay graph for the tensor products of what is called canonical (faithful) 2-D representation V of G with irreps ξ_i of G corresponds to an extended Dynkin diagram with one node added to a Dynkin diagram. Note that V need not be always irreducible.

   The constraints on the graph come from the conditions for the dimension $d = 2d_j$ of the tensor product $V \otimes \xi_i$ satisfies $2d_i = \sum_j a_{ij} d_j$, where the sum is over all vertices directed away from the vertex i. If arrows in both directions are present, there is no arrow. This implies that the dimensions d_j associated with the vertex have G.C.D equal to 1.

2. Dynkin diagram in turn describes the minimal set of roots from which the roots of Lie algebra can be generated by repeated reflections with respect to roots. EDDs can be assigned to affine algebras and for them the eigenvalues of the adjacency matrix are not larger than 2. The maximum of the eigenvalues measures the complexity of the graph.

3. The Weyl group characterizes the symmetries of the root diagram and is generated by reflections of roots with respect to other roots. The Dynkin diagram contains a minimal number of roots needed to generate all roots by reflections as Weyl orbits of the roots of the Dynkin diagram. The action of the Weyl group leads away from the Dynkin diagram since otherwise this set of roots would not be minimal.

   The number of lines characterizes the angle between the roots i and j. For ADE groups $a_{ij} = 1$ codes for angle of 120 degrees $2\pi/3$, $a_{ij} = 2$ corresponds to 135 degrees, and $a_{ij} = 3$ to 150 degrees. $a_{ij} = 0$ means either angle π or $\pi/2$. In the general case, there are 2-valent and 3-valent nodes depending on the number of oriented lines emerging from the node.

   For instance, in the case of a triangle group with 6 elements with irreps $1, 1_1, 1_2$. The canonical representation to 2-D reducible representation decomposes to $1_1 + 1_2$ so that there are 3 vertices involved corresponding to 1_1 and 1_2 and 1. It is easy to see that the adjacency matrix is symmetric and gives rise to an EDD with 3 vertices. From the corresponding Dynkin diagram, representing 2 neighboring roots of the root diagram one obtains the entire root diagram by repeated reflections having 6 roots characterizing the octet representation of A_2 ($SU(3)$).

4. What kind of McKay graphs are associated with other than canonical 2-D representations in the case of rotation groups? Every representation of G belongs to some minimal tensor power $V^{\otimes k}$ and one can study the MacKay diagrams assignable to $V^{\otimes k}$. It is easy to see that the number of paths connecting vertices i and j in the McKay graph $M^k(V)$ for $V^{\otimes k}$ can be understood in terms of the McKay graph $M(V)$ for V. The paths leading from i to j are all k-edged paths along $M(V)$ leading from i to j.

   The symmetry of the adjacency matrix A implies that forth and back movement along $M(V)$ is possible. The adjacency matrix has the same number of nodes and equals the $k:th$ power A^k of A so that extended ADE type Dynkin diagrams are not in question.

5.3.2 Questions

McKay correspondence raises a series of questions which I have discussed several times from the TGD point of view several times [L17, L42, L41]. In the following these questions are discussed by introducing the possibility of quantum arithmetics and cognitive representations as new elements.

Why would $SL(2,C)$ be so special?

$SL(2,C)$ is in a very special role in McKay correspondence. Of course, also the finite subgroups of other groups could have a special role and it is actually known that $SL(n,C)$ $n<5$ are in the same role, which suggests that all groups $SL(n,C)$ have this role [A65, A64].

Why? In the TGD framework, a possible reason for the special role of $SL(2,C)$ acts as the double covering group of the isometries of the mass shell $H^3 \subset M^4 \subset M^8$ and its counterpart in $M^4 \times CP_2$ obtained by M^8-H correspondence. $SL(2,C)$ has also natural action on the spinors of H. The finite subgroups relate naturally to the tessellations of the mass shell H^3 leaving the basic unit of tessellation invariant.

The tessellations could naturally force the emergence of ADE type affine algebras as dynamical symmetries in the TGD framework. In fact, the icosa-tetrahedral tessellation plays a key role in the proposed model of the genetic code based on Hamiltonian cycles at icosahedron and tetrahedron [L54].

Why does the faithful representation have a special role?

The mathematical reason for the special role of the faithful canonical representation V is that its tensor powers contain all irreps of the finite group: the tensor product structure for other choices of V can be deduced from that for canonical representations. It is known that any irrep V, which is faithful irrep of G, generates the fusion algebra.

However, this kind of irrep might fail to exist. If G has a normal subgroup H and the irrep χ has H as kernel then the powers of χ contain only the irreps of G/H. In the article "McKay Connectivity Properties of McKay Quivers" by Hazel Brown [A72] (`https://arxiv.org/pdf/2003.09502.pdf`) it was shown that the number of connected components of the McKay quiver is the number of classes of the G, which are contained in H. For instance, the classes associated with the center of G are such (Z_n for $SL(n,C)$).

For simple groups this does not happen but in the case of Galois groups assignable to composite polynomials one has a hierarchy of normal subgroups and this kind of situation can occur since the number of classes of G contained in normal subgroups can be non-vanishing.

2-D representation is also in a special role physically in the TGD framework, the ground states of affine representation correspond to a 2-D spinor representation since quarks are the fundamental particles.

The irreps of the affine representation are obtained as tensor products of the irrep associated with the affine generators with it. Cognitive representations imply a unique discretization and this forces discrete subgroups of $SL(2,C)$ and implies that the irreps of $SL(2,C)$ decompose to irreps of a discrete subgroup. Therefore the quivers for their tensor products appear naturally.

Electroweak gauge group $U(2)$ corresponds to the holonomy group $U(2)$ for CP_2 and for $SU(2)_w$ the McKay correspondence holds true. Also the isometry group $SU(3)$ of CP_2 is assumed to appear as affine algebra. Discretization due to cognitive representations in M^8 induces discretization in H and CP_2. The replacement of $SU(3)$ with its discrete subgroups would decompose irreps for $SU(3)$ to irreps of $SU(3)$. $SL(3,C)$ allows analog of McKay correspondence [A65] so that also the finite subgroups of $SU(3)$ allow it.

What about McKay graphs for more general finite groups?

The obvious question concerns the generality of McKay correspondence. What finite groups and therefore corresponding Galois groups correspond to representations of affine type algebras.

In the general case, the McKay graphs look very different from Dynkin diagrams. The article "Spectral measures for G_2" of Evans and Pugh [A51] (`https://cutt.ly/hLQO7HE`) is of special interest from the TGD point of view since G_2 is the automorphism group of octonions. G_2 however naturally reduces to $SU(3)$ corresponding to color isometries in H. The article discusses in detail McKay graphs for the finite subgroups of G_2. These finite subgroups correspond to those for $SU(2)\times SU(2)$ and $SU(3)$ plus 7 other groups. The McKay graphs for the latter groups contain loops are very complex and contain loops.

What can one say about finite groups, which allow McKay correspondence.

1. ADE diagrams are known to classify the following three finite simple groups, the derived group F'_{24} of the Fischer F_{24}, the Baby monster B and the Monster M are related with E_6, E_7 and E_8 respectively [A82] (`https://cutt.ly/5LQPyhy`). In the TGD framework, this finding inspires the question whether these groups could appear as Galois groups of some polynomial and give rise to E_6, E_7 and E_8 as dynamical symmetries.

   In the TGD framework, one can ask whether also the above mentioned simple groups could appear as Galois groups. What is fascinating that monster would relate to icosahedron and dodecahedron: icosahedron and tetrahedron play key role in TGD inspired model of genetic code, in particular in the proposal that it relates to tetra-icosahedral tessellation of hyperbolic space H^3 [L54].

2. The article [A136](`https://cutt.ly/jLQPgkQ`) mentioned the conjecture that the tensor product structure for the finite subgroups of $SU(3)$ could relate to the integrable characters for some representations of affine algebra associated with $SU(3)$. This encourages the conjecture that this is true also for $SU(n)$.

   In TGD, this inspires the question whether finite Galois groups representable as subgroups of $SU(3)$ could give rise to corresponding affine algebras as dynamical symmetries of TGD.

3. Butin and Perets demonstrated McKay correspondence in the article "Branching law for finite subgroups of $SL(3,C)$ and McKay correspondence" [A65] (`https://cutt.ly/CLQPvp2`) for finite subgroups of $SL(3,C)$ in the sense that branching law defines a generalized Cartan matrix. In the article "Branching Law for the Finite Subgroups of SL(4,C) and the Related Generalized Poincare Polynomials" [A64] (`https://cutt.ly/mLQPQnT`) shows that the same result holds true for $SL(4,C)$, which suggests that it is true for all $SL(n,C)$.

   A generalization to finite subgroups of $SL(n,C)$ is a natural guess. Therefore Galois groups with this property could be assigned with affine algebras characterized by the generalized Cartan matrices and could correspond to physically very special kind of extensions of rationals,

5.3.3 TGD view about McKay correspondence

The key idea is that one replaces quantum numbers representable as sums of the roots of Lie algebra with representations of the isotropy group of Galois group which is same as a finite subgroup of say $SL(2,C)$ and that Galois groups acts as Weyl group. The Weyl group codes for the differential geometric notion of symmetry realized by Lie groups and Galois group codes for the number theoretic view of symmetry. This correspondence would represent a facet of the duality between number theory and differential geometry.

Quantum roots as direct sums of irreps

Consider first the correspondence between quantum roots (or more generally weights defined as dual space of roots) and ordinary roots (weights) as quantum numbers.

1. The representations of finite group G (say subgroup of $SL(2,C)$) represented by the isotropy group Gal_I of Galois group for a given root, would appear as labels of states rather than as counterparts of states. Galois group Gal itself would act as Weyl group on the roots.

2. Quantum numbers as labels of quantum states would be replaced with representations of Gal_I. The additivity of quantum numbers would correspond to the additivity of representations with respect to $\oplus$. Tensor product for the representations would be analogous to multiplication of quantum quantum numbers so that they would form an algebra. An abstraction or cognitive representation would be in question. Since the roots of the Dynkin diagram correspond to roots of a monic polynomial, one could map them to ordinary algebraic numbers. Same applies to the root of affine representations.

Could also the quantal version of the coeffient field of the state space make sense?

Could also the coefficient field of state space be replaced with a quantum variant of p-adic numbers or of finite field?

1. Here one encounters a technical problem that is encountered already at the level of ordinary p-adics and finite fields. Inner products are bilinear. If norm squared is defined as a sum for the squares of the coefficients of the state in the basis of n states, the non-well-ordered character of p-adics implies that one can have states for which this sum vanishes in p-adic and finite fields.

   In the p-adic case, allowance of only finite number of non-vanishing binary digits for the coefficients might help and would conform with the idea about finite measurement resolution as a pinary cutoff. One could even allow negatives of integers with finite number of pinary digits if the p-adic quantum integers are mapped to the real counterparts.

2. There is also a problem associated with the normalization factors of the states, which cannot be p-adic integers in general. Overall normalization does not however matter so that this problem might be circumvented.

   Physical predictions would require the map of the quantum integers to real ones. The fact that quantum integers are $\oplus$ sums of quantum roots of ordinary monic polynomials, makes this possible. The irreps appearing as coefficients of states would be mapped to ordinary algebraic numbers and the normalization of the states could be carried out at the level of the ordinary algebraic numbers.

What about negative multiples of quantum roots

If the quantum roots of a polynomial correspond to irreps of the Galois group, one encounters a technical problem with negative multiples of quantum roots.

1. The negatives of positive roots correspond to -1 multiples of irreps. This does not make sense in ordinary arithmetics. p-Adically -1 corresponds to $(p-1)(1+p+p^2+...)$ and would correspond to infinite $\oplus$-multiple of root but decompose to p^n multiples to which one can assign norm p^{-k} so that the sum converges: $-\xi_i = (p-1)(Id \oplus pId \oplus p^2Id \oplus ...)\xi_i$.

   One has finite measurement resolution so that the appearance of strictly infinite sums is highly questionable. Should one consider only finite sums of positive roots and their negatives but how should one deal with the negatives?

   Could the creation operators labelled by negative roots correspond to annihilation operators with positive roots as in the case of super-Virasoro and affine algebras. Note that if one restricts to ordinary integers at the level of algebra as one must to for supersymplectic and Yangian algebras, one must consider only half-algebras with generators, which have only non-negative conformal weights. This does not make sense for ordinary affine generators.

2. The most plausible solution of the problem relies on the proposed categorical correspondence between quantum roots and ordinary roots as roots of the same monic polynomial. One could map the quantum roots and their direct summands to sums of ordinary roots and this would make sense also for the negatives of positive roots with a finite number of summands. It would be essential that p-adic integers correspond to finite ordinary integers and to their negatives and are mapped to numbers in an extension of rationals. As found, this map would also allow us to circumvent the objections against the quantum variant of the state space.

3. Could zero energy ontology (ZEO) come to the rescue? In zero energy ontology creation and annihilation operators are assigned with the opposite boundaries of causal diamond (CD). Could one assign the negative conformal weights and roots with the members of state pairs located at the opposite boundary of CD?

   This works for the Virasoro and affine generators but this kind of restriction is unphysical in the case of eigenvalues of L_z with both signs? Why would opposite values of L_z be assigned to opposite boundaries of CD?

Wheels and quantum arithmetics

Gary Ehlenberg gave a link to a Wikipedia article telling of Wheel theory (`https://cutt.ly/RZnUB5y`). Wheel theory could be very relevant to the TGD inspired idea about quantum arithmetics.

I understood that Wheel structure is special in the sense that division by zero is well defined and multiplication by zero gives a non-vanishing result. The wheel of fractions, discussed in the Wikipedia article as an example of wheel structure, brings into mind a generalization of arithmetics and perhaps even of number theory to its quantum counterpart obtained by replacing + and - with direct sum $\oplus$ and tensor product $\otimes$ for irreps of finite groups with trivial representation as multiplicative unit: Galois group is the natural group in TGD framework.

Could wheel structure provide a more rigorous generalization of the notions of the additive and multiplicative inverse of the representation in order to build quantum counterparts of rationals, algebraic numbers and p-adics and their extensions?

1. One way to achieve this is to restrict consideration to the quantum analogs of finite fields $G(p,n)$: + and x would be replaced with $\oplus$ and $\otimes$ obtained as extensions by the irreps of the Galois group in TGD picture. There would be quantum-classical correspondence between roots of quantum polynomials and ordinary monic polynomials.

2. The notion of rational as a pair of integers (now representations) would provide at least a formal solution of the problem, and one could define non-negative rationals.

   p-Adically one can also define quite concretely the inverse for a representation of form $R = 1 \oplus O(p)$ where the representation $O(p)$ is proportional to p (p-fold direct sum) as a geometric series.

3. Negative integers and rationals pose a problem for ordinary integers and rationals: it is difficult to imagine what direct sum of -n irreps could mean.

   The definition of the negative of representation could work in the case of p-adic integers: $-1 = (p-1) \otimes (1 \oplus p * 1 \oplus p^2 * 1 \oplus ...)$ would be generalized by replacing 1 with trivial representation. Infinite direct sum would be obtained but it would converge rapidly in p-adic topology.

4. Could $1/p^n$ make sense in the Wheel structure so that one would obtain the quantum analog of a p-adic number field? The definition of rationals as pairs might allow this since only non-negative powers of p need to be considered. p would represent zero in the sense of Wheel structure but multiplication by p would give a non-vanishing result and also division with p would be well-defined operation.

Galois group as Weyl group?

The action of the Weyl group as reflections could make sense in the quantum arithmetics for quantum variants of extensions of p-adics and finite fields. The generalized Cartan matrix $C_{ij} = d\delta_{ij} - n_{ij}$, where n_{ij} is the number of lines connecting the nodes i and j and d is the dimension of V, is indeed well-defined for any finite group and has integer valued coefficients so that Weyl reflection makes sense also in quantum case.

Can one identify the Weyl group giving the entire root diagram number theoretically? The natural guess is $Gal = W$: Gal would define the Weyl group giving the entire root diagram from the Dynkin diagram by reflections of the roots of the EDD. One can assign to Gal an extension defined by a monic polynomial P with Galois group Gal.

How the group defining the McKay graph is represented?

How the group G defining the McKay graph is represented? The irreps of G should have natural realization and the quarks at mass shells would provide these representations.

One can consider two options. The first option is based on the isotropy group G_I of $Gal = W$ leaving a given root invariant. Second option is based on the finite subgroup of $SU(2)$ as a covering group of quaternion automorphisms.

1. The subgroup $Gal_I \subset Gal$ acting as an isotropy group of a given root of Gal would naturally define the EDD since the action of $Gal = W$ would not leave its nodes as irreps of Gal_I invariant.

   The root diagram should be the orbit of the EDD under $Gal = W$. The irreps of the EDD would correspond to the roots of a monic polynomial P_I associated with Gal_I and having $n_c + 1$ quantum roots. The quantum roots would be in the quantum extension defined by a monic polynomial P for Gal so that the action of Gal on EDD would be well-defined and non-trivial.

2. In the TGD framework, the mass mass squared values assignable to the monic polynomial representing the EDD correspond to different mass squared values. There is no deep reason for why the irreps of Gal_I could not correspond to different mass squared values and in the TGD framework the symmetry breaking $Gal \to Gal_I$ is the analog for the symmetry breaking in the Higgs mechanism.

   In the recent case this symmetry breaking would be associated with $Gal_I \to Gal_{I,I}$ and imply that quantum roots correspond to different mass squared values. At the level of affine algebra this could mean symmetry breaking since the different roots would not have different mass squared values.

   If Gal acts as a Weyl group, the McKay graph associated with Gal_I corresponds to the EDD. Gal_I is a subgroup of Gal so that the action of $Gal = Weyl$ on the quantum roots of the monic polynomial P_I would be non-trivial and natural. Could Gal_I be a normal subgroup in which case Gal/Gal_I would be a group and one would have a composite polynomial $P = Q \circ P_1$? This cannot be true generally: for instance for A_p, p prime and E_6 the W is simple. For E^7 and E^8 W is a semidirect product.

3. There is an additional restriction coming from the fact that Gal_I does not affect the rational parts of the 4-momenta. Is it possible to have construct irreps for a finite subgroup of $SL(2,C)$ or even $SL(n,C)$ using many quark states at a given mass shell? The non-rational part of 4-momentum corresponds to the "genuinely" virtual part of virtual momentum and for Galois confined states only the rational parts contribute to the total 4-momentum. Could one say that these representations are possible but only for the virtual states which do not appear as physical states: cognition remains physically hidden.

The very cautious, and perhaps over-optimistic conclusion, would be that only Galois groups, which act as Weyl groups, can give rise to affine algebras as dynamical symmetries. For this option, one would obtain cognitive representations for the isotropy groups of all Galois groups. For Galois groups acting as Weyl groups, EDDs could define cognitive representations of affine algebras. Also cognitive representations for finite subgroups of $SL(n,C)$ and groups like Monster would be obtained.

For the second option in which the subgroup G of quaternionic automorphisms affecting the real parts of 4-momenta is involved. This representation would be possible only for the subgroups of $SL(2,C)$. In this case one would have 3 different groups $Gal = W$, Gal_I and G rather than $Gal = W$ and Gal_I.

1. Quaternionic automorphisms are analogous to the Galois group and one can ask whether the finite subgroups G of quaternionic automorphisms could be directly involved with cognitive representations. This would give McKay correspondence for $SL(2,C)$ only. The quaternionic automorphism would affect the rational part of the 4-momentum in an extension of rationals unlike the Galois group which leaves it invariant. The irrep of G would be realized as many-quark states at a fixed mass shell. Different irreps would correspond to different masses having interpretation in terms of symmetry breaking.

2. Also now one would consider the extension defined by the roots of a monic polynomial P having Galois group $Gal = W$ associated with the corresponding EDD. P_I would give quantum roots defining the Dynkin diagram and define the mass squared values assignable to irreps of G.

3. The situation would differ from the previous one in that the action of G_I on irreps would be replaced by the action of G. Indeed, since G_I leaves the rational part of the 4-momentum invariant, G_I cannot represent G as a genuine subgroup of rotations.

4. The roots would correspond to irreps of a subgroup G of quaternionic automorphisms, which would affect the 4-momenta with a given mass shell and define an irrep of G. Different roots of P would define the mass shells and irreps of G associated with EDD as a McKay graph.

Information about Weyl groups of ADE groups

The Wikipedia article about Coxeter groups (https://en.wikipedia.org/wiki/Coxeter_group#Properties), which include Weyl groups, lists some properties of finite irreducible Coxter groups and contains information about Weyl groups. This information might be of interest in the proposed realization as a Galois group.

- $W(A_n) = S_{n+1}$, which is the maximal Galois group associated with a polynomial of degree $n+1$.
- $W(D_n) = Z_2^{n-1} \rtimes S_n$.
- $W(E_6)$ is a unique simple group of order 25920.
- $W(E^7)$ is a direct product of a unique simple group of order 2903040 with Z_2.
- $W(E_8)$ acts as an orthogonal group for F_2 linear automorphisms preserving a norm in Ω/Z_2, where Ω is E_8 lattice (https://mathoverflow.net/questions/230120/the-weyl-group-of-e8-versus-o-82/230130#230130)
- $W(B_n) = W(C_n) = Z_2^n \rtimes S_n$.
- $W(F_4)$ is a solvable group of order 1152 , and is isomorphic to the orthogonal group $O_4(F_3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the field F_3.
- $W(G_2) = D_6 = Z_2 \rtimes Z_6$.

Candidates for symmetry algebras of WCW, inclusions of hyperfinite factors, and Galois groups acting as Weyl groups

TGD allows several candidates for the symmetry algebras acting in WCW. The intuitive guess is that the isometries and possibly also symplectic transformations of the light-cone boundary $\delta M^4_+ \times CP_2$ define isometries of WCW whereas holonomies of H induce holonomies of WCW.

1. In TGD, supersymplectic algebra SSA could replace affine algebras of string models.

2. By the metric 2-dimensionality of the light-cone boundary δM^4_+, one can assign to it an infinite-dimensional conformal group of sphere S^2 in well-defined sense local with respect to the complex coordinate z of S^2. These transformations can be made local with respect to the light-like coordinate r of δM^4_+. Also a S^2-local radial scaling making these transformations isometries is possible. This is possible only for M^4 and makes it unique.

   Whether SSA or this algebra or both act as isometries of WCW is not clear: see the more detailed discussion in the Appendix of [L65].

3. One can assign this kind of hierarchy also to affine algebras assignable to the holomies of H and Virasoro algebras and their super counterparts. The geometric interpretation of these algebras would be as analogs of holonomy algebras, which serve at the level of H as the counterparts of broken gauge symmetries: isometries would correspond to non-broken gauge symmetries.

All these algebras, refer to them collectively by A, define inclusion hierarchies of sub-algebras A_n with the radial conformal weights given by n-ples of the weights of A.

1. I have proposed that the hierarchy of inclusions of hyperfinite factors of type II_1 to which one could perhaps assign ADE hierarchy could correspond to the hierarchies of subalgebras assignable to SSA and labelled by integer n: the radial conformal weights would be multiples of n. Only non-negative values of n would be allowed.

2. For a given hierarchy A_n, one has $n_1 \mid n_2 \mid .....$, where $\mid$ means "divides". At the n:th level of the hierarchy physical states are annihilated by A_n and $[A_n, A]$. For isometries, the corresponding Noether charges vanish both classically and quantally.

3. The algebra A_n effectively reduces to a finite-D algebra and A_n would be analogous to normal subgroup, which suggests that this hierarchy relates to a hierarchy of Galois groups associated with composite polynomials and having a decomposition to a product of normal subgroups.

4. These hierarchies could naturally relate to the hierarchies of inclusions of hyperfinite factors of type II_1 and also to hierarchies of Galois groups for extensions of rationals defined by composites $P_n \circ P_{n-1} \circ ...P_1$ of polynomials.

   The Galois correspondence raises questions.

1. Could the Dynkin diagrams for A_n be assigned to the McKay graphs of Galois groups acting as Weyl groups?

2. The Galois groups acting as Weyl group could be assigned to finite subgroups of $SU(2)$ acting as the covering group of quaternion automorphisms and of $SL(2,C)$ as covering group of H^3 isometries acting on tessellations of H^3. Also the finite subgroups of $SL(n,C)$ can be considered.

The proposed interpretation for the hierarchies of inclusions of HFFs is that they correspond to hierarchies for the inclusions of Galois groups defined by hierarchies of composite polynomials $P_n \circ ... \circ P_1$ interpreted as number theoretical evolutionary hierarchies.

If the relative Galois groups act as Weyl groups, they would be associated with the inclusions of HFFs naturally and the corresponding affine algebra (perhaps its finite field or p-adic variant) would characterize the inclusion. The proposed interpretation of the inclusion is in terms of measurement resolution defined by the included algebra. This suggests that a finite field version of the affine algebra could be in question.

This picture would suggest that hierarchies of polynomials for which the relative Galois groups act as Weyl groups are very special and could be selected in the number theoretical fight for survival.

One could argue that since number theoretic degrees of freedom relate to cognition, the quantum arithmetics for the irreps of Galois groups could make possible cognitive representations of the ordinary quantum states: roots would be represented by irreps. Irreps as quantum roots would correspond to ordinary roots as roots of the same monic polynomial and the direct sums of irreps would correspond to ordinary algebraic numbers.

About the interpretation of EDDs

An innocent layman can wonder whether the tensor products for 2-D spinor ground states for the discrete subgroups of the covering group of quaternionic automorphisms or of $SL(2,C)$ as covering group of H^3 isometries could give rise to representations contained by ADE type affine algebras characterized by the same EDD. These representations would be only a small part of the representations and perhaps define representation from which all states can be generated.

1. The reflections for the roots represented as irreps of Gal_I by Weyl group represented as Gal should assign to the irreps of G new copies so that the nodes of the entire root diagram would correspond to a set of representations obtained from the ground state. Infinite number of states labelled by conformal weight n is obtained.

2. Adjacency matrix A should characterize the angles between the roots represented as irreps? If the irreps of Gal_I and their Weyl images correspond to roots of a monic polynomial, they can be mapped to roots of an ordinary algebraic extension of rationals and the angles could correspond to angles between the points of extension regarded as vectors.

How the EDD characterizing the tensor products of the irreps of finite subgroups G with 2-D canonical representation V could define an ADE type affine algebra?

1. Roots are replaced with representations of G, which are in the general case direct sums of irreps. The identity representation should correspond to the scaling generator L_0, whose eigenvalues define integer value conformal weights.

   The inner products between the roots appearing in the Cartan matrix would correspond to the symmetric matrices defined by the structure constant n_{2ij} characterizing the tensor product. One might say that the inner products are matrix elements of the operator $\langle \xi_j | V \otimes \xi_i \rangle$ defined by the tensor product action of V. The diagonal elements of the Cartan matrix have value $+2$ and non-diagonal elements are negative integers or vanish.

2. Weyl reflections of roots with respect to roots involve negatives of the non-diagonal elements of Cartan matrix, which are negative so that the coefficient of the added root is positive represented as a direct sum. The negatives of the positive roots would correspond to negative integers and make sense only p-adically or for finite fields.

   The expression for the generalized Cartan matrix for McKay graph is known (`https://cuttly/QLRqrGt`) for the tensor products of representation with dimension d and multiplicities n^d_{ij} and is given by

   $$C^d_{ij} = d\delta_{ij} - n^d_{ij} \quad .$$

   For Dynkin diagrams the Cartan matrix satisfies additional conditions.

   Weyl reflection (`https://cutt.ly/kLRuXBP`) of the root v with respect to root α in the space of roots is defined as

   $$s_\alpha v = v - 2\frac{(v, \alpha)}{(\alpha, \alpha)}\alpha \quad .$$

   where $(.,.)$ is the inner product in V, which now corresponds to extension of rationals associated with Gal.

   The Weyl chamber is identified as the set of points of V for which the inner products (α, v) are positive. The Weyl group permutes the Weyl chambers.

3. The root system would be obtained from the roots of the quantum Dynkin diagram by Weyl reflections (Galois group as Weyl group) with respect to other roots. The number N of these roots is $n = d_C + 1$,where d_C is the dimension of Cartan algebra of the Dynkin diagram. The number N_I of irreps is the same: $N = N_I$. The Cartan matrix defines metric in the roots so that the reflections are well-defined also in the generalized picture.

4. It would seem that one must introduce an infinite number of copies of the Lie algebra realized in the usual manner (in terms of oscillator operators) with copies labelled by the conformal weight n. The commutators of these copies would be like for an ordinary affine algebra. Only the roots as labels of generators and possibly also the coefficient field would be replaced with their quantum variants.

5. What about the realization of the scaling generator L_0, whose Sugawara representation involves bilinears of the generators and their Hermitian conjugates with negative conformal weight? In the case of finite fields there are no obvious problems. Also the analog of Virasoro algebra can be realized in the case of finite fields. If one restricts consideration to finite quantum integers and their negatives as conformal weights, the map of the roots to algebraic numbers in extension of rationals is well defined.

5.3.4 Could the inclusion hierarchies of extensions of rationals correspond to inclusion hierarchies of hyperfinite factors?

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (https://cutt.ly/lXp6MDB) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

   It is. The number of conjugacy classes $g_i\sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD comes to rescue.

1. In the TGD framework, I am primarily interested in Galois groups, which are finite groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

   One would have a deep connection between number theory and HFFs.

5.4 Appendix: Isometries and holonomies of WCW as counterparts of exact and broken gauge symmetries

The detailed interpretation of various candidates for the symmetries of WCW [L37] has remained somewhat obscure. At the level of H, isometries are exact symmetries and analogous to unbroken gauge symmetries assignable to color interactions. Holonomies do not give rise to Noether charges and are analogous to broken gauge symmetries assignable to electroweak interactions. This observation can serve as a principle in attempts to understand WCW symmetries.

The division to isometries and holonomies is expected to take place at the level of WCW and this decomposition would naturally correspond to exact and broken gauge symmetries.

5.4.1 Isometries of WCW

The identification of the isometries of WCW is still on shaky ground.

1. In the H picture, the conjecture has been that symplectic transformations of δM^4_+ act as isometries. The hierarchies of dynamically emerging symmetries could relate to the hierarchies of sub-algebras (SSA_n) of super symplectic algebra SSA [L37] acting as isometries of the "world of classical worlds" (WCW) [K63] [L60].

   Each level in the hierarchy of subalgebras SSA_n of SSA corresponds to a transformation in which SSA_n acts as a gauge symmetry and its complement acts as genuine isometries of WCW: gauge symmetry breaking in the complement generates a genuine symmetry, which could correspond to Kac-Moody symmetry. By Noether's theorem, the isometries of WCW would give rise to local integrals of motion: also super-charges are involved. These charges are well-defined but they need not be conserved so that the interpretation as dynamically emerging symmetries must be considered.

   The symmetries would naturally correspond to a long range order. The hierarchies of SSA_n:s, of relative Galois groups and of inclusions of hyperfinite factors [K87, K28] could relate to each other as M^8-H duality suggests [L68].

   What can one say about the algebras SSA_n and the corresponding affine analogs KM_n (for affine algebras the generalized Cartan matrix is a product of a diagonal matrix with integer entries with a symmetric matrix). If n is prime, one can regard these algebras as local algebras in a finite field $G(p)$. Also extensions $G(p,n)$ of $G(p)$ induced by extensions of rationals can be considered. KM algebras in finite fields define what are called the incomplete Kac-Moody groups. Some of their aspects are discussed in the article "Abstract simplicity of complete Kac-Moody groups over finite fields" [A43]. It is shown that for $p>3$, affine groups are abstractly simple, that is, have no proper non-trivial closed subgroups. Complete KM groups are obtained as completions of incomplete KM groups and are totally disconnected: this suggests that they define p-adic analogs of Kac-Moody groups. Complete KM groups are known to be simple.

2. There are also different kinds of isometries. Consider first the light-cone boundary $\delta M^4_+ \times CP_2$ as an example of a light-like 3-surface. The isometries of CP_2 are symmetries. ΔM^4_+ is metrically equivalent with sphere S^2. Conformal transformations of S^2, which are made local with light-like coordinate r of δM^4_+, induce a conformal scaling of the metric of S^2 depending on r. It is possible to compensate for this scaling by a local radial scaling of r depending on S^2 coordinates such that the transformation acts as an isometry of δM^4_+.

   These isometries of ΔM^4_+ form an infinite-D group. The transformations of this group differ from those of the symplectic group in that the symplectic group of δM^4_+ is replaced with the isometries of δM^4_+ consisting of r-local conformal transformations of S^2 involving S^2-local radial scaling. There are no localizat of CP_2 isometries. This yields an analog of KM algebra.

   This group induces local spinor rotations defining a realization of KM algebra. Also super-KM algebra defined in terms of conserved super-charges associated with the modified Dirac action is possible. These isometries would be Noether symmetries just like those defined by SSA.

3. What about light-like partonic orbits analogous to $\delta M^4_+ \times CP_2$. Can one assign with them Kac-Moody type algebras acting as isometries?

   The infinite-D group of isometries of the light-cone boundary could generalize. If they leave the partonic 2-surfaces at the ends of the orbit X^3_L, they could be seen as 3-D general coordinate transformations acting as internal isometries of the partonic 3-surface, which cannot be regarded as isometries of a fixed subspace of H. These isometries do not affect the partonic 3-surface as a whole and cannot induce isometries of WCW.

   However, if X^3_L is connected by string world sheets to other partonic orbits, these transformations affect the string world sheets and there is a real physical effect, and one has genuine isometries. Same is true if these transformations do not leave the partonic 2-surfaces at the ends of X^3_L invariant.

5.4.2 Holonomies of WCW

What about holonomies at the level of WCW? The holonomies of H acting on spinors induces a holonomy at the level of WCW: WCW spinors identified as Fock states created by oscillator operators of the second quantized H spinors. This would give a generalized KM-type algebra decomposing to sub-algebras corresponding to spin and electroweak quantum numbers. This algebra would have 3 tensor-factors. p-Adic mass calculations imply that the optimal number of tensor factors in conformal algebra is 5 [K41]. 2 tensor factors are needed.

1. SSA would give 2 tensor factors corresponding to δM^4_+ (effectively S^2) and CP_2. This gives 5 tensor factors which is the optimal number of tensor factors in p-adic mass calculations [K41]. SSA Noether charges are well-defined but not conserved. Could SSA only define a hierarchy of dynamical symmetries. Note however that for isometries of H conservation holds true.

2. Also the isometries of δM^4 and of light-like orbits of partonic 2-surfaces give the needed 2 tensor factors. Also this alternative would give inclusion hierarchies of KM sub-algebras with conformal weights coming as multiples of the full algebra. The corresponding Noether charges are well-defined but can one speak of conservation only in the partonic case? One can even argue that the isometries of $\delta M^4_+ \times CP_2$ define a more plausible candidate for inducing WCW isometries than the symplectic transformations. p-Adic mass calculations conform with this option.

To sum up, WCW symmetries would have a nice geometric interpretation as isometries and holonomies. The details of the interpretation are however still unclear and one must leave the status of SSA open.

Chapter 6

Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole

6.1 Introduction

I have had a very interesting discussions with Baba Ilya Iyo Azza about von Neumann algebras [A89]. I have a background of physicist and have suffered a lot of frustration in trying to understand hyperfinite factors of type II_1 (HFFs, `https://cutt.ly/OX8uP32`) by trying to read mathematicians' articles.

I cannot understand without a physical interpretation and associations to my own big vision TGD. Again I stared at the basic definitions, ideas and concepts trying to build a physical interpretation. This is not my first attempt to understand the possible role of HFFs in TGD: I have written already earlier of the possible role of von Neumann algebras in the TGD framework [K87, K28]. In the sequel I try to summarize what I have possibly understood with my meager technical background.

In the first section I will redescribe the basic notions and ideas related to von Neumann algebras as I see them now, in particular HFFs, which seem to be especially relevant for TGD because of their "hyperfiniteness" property implying that they are effectively finite-D matrix algebras.

There are also more general factors of type II_1, in particular those related to the notion of free probability (`hhttps://cutt.ly/SX2ftyx`), which is a notion related to a theory of non-commutative random variables. The free group generated by a finite number of generators is basic notion and the group algebras associated with free groups are factors of type II_1. The isomorphism problem asks whether these algebras are isomorphic for different numbers of generators. These algebras are not hyperfinite and from the physics point of view this is not a good news.

6.1.1 Basic notions of HFFs from TGD perspective

In this section I will describe my recent, still rather primitive physicist's understanding of HFFs. Factor M and its commutant M' are central notions in the theory of von Neumann algebras. An important question, not discussed earlier, concerns the physical counterparts of M and M'. I will not discuss technical details: I have made at least a noble attempt to do this earlier [K87, K28].

1. In the TGD framework, one can distinguish between quantum degrees of freedom and classical ones, and classical physics can be said to be an exact part of quantum physics.

2. The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) is briefly summarized in the Appendix. The formulation involves hierarchies A_n of 3 kinds of algebras; supersymplectic algebras SSA_n acting on $\delta M^4_+ \times CP_2$ and assumed to induce isometries of WCW, affine algebras Aff_n associated with isometries and holonomies of $H = M^4 \times CP_2$ acting on light-like partonic orbits, and isometries I_n of the light-cone boundary δM^4_+.

At the H-side, quantum degrees of freedom are assignable to A_n, which would correspond to M.

In zero energy ontology (ZEO) [K89] states are quantum superpositions of preferred extremals. Preferred extremals depend on zero modes, which are symplectic invariants and do not appear in the line element of WCW. Zero modes serve as classical variables, which commute with super symplectic transformations and could correspond to M' for SSA_n at H-side. Similar identification of analogs of zero modes should be possible for Aff_n and I_n.

3. In the number theoretic sector at the M^8-side, braided group algebras would correspond to quantum degrees of freedom, that is M. M' would correspond to some number theoretic invariants of polynomials P determining the space-time surface in H by $M^8 - H$ duality [L45, L46]. The set of roots of P and ramified primes dividing the discriminant of P are such invariants.

6.1.2 Bird's eye view of HFFs in TGD

A rough bird's eye view of HFFs is discussed with an emphasis on their physical interpretation. There are two visions of TGD: the number theoretic view [L23, L24] and the geometric view [K35, K20, K88, K63] and $M^8 - H$ duality relates these views [L60, L45, L46].

1. At the M^8 side, the p-adic representations of braided group algebras of Galois groups associated with hierarchies of extensions of rationals define natural candidates for the inclusion hierarchies of HFFs.

   Braid groups represent basically permutations of tensor factors and the same applies to the braided Galois groups with S_n restricted to the Galois group.

   A good guess is that braid strands correspond to the roots of a polynomial labelling mass shells H^3 in $M^4 \subset M^8$.

2. The 3-D mass shells define a 4-surface in M^8 by holography based on associativity, which makes possible holography.

   The condition that the normal space N of the 4-surface X^4 in M^8 is associative and contains a 2-D commutative sub-space X^2, guarantees both holography and $M^8 - H$ duality mapping this 4-surface $X^4 \subset M^8$ to a space-time surface $Y^4 \subset H$.

   The 2-D commutative space $X^2 \subset N$ can be regarded as a normal space of the 6-D counterpart of twistor space $T(M^4)$. $T(M^4)$ is mapped by $M^8 - H$ duality to a point of the twistor space $T(CP_2) = SU(3)/U(1) \times U(1)$ of CP_2. This map is assumed to define the twistor space $T(Y^4) \subset T(M^4) \times T(CP_2)$ as a preferred extremal [L62, L63].

3. The physical picture strongly suggests that also string world sheets and partonic 2-surfaces in $Y^4 \subset H$ are needed. They are are assumed to correspond to singularities for the map to H. A natural conjecture is that the 2-D subspace $X^2 \subset N$ is mapped to a 2-D subspace $Y^2 \subset T$ of the tangent space T of X^4 by a multiplication with a preferred octonionic imaginary unit in T.

   How could this preferred octonionic unit be determined?

   (a) Complexified octonionic units in the tangent space of M^8_c decomposes under $SU(3) \subset G_2$, having interpretation as color group, to representations $1_1 \oplus 1_2 \oplus 3 \oplus \overline{3}$. 1_1 and 1_2 correspond to the real unit I_0 and imaginary unit I_1 and 3 and $\overline{3}$ correspond to color triplets analogous to quarks and antiquarks.

   (b) Complexified quaternionic sub-space defining N corresponds to color singlets I_0, I_1, and quarks I_2, I_3 with $(Y = -1/3, I_3 = 1/2)$ and $Y = 2/3, I_3 = 0)$. The complement T corresponds to quark I_4 $(Y = -1/3, I_3 = 1/2)$ and 3 antiquarks (I_5, I_6, I_7). The octonionic multiplication of the units of quaternionic subspace by quark I_4 gives T as the orthogonal complement of the quaternionic sub-space N.

(c) This multiplication would assign to $X^2 \subset N$ 2-D subspace of T and also its orthogonal complement Y^2 in T. If the distributions of X^2 and Y^2 are integrable, they define the slicing of X^4 by partonic 2-surfaces and string world sheets. The tangent spaces for them would correspond to the local choice of I_0, I_1 and I_2, I_3. X^2 and Y^2 at different points would differ by a local SU(3) transformation. In fact, the 4-surface in M^8 would correspond to a complex color gauge transformation [L45, L46].

This choice could correspond to what I have called Hamilton-Jacobi (H-J) structure [K7] in X^4 defining a slicing of X^4 defined by an integrable distribution of pairs of orthonormal 2-surfaces analogous to the choice of massless wave vector and orthogonal polarization plane depending on the point of X^4 or equivalently on the point of M^4 as its projection. H-J structure would also define the analog of Kähler structure in M^4 strongly suggested by twistor lift.

The original proposal was that the H-J structure is associated with M^4, and one cannot completely exclude the possibility that the projection of the proposed slicing to M^4 defines H-J. The idea about single H-J structure is not physical. Dynamical H-J structure does not conform with the idea that M^4 is completely non-dynamical. However, if the H-J structure is determined by the choice $X^2 \subset N$ and defines H-J structure in X^4, this objection can be circumvented.

At the H side there are 3 algebras.

1. The subalgebras SSA_n of super-symplectic algebra (SSA) are assumed to induce isometries of WCW. Since SSA and also other algebras have non-negative conformal weights, it has a hierarchy of subalgebras SSA_n with conformal weights coming as n-multiples of those for SSA.

2. There are also affine algebras Aff associated with H isometries acting on light-like orbits of partonic 2-surfaces and having similar hierarchy of Aff_n. Both isometries and holonomies of H are involved.

3. Light-cone boundary allows infinite dimensional isometry group I consisting of generalized conformal transformation combined with a local scaling allowing similar hierarchy I_n.

One should understand how the number theoretic and geometric hierarchies relate to each other and a good guess is that braided group algebras act on braids assignable to SSA_n with n interpreted as the number of braid strands and thus the degree n of P.

Also the interpretational problems related to quantum measurement theory and probability interpretation are discussed from the TGD point of view, in which zero energy ontology (ZEO) allows us to solve the basic problem of quantum measurement theory.

6.1.3 $M^8 - H$ duality and HFFS

$M^8 - H$ duality [L45, L46] suggests that the hierarchies of extensions of rationals at the number theoretic side and hierarchies of HFFs at the geometric side are closely related.

The key idea is that the braided Galois groups at M^8-side interact on algebras $A_n \in \{SSA_n, Aff_n, I_n\}$ at H level as number theoretic braid groups permuting the tensor factors assignable to the braid strands, which correspond to the roots of the polynomial P.

The basic notions associated with a polynomial P with rational coefficients having degree n are its n roots, ramified primes as factors of the discriminant defined by the difference of its roots, and Galois group plus a set of Galois invariants such as symmetric polynomials of roots. The Galois group is the same for a very large number of polynomials P. The question concerns the counterparts of these notions at the level of H?

An educated guess is that the n roots of P label the strands of an n-braid in H assignable to A_n, ramified primes correspond to physically preferred p-adic primes in the adelic structure formed by various p-adic representations $A_{n,p}$ of the algebras A_n and the Galois group algebra associated with the polynomial P with degree n.

This picture suggests a generalization of arithmetics to quantum arithmetics based on the replacement of + and × with ⊕ and ⊗ and replacement of numbers with representations of groups or algebras [L68]. This implies a generalization of adele by replacing p-adic numbers with the p-adic quantum counterparts of algebras A_n.

The mysterious McKay correspondence [A108] has inspired several articles during years [L17, L42, L41, L68] but it is fair to say that I do not really understand it. Hence I could not avoid the temptation to attack this mystery also in this article.

6.1.4 Infinite primes

The notion of infinite primes [K72, K44] is one of the ideas inspired by TGD, which has waited for a long time for its application. Their construction is analogous to a quantization of supersymmetric arithmetic quantum field theory.

1. The analog of Dirac sea X is defined by the product of finite primes and one "kicks" from sea a subset of primes defining a square free integer n_F to get the sum $X/n_F + n_F$. One can also add bosons to X/n_F *resp.* n_F multiplying it with integer n_{B_1} *resp.* n_{B_2}, which is divisible only by primes dividing Z/n_F *resp.* n_F.

2. This construction generalizes and one can form polynomials of X to get infinite primes analogous to bound states. One can consider instead of $P(X)$ a polynomial $P(X,Y)$, where Y is the product of all primes at the first level thus involving the product of all infinite primes already constructed, and repeat the procedure. One can repeat the procedure indefinitely and the formal interpretation is as a repeated quantization. The interpretation could be in terms of many-sheeted space-time or abstraction process involving formation of logical statements about statements about ...

3. The polynomials Q could also be interpreted as ordinary polynomials. If $Q(X) = P(X)$, where $P(X)$ is the polynomial defining a 4-surface in M^8, the space-time surface X^4 in H would correspond to infinite prime. This would give a "quantization" of P defining the space-time surface.

   The polynomial P defining 4-surface in H would fix various quantum algebras associated with it. The polynomials $P(X_1, X_2, ...X_n)$ could be interpreted as $n-1$-parameter families defining surfaces in the "world of classical worlds" (WCW) [L60] (for the development of the notion see [K35, K20, K88, K63]).

4. X is analogous to adele and infinite primes could be perhaps seen as a generalization of the notion of adele. One could assign p-adic variants of various HFFs to the primes defining the adele and + and × could be replaced with ⊕ and ⊗. The physical interpretation of ramified primes of P is highly interesting.

In the last section, I try to guess how the fusion of these building blocks by using the ideas introduced in the previous sections could give rise to what might be called quantum TGD. It must be made clear that the twistor lift of TGD [L62, L63] is not considered in this work.

6.2 Basic notions related to hyperfinite factors of type II_1 from TGD point of view

In this section, the basic notions of hyperfinite factors (HFFs) as a physicists from the TGD point of view will be discussed. I have considered HFFs earlier several times [K87, K28] and will not discuss here the technical details of various notions.

6.2.1 Basic concepts related to von Neumann algebras

John von Neumann proposed that the algebras, which now carry his name are central for quantum theory [A89]. Von Neumann algebra decomposes to a direct integral of factors appearing and there are 3 types of factors corresponding to types I, II, and III.

Inclusion/embedding as a basic aspect of physics

Inclusion (https://cutt.ly/NX8eWwa, https://cutt.ly/cX8eUuf, https://cutt.ly/4X8ePn6)) is a central notion in the theory of factors. Inclusion/embedding involving induction of various geometric structures is a key element of classical and quantum TGD.

One starts from the algebra $B(H)$ of bounded operators in Hilbert space. This algebra has naturally hermitian conjugation $*$ as an antiunitary operation and therefore one talks of C_* algebras. von Neumann algebra is a subalgebra of $B(H)$. Already here an analog of inclusion is involved (https://cutt.ly/3XkPO2s). There are also inclusions between von Neumann algebras, in particular HFFs.

What could the inclusion of von Neumann algebra to $B(H)$ as subalgebra mean physically? In the TGD framework, one can identify several analogies.

1. Space-time is a 4-surface in $H = M^4 \times CP_2$: analog of inclusion reducing degrees of freedom.

2. Space-time is not only an extremal of an action [K7] [L61] but a preferred extremal (PE), which satisfies holography so that it is almost uniquely defined by a 3-surface. This guarantees general coordinate invariance at the level of H without path integral. I talk about preferred extremals (PEs) analogous to Bohr orbits. Space-time surface as PE is a 4-D minimal surface with singularities [L61]: there is an analogy with a soap film spanned by frames. This implies a small failure of determinism localizable at the analogs of frames so that holography is not completely unique.

   Holography means that very few extremals are physically possible. This Bohr orbit property conforms with the Uncertainty Principle. Also HFFs correspond to small sub-spaces of $B(H)$. Quantum classical correspondence suggests that this analogy is not accidental.

The notion of commutant and its physical interpretation in the TGD framework

The notion of the commutant M' of M, which also defines HFF, is also essential. What could be the physical interpretation of M'? TGD suggests 3 important hierarchies of HFFs as algebras A_n. A_n could correspond to super-symplectic algebras SSA_n acting at $\delta M^4_- \times CP_2$; to an affine algebras Aff_n acting at the light-like partonic orbits; or to an isometry algebra I_n acting at δM^4_+. All these HFF candidates have commutants and would have interpretation in terms of quantum-classical correspondence.

One can consider SSA as an example.

1. In TGD, one has indeed an excellent candidate for the commutant. Supersymplectic symmetry algebra (SSA) of $\delta M^4_+ \times CP_2$ (δM^4_+ denotes the boundary of a future directed light-cone) is proposed to act as isometries of the "world of classical worlds" (WCW) consisting of space-time surfaces as PEs (very, very roughly).

   Symplectic symmetries would be generated by Hamiltonians, which are products of Hamiltonians associated with δM^4_+ (metrically sphere S^2) and CP_2. Symplectic symmetries are conjectured to act as isometries of WCW and gamma matrices of WCW extend symplectic symmetries to super-symplectic ones.

   Hamiltonians and their super-counterparts generate the super-symplectic algebra (SSA) and quantum states are created by using them. SSA is a candidate for HFF. Call it M. What about M?

2. The symplectic symmetries leave invariant the induced Kähler forms of CP_2 and contact form of δM^4_+ (assignable to the analog of Kähler structure in M^4).

3. The wave functions in WCW depending of magnetic fluxes defined by these Kähler forms over 2-surfaces are physically observables which commute SSA and with M. These fluxes are in a central role in the classical view about TGD and define what might perhaps be regarded as a dual description necessary to interpret quantum measurements.

   Could M' correspond or at least include the WCW wave functions (actually the scalar parts multiplying WCW spinor fields with WCW spinor for a given 4-surface a fermionic Fock state) depending on these fluxes only? I have previously talked of these degrees of freedom as zero modes commuting with quantum degrees of freedom and of quantum classical correspondence.

4. There is $M-M'$ correspondence also for number theoretic degrees of freedom, which naturally appear from the number theoretic M^8 description mapped to H-description. Polynomials P associated with a given Galois group are analogous to symplectic degrees of freedom with given fluxes as symplectic invariants. Galois groups and Galois invariants are "classical" invariants at the M^8 side and should have counterparts on the H side. For instance, the degree n of polynomial P could correspond to the number of braid stran

More algebraic notions

There are further algebraic notions involved. The article of John Baez (`https://cutt.ly/VXlQyqD`) describes these notions nicely.

1. The condition $M'' = M$ is a defining algebraic condition for von Neumann algebras. What does this mean? Or what could its failure mean? Could M'' be larger than M? It would seem that this condition is achieved by replacing M with M''.

   $M'' = M$ codes algebraically the notion of weak continuity, which is motivated by the idea that functions of operators obtained by replacing classical observable by its quantum counterpart are also observables. This requires the notion of continuity. Every sequence of operators must approach an operator belonging to the von Neumann algebra and this can be required in a weak sense, that is for matrix elements of the operators.

   Does $M'' = M$ mean that the classical descriptions and quantum descriptions are somehow equivalent? At first, this looks nonsensical but when one notices that the scalar parts of WCW spinor fields correspond to wave functions in the zero mode of WCW which do not appear in the line element of WCW, this idea starts to look more sensible. In quantum measurements the outcome is indeed expressed in terms of classical variables. Zero modes and quantum fluctuating modes would provide dual descriptions of physics.

2. There is also the notion of hermitian conjugation defined by an antiunitary operator J: $a^\dagger = JAJ$. This operator is absolutely essential in quantum theory and in the TGD framework it is geometrized in terms of the Kähler form of WCW. The idea is that entire quantum theory, rather than only gravitation or gravitation and gauge interactions should be geometrized. Left multiplication by JaJ corresponds to right multiplication by a.

Connes tensor product and category theoretic notions

Connes tensor product (Connes fusion) [A29] appears in the construction of the hierarchy of inclusions of HFFs. For instance, matrix multiplication has an interpretation as Connes tensor product reduct tensor product of matrices to a matrix product. The number of degrees of freedom is reduced. The tensor product $A \otimes_R B$ depends on the coefficient ring R acting as right multiplication in A and left multiplication in B. If the dimension of R increases, the dimension of A (B) as a left/right R module is reduced. For instance, A as an A-module is 1-dimensional.

Also category theory related algebraic notions appear. I still do not have an intuitive grasp about category theory. In any case, one would have a so-called 2-category (`https://cutt.ly/3XkPO2s`). M and N correspond to 0-morphisms (objects). One can multiply arguments of functions in $L^2(M)$ and $L^2(N)$ by M or N.

Bimodule (`https://cutt.ly/EX885WA`) is a key notion. For instance the set of $R_{m,n}$ of $m \times n$ matrices is a bimodule, which is a left (right) module with respect to $m \times m$ ($n \times n$) matrices. One can replace matrices with algebras. The bimodule ${}_M M_M$ *resp.* ${}_N N_N$ is analogous to $m \times m$ *resp.* $n \times n$ matrices. They correspond to 1-morphisms, which behave like units. The bimodule ${}_M N_N$ *resp.* ${}_N N_M$ is analogous to $m \times n$ *resp.* $n \times m$ matrices. These two bimodules correspond to a generating 1-morphisms mapping N to M *resp.* M to N. Bimodule map corresponds to 2-morphisms. Connes tensor product defines what category theorists call a tensor functory.

The notions of factor and trace

The notion of factor as a building block of more complex structures is central and analogous to the notion of simple group or prime. Factor is a von Neumann algebra, which is simple in the sense that it has a trivial center consisting of multiples of unit operators. The algebra is direct sum or integral over different factors.

The notion of trace is fundamental and highly counter intuitive. For the factors of type I, it is just the ordinary trace and the trace $Tr(I)$ of the unit operator is equal to the dimension n of the Hilbert space. This notion is natural when direct sum is the key notion. For the other factors, the situation is different.

Factors can be classified into three types: $I, II, andIII$.

1. For factors of type I associated with three bosons, the trace equals n in the n-D case and ∞ in the infinite-D case.

2. A highly non-intuitive and non-trivial axiom relating to HFFs as hyperfinite factors of type II_1 is that the trace of the unit operator satisfies $Tr(Id) = 1$: for factors of type II (see the article of Popa at `https://cutt.ly/KX8yOFs`). This definition is natural in the sense that being a subsystem means being a tensor factor rather than subspace.

   The intuitive idea is that the density matrix for an infinite-D system identified as a unit operator gives as its trace total probability equal to one. These factors emerge naturally for free fermions. "Hyperfinite" expresses the fact that the approximation of a factor with its finite-D cutoff is an excellent approximation.

   HFFs are extremely flexible and can look like arbitrarily high-dimensional factor I_n. For instance, one can extract any matrix algebra $M^n(C)$ as a tensor factor so that one has $M = M^n(C) \otimes M^{1/n}$ by the multiplicativity of dimensions in the tensor product. Should one interpret this by saying that measurement can separate from a factor an $n - D$ Hilbert space and that $M^{1/n}$ is something that remains inaccessible to the measurements considered? If one introduces the notion of measurement resolution in this manner, the description of measurement could be based on factors of tyoe I_n.

3. The factors of type II_∞ are tensor products of infinite-D factors of type I and HFFs and could describe free bosons and fermions.

4. In quantum field theory (QFT), factors of type III appear and in this case the notion of trace becomes useless. These factors are pathological and in QFT they lead to divergence difficulties. The physical reason is the idea about point-like particles, which is too strong an idealization.

   In the TGD framework, the generalization of a point-like particle to 3-surface saves from these difficulties and leads to factors of type I and HFFs. In TGD, finite measurement resolution is realized in terms of a unique number theoretic discretization, which further simplifies the situation in the TGD framework.

6.2.2 Standard construction for the hierarchy of HFFs

Consider now the standard construction leading to a hierarchy of HFFs and their inclusions.

1. One starts from an inclusion $M \subset N$ of HFFs. I will later consider what these algebras could be in the TGD framework.

2. One introduces the spaces $L^2(M)$ *resp.* $L^2(N)$ of square integrable functions in M *resp.* N.

   From the physics point of view, bringing in "L^2" is something extremely non-trivial. Space is replaced with wave functions in space: this corresponds to what is done in wave mechanics, that is quantization! One quantizes in M, particles as points of M are replaced by wave functions in M, one might say.

3. At the next step one introduces the projection operator e as a projection from $L^2(N)$ to $L^2(M)$: this is like projecting wave functions in N to wave functions in M. I wish I could really understand the physical meaning of this. The induction procedure for second quantized spinor fields in H to the space-time surface by restriction is completely analogous to this procedure.

After that one generates a HFF as an algebra generated by e and $L^2(N)$: call it $\langle L^2(N), e\rangle$. One has now 3 HFFs and their inclusions: $M_0 \equiv M$, $M_1 \equiv N$, and $\langle L^2(N), e\rangle \equiv M_2$.

An interesting question is whether this process could generalize to the level of induced spinor fields?

4. Even this is not enough! One constructs $L^2(M_2) \equiv M_3$ including M_2. One can continue this indefinitely. Physically this means a repeated quantization.

   One could ask whether one could build a hierarchy M_0, $L^2(M_0)$,..., $L^2(L^2...(M_0))..$): why is this not done?

   The hierarchy of projectors e_i to M_i defines what is called Temperley-Lieb algebra [A129] involving quantum phase $q = exp(i\pi/n)$ as a parameter. This algebra resembles that of S_∞ but differs from it in that one has projectors instead of group elements. For the braid group $e_i^2 = 1$ is replaced with a sum of terms proportional to e_i and unit matrix: mixture of projector and permutation is in question.

5. There is a fascinating connection in TGD and theory of consciousness. The construction of what I call infinite primes [K72, K44] is structurally like a repeated second quantization of a supersymmetric arithmetic quantum field theory involving fermions and bosons labelled by the primes of a given level I conjectured that it corresponds physically to quantum theory in the many-sheeted space-time.

   Also an interpretation in terms of a hierarchy of statements about statements about .... bringing in mind hierarchy of logics comes to mind. Cognition involves generation of reflective levels and this could have the quantization in the proposed sense as a quantum physical correlate.

6.2.3 Classification of inclusions of HFFs using extended ADE diagrams

Extended ADE Dynkin diagrams for ADE Lie groups, which correspond to finite subgroups of $SU(2)$ by McKay correspondence [A108, A105, A82], discussed from the TGD point of view in [L68], characterize inclusions of HFFs.

For a subset of ADE groups not containing E_7 and D_{2n+1}, there are inclusions, which correspond to Dynkin diagrams corresponding quantum groups. What is interesting that E_6 (tetrahedron) and E_8 (icosahedron/dodecahedron) appear in the TGD based model of bioharmony and genetic code but not E_7 (cube and octahedron) [L54].

1. Why finite subgroups of $SU(2)$ (or $SU(2)_q$) should characterize the inclusions in the tunnel hierarchies with the same value of the quantum dimension $M_{n+1} : M_n$ of quantum group?

   In the TGD interpretation M_{n+1} reduces to a tensor product of M_n and quantum group, when M_n represents reduced measurement resolution and quantum group the added degrees of freedom. Quantum groups would represent the reduced degrees of freedom. This has a number theoretical counterpart in terms of finite measurement resolution obtained when an extension of ... of rationals is reduced to a smaller extension. The braided relative Galois group would represent the new degrees of freedom.

2. One can algebraically identify HFF as a "tunnel" obtained by iterated standard construction as an infinite tensor power of $GL(2,c)$ or $GL(n,C)$. The analog of the McKay graph for the irreps of a closed subgroup of $GL(2,C)$ defines an invariant characterizing the fusion rules involved with the reduction of the Connes tensor products. This invariant reduces to the McKay graph for the tensor products of the canonical 2-D representation with the irreps of a *finite* rather than only closed subgroups of $SU(2)$. This must take place also for $GL(n,C)$. Why?

   The reduction of degrees of freedom implied by the Connes tensor product seems to imply a discretization at the level of $SU(2)$ and replace closed subgroups of $SU(2)$ with finite subgroups. This conforms with the similarity of the representation theories of discrete and closed groups. In the case of quantum group representations only a finite number of ordinary finite-D group representations survive.

All this conforms with the TGD view about the equivalence of number-theoretic discretization and inclusions as descriptions of finite measurement resolution.

In the TGD framework, $SU(2)$ could correspond to a covering group of quaternionic automorphisms and number theoretic discretization (cognitive representations) would naturally lead to discrete and finite subgroups of $SU(2)$.

6.3 TGD and hyperfinite factors of type II_1: a bird's eye of view

In this section, a tentative identification of hyperfinite factors of type II_1 (HFFS) in the TGD framework [K87, K28] is discussed. Also some general related to the interpretation of HFFs and their possible resolution in the TGD framework are considered.

6.3.1 Identification of HFFs in the TGD framework

Inclusion hierarchies of extensions of rationals and of HFFs

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (`https://cutt.ly/lXp6MDB`) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

   It is. The number of conjugacy classes $g_i\sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD could comes to the rescue.

1. In the TGD framework, I am primarily interested in Galois groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups, which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

   One would have a deep connection between number theory and HFFs.

How could HFFs emerge in TGD?

What could HFFs correspond to in the TGD framework? Consider first the situation at the level of M^8.

1. Braid group $B(G)$ of group (say Galois group as subgroup of S_n) and its group algebra would correspond to $B(G)$ and $L^2(B(G)\ )$. Braided Galois group and its group algebra could correspond to $B(G)$ and $L^2(B(G))$.

   The inclusion of Galois group algebra of extension to its extension could naturally define a Connes tensor product. The additional degrees of freedom brought in by extension of extension would be below measurement resolution.

2. Composite polynomials $P_n \circ .... \circ P_1$ are used instead of a product of polynomials naturally characterizing free n-particle states. Composition would describe interaction physically: the degree is the product of degrees of factors for a composite polynomial and sum for the product of polynomials.

   The multiplication rule for the dimensions holds also for the tensor product so that functional composition could be also seen as a number theoretic correlate for the formation of interacting many particle states.

3. Compositeness implies correlations and formation of bound states so that the number of degrees of freedom is reduced. The interpretation as bound state entanglement is suggestive. This hierarchical entanglement could be assigned with directed attention in the TGD inspired theory of consciousness [L50].

   An alternative interpretation is in terms of braids of braids of ... of braids with braid strands at a given level characterized by the roots of P_i. These interpretations could be actually consistent with each other.

4. Composite polynomials define hierarchies of Galois groups such that the included Galois group is a normal subgroup. This kind of hierarchy could define an increasing sequence of inclusions of braided Galois groups.

Consider the situation at the H level.

1. At the level of H, elements of the algebras $A \in \{SSA, Aff, I\}$a associated with super-symplectic symmetries acting at δM^4_+, affine isometries acting at light-like partonic orbites, isometries of δM^4_+, are labelled by conformal weights coming as non-negative integers. Also algebraic integers can be considered but for physical states conformal confinement requires integer valued conformal weights.

2. One can construct a hierarchy of representations of A such that subalgebras A_n with conformal weights $h \geq 0$ coming as multiples of n and the commutator $[A_n, A]$ annihilate the physical states. These representations are analogous to quantum groups and one can say that A_n defines a finite measurement resolution in A. A_{nk}, $k \geq 1$ is included by A_n for and one has a reversed sequence of inclusions.

   One can construct inclusion hierarchies defined by the sequences $1 \div n_1 \div n_2 \div ....$ "$n_{-1} = 1$" corresponds to SSA. The factor spaces $A_{n_k}/A_{n_{k+1}}$ are analogs of quantum group-like objects associated with Jones inclusions and the interpretation is in terms of finite measurement resolution defined by $A_{n_{k+1}}$.

The factor spaces A/A_{n_k} define inclusion hierarchies with an increasing measurement resolution.

6.3.2 Could the notion of free probability be relevant in TGD?

In discussions with Baba Ilya Iyo Azza, I learned about the notion of free probability (`https://cutt.ly/LCY51sy`) assignable to von Neumann algebras. This algebra is II_1 factor. Originally, the notion was discovered by Voiolescu [A128] in order to attack some operator algebra problems, in particular free group factor isomorphism problem and Voiolescu demonstrated there is an infinity of von Neumann free group factors, which can be isomorphic. One can ask whether the free probability could have physical applications. In particular, whether the HFFs emerging naturally in TGD are consistent with this notion.

I try first to describe the notion of free probability as I understand it, on basis of what I learned in the discussions.

1. Free probability theory and classical probability theory differ because the latter is commutative and the former is highly noncommutative, and the notion of independence differs for them.

   In the classical theory, the expectations of the variables $X, Y, ...$ are commutative whereas in free probability theory they become observables represented by operators, which in general are non-commutative. The expectations for independent variables satisfy $E(XY) = E(X)E(Y)$ and more generally $E(X^mY^n) = E(X^m)E(Y^n)$. The expectations for powers are called moments.

   The free probability theory generalizes independent variables to free, in general non-commutative, operators $a, b, ...$ of von Neumann algebra M. The mean value $E(X)$ is replaced with a vacuum expectation value $\tau(a)$ of a, as physicists would call it. The expectations define what mathematicians call a normal state. Here τ, defining the vacuum expectation, denotes a linear functional in M.

   The random variable a can act on the argument of square integrable functions $F(m)$ defined in non-commutative von Neumann algebra M defining a commutative algebra $L^2(M)$. The action of a is non-commutative right or left multiplication of the argument of $F(m)$. One can speak of non-commutative probability space.

   Could group algebras and braid group algebra represent free algebras? Unfornatunately not. It is known that HFF probability is not consistent with free algebra property.

2. In the classical theory of independent random variables, one has $E(XY) = E(X)E(Y)$ and it is possible to express all expectations of monomials of $X_1, X_2, ...$ of polynomials of variables $X_1, X_2, ..$ in terms of moments $E(X_i^n)$.

   For free probability theory an analogous situation prevails although the formulas are not identical. Consider factor M of type II_1, which in the case ff free algebras cannot be hyperfinite. A linear functional $\tau(a)$ corresponds to vacuum expectation value, using the language of physicists. One has $\tau(1) = 1$. This corresponds to the condition $Tr(Id) = 1$. One has pointless space in the sense that the projector to a ray of Hilbert state defined by M has a vanishing trace. This corresponds to a finite measurement resolution requiring that the trace of the projector characterizing quantum measurement is a nonvanishing number.

   $\tau(ab) = \tau(a)\tau(b)$ is true for the generators of the free algebra and states that there is no correlation between a and b. This is however not true in general.

   [Note that an analogous condition holds true for the correlators of free quantum field fields at the level of momentum space: the n-point correlation functions reduce to products of momentum space propagators.]

   For instance, one would have

   $$\tau(abab) = \tau(a^2)\tau(b)^2 + \tau(a)^2\tau(b^2) - 2\tau(a)^2\tau(b)^2$$

instead of $E(XYXY) = E(X^2)E(Y^2)$ for classical independent variables. Also now however only powers of a and b appear in the formula. This reduction of the expectations to the momenta $\tau(a^n)$ would hold quite generally.

3. A more precise definition is as follows (`https://cutt.ly/LCY51sy`). Unital subalgebras $A_1, ..., A_m$ are said to be freely independent if the expectation of the product $a_1...a_n$ is zero whenever each a_j has zero expectation, lies in an A_k, no *adjacent* a_j's come from the same subalgebra A_k, and n is nonzero. Random variables are freely independent if they generate freely independent unital subalgebras.

4. The lattice of non-crossing partitions (`https://cutt.ly/jCY6jNe`) for a finite set ordered cyclically, distinguishes free probability theory from lattice of all partition in the theory if independent random variables. Two partitions ab and xy are non-crossing if their elements do not correspond to order $axby$. The subsets of a non-crossing partition consist of elements, which are adjacent in this ordering and form connected subsets with k_i elements in which a cyclic subgroup $Z_{k_i} \subset Z_n$ acts. The expression of an element of S_n as a product of elements of cyclic subgroups Z_{n_k} of S_n corresponds to this kind of partition.

   Interestingly, in the construction of the non-planar parts of the twistor amplitudes similar cyclic ordering plays an important role. The problem of the twistor constructed are non-planar amplitudes which do not allow cyclic ordering. Could it be that the non-planar parts of the amplitudes do not have counterparts in a deeper theory utilizing HFFs? If so, free probability could code a very profound aspect of quantum theory.

Free random variables could correspond to the generators of von Neumann algebra of type II_1. My un-educated guess was that also HFFs realize free probability. I was wrong: thanks for Baba Ilya Iyo Azza for noticing this. It however seems that something highly reminiscent of free probability emerges for the algebras involved with TGD.

1. In the TGD framework, the generators typically generate an algebra of observables having interpretation as algebra of symmetries, such as affine algebra or super symplectic algebra.

2. ab would correspond to the product of say affine algebra generators a and b labelled by quantum numbers which are additive in the product. $\tau(a)$ would vanish as a vacuum expectation value of a generator with non-vanishing quantum numbers so that for generators one should have $\tau(ab) = \tau(a)\tau(b) = 0$.

   ab is expressible in terms of commutator and anticommutator as the sum $[a, b]/2 + \{a, b\}/2$. Both terms vanish if the quantum numbers of ab are non-vanishing. Only when the quantum numbers of a and b are opposite, the vanishing need not take place. If a and its Hermitian conjugate $a^\dagger$ with opposite quantum numbers belong to the set of generators of the free algebra, one has $\tau(aa^\dagger) > 0$ and is different from $\tau(a)\tau(a^\dagger) = 0$.

   Therefore the hermitian conjugates of generators cannot belong to the generators of the algebra creating the physical states. This algebra is highly reminiscent of free algebra since all vacuum expectations for the products vanish.

3. For affine and conformal algebras this condition corresponds to the requirement that physical states are created using only the generators with non-negative conformal weight $n \geq 0$ analogous to the algebra of creation operators. Also the generators of this algebra, whose number is finite, satisfy this condition. One could speak of half-algebra.

4. In TGD half-algebras appear for a different reason. The TGD Universe is fractal in several senses of the word. Also the algebra A of observables is fractal in the sense that it contains an infinite hierarchy of sub-algebras A_n for which the conformal weights are n-multiples of those for A. The finite measurement resolution is realized by the conditions that A_n and $[A_n, A]$ annihilate physical states and that also the corresponding classical Noether charges vanish, which gives strong conditions on space-time surfaces. These sub-algebras define hierarchies of measurement resolutions related to inclusions of HFFs.

   If the generators of super symplectic algebra and extensions of affine algebras indeed define free algebras, the rules of free probability theory could bring in dramatic computational simplifications if the scattering amplitudes correspond to expectations for the polynomials of the free-algebra generators.

5. In ZEO, zero energy states are generated by this kind of half-algebra and its hermitian conjugate as superpositions of state pairs assigned to the opposite boundaries of causal diamond (CD= $cd \times CP_2$, where cd is the intersection of future and past directed light-ones).

   The members of the state pair are created by the half-algebra *resp.* its hermitian conjugate and are assigned with opposite boundaries of CD (intersection of light-ones with opposite time directions). The corresponding vacua are analogous to Dirac sea of negative energy fermions and its hermitian conjugate consisting of positive energy fermions. The zero energy states are analogous to pairs formed by Dirac's bras and kets. This allows to code the scattering matrix elements [L62, L63] as zero energy states.

6.3.3 Some objections against HFFs

One cannot avoid philosophical considerations related to the notion of probability and to the interpretations of quantum measurement theory (`https://cutt.ly/YXxSLS1`).

Standard measurement theory and HFFs

The standard interpretations of quantum measurement theory are known to lead to problems in the case of HFFs.

1. An important aspect related to the probabilistic interpretation is that physical states are characterized by a density matrix so that quantum theory reduces to a purely statistical theory. Therefore the phenomenon of interference central in the wave mechanics does not have a direct description.

   Another problem is that for HFFs, pure states do not exist as so-called normal states, which are such that it is possible to assign a density operator to them. This is easy to understand intuitively since by the $Tr(Id) = 1$ property of the unit matrix, there is no minimal projection. Selection of a ray would correspond to an infinite precision and delta function type density operator. The axiom of choices in mathematics is quite a precise analogy.

   One can of course argue that even if pure states as normal states are possible, in practice the studied system is entangled with the environment and that this forces the description in terms of a density matrix even when pure states are realized at the fundamental level.

2. In the purely statistical approach, the notion of quantum measurement must be formulated in terms of what occurs for the density matrix in quantum measurement. The expectation value of any observable A for the new density matrix generated in the measurement of observable O with a discrete spectrum must be a weighted sum for the expectations for the eigenstates of the observable with weights given by the state function reduction probabilities.

   Problems are however encountered when the spectrum contains discrete parts. In the TGD framework, the number theoretic discretization would make it possible to avoid these problems.

Should density matrix be replaced with a more quantal object?

These problems force us to ask whether there could be something deeply wrong with the notion of density matrix? The TGD inspired view of HFFs [K87, K28] suggests a generalization of the state as a density matrix to a "complex square root" of the density matrix. At the level of WCW as vacuum functional, it would be proportional to exponent of a real valued Kähler function of WCW identified as Kähler action for the space-time region as a preferred extremal and a phase factor defined by the analog of of action exponential. Zero energy state would be proportional to an exponent of Kähler function of WCW identified as Kähler action for space-time surface as a preferred extrema.

Problems with the interpretations of quantum theory

HFFs based probability concept has also problems with the interpretations of quantum theory, which actually strongly suggest that something is badly wrong with the standard ontology.

1. In TGD, this requires a generalization of quantum measurement theory [L38] [K89] based on zero energy ontology (ZEO) and Negentropy Maximization Principle (NMP) [K45] [L12], which is consistent with the second law [L56]. What is essential is that physics is extended to what I call adelic physics [L23, L24] to describe also the correlates of cognition. This brings in a measure for conscious information based on a p-adic generalization of Shannon entropy.

2. ZEO [K89] is forced by an almost exact holography in turn implied by general coordinate invariance for space-time as 4-surface. States in ZEO are superpositions of classical time evolutions and and is replaced by a new one in a state function reduction (SFR) [L38, L64]. The determinism of the unitary time evolution is consistent with the non-determinism of SFR. The basic problem of quantum measurement theory disappears since there are two times and two causalities. Causality of field equations and geometric time of physicists can be assigned to the classical time evolutions. The causality of free will and flow of experienced time can be assigned to a sequence of SFRs. The findings of Minev et al [L32] provide support for ZEO [L32].

   Quantum measurement as a reduction of entanglement can in principle occur for any entangled system pair if NMP favors it. There is no need to assume mysterious decoherence as a separate postulate. By NMP, entanglement negentropy can also increase by the formation of entangled states. Since entanglement negentropy is the sum of positive p-adic contribution and negative contribution from real entanglement and is positive, the increase of negentropy is consistent with the increase of real entanglement entropy.

   However, since classical determinism is slightly broken [L61] (there is analogy with the non-uniqueness of the minimal surfaces spanned by frames), the holography is not quite exact. This has important implications for the understanding of the space-time correlates of cognition and intentionality in the TGD framework.

The notion of finite measurement resolution and probabilistic interpretation

One can also ask whether something could go wrong with the quantum measurement theory itself. This notion of quantum measurement does not take into account the fact that the measurement resolution is finite.

The notion of finite measurement resolution realized in terms of inclusion, replacing Hilbert space ray with the included factor and reducing state space to quantum group like object, could allow us to overcome the problems due to the absence of minimal projectors for HFFs implying that the notion of Hilbert space ray does not make sense.

Quantum group like object would represent the degrees of freedom modulo finite measurement resolution described by the included factor. The quantum group representations form a finite subset of corresponding group representations and the state function reductions could occur to quantum group representations and the standard quantum measurement theory for factors of type I would generalize.

Connes tensor product and finite measurement resolution

In the TGD framework Connes tensor product could provide a description of finite measurement resolution in terms of inclusion.

1. In the TGD framework, inclusion of HFFs are interpreted in terms of measurement resolution. The included factor $M \subset N$ would represent the degrees of freedom below measurement resolution. N as M module would mean that M degrees of freedom are absorbed to the coefficient ring and are not visible in the physical states. Complex numbers as a coefficient ring of the Hilbert space are effectively replaced with M. In the number theoretic description of the measurement resolution, the extension of extension is replaced with the extension. The

quantum group, N as M, quantum group with quantum dimension $N : M$ would characterize the observable degrees of freedom.

This fits with the hierarchy of SSA_n:s. SSA_{n+1} would take the role of M and SSA_n that of N. This conforms with the physical intuition. Since n corresponds to conformal weight, the large values of n would naturally correspond to degrees of freedom below UV cutoff.

Could also IR cutoff have a description in the super symplectic hierarchy of SSA_n:s. It should correspond to a minimal value for conformal weight. The finite size of CD defining a momentum unit gives a natural IR cutoff. The proposal is that the total momentum assignable to the either half-cone of CD defines by $M^8 - H$ duality the size scale L as $L = h_{eff}/M$ [L45, L46].

2. For the hierarchies of extensions of rationals the upper levels of the extension hierarchy would not be observed. The larger the value of $n = h_{eff}/h_0$, n a dimension of extension of rationals associated with polynomial P defining the space-time region by $M^8 - H$ duality, the longer the quantum coherence scale.

   In this case large values for the dimension of extension would correspond to IR cutoff. Therefore UV and IR cutoffs would correspond to number theoretic and geometric cutoffs. This conforms with the view that $M^8 - H$ duality as an analog of Langlands duality is between number theoretic and geometric descriptions.

3. Duality suggests that also UV cutoff should have a number theoretic description. In the number theoretic situation, Galois confinement for these levels might imply that they are indeed unobservable, just like color-confined quarks. In fact, the hypothesis $n = h_{eff}/h_0$, n a dimension of extension of rationals associated with polynomial P defining the space-time region by $M^8 - H$ duality, for the effective Planck constant leads to estimate for ordinary Planck constant as $h = n_0 h_0$ where n_0 corresponds to the order of permutation group S_7.

   Could the interpretation be that these degrees of freedom are Galois confined and unobservable in the scales at which measurements are performed. Smaller values of h_{eff} would appear only in length scales much below the electroweak scale and at the limit of CP_2 scale?

How finite measurement resolution could be realized using inclusions of HFFs?

The basic ideas are that finite measurement resolution corresponds to inclusions of HFFs on one hand, and to number theoretic discretizations defined by extensions of rationals. In both cases one has inclusion hierarchies.

One can consider realizations at the level of WCW (geometry) and at the level of number theory in terms of adelic structures assignable to the extensions of rationals. Space-time surfaces can be discretized and this induces discretization of WCW. Even more, WCW should be in some natural manner effectively discrete.

In [K35, K20, K63] the construction WCW Kähler metric is considered and the mere existence of the K "ahler metric is expected to require infinite-D isometry group and imply constant curvature property. The Kähler function K is defined in terms of action consisting of the Kähler action and volume part for a preferred extremal (PE). There are however zero modes present and the metric depends on the zero modes. Twistor lift fixes the choices of H uniquely [L62, L63].

How to define WCW functional integral and how to discretize it? I have proposed that the Gaussian approximation to WCW integration is exact and allows to define a discretization in terms of the maxima (maybe also other extrema) of Kähler function. The proposal is that the exponential of Kähler function should correspond to a number theoretic invariant, perhaps the discriminant of the polynomial P defining PE by $M^8 - H$ duality.

Consider first the standard realization of the restriction $P : N \to M$ reducing the measurement resolution.

1. The definition of a unitary S-matrix for HFFs is non-trivial. Usually one considers only density matrices expressible in terms of projection operators P to subspaces of HFF.

   I have earlier proposed the notion of a complex square root of the density matrix as a generalization of the density matrix. In a direct sum representation of S over projections, in which S-matrix is diagonal, and the projection operators would be multiplied by phase factors. This definition looks sensible at the level of WCW but perhaps as a generalization of the density matrix rather than the S-matrix.

The exponent of Kähler function could have a modulus multiplied by a phase factor. Also an additional state dependent phase factor can be considered. The mathematical existence of the WCW integral fixes the modulus essentially uniquely to an exponent of Kähler function K multiplied by the metric volume element. K could also have an imaginary part.

2. The projected S-matrix PSP is unitary if the projection operator P must commute with S. S-matrix is realized at the level of HFFs so that the matrix representation does not make sense in a strict sense since the notion of ray is not sensical.

3. Projection $N \rightarrow M$ respects unitarity only if P commutes with S and $S^\dagger$. The S-matrix does not have matrix elements between M and N. This is a very tough condition.

How the finite measurement resolution could be realized in the TGD framework?

1. In WCW spin degrees of freedom plus algebras A_n. Number theoretic degrees of freedom are discrete and correspond to various p-adic degrees of freedom. Continuous WCW is associated with the real part of the adelic structure. Its number theoretic parts correspond to the p-adic degrees of freedom, which are discrete.

2. Discretization could be a natural and necessary part of the definition of WCW. Could discrete WCW degrees of freedom be identified in terms of symplectic and number theoretic invariants? They would represent for WCW spinor fields scalar degrees scalar degrees of freedom separable from spin degrees of freedom representable in terms of algebras A_n. These two kinds of degrees of freedom correspond to M and M' if the proposed general picture is correct.

Measurement resolution would be realized in terms of braid group algebras and algebras A_n defining the measurement resolution. What does this mean at the level of WCW?

1. Bosonic generators of SSA_n and possible other algebras A_n define tangent space basis for WCW. The gauge conditions stating that A_n and $[A_n, A]$ annihilate WCW spinor fields define a finite measurement resolution selecting only a subset of tangent space-generators and their super counterparts.

2. Consider first ideal measurement resolution in a function space. There is a complete basis of scalar functions Φ_m in a given space. The sum $\overline{\Phi}_m(x)\Phi_m(y) = \delta(x,y)$ would hold true for an infinite measurement resolution.

In a finite measurement resolution one uses only a finite subset of the scalar function basis, and completeness relation becomes non-local and is smoothed out: $\delta(x,y) \rightarrow D(x,y)$, which is non-vanishing for different point pairs x, y.

3. The condition of finite measurement resolution should define a partition of WCW to disjoint sets. In real topology, the condition $|x-y|^2$ would define a natural measurement resolution but would not define a partition.

In p-adic topology, the situation is different: the p-adic distance function $d(x-y)$ has values p^{-n} and the sets $d(x-y) < d$ are either disjoint or identical. One would have the desired partition. Therefore it seems that p-adicization is essential and the p-adic variants of WCW, or rather regions of WCW, obtained by discretization could allow partitions corresponding to various p-adic number fields forming the adele. Different p-adic representations of algebras A_n would define measurement resolutions.

There is a connection with spin glasses where spin energy landscape consisting of free energy minima allows ultrametric topology: p-adic topologies are indeed ultrametric. The TGD view of spin glasses is discussed in [L58]. One expects the decomposition of WCW to different p-adic topologies with ramified primes of polynomial P defining the p-adic sectors to which a given space-time surface can belong.

4. The consistency condition is that the transition probabilities $P(m \to n)$ between the states satisfying the gauge conditions representing finite measurement resolution, predicted by S-matrix or its TGD counterpat, should be constant should be constant in the subsets of WCW for which the completeness relation gives a non-vanishing $D(x,y)$ for the point pairs (x,y).

5. Does WCW have hierarchies of partitions such that the constancy of $P(m \to n)$ holds true within each partition?

   Do these partitions correspond to hierarchies of inclusions of HFFs defining increasing resolution? M^8-H duality does not allow all kinds of hierarchies. The hierarchies should be induced by the hierarchies of extensions of rationals. As the measurement precision increases, the partition involves an increasing number of sets and at the limit of ideal measurement resolution, the partition consists of algebraic points of WCW and of space-time surfaces.

6. $P=Q$ condition implying that space-time surfaces correspond to infinite prime, could appear as a consistency condition for allowed hierarchies. Preferred extremals and preferred polynomials would correspond to each other. Note that $P=Q$ conditions fixes the scaling of P.

In the TGD framework, one can challenge the idea, originally due to Wheeler, that transition probabilities are given by a unitary S-matrix.

1. The TGD based proposal is that in spin degrees of freedom, that is for many-fermion states for a given space-time surface, the counterpart of S-matrix could be be given by the analog of Kähler metric in the fermionic Hilbert space [L52]. This would mean a geometrization of quantum theory, at least in fermionic degrees of freedom.

   The transition probabilities would be given by $P(m \to n) = K_{\overline{m}n}K^{\overline{n}m}$ and the properties of Kähler metric K give analogs of unitary conditions and probability conservation plus some prediction distinguishing the proposal from the standard view.

2. In the infinite-D situation, the existence of Hilbert space Kähler metric in the fermionic sector is an extremely powerful condition and one expects that the Kähler metric is a unique constant curvature metric allowing a maximal group of isometries. This, together with p-adization, would help to satisfy the constancy conditions for $P(m \to n)$ inside the sets for which $D(x,y)$ is non-vanishing. In fact, one expects that since super-generators are proportional to isometry generators contracted with WCW gamma matrices the metric in the fermionic degrees of freedom is induced by Kähler metric in the basis of isometry generators.

3. This condition could allow a generalization to include the states obtained by application of the bosonic generations of A_n the to ground state. This would mean that in bosonic degrees of freedom Kähler metric of WCW in the isometry basis defines the transition probabilities. Tangent vectors of WCW correspond to the isometry generators. An arbitrary number of isometry generators is involved in the definition of the state. However, the Kähler metric of WCW induces a Kähler metric in the algebra generated by the isometry generators, which is analogous to the algebra of tensors.

6.4 M^8-H duality and HFFs

M^8-H duality [L45, L46] gives strong constraints on the interpretation of HFFs at the number theoretic M^8 side and the geometric H side of the duality. One must also understand the relation between M^8-H duality and $M-M'$ duality, identifiable as quantum-classical correspondence (QCC).

Although McKay correspondence [A108, A105, A82, A65, A64] is not quite at the core of M^8-H duality, it is difficult to avoid its discussion. I have considered McKay correspondence also before [L17, L41, L42, L68].

6.4.1 Number theoretical level: M^8 picture

Braided Galois group algebras

For n-braids the permutation group has extension to a braid group B_n defining an infinite covering of S_n for which permutation corresponds to a geometric operation exchanging the two strands of a braid. There are also hierarchies of finite coverings.

S_n is replaced with the Galois group which is a subgroup of S_n and the property of being a subgroup of S_n allows to identify a braided Galois group as a braided Galois subgroup of braided S_n. In the same way one can identify the braided Galois group algebra defining HFF as a sub-algebra of HFF associated with braid group algebra defined by S_n. One can ask whether the property of being a number theoretic braid could be interpreted as a kind of symmetry breaking to S_n to the Galois group of P.

$M^8 - H$ duality [L45, L46] suggests that the roots correspond to braid strands of geometric braids in H. If so, the braided Galois group would be both topological and number theoretic: topology, natural at the level of H, and number theory, natural at the level of M^8, would meet by $M^8 - H$ duality.

This picture looks nice but one can make critical questions.

1. Can the n roots really correspond to n braid strands at the level of H? The n roots correspond to, in general complex, algebraic numbers associated with the extension of rationals. The real projections correspond to mass shells with different mass values mapped to light-cone proper time surfaces in H by $M^8 - H$ duality. Therefore the action of the Galois group changes mass squared values and does not commute with Lorentz transformations. This suggests a violation of causality.

   Should one restrict the Galois group to the isotropy group of a given root? This would mean number theoretic symmetry breaking and could relate to massivation. This restriction would however trivialize the braid.

2. Zero energy ontology (ZEO) could come to the rescue here. In fact, ZEO implies space-time surfaces are the basic objects rather than 3-surfaces so that quantum states are superpositions of space-time surfaces as preferred extremals (PEs). This is forced by the slight violation of determinism of field equations implying also a slight violation of ideal holography.

   Space-time surfaces are minimal surfaces [L61] analogous to soap films spanned by frames and there can be a slight violation of the strict determinism localized to frames as already 2-D case suggests. This could be also seen as violation of classical causality. At the level of consciousness theory it would be a classical correlate for the non-determinism of intentional free will.

   In particular, time-like braids for which the braiding is time-like and corresponds to a dynamical dance pattern, make sense. For these braids one can in principle select the mass squared value mapped to a value of light-cone proper time a to belong to the braid. The values of a need not be the same.

Also Galois confinement, which is a key aspect of the number theoretic vision, is involved.

1. Galois confinement states that physical states transform trivially under the Galois group of extension. This condition for physical states follows as a consequence of periodic boundary conditions for causal diamond (CD), which takes the role of box for a particle in a box.

   A weaker condition would be that singlet property holds only for the isotropy group of a given root of the polynomial P characterizing the space-time region and corresponding to mass squared value and at the level of H to a value of the light-cone proper time a.

2. In M^8, the momenta of particles are points at the mass shells of $M^4 \subset M^8$ identifiable as hyperbolic spaces $H^3 \subset M^4$ defined with mass squared values defined as the roots of P. The momenta correspond to algebraic integers (the momentum unit is defined by CD) for the extension defined by P, and in general they are complex. The interpretation is as virtual particles which form physical particles as composites. The physical states must have total momenta, which are ordinary integers. This gives the simplest form of Galois confinement.

3. Commutativity with the Lorentz group would favor the isotropy group instead of the full Galois group. One must be however very cautious since in zero energy ontology (ZEO) physical states correspond to a superposition of space-time surfaces and time-like braids are natural. There is a small violation of strict determinism at the level of preferred extremas. The labelling of braid strands based on the images of roots as mass squared values at level of H is quite natural and is not in conflict with causality.

The Galois group for a polynomial $P_n \circ ... \circ P_1$ has a decomposition to normal subgroups GA_i acting as Galois groups for the i:th sub-extension.

1. The number of roots is a product of the numbers of roots for P_i. Therefore the natural identification is that number theoretic braid groups allow a natural interpretation in terms of braids of braids ... of braids.

2. This hierarchy defines an inclusion hierarchy for the braided HFFs assignable to the polynomials $P_k \circ ... \circ P_1$, $k = 1, ..., n$. It is not quite clear to me whether these inclusions reduce to Jones inclusions and whether one can characterize the inclusions in the sequence by the same invariants as in the case of Jones inclusions.

3. In this picture the Connes tensor product would correspond to formation of composite polynomials $P \circ Q$. The reduction in the number of degrees of freedom from that for the ordinary tensor product of braided Galois group algebras would be due to interactions described in terms of polynomial decomposition. Various braids in the hierarchy could correspond to braids at different sheets of the many-sheeted space-time.

4. Any normal subgroup Gal_i of Galois group Gal defining a sequence of inclusions of normal sub-groups Gal_i can be trivially represented. By normal subgroup property, the elements of Gal can be represented as semidirect products of elements of the factor groups $G_i = Gal_i/Gal_{i-1}$. Any representation of Gal can be decomposed to a direct sum of tensor products of representations of G_i.

   From this decomposition it is clear that any group G_i in the decomposition can be trivially represented so that one obtains a rich structure of representation in which some G_i:s are trivially represented.

A possible interpretation is that in TGD, rational polynomials give discrete cognitive representations as approximations for physics. Cognitive representations are in the intersection of p-adicities and reality defined by the intersection of reals and extension of p-adics defined by the algebraic extension of the polynomial P defining a given space-time surface. Continuum theory would represent real numbers as a factor of the adele.

One can ask whether the various zeta functions consistent with the integer spectrum for the conformal weights and possibly also with conformal confinement, appear at the continuum limit and provide representations for the space-time surfaces at this limit? In this framework, it would be natural for the roots of zeta to be algebraic numbers [K64]. Also in the case of ζ, the virtual momenta of fermions would be algebraic integers for virtual fermions and integers for the physical states. This makes sense if the notions of Galois group and Galois confinement are sensible for ζ.

As noticed, the notion of ζ generalizes. The so-called global L-functions (`https://cutt.ly/3VNPYmp`) are formally similar to ζ and the extended Riemann Hypothesis could be true for them. The physical motivation for RH would be that it would allow fermion with any conformal weight to appear in a state which is conformal singlet. Algebraic integers for a finite extension of rationals replace integers in the ordinary ζ and one has an entire hierarchy of L-functions. Could one think that the global L-functions could define preferred extremals at the continuum limit?

How could the degrees of prime polynomials associated with simple Galois groups and ramified primes relate to the symmetry algebras acting in H?

The goal is to relate various parameters characterizing polynomials P for which braided Galois group algebras define HFFs to the parameters labelling the symmetry algebras defining hierarchies of HFFs at the level H. There are good reasons to believe that polynomial composition defines inclusion of HFFs and that this inclusion induces the inclusions for the symmetry algebras A_n at the level of H.

One can identify simple Galois groups as prime groups having no normal subgroups. The polynomial P associated with a simple Galois group cannot have no non-trivial functional decomposition $P_n \circ ... \circ P_1$ if one stays in the field of rationals (say). This leads to the notion of prime polynomials. Note that this notion of primeness does not correspond to the irreducibility stating that polynomials with coefficients in a given number field do not allow decomposition to lower degree polynomials.

A polynomial P is also partially characterized by ramified primes and discriminant defines a Galois invariant for the polynomial as also the symmetric polynomials formed from the roots.

How do these two notions of primeness relate to the p-adic prime decomposition of adelic structures defined by the algebras A_n, which act at the level of H and decomposed adelically to a tensor product of all $A_{n,p}$:s?

Simple Galois groups correspond to prime polynomials. This notion looks fundamental concerning the understanding of the situation at the level of H.

1. Polynomials can be factorized into composites of prime polynomials [A33, A96] (`https://cutt.ly/HXAKDzT` and `https://cutt.ly/5XAKCe2`). A polynomial, which does not have a functional composition to lower degree polynomials, is called a prime polynomial. It is not possible to assign to prime polynomials prime degrees except in special cases. Simple Galois groups with no normal subgroups must correspond to prime polynomials.

2. For a non-prime polynomial, the number N of the factors P_i, their degrees n_i are fixed and only their order can vary so that n_i and $n = \prod n_i$ is an invariant of a prime polynomial and of simple Galois group [A33, A96]. Note that this composition need not exist for monic polynomials even if the Galois group is not simple so that polynomial primes in the monic sense need not correspond to simple Galois groups.

3. The number of the roots of P_i is given by its order n_i, and since Galois group and its braided variant permute the roots as subgroup of S_{n_i}, it is natural to assume that the roots define an n_i-braid. The composite polynomial would define braid of braids of ... of braids. At the level of H the braid strands would correspond to flux tubes and braiding would have a geometric interpretation.

4. The integer n characterizing the algebra A_n acting in H would naturally correspond to the degree of n of P and the decomposition of P to polynomial primes would naturally correspond to an inclusion hierarchy A_{n_i} $A_{n_1} \subset A_{n_1 n_2} \subset ... \subset A_n$ with improving resolution allowing to see braids and braids of braids.

   The corresponding factor spaces realizing the notion of finite measurement resolution, would be analogous to quantum groups obtained when some number of the highest levels in the hierarchy of braids in the braid of braids of ... braids are neglected and the entire algebra is replaced with a quantum group-like structure. This means cutting off some number of the highest levels in the tree-like hierarchy. The trunk is described by a quantum group-like object.

5. This hierarchy corresponds to the hierarchy of Galois groups as normal subgroups assignable to braids in the decomposition and the hierarchy of corresponding braided Galois group algebras defining inclusions of HFFs. Galois group algebras would act as braid groups inc corresponding algebras A_n. Therefore number theoretic and geometric views would fuse together.

6. Connes tensor product is a central notion in the theory of HFFs and it could be naturally associated with the inclusions of brided Galois group algebras. The counterpart for the quantum group as factor space N/M of the factors would correspond to the inclusion $Gal_{i-1} \subset Gal_i$ as a normal subgroup. The inclusion defines group $G_i = Gal_i/Gl_{i-1}$. Also its braided variant is defined. The factor space of braided group algebras would be the counterpart of the quantum group G_i.

   Note that these quantum group-like objects could be much more general than the quantum groups defined by subgroups of $SU(2)$ appearing in Jones inclusions.

What about the interpretation of the ramified primes, which are Galois invariants as also the root spectrum (but not the roots themselves) and depends on the polynomial.

In accordance with the proposed physical interpretation of the ramified primes as preferred p-adic primes labelling particles in p-adic thermodynamics, ramified primes p_i would define preferred p-adic primes for the p-adic variants of the algebras A_n in the adelic generalization of A_n as tensor product of p-adic representations of $A_{n,p}$ of A_n. A_{n,p_i} would be physically and also mathematically special.

Both the degree n as the number of braids of P and the ramified primes of P would dictate the physically especially relevant algebras A_{n,p_i}. For instance, un-ramified primes could be such that corresponding p-adic degrees of freedom are not excited.

6.4.2 Geometric level: H picture

The hierarchies of algebras SSA_n, Aff_n and I_n

The algebras $A_n \in \{SSA_n, Aff_n, I_n\}$ for $n = p$ acting at the level of WCW seem to have special properties since the values of the conformal weights for the factor algebras defined by the conditions that A_n and $[A_n, A]$ annihilate physical states, allow the structure of finite field $G(p)$ or even its extension $G(p, k)$ for conformal weights in extension of rationals. The representations would be finite-D. Also the values $n = p^k$ seem special and the finite field representations of SSA_p could be extended to p-adic representations.

This raises the question, whether one could regard n as a p-adic number? The interpretation of n as the number of braid strands assignable to roots of the polynomial P with degree n defining the space-time surface, looks more approriate since it allows braid group algebra of P to act in SSA_n, This identification does not favor this interpretation.

A more plausible interpretation is that the p-adic primes, identifiable as ramified primes of P, characterize the p-adic representations of SSA_n. This also conforms with the interpretation of preferred p-adic primes characterizing elementary particles as ramified primes.

The polynomials with prime degree could be however physically special. The algebras SSA_p, with p defining the degree of polynomial p allow finite field representations, which extend to p-adic representations and one can ask whether the prime decomposition of n could allow some kind of inclusion hierarchy of representations.

This would also give a possible content for the p-adic length scale hypothesis $p \simeq 2^k$, k prime, or its generalization involving primes near powers of prime $q = 2, 3, 5, ...$ A more general form of p-adic length scale hypothesis would be $p \simeq q^n$, n the degree of P.

Commutants for algebras A_n and braid group algebras

For the super $A \in \{SSA, Aff, I\}$, the inclusion An_{nk} to SSA_n should define a Connes tensor product. One would obtain inclusion hierarchies labelled by divisibility hierarchies $n_1 \div n_2 \div ....$. For braid group algebras one obtains similar hierarchies realized in terms of composite polynomials.

What about the already mentioned "classical" degrees of freedom associated with the fluxes of the induced Kähler form? They should be included to M' at the level of H. The hierarchies of flux tubes within ... within flux tubes correspond to the hierarchies assignable to M' at the level of H.

The number theoretic degrees of freedom identifiable as invariants of Galois groups should be included to M' at the number theoretical level. The hierarchies of roots assignable to composite polynomials $P_n \circ ... \circ P_1$ with roots assigned to the strands of time like braid strands could correspond to these hierarchies at the level of M^8.

6.4.3 Wild speculations about McKay correspondence

McKay correspondence is loosely related to the HFFs in TGD framework [L17, L42, L41, L68] and I cannot avoid the temptation to try to understand it in TGD framework.

1. The origin of the McKay graphs for inclusions is intuitively understood. Representations of finite subgroups of $SU(2)$ are assignable to 2-D factors. These representations could correspond to closed subgroups of quaternionic $SU(2)$ on the basis of the reduction to

$M_2(C) \otimes M_2(C) \otimes$ ..... A reduction of degrees of freedom happens for HFFs since they are subalgebras of $B(H)$ and this could reduce the closed subgroup to a finite subgroup.

Also the interpretation N as tensor product of M and quantum group $SU(2)$ suggests the same since quantum groups have a finite number of irreps, when q equal is a root of unity. The analog of McKay graph coding fusion rules for the quantum group tensor products would reduce to McKay graphs.

2. Why would the McKay graphs for finite subgroups of $U(2)$ correspond to those for affine or ordinary Lie algebras? Could these Lie-algebras emerge from the inclusions. This is a mystery, at least to me.

3. In the TGD framework one can ask why there should be Weyl group of extended ADE Dynkin diagram assignable to SSA_n? SSA_n defines a representation of SSA with SSA_n and $[SSA_n, SSA]$ acting trivially. Could this representation correspond to an affine or ordinary ADE algebra? Similar question makes sense for all algebras $A_n \in \{SSA_n, Aff_n, I_n\}$. A_nn would define a cutoff of the SSA so that all generators with conformal weight larger than n would be represented trivially.

   Note that for $n = p$, the conformal weights of A_n would define a finite field and if algebraic integers also its extension. This case could correspond to polynomials defining cyclic extension of order p with roots coming as roots of unity.

4. The Weyl groups assignable to the "factor algebra" of SSA_n defined by the gauge conditions for A_n and $[A_n, A]$ and proposed to reduce to ADE type affine or ordinary Lie algebra should relate to Galois groups for polynomial P with degree n as number of braid strands.

   (a) Could the braid strands correspond to the roots of ADE algebra so that roots in the number theoretic sense would correspond to the roots in the group theoretic sense? This would conform with Langlands correspondence [K38, A57, A56] discussed from the TGD perspective in [K38] [L4, L11].

   (b) Could the Weyl groups allow identification as subgroups of corresponding Galois groups?

   Note that simple Galois groups correspond to so-called prime polynomials [A33, A96] allowing no decomposition to polynomials of lower degree so that the preferred values of n would correspond to prime polynomials.

5. Affine electroweak and color algebras an their M^4 counterparts would be special since they wuuld not emerge a dynamical symmetries of SSA_n but define algebras Aff_n and I_n related to the light-like partonic orbits. They would also correspond to symmetries both at the level of M^8 and H.

This inspires the following questions, which of course look very naive from the point of view of a professional mathematician. My only excuse is the strong conviction that the proposed picture is on the right track. I might be wrong.

1. The Jones inclusion of HFFs [A66, A131, A132] involves an extended or ordinary ADE Dynkin diagram assignable also to finite subgroups of $SU(2)$ by McKay correspondence [A108].

   Could the Weyl group of an extended ADE diagram really correspond to an affine algebra or quantum group assignable to A_n? If so, one would have dynamical symmetries and should relate to the "factor" space SSA/SSA_n in which SSA_n defines a measurement resolution.

2. HFF can be regarded algebraically as an infinite tensor power of $M_2(C)$. Does the representation as a 2×2 matrix imply the emergence of representations of a closed subgroup of $SU(2)$ or its quantum counterpart. Could the reduction of degrees of freedom due to the finite measurement resolution imply that the closed subgroup reduces to a finite subgroup?

3. The algebraic decomposition of HFF to an infinite tensor power of $M^2(C)$ would suggest that the including factor N with dimension 1 is equal to $M^{d_q} \otimes M^{1/d_q}$, where d_q is the quantum dimension characterizing eith M or N. Could these two objects correspond to an ADE type affine algebra and quantum group with inverse quantum dimensions? Or could either of them correspond to ADE type affine algebra or quantum group?

4. Could one think that the analog of McKay graph for the quantum group-like object assignable to affine group by a finite measurement resolution reduces to the McKay graph for a finite subgroup of $SU(2)$ because only a finite number of representations survives?

5. Could the finite subgroups of $SU(2)$ correspond to finite subgroups for the covering group of quaternion automorphisms acting naturally in M^8? Could these finite subgroups correspond to finite subgroups of the rotation group $SU(2)$ at H side?

Could only the n_C (dimension of Cartan algebra) roots appearing in the Dynkin diagram be represented as roots of a polynomial P in extension of rationals or its quantum variant? This option fails since the Dynkin diagram does not allow a symmetry group identifiable as the Galois group. The so called Steinberg symmetry groups (`https://cutt.ly/GXMb8Si`) act as automorphisms of Dynkin diagrams of ADE type groups and seem quite too small and fail to be transitive as action of the Galois group of an irreducible polynomial is.

$M^8 - H$ duality inspires the question whether a subgroup of Galois group could act as the Weyl group of ADE type affine or ordinary Lie algebra at H side.

1. The Galois group acts as a braid group and permutes the roots of P represented as braid strands. Weyl group permutes the roots of Lie algebra

   The crazy question is whether the roots of P and roots of the ADE type Lie-algebra could correspond to each other. Could the roots of P in $N \rightarrow 1$-correspondence with the non-vanishing roots of the representation of Lie algebra or of its affine counterpart containing an additional root corresponding to the central extension?

   If the roots appearing in the Dynkin diagram correspond to a subset of roots of polynomial P, the Weyl group could correspond to a minimal subgroup of the Galois group generated by reflections and generating all non-vanishing roots of the Lie algebra.

2. The action of the Weyl group should give all roots for the representation of G. Could the Weyl group, which is generated by reflections, correspond to a minimal subgroup of Gal giving all roots as roots of P when applied to the McKay graph?

   The obvious objection is that the order of the Weyl group increases rapidly with the order of the Cartan group so that also the Gal and also the order of corresponding polynomials P would increase very rapidly. Gal is a subgroup of S_n having order $n!$ for a polynomial of degree n so that the degree of P need not be large and this is what matters.

   If the m braid strands labelled by the m roots correspond to the roots of the affine algebra, it would be natural to assign affine algebra generators to these roots with the braid strands. The condition $n = Nm$ implies that m divides n. For $Gal = S_n$ with order $n!$ this condition is very mild. $Gal = Z_p$ fixes the Lie algebra to A_p.

   The root space of the dynamical symmetry group would have dimension m, which is a factor of n. For Lie algebras A_n and D_{2n} (with $n \geq 4$) appear besides E_6 and E_8. For affine Lie algebras $\hat{A}_n$ or $hatD_n$ (with $n \geq 3$) and $\hat{E}_6$, $\hat{E}_7$ and $\hat{E}_8$ appear. For large values of n, there are two alternatives for even values of n.

3. One can also consider quantum arithmetics based on $\oplus$ and $\otimes$ and replace P with its quantum counterpart and solve it in the space of irreps of the finite subgroup G of $U(2)$ defining a quantum analog for an extension of rationals. The roots of the quantum variant of P would be direct sums of irreps of G.

   These quantum roots define nodes of a diagram. This diagram should include as nodes the roots of the Dynkin diagram defined by positive roots, whose number is the dimension n_C of Cartan algebra.

Could the missing edges correspond to the edges of the Mac-Kay graph in the tensor product with a 2-D representation of $SU(2)$ restricted to a subgroup? The action of 2-D representation would generate the (extended) Dynkin diagram ADE type.

One can look this option in more detail.

1. Assume that adjoint representation Adj of an affine or ordinary ADE Lie group L emerges in the tensor product $M^2(C)\otimes ...\otimes M^2(C)$ allowing embedding of $SU(2)$ as diagonal embedding. One can imbed the finite subgroup $G \subset SU(2)$ as a diagonal group $G \times G \times ... \times G$ to $M^2(C) \otimes ... \otimes M^2(C)$.

   Also a given representation of G can be embedded as a direct sum of the copies of the representation, each acting in one factor of $M^2(C) \otimes ... \otimes M^2(C)$. The $2-D$ canonical representation of $G \subset SU(2)$ has a natural action in the $G\times G\times ...\times G$ to $M^2(C)\otimes ...\otimes M^2(C)$ and would generate a McKay graph.

   One can also embed G to L as $G \subset SU(2) \subset L$. Adj can be decomposed to irreps G. Therefore the tensor product action of various irreps of G, in particular the canonical 2-D representation, in Adj is well-defined. The tensor action of the 2-D canonical representation of G gives a McKay graph such that the nodes have weights telling how many times a given irrep appears in the decomposition of Adj to irreps of G. The weighted sum of the dimensions of irreps of G is equal to the dimension of Adj.

2. This construction is possible for any Lie group and some consistency conditions should be satisfied. That McKay graph is the same as the generalized Dynkin diagram would be such a consistency condition and leave only simply laced Lie groups.

3. What can one say about the weights of the weighted McKay graph? Could the weights be the number of the images of the positive root under the action of the Weyl group W of L.

   The McKay graph would correspond only to the n_C (dimension of the Cartan algebra) positive roots appearing in the Dynkin diagram of Adj. How to continue the Dynkin dynkin to a root diagram of Adj?

4. Could the n_C roots in the Dynkin diagram correspond to the roots of a polynomial P in a quantum extension of rationals with roots as irreps of G appearing in the McKay graph. The multiple of a given root would correspond to its orbit under W. The action of W as reflections in the quantum extension of rationals, spanned by the roots of Adj, as vectors with integer components would generate all roots of Adj as quantum algebraic integers in the quantum extension of rationals.

5. As proposed, one could interpret the Dynkin diagram as a subdiagram of the root diagram of Adj and identify its nodes as roots of Gal for a suitable polynomial P. The Weyl group could be the minimal transitive subgroup of Gal.

6. The Galois group of extension of ... of rationals is a semidirect of Galois groups which can be chosen to be simple so that the polynomials considered are prime polynomials unless one poses additional restrictions. What does this restriction mean for the ADE type Weyl group of assignable to the exrtension

6.5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K35, K63].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

6.5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [L26] [L60, L62, L63] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and $T(CP_2)$ of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A79] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

1. Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

2. Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

3. The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

   For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

1. The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

2. Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

   For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

6.5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

1. If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

2. WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L72]. There are also other important algebras but these will not be discussed now.

1. The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

   The super symplectic algebra A has an infinite hierarchy of sub-algebras [L72] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

   This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

2. The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

   The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on M^8-H duality [L72] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

1. The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by M^8-H duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

2. This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L72] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P) = n(braid) = n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

3. The actions related to different values of $n(P) = n(braid) = n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

   WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

4. Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

   A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L66] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i = 2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r = 1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

   A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

5. A subset of p-adic primes allowed by the p-adic length scale hypothesis $p \simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K46, K47]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

1. Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

   P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L66], and the forward scattering amplitude corresponds to the propagator.

   This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

2. According to [L66], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

   A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L66].

3. p-Adic length scale hypothesis [L73] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

   For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

   Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

1. The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

   In [L72] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

2. The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K35, K20]. As isometries they would naturally permute the maxima with each other.

6.6 About the TGD based notions of mass, of twistors and hyperbolic counterpart of Fermi torus

The notion of mass in the TGD framework is discussed from the perspective of $M^8 - H$ duality [L45, L46, L73, L63].

1. In TGD, space-time regions are characterized by polynomials P with rational coefficients [L45, L46]. Galois confinement defines a universal mechanism for the formation of bound states. Momenta for virtual fermions have components, which are algebraic integers in an extension of rationals defined by a polynomial P characterizing a space-time region. For the physical many fermion states, the total momentum as the sum of fermion momenta has components, which are integers using the unit defined by the size of the causal diamond (CD) [L38, L57, L64].

2. This defines a universal number theoretical mechanism for the formation of bound states as Galois singlets. The condition is very strong but for rational coefficients it can be satisfied since the sum of all roots is always a rational number as the coefficient of the first order term.

3. Galois confinement implies that the sum of the mass squared values, which are in general complex algebraic numbers in E, is also an integer. Since the mass squared values correspond to conformal weights as also in string models, one has conformal confinement: states are conformal singlets. This condition replaces the masslessness condition of gauge theories [L73].

Also the TGD based notion of twistor space is considered at concrete geometric level.

1. Twistor lift of TGD means that space-time surfaces X^4 is $H = M^4 \times CP_2$ are replaced with 6-surfaces in the twistor space with induced twistor structure of $T(H) = T(M^4) \times T(CP_2)$ identified as twistor space $T(X^4)$. This proposal requires that T(H) has Kähler structure and this selects $M^4 \times CP_2$ as a unique candidate [A79] so that TGD is unique.

2. One ends up to a more precise understanding of the fiber of the twistor space of CP_2 as a space of "light-like" geodesics emanating from a given point. Also a more precise view of the induced twistor spaces for preferred extremals with varying dimensions of M^4 and CP_2 projections emerges. Also the identification of the twistor space of the space-time surface as the space of light-like geodesics itself is considered.

3. Twistor lift leads to a concrete proposal for the construction of scattering amplitudes. Scattering can be seen as a mere re-organization of the physical many-fermion states as Galois singlets to new Galois singlets. There are no primary gauge fields and both fermions and bosons are bound states of fundamental fermions. 4-fermion vertices are not needed so that there are no divergences.

4. There is however a technical problem: fermion and antifermion numbers are separately conserved in the simplest picture, in which momenta in $M^4 \subset M^8$ are mapped to geodesics of $M^4 \subset H$.The led to a proposal for the modification of $M^8 - H$ duality [L45, L46]. The modification would map the 4-momenta to geodesics of X^4. Since X^4 allows both Minkowskian and Euclidean regions, one can have geodesics, whose M^4 projection turns backwards in time. The emission of a boson as a fermion-antifermion pair would correspond to a fermion turning backwards in time. A more precise formulation of the modification shows that it indeed works

The third topic of this article is the hyperbolic generalization of the Fermi torus to hyperbolic 3-manifold H^3/Γ. Here $H^3 = SO(1,3)/SO(3)$ identifiable the mass shell $M^4 \subset M^8$ or its $M^8 - H$ dual in $H = M^4 \times CP_2$. Γ denotes an infinite subgroup of $SO(1,3)$ acting completely discontinuously in H^3. For virtual fermions also complexified mass shells are required and the question is whether the generalization of H^3/Γ, defining besides hyperbolic 3-manifold also tessellation of H^3 analogous to a cubic lattice of E^3.

6.6.1 Conformal confinement

The notion of mass distinguishes TGD from QFT. As in string models, mass squared corresponds to a conformal weight in TGD. However, in the TGD framework tachyonic states are not a curse but an essential part of the physical picture and conformal confinement, generalizing masslessness condition, states that the sum of conformal weights for physical states vanishes. This view conforms with the fact that Euclidean space-time regions are unavoidable at the level of H. Positive *resp.* negative *resp.* vanishing conformal weights can be assigned with Minkowskian *resp.* Euclidean space-time regions *resp.* light-like boundaries associated with them.

Mass squared as conformal weight, conformal confinement and its breaking

At the level of M^8, the momentum components for momenta as points of $H^3_c \subset M^4_c \subset M^8_c$ are (in general complex) algebraic integers in an extension of rationals defined by the polynomial P defining the space-time region. For physical states the momentum components for the sum of the momenta are ordinary integers when the momentum unit is defined by the size scales of causal diamond (CD). This scale corresponds to a p-adic length scale for p-adic prime, which is a ramified prime of the extension of rationals defined by the polynomial P.

For virtual many-fermion states the mass squared is an algebraic integer but an ordinary integer for the physical states [L73]. The question is whether the mass squared for the physical states can be negative so that one would have tachyons. The p-adic mass calculations require the presence of tachyonic mass squared values and the proposal is conformal confinement in the sense that the sum of mass squared values for the particles present in state and identifiable as conformal weights sum up to zero. Conformal confinement would generalize the masslessness condition of gauge field theories.

The observed mass squared values would correspond to the Minkowskian non-tachyonic parts of the mass squared values assignable to states, which in general are entangled states formed from tachyonic and non-tachyonic states. p-Adic thermodynamics would describe the entanglement in terms of the density matrix and observed mass squared would be thermal average. p-Adic thermodynamics leads to a breaking of the generalized conformal invariance and explains why different values of the Virasoro scaling generator L_0 are involved. Since complex mass squared values with a negative real part are allowed as roots of polynomials, the condition is highly non-trivial.

Association of mass squared values to space-time regions

$M^8 - H$ duality [L45, L46] would make it natural to assign tachyonic masses with CP_2 type extremals and with the Euclidean regions of the space-time surface. Time-like masses would be assigned with time-like space-time regions. In [L71] it was found that, contrary to the beliefs held hitherto, it is possible to satisfy boundary conditions for the action action consisting of the Kähler action, volume term and Chern-Simons term, at boundaries (genuine or between Minkowskian and Euclidean space-time regions) if they are light-like surfaces satisfying also $detg_4 = 0$. Masslessness, at least in the classical sense, would be naturally associated with light-like boundaries (genuine or between Minkowskian and Euclidean regions).

Riemann zeta, quantum criticality, and conformal confinement

The assumption that the space-time surface corresponds to rational polynomials in TGD is not necessary. One can also consider real analytic functions f [L63]. The condition that momenta of physical states have integer valued momentum components implies integer valued conformal weights poses extremely strong conditions on this kind of functions since the sum of the real parts of the roots of f must be an integer as a conformal weight identified as the sum of in general complex virtual mass squared values.

There are strong indications Riemann zeta (`https://cutt.ly/iVTV1kqs`) has a deep role in physics, in particular in the physics of critical systems. TGD Universe is quantum critical. What quantum criticality would mean at the space-time level is discussed in [L71]. This raises the question whether Riemann zeta could have a deep role in TGD.

First some background relating to the number theoretic view of TGD.

1. In TGD, space-time regions are characterized by polynomials P with rational coefficients [L45, L46]. Galois confinement defines a universal mechanism for the formation of bound states. Momenta for virtual fermions have components, which are algebraic integers in an extension of rationals defined by a polynomial P characterizing space-time region. For the physical many fermion states, the total momentum as the sum of fermion momenta has components, which are integers using the unit defined by the size of the causal diamond (CD).

   This defines a universal number theoretical mechanism for the formation of bound states. The condition is very strong but for rational coefficients it can be satisfied since the sum of all roots is always a rational number as the coefficient of the first order term.

2. Galois confinement implies that the sum of the mass squared values, which are in general complex algebraic numbers in E, is also an integer. Since the mass squared values correspond to conformal weights as also in string models, one can have conformal confinement: states would be conformal singlets. This condition replaces the masslessness condition of gauge theories [L73].

Riemann zeta [A60] (`https://cutt.ly/oVNSltD`)is not a polynomial but has infinite number of roots. How could one end up with Riemann zeta in TGD? One can also consider the replacement of the rational polynomials with analytic functions with rational coefficients or even more general functions [L63].

1. For real analytic functions roots come as pairs but building many-fermion states for which the sum of roots would be a real integer, is very difficult and in general impossible.

2. Riemann zeta and the hierarchy of its generalizations to extensions of rationals (Dedekind zeta functions, and L-functions in general) is however a complete exception! If the roots are at the critical line as the generalization of Riemann Hypothesis (RH) assumes, the sum of the root and its conjugate is equal to 1 and it is easy to construct many fermion states as 2N fermion states, such that they have integer value conformal weight.

   Since zeta has also trivial zeros for even negative integers interpretable in terms of tachyonic states, also conformal confinement with vanishing net conformal weight for physical states is possible. The trivial zeros would be associated with Euclidean space-time regions and non-trivial ones to Minkowskian ones.

   One can wonder whether one could see Riemann zeta as an analog of a polynomial such that the roots as zeros are algebraic numbers. This is however not necessary. Could zeta and its analogies allow it to build a very large number of Galois singlets and they would form a hierarchy corresponding to extensions of rationals. Could they represent a kind of second abstraction level after rational polynomials?

A possible interpretation is that in TGD, rational polynomials give discrete cognitive representations as approximations for physics. Cognitive representations are in the intersection of p-adicities and reality defined by the intersection of reals and extension of p-adics defined by the algebraic extension of the polynomial P defining a given space-time surface. Continuum theory would represent real numbers as a factor of the adele.

One can ask whether the various zeta functions consistent with the integer spectrum for the conformal weights and possibly also with conformal confinement, appear at the continuum limit and provide representations for the space-time surfaces at this limit? In this framework, it would be natural for the roots of zeta to be algebraic numbers [K64]. Also in the case of ζ, the virtual momenta of fermions would be algebraic integers for virtual fermions and integers for the physical states. This makes sense if the notions of Galois group and Galois confinement are sensible for ζ.

As noticed, the notion of ζ generalizes. The so-called global L-functions (`https://cutt.ly/3VNPYmp`) are formally similar to ζ and the extended Riemann Hypothesis (RH) could be true for them. The physical motivation for RH would be that it would allow a fermion with any conformal weight to appear in a state which is conformal singlet. Algebraic integers for a finite extension of rationals replace integers in the ordinary ζ and one has an entire hierarchy of L-functions. Could one think that the global L-functions could define preferred extremals at the continuum limit?

6.6.2 About the notion of twistor space

For the twistor lift of TGD, twistor space $T(X^4)$ of the space-time surface X^4 is identified an S^2 bundle over X^4 obtained by the induction of the twistor bundle $T(H) = T(M^4) \times T(CP_2)$. The definition of the $T(X^4)$ as 6-surface in $T(H)$ identifies the twistor spheres of $T(M^4)$ and $T(CP_2)$ and identifies it as a twistor sphere of $T(X^4)$.

The notion of twistor space for different different types of preferred extremals

I have not previously considered the notion of the induced twistor space for the different types of preferred extremals. Here some technical complications emerge.

1. Since the points of the twistor spaces $T(M^4)$ and $T(CP_2)$ are in 1-1 correspondence, one can use either $T(M^4)$ or $T(CP_2)$ so that the projection to M^4 or CP_2 would serve as the base space of $T(X^4)$. One could use either CP_2 coordinates or M^4 coordinates as space-time coordinates if the dimension of the projection is 4 to either of these spaces. In the generic case, both dimensions are 4 but one must be very cautious with genericity arguments, which turned out to fail at the level of M^8 [L45, L46].

2. There are exceptional situations in which genericity fails at the level of H. String-like objects of the form $X^2 \times Y^2 \subset M^4 \subset CP_2$ is one example of this. In this case, X^6 would not define 1-1 correspondence between $T(M^4)$ or $T(CP_2)$.

   Could one use partial projections to M^2 and S^2 in this case? Could $T(X^4)$ be divided locally into a Cartesian product of 3-D M^4 part projecting to $M^2 \subset M^4$ and of 3-D CP_2 part projected to $Y^2 \subset CP_2$?

3. One can also consider the possibility of defining the twistor space $T(M^2 \times S^2)$. Its fiber at a given point would consist of light-like geodesics of $M^2 \times S^2$. The fiber consists of direction vectors of light-like geodesics. S^2 projection would correspond to a geodesic circle $S^1 \subset S^2$ going through a given point of S^2 and its points are parametrized by azimuthal angle Φ. Hyperbolic tangent $tanh(\eta)$ with range $[-1, 1]$ would characterize the direction of a time like geodesic in M^2. At the limit of $\eta \to \pm\infty$ the S^2 contribution to the S^2 tangent vector to length squared of the tangent vector vanishes so that all angles in the range $(0, 2\pi)$ correspond to the same point. Therefore the fiber space has a topology of S^2.

   There are also other special situations such as $M^1 \times S^3$, $M^3 \times S^1$ for which one must introduce specific twistor space and which can be treated in the same way.

To deal with these special cases in which the dimensions of both M^4 and CP_2 are not equal to 4, one must allow also 6-surfaces X^6 which can have dimension of M^4 and CP_2 projections which are different from the canonical value 4. For CP_2 type extremals the dimension of CP_2

projection would be 6 and the dimension of M^4 projection would be 1. For cosmic strings the dimensions of M^4 projection and CP_2 projection would be 2.

The concrete definition of the twistor space of H as the space of light-like geodesics

During the writing of this article I realized that the twistor space of H defined geometrically as a bundle, which has as H as base space and fiber as the space of light-like geodesic starting from a given point of H, need not be equal to $T(M^4) \times T(CP_2)$, where $T(CP_2)$ is identified as $SU(3)/U(1) \times U(1)$ characterizing the choices of color quantization axes. Is this really the case?

1. The definition of $T(CP_2)$ as the space of light-like geodesics from a given point of CP_2 is not possible. One could also define the fiber space of $T(CP_2)$ geometrically as the space of geodesics emating from origin at $r = 0$ in the Eguchi-Hanson coordinates [K11] and connecting it to the homologically non-trivial geodesic sphere S^2_G $r = \infty$. This relation is symmetric.

   In fact, all geodesics from $r = 0$ end up to S^2. This is due to the compactness and symmetries of CP_2. In the same way, the geodesics from the North Pole of S^2 end up to the South Pole. If only the endpoint of the geodesic of CP_2 matters, one can always regard it as a point S^2_G.

   The two homologically non-trivial geodesic spheres associated with distinct points of CP_2 always intersect at a single point, which means that their twistor fibers contain a common geodesic line of this kind. Also the twistor spheres of $T(M^4)$ associated with distinct points of M^4 with a light-like distance intersect at a common point identifiable as a light-like geodesic connecting them.

2. Geometrically, a light-like geodesic of H is defined by a 3-D momentum vector in M^4 and 3-D color momentum along CP_2 geodesic. The scale of the 8-D tangent vector does not matter and the 8-D light-likeness condition holds true. This leaves 4 parameters so that $T(H)$ identified in this way is 12-dimensional.

   The M^4 momenta corresponds to a mass shell H^3. Only the momentum direction matters so that also in the M^4 sector the fiber reduces to S^2. If this argument is correct, the space of light-like geodesics at point of H has the topology of $S^2 \times S^2$ and $T(H)$ would reduce to $T(M^4) \times T(CP_2)$ as indeed looks natural.

The twistor space of the space-time surface

The twistor lift of TGD allows to identify the twistor space of the space-time surface X^4 as the base space of the S^2 bundle induced from the 12-D twistor space $T(8) = T(M^4) \times CP(2)$ to the 6-surface $X^6 \subset T(H)$ by a local dimensional reduction to $X^4 \times S^2$ occurring for the preferred extremals of 6-D Kähler action existing only in case of $H = M^4 \times CP_2$.

Could the geometric definition of $T(X^4)$ as the space of light-like geodesics make sense in the Minkowskian regions of X^4?

1. By their definition, stating that the length of the tangent vector of the geodesic is conserved, the geodesic equations conserve the value of the velocity squared so that light-likeness can be forced via the initial values. This allows the assignment of a twistor sphere to a given point of a Minkowskian space-time region. Whether this assignment can be made global is not at all trivial and the difficulties related to the definition of twistor space in general relativity probably reflects this problem. If this is the case, then the direct geometric definition might not make sense unless the very special properties of the PEs come to rescue.

2. The twistor lift of TGD is proposed to modify the definition of the twistor space so that one can assign twistor structure to the space-time surface by inducing the twistor structure of H just as one can assign spinor structure with the space-time surface by inducing the spinor structure of H.

   Could the generalized holomorphic structure, implying that PEs are extremals of both volume and of 4-D Kähler action, make possible the existence of light-like geodesics and even allow to assign to a given point of the space-time surface sphere parametrizing light-like geodesics?

3. The light-like 3-surfaces X^3 representing partonic orbits carry fermionic lines as light-like geodesics and are therefore especially interesting. They are metrically 2-D and boundary conditions for the field equations force the vanishing of the determinant $det(g_4)$ of the induced metric at them so that the dimension of the tangent space is effectively reduced. Light-like 3-surfaces allow a generalization of isometries such that conformal symmetries accompanied by scaling of the light-like radial coordinate depending on transversal complex coordinates is isometry.

   It seems that to a given point of the space-like intersection, only a single light-like geodesic can be assigned so that the twistor space at a given point would consist of a single light-like geodesic. This would be caused by the light-likeness of X^3.

The geometric definition of the twistor space for CP_2

In the case of the Euclidean regions, the notion of a light-like geodesic does not make sense. The closed geodesics and the presence of pairs of points analogous to North pole-South pole pairs, where diverging geodesics meet, would be required. This condition is very strong and the minimal requirement is that the space has a positive curvature so that the geodesics do not diverge. Also symmetries seem to be necessary. Clearly, something new is required.

1. The addition of Kähler coupling term equal to an odd multiple of the induced Kähler gauge potential A to the spinor connection is an essential element in the definition of a generalized spinor structure of CP_2.

2. Should one replace the light-like geodesics with orbits of Kähler charged particles for which CP_2 has been replaced with $p - q_K A$. For the counterparts of light-like geodesics $p - q_K A$ would vanish and the analog of mass squared would vanish but one would have a line. For a geodesic p would be constant.

   Is it possible to have $A = constant$ along a closed geodesic? In the case of sphere, the Kähler gauge potential in the spherical coordinates is $(A_\theta = A_\phi = kcos(\theta)$ and is constant along the geodesics going through South and North Poles. Something like this could happen in the case of CP_2 but it seems that a special pair of homological non-trivial spheres S^2 invariant under $U(2) \subset SU(3)$ is selected. One might perhaps speak of symmetry breaking.

   To obtain entire S^2 of light-like geodesics in this sense, the geodesics must emanate from a coordinate singularity, the origin of Eguchi-Hanson coordinates at $r = 0$, where the values of the coordinates (θ, ϕ, ψ) correspond to the same point. The space for the light-like geodesics must be 2-D rather than 3-D. This must be forced by the $p - A = 0$ condition. For the homologically trivial geodesic sphere $r = \infty$, Ψ coordinate is redundant so that the conserved value of A_ψ must vanish for the light-like geodesics and the associated velocities cannot have component in the direction of Ψ.

3. Note that this definition could apply also in Minkowskian regions of space-time surface.

The description of particle reactions without vertices

In standard field theory, particles are point-like and particle reactions are described using vertices assignable to non-linear interaction terms in the action.

1. In the TGD framework, particles are replaced with 3-surfaces and elementary particles are assigned to partonic 2-surface whose orbits correspond to light-like 3-surfaces identifiables as the boundary regions between Minkowskian and Euclidean space-time regions and modelled as wormhole contacts between two space-time sheets with a Minkowskian signature. Vertices are replaced with topological vertices at which incoming partonic 2-surfaces, whose orbits are light-like 3-surfaces, meet at partonic 2-surfaces.

2. In TGD, all particles are composites of fundamental fermions assignable to the wormhole throats identified as partonic orbits. In particular, bosons consist of fermions and antifermions assignable to the throats of wormholes. Since wormhole contact contains homologically trivial 2-surface of CP_2, there is a monopole flux throwing out of the throat and one must have at least two wormhole contacts so that one obtains a closed monopole flux flowing between the sheets and forming a closed flux tube.

3. The light-like orbits of the partonic 2-surfaces contain fermionic lines defined at the ends of string world sheets connecting different partonic orbits. In QFT description, this would require a 4-fermion vertex as a fundamental vertex involving dimensional coupling constant and leading to a non-renormalizable QFT. Therefore there can be no vertices at the level of fermion lines.

In the number theoretic vision based on Galois confinement [L62, L63], the interactions correspond at the level of M^8 to re-arrangements of virtual fermions, having virtual momentum components in the extension of rationals defined by P, to new combinations required to be Galois singlets and therefore having momentum components, which are ordinary integers. Note that P fixes by holography the 4-surface in M^8 in turn defining the space-time surface in H by M^8-H duality based on associativity.

There is however a problem. If the particle reactions are mere re-arrangements of fundamental fermions and antifermions, moving along light-like geodesic lines in fixed time direction, the total numbers of fermions and antifermions are separately conserved. How can one overcome this problem without introducing the disastrous 4-fermion vertex?

Consider FFB vertex describing boson emission by fermion as a concrete example.

1. B is described as a pair of partonic surfaces containing at least one fermion-antifermion pair, which must be created in the vertex. Incoming particles for the topolocal FFB 3-vertex correspond to partonic orbits for incoming F and outgoing F, each containing one fermion line and possibly a pair of fermion and antifermion.

2. The idea is that boson emission as a pair creation could be described geometrically as a turning of fermion backwards in time. This forces us to reconsider the definition of M^8-H duality. The simplest view of M^8-H duality is that momenta of $M^4\subset M^8$ are mapped to the geodesic lines of M^4. Tachyonic momenta in $M^4\subset M^8$ would be mapped to space-like geodesics in H emanating from the center of CD which is a sub-CD of a larger CD in general. It seems that this definition does not allow us to understand boson emission by fermion in the way proposed in [L63].

3. This led to a proposal that the images of momenta could be geodesics of the space-time surface X^4, rather than H. Since X^4 allows also Euclidean regions and the interiors of the deformed CP_2 type extremals are Euclidean, one ends up with the idea that the geodesics lines of X^4 can have M^4 projections, which turn backwards in the time direction [L45, L46, L60].

   This would allow us to interpret the emission of a boson as a fermion-antifermion pair as the turning of a fermionic line backwards in time. Fermions lines would be identified as the boundaries of string world sheets. Sub-manifold gravitation would play a key role in the elimination of 4-fermion vertex and thus of QFT type divergences.

4. But is it possible to have a light-like geodesic arriving at the partonic 2-surface and continuing as a light-like geodesic in the Euclidean wormhole contact and returning back? The problem is that in Euclidean regions, ordinary light-like geodesics degenerate to points. The generalization of the light-like geodesics satisfying $p=qA$ implying $(p-qA)^2=0$ is possible. At the space-time level, these conditions could be true quite generally and give as a special case light-like geodesics with $p^2=0$ in the Minkowskian regions.

6.6.3 About the analogies of Fermi torus and Fermi surface in H^3

Fermi torus (cube with opposite faces identified) emerges as a coset space of E^3/T^3, which defines a lattice in the group E^3. Here T^3 is a discrete translation group T^3 corresponding to periodic boundary conditions in a lattice.

In a realistic situation, Fermi torus is replaced with a much more complex object having Fermi surface as boundary with non-trivial topology. Could one find an elegant description of the situation?

Hyperbolic manifolds as analogies for Fermi torus?

The hyperbolic manifold assignable to a tessellation of H^3 defines a natural relativistic generalization of Fermi torus and Fermi surface as its boundary. To understand why this is the case, consider first the notion of cognitive representation.

1. Momenta for the cognitive representations [L72] define a unique discretization of 4-surface in M^4 and, by M^8-H duality, for the space-time surfaces in H and are realized at mass shells $H^3 \subset M^4 \subset M^8$ defined as roots of polynomials P. Momentum components are assumed to be algebraic integers in the extension of rationals defined by P and are in general complex.

   If the Minkowskian norm instead of its continuation to a Hermitian norm is used, the mass squared is in general complex. One could also use Hermitian inner product but Minkowskian complex bilinear form is the only number-theoretically acceptable possibility. Tachyonicity would mean in this case that the real part of mass squared, invariant under $SO(1,3)$ and even its complexification $SO_c(1,3)$, is negative.

2. The active points of the cognitive representation contain fermion. Complexification of H^3 occurs if one allows algebraic integers. Galois confinement [L72, L68] states that physical states correspond to points of H^3 with integer valued momentum components in the scale defined by CD.

   Cognitive representations are in general finite inside regions of 4-surface of M^8 but at H^3 they explode and involve all algebraic numbers consistent with H^3 and belonging to the extension of rationals defined by P. If the components of momenta are algebraic integers, Galois confinement allows only states with momenta with integer components favored by periodic boundary conditions.

Could hyperbolic manifolds as coset spaces $SO(1,3)/\Gamma$, where Γ is an infinite discrete subgroup $SO(1,3)$, which acts completely discontinuously from left or right, replace the Fermi torus? Discrete translations in E^3 would thus be replaced with an infinite discrete subgroup Γ. For a given P, the matrix coefficients for the elements of the matrix belonging to Γ would belong to an extension of rationals defined by P.

1. The division of $SO(1,3)$ by a discrete subgroup Γ gives rise to a hyperbolic manifold with a finite volume. Hyperbolic space is an infinite covering of the hyperbolic manifold as a fundamental region of tessellation. There is an infinite number of the counterparts of Fermi torus [L54]. The invariance respect to Γ would define the counterpart for the periodic boundary conditions.

   Note that one can start from $SO(1,3)/\Gamma$ and divide by $SO(3)$ since Γ and $SO(3)$ act from right and left and therefore commute so that hyperbolic manifold is $SO(3)\setminus SO(1,3)/\Gamma$.

2. There is a deep connection between the topology and geometry of the Fermi manifold as a hyperbolic manifold. Hyperbolic volume is a topological invariant, which would become a basic concept of relativistic topological physics (`https://cutt.ly/RVsdNl3`).

   The hyperbolic volume of the knot complement serves as a knot invariant for knots in S^3. Could this have physical interpretation in the TGD framework, where knots and links, assignable to flux tubes and strings at the level of H, are central. Could one regard the effective hyperbolic manifold in H^3 as a representation of a knot complement in S^3?

   Could these fundamental regions be physically preferred 3-surfaces at H^3 determining the holography and M^8-H duality in terms of associativity [L45, L46]. Boundary conditions at the boundary of the unit cell of the tessellation should give rise to effective identifications just as in the case of Fermi torus obtained from the cube in this way.

De Sitter manifolds as tachyonic analogies of Fermi torus do not exist

Can one define the analogy of Fermi torus for the real 4-momenta having negative, tachyonic mass squared? Mass shells with negative mass squared correspond to De-Sitter space $SO(1,3)/SO(1,2)$ having a Minkowskian signature. It does not have analogies of the tessellations of H^3 defined by discrete subgroups of $SO(1,3)$.

The reason is that there are no closed de-Sitter manifolds of finite size since no infinite group of isometries acts completely discontinuously on de Sitter space: therefore these is no group replacing the Γ in H^3/Γ. (`https://cutt.ly/XVsdLwY`).

Do complexified hyperbolic manifolds as analogies of Fermi torus exist?

The momenta for virtual fermions defined by the roots defining mass squared values can also be complex. Tachyon property and complexity of mass squared values are not of course not the same thing.

1. Complexification of H^3 would be involved and it is not clear what this could mean. For instance, does the notion of complexified hyperbolic manifold with complex mass squared make sense.

2. $SO(1,3)$ and its infinite discrete groups Γ act in the complexification. Do they also act completely discontinuously? p^2 remains invariant if $SO(1,3)$ acts in the same way on the real and imaginary parts of the momentum leaves invariant both imaginary and complex mass squared as well as the inner product between the real and imaginary parts of the momenta. So that the orbit is 5-dimensional. Same is true for the infinite discrete subgroup Γ so that the construction of the coset space could make sense. If Γ remains the same, the additional 2 dimensions can make the volume of the coset space infinite. Indeed, the constancy of $p_1 \cdot p_2$ eliminates one of the two infinitely large dimensions and leaves one.

   Could one allow a complexification of $SO(1,3)$, $SO(3)$ and $SO(1,3)_c/SO(3)_c$? Complexified $SO(1,3)$ and corresponding subgroups Γ satisfy $OO^T = 1$. Γ_c would be much larger and contain the real Γ as a subgroup. Could this give rise to a complexified hyperbolic manifold H^3_c with a finite volume?

3. A good guess is that the real part of the complexified bilinear form $p \cdot p$ determines what tachyonicity means. Since it is given by $Re(p)^2 - Im(p)^2$ and is invariant under $SO_c(1,3)$ as also $Re(p) \cdot Im(p)$, one can define the notions of time-likeness, light-likeness, and space-likeness using the sign of $Re(p)^2 - Im(p^2)$ as a criterion. Note that $Re(p)^2$ and $Im(p)^2$ are separately invariant under $SO(1,3)$.

   The physicist's naive guess is that the complexified analogies of infinite discrete and discontinuous groups and complexified hyperbolic manifolds as analogies of Fermi torus exist for $Re(P^2) - Im(p^2) > 0$ but not for $Re(P^2) - Im(p^2) < 0$ so that complexified dS manifolds do not exist.

4. The bilinear form in H^3_c would be complex valued and would not define a real valued Riemannian metric. As a manifold, complexified hyperbolic manifold is the same as the complex hyperbolic manifold with a hermitian metric (see `https://cutt.ly/qVsdS7Y` and `https://cutt.ly/kVsd3Q2`) but has different symmetries. The symmetry group of the complexified bilinear form of H^3_c is $SO_c(1,3)$ and the symmetry group of the Hermitian metric is $U(1,3)$ containing $SO(1,3)$ as a real subgroup. The infinite discrete subgroups Γ for $U(1,3)$ contain those for $SO(1,3)$. Since one has complex mass squared, one cannot replace the bilinear form with hermitian one. The complex H^3 is not a constant curvature space with curvature -1 whereas H^3_c could be such in a complexified sense.

6.7 The notion of generalized integer

This chapter was inspired by the article "Space Element Reduction Duplication (SERD) model produces photon-like information packets and light-like cosmological horizons" by Thomas L.

Wood, published in Metodologia IV B: Journal of International and Finnish Methodology, expresses the basic assumptions of the SERD approach very coherently and in a systematic way so that it easy to criticize them and compare with other views, in my case the TGD view.

My criticism, summarized below, is based on a different interpretations of the discreteness. In TGD framework would be assignable to cognitive representations based on p-adic numbers fields involving extensions of rationals rather than being a feature of space-time. The introduction of continuous number fields (reals, complex numbers, quaternions, octonions) besides p-adic number fields brings in real space-time as sensory representation and one ends up to a generalization of the standard model proving a number theoretic interpretation for its symmetries.

The approach of Wood looks is essentially topological: for instance, the information propagating in the hypergraph is assumed to be topological and characterize the graph. In TGD, discrete structures analogs define cognitive representations of the continuous sensory world and are basically number theoretic. The description of the sensory world involves both topology and geometry.

6.7.1 The first reactions to the abstract

The abstract gives a very concise summary of the approach and I have added below my reactions to it. The following commentary is my attempt to understand the basic ideas of SERD. I have also used the third section of the article to clarify my views. I must admit that I didn't quite get the two basic principles in the beginning of the third section. I have slightly re-organized the abstract and hope that I have not done any damage.

[TW] This document describes a correspondence between photons and propagating information packets (PIPs) that are emergent out of the Space Element Reduction Duplication (SERD) model introduced in a rudimentary form in [1, 2]. The SERD model is a discrete background independent microscopic space-time description.

[MP] The assumption of discreteness at the fundamental space-time level raises several challenges. 4-D space-time with Minkowskian signature should somehow emerge. The mere hypergraph might possess under additional assumptions a local dimension defined homologically/combinatorially but would vary. Note that in standard homology theory an embedding to some space is required and would give a metric. Now the distance and other geometric notions look problematic to me. One can also ask what kind of dynamics for hypergraphs could select the 4-D space-time? Should one have a variational principle of some kind?

The notion of symmetries is central in physics. Lorentz invariance or even Poincare invariance should emerge as approximate symmetries at least. Only discrete subgroups of these groups can emerge in the hypergraph approach. Lorentz invariance poses very, perhaps too, powerful constraints on the hypergraphs. The notion of discretized time is introduced. It should be Lorentz invariant and here the light-cone proper time a serves as an analog. a=constant sections would be analogs of hyperbolic 3-space H^3.

[TW] By observation of physically comparable behaviour emerging from this system, through analysis and computer simulation, we draw conclusions of what the form and dynamics of the true underlying space-time may be.

By treating elements of the system as fundamental observers, mathematical and empirical evidence is obtained of the existence of fully emergent light-like cosmological horizons, implying the existence of causally separated 'pocket universes'.

[MP] The emergence of the analogy with expanding cosmology presumably reflects the underlying dynamics implying the increase of the size of the hypergraph. The emergence of light-like causal horizons is natural if the dynamics involves maximal velocity of propagation for the signals. This is probably due to the locality of the basic dynamics involving only local changes of the hypergraph topology. Locality and classicality raise challenges if one wants to describe phenomena like quantum entanglement.

[TW] The SERD model is a hypergraph of connected hyperedges called Point Particles (PP) which represent the fundamental constituents of all matter and particles (and therefore observers) separated by strings of consecutive and fundamental elements or edges called Space Elements (SE).

[MP] I had to clarify myself what a hypergraph is. Hypergraph is a generalization of graphs. Also it contains the set of vertices/nodes. The notion of edge connecting a pair of vertices is however generalized to a hyperedge (PP) as a pair of subsets of vertices. PPs correspond to hyperedges as fundamental constituents of matter and formed by pairs of subsets of the set of nodes.

One could interpret this as a combinatorial counterpart for a length scale hierarchy of TGD in which a set can be approximated as a point. One might also interpret subsets of vertices as analogs of bound states of fundamental particles. In the TGD framework, many-sheeted space-time and various other hierarchies serve as its analogs.

Space elements (SEs) would bring in basic aspects of 3-space. It is said that they are infinitesimal or maximally small. SEs would be like edges (not hyperedges) of the hypergraph. Consecutive SEs in turn form interaction edges (IEs) connecting PPs. IEs store and transmit information relating to the structure space. What comes to mind is that functionally PPs are like neutrons and neuron groups and IEs are like axons.

[TW] All elements are separated by nodes called Information Gaps (IGs), that store propagating topological information of the hypergraph. Information gaps (IGs) are between PPs and SEs, between SEs and between PPs themselves.

[MP] What distinguishes the SERD model from physical theories, is that information takes the role of matter. Information is treated as some kind of substance. The basic objection is that conscious information is always about something, whereas matter just is.

IGs have the role of interfaces somewhat analogous to black-hole horizon assumed to store information in the holographic picture. One could see PPs as the nodes and IEs as the edges or SEs as the edges and IGs as the nodes. IGs could have synaptic contacts as analogs.

[TW] In time step (TS), SE can duplicate and reduce (disappear) while the PPs split and merge through discrete time. These processes create space or destroy it and increase or reduce the effective distance between PPs. Splitting generates an SE between the resulting PPs. These are known as the actions of the elements and create a highly dynamic multi-way system.

[MP] Time step (TS) is a further basic notion and corresponds to an elementary event as nearest neighbor interaction taking during the time chronon. The propagation rate for information is CS/TS and is analogous to maximal signal velocity. The counterpart of the space-time metric is thus brought in by the introduction of TS and CS.

SEs emerge or disappear so that the effective distances of the nearby points change: this would the counterpart for the dynamics of space-time metric in General Relativity. I understood that duplication and reduction effectively corresponds to the duplication or halving of the distance assignable to SE.

[TW] Elements have an 'awareness' of the information around them and communicate with their nearest neighbours through time.

[MP] The treatment of elements as fundamental observers is an interesting idea but can be criticized. Why not PPs? One could also argue that the SEs become conscious observers only under some additional assumptions. For instance, one can imagine that they represent matter and become fundamental conscious observers if fermions or fermion pairs can be assigned to them.

The abstract says nothing about quantum theory. To my view it is very difficult to imagine how quantum theory could emerge from an approach based on classical probability and some kind of quantum approach would be required to understand entanglement and state function reduction.

6.7.2 Fundamental discretization as a cognitive representation?

In the sequel TGD view of the discretization interpreted as cognitive representation is described. The surprise was the discovery of what I call generalized integers and rationals as a union of various p-adic number fields with different p-adic number fields glued together along numbers which belong to both p-adic number fields. I do not know whether mathematicians have played with this thought. This space has an ultrametric topology and could have application to the description of spin glass type systems [L58]. In TGD it could have application in the mathematical description of processes in which the p-adic prime associated with the particle changes.

Something is discrete but what it is?

Something is discrete at the fundamental level: is it space-time or only a discrete cognitive representation, a discretization of a continuous space-time? The essential assumption of SERD is that it is space-time, which is fundamentally discrete and realized as hypergraph. The basic problem is that it is not clear whether the notions of space-time dimensions, distance, angle, and curvature can emerge in a purely combinatorial approach in which only distance between nearby nodes is a metric notion. These notions also have a formal generalization to gauge theories.

The alternative approach would be based on the observation that cognition is discrete and finite. Cognition provides representations of the physical world. Could one assume that the physical world has continuous geometry and that only cognition is discrete?

Could the cognitive Universe consist of generalized integers?

Integers (and rationals) are the simplest discrete but infinite systems. Integers/rationals are usually assumed to have real topology. One can however imagine an infinite number of p-adic topologies, which are ultrametric and are defined by a p-adic norm having values coming as powers of prime p. p-Adic primes typically have an infinite expansion in powers of p and large powers of p have small p-adic norm in contrast to the real norm.

p-Adic integer/rational has expansion in powers of p and the inverse of the smallest power in the expansion determines the norm so that the notion of size is completely different for p-adic and real integers. Note that also the p-adic expansion of rationals involves an infinite number of powers of p but is periodic. p-Adic transcendentals do not have this property. Note also that p-adic integers modulo p define a finite field $G(p)$.

p-Adic integers are only weakly ordered. Only if two p-adic integers/ rationals have different p-adic norms, can one tell which is the larger one. One can however construct continuous maps from p-adics to reals to approximately preserve the norm. p-Adic norm is ultrametric and this property is essential in the thermodynamic models of spin glass energy landscape [L58].

One could, at least as the first guess, imagine that the Universe of cognition consists of integers/rationals or a finite subset of them and that one also allows integers/rationals, which are infinite as real integers but finite as p-adic integers for some prime p.

One can decompose generalized integers to subsets with different p-adic topologies.

1. Regions corresponding to two different p-adic topologies p_1 and p_2 have as an interface as the set integers, which have an expansion in powers of $n_{12} = p_1p_2$. Therefore the cognitive world decomposes into p-adic regions having interfaces, which consist of power series of $n_{12..k} = p_1^{k_1}... \times p_k^{n_k}$. Ordinary integer n with a decomposition to primes belongs to the interface of the p-adic worlds corresponding to the prime factors.

   How does this decomposition relate to adeles [L23, L22], which can be regarded as a Cartesian product of p-adic number fields defining and of reals [L23, L22]? Adeles correspond to a Cartesian product but now one has a union so that these concepts seems to be different. I do not know whether mathematicians have encountered the notion of generalized integers and rationals.

2. Each p-adic region decomposes into shells, kinds of analogs of mass shells, consisting of p-adic integers with p-adic norm given by a power of p.

3. The distance between the points of the cognitive sub-landscape corresponding to p would be defined by the p-adic norm. The points with the same p-adic norm would have a distance defined as the p-adic norm of their difference. This distance is the same for several point pairs so that p-adic topology is much rougher than the real topology. For instance the p-adic norm of numbers $1, ..., p-1$ is the same.

4. One could define a distance between points associated with p-adic topologies p_1 and p_2 as the shortest distance between them identified as the sum of the distances to the interface between these regions.

In this framework, the analog of a hypergraph would be simply a subset of generalized integers decomposing to p-adic integers labeled by some subset of primes.

1. The simplest dynamical operation, having now an interpretation as a cognitive operation, would be addition or removal of a p-adic integer corresponding to some value of p-adic prime or several of them. The addition would have an interpretation or worsening or improving the cognitive representation for some prime p.

2. Arithmetic operations for the points inside a region corresponding to a given p are possible. Arithmetic operations of finite integers are basic elements of at least human cognition and their sum and product would correspond to "particle reactions" in which two points fuse together to form a sum or product. If infinite integers can be expressed as power series of integers n_1 and n_2, they can be regarded as p-adic integers for the factors of n_1 and n_2 and both sum and product make sense for common prime factors. Note that the operations are well-defined also for generalized rationals.

3. What happens in the arithmetic operations information theoretically? In the product operation, the outcome is in the interface region associated with n_1 and n_2 and the information about factors is not lost since a measurement revealing prime factors can be done repeatedly.

   The projection operator applied to a quantum superposition of integers would project to a subspace of integers, which are divisible by a given prime p. This operation could be repeated for different primes and eventually give the prime number decomposition for some integer n in the superposition.

   One strange fact about idiot savants described by Oliver Sacks (this is discussed from the TGD point of view in [K65]) is that they can decompose integers into prime factors and obviously see the emergence of the prime factors. Could this kind of cognitive measurement be in question?

   Sum does not in general belong to the interface region of either integer and information is lost since many number pairs give rise to the same sum. Therefore sum and product are information-theoretically very different operations.

Could there be a quantum physical realization for the arithmetic operations? Could they relate to our conscious arithmetic thinking?

1. Consider first the sum operation. Quantum numbers, such as momenta, represented as integers or even algebraic integers are conserved in the physical reaction vertices. The conserved quantum numbers for the final state for a fusion reaction are sums of integers so that these reactions have an arithmetic interpretation.

2. In the case of a product, the fusion reaction should give a product of integers n_1 and n_2 or a representation of it? One should have conserved multiplicative quantum numbers in the vertex.

   Phase factors as eigenvalues of unitary operators are such. They should form a multiplicative group as representation of integers or even rationals. Integer scalings define such a group. One can also consider eigenvalues $n^{i\phi}$, ϕ some fixed phase angle. The operator would therefore be a scaling represented unitarily by these phase factors.

   Initial state would be a product of eigenstates of the scaling operator with eigenphases $n_1^{i\phi}$ and $n_2^{i\phi}$ and the final state would be a single particle state with the eigenvalue $n_1^{i\phi}n_2^{i\phi} = (n_1n_2)^{i\phi}$. One can say that n_1 acts on n_2 by scaling or vice versa. Interestingly, at the fundamental level scalings replace time translations in the TGD framework (and also in superstring theory), and this is especially so for spin glass phase [L58].

   Interestingly, sum appears at the level of Lie algebras and product at the level of Lie groups.

In quantum groups also the reverse operations, co-product and co-sum, having pair creation as analog, are possible. For the co-sum the information increases for the product. These operations would be time reversals of each other. In the zero energy ontology (ZEO) of TGD time reversal occurs in "big" (ordinary) state function reductions (BSFRs) [L38, L64] [K89]. What comes to mind is that the idiot savants described by Sacks might perform a time reversal decomposing product to prime factors. The cognitive measurement would correspond to BSFR.

Note that ZEO also predicts "small" state function reductions (SSFRs), which do not change the arrow of time and give rise to the flow of consciousness whereas BSFR corresponds to a universal counterpart of death or of falling asleep. It is the TGD counterpart of repeated measurements in the Zeno effect and of weak measurements of quantum optics.

This cognitive world would in TGD correspond physically to the most general spin glass energy landscape having an ultrametric topology [L58].

The algebraic extensions of p-adic number fields are discrete

The proposed structure does not have any natural notion of dimension. We are however able to cognize higher dimensional spaces using formulas.

1. p-Adic number fields indeed allow infinite hierarchies of algebraic extensions obtained by adding to them roots of polynomials, which are algebraic numbers. These induce extensions of p-adic number fields as finite fields $G(p, k)$ having algebraic dimension, which is at most the dimension of the corresponding extensions of rationals.

2. It is natural to assume that cognitive representations are always finite. This suggests that the set of "populated" points of the cognitive space is discrete and even finite. Being "populated" could mean that a fermion, having an interpretation as a generator of Boolean algebra, is labelled by the algebraic number defining the point. In a more general formulation bringing in quaternions and octonions as number fields: algebraic complexified quaternions would define the momentum components of fermions.

   What has been said above, generalizes almost as such and one obtains a hierarchy of generalized integers as algebraic extensions of generalized integers at the lowest level. This could generalize the rational number based computationalism (Turing paradigm) to an entire hierarchy of cognitive computationalisms. The hierarchy of algebraic extensions suggests the same.

3. The algebraic complexity of generalized integers increases with the dimension of extension and in the TGD framework it corresponds to an evolutionary hierarchy. The dimension of extension defines what is identified in terms of an effective Planck constant.

But what about the real world?

A hierarchy of p-adicities and hierarchies of the algebraic extensions of p-adicities have been obtained. The 4-D world of sensory perceptions with its fundamental symmetries is however still missing. Could number theory come to rescue also here? This is indeed the case.

1. The fundamental continuous number fields consist of reals, complex numbers, quaternions and octonions with dimensions 1,2,4, 8 [L45, L46, L75]. Quaternions cannot as such correspond to 4-D space-time since the number theoretic purely algebraic norm defines the Euclidean metric.

2. This norm can be however algebraically continued to the complexification of quaternions obtained by adding a commuting imaginary unit i commuting with quaternionic and octonionic imaginary units. This algebraic norm squared does not involve complex conjugation as the Hilbert space norm and is in general complex but real for the subspaces corresponding to various metric signatures (a given component of quaternion are either real or imaginary). One obtains therefore Minkowski space and even more: its variants with various metric signatures.

3. One can imagine a generalization of the notion of generalized integer so that one would have hierarchies of generalized complex numbers, quaternions and octonions and their complexifications for various extensions of rationals.

   A possible problem relates to the p-adic variants of quaternions, octonions and complex numbers. Consider the inverse $z^{-1} = (x-iy)/(x^2+y^2)$ of p-adic complex numbers $z = x+iy$. The problem is that $x^2 + y^2$ can vanish since there is no notion of sign of the number. For $p \ mod \ 4 = 1$, $\sqrt{-1}$ is an ordinary p-adic number, albeit with an infinite pinary expansion so that for $y = \sqrt{-1}x$, one has this problem.

   Could the finiteness of cognition solve the problem? If only finite p-adic integers and rationals can define momentum components of fermions (finite cognitive and measurement resolution), the problem disappears.

   Could one give up the field property for the p-adic variants of classical number fields? Already the complexification by i forces to give up the field property but has physical meaning since it makes Minkowski signature possible.

This would give Minkowski space M^4 as a special case. This is however not enough. One wants curved 4-D space-times. The basic structure is complexified octonions.

1. One should obtain 4-D surfaces of M^8 generalizing empty Minkowski space M^4. Octonions fail to be associative and at the level of M_c^8 the natural proposal is that there is number theoretic dynamics based on associativity. The 4-D surfaces must be associative in some sense. The geometric vision predicts holography and this holography should have a number theoretic counterpart based on associativity.

2. The first guess is that the tangent space of 4-surface is associative and thus quaternionic. This gives only M^4 and is therefore trivial [L45, L46, L75].

   The requirement that the normal space of the 4-surface Y^4 in M_c^8 is associative/quaternionic however works. If one requires that the normal subspace contains also a commutative (complex) subspace, one ends up to M^8-H-duality ($H=M^4\times CP_2$ mapping the associative 4-D surfaces Y^4 of M_c^8 to space-time surfaces X^4 in H determined by holography forced by generalized coordinate invariance. The symmetries of H include Poincare symmetries and standard model symmetries.

3. At the level of M^8, associativity of the normal space allows also 6-D surfaces with 2-D commutative normal space and they can be interpreted in terms of analogs of 6-D twistor spaces of 4-D surfaces Y^4. They can be mapped to to the twistor space of H by M^8-H duality and define 6-D twistor spaces of space-time surfaces X^4 of H. What is beautiful is that the Kähler structure for the twistor space of H exists only for the choice $H=M^4\times CP_2$, which is also forced by the associative dynamics [A79]! TGD is unique!

4. The dynamics would rely on holography but how to get the algebraic extensions? The roots of a polynomial P with rational or even integer coefficients satisfying some additional conditions would define the needed extension of rationals. The roots would in the general case define complex mass shells H_c^3 as complex variants of hyperbolic 3-spaces H^3 in $M_c^4\subset M_c^8$ having interpretation as a momentum space. M^8-H duality serves as a generalization of momentum position duality. The 3-D surfaces as subsets of these H^3:s define the data of the associative holography and are contained by the 4-surface Y^4.

5. Cognitive representation would be defined as a unique number theoretic discretization of the 4-surface Y^4 of M_c^8 consisting of points, whose number theoretically preferred linear Minkowski coordinates are algebraic integers in an extension defining the 4-surface in question. This discretization induces discretization of the space-time surface via M^8-H duality. The cognitive representations are number-theoretically universal and belong to the intersections of realities and p-adicities.

6. The mass shells H_c^3 are very special since in the preferred Minkowski coordinates a cognitive explosion takes place. All algebraic rationals, in particular integers, are points of H_c^3. Algebraic integers are physically favored and define components of four-momenta. Galois confinement [L59] states that the total momenta have components which are ordinary integers when a suitable momentum unit is used.

6.8 Infinite primes as a basic mathematical building block

Infinite primes [K72, K36, K44] are one of the key ideas of TGD. Their precise physical interpretation and the role in the mathematical structure of TGD has however remained unclear.

3 new ideas are be discussed. Infinite primes could define a generalization of the notion of adele; quantum arithmetics could replace $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary primes with p-adic representations of say HFFs; the polynomial Q defining an infinite prime could be identified with the polynomial P defining the space-time surface: $P=Q$.

6.8.1 Construction of infinite primes

Consider first the construction of infinite primes [K72].

1. At the lowest level of hierachy, infinite primes (in real sense, p-adically they have unit norm) can be defined by polynomials of the product X of all primes as an analog of Dirac vacuum.

   The decomposition of the simplest infinite primes at the lowest level are of form $aX + b$, where the terms have no common prime divisors. More concretely $a = m_1/n_F$ $b = m_0 n_F$, where n_F is square free integer analogous and the integer m_1 and n_F have no common prime divisors divisors. The divisors of m_2 are divisors of n_F and m_i has interpretation as n-boson state. Power p^k corresponds to k-boson state with momenta p. $n_F = \prod p_i$ has interpretation as many-fermion state satisfying Fermi-Dirac statistics.

   The decomposition of lowest level infinite primes to infinite and finite part has a physical analogy as kicking of fermions from Dirac sea to form the finite part of infinite prime. These states have interpretation as analogs of free states of supersymmetric arithmetic quantum field theory (QFT) There is a temptation to interpret the sum $X/n_F + n_F$ as an analog of quantum superposition. Fermion number is well-defined if one assigns the number of factors of n_F to both n_F and X/n_F.

2. More general infinite primes correspond to polynomials $Q(X) = \sum_n q_n X^n$ required to define infinite integers which are not divisible by finite primes. Each summand $q_n X^n$ must be a infinite integer. This requires that q_n is given by $q_n = m_{B,n}/\prod_{i_1}^n n_{F,i}$ of square free integers $n_{F,i}$ having no common divisors.

   The coefficients $m_{B,n}$ representing bosonic states have no common primes with $\prod n_{F,i}$ and there exists no prime dividing all coefficients $m_{B,n}$: there is no boson with momentum p present in all states in the sum.

   These states have a formal interpretation as bound states of arithmetic supersymmetric QFT. The degree k of Q determines the number of particles in the bound states.

   The products of infinite primes at given level are infinite primes with respect to the primes at the lower levels but infinite integers at their own level. Sums of infinite primes are not in general infinite primes. For instance the sum and difference of $X/n_F + n_F$ and $X/n_F - n_F$ are not infinite primes.

3. At the next step one can form the product of all finite primes and infinite primes constructed in this manner and repeat the process as an analog to second quantization. This procedure can be repeated indefinitely. This repeated quantization a hierarchy of infinite primes, which could correspond to the hierarchy of space-time sheets.

   At the n:th hierarchy level the polynomials are polynomials of n variables X_i. A possible interpretation would be that one has families of infinite primes at the first level labelled by n_1 parameters. If the polynomials $P(x)$ at the first level define space-time surfaces, the interpretation at the level of WCW could be that one has an $n-1$-D surface in WCW parametrized by $n-1$ parameters with rational values and defining a kind of sub-WCW. The WCW spinor fields would be restricted to this surface of WCW.

The Dirac vacuum X brings in mind adele, which is roughly a product of p-adic number fields. The primes of infinite prime could be interpreted as labels for p-adic number fields. Even more generally, they could serve as labels for p-adic representations of various algebras and one could even consider replacing the arithmetic operations with $\oplus$ and $\otimes$ to get the quantum variants of various number fields and of adeles.

The quantum counterparts of nfinite primes at the lowest and also at the higher levels of hierarchy could be seen as a generalization of adeles to quantum adeles.

6.8.2 Questions about infinite primes

One can ask several questions about infinite primes.

1. Could $\oplus$ and $\otimes$ replace $+$ and $-$ also for infinite primes. This would allow us to interpret the primes p as labels for algebras realized p-adically. This would give rise to quantal counterparts of infinite primes.

2. What could $+ \rightarrow \oplus$ for infinite primes mean physically? Could it make sense in adelic context? Infinite part has finite p-adic norms. The interpretation as direct sum conforms with the fermionic interpretation if the product of all finite primes is interpreted as Dirac sea. In this case, the finite and infinite parts of infinite prime would have the same fermion number.

3. Could adelization relate to the notion of infinite primes? Could one generalize quantum adeles based on $\oplus$ and $\otimes$ so that they would have parts with various degrees of infinity?

6.8.3 $P = Q$ hypothesis

One cannot avoid the idea that that polynomial, call it $Q(X)$, defining an infinite prime at the first level of the hierarchy, is nothing but the polynomial P defining a 4-surface in M^4 and therefore also a space-time surface. $P = Q$ would be a condition analogous to the variational principle defining preferred extremals (PEs) at the level of H.

There is however an objection.

1. $P = Q$ gives very powerful constraints on Q since it must define an infinite integer. The prime polynomials P are expected to be highly non-unique and an entire class of polynomials of fixed degree characterized by the Galois group as an invariant is in question. The same applies to polynomials Q as is easy to see: the only condition is that powers of $a_k X^k$ defining infinite integers have no common prime factors.

2. It seems that a composite polynomial $P_n \circ ... \circ P_1$ satisfying $P_i = Q_i$ cannot define an infinite prime or even infinite integer. Even infinite integer property requires very special conditions.

3. There is however no need to assume $P_i = Q_i$ conditions. It is enough to require that there exists a composite $P_n \circ ... \circ P_1$ of prime polynomials satisfying $P_n \circ ... \circ P_1 = Q$ defining an infinite prime.

   The physical interpretation would be that the interaction spoils the infinite prime property of the composites and they become analogs of off-mass-shell particles. Exactly this occurs for bound many-particle states of particles represented by P_i represented composite polynomials $P_1 \circ ... P_n$. The roots of the composite polynomials are indeed affected for the composite. Note that also products of Q_i are infinite primes and the interpretation is as a free many-particle state formed by bound states Q_i.

There is also a second objection against $P = Q$ property.

1. The proposed physical interpretation is that the ramified primes associated with $P = Q$ correspond to the p-adic primes characterizing particles. This would mean that the ramimied primes appearing in the infinite primes at the first level of the hierarchy should be physically special.

2. The first naive guess is that for the simplest infinite primes $Q(X) = (m_1/n_F)X + m_2 n_F$ at the first level, the finite part $m_2 n_F$ has an identification as the discriminant D of the polynomial $P(X)$ defining the space-time surface. This guess has no obvious generalization to higher degree polynomials $Q(X)$ and the following argument shows that it does not make sense.

   Since Q is a rational polynomial of degree 1 there is only a single rational root and discriminant defined by the differences of distinct roots is ill-defined that $Q = P$ condition would not allow the simplest infinite primes.

   Therefore one must give either of these conjectures and since $P = Q$ conjecture dictates the algebraic structure of the quantum theory for a given space-time surface, it is much more attractive.

The following argument gives $P = Q$. One can assign to polynomial P invariants as symmetric functions of the roots. They are invariants under permutation group S_n of roots containing Galois group and therefore also Galois invariants (for polynomials of second order correspond to sum and product of roots appearing as coefficients of the polynomial in the representation $x^2 + bx + cx$). The polynomial Q having as coefficients these invariants is the original polynomial. This interpretation gives $P = Q$.

6.9 Summary of the proposed big picture

In the previous sections the plausible looking building blocks of the bigger picture of the TGD were discussed. Here I try to summarize a guess for the big picture.

6.9.1 The relation between $M^8 - H$ and $M - M'$ dualities

The first question is whether $M^8 - H$ duality between number theoretical and geometric physics, very probably relating to Langlands duality, corresponds to a duality between M and its commutant M'. Physical intuition suggests that these dualities are independent. M' would more naturally correspond to classical description as dual to quantum description using M. One would assign classical and quantum views to both number theoretic (M^8) and geometric (H) descriptions.

1. At the geometric side M would be realized in terms of HFFs associated with SSA_n, Aff_n and I acting in H. At the number theoretic side, braided Galois group algebras would define the HFFs and have natural action in SSA_n, A_n and I .

2. The descriptions in terms of preferred extremals in H and of polynomials P defining 4-surfaces in M^8 would correspond to classical descriptions. $P = Q$ condition would define preferred polynomials and infinite primes.

3. At the geometric side, M' would correspond to scalar factors of WCW wave functions symplectic invariants identifiable as Kähler magnetic fluxes at both M^4 and CP_2 sectors. They are zero modes and therefore do not contribute to the WCW line element.

4. At the number theoretic side, the wave functions would depend on Galois invariants. Discriminant D, set of roots to which braid strands can be assigned to define n-braid, and ramified primes dividing it in the case of polynomials with rational/integer coefficients are Galois invariants analogous to Kähler fluxes. They code information about the spectrum of virtual mass squared values as roots of P. The strands of braid as Galois invariant correspond to (possibly) monopole flux tubes and one assign them quantized magnetic fluxes as integer valued symplectic invariants.

6.9.2 Basic mathematical building blocks

The basic mathematical building blocks of quantum aspects of TGD involve at least the following ones.

1. The generalization of arithmetics and even number theory by replacing sum and product by direct sum and tensor product for various algebras and associated representations is a mathematical notion expected to be important and a straightforward generalization of adeles and infinite primes to their quantum counterparts is highly suggestive.

2. Quantum version of adelic physics obtained by replacing ordinary arithmetic operations with direct sum and tensor product relates closely to the fusion of real and various p-adic physics at quantum level.

3. The hierarchy of infinite primes suggested by the many-sheeted space-time suggests a profound generalization of the notion of adelic physics. Infinite primes are defined by polynomials of several variables the basic equation in the general form would be $Q(X_1, ..., X_n) = P(X_1, ..., X_n)$.

6.9.3 Basic algebraic structures at number theoretic side

Number theoretic side involves several key notions that must have counterparts at the geometric side.

1. Number theoretic side involves Galois groups as counterparts of symplectic symmetries and can be regarded as number theoretic variants of permutation symmetries and lead to the notion of braided Galois group, whose group algebra defines HFF.

2. Galois groups can be decomposed to a hierarchy of normal subgroups, which are simple and therefore primes in group theoretic sense. Simple Galois groups correspond to polynomial primes with respect to functional composition, and one can assign to a given Galois group a set of polynomials with fixed degrees although the polynomials and their order of polynomials in composition are not unique.

3. There is a large class of polynomials giving rise to a given Galois group and they bring in additional degrees of freedom. The variation of the polynomial coefficients corresponding to the same Galois group is analogous to symplectic transformations leaving the induced Kähler form invariant.

   The roots of polynomials define analogs for the strands of n-braid, discriminant D, and ramified primes dividing the discriminant. They are central Galois invariants analogous to Kähler magnetic fluxes at the geometry side.

4. Ramified primes characterize polynomials P but are not fixed by the Galois group, are analogous to the zero modes at the level of H. Magnetic fluxes are their counterparts at the level of H. I have proposed the interpretation of ramified primes p as p-adic primes characterizing elementary particles in the model of particle masses based on p-adic thermodynamics. These primes are rather large: for instance, $M_{127} = 2^{127} - 1$ would characterize electrons. It would however seem that the prime k in SSA_k corresponds to the prime characterizing simple Galois group.

   Also affine algebras Aff_n assignable to the light-like partonic orbits and isometries of H are present and also they appear in p-adic mass calculations based on p-adic thermodynamics. Could the adelic hierarchy p-adic variants of algebras SSA, Aff and I have adelic factors labelled by ramified primes p form also an adelic structure with respect to $\oplus$ and $\otimes$?

6.9.4 Basic algebraic structures at the geometric side

The symmetry algebras at the level of H define the key quantal structures.

1. The symmetries at the geometric side involve hierarchies A_n of algebras $A_n \in SSA_n, A_n, I_n$ defining hierarchies of factor algebras. The condition that subalgebras A_n and $[A_n, A]$ annihilate physical states gives rise to hierarchies of algebras, which would correspond to those for Galois groups for multiple extensions of rationals. The braided Galois groups for polynomials of degree n n roots/braids would act naturally in A_n so that it would have number theoretic braiding.

2. The decomposition of the Galois group to simple normal subgroups would correspond to a functional composite of prime polynomials, which corresponds to the inclusion hierarchy of HFFs associated with A_n with n identified as the degree of polynomial.

   The polynomials $Q(X)$ defining infinite prime have decomposition to polynomial primes but the polynomial primes in the decomposition cannot define infinite primes.

   Kähler magnetic fluxes for CP_2 and M^4 Kähler forms are symplectic invariants and represent zero modes. At the number theoretic side the discriminant and root spectrum (mass squared spectrum) are classical Galois invariants. States as Galois singlets are Galois invariants at quantum level.

The key equation, not encountered before in the TGD framework, is $P = Q$ motivated by the notion of infinite prime. It would assign to polynomial P unique algebraic structures defining what might be called its quantization. Without this structure one should give up the notion of infinite prime and lose the notion of preferred P as analog of preferred extremal.

6.10 Appendix: The reduction of quantum TGD to WCW geometry and spinor structure

The first attempts to build quantum TGD were based on the standard method used to quantize quantum field theories. The path integral over all possible space-time surfaces connecting initial and final 3-surfaces for an action exponential using for instance Kähler action, would have given the scattering amplitudes.

6.10.1 The problems

The first problem is that the integrand is a phase factor $exp(iS)$, where S could be the Kähler action. Phase factor has modulus 1 and the integral does not converge even formally. One would need a real exponent to have any hopes of convergence. This problem can be circumvented in free quantum field theory by algebraic tricks.

The second problem is that all conceivable actions are extremely nonlinear and new kinds of divergences appear in each order of perturbation theory. This is essentially due to the locality of the action principle involving interaction vertices with arbitrarily high numbers of particles. Also ordinary QFTs meet the same problem and for renormalizable theories the addition of counterterms with suitably infinite coefficients can cancel the divergences without the addition of an infinite number of counter terms. It became clear that there are no hopes of getting rid of the divergences in TGD by addition of counterterms. The situation is the same in general relativity although heroic and ingenious attempts to calculate scattering amplitudes have been made.

Only $\mathcal{N} = 4$ SUSY is a QFT that is hoped to be free of divergences without renormalization but here the problem is caused by the non-planar Feynman diagrams, to which the twistor approach does not apply.

6.10.2 3-D surfaces or 4-surfaces associated to them by holography replace point-like particles

The key idea of TGD is that point-like particles are replaced with 3-surfaces. This idea does not favour path integral approach.

1. In TGD, point-like particles are replaced with 3-surfaces. Local interaction vertices are smoothed out to non-local ones so that there should be no local divergences. Perhaps the path integral, derived originally as a representation of Schrödinger equation, is not only unnecessary but also a wrong way to compute anything in TGD. In superstring models, the replacement of a point-like particle with string indeed allows elimination of the local divergences.

   3-D surface should be the basic dynamical object. One should therefore have a functional integral over 3-surfaces, which is analogous to the Gaussian integral and converges.

2. This problem led to the idea of the "world of classical worlds" (WCW). 4-D General Coordinate Invariance implies that to a given 3-surface X^3 one must be able to assign a 4-surface $X^4(X^3)$ at which the 4-D general coordinate transformations act.

   Either 3-surfaces X^3 or almost unique 4-surfaces $X^4(X^3)$ are the fundamental objects so that holography holds true. At that time I did not talk about holography, which was introduced by Susskind much later, around 1995. Therefore the introduction of the path integral is not necessary.

   Later it became clear that the exact determinism of the classical dynamics can be lost, at least for Kähler action having huge spin glass degeneracy. Later 4-D Käehler action replaced in twistor lift of TGD by its sum with a volume term, and for this action the non-determinism is analogous to that for soap films spanned by frames, that is finite, and has physical interpretation.

6.10.3 WCW Kähler geometry as s geometrization of the entire quantum physics

This argument led to the vision about quantum TGD as WCW geometry, which generalizes Einstein's vision of geometrization of gravitational interaction to geometrization of all classical interactions and then to the geometrization of the entire quantum theory.

1. WCW is the space of all 3-surfaces or almost equivalently the space of 4-surfaces. Physical states correspond to WCW spinor fields.

2. WCW must have Kähler geometry since Kähler structure allows to geometrize the hermitian conjugation which is fundamental for quantum theory. Imaginary unit is represented geometrically by the Kähler form and the real unit by the Kähler metric. The tensor square Kähler form as an imaginary unit is equal to the negative of the real unit, that is the negative of the metric.

3. The construction of loop space geometries by Dan Freed [A54] led to a unique geometry of loop space. The mere existence of Riemann connection requires that the metric has maximal isometries and is unique apart from scaling. When basic objects are 3-D this condition is even more stringent. The Kähler geometry of WCW and thus physics could be unique from its mere mathematical existence!

   Why $H = M^4 \times CP_2$? The existence of the twistor lift fixes H uniquely since only M^4 (E^4) and CP_2 allow a twistor space with Kähler structure [A79]. The necessarily dimensionally reduced Kähler action at the twistor space level adds to the 4-D Kähler action a volume term removing the non-determinism and explaining cosmological constant and its smallness in long scales.

4. How is the Kähler geometry of WCW determined? The definition of the Kähler metric of WCW must assign to a 3-surface X^3 a more or less unique space-time surface $X^4(X^3)$ in order to have a general coordinate invariance. One must also have a connection with classical physics: classical physics must be an exact part of quantum physics and thus the definition of WCW Kähler geometry involves a classical action principle.

   The Kähler metric is defined by the Kähler function K. The idea is that K is the value of Kähler action S_K or of a more general action for a more or less unique space-time surface $X^4(X^3)$ containing a given 3-surface X^3.

5. It is convenient to speak of preferred extremal (PE) and there are several characterizations of what PE is. $M^8 - H$ duality gives the most concrete one. Twistor lift gives the second one and the gauge conditions associated with the WCW Dirac equation provide the third characterization.

6.10.4 Quantum physics as physics of free, classical spinor fields in WCW

How to develop quantum physics in WCW? The idea is that free, classical WCW spinor fields define all possible quantum states of the Universe and interactions reduce to topology. There would be no quantization at the level of WCW and the only genuinely quantal element of quantum theory would be state function reduction giving rise to conscious experience.

1. In order to have spinor fields in WCW, one must have the notion of spinor structure. Spinor structure is almost uniquely fixed by the metric and involves in an essential manner gamma matrices, which anticommute to metric.

2. The second quantization of H spinor fields assigns to the modes of H spinor fields fermionic oscillator operators. Why not build the conplexified gamma matrices of WCW (their hermitian conjugates) as linear combinations of the creation (annihilation) operators?! Second quantization for the *free* H spinor field, is completely unique and straightforward and avoids all problems of quantization in curved space-time.

One could interpret the second quantization of free fermions and fermionic statistics in terms of WCW geometry, which is something completely new.

3. WCW spinors (for given 4-surface as point of WCW) would be fermionic Fock states created using fermionic oscillator operators and depend on the space-time surface $X^4(X^3)$ as a 4-surface almost uniquely determined by 3-surface X^3.

   The fermionic Fock state basis can be interpreted as a representation of Boolean logic so that Boolean logic could be seen as a "square root" of Kähler geometry.

   The WCW spinor field would correspond to a superposition of preferred extremals X^4 with a WCW spinor assigned with each X^4.

6.10.5 Dirac equation for WCW spinor fields

Free Dirac equation is the key equation for classical spinor fields.

1. In string models it corresponds to the analogs of super-Virasoro and super-Kac-Moody conditions stating conformal invariance and Kac-Moody invariance analogous but not quite equivalent with gauge symmetry.

2. In TGD, these conditions as a counterpart of the WCW Dirac equation generalize. Super symplectic algebra associated with $\delta M^4_+ \times CP_2$ (δM^4_+ denotes light-cone boundary) SSA, the infinite-D algebras of conformal symmetries ($Conf$) and isometries (I) of δM^4_+ (unique to the 4-D Minkowski space), and the affine algebras Aff associated with the light-like orbits of partonic 2-surfaces would be the basic algebras.

3. To each of these algebras, one can assign a generalization of the gauge conditions of conformal field theories. What is new is that one obtains a hierarchy of gauge conditions. The algebra in question, call it A, and sub-algebra A_n, $n \geq 0$, with conformal weights coming as n-multiples of weights for A, and the commutator $[A_n, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish, which gives strong conditions on space-time surfaces and decomposes WCW to sectors characterized by n.

4. In superstring models one has only $n = 0$. In the number theoretic vision, the hierarchy of values of n would actually correspond to the hierarchy of extensions of rationals. If $M^8 - H$ duality holds true, n corresponds to the degree of polynomial P defining the space-time surface and polynomials P would decompose WCW to sectors.

6.10.6 $M^8 - H$ duality at the level of WCW

WCW emerges in the geometric view of quantum TGD. $M^8 - H$ duality should lso work for WCW. What is the number theoretic counterpart of WCW? What is the geometric counterpart of the discretization characteristic to the number theoretic approach?

In the number theoretic vision in which WCW is discretized by replacing space-time surfaces with their number theoretical discretizations determined by the points of $X^4 \subset M^8$ having the octonionic coordinates of M^8 in an extension of rationals and therefore making sense in all p-adic number fields? How could an effective discretization of the real WCW at the geometric H level, making computations easy in contrast to all expectations, take place?

1. The key observation is that any functional or path integral with integrand defined as exponent of action, can be *formally* calculated as an analog of Gaussian integral over the extrema of the action exponential $exp(S)$. The configuration space of fields would be effectively discretized. Unfortunately, this holds true only for the so called integrable quantum field theories and there are very few of them and they have huge symmetries. But could this happen for WCW integration thanks to the maximal symmetries of the WCW metric?

2. For the Kähler function K, its maxima (or maybe extrema) would define a natural effective discretization of the sector of WCW corresponding to a given polynomial P defining an extension of rationals.

The discretization of the WCW defined by polynomials P defining the space-time surfaces should be equivalent with the number theoretical discretization induced by the number theoretical discretization of the corresponding space-time surfaces. Various p-adic physics and corresponding discretizations should emerge naturally from the real physics in WCW.

3. The physical interpretation is clear. The TGD Universe is analogous to the spin glass phase [L58]. The discretized WCW corresponds to the energy landscape of spin glass having an ultrametric topology. Ultrametric topology of WCW means that discretized WCW decomposes to p-adic sectors labelled by polynomials P. The ramified primes of P label various p-adic topologies associated with P.

Part II

CATEGORY THEORY AND QUANTUM TGD

Chapter 7

Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness

7.1 Introduction

Goro Kato has proposed an ontology of consciousness relying on category theory [A67, A98]. Physicist friendly summary of the basic concepts of category theory can be found in [A83]) whereas the books [A36, A74] provide more mathematically oriented representations. Category theory has been proposed as a new approach to the deep problems of modern physics, in particular quantization of General Relativity. To mention only one example, C. J. Isham [A83] has proposed that topos theory could provide a new approach to quantum gravity in which space-time points would be replaced by regions of space-time and that category theory could geometrize and dynamicize even logic by replacing the standard Boolean logic with a dynamical logic dictated by the structure of the fundamental category purely geometrically [A116].

Although I am an innocent novice in this field and know nothing about the horrible technicalities of the field, I have a strong gut feeling that category theory might provide the desired systematic approach to quantum TGD proper, the general theory of consciousness, and the theory of cognitive representations [K52].

7.1.1 Category Theory As A Purely Technical Tool

Category theory could help to disentangle the enormous technical complexities of the quantum TGD and to organize the existing bundle of ideas into a coherent conceptual framework. The construction of the geometry of the configuration space ("world of classical worlds") [K35, K20]. of classical configuration space spinor fields [K88]. and of S-matrix [K18] using a generalization of the quantum holography principle are especially natural applications. Category theory might also help in formulating the new TGD inspired view about number system as a structure obtained by "gluing together" real and p-adic number fields and TGD as a quantum theory based on this generalized notion of number [K73, K74, K72].

7.1.2 Category Theory Based Formulation Of The Ontology Of TGD Universe

It is interesting to find whether also the ontology of quantum TGD and TGD inspired theory of consciousness based on the trinity of geometric, objective and subjective existences [**?**] could be expressed elegantly using the language of the category theory.

There are indeed natural and non-trivial categories involved with many-sheeted space-time and the geometry of the configuration space ("the world of classical worlds"); with configuration space spinor fields; and with the notions of quantum jump, self and self hierarchy. Functors between these categories could express more precisely the quantum classical correspondences and self-referentiality of quantum states allowing them to express information about quantum jump sequence.

1. Self hierarchy has a structure of category and corresponds functorially to the hierarchical structure of the many-sheeted space-time.

2. Quantum jump sequence has a structure of category and corresponds functorially to the category formed by a sequence of maximally deterministic regions of space-time sheet. Even the quantum jump could have space-time correlates made possible by the generalization of the Boolean logic to what might be space-time correlate of quantum logic and allowing to identify space-time correlate for the notion of quantum superposition.

3. The category of light cones with inclusion as an arrow defining time ordering appears naturally in the construction of the configuration space geometry and realizes the cosmologies within cosmologies scenario. In particular, the notion of the arrow of psychological time finds a nice formulation unifying earlier two different explanations.

4. In zero energy ontology (ZEO), which emerged many years after writing the first version of this chapter, causal diamonds (CDs) defined in terms of intersection of future and past directed light-cones form a category with arrow identified as inclusion.

5. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K74]. The duality would allow to construct new preferred extremals of Kähler action.

7.1.3 Other Applications

One can imagine also other applications.

1. Categories posses inherent logic [A116] based on the notion of sieves relying on the notion of presheaf which generalizes Boolean logic based on inclusion. In TGD framework inclusion is naturally replaced by topological condensation and this leads to a two-valued logic realizing space-time correlate of quantum logic based on the notions of quantum sieve and quantum topos.

   This suggests the possibility to geometrize the logic of both geometric, objective and subjective existences and perhaps understand why ordinary consciousness experiences the world through Boolean logic and Zen consciousness experiences universe through logic in which the law of excluded middle is not true. Interestingly, the p-adic logic of cognition is naturally 2-valued whereas the real number based logic of sensory experience allows excluded middle (is the person at the door in or out, in and out, or neither in nor out?). The quantum logic naturally associated with spinors (in the "world of classical worlds") is consistent with the logic based on quantum sieves.

2. Simple Boolean logic of right and wrong does not seem to be ideal for understanding moral rules. Same applies to the beauty-ugly logic of aesthetic experience. The logic based on quantum sieves would perhaps provide a more flexible framework.

3. Cognition is categorizing and category theory suggests itself as a tool for understanding cognition and self hierarchies and the abstraction processes involved with conscious experience. Here the new elements associated with the ontology of space-time due to the generalization of number concept would be central. Category theory could be also helpful in the modelling of conscious communications, in particular the telepathic communications based on sharing of mental images involving the same mechanism which makes possible space-time correlates of quantum logic and quantum superposition.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L5]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

7.2 What Categories Are?

In the following the basic notions of category theory are introduced and the notion of presheaf and category induced logic are discussed.

7.2.1 Basic Concepts

Categories [A36, A74, A83] are roughly collections of objects A, B, C... and morphisms $f(A \to B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Topological/linear spaces form a category with continuous/linear maps acting as morphisms. Also algebraic structures of a given type form a category: morphisms are now homomorphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can can be very general: for instance, partial ordering $a \leq b$ can define morphism $f(A \to B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Group representation is example of this kind of a functor: now group action in group is mapped to a linear action at the level of the representations. Commuting square is an easy visual manner to understand the basic properties of a functor, see **Fig. 7.1**.

The product $C = AB$ for objects of categories is defined by the requirement that there are projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \to A)$ and $g(D \to B)$ there exist morphism $h(D \to C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically (see **Fig. 7.1**) this corresponds to a square diagram in which pairs A, B and C, D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC.

Examples of product categories are Cartesian products of topological and linear spaces, of differentiable manifolds, groups, etc. Also tensor products of linear spaces satisfies these axioms. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

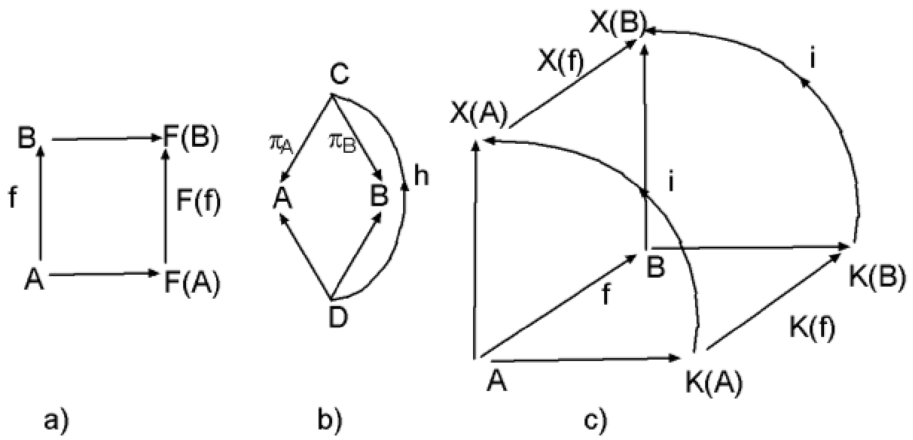

Figure 7.1: Commuting diagram associated with the definition of a) functor, b) product of objects of category, c) presheaf K as sub-object of presheaf X ("two pages of book".)

7.2.2 Presheaf As A Generalization For The Notion Of Set

Presheafs can be regarded as a generalization for the notion of set. Presheaf is a functor X that assigns to any object of a category **C** an object in the category **Set** (category of sets) and maps morphisms to morphisms (maps between sets for **C**). In order to have a category of presheafs, also morphisms between presheafs are needed. These morphisms are called natural transformations $N : X(A) \to Y(A)$ between the images $X(A)$ and $Y(A)$ of object A of **C**. They are assumed to obey the commutativity property $N(B)X(f) = Y(f)N(A)$ which is best visualized as a commutative square diagram. Set theoretic inclusion $i : X(A) \subset Y(A)$ is obviously a natural transformation.

An easy manner to understand and remember this definition is commuting diagram consisting of two pages of book with arrows of natural transformation connecting the corners of the pages: see **Fig. ??**.

As noticed, presheafs are generalizations of sets and a generalization for the notion of subset to a sub-object of presheaf is needed and this leads to the notion of topos [A116, A83]. In the classical set theory a subset of given sets X can be characterized by a mapping from set X to the set $\Omega = \{true, false\}$ of Boolean statements. Ω itself belongs to the category **C**. This idea generalizes to sub-objects whose objects are collections of sets: Ω is only replaced with its Cartesian power. It can be shown that in the case of presheafs associated with category **C** the sub-object classifier Ω can be replaced with a more general algebra, so called Heyting algebra [A116, A83] possessing the same basic operations as Boolean algebra (and, or, implication arrow, and negation) but is not in general equivalent with any Boolean algebra. What is important is that this generalized logic is inherent to the category **C** so that many-valued logic ceases to be an ad hoc construct in category theory.

In the theory of presheafs sub-object classifier Ω, which belongs to **Set**, is defined as a particular presheaf. Ω is defined by the structure of category **C** itself so that one has a geometrization of the notion of logic implied by the properties of category. The notion of sieve is essential here. A sieve for an object A of category **C** is defined as a collection of arrows $f(A \to ...)$ with the property that if $f(A \to B)$ is an arrow in sieve and if $g(B \to C)$ is any arrow then $gf(A \to C)$ belongs to sieve.

In the case that morphism corresponds to a set theoretic inclusion the sieve is just either empty set or the set of all sets of category containing set A so that there are only two sieves corresponding to Boolean logic. In the case of a poset (partially ordered set) sieves are sets for which all elements are larger than some element.

7.2.3 Generalized Logic Defined By Category

The presheaf $\Omega : \mathbf{C} \to \mathbf{Set}$ defining sub-object classifier and a generalization of Boolean logic is defined as the map assigning to a given object A the set of all sieves on A. The generalization of maps $X \to \Omega$ defining subsets is based on the notion of sub-object K. K is sub-object of presheaf X in the category of presheaves if there exist natural transformation $i : K \to X$ such that for each A one has $K(A) \subset X(A)$ (so that sub-object property is reduced to subset property).

The generalization of the map $X \to \Omega$ defining subset is achieved as follows. Let K be a sub-object of X. Then there is an associated characteristic arrow $\chi^K : X \to \Omega$ generalizing the characteristic Boolean valued map defining subset, whose components $\chi^K_A : X(A) \to \Omega(A)$ in **C** is defined as

$$\chi^K_A(x) = \{f(A \to B) | X(f)(x) \in K(B)\} \ .$$

By using the diagrammatic representation of **Fig. 7.1** for the natural transformation i defining sub-object, it is not difficult to see that by the basic properties of the presheaf K $\chi^K_A(x)$ is a sieve. When morphisms f are inclusions in category Set, only two sheaves corresponding to all sets containing X and empty sheaf result. Thus binary valued maps are replaced with sieve-valued maps and sieves take the role of possible truth values. What is also new that truths and logic are in principle context dependent since each object A of **C** serves as a context and defines its own collection of sieves.

The generalization for the notion of point of set X exists also and corresponds to a selection of single element γ_A in the set $X(A)$ for each A object of **C**. This selection must be consistent with the action of morphisms $f(A \to B)$ in the sense that the matching condition $X(f)(\gamma_A) = \gamma_B$ is satisfied. It can happen that category of presheaves has no points at all since the matching condition need not be satisfied globally.

It turns out that TGD based notion of subsystem leads naturally to what might be called quantal versions of topos, presheaves, sieves and logic.

7.3 More Precise Characterization Of The Basic Categories And Possible Applications

In the following the categories associated with self and quantum jump are discussed in more precise manner and applications to communications and cognition are considered.

7.3.1 Intuitive Picture About The Category Formed By The Geometric Correlates Of Selves

Space-time surface $X^4(X^3)$ decomposes into regions obeying either real or p-adic topology and each region of this kind corresponds to an unentangled subsystem or self lasting at least one quantum jump. By the localization in the zero modes these decompositions are equivalent for all 3-surfaces X^3 in the quantum superposition defined by the prepared WCW spinor fields resulting in quantum jumps. There is a hierarchy of selves since selves can contain sub-selves. The entire space-time surface $X^4(X^3)$ represents the highest level of the self hierarchy.

This structure defines in a natural manner a category. Objects are all possible sub-selves contained in the self hierarchy: sub-self is set consisting of lower level sub-selves, which in turn have a further decomposition to sub-selves, etc... The naïve expectation is that geometrically sub-self belongs to a self as a subset and this defines an inclusion map acting as a natural morphism in this category. This expectation is not quite correct. More natural morphisms are the arrows telling that self as a set of sub-selves contains sub-self as an element. These arrows define a structure analogous to a composite of hierarchy trees.

To be more precise, for a single space-time surface $X^4(X^3)$ this hierarchy corresponds to a subjective time slice of the self hierarchy defined by a single quantum jump. The sequence of hierarchies associated with a sequence of quantum jumps is a natural geometric correlate for the self hierarchy. This means that the objects are now sequences of submoments of consciousness. Sequences are not arbitrary. Self must survive its lifetime although sub-selves at various levels can disappear and reappear (generation and disappearance of mental images). Geometrically this means typically a phase transition transforming real or p_1-adic to p_2-adic space-time region with same topology as the environment. Also sub-selves can fuse to single sub-self. The constraints on self sequences must be such that it takes these processes into account. Note that these constraints emerge naturally from the fact that quantum jumps sequences define the sequences of surfaces $X^4(X^3)$.

By the rich anatomy of the quantum jump there is large number of quantum jumps leading from a given initial quantum history to a given final quantum history. One could envisage quantum jump also as a discrete path in the space of WCW spinor fields leading from the initial state to the final state. In particular, for given self there is an infinite number of closed elementary paths leading from the initial quantum history back to the initial quantum history and these paths in principle give all possible conscious information about a given quantum history/idea: kind of self morphisms are in question (analogous to, say, group automorhisms). Information about point of space is obtained only by moving around and coming back to the point, that is by studying the surroundings of the point. Self in turn can be seen as a composite of elementary paths defined by the quantum jumps. Selves can define arbitrarily complex composite closed paths giving information about a given quantum history.

7.3.2 Categories Related To Self And Quantum Jump

The categories defined by moments of consciousness and the notion of self

Since quantum jump involves state reduction and the sequence of self measurement reducing all entanglement except bound state entanglement, it defines a hierarchy of unentangled subsystems allowing interpretation as objects of a category. Arrows correspond to subsystem-system relationship and the two subsystems resulting in self measurement to the system. What subsystem corresponds mathematically is however not at all trivial and the naïve description as a tensor factor does not work. Rather, a definition relying on the notion of p-adic length scale cutoff identified as a fundamental aspect of nature and consciousness is needed.

It is not clear what the statement that self corresponds to a subsystem which remains unentangled in subsequent quantum jump means concretely since subsystem can certainly change in some limits. What is clear that bound state entanglement between selves means a loss of consciousness. Category theory suggests that there should exist a functor between categories defined by two subsequent moments of consciousness. This functor maps submoments of consciousness to submoments of consciousness and arrows to arrows. Two subsequent submoments of consciousness belong to same sub-self is the functor maps the first one to the latter one. Thus category theory would play essential role in the precise definition of the notion of self.

The sequences of moments of consciousness form a larger category containing sub-selves as sequences of unentangled subsystems mapped to each other by functor arrows functoring subsequent quantum jumps to each other.

What might then be the ultimate characterizer of the self-identity? The theory of infinite primes suggests that space-time surface decomposes into regions labelled by finite p-adic primes. These primes must label also real regions rather than only p-adic ones. A p-adic space-time region characterized by prime p can transform to a real one or vice versa in quantum jump if the sizes of real and p-adic regions are characterized by the p-adic length scale L_p (or n-ary p-adic length scale $L_p(n)$. One can also consider the possibility that real region is accompanied by a p-adic region characterized by a definite prime p and providing a cognitive self-representation of the real region.

If this view is correct, the p-adic prime characterizing a given real or p-adic space-time sheet could be one characterizer of the self-identity. Self identity is lost in bound state entanglement with another space-time sheet (at least when a space-time sheet with smaller value of the p-adic prime joins by flux tube to a one with a higher value of the p-adic prime). Self identity is also lost if a space-time sheet characterized by a given p-adic prime disappears in quantum jump.

The category associated with quantum jump sequences

There are several similarities between the ontologies and epistemologies of TGD and of category theory. Conscious experience is always determined by the discrete paths in the space of configuration space spinor fields defined by a quantum jump connecting two quantum histories (states) and is never determined by single quantum history as such (quantum states are unconscious). Also category theory is about relations between objects, not about objects directly: self-morphisms give information about the object of category (in case of group composite paths would correspond to products of group automorphisms). Analogously closed paths determined by quantum jump sequences give information about single quantum history. The point is however that it is impossible to have direct knowledge about the quantum histories: they are not conscious.

One can indeed define a natural category, call it **QSelf**, applying to this situation. The objects of the category **QSelf** are initial quantum histories of quantum jumps and correspond to prepared quantum states. The discrete path defining quantum jump can be regarded as an elementary morphism. Selves are composites of elementary morphisms of the initial quantum history defined by quantum jumps: one can characterize the morphisms by the number of the elementary morphisms in the product. Trivial self contains no quantum jumps and corresponds to the identity morphism, null path. Thus the collection of all possible sequences of quantum jumps, that is collections of selves allows a description in terms of category theory although the category in question is not a subcategory of the category **Set**.

Category **QSelf** does not possess terminal and initial elements (for terminal (initial) element T there is exactly one arrow $A \rightarrow T$ ($T \rightarrow A$) for every A: now there are always many paths between quantum histories involved).

7.3.3 Communications In TGD Framework

Goro Kato identifies communications between conscious entities as natural maps between them whereas in TGD natural maps bind submoments of consciousness to selves. In TGD framework quantum measurement and the sharing of mental images are the basic candidates for communications. The problem is that the identification of communications as sharing of mental images is not consistent with the naïve view about subsystem as a tensor factor. Many-sheeted space-time however forces length scale dependent notion of subsystem at space-time level and this saves the situation.

What communications are?

Communication is essentially generation of desired mental images/sub-selves in receiver. Communication between selves need not be directly conscious: in this case communication would generate mental images at some lower level of self hierarchy of receiver: for instance generate large number of sub-sub-selves of similar type. This is like communications between organizations. Communication can be also vertical: self can generate somehow sub-self in some sub-sub....sub-self or sub-sub...sub-self can generate sub-self of self somehow. This is communication from boss to the lower levels organization or vice versa.

These communications should have direct topological counterparts. For instance, the communication between selves could correspond to an exchange of mental image represented as a space-time region of different topology inside sender self space-time sheet. The sender self would simply throw this space-time region to a receiver self like a ball. This mechanism applies also to vertical communications since the ball could be also thrown from a boss to sub...sub-self at some lower level of hierarchy and vice versa.

The sequence of space-time surfaces provides a direct topological counterpart for communication as throwing balls representing sub-selves. Quantum jump sequence contains space-time surfaces in which the regions corresponding to receiver and sender selves are connected by a flux tube (perhaps massless extremal) representing classically the communication: during the communication the receiver and sender would form single self. The cartoon vision about rays connecting the eyes of communication persons would make sense quite concretely.

More refined means of communication would generate sub-selves of desired type directly at the end of receiver. In this case it is not so obvious how the sequence $X^(X^3)$ of space-time surfaces could represent communication. Of course, one can question whether communication is really what happens in this kind of situation. For instance, sender can affect the environment of receiver to be such that receiver gets irritated (computer virus is good manner to achieve this!) but one can wonder whether this is real communication.

Communication as quantum measurement?

Quantum measurement generates one-one map between the states of the entangled systems resulting in quantum measurement. Both state function reduction and self measurement give rise to this kind of map. This map could perhaps be interpreted as quantum communication between unentangled subsystems resulting in quantum measurement. For the state reduction process the space-time correlates are the values of zero modes. For state preparation the space-time correlates should correspond to classical spinor field modes correlating for the two subsystems generated in self measurement.

Communication as sharing of mental images

It has become clear that the sharing of mental images induced by quantum entanglement of sub-selves of two separate selves represents genuine conscious communication which is analogous telepathy and provides general mechanism of remote mental interactions making possible even molecular recognition mechanisms.

1. The sharing of mental images is not possible unless one assumes that self hierarchy is defined by using the notion of length scale resolution defined by p-adic length scale. The notion of scale of resolution is indeed fundamental for all quantum field theories (renormalization group invariance) for all quantum field theories and without it the practical modelling of physics would not be possible. The notion reflects directly the length scale resolution of conscious experience. For a given sub-self of self the resolution is given by the p-adic length scale associated with the sub-self space-time sheet.

2. Length scale resolution emerges naturally from the fact that sub-self space-time sheets having Minkowskian signature of metric are separated from the one representing self by wormhole contacts with Euclidian signature of metric. The signature of the induced metric changes from Minkowskian signature to Euclidian signature at "elementary particle horizons" surrounding the throats of the wormhole contacts and having degenerate induced metric. Elementary

particle horizons are thus metrically two-dimensional light like surfaces analogous to the boundary of the light cone and allow conformal invariance. Elementary particle horizons act as causal horizons. Topologically condensed space-time sheets are analogous to black hole interiors and due to the lack of the causal connectedness the standard description of sub-selves as tensor factors of the state space corresponding to self is not appropriate.

Hence systems correspond, not to the space-time sheets plus entire hierarchy of space-time sheets condensed to it, but rather, to space-time sheets with holes resulting when the space-time sheets representing subsystems are spliced off along the elementary particle horizons around wormhole contacts. This does not mean that all information about subsystem is lost: subsystem space-time sheet is only replaced by the elementary particle horizon. In analogy with the description of the black hole, some parameters (mass, charges, ...) characterizing the classical fields created by the sub-self space-time sheet characterize sub-self.

One can say that the state space of the system contains "holes". There is a hierarchy of state spaces labelled by p-adic primes defining length scale resolutions. This picture resolves a longstanding puzzle relating to the interpretation of the fact that particle is characterize by both classical and quantum charges. Particle cannot couple simultaneously to both and this is achieved if quantum charge is associated with the lowest level description of the particle as CP_2 extremal and classical charges to its description at higher levels of hierarchy.

3. The immediate implication indeed is that it is possible to have a situation in which two selves are unentangled although their sub-selves (mental images) are entangled. This corresponds to the fusion and sharing of mental images. The sharing of the mental images means that union of disjoint hierarchy trees with levels labelled by p-adic primes p is replaced by a union of hierarchy threes with horizontal lines connecting subsystems at the same level of hierarchy. Thus the classical correspondence defines a category of presheaves with both vertical arrows replaced by sub-self-self relationship, horizontal arrows representing sharing of mental images, and natural maps representing binding of submoments of consciousness to selves.

Comparison with Goro Kato's approach

It is of interest to compare Goro Kato's approach with TGD approach. The following correspondence suggests itself.

1. In TGD each quantum jumps defines a category analogous to the Goro Kato's category of open sets of some topological space but set theoretic inclusion replaced by topological condensation. The category defined by a moment of consciousness is dynamical whereas the category of open sets is non-dynamical.

2. The assignment of a 3-surface acting as a causal determinant to each unentangled subsystem defined by a moment of consciousness defines a unique "quantum presheaf" which is the counterpart of the presheaf in Goro Kato's theory. The conscious entity of Kato's theory corresponds to the classical correlate for a moment of consciousness.

3. Natural maps between the causal determinants correspond to the space-time correlates for the functor arrows defining the threads connecting submoments of consciousness to selves. In Goro Kato's theory natural maps are interpreted as communications between conscious entities. The sharing of mental images by quantum entanglement between subsystems of unentangled systems defines horizontal bi-directional arrows between subsystems associated with same moment of consciousness and is counterpart of communication in TGD framework. It replaces the union of disjoint hierarchy trees associated with various unentangled subsystems with hierarchy trees having horizontal connections defining the bi-directional arrows. The sharing of mental images is not possible if subsystem is identified as a tensor factor and thus without taking into account length scale resolution.

7.3.4 Cognizing About Cognition

There are close connections with basic facts about cognition.

1. Categorization means classification and abstraction of common features in the class formed by the objects of a category. Already quantum jump defines category with hierarchical structure and can be regarded as consciously experienced analysis in which totally entangled entire universe $U\Psi_i$ decomposes to a product of maximally unentangled subsystems. The sub-selves of self are like elements of set and are experienced as separate objects whereas sub-sub-selves of sub-self self experiences as an average: they belong to a class or category formed by the sub-self. This kind of averaging occurs also for the contributions of quantum jumps to conscious experience of self.

2. The notions of category theory might be useful in an attempt to construct a theory of cognitive structures since cognition is indeed to high degree classification and abstraction process. The sub-selves of a real self indeed have p-adic space-time sheets as geometric correlates and thus correspond to cognitive sub-selves, thoughts. A meditative experience of empty mind means in case of real self the total absence of thoughts.

3. Predicate logic provides a formalization of the natural language and relies heavily on the notion of n-ary relation. Binary relations $R(a,b)$ corresponds formally to the subset of the product set $A \times B$. For instance, statements like "A does something to B" can be expressed as a binary relation, particular kind of arrow and morphism ($A \leq B$ is a standard example). For sub-selves this relation would correspond to a dynamical evolution at space-time level modelling the interaction between A and B. The dynamical path defined by a sequence of quantum jumps is able to describe this kind of relationships too at level of conscious experience. For instance, "A touches B" would involve the temporary fusion of sub-selves A and B to sub-self C.

7.4 Logic And Category Theory

Category theory allows naturally more general than Boolean logics inherent to the notion of topos associated with any category. Basic question is whether the ordinary notion of topos algebra based on set theoretic inclusion or the notion of quantum topos based on topological condensation is physically appropriate. Starting from the quasi-Boolean algebra of open sets one ends up to the conclusion that quantum logic is more natural. Also WCW spinor fields lead naturally to the notion of quantum logic.

7.4.1 Is The Logic Of Conscious Experience Based On Set Theoretic Inclusion Or Topological Condensation?

The algebra of open sets with intersections and unions and complement defined as the interior of the complement defines a modification of Boolean algebra having the peculiar feature that the points at the boundary of the closure of open set cannot be said to belong to neither interior of open set or of its complement. There are two options concerning the interpretation.

1. 3-valued logic could be in question. It is however not possible to understand this three-valuedness if one defines the quasi-Boolean algebra of open sets as Heyting algebra. The resulting logic is two-valued and the points at boundaries of the closure do not correspond neither to the statement or its negation. In p-adic context the situation changes since p-adic open sets are also closed so that the logic is strictly Boolean. That our ordinary cognitive mind is Boolean provides a further good reason for why cognition is p-adic.

2. These points at the boundary of the closure belong to both interior and exterior in which case a two-valued "quantum logic" allowing superposition of opposite truth values is in question. The situation is indeed exactly the same as in the case of space-time sheet having wormhole contacts to several space-time sheets.

The quantum logic brings in mind Zen consciousness [J6] (which I became fascinated of while reading Hofstadter's book "Gödel, Escher, Bach" [A53]) and one can wonder whether selves having real space-time sheets as geometric correlates and able to live simultaneously in many parallel worlds correspond to Zen consciousness and Zen logic. Zen logic would be also logic of sensory experience whereas cognition would obey strictly Boolean logic.

The causal determinants associated with space-time sheets correspond to light like 3-surfaces which could elementary particle horizons or space-time boundaries and possibly also 3-surfaces separating two maximal deterministic regions of a space-time sheet. These surfaces act as 3-dimensional quantum holograms and have the strange Zen property that they are neither space-like nor time-like so that they represent both the state and the process. In the TGD based model for topological quantum computation (TQC) light-like boundaries code for the computation so that TQC program code would be equivalent with the running program [K4].

7.4.2 Do WCW Spinor Fields Define Quantum Logic And Quantum Topos

I have proposed already earlier that WCW spinor fields define what might be called quantum logic. One can wonder whether WCW spinor s could also naturally define what might be called quantum topos since the category underlying topos defines the logic appropriate to the topos. This question remains unanswered in the following: I just describe the line of though generalizing ordinary Boolean logic.

Finite-dimensional spinors define quantum logic

Spinors at a point of an $2N$-dimensional space span 2^N-dimensional space and spinor basis is in one-one correspondence with Boolean algebra with N different truth values (N bits). 2N=2-dimensional case is simple: Spin up spinor= true and spin-dow spinor=false. The spinors for $2N$-dimensional space are obtained as an N-fold tensor product of 2-dimensional spinors (spin up, spin down): just like in the case of Cartesian power of Ω.

Boolean spinors in a given basis are eigen states for a set N mutually commuting sigma matrices providing a representation for the tangent space group acting as rotations. Boolean spinors define N Boolean statements in the set Ω^N so that one can in a natural manner assign a set with a Boolean spinor. In the real case this group is $SO(2N)$ and reduces to $SU(N)$ for Kähler manifolds. For pseudo-euclidian metric some non-compact variant of the tangent space group is involved. The selections of N mutually commuting generators are labelled by the flag-manifold $SO(2N)/SO(2)^N$ in real context and by the flag-manifold $U(N)/U(1)^N$ in the complex case. The selection of these generators defines a collection of N 2-dimensional linear subspaces of the tangent space.

Spinors are in general complex superpositions of spinor basis which can be taken as the product spinors. The quantum measurement of N spins representing the Cartan algebra of $SO(2N)$ ($SU(N)$) leads to a state representing a definite Boolean statement. This suggests that quantum jumps as moments of consciousness quite generally make universe classical, not only in geometric but also in logical sense. This is indeed what the state preparation process for WCW spinor field seems to do.

Quantum logic for finite-dimensional spinor fields

One can generalize the idea of the spinor logic also to the case of spinor fields. For a given choice of the local spinor basis (which is unique only modular local gauge rotation) spinor field assigns to each point of finite-dimensional space a quantum superposition of Boolean statements decomposing into product of N statements.

Also now one can ask whether it is possible to find a gauge in which each point corresponds to definite Boolean statement and is thus an eigen state of a maximal number of mutually commuting rotation generators Σ_{ij}. This is not trivial if one requires that Dirac equation is satisfied. In the case of flat space this is certainly true and constant spinors multiplied by functions which solve d'Alembert equation provide a global basis.

The solutions of Dirac equation in a curved finite-dimensional space do not usually possess a definite spin direction globally since spinor curvature means the presence of magnetic spin-flipping interaction and since there need not exist a global gauge transformation leading to an eigen state of the local Cartan algebra everywhere. What might happen is that the local gauge transformation becomes singular at some point: for instance, the direction of spin would be radial around given point and become ill defined at the point. This is much like the singularities for vector fields on sphere. The spinor field having this kind of singularity should vanish at singularity but the local gauge rotation rotating spin in same direction everywhere is necessarily ill-defined at the singularity.

In fact, this can be expressed using the language of category theory. The category in question corresponds to a presheaf which assigns to the points of the base space the fiber space of the spinor bundle. The presence of singularity means that there are no global section for this presheaf, that is a continuous choice of a non-vanishing spinor at each point of the base space. The so called Kochen-Specker theorem discussed in [A83] is closely related to a completely analogous phenomenon involving non-existence of global sections and thus non-existence of a global truth value.

Thus in case of curved spaces is not necessarily possible to have spinor field basis representing globally Boolean statements and only the notion of locally Boolean logic makes sense. Indeed, one can select the basis to be eigen state of maximal set of mutually commuting rotation generators in single point of the compact space. Any such choice does.

Quantum logic and quantum topos defined by the prepared WCW spinor fields

The prepared WCW spinor fields occurring as initial and final states of quantum jumps are the natural candidates for defining quantum logic. The outcomes of the quantum jumps resulting in the state preparation process are maximally unentangled states and are as close to Boolean states as possible.

WCW spinors correspond to fermionic Fock states created by infinite number of fermionic (leptonic and quarklike) creation and annihilation operators. The spin degeneracy is replaced by the double-fold degeneracy associated with a given fermion mode: given state either contains fermion or not and these two states represent true and false now. If WCW were flat, the Fock state basis with definite fermion and anti-fermion numbers in each mode would be in one-one correspondence with Boolean algebra.

Situation is however not so simple. Finite-dimensional curved space is replaced with the fiber degrees of freedom of WCW in which the metric is non-vanishing. The precise analogy with the finite-dimensional case suggests that if the curvature form of the WCW spinor connection is nontrivial, it is impossible to diagonalize even the prepared maximally unentangled WCW spinor fields Ψ_i in the entire fiber of WCW (quantum fluctuating degrees of freedom) for given values of the zero modes. Local singularities at which the spin quantum numbers of the diagonalized but vanishing WCW spinor field become ill-defined are possible also now.

In the infinite-dimensional context the presence of the fermion-anti-fermion pairs in the state means that it does not represent a definite Boolean statement unless one defines a more general basis of WCW spinor s for which pairs are present in the states of the state basis: this generalization is indeed possible. The sigma matrices of the WCW appearing in the spinor connection term of the Dirac operator of WCW indeed create fermion-fermion pairs. What is decisive, is not the absence of fermion-anti-fermion pairs, but the possibility that the spinor field basis cannot be reduced to eigen states of the local Cartan algebra in fiber degrees of freedom globally.

Also for bound states of fermions (say leptons and quarks) it is impossible to reduce the state to a definite Boolean statement even locally. This would suggest that fermionic logic does not reduce to a completely Boolean logic even in the case of the prepared states.

Thus WCW spinor fields could have interpretation in terms of non-Boolean quantum logic possessing Boolean logics only as sub-logics and define what might be called quantum topos. Instead of Ω^N-valued maps the values for the maps are complex valued quantum superpositions of truth values in Ω^N.

An objection against the notion of quantum logic is that Boolean algebra operationsandOR do not preserve fermion number so that quantum jump sequences leading from the product state defined by operands to the state representing the result of operation are therefore not possible. One manner to circumvent the objection is to consider the sub-algebra spanned by fermion and anti-fermion pairs for given mode so that fermion number conservation is not a problem. The objection can be also circumvented for pairs of space-time sheets with opposite time orientations and thus opposite signs of energies for particles. One can construct the algebra in question as pairs of many fermion states consisting of positive energy fermion and negative energy anti-fermion so that all states have vanishing fermion number and logical operations become possible. Pairs of MEs with opposite time orientations are excellent candidates for carries of these fermion-anti-fermion pairs.

Quantum classical correspondence and quantum logic

The intuitive idea is that the global Boolean statements correspond to sections of Z^2 bundle. Möbius band is a prototype example here. The failure of a global statement would reduce to the non-existence of global section so that true would transforms to false as one goes around full 2π rotation.

One can ask whether fermionic quantum realization of Boolean logic could have space-time counterpart in terms of Z_2 fiber bundle structure. This would give some hopes of having some connection between category theoretical and fermionic realizations of logic. The following argument stimulated by email discussion with Diego Lucio Rapoport suggests that this might be the case.

1. The hierarchy of Planck constants realized using the notion of generalized embedding space involves only groups $Z_{n_a} \times Z_{n_b}$, $n_a, n_b \neq 2$ if one takes Jones inclusions as starting point. There is however no obvious reason for excluding the values $n_a = 2$ and $n_b = 2$ and the question concerns physical interpretation. Even if one allows only $n_i \geq 3$ one can ask for the physical interpretation for the factorization $Z_{2n} = Z_2 \times Z_n$. Could it perhaps relate to a space-time correlates for Boolean two-valuedness?

2. An important implication of fiber bundle structure is that the partonic 2-surfaces have $Z_{n_a} \times Z_{n_b} = Z_{n_a n_b}$ as a group of conformal symmetries. I have proposed that n_a or n_b is even for fermions so that Z_2 acts as a conformal symmetry of the partonic 2-surface. Both n_a and n_b would be odd for truly elementary bosons. Note that this hypothesis makes sense also for $n_i \geq 3$.

3. Z_2 conformal symmetry for fermions would imply that all partonic 2-surfaces associated with fermions are hyper-elliptic. As a consequence elementary particle vacuum functionals defined in modular degrees of freedom would vanish for fermions for genus $g > 2$ so that only three fermion families would be possible in accordance with experimental facts. Since gauge bosons and Higgs correspond to pairs of partonic 2-surfaces (the throats of the wormhole contact) one has 9 gauge boson states labelled by the pairs (g_1, g_2) which can be grouped to SU(3) singlet and octet. Singlet corresponds to ordinary gauge bosons.

   super-symplectic bosons are truly elementary bosons in the sense that they do not consist of fermion-anti-fermion pairs. For them both n_a and n_b should be odd if the correspondence is taken seriously and all genera would be possible. The super-conformal partners of these bosons have the quantum numbers of right handed neutrino. Since both spin directions are possible, one can ask whether Boolean Z_2 must be present also now. This need not be the case, ν_R generates only super-symmetries and does not define a family of fermionic oscillator operators. The electro-weak spin of ν_R is frozen and it does not couple at all to electro-weak intersections. Perhaps (only) odd values of n_i are possible in this case.

4. If fermionic Boolean logic has a space-time correlate, one can wonder whether the fermionic Z_2 conformal symmetry might correspond to a space-time correlate for the Boolean true-false dichotomy. If the partonic 2-surface contains points which are fixed points of Z_2 symmetry, there exists no everywhere non-vanishing sections. Furthermore, induced spinor fields should vanish at the fixed points of Z_2 symmetry since they correspond to singular orbifold points so that one could not actually have a situation in which true and false are true simultaneously. Global sections could however fail to exist since CP_2 spinor bundle is non-trivial.

7.4.3 Category Theory And The Modelling Of Aesthetic And Ethical Judgements

Consciousness theory should allow to model model the logics of ethics and aesthetics. Evolution (representable as p-adic evolution in TGD framework) is regarded as something positive and is a good candidate for defining universal ethics in TGD framework. Good deeds are such that they support this evolution occurring in statistical sense in any case. Moral provides a practical model for what good deeds are and moral right-wrong statements are analogous to logical statements. Often however the two-valued right-wrong logic seems to be too simplistic in case of moral statements. Same applies to aesthetic judgements. A possible application of the generalized logics defined by the inherent structure of categories relates to the understanding of the dilemmas associated with the moral and aesthetic rules.

As already found, quantum versions of sieves provide a formal generalization of Boolean truth values as a characteristic of a given category. Generalized moral rules could perhaps be seen as sieve valued statements about deeds. Deeds are either right or wrong in what might be called Boolean moral code. One can also consider Zen moral in which some deeds can be said to be right and wrong simultaneously. Some deeds could also be such that there simply exists no globally consistent moral rule: this would correspond to the non-existence of what is called global section assigning to each object of the category consisting of the pairs formed by a moral agents and given deed) a sieve simultaneously.

7.5 Platonism, Constructivism, And Quantum Platonism

During years I have been trying to understand how Category Theory and Set Theory relate to quantum TGD inspired view about fundamentals of mathematics and the outcome section is added to this chapter several years after its first writing. I hope that reader does not experience too unpleasant discontinuity. I managed to clarify my thoughts about what these theories are by reading the article Structuralism, Category Theory and Philosophy of Mathematics by Richard Stefanik [A119]. Blog discussions and email correspondence with Sampo Vesterinen have been very stimulating and inspired the attempt to represent TGD based vision about the unification of mathematics, physics, and consciousness theory in a more systematic manner.

Before continuing I want to summarize the basic ideas behind TGD vision. One cannot understand mathematics without understanding mathematical consciousness. Mathematical consciousness and its evolution must have direct quantum physical correlates and by quantum classical correspondence these correlates must appear also at space-time level. Quantum physics must allow to realize number as a conscious experience analogous to a sensory quale. In TGD based ontology there is no need to postulate physical world behind the quantum states as mathematical entities (theory is the reality). Hence number cannot be any physical object, but can be identified as a quantum state or its label and its number theoretical anatomy is revealed by the conscious experiences induced by the number theoretic variants of particle reactions. Mathematical systems and their axiomatics are dynamical evolving systems and physics is number theoretically universal selecting rationals and their extensions in a special role as numbers, which can can be regarded elements of several number fields simultaneously.

7.5.1 Platonism And Structuralism

There are basically two philosophies of mathematics.

1. Platonism assumes that mathematical objects and structures have independent existence. Natural numbers would be the most fundamental objects of this kind. For instance, each natural number has its own number-theoretical anatomy decomposing into a product of prime numbers defining the elementary particles of Platonia. For quantum physicist this vision is attractive, and even more so if one accepts that elementary particles are labelled by primes (as I do)! The problematic aspects of this vision relate to the physical realization of the Platonia. Neither Minkowski space-time nor its curved variants understood in the sense of set theory have no room for Platonia and physical laws (as we know them) do not seem to allow the realization of all imaginable internally consistent mathematical structures.

2. Structuralist believes that the properties of natural numbers result from their relations to other natural numbers so that it is not possible to speak about number theoretical anatomy in the Platonic sense. Numbers as such are structureless and their relationships to other numbers provide them with their apparent structure. According to [A119] structuralism is

however not enough for the purposes of number theory: in combinatorics it is much more natural to use intensional definition for integers by providing them with inherent properties such as decomposition into primes. I am not competent to take any strong attitudes on this statement but my physicist's intuition tells that numbers have number theoretic anatomy and that this anatomy can be only revealed by the morphisms or something more general which must have physical counterparts. I would like to regard numbers are analogous to bound states of elementary particles. Just as the decays of bound states reveal their inner structure, the generalizations of morphisms would reveal to the mathematician the inherent number theoretic anatomy of integers.

7.5.2 Structuralism

Set theory and category theory represent two basic variants of structuralism and before continuing I want to clarify to myself the basic ideas of structuralism: the reader can skip this section if it looks too boring.

Set theory

Structuralism has many variants. In set theory [A20] the elements of set are treated as structureless points and sets with the same cardinality are equivalent. In number theory additional structure must be introduced. In the case of natural numbers one introduces the notion of successor and induction axiom and defines the basic arithmetic operations using these. Set theoretic realization is not unique. For instance, one can start from empty set Φ identified as 0, identify 1 as $\{\Phi\}$, 2 as $\{0,1\}$ and so on. One can also identify 0 as Φ, 1 as $\{0\}$, 2 as $\{\{0\}\}$, ... For both physicist and consciousness theorist these formal definitions look rather weird.

The non-uniqueness of the identification of natural numbers as a set could be seen as a problem. The structuralist's approach is based on an extensional definition meaning that two objects are regarded as identical if one cannot find any property distinguishing them: object is a representative for the equivalence class of similar objects. This brings in mind gauge fixing to the mind of physicists.

Category theory

Category theory [A3] represents a second form of structuralism. Category theorist does not worry about the ontological problems and dreams that all properties of objects could be reduced to the arrows and formally one could identify even objects as identity morphisms (looks like a trick to me). The great idea is that functors between categories respecting the structure defined by morphisms provide information about categories. Second basic concept is natural transformation which maps functors to functors in a structure preserving manner. Also functors define a category so that one can construct endless hierarchy of categories. This approach has enormous unifying power since functors and natural maps systemize the process of generalization. There is no doubt that category theory forms a huge piece of mathematics but I find difficult to believe that arrows can catch all of it.

The notion of category can be extended to that of n-category. In the blog post "First edge of the cube" (see `http://tinyurl.com/yydjavv8`) I have proposed a geometric realization of this hierarchy in which one defines 1-morphisms by parallel translations, 2-morphisms by parallel translations of parallel translations, and so on. In infinite-dimensional space this hierarchy would be infinite. Abstractions about abstractions about.., thoughts about thoughts about, statements about statements about..., is the basic idea behind this interpretation. Also the hierarchy of logics of various orders corresponds to this hierarchy. This encourages to see category theoretic thinking as being analogous to higher level self reflection which must be distinguished from the direct sensory experience.

In the case of natural numbers category theoretician would identify successor function as the arrow binding natural numbers to an infinitely long string with 0 as its end. If this approach would work, the properties of numbers would reflect the properties of the successor function.

7.5.3 The View About Mathematics Inspired By TGD And TGD Inspired Theory Of Consciousness

TGD based view might be called quantum Platonism. It is inspired by the requirement that both quantum states and quantum jumps between them are able to represent number theory and that all quantum notions have also space-time correlates so that Platonia should in some sense exist also at the level of space-time. Here I provide a brief summary of this view as it is now.

Physics is fixed from the uniqueness of infinite-D existence and number theoretic universality

1. The basic philosophy of quantum TGD relies on the geometrization of physics in terms of infinite-dimensional Kähler geometry of WCW , whose uniqueness is forced by the mere mathematical existence. Space-time dimension and embedding space $H = M^4 \times CP_2$ are fixed among other things by this condition and allow interpretation in terms of classical number fields. Physical states correspond to WCW spinor fields with WCW spinor s having interpretation as Fock states. Rather remarkably, WCW Clifford algebra defines standard representation of so called hyper finite factor of II_1, perhaps the most fascinating von Neumann algebra.

2. Number theoretic universality states that all number fields are in a democratic position. This vision can be realized by requiring generalization of notions of embedding space by gluing together real and p-adic variants of embedding space along common algebraic numbers. All algebraic extensions of p-adic numbers are allowed. Real and p-adic space-time sheets intersect along common algebraics. The identification of the p-adic space-time sheets as correlates of cognition and intentionality explains why cognitive representations at space-time level are always discrete. Only space-time points belonging to an algebraic extension of rationals associated contribute to the data defining S-matrix. These points define what I call number theoretic braids. The interpretation in of algebraic discreteness terms of a physical realization of axiom of choice is highly suggestive. The axiom of choice would be dynamical and evolving quantum jump by quantum jump as the algebraic complexity of quantum states increases.

Holy trinity of existence

In TGD framework one would have 3-levelled ontology numbers should have representations at all these levels [L1].

1. Subjective existence as a sequence of quantum jumps giving conscious sensory representations for numbers and various geometric structures would be the first level.

2. Quantum states would correspond to Platonia of mathematical ideas and mathematician- or if one is unwilling to use this practical illusion- conscious experiences about mathematic ideas, would be in quantum jumps. The quantum jumps between quantum states respecting the symmetries characterizing the mathematical structure would provide conscious information about the mathematical ideas not directly accessible to conscious experience. Mathematician would live in Plato's cave. There is no need to assume any independent physical reality behind quantum states as mathematical entities since quantum jumps between these states give rise to conscious experience. Theory-reality dualism disappears since the theory is reality or more poetically: painting is the landscape.

3. The third level of ontology would be represented by classical physics at the space-time level essential for quantum measurement theory. By quantum classical correspondence space-time physics would be like a written language providing symbolic representations for both quantum states and changes of them (by the failure of complete classical determinism of the fundamental variational principle). This would involve both real and p-adic space-time sheets corresponding to sensory and cognitive representations of mathematical concepts. This representation makes possible the feedback analogous to formulas written by mathematician

crucial for the ability of becoming conscious about what one was conscious of and the dynamical character of this process allows to explain the self-referentiality of consciousness without paradox.

This ontology releases a deep Platonistic sigh of relief. Since there are no physical objects, there is no need to reduce mathematical notions to objects of the physical world. There are only quantum states identified as mathematical entities labelled naturally by integer valued quantum numbers; conscious experiences, which must represent sensations giving information about the number theoretical anatomy of a given quantum number; and space-time surfaces providing space-time correlates for quantum physics and therefore also for number theory and mathematical structures in general.

Factorization of integers as a direct sensory perception?

Both physicist and consciousness theorist would argue that the set theoretic construction of natural numbers could not be farther away from how we experience integers. Personally I feel that neither structuralist's approach nor Platonism as it is understood usually are enough. Mathematics is a conscious activity and this suggests that quantum theory of consciousness must be included if one wants to build more satisfactory view about fundamentals of mathematics.

Oliver Sack's book *The man who mistook his wife for a hat* [J5] (see also [K65]) contains fascinating stories about those aspects of brain and consciousness which are more or less mysterious from the view point of neuroscience. Sacks tells in his book also a story about twins who were classified as idiots but had amazing number theoretical abilities. I feel that this story reveals something very important about the real character of mathematical consciousness.

The twins had absolutely no idea about mathematical concepts such as the notion of primeness but they could factorize huge numbers and tell whether they are primes. Their eyes rolled wildly during the process and suddenly their face started to glow of happiness and they reported a discovery of a factor. One could not avoid the feeling that they quite concretely saw the factorization process. The failure to detect the factorization served for them as the definition of primeness. For them the factorization was not a process based on some rules but a direct sensory perception.

The simplest explanation for the abilities of twins would in terms of a model of integers represented as string like structures consisting of identical basic units. This string can decay to strings. If string containing n units decaying into $m > 1$ identical pieces is not perceived, the conclusion is that a prime is in question. It could also be that decay to units smaller than 2 was forbidden in this dynamics. The necessary connection between written representations of numbers and representative strings is easy to build as associations.

This kind theory might help to understand marvellous feats of mathematicians like Ramanujan who represents a diametrical opposite of Groethendienck as a mathematician (when Groethendienck was asked to give an example about prime, he mentioned 57 which became known as Groethendienck prime!).

The lesson would be that one very fundamental representation of integers would be, not as objects, but conscious experiences. Primeness would be like the quale of redness. This of course does not exclude also other representations.

Experience of integers in TGD inspired quantum theory of consciousness

In quantum physics integers appear very naturally as quantum numbers. In quantal axiomatization or interpretation of mathematics same should hold true.

1. In TGD inspired theory of consciousness [L1] quantum jump is identified as a moment of consciousness. There is actually an entire fractal hierarchy of quantum jumps consisting of quantum jumps and this correlates directly with the corresponding hierarchy of physical states and dark matter hierarchy. This means that the experience of integer should be reducible to a certain kind of quantum jump. The possible changes of state in the quantum jump would characterize the sensory representation of integer.

2. The quantum state as such does not give conscious information about the number theoretic anatomy of the integer labelling it: the change of the quantum state is required. The above

geometric model translated to quantum case would suggest that integer represents a multiplicatively conserved quantum number. Decays of this this state into states labelled by integers n_i such that one has $n = \prod_i n_i$ would provide the fundamental conscious representation for the number theoretic anatomy of the integer. At the level of sensory perception based the space-time correlates a string-like bound state of basic particles representing n=1.

3. This picture is consistent with the Platonist view about integers represented as structured objects, now labels of quantum states. It would also conform with the view of category theorist in the sense that the arrows of category theorist replaced with quantum jumps are necessary to gain conscious information about the structure of the integer.

Infinite primes and arithmetic consciousness

Infinite primes [K72] were the first mathematical fruit of TGD inspired theory of consciousness and the inspiration for writing this posting came from the observation that the infinite primes at the lowest level of hierarchy provide a representation of algebraic numbers as Fock states of a super-symmetric arithmetic QFT so that it becomes possible to realize quantum jumps revealing the number theoretic anatomy of integers, rationals, and perhaps even that of algebraic numbers.

1. Infinite primes have a representation as Fock states of super-symmetric arithmetic QFT and at the lowest level of hierarchy they provide representations for primes, integers, rationals and algebraic numbers in the sense that at the lowest level of hierarchy of second quantizations the simplest infinite primes are naturally mapped to rationals whereas more complex infinite primes having interpretation as bound states can be mapped to algebraic numbers. Conscious experience of number can be assigned to the quantum jumps between these quantum states revealing information about the number theoretic anatomy of the number represented. It would be wrong to say that rationals only label these states: rather, these states represent rationals and since primes label the particles of these states.

2. More concretely, the conservation of number theoretic energy defined by the logarithm of the rational assignable with the Fock state implies that the allowed decays of the state to a product of infinite integers are such that the rational can decompose only into a product of rationals. These decays could provide for the above discussed fundamental realization of multiplicative aspects of arithmetic consciousness. Also additive aspects are represented since the exponents k in the powers p^k appearing in the decomposition are conserved so that only the partitions $k = \sum_i k_i$ are representable. Thus both product decompositions and partitions, the basic operations of number theorist, are represented.

3. The higher levels of the hierarchy represent a hierarchy of abstractions about abstractions bringing strongly in mind the hierarchy of n-categories and various similar constructions including n: th order logic. It also seems that the n+1: th level of hierarchy provides a quantum representation for the n: th level. Ordinary primes, integers, rationals, and algebraic numbers would be the lowest level, -the initial object- of the hierarchy representing nothing at low level. Higher levels could be reduced to them by the analog of category theoretic reductionism in the sense that there is arrow between n: th and n+1: th level representing the second quantization at this level. On can also say that these levels represent higher reflective level of mathematical consciousness and the fundamental sensory perception corresponds the lowest level.

4. Infinite primes have also space-time correlates. The decomposition of particle into partons can be interpreted as a infinite prime and this gives geometric representations of infinite primes and also rationals. The finite primes appearing in the decomposition of infinite prime correspond to bosonic or fermionic partonic 2-surfaces. Many-sheeted space-time provides a representation for the hierarchy of second quantizations: one physical prediction is that many particle bound state associated with space-time sheet behaves exactly like a boson or fermion. Nuclear string model is one concrete application of this idea: it replaces nucleon reductionism with reductionism occurs first to strings consisting of $A \leq 4$ nuclei and which in turn are strings consisting of nucleons. A further more speculative representation of infinite rationals as space-time surfaces is based on their mapping to rational functions.

Number theoretic Brahman=Atman identity

The notion of infinite primes leads to the notion of algebraic holography in which space-time points possess infinitely rich number-theoretic anatomy. This anatomy would be due to the existence of infinite number of real units defined as ratios of infinite integers which reduce to unit in the real sense and various p-adic senses. This anatomy is not visible in real physics but can contribute directly to mathematical consciousness [K72].

The anatomies of single space-time point could represent the entire world of classical worlds and quantum states of universe: the number theoretic anatomy is of course not visible in the structure of these these states. Therefore the basic building brick of mathematics - point- would become the Platonia able to represent all of the mathematics consistent with the laws of quantum physics. Space-time points would evolve, becoming more and more complex quantum jump by quantum jump. WCW and quantum states would be represented by the anatomies of space-time points. Some space-time points are more "civilized" than others so that space-time decomposes into "civilizations" at different levels of mathematical evolution.

Paths between space-time points represent processes analogous to parallel translations affecting the structure of the point and one can also define n-parallel translations up to $n = 4$ at level of space-time and $n = 8$ at level of embedding space. At level of world of classical worlds whose points are representable as number theoretical anatomies arbitrary high values of n can be realized.

It is fair to say that the number theoretical anatomy of the space-time point makes it possible self-reference loop to close so that structured points are able to represent the physics of associated with with the structures constructed from structureless points. Hence one can speak about algebraic holography or number theoretic Brahman=Atman identity.

Finite measurement resolution, Jones inclusions, and number theoretic braids

In the history of physics and mathematics the realization of various limitations have been the royal road to a deeper understanding (Uncertainty Principle, Gödel's theorem). The precision of quantum measurement, sensory perception, and cognition are always finite. In standard quantum measurement theory this limitation is not taken into account but forms a corner stone of TGD based vision about quantum physics and of mathematics too as I want to argue in the following.

The finite resolutions has representation both at classical and quantum level.

1. At the level of quantum states finite resolution is represented in terms of Jones inclusions N subset M of hyper-finite factors of type II_1 (HFFs) [K27]. N represents measurement resolution in the sense that the states related by the action of N cannot be distinguished in the measurement considered. Complex rays are replaced by N rays. This brings in non-commutativity via quantum groups [K9]. Non-commutativity in TGD Universe would be therefore due to a finite measurement resolution rather than something exotic emerging in the Planck length scale. Same applies to p-adic physics: p-adic space-time sheets have literally infinite size in real topology!

2. At the space-time level discretization implied by the number theoretic universality could be seen as being due to the finite resolution with common algebraic points of real and p-adic variant of the partonic 3-surface chosen as representatives for regions of the surface. The solutions of Kähler-Dirac equation are characterized by the prime in question so that the preferred prime makes itself visible at the level of quantum dynamics and characterizes the p-adic length scale fixing the values of coupling constants. Discretization could be also understood as effective non-commutativity of embedding space points due to the finite resolution implying that second quantized spinor fields anti-commute only at a discrete set of points rather than along stringy curve.

In this framework it is easy to imagine physical representations of number theoretical and other mathematical structures.

1. Every compact group corresponds to a hierarchy of Jones inclusions corresponding to various representations for the quantum variants of the group labelled by roots of unity. I would be surprised if non-compact groups would not allow similar representation since HFF can be

regarded as infinite tensor power of n-dimensional complex matrix algebra for any value of n. Somewhat paradoxically, the finite measurement resolution would make possible to represent Lie group theory physically [K27].

2. There is a strong temptation to identify the Galois groups of algebraic numbers as the infinite permutation group S_∞ consisting of permutations of finite number of objects, whose projective representations give rise to an infinite braid group B_∞. The group algebras of these groups are HFFs besides the representation provided by the spinors of the world of classical worlds having physical identification as fermionic Fock states. Therefore physical states would provide a direct representation also for the more abstract features of number theory [K38].

3. Number theoretical braids crucial for the construction of S-matrix provide naturally representations for the Galois groups G associated with the algebraic extensions of rationals as diagonal embeddings $G \times G \times ....$ to the completion of S_∞ representable also as the action on the completion of spinors in the world of classical worlds so that the core of number theory would be represented physically [K38]. At the space-time level number theoretic braid having G as symmetries would represent the G. These representations are analogous to global gauge transformations. The elements of S_∞ are analogous to local gauge transformations having a natural identification as a universal number theoretical gauge symmetry group leaving physical states invariant.

Hierarchy of Planck constants and the generalization of embedding space

Jones inclusions inspire a further generalization of the notion of embedding space obtained by gluing together copies of the embedding space H regarded as coverings $H \to H/G_a \times G_b$. In the simplest scenario $G_a \times G_b$ leaves invariant the choice of quantization axis and thus this hierarchy provides embedding space correlate for the choice of quantization axes inducing these correlates also at space-time level and at the level of world of classical worlds [K27].

Dark matter hierarchy is identified in terms of different sectors of H glued together along common points of base spaces and thus forming a book like structure. For the simplest option elementary particles proper correspond to maximally quantum critical systems in the intersection of all pages. The field bodies of elementary particles are in the interiors of the pages of this “book”.

One can assign to Jones inclusions quantum phase $q = exp(i2\pi/n)$ and the groups Z_n acts as exact symmetries both at level of M^4 and CP_2. In the case of M^4 this means that space-time sheets have exact Z_n rotational symmetry. This suggests that the algebraic numbers q^m could have geometric representation at the level of sensory perception as Z_n symmetric objects. We need not be conscious of this representation in the ordinary wake-up consciousness dominated by sensory perception of ordinary matter with $q = 1$. This would make possible the idea about transcendentals like π, which do not appear in any finite-dimensional extension of even p-adic numbers (p-adic numbers allow finite-dimensional extension by since e^p is ordinary p-adic number). Quantum jumps in which state suffers an action of the generating element of Z_n could also provide a sensory realization of these groups and numbers $exp(i2\pi/n)$.

Planck constant is identified as the ratio n_a/n_b of integers associated with M^4 and CP_2 degrees of freedom so that a representation of rationals emerge again. The so called ruler and compass rationals whose definition involves only a repeated square root operation applied on rationals are cognitively the simplest ones and should appear first in the evolution of mathematical consciousness. The successful [K25] quantum model for EEG is only one of the applications providing support for their preferred role. Other applications are to Bohr quantization of planetary orbits interpreted as being induced by the presence of macroscopically quantum coherent dark matter [K68].

7.5.4 Farey Sequences, Riemann Hypothesis, Tangles, And TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *“Platonia as the best possible world in the* *sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

Farey sequences

Some basic facts about Farey sequences [A4] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \tag{7.5.1}$$

   Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p+1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p+1)$ to the sequence.

   The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the embedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [K19] would naturally relate to the Farey sequence.

Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \ .$$

In other words, $d_{n,N}$ is the difference between the n: th term of the N: th Farey sequence, and the n: th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\sum_{n=1,\ldots,|F_N|} |d_{n,N}| = O(N^r) \ \text{ for any } \ r > 1/2 \ ,$$
$$\sum_{n=1,\ldots,|F_N|} d_{n,N}^2 = O(N^r) \ \text{ for any } \ r > 1 \ . \tag{7.5.2}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [K19, K17].

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [K19].

Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$: th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [A99] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2, Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The

so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles $[0]$, $[\infty]$, $\pm[1]$, $\pm 1/[1]$, $\pm[2]$, $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M+N-2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M+N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N>2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N>>2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N>2$ case.

1. Could the commutativity of tangle product allow to characterize the $N>2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N>2$. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N: th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a,b]^T\rightarrow a/b$ from a rational 2-spinor $[a,b]^T$ to which SL(2(N-1), Z) acts. Equivalence means that the columns $[a,b]^T$ and $[c,d]^T$ combine to form element of SL(2, Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N-1$ $2(N-1)$-component projective column-spinors $[a_i^1,a_i^2,..,a_i^{2(N-1)}]^T$, $i=1,...N-1$ so that only the ratios $a_i^k/a_i^{2(N-1)}\leq 1$ matter? Could equivalence for them mean that the $N-1$ spinors combine to form $N-1+N-1$ columns of $SL(2(N-1),Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc=\pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g, Z) so that $N-1$ would be analogous to g and $1\leq N\geq 3$- braids would correspond to $g\leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2,Q)$ labelled by N (the generator $\tau\rightarrow\tau+2$ of modular group is replaced with $\tau\rightarrow\tau+2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N-1),Q)$.

Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N: th root of unity instead of phase U satisfying $U^2 = 1$?

How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N: th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

7.6 Quantum Quandaries

John Baez's [A87] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

7.6.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

7.6.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups [K9] too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

7.6.3 Tqft As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

7.6.4 The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A50] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In

TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of 2→2 reaction open string is pinched to a point at vertex. 1→2 vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. There is U-matrix acting in zero energy states. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K48].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics.

7.7 How To Represent Algebraic Numbers As Geometric Objects?

Physics blogs are also interesting because they allow to get some grasp about very different styles of thinking of a mathematician and physicist. For mathematician it is very important that the result is obtained by a strict use of axioms and deduction rules. Physicist is a cognitive opportunist: it does not matter how the result is obtained by moving along axiomatically allowed paths or not, and the new result is often more like a discovery of a new axiom and physicist is ever-grateful for Gödel for giving justification for what sometimes admittedly degenerates to a creative hand-waving. For physicist ideas form a kind of bio-sphere and the fate of the individual idea depends on its ability to survive, which is determined by its ability to become generalized, its consistency with other ideas, and ability to interact with other ideas to produce new ideas.

7.7.1 Can One Define Complex Numbers As Cardinalities Of Sets?

During few days before writing this we have had in Kea's blog a little bit of discussion inspired by the problem related to the categorification of basic number theoretical structures. I have learned that sum and product are natural operations for the objects of category. For instance, one can define sum as in terms of union of sets or direct sum of vector spaces and product as Cartesian product of sets and tensor product of vector spaces: rigs [A27] are example of categories for which natural numbers define sum and product.

Subtraction and division are however problematic operations. Negative numbers and inverses of integers do not have a realization as a number of elements for any set or as dimension of vector space. The naïve physicist inside me asks immediately: why not go from statics to dynamics and take operations (arrows with direction) as objects: couldn't this allow to define subtraction and division? Is the problem that the axiomatization of group theory requires something which purest categorification does not give? Or aren't the numbers representable in terms of operations of finite groups not enough? In any case cyclic groups would allow to realize roots of unity as operations (Z_2 would give -1).

One could also wonder why the algebraic numbers might not somehow result via the representations of permutation group of infinite number of elements containing all finite groups and thus Galois groups of algebraic extensions as subgroups? Why not take the elements of this group as objects of the basic category and continue by building group algebra and hyper-finite factors of type II_1 isomorphic to spinors of world of classical worlds, and so on.

After having written the first half of the section, I learned that something similar to the transition from statics to dynamics is actually carried out but by manner which is by many orders of magnitudes more refined than the proposal above and that I had never been able to imagine. The article *Objects of categories as complex numbers* of Marcelo Fiore and Tom Leinster [A27] describes a fascinating idea summarized also by John Baez [A25] about how one can assign to the objects of a category complex numbers as roots of a polynomial $Z = P(Z)$ defining an isomorphism of object. Z is the element of a category called rig, which differs from ring in that integers are replaced with natural numbers. One can replace Z with a complex number $|Z|$ defined as a root of polynomial. $|Z|$ is interpreted formally as the cardinality of the object. It is essential to have natural numbers and thus only product and sum are defined. This means a restriction: for instance, only complex algebraic numbers associated with polynomials having natural numbers as coefficients are obtained. Something is still missing.

Note that this correspondence assumes the existence of complex numbers and one cannot say that complex numbers are categorified. Maybe basic number fields must be left outside categorification. One can however require that all of them have a concrete set theoretic representation rather than only formal interpretation as cardinality so that one still encounters the problem how to represent algebraic complex number as a concrete cardinality of a set.

7.7.2 In What Sense A Set Can Have Cardinality -1?

The discussion in Kea's blog led me to ask what the situation is in the case of p-adic numbers. Could it be possible to represent the negative and inverse of p-adic integer, and in fact any p-adic number, as a geometric object? In other words, does a set with -1 or $1/n$ or even $\sqrt{-1}$ elements exist? If this were in some sense true for all p-adic number fields, then all this wisdom combined together might provide something analogous to the adelic representation for the norm of a rational number as product of its p-adic norms. As will be found, alternative interpretations of complex algebraic numbers as p-adic numbers representing cardinalities of p-adic fractals emerge. The fractal defines the manner how one must do an infinite sum to get an infinite real number but finite p-adic number.

Of course, this representation might not help to define p-adics or reals categorically but might help to understand how p-adic cognitive representations defined as subsets for rational intersections of real and p-adic space-time sheets could represent p-adic number as the number of points of p-adic fractal having infinite number of points in real sense but finite in the p-adic sense. This would also give a fundamental cognitive role for p-adic fractals as cognitive representations of numbers.

How to construct a set with -1 elements?

The basic observation is that p-adic -1 has the representation

$$-1 = (p-1)/(1-p) = (p-1)(1+p+p^2+p^3....)$$

As a real number this number is infinite or -1 but as a p-adic number the series converges and has p-adic norm equal to 1. One can also map this number to a real number by canonical identification taking the powers of p to their inverses: one obtains p in this particular case. As a matter fact, any rational with p-adic norm equal to 1 has similar power series representation.

The idea would be to represent a given p-adic number as the infinite number of points (in real sense) of a p-adic fractal such that p-adic topology is natural for this fractal. This kind of fractals can be constructed in a simple manner: from this more below. This construction allows to represent any p-adic number as a fractal and code the arithmetic operations to geometric operations for these fractals.

These representations - interpreted as cognitive representations defined by intersections of real and p-adic space-time sheets - are in practice approximate if real space-time sheets are assumed to have a finite size: this is due to the finite p-adic cutoff implied by this assumption and the meaning a finite resolution. One can however say that the p-adic space-time itself could by its necessarily infinite size represent the *idea* of given p-adic number faithfully.

This representation applies also to the p-adic counterparts of algebraic numbers in case that they exist. For instance, roughly one half of p-adic numbers have square root as ordinary p-adic number and quite generally algebraic operations on p-adic numbers can give rise to p-adic numbers so that also these could have set theoretic representation. For $p\ mod\ 4 = 1$ also $\sqrt{(-1)}$ exists: for instance, for $p = 5$: $2^2 = 4 = -1\ mod\ 5$ guarantees this so that also imaginary unit and complex numbers would have a fractal representation. Also many transcendentals possess this kind of representation. For instance $exp(xp)$ exists as a p-adic number if x has p-adic norm not larger than 1: also $log(1 + xp)$ does so.

Hence a quite impressive repertoire of p-adic counterparts of real numbers would have representation as a p-adic fractal for some values of p. Adelic vision would suggest that combining these representations one might be able to represent quite a many real numbers. In the case of π I do not find any obvious p-adic representation (for instance $sin(\pi/6) = 1/2$ does not help since the p-adic variant of the Taylor expansion of $\pi/6 = arcsin(1/2)$ does not converge p-adically for any value of p). It might be that there are very many transcendentals not allowing fractal representation for any value of p.

Conditions on the fractal representations of p-adic numbers

Consider now the construction of the fractal representations in terms of rational intersections of real real and p-adic space-time sheets. The question is what conditions are natural for this representation if it corresponds to a cognitive representation is realized in the rational intersection of real and p-adic space-time sheets obeying same algebraic equations.

1. Pinary cutoff is the analog of the decimal cutoff but is obtained by dropping away high positive rather than negative powers of p to get a finite real number: example of pinary cutoff is $-1 = (p-1)(1+p+p^2+...) \rightarrow (p-1)(1+p+p^2)$. This cutoff must reduce to a fractal cutoff meaning a finite resolution due to a finite size for the real space-time sheet. In the real sense the p-adic fractal cutoff means not forgetting details below some scale but cutting out all above some length scale. Physical analog would be forgetting all frequencies below some cutoff frequency in Fourier expansion.

   The motivation comes from the fact that TGD inspired consciousness assigns to a given biological body there is associated a field body or magnetic body containing dark matter with large $\hbar$ and quantum controlling the behavior of biological body and so strongly identifying with it so as to belief that this all ends up to a biological death. This field body has an onion like fractal structure and a size of at least order of light-life. Of course, also larger onion layers could be present and would represent those levels of cognitive consciousness not depending on the sensory input on biological body: some altered states of consciousness could relate to these levels. In any case, the larger the magnetic body, the better the numerical skills of the p-adic mathematician.

2. Lowest pinary digits of $x = x_0 + x_1 p + x_2 p^2 + ...$, $x_n \leq p$ must have the most reliable representation since they are the most significant ones. The representation must be also highly redundant to guarantee reliability. This requires repetitions and periodicity. This is guaranteed if the representation is hologram like with segments of length p^n with digit x_n represented again and again in all segments of length p^m, $m > n$.

3. The TGD based physical constraint is that the representation must be realizable in terms of induced classical fields assignable to the field body hierarchy of an intelligent system interested in artistic expression of p-adic numbers using its own field body as instrument. As a matter, sensory and cognitive representations are realized at field body in TGD Universe and EEG is in a fundamental role in building this representation. By p-adic fractality fractal wavelets are the most natural candidate. The fundamental wavelet should represent the p different pinary digits and its scaled up variants would correspond to various powers of p so that the representation would reduce to a Fourier expansion of a classical field.

Concrete representation

Consider now a concrete candidate for a representation satisfying these constraints.

1. Consider a p-adic number

   $$y = p^{n_0} x , \quad x = \sum x_n p^n \quad , \quad n \geq n_0 = 0 \quad .$$

   If one has a representation for a p-adic unit x the representation of is by a purely geometric fractal scaling of the representation by p^n. Hence one can restrict the consideration to p-adic units.

2. To construct the representation take a real line starting from origin and divide it into segments with lengths $1, p, p^2, ...$ In TGD framework this scalings come actually as powers of $p^{1/2}$ but this is just a technical detail.

3. It is natural to realize the representation in terms of periodic field patterns. One can use wavelets with fractal spectrum $p^n\lambda_0$ of "wavelet lengths", where λ_0 is the fundamental wavelength. Fundamental wavelet should have p different patterns correspond to the p values of pinary digit as its structures. Periodicity guarantees the hologram like character enabling to pick n: th digit by studying the field pattern in scale p^n anywhere inside the field body.

4. Periodicity guarantees also that the intersections of p-adic and real space-time sheets can represent the values of pinary digits. For instance, wavelets could be such that in a given p-adic scale the number of rational points in the intersection of the real and p-adic space-time sheet equals to x_n. This would give in the limit of an infinite pinary expansion a set theoretic realization of any p-adic number in which each pinary digit x_n corresponds to infinite copies of a set with x_n elements and fractal cutoff due to the finite size of real space-time sheet would bring in a finite precision. Note however that p-adic space-time sheet necessarily has an infinite size and it is only real world realization of the representation which has finite accuracy.

5. A concrete realization for this object would be as an infinite tree with $x_n+1 \leq p$ branches in each node at level n (x_n+1 is needed in order to avoid the splitting tree at $x_n = 0$). In 2-adic case -1 would be represented by an infinite pinary tree. Negative powers of p correspond to the of the tree extending to a finite depth in ground.

7.7.3 Generalization Of The Notion Of Rig By Replacing Naturals With P-Adic Integers

Previous considerations do not relate directly to category theoretical problem of assigning complex numbers to objects. It however turns out that p-adic approach allows to generalize the proposal of [A27] by replacing natural numbers with p-adic integers in the definition of rig so that any algebraic complex number can define cardinality of an object of category allowing multiplication and sum and that these complex numbers can be replaced with p-adic numbers if they make sense as such so that previous arguments provide a concrete geometric representation of the cardinality. The road to the realization this simple generalization required a visit to the John Baez's Weekly Finds (Week 102) [A25].

The outcome was the realization that the notion of rig used to categorify the subset of algebraic numbers obtained as roots of polynomials with *natural number* valued coefficients generalizes trivially by replacing natural numbers by *p-adic integers*. As a consequence one obtains beautiful p-adicization of the generating function F(x) of structure as a function which converges p-adically for any rational $x = q$ for which it has prime p as a positive power divisor.

Effectively this generalization means the replacement of natural numbers as coefficients of the polynomial defining the rig with all rationals, also negative, and *all* complex algebraic numbers find a category theoretical representation as "cardinalities". These cardinalities have a dual interpretation as p-adic integers which in general correspond to infinite real numbers but are mappable to real numbers by canonical identification and have a geometric representation as fractals.

Mapping of objects to complex numbers and the notion of rig

The idea of rig approach is to categorify the notion of cardinality in such a way that one obtains a *subset* of algebraic complex numbers as cardinalities in the category-theoretical sense. One can assign to an object a polynomial with coefficients, which are *natural numbers* and the condition $Z = P(Z)$ says that $P(Z)$ acts as an isomorphism of the object. One can interpret the equation also in terms of complex numbers. Hence the object is mapped to a complex number Z defining a root of the polynomial interpreted as an ordinary polynomial: it does not matter which root is chosen. The complex number Z is interpreted as the "cardinality" of the object but I do not really understand the motivation for this. The deep further result is that also more general polynomial equations $R(|Z|) = Q(|Z|)$ satisfied by the generalized cardinality Z imply $R(Z) = Q(Z)$ as isomorphism.

I try to reproduce what looks the most essential in the explanation of John Baez and relate it to my own ideas but take this as my talk to myself and visit This Week's Finds [A25], one of the many classics of Baez, to learn of this fascinating idea.

1. Baez considers first the ways of putting a given structure to n-element set. The set of these structures is denoted by F_n and the number of them by $|F_n|$. The generating function $|F|(x) = \sum_n |F_n| x^n$ packs all this information to a single function.

   For instance, if the structure is binary tree, this function is given by $T(x) = \sum_n C_{n-1} x^n$, where C_{n-1} are Catalan numbers and n¿0 holds true. One can show that T satisfies the formula

   $$T = X + T^2 \ ,$$

   since any binary tree is either trivial or decomposes to a product of binary trees, where two trees emanate from the root. One can solve this second order polynomial equation and the power expansion gives the generating function.

2. The great insight is that one can also work directly with structures. For instance, by starting from the isomorphism $T = 1 + T^2$ applying to an object with cardinality 1 and substituting T^2 with $(1 + T^2)^2$ repeatedly, one can deduce the amazing formula $T^7(1) = T(1)$ mentioned by Kea, and this identity can be interpreted as an isomorphism of binary trees.

3. This result can be generalized using the notion of rig category [A27]. In rig category one can add and multiply but negatives are not defined as in the case of ring. The lack of subtraction and division is still the problem and as I suggested in previous posting p-adic integers might resolve the problem.

   Whenever Z is object of a rig category, one can equip it with an isomorphism $Z = P(Z)$ where $P(Z)$ is polynomial with *natural numbers* as coefficients and one can assign to object "cardinality" as any root of the equation $Z = P(Z)$. Note that set with n elements corresponds to $P(|Z|) = n$. Thus subset of algebraic complex numbers receive formal identification as cardinalities of sets. Furthermore, if the cardinality satisfies another equation $Q(|Z|) = R(|Z|)$ such that neither polynomial is constant, then one can construct an isomorphism $Q(Z) = R(Z)$. Isomorphisms correspond to equations!

4. This is indeed nice that there is something which is not so beautiful as it could be: why should we restrict ourselves to *natural numbers* as coefficients of $P(Z)$? Could it be possible to replace them with integers to obtain *all complex algebraic numbers* as cardinalities? Could it be possible to replace natural numbers by p-adic integers?

p-Adic rigs and Golden Object as p-adic fractal

The notions of generating function and rig generalize to the p-adic context.

1. The generating function $F(x)$ defining isomorphism Z in the rig formulation converges p-adically for any p-adic number containing p as a factor so that the idea that all structures have p-adic counterparts is natural. In the real context the generating function typically diverges and must be defined by analytic continuation. Hence one might even argue that p-adic numbers are more natural in the description of structures assignable to finite sets than reals.

2. For rig one considers only polynomials $P(Z)$ (Z corresponds to the generating function F) with coefficients which are natural numbers. Any p-adic integer can be however interpreted as a non-negative integer: natural number if it is finite and "super-natural" number if it is infinite. Hence can generalize the notion of rig by replacing natural numbers by p-adic integers. The rig formalism would thus generalize to arbitrary polynomials with integer valued coefficients so that all complex algebraic numbers could appear as cardinalities of category theoretical objects. Even rational coefficients are allowed. This is highly natural number theoretically.

3. For instance, in the case of binary trees the solutions to the isomorphism condition $T = p+T^2$ giving $T = [1 \pm (1-4p)^{1/2}]/2$ and T would be complex number $[p \pm (1-4p)^{1/2}]/2$. $T(p)$ can be interpreted also as a p-adic number by performing power expansion of square root in case that the p-adic square root exists: this super-natural number can be mapped to a real number by the canonical identification and one obtains also the set theoretic representations of the category theoretical object $T(p)$ as a p-adic fractal. This interpretation of cardinality is much more natural than the purely formal interpretation as a complex number. This argument applies completely generally. The case $x = 1$ discussed by Baez gives $T = [1 \pm (-3)^{1/2}]/2$ allows p-adic representation if $-3 == p-3$ is square mod p. This is the case for $p = 7$ for instance.

4. John Baez [A25] poses also the question about the category theoretic realization of "Golden Object", his big dream. In this case one would have $Z = G = -1 + G^2 = P(Z)$. The polynomial on the right hand side does not conform with the notion of rig since -1 is not a natural number. If one allows p-adic rigs, $x = -1$ can be interpreted as a p-adic integer $(p-1)(1+p+...)$, positive and infinite and "super-natural", actually largest possible p-adic integer in a well defined sense.

   A further condition is that Golden Mean converges as a p-adic number: this requires that $\sqrt{5}$ must exist as a p-adic number: $(5 = 1+4)^{1/2}$ certainly converges as power series for $p = 2$ so that Golden Object exists 2-adically. By using [A18] of Euler, one finds that 5 is square mod p only if p is square mod 5. To decide whether given p is Golden it is enough to look whether p mod 5 is 1 or 4. For instance, $p = 11, 19, 29, 31$ ($=M_5$) are Golden. Mersennes M_k, $k = 3, 7, 127$ and Fermat primes are not Golden. One representation of Golden Object as p-adic fractal is the p-adic series expansion of $[1/2 \pm 5^{1/2}]/2$ representable geometrically as a binary tree such that there are $0 \leq x_n + 1 \leq p$ branches at each node at height n if n: th p-adic coefficient is x_n. The "cognitive" p-adic representation in terms of wavelet spectrum of classical fields is discussed in the previous posting.

5. It would be interesting to know how quantum dimensions of quantum groups assignable to Jones inclusions [K87, K27, K9] relate to the generalized cardinalities. The root of unity property of quantum phase ($q^{n+1} = q$) suggests $Q = Q^{n+1} = P(Q)$ as the relevant isomorphism. For Jones inclusions the cardinality $q = exp(i2\pi/n)$ would not be however equal to quantum dimension $D(n) = 4cos^2(\pi/n)$.

Is there a connection with infinite integers?

Infinite primes [K72] correspond to Fock states of a super-symmetric arithmetic quantum field theory and there is entire infinite hierarchy of them corresponding to repeated second quantization. Also infinite primes and rationals make sense. Besides free Fock states spectrum contains at each level also what might be identified as bound states. All these states can be mapped to polynomials. Since the roots of polynomials represent complex algebraic numbers and as they seem to characterize objects of categories, there are reasons to expect that infinite rationals might allow also interpretation in terms of say rig categories or their generalization. Also the possibility to identify space-time coordinate as isomorphism of a category might be highly interesting concerning the interpretation of quantum classical correspondence.

7.8 Gerbes And TGD

The notion of gerbes has gained much attention during last years in theoretical physics and there is an abundant gerbe-related literature in hep-th archives. Personally I learned about gerbes from the excellent article of Jouko Mickelson [A91] (Jouko was my opponent in PhD dissertation for more than two decades ago: so the time flows!).

I have already applied the notion of bundle gerbe in TGD framework in the construction of the Dirac determinant which I have proposed to define the Kähler function for the WCW (see [K88]). The insights provided by the general results about bundle gerbes discussed in [A91] led, not only to a justification for the hypothesis that Dirac determinant exists for the Kähler-Dirac action, but also to an elegant solution of the conceptual problems related to the construction of Dirac determinant in the presence of chiral symmetry. Furthermore, on basis of the special properties of the Kähler-Dirac operator there are good reasons to hope that the determinant exists even without zeta function regularization. The construction also leads to the conclusion that the space-time sheets serving as causal determinants must be geodesic sub-manifolds (presumably light like boundary components or "elementary particle horizons"). Quantum gravitational holography is realized since the exponent of Kähler function is expressible as a Dirac determinant determined by the local data at causal determinants and there would be no need to find absolute minima of Kähler action explicitly.

In the sequel the emergence of 2-gerbes at the space-time level in TGD framework is discussed and shown to lead to a geometric interpretation of the somewhat mysterious cocycle conditions for a wide class of gerbes generated via the $\wedge d$ products of connections associated with 0-gerbes. The resulting conjecture is that gerbes form a graded-commutative Grassmman algebra like structure generated by -1- and 0-gerbes. 2-gerbes provide also a beautiful topological characterization of space-time sheets as structures carrying Chern-Simons charges at boundary components and the 2-gerbe variant of Bohm-Aharonov effect occurs for perhaps the most interesting asymptotic solutions of field equations especially relevant for anyonics systems, quantum Hall effect, and living matter [K4].

7.8.1 What Gerbes Roughly Are?

Very roughly and differential geometrically, gerbes can be regarded as a generalization of connection. Instead of connection 1-form (0-gerbe) one considers a connection $n+1$-form defining n-gerbe. The curvature of n-gerbe is closed $n+2$-form and its integral defines an analog of magnetic charge. The notion of holonomy generalizes: instead of integrating n-gerbe connection over curve one integrates its connection form over n+1-dimensional closed surface and can transform it to the analog of magnetic flux.

There are some puzzling features associated with gerbes. Ordinary $U(1)$-bundles are defined in terms of open sets U_α with gauge transformations $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ defined in $U_\alpha \cap U_\beta$ relating the connection forms in the patch U_β to that in patch U_α. The 3-cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \tag{7.8.1}$$

makes it possible to glue the patches to a bundle structure.

In the case of 1-gerbes the transition functions are replaced with the transition functions $g_{\alpha\beta\gamma} = g_{\gamma\beta\alpha}^{-1}$ defined in triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and 3-cocycle must be replaced with 4-cocycle:

$$g_{\alpha\beta\gamma}g_{\beta\gamma\delta}g_{\gamma\delta\alpha}g_{\delta\alpha\beta} = 1 \ \ . \tag{7.8.2}$$

The generalizations of these conditions to n-gerbes is obvious.

In the case of 2-intersections one can build a bundle structure naturally but in the case of 3-intersections this is not possible. Hence the geometric interpretation of the higher gerbes is far from obvious. One possible interpretation of non-trivial 1-gerbe is as an obstruction for lifting projective bundles with fiber space CP_n to vector bundles with fiber space C^{n+1} [A91]. This involves the lifting of the holomorphic transition functions g_α defined in the projective linear group $PGL(n+1,C)$ to $GL(n+1,C)$. When the 3-cocycle condition for the lifted transition functions $\overline{g}_{\alpha\beta}$ fails it can be replaced with 4-cocycle and one obtains 1-gerbe.

7.8.2 How Do 2-Gerbes Emerge In TGD?

Gerbes seem to be interesting also from the point of view of TGD, and TGD approach allows a geometric interpretation of the cocycle conditions for a rather wide class of gerbes.

Recall that the Kähler form J of CP_2 defines a non-trivial magnetically charged and self-dual $U(1)$-connection A. The Chern-Simons form $\omega = A\wedge J = A\wedge dA$ having CP_2 Abelian instanton density $J\wedge J$ as its curvature form and can thus be regarded as a 3-connection form of a 2-gerbe. This 2-gerbe is induced by 0-gerbe.

The coordinate patches U_α are same as for $U(1)$ connection. In the transition between patches A and ω transform as

$$\begin{aligned} A &\rightarrow A + d\phi \ , \\ \omega &\rightarrow \omega + dA_2 \ , \\ A_2 &= \phi \wedge J \ . \end{aligned} \tag{7.8.3}$$

The transformation formula is induced by the transformation formula for $U(1)$ bundle. Somewhat mysteriously, there is no need to define anything in the intersections of U_α in the recent case.

The connection form of the 2-gerbe can be regarded as a second $\wedge d$ power of Kähler connection:

$$A_3 \equiv A \wedge dA \ . \tag{7.8.4}$$

The generalization of this observation allows to develop a different view about n-gerbes generated as $\wedge d$ products of 0-gerbes.

The hierarchy of gerbes generated by 0-gerbes

Consider a collection of $U(1)$ connections $A^{i)}$. They generate entire hierarchy of gerbe-connections via the $\wedge d$ product

$$A_3 = A^{1)} \wedge dA^{2)} \tag{7.8.5}$$

defining 2-gerbe having a closed curvature 4-form

$$F_4 = dA^{1)} \wedge dA^{2)} \ . \tag{7.8.6}$$

$\wedge d$ product is commutative apart from a gauge transformation and the curvature forms of $A^{1)} \wedge dA^{2)}$ and $A^{2)} \wedge dA^{1)}$ are the same.

Quite generally, the connections A_m of $m-1$ gerbe and A_n of $n-1$-gerbe define $m+n+1$ connection form and the closed curvature form of $m+n$-gerbe as

$$\begin{aligned} A_{m+n+1} &= A_m^{1)} \wedge dA_n^{2)} \ , \\ F_{m+n+2} &= dA_m^{1)} \wedge dA_n^{2)} \ . \end{aligned} \tag{7.8.7}$$

The sequence of gerbes extends up to $n = D-2$, where D is the dimension of the underlying manifold. These gerbes are not the most general ones since one starts from 0-gerbes. One can of course start from $n > 0$-gerbes too.

The generalization of the $\wedge d$ product to the non-Abelian situation is not obvious. The problems stem from the that the Lie-algebra valued connection forms $A^{1)}$ and $A^{2)}$ appearing in the covariant version $D = d + A$ do not commute.

7.8.3 How To Understand The Replacement Of 3-Cycles With N-Cycles?

If n-gerbes are generated from 0-gerbes it is possible to understand how the intersections of the open sets emerge. Consider the product of 0-gerbes as the simplest possible case. The crucial observation is that the coverings U_α for $A^{1)}$ and V_β for $A^{2)}$ need not be same (for CP_2 this was the case). One can form a new covering consisting of sets $U_\alpha \cap V_{\alpha_1}$. Just by increasing the index range one can replace V with U and one has covering by $U_\alpha \cap U_{\alpha_1} \equiv U_{\alpha\alpha_1}$.

The transition functions are defined in the intersections $U_{\alpha\alpha_1} \cap U_{\beta\beta_1} \equiv U_{\alpha\alpha_1\beta\beta_1}$ and cocycle conditions must be formulated using instead of intersections $U_{\alpha\beta\gamma}$ the intersections $U_{\alpha\alpha_1\beta\beta_1\gamma\gamma_1}$.

Hence the transition functions can be written as $g_{\alpha\alpha_1\beta\beta_1}$ and the 3-cocycle are replaced with 5-cocycle conditions since the minimal co-cycle corresponds to a sequence of 6 steps instead of 4:

$$U_{\alpha\alpha_1\beta\beta_1} \rightarrow U_{\alpha_1\beta\beta_1\gamma} \rightarrow U_{\beta\beta_1\gamma\gamma_1} \rightarrow U_{\beta_1\gamma\gamma_1\alpha} \rightarrow U_{\gamma\gamma_1\alpha\alpha_1} \quad .$$

The emergence of higher co-cycles is thus forced by the modification of the bundle covering necessary when gerbe is formed as a product of lower gerbes. The conjecture is that any even gerbe is expressible as a product of 0-gerbes.

An interesting application of the product structure is at the level of WCW ("world of classical worlds"). The Kähler form of WCW defines a connection 1-form and this generates infinite hierarchy of connection $2n+1$-forms associated with $2n$-gerbes.

7.8.4 Gerbes As Graded-Commutative Algebra: Can One Express All Gerbes As Products Of -1 And 0-Gerbes?

If one starts from, say 1-gerbes, the previous argument providing a geometric understanding of gerbes is not applicable as such. One might however hope that it is possible to represent the connection 2-form of any 1-gerbe as a $\wedge d$ product of a connection 0-form ϕ of "-1" -gerbe and connection 1-form A of 0-gerbe:

$$A_2 = \phi dA \equiv A \wedge d\phi \quad ,$$

with different coverings for ϕ and A. The interpretation as an obstruction for the modification of the underlying bundle structure is consistent with this interpretation.

The notion of -1-gerbe is not well-defined unless one can define the notion of -1 form precisely. The simplest possibility that 0-form transforms trivially in the change of patch is not consistent. One could identify contravariant n-tensors as $-n$-forms and d for them as divergence and d^2 as the antisymmetrized double divergence giving zero. ϕ would change in a gauge transformation by a divergence of a vector field. The integral of a divergence over closed M vanishes identically so that if the integral of ϕ over M is non-vanishing it corresponds to a non-trivial 0-connection. This interpretation of course requires the introduction of metric.

The requirement that the minimal intersections of the patches for 1-gerbes are of form $U_{\alpha\beta\gamma}$ would be achieved if the intersections patches can be restricted to the intersections $U_{\alpha\beta\gamma}$ defined by $U_\alpha \cap V_\gamma$ and $U_\beta \cap V_\gamma$ (instead of $U_\beta \cap V_\delta$), where the patches V_γ would be most naturally associated with -1-gerbe. It is not clear why one could make this restriction. The general conjecture is that any gerbe decomposes into a multiple $\wedge d$ product of -1 and 0-gerbes just like integers decompose into primes. The $\wedge d$ product of two odd gerbes is anti-commutative so that there is also an analogy with the decomposition of the physical state into fermions and bosons, and gerbes for a graded-commutative super-algebra generalizing the Grassmann algebra of manifold to a Grassmann algebra of gerbe structures for manifold.

7.8.5 The Physical Interpretation Of 2-Gerbes In TGD Framework

2-gerbes could provide some insight to how to characterize the topological structure of the many-sheeted space-time.

1. The cohomology group H^4 is obviously crucial in characterizing 2-gerbe. In TGD framework many-sheetedness means that different space-time sheets with induced metric having Minkowski signature are separated by elementary particle horizons which are light like 3-surfaces at which the induced metric becomes degenerate. Also the time orientation of the space-time sheet can change at these surfaces since the determinant of the induced metric vanishes.

   This justifies the term elementary particle horizon and also the idea that one should treat different space-time sheets as generating independent direct summands in the homology group of the space-time surface: as if the space-time sheets not connected by join along boundaries bonds were disjoint. Thus the homology group H^4 and 2-gerbes defining instanton numbers would become important topological characteristics of the many-sheeted space-time.

2. The asymptotic behavior of the general solutions of field equations can be classified by the dimension D of the CP_2 projection of the space-time sheet. For $D = 4$ the instanton density defining the curvature form of 2-gerbe is non-vanishing and instanton number defines a topological charge. Also the values of the Chern-Simons invariants associated with the boundary components of the space-time sheet define topological quantum numbers characterizing the space-time sheet and their sum equals to the instanton charge. CP_2 type extremals represent a basic example of this kind of situation. From the physical view point $D = 4$ asymptotic solutions correspond to what might be regarded chaotic phase for the flow lines of the Kähler magnetic field. Kähler current vanishes so that empty space Maxwell's equations are satisfied.

3. For $D = 3$ situation is more subtle when boundaries are present so that the higher-dimensional analog of Aharonov-Bohm effect becomes possible. In this case instanton density vanishes but the Chern-Simons invariants associated with the boundary components can be non-vanishing. Their sum obviously vanishes. The space-time sheet can be said to be a neutral C-S multipole. Separate space-time sheets can become connected by flux tubes in a quantum jump replacing a space-time surface with a new one. This means that the cohomology group H^4 as well as instanton charges and C-S charges of the system change.

Concerning the asymptotic dynamics of the Kähler magnetic field, $D = 3$ phase corresponds to an extremely complex but highly organized phase serving as an excellent candidate for the modelling of living matter. Both the TGD based description of anyons and quantum Hall effect and the model for topological quantum computation based on the braiding of magnetic flux tubes rely heavily on the properties $D = 3$ phase [K4].

The non-vanishing of the C-S form implies that the flow lines of the Kähler magnetic are highly entangled and have as an analog mixing hydrodynamical flow. In particular, one cannot define non-trivial order parameters, say phase factors, which would be constant along the lines. The interpretation in terms of broken super-conductivity suggests itself. Kähler current can be non-vanishing so that there is no counterpart for this phase at the level of Maxwell's equations.

7.9 Appendix: Category Theory And Construction Of S-Matrix

The construction of WCW geometry, spinor structure and of S-matrix involve difficult technical and conceptual problems and category theory might be of help here. As already found, the application of category theory to the construction of WCW geometry allows to understand how the arrow of psychological time emerges.

The construction of the S-matrix involves several difficult conceptual and technical problems in which category theory might help. The incoming states of the theory are what might be called free states and are constructed as products of the WCW spinor fields. One can effectively regard them as being defined in the Cartesian power of WCW divided by an appropriate permutation group. Interacting states in turn are defined in the WCW .

Cartesian power of WCW of 3-surfaces is however in geometrical sense more or less identical with WCW since the disjoint union of N 3-surfaces is itself a 3-surface in WCW . Actually it differs from WCW itself only in that the 3-surfaces of many particle state can intersect each other and if one allows this, one has paradoxical self-referential identification $CH = \overline{CH^2}/S_2 = ... = \overline{CH^N}/S_N ...$, where over-line signifies that intersecting 3-surfaces have been dropped from the product.

Note that arbitrarily small deformation can remove the intersections between 3-surfaces and four-dimensional general coordinate invariance allows always to use non-intersecting representatives. In case of the spinor structure of the Cartesian power this identification means that the tensor powers SCH^N of the WCW spinor structure are in some sense identical with the spinor structure SCH of the WCW . Certainly the oscillator operators of the tensor factors must be assumed to be mutually anti-commuting.

The identities $CH = \overline{CH^2}/S_2 = ..$ and corresponding identities $SCH = SCH^2 = ...$ for the space SCH of WCW spinor fields might imply very deep constraints on S-matrix. What comes into mind are counterparts for the Schwinger-Dyson equations of perturbative quantum field theory providing defining equations for the n-point functions of the theory [A84]. The isomorphism between SCH^2 and SCH is actually what is needed to calculate the S-matrix elements. Category theory might help to understand at a general level what these self-referential and somewhat paradoxical looking identities really imply and perhaps even develop TGD counterparts of Schwinger-Dyson equations.

There is also the issue of bound states. The interacting states contain also bound states not belonging to the space of free states and category theory might help also here. It would seem that the state space must be constructed by taking into account also the bound states as additional "free" states in the decomposition of states to product states.

A category naturally involved with the construction of the S-matrix (or U-matrix) is the space of preferred extremals of the Kähler action which might be called interacting category. The symplectic transformations acting as isometries of the configuration space geometry act naturally as the morphisms of this category. The group $Diff^4$ of general coordinate transformations in turn acts as gauge symmetries.

S-matrix relates free and interacting states and is induced by the classical long range interactions induced by the criticality of the preferred extremals in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes S-matrix elements are essentially Glebch-Gordan coefficients relating the states in the tensor power of the interacting super-symplectic representation with the interacting super-symplectic representation itself. More concretely, N-particle free states can be seen as WCW spinor fields in CH^N obtained as tensor products of ordinary WCW spinor fields. Free states correspond classically to the unions of space-time surfaces associated with the 3-surfaces representing incoming particles whereas interacting states correspond classically to the space-time surfaces associated with the unions of the 3-surfaces defining incoming states. These two states define what might be called free and interacting categories with canonical transformations acting as morphisms.

The classical interaction is represented by a functor $S : \overline{CH^N}/S_N \rightarrow CH$ mapping the classical free many particle states, that is objects of the product category defined by $\overline{CH^N}/S_N$ to the interacting category CH. This functor assigns to the union $\cup_i X^4(X_i^3)$ of the absolute minima $X^4(X_i^3)$ of Kähler action associated with the incoming, free states X_i^3 the preferred extreal $X^4(\cup X_i^3)$ associated with the union of 3-surfaces representing the outgoing interacting state. At quantum level this functor maps the state space SCH^N associated with $\cup_i X^4(X_i^3)$ to SCH in a unitary manner. An important constraint on S-matrix is that it acts effectively as a flow in zero modes correlating the quantum numbers in fiber degrees of freedom in one-to-one manner with the values of zero modes so that quantum jump $U\Psi_i \rightarrow \Psi_0$... gives rise to a quantum measurement.

Chapter 8

Category Theory and Quantum TGD

8.1 Introduction

TGD predicts several hierarchical structures involving a lot of new physics. These structures look frustratingly complex and category theoretical thinking might help to build a bird's eye view about the situation. I have already earlier considered the question how category theory might be applied in TGD [K18, K15]. Besides the far from complete understanding of the basic mathematical structure of TGD also my own limited understanding of category theoretical ideas have been a serious limitation. During last years considerable progress in the understanding of quantum TGD proper has taken place and the recent formulation of TGD is in terms of light-like 3-surfaces, zero energy ontology and number theoretic braids [K85, ?]. There exist also rather detailed formulations for the fusion of p-adic and real physics and for the dark matter hierarchy. This motivates a fresh look to how category theory might help to understand quantum TGD.

The fusion rules for the symplectic variant of conformal field theory, whose existence is strongly suggested by quantum TGD, allow rather precise description using the basic notions of category theory and one can identify a series of finite-dimensional nilpotent algebras as discretized versions of field algebras defined by the fusion rules. These primitive fusion algebras can be used to construct more complex algebras by replacing any algebra element by a primitive fusion algebra. Trees with arbitrary numbers of branches in any node characterize the resulting collection of fusion algebras forming an operad. One can say that an exact solution of symplectic scalar field theory is obtained.

Conformal fields and symplectic scalar field can be combined to form symplecto-formal fields. The combination of symplectic operad and Feynman graph operad leads to a construction of Feynman diagrams in terms of n-point functions of conformal field theory. M-matrix elements with a finite measurement resolution are expressed in terms of a hierarchy of symplecto-conformal n-point functions such that the improvement of measurement resolution corresponds to an algebra homomorphism mapping conformal fields in given resolution to composite conformal fields in improved resolution. This expresses the idea that composites behave as independent conformal fields. Also other applications are briefly discussed.

Years after writing this chapter a very interesting new TGD related candidate for a category emerged. The preferred extremals would form a category if the proposed duality mapping associative (co-associative) 4-surfaces of embedding space respects associativity (co-associativity) [K74]. The duality would allow to construct new preferred extremals of Kähler action.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. There are concept maps about topics related to the contents of the chapter prepared using CMAP realized as html files. Links to all CMAP files can be found at `http://tgdtheory.fi/cmaphtml.html` [L5]. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

8.2 S-Matrix As A Functor

John Baez's [A94] discusses in a physicist friendly manner the possible application of category theory to physics. The lessons obtained from the construction of topological quantum field theories (TQFTs) suggest that category theoretical thinking might be very useful in attempts to construct theories of quantum gravitation.

The point is that the Hilbert spaces associated with the initial and final state n-1-manifold of n-cobordism indeed form in a natural manner category. Morphisms of Hilb in turn are unitary or possibly more general maps between Hilbert spaces. TQFT itself is a functor assigning to a cobordism the counterpart of S-matrix between the Hilbert spaces associated with the initial and final n-1-manifold. The surprising result is that for $n \leq 4$ the S-matrix can be unitary S-matrix only if the cobordism is trivial. This should lead even string theorist to raise some worried questions.

In the hope of feeding some category theoretic thinking into my spine, I briefly summarize some of the category theoretical ideas discussed in the article and relate it to the TGD vision, and after that discuss the worried questions from TGD perspective. That space-time makes sense only relative to embedding space would conform with category theoretic thinking.

8.2.1 The *-Category Of Hilbert Spaces

Baez considers first the category of Hilbert spaces. Intuitively the definition of this category looks obvious: take linear spaces as objects in category Set, introduce inner product as additional structure and identify morphisms as maps preserving this inner product. In finite-D case the category with inner product is however identical to the linear category so that the inner product does not seem to be absolutely essential. Baez argues that in infinite-D case the morphisms need not be restricted to unitary transformations: one can consider also bounded linear operators as morphisms since they play key role in quantum theory (consider only observables as Hermitian operators). For hyper-finite factors of type II_1 inclusions define very important morphisms which are not unitary transformations but very similar to them. This challenges the belief about the fundamental role of unitarity and raises the question about how to weaken the unitarity condition without losing everything.

The existence of the inner product is essential only for the metric topology of the Hilbert space. Can one do without inner product as an inherent property of state space and reduce it to a morphism? One can indeed express inner product in terms of morphisms from complex numbers to Hilbert space and their conjugates. For any state Ψ of Hilbert space there is a unique morphisms T_Ψ from C to Hilbert space satisfying $T_\Psi(1) = \Psi$. If one assumes that these morphisms have conjugates T_Ψ^* mapping Hilbert space to C, inner products can be defined as morphisms $T_\Phi^* T_\Psi$. The Hermitian conjugates of operators can be defined with respect to this inner product so that one obtains *-category. Reader has probably realized that T_Ψ and its conjugate correspond to ket and bra in Dirac's formalism.

Note that in TGD framework based on hyper-finite factors of type II_1 (HFFs) the inclusions of complex rays might be replaced with inclusions of HFFs with included factor representing the finite measurement resolution. Note also the analogy of inner product with the representation of space-times as 4-surfaces of the embedding space in TGD.

8.2.2 The Monoidal *-Category Of Hilbert Spaces And Its Counterpart At The Level Of Ncob

One can give the category of Hilbert spaces a structure of monoid by introducing explicitly the tensor products of Hilbert spaces. The interpretation is obvious for physicist. Baez describes the details of this identification, which are far from trivial and in the theory of quantum groups very interesting things happen. A non-commutative quantum version of the tensor product implying braiding is possible and associativity condition leads to the celebrated Yang-Baxter equations: inclusions of HFFs lead to quantum groups too.

At the level of nCob the counterpart of the tensor product is disjoint union of n-1-manifolds. This unavoidably creates the feeling of cosmic loneliness. Am I really a disjoint 3-surface in emptiness which is not vacuum even in the geometric sense? Cannot be true!

This horrifying sensation disappears if n-1-manifolds are n-1-surfaces in some higher-dimensional embedding space so that there would be at least something between them. I can emit a little baby manifold moving somewhere perhaps being received by some-one somewhere and I can receive radiation from some-one at some distance and in some direction as small baby manifolds making gentle tosses on my face!

This consoling feeling could be seen as one of the deep justifications for identifying fundamental objects as light-like partonic 3-surfaces in TGD framework. Their ends correspond to 2-D partonic surfaces at the boundaries of future or past directed light-cones (states of positive and negative energy respectively) and are indeed disjoint but not in the desperately existential sense as 3-geometries of General Relativity.

This disjointness has also positive aspect in TGD framework. One can identify the color degrees of freedom of partons as those associated with CP_2 degrees of freedom. For instance, SU(3) analogs for rotational states of rigid body become possible. 4-D space-time surfaces as preferred extremals of Kähler action connect the partonic 3-surfaces and bring in classical representation of correlations and thus of interactions. The representation as sub-manifolds makes it also possible to speak about positions of these sub-Universes and about distances between them. The habitants of TGD Universe are maximally free but not completely alone.

8.2.3 TSFT As A Functor

The category theoretic formulation of TQFT relies on a very elegant and general idea. Quantum transition has as a space-time correlate an n-dimensional surface having initial final states as its n-1-dimensional ends. One assigns Hilbert spaces of states to the ends and S-matrix would be a unitary morphism between the ends. This is expressed in terms of the category theoretic language by introducing the category nCob with objects identified as n-1-manifolds and morphisms as cobordisms and *-category Hilb consisting of Hilbert spaces with inner product and morphisms which are bounded linear operators which do not however preserve the unitarity. Note that the morphisms of nCob cannot anymore be identified as maps between n-1-manifolds interpreted as sets with additional structure so that in this case category theory is more powerful than set theory.

TQFT is identified as a functor nCob $\rightarrow$ Hilb assigning to n-1-manifolds Hilbert spaces, and to cobordisms unitary S-matrices in the category Hilb. This looks nice but the surprise is that for $n \leq 4$ unitary S-matrix exists only if the cobordism is trivial so that topology changing transitions are not possible unless one gives up unitarity.

This raises several worried questions.

1. Does this result mean that in TQFT sense unitary S-matrix for topology changing transitions from a state containing n_i closed strings to a state containing $n_f \neq n_i$ strings does not exist? Could the situation be same also for more general non-topological stringy S-matrices? Could the non-converging perturbation series for S-matrix with finite individual terms matrix fail to no non-perturbative counterpart? Could it be that M-theory is doomed to remain a dream with no hope of being fulfilled?

2. Should one give up the unitarity condition and require that the theory predicts only the relative probabilities of transitions rather than absolute rates? What the proper generalization of the S-matrix could be?

3. What is the relevance of this result for quantum TGD?

8.2.4 The Situation Is In TGD Framework

The result about the non-existence of unitary S-matrix for topology changing cobordisms allows new insights about the meaning of the departures of TGD from string models.

Cobordism cannot give interesting selection rules

When I started to work with TGD for more than 28 years ago, one of the first ideas was that one could identify the selection rules of quantum transitions as topological selection rules for cobordisms. Within week or two came the great disappointment: there were practically no selection rules. Could one revive this naïve idea? Could the existence of unitary S-matrix force the topological selection rules after all? I am skeptic. If I have understood correctly the discussion of what happens in 4-D case [A50] only the exotic diffeo-structures modify the situation in 4-D case.

Light-like 3-surfaces allow cobordism

In the physically interesting GRT like situation one would expect the cobordism to be mediated by a space-time surface possessing Lorentz signature. This brings in metric and temporal distance. This means complications since one must leave the pure TQFT context. Also the classical dynamics of quantum gravitation brings in strong selection rules related to the dynamics in metric degrees of freedom so that TQFT approach is not expected to be useful from the point of view of quantum gravity and certainly not the limit of a realistic theory of quantum gravitation.

In TGD framework situation is different. 4-D space-time sheets can have Euclidian signature of the induced metric so that Lorentz signature does not pose conditions. The counterparts of cobordisms correspond at fundamental level to light-like 3-surfaces, which are arbitrarily except for the light-likeness condition (the effective 2-dimensionality implies generalized conformal invariance and analogy with 3-D black-holes since 3-D vacuum Einstein equations are satisfied). Field equations defined by the Chern-Simons action imply that CP_2 projection is at most 2-D but this condition holds true only for the extremals and one has functional integral over all light-like 3-surfaces. The temporal distance between points along light-like 3-surface vanishes. The constraints from light-likeness bring in metric degrees of freedom but in a very gentle manner and just to make the theory physically interesting.

Feynman cobordism as opposed to ordinary cobordism

In string model context the discouraging results from TQFT hold true in the category of nCob, which corresponds to trouser diagrams for closed strings or for their open string counterparts. In TGD framework these diagrams are replaced with a direct generalization of Feynman diagrams for which 3-D light-like partonic 3-surfaces meet along their 2-D ends at the vertices. In honor of Feynman one could perhaps speak of Feynman cobordisms. These surfaces are singular as 3-manifolds but vertices are nice 2-manifolds. I contrast to this, in string models diagrams are nice 2-manifolds but vertices are singular as 1-manifolds (say eye-glass type configurations for closed strings).

This picture gains a strong support for the interpretation of fermions as light-like throats associated with connected sums of CP_2 type extremals with space-time sheets with Minkowski signature and of bosons as pairs of light-like wormhole throats associated with CP_2 type extremal connecting two space-time sheets with Minkowski signature of induced metric. The space-time sheets have opposite time orientations so that also zero energy ontology emerges unavoidably. There is also consistency TGD based explanation of the family replication phenomenon in terms of genus of light-like partonic 2-surfaces.

One can wonder what the 4-D space-time sheets associated with the generalized Feynman diagrams could look like? One can try to gain some idea about this by trying to assign 2-D surfaces to ordinary Feynman diagrams having a subset of lines as boundaries. In the case of $2 \rightarrow 2$ reaction open string is pinched to a point at vertex. $1 \rightarrow 2$ vertex, and quite generally, vertices with odd number of lines, are impossible. The reason is that 1-D manifolds of finite size can have either 0 or 2 ends whereas in higher-D the number of boundary components is arbitrary. What one expects to happen in TGD context is that wormhole throats which are at distance characterized by CP_2 fuse together in the vertex so that some kind of pinches appear also now.

Zero energy ontology

Zero energy ontology gives rise to a second profound distinction between TGD and standard QFT. Physical states are identified as states with vanishing net quantum numbers, in particular energy. Everything is creatable from vacuum - and one could add- by intentional action so that zero energy ontology is profoundly Eastern. Positive *resp.* negative energy parts of states can be identified as states associated with 2-D partonic surfaces at the boundaries of future *resp.* past directed light-cones, whose tips correspond to the arguments of n-point functions. Each incoming/outgoing particle would define a mini-cosmology corresponding to not so big bang/crunch. If the time scale of perception is much shorter than time interval between positive and zero energy states, the ontology looks like the Western positive energy ontology. Bras and kets correspond naturally to the positive and negative energy states and phase conjugation for laser photons making them indeed something which seems to travel in opposite time direction is counterpart for bra-ket duality.

The new element would be quantum measurements performed separately for observables assignable to positive and negative energy states. These measurements would be characterized in terms of Jones inclusions. The state function reduction for the negative energy states could be interpreted as a detection of a particle reaction.

Finite temperature S-matrix defines genuine quantum state in zero energy ontology

In TGD framework one encounters two S-matrix like operators.

1. U-matrix is the analog of the ordinary S-matrix and constructible in terms of it and orthonormal basis of square roots of density matrices expressible as products of hermitian operators multiplied by unitary S-matrix [K48].

2. The S-matrix like operator describing what happens in laboratory corresponds to the time-like entanglement coefficients between positive and negative energy parts of the state. Measurement of reaction rates would be a measurement of observables reducing time like entanglement and very much analogous to an ordinary quantum measurement reducing space-like entanglement. There is a finite measurement resolution described by inclusion of HFFs and this means that situation reduces effectively to a finite-dimensional one.

p-Adic thermodynamics strengthened with p-adic length scale hypothesis predicts particle masses with an amazing success. At first the thermodynamical approach seems to be in contradiction with the idea that elementary particles are quantal objects. Unitarity is however *not* necessary if one accepts that only relative probabilities for reductions to pairs of initial and final states interpreted as particle reactions can be measured.

The beneficial implications of unitarity are not lost if one replaces QFT with thermal QFT. Category theoretically this would mean that the time-like entanglement matrix associated with the product of cobordisms is a product of these matrices for the factors. The time parameter in S-matrix would be replaced with a complex time parameter with the imaginary part identified as inverse temperature. Hence the interpretation in terms of time evolution is not lost.

In the theory of hyper-finite factors of type III_1 the partition function for thermal equilibrium states and S-matrix can be neatly fused to a thermal S-matrix for zero energy states and one could introduce p-adic thermodynamics at the level of quantum states. It seems that this picture applies to HFFs by restriction. Therefore the loss of unitarity S-matrix might after all turn to a victory by more or less forcing both zero energy ontology and p-adic thermodynamics. Note that also the presence of factor of type I coming from embedding space degrees of freedom forces thermal S-matrix.

Time-like entanglement coefficients as a square root of density matrix?

All quantum states do not correspond to thermal states and one can wonder what might be the most general identification of the quantum state in zero energy ontology. Density matrix formalism defines a very general formulation of quantum theory. Since the quantum states in zero energy ontology are analogous to operators, the idea that time-like entanglement coefficients in some sense define a square root of density matrix is rather natural. This would give the defining conditions

$$\begin{aligned} \rho^+ &= SS^\dagger \ , \rho^- = S^\dagger S \ , \\ Tr(\rho^\pm) &= 1 \ . \end{aligned} \tag{8.2.1}$$

$\rho^\pm$ would define density matrix for positive/negative energy states. In the case HFFs of type II_1 one obtains unitary S-matrix and also the analogs of pure quantum states are possible for factors of type I. The numbers $p^+_{m,n} = |S^2_{m,n}|/\rho^+_{m,m}$ and $p^-_{m,n} = |S^2_{n,m}|/\rho^-_{m,m}$ give the counterparts of the usual scattering probabilities.

A physically well-motivated hypothesis would be that S has expression $S = \sqrt{\rho}S_0$ such that S_0 is a universal unitary S-matrix, and $\sqrt{\rho}$ is square root of a state dependent density matrix. Note that in general S is not diagonalizable in the algebraic extension involved so that it is not possible to reduce the scattering to a mere phase change by a suitable choice of state basis.

What makes this kind of hypothesis aesthetically attractive is the unification of two fundamental matrices of quantum theory to single one. This unification is completely analogous to the combination of modulus squared and phase of complex number to a single complex number: complex valued Schrödinger amplitude is replaced with operator valued one.

S-matrix as a functor and the groupoid structure formed by S-matrices

In zero energy ontology S-matrix can be seen as a functor from the category of Feynman cobordisms to the category of operators. S-matrix can be identified as a "square root" of the positive energy density matrix $S = \rho_+^{1/2}S_0$, where S_0 is a unitary matrix and ρ_+ is the density matrix for positive energy part of the zero energy state. Obviously one has $SS^\dagger = \rho_+$. $S^\dagger S = \rho_-$ gives the density matrix for negative energy part of zero energy state. Clearly, S-matrix can be seen as matrix valued generalization of Schrödinger amplitude. Note that the "indices" of the S-matrices correspond to WCW spinor s (fermions and their bound states giving rise to gauge bosons and gravitons) and to WCW degrees of freedom. For hyper-finite factor of II_1 it is not strictly speaking possible to speak about indices since the matrix elements are traces of the S-matrix multiplied by projection operators to infinite-dimensional subspaces from right and left.

The functor property of S-matrices implies that they form a multiplicative structure analogous but not identical to groupoid [A6]. Recall that groupoid has associative product and there exist always right and left inverses and identity in the sense that ff^{-1} and $f^{-1}f$ are always defined but not identical and one has $fgg^{-1} = f$ and $f^{-1}fg = g$.

The reason for the groupoid like property is that S-matrix is a map between state spaces associated with initial and final sets of partonic surfaces and these state spaces are different so that inverse must be replaced with right and left inverse. The defining conditions for groupoid are replaced with more general ones. Also now associativity holds but the role of inverse is taken by hermitian conjugate. Thus one has the conditions $fgg^\dagger = f\rho_{g,+}$ and $f^\dagger fg = \rho_{f,-}g$, and the conditions $ff^\dagger = \rho_+$ and $f^\dagger f = \rho_-$ are satisfied. Here $\rho_\pm$ is density matrix associated with positive/negative energy parts of zero energy state. If the inverses of the density matrices exist, groupoid axioms hold true since $f_L^{-1} = f^\dagger\rho_{f,+}^{-1}$ satisfies $ff_L^{-1} = Id_+$ and $f_R^{-1} = \rho_{f,-}^{-1}f^\dagger$ satisfies $f_R^{-1}f = Id_-$.

There are good reasons to believe that also tensor product of its appropriate generalization to the analog of co-product makes sense with non-triviality characterizing the interaction between the systems of the tensor product. If so, the S-matrices would form very beautiful mathematical structure bringing in mind the corresponding structures for 2-tangles and N-tangles. Knowing how incredibly powerful the group like structures have been in physics one has good reasons to hope that groupoid like structure might help to deduce a lot of information about the quantum dynamics of TGD.

A word about nomenclature is in order. S has strong associations to unitarity and it might be appropriate to replace S with some other letter. The interpretation of S-matrix as a generalized Schrödinger amplitude would suggest Ψ-matrix. Since the interaction with Kea's M-theory blog at (see `http://tinyurl.com/yb3lsbjq` (M denotes Monad or Motif in this context) was led ot the realization of the connection with density matrix, also M-matrix might be considered. S-matrix as a functor from the category of Feynman cobordisms in turn suggests C or F. Or could just Matrix denoted by M in formulas be enough? Certainly it would inspire feeling of awe!

8.3 Further Ideas

The work of John Baez and students has inspired also the following ideas about the role of category theory in TGD.

8.3.1 Operads, Number Theoretical Braids, And Inclusions Of HFFs

The description of braids leads naturally to category theory and quantum groups when the braiding operation, which can be regarded as a functor, is not a mere permutation. Discreteness is a natural notion in the category theoretical context. To me the most natural manner to interpret discreteness is - not something emerging in Planck scale- but as a correlate for a finite measurement resolution and quantum measurement theory with finite measurement resolution leads naturally to number theoretical braids as fundamental discrete structures so that category theoretic approach becomes well-motivated. Discreteness is also implied by the number theoretic approach to quantum TGD from number theoretic associativity condition [L2] central also for category theoretical thinking as well as from the realization of number theoretical universality by the fusion of real and p-adic physics to single coherent whole.

Operads are formally single object multi-categories [A12, A112]. This object consist of an infinite sequence of sets of n-ary operations. These operations can be composed and the compositions are associative (operations themselves need not be associative) in the sense that the is natural isomorphism (symmetries) mapping differently bracketed compositions to each other. The coherence laws for operads formulate the effect of permutations and bracketing (association) as functors acting as natural isomorphisms. A simple manner to visualize the composition is as an addition of $n_1, ...n_k$ leaves to the leaves $1, ..., k$ of k-leaved tree.

An interesting example of operad is the braid operad formulating the combinatorics for a hierarchy of braids formed from braids by grouping subsets of braids having $n_1, ...n_k$ strands and defining the strands of a k-braid. In TGD framework this grouping can be identified in terms of the formation bound states of particles topologically condensed at larger space-time sheet and coherence laws allow to deduce information about scattering amplitudes. In conformal theories braided categories indeed allow to understand duality of stringy amplitudes in terms of associativity condition.

Planar operads [A39] define an especially interesting class of operads. The reason is that the inclusions of HFFs give rise to a special kind of planar operad [A15]. The object of this multi-category [A11] consists of planar k-tangles. Planar operads are accompanied by planar algebras. It will be found that planar operads allow a generalization which could provide a description for the combinatorics of the generalized Feynman diagrams and also rigorous formulation for how the arrow of time emerges in TGD framework and related heuristic ideas challenging the standard views.

8.3.2 Generalized Feynman Diagram As Category?

John Baez has proposed a category theoretical formulation of quantum field theory as a functor from the category of n-cobordisms to the category of Hilbert spaces [A94, A38]. The attempt to generalize this formulation looks well motivated in TGD framework because TGD can be regarded as almost topological quantum field theory in a well defined sense and braids appear as fundamental structures. It however seems that formulation as a functor from nCob to Hilb is not general enough.

In zero energy ontology events of ordinary ontology become quantum states with positive and negative energy parts of quantum states localizable to the upper and lower light-like boundaries of causal diamond (CD).

1. Generalized Feynman diagrams associated with a given CD involve quantum superposition of light-like 3-surfaces corresponding to given generalized Feynman diagram. These superpositions could be seen as categories with 3-D light-like surfaces containing braids as arrows and 2-D vertices as objects. Zero energy states would represent quantum superposition of categories (different topologies of generalized Feynman diagram) and M-matrix defined as Connes tensor product would define a functor from this category to the Hilbert space of zero energy states for given CD (tensor product defines quite generally a functor).

2. What is new from the point of view of physics that the sequences of generalized lines would define compositions of arrows and morphisms having identification in terms of braids which replicate in vertices. The possible interpretation of the replication is in terms of copying of information in classical sense so that even elementary particles would be information carrying and processing structures. This structure would be more general than the proposal of John

Baez that S-matrix corresponds to a function from the category of n-dimensional cobordisms to the category Hilb.

3. p-Adic length scale hypothesis follows if the temporal distance between the tips of CD measured as light-cone proper time comes as an octave of CP_2 time scale: $T = 2^nT_0$. This assumption implies that the p-adic length scale resolution interpreted in terms of a hierarchy of increasing measurement resolutions comes as octaves of time scale. A weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs.

This preliminary picture is of course not far complete since it applies only to single CD. There are several questions. Can one allow CDs within CDs and is every vertex of generalized Feynman diagram surrounded by this kind of CD. Can one form unions of CDs freely?

1. Since light-like 3-surfaces in 8-D embedding space have no intersections in the generic position, one could argue that the overlap must be allowed and makes possible the interaction of between zero energy states belonging to different CDs. This interaction would be something new and present also for sub-CDs of a given CD.

2. The simplest guess is that the unrestricted union of CDs defines the counterpart of tensor product at geometric level and that extended M-matrix is a functor from this category to the tensor product of zero energy state spaces. For non-overlapping CDs ordinary tensor product could be in question and for overlapping CDs tensor product would be non-trivial. One could interpret this M-matrix as an arrow between M-matrices of zero energy states at different CDs: the analog of natural transformation mapping two functors to each other. This hierarchy could be continued ad infinitum and would correspond to the hierarchy of n-categories.

This rough heuristics represents of course only one possibility among many since the notion of category is extremely general and the only limits are posed by the imagination of the mathematician. Also the view about zero energy states is still rather primitive.

8.4 Planar Operads, The Notion Of Finite Measurement Resolution, And Arrow Of Geometric Time

In the sequel the idea that planar operads or their appropriate generalization might allow to formulate generalized Feynman diagrammatics in zero energy ontology will be considered. Also a description of measurement resolution and arrow of geometric time in terms of operads is discussed.

8.4.1 Zeroth Order Heuristics About Zero Energy States

Consider now the existing heuristic picture about the zero energy states and coupling constant evolution provided by CDs.

1. The tentative description for the increase of the measurement resolution in terms CDs is that one inserts to the upper and/or lower light-like boundary of CD smaller CDs by gluing them along light-like radial ray from the tip of CD. It is also possible that the vertices of generalized Feynman diagrams belong inside smaller CD: s and it turns out that these CD: s must be allowed.

2. The considerations related to the arrow of geometric time suggest that there is asymmetry between upper and lower boundaries of CD. The minimum requirement is that the measurement resolution is better at upper light-like boundary.

3. In zero energy ontology communications to the direction of geometric past are possible and phase conjugate laser photons represent one example of this.

4. Second law of thermodynamics must be generalized in such a way that it holds with respect to subjective time identified as sequence of quantum jumps. The arrow of geometric time can however vary so that apparent breaking of second law is possible in shorter time scales at least. One must however understand why second law holds true in so good an approximation.

5. One must understand also why the contents of sensory experience is concentrated around a narrow time interval whereas the time scale of memories and anticipation are much longer. The proposed mechanism is that the resolution of conscious experience is higher at the upper boundary of CD. Since zero energy states correspond to light-like 3-surfaces, this could be a result of self-organization rather than a fundamental physical law.

   (a) CDs define the perceptive field for self. Selves are curious about the space-time sheets outside their perceptive field in the geometric future of the embedding space and perform quantum jumps tending to shift the superposition of the space-time sheets to the direction of geometric past (past defined as the direction of shift!). This creates the illusion that there is a time=snapshot front of consciousness moving to geometric future in fixed background space-time as an analog of train illusion.

   (b) The fact that news come from the upper boundary of CD implies that self concentrates its attention to this region and improves the resolutions of sensory experience and quantum measurement here. The sub-CD: s generated in this manner correspond to mental images with contents about this region. As a consequence, the contents of conscious experience, in particular sensory experience, tend to be about the region near the upper boundary.

   (c) This mechanism in principle allows the arrow of the geometric time to vary and depend on p-adic length scale and the level of dark matter hierarchy. The occurrence of phase transitions forcing the arrow of geometric time to be same everywhere are however plausible for the reason that the lower and upper boundaries of given CD must possess the same arrow of geometric time.

   (d) If this is the mechanism behind the arrow of time, planar operads can provide a description of the arrow of time but not its explanation.

This picture is certainly not general enough, can be wrong at the level of details, and at best relates to the whole like single particle wave mechanics to quantum field theory.

8.4.2 Planar Operads

The geometric definition of planar operads [A16, A12, A15, A39] without using the category theoretical jargon goes as follows.

1. There is an external disk and some internal disks and a collection of disjoint lines connecting disk boundaries.

2. To each disk one attaches a non-negative integer k, called the color of disk. The disk with color k has k points at each boundary with the labeling $1, 2, ...k$ running clockwise and starting from a distinguished marked point, decorated by "*". A more restrictive definition is that disk colors are correspond to even numbers so that there are $k = 2n$ points lines leaving the disk boundary boundary. The planar tangles with $k = 2n$ correspond to inclusions of HFFs.

3. Each curve is either closed (no common points with disk boundaries) or joins a marked point to another marked point. Each marked point is the end point of exactly one curve.

4. The picture is planar meaning that the curves cannot intersect and diks cannot overlap.

5. Disks differing by isotopies preserving *'s are equivalent.

Given a planar k-tangle-one of whose internal disks has color k_i- and a k_i-tangle S, one can define the tangle $T \circ_i S$ by isotoping S so that its boundary, together with the marked points and the *'s co-incides with that of D_i and after that erase the boundary of D_i. The collection of planar tangle together with the composition defined in this manner- is called the colored operad of planar tangles.

One can consider also generalizations of planar operads.

1. The composition law is not affected if the lines of operads branch outside the disks. Branching could be allowed even at the boundaries of the disks although this does not correspond to a generic situation. One might call these operads branched operads.

2. The composition law could be generalized to allow additional lines connecting the points at the boundary of the added disk so that each composition would bring in something genuinely new. Zero energy insertion could correspond to this kind of insertions.

3. TGD picture suggests also the replacement of lines with braids. In category theoretical terms this means that besides association one allows also permutations of the points at the boundaries of the disks.

The question is whether planar operads or their appropriate generalizations could allow a characterization of the generalized Feynman diagrams representing the combinatorics of zero energy states in zero energy ontology and whether also the emergence of arrow of time could be described (but probably not explained) in this framework.

8.4.3 Planar Operads And Zero Energy States

Are planar operads sufficiently powerful to code the vision about the geometric correlates for the increase of the measurement resolution and coupling constant evolution formulated in terms of CDs? Or perhaps more realistically, could one improve this formulation by assuming that zero energy states correspond to wave functions in the space of planar tangles or of appropriate modifications of them? It seems that the answer to the first question is almost affirmative.

1. Disks are analogous to the white regions of a map whose details are not visible in the measurement resolution used. Disks correspond to causal diamonds (CDs) in zero energy ontology. Physically the white regions relate to the vertices of the generalized Feynman diagrams and possibly also to the initial and final states (strictly speaking, the initial and final states correspond to the legs of generalized Feynman diagrams rather than their ends).

2. The composition of tangles means addition of previously unknown details to a given white region of the map and thus to an increase of the measurement resolution. This conforms with the interpretation of inclusions of HFFs as a characterization of finite measurement resolution and raises the hope that planar operads or their appropriate generalization could provide the proper language to describe coupling constant evolution and their perhaps even generalized Feynman diagrams.

3. For planar operad there is an asymmetry between the outer disk and inner disks. One might hope that this asymmetry could explain or at least allow to describe the arrow of time. This is not the case. If the disks correspond to causal diamonds (CDs) carrying positive *resp.* negative energy part of zero energy state at upper *resp.* lower light-cone boundary, the TGD counterpart of the planar tangle is CD containing smaller CD: s inside it. The smaller CD: s contain negative energy particles at their upper boundary and positive energy particles at their lower boundary. In the ideal resolution vertices represented 2-dimensional partonic at which light-like 3-surfaces meet become visible. There is no inherent asymmetry between positive and negative energies and no inherent arrow of geometric time at the fundamental level. It is however possible to model the arrow of time by the distribution of sub-CD: s. By previous arguments self-organization of selves can lead to zero energy states for which the measurement resolution is better near the upper boundary of the CD.

4. If the lines carry fermion or anti-fermion number, the number of lines entering to a given CD must be even as in the case of planar operads as the following argument shows.

(a) In TGD framework elementary fermions correspond to single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat.

(b) Elementary bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets of opposite time orientation and modellable as a piece of CP_2 type extremal. Each boson therefore corresponds to 2 lines within CP_2 radius.

(c) As a consequence the total number of lines associated with given CD is even and the generalized Feynman diagrams can correspond to a planar algebra associated with an inclusion of HFFs.

5. This picture does not yet describe zero energy insertions.

   (a) The addition of zero energy insertions corresponds intuitively to the allowance of new lines inside the smaller CD: s not coming from the exterior. The addition of lines connecting points at the boundary of disk is possible without losing the basic geometric composition of operads. In particular one does not lose the possibility to color the added tangle using two colors (colors correspond to two groups G and H which characterize an inclusion of HFFs [A39]).

   (b) There is however a problem. One cannot remove the boundaries of sub-CD after the composition of CDs since this would give lines beginning from and ending to the interior of disk and they are invisible only in the original resolution. Physically this is of course what one wants but the inclusion of planar tangles is expected to fail in its original form, and one must generalize the composition of tangles to that of CD: s so that the boundaries of sub-CD: s are not thrown away in the process.

   (c) It is easy to see that zero energy insertions are inconsistent with the composition of planar tangles. In the inclusion defining the composition of tangles both sub-tangle and tangle induce a color to a given segment of the inner disk. If these colors are identical, one can forget the presence of the boundary of the added tangle. When zero energy insertions are allowed, situation changes as is easy to see by adding a line connecting points in a segment of given color at the boundary of the included tangle. There exists no consistent coloring of the resulting structure by using only two colors. Coloring is however possible using four colors, which by four-color theorem is the minimum number of colors needed for a coloring of planar map: this however requires that the color can change as one moves through the boundary of the included disk - this is in accordance with the physical picture.

   (d) Physical intuition suggests that zero energy insertion as an improvement of measurement resolution maps to an improved color resolution and that the composition of tangles generalizes by requiring that the included disk is colored by using new nuances of the original colors. The role of groups in the definition of inclusions of HFFs is consistent with idea that G and H describe color resolution in the sense that the colors obtained by their action cannot be resolved. If so, the improved resolution means that G and H are replaced by their subgroups $G_1 \subset G$ and $H_1 \subset H$. Since the elements of a subgroup have interpretation as elements of group, there are good hopes that by representing the inclusion of tangles as inclusion of groups, one can generalize the composition of tangles.

6. Also CD: s glued along light-like ray to the upper and lower boundaries of CD are possible in principle and -according the original proposal- correspond to zero energy insertions according. These CD: s might be associated with the phase transitions changing the value of $\hbar$ leading to different pages of the book like structure defined by the generalized embedding space.

7. p-Adic length scale hypothesis is realized if the hierarchy of CDs corresponds to a hierarchy of temporal distances between tips of CDs given as $a = T_n = 2^{-n}T_0$ using light-cone proper time.

8. How this description relates to braiding? Each line corresponds to an orbit of a partonic boundary component and in principle one must allow internal states containing arbitrarily high fermion and anti-fermion numbers. Thus the lines decompose into braids and one must allow also braids of braids hierarchy so that each line corresponds to a braid operad in improved resolution.

8.4.4 Relationship To Ordinary Feynman Diagrammatics

The proposed description is not equivalent with the description based on ordinary Feynman diagrams.

1. In standard physics framework the resolution scale at the level of vertices of Feynman diagrams is something which one is forced to pose in practical calculations but cannot pose at will as opposed to the measurement resolution. Light-like 3-surfaces can be however regarded only locally orbits of partonic 2-surfaces since generalized conformal invariance is true only in 3-D patches of the light-like 3-surface. This means that light-like 3-surfaces are in principle the fundamental objects so that zero energy states can be regarded only locally as a time evolutions. Therefore measurement resolution can be applied also to the distances between vertices of generalized Feynman diagrams and calculational resolution corresponds to physical resolution. Also the resolution can be better towards upper boundary of CD so that the arrow of geometric time can be understood. This is a definite prediction which can in principle kill the proposed scenario.

2. A further counter argument is that generalized Feynman diagrams are identified as light-like 3-surfaces for which Kähler function defined by a preferred extremal of Kähler action is maximum. Therefore one cannot pose any ad hoc rules on the positions of the vertices. One can of course insist that maximum of Kähler function with the constraint posed by $T_n = 2^n T_0$ (or $T_p = p^n T_0$) hierarchy is in question.

It would be too optimistic to believe that the details of the proposal are correct. However, if the proposal is on correct track, zero energy states could be seen as wave functions in the operad of generalized tangles (zero energy insertions and braiding) as far as combinatorics is involved and the coherence rules for these operads would give strong constraints on the zero energy state and fix the general structure of coupling constant evolution.

8.5 Category Theory And Symplectic QFT

Besides the counterpart of the ordinary Kac-Moody invariance quantum TGD possesses so called super-symplectic conformal invariance. This symmetry leads to the proposal that a symplectic variant of conformal field theory should exist. The n-point functions of this theory defined in S^2 should be expressible in terms of symplectic areas of triangles assignable to a set of n-points and satisfy the duality rules of conformal field theories guaranteeing associativity. The crucial prediction is that symplectic n-point functions vanish whenever two arguments co-incide. This provides a mechanism guaranteeing the finiteness of quantum TGD implied by very general arguments relying on non-locality of the theory at the level of 3-D surfaces.

The classical picture suggests that the generators of the fusion algebra formed by fields at different point of S^2 have this point as a continuous index. Finite quantum measurement resolution and category theoretic thinking in turn suggest that only the points of S^2 corresponding the strands of number theoretic braids are involved. It turns out that the category theoretic option works and leads to an explicit hierarchy of fusion algebras forming a good candidate for so called little disk operad whereas the first option has difficulties.

8.5.1 Fusion Rules

Symplectic fusion rules are non-local and express the product of fields at two points s_k an s_l of S^2 as an integral over fields at point s_r, where integral can be taken over entire S^2 or possibly also over a 1-D curve which is symplectic invariant in some sense. Also discretized version of fusion rules makes sense and is expected serve as a correlate for finite measurement resolution.

By using the fusion rules one can reduce n-point functions to convolutions of 3-point functions involving a sequence of triangles such that two subsequent triangles have one vertex in common. For instance, 4-point function reduces to an expression in which one integrates over the positions of the common vertex of two triangles whose other vertices have fixed. For n-point functions one has n-3 freely varying intermediate points in the representation in terms of 3-point functions.

The application of fusion rules assigns to a line segment connecting the two points s_k and s_l a triangle spanned by s_k, s_l and s_r. This triangle should be symplectic invariant in some sense and its symplectic area A_{klm} would define the basic variable in terms of which the fusion rule could be expressed as $C_{klm} = f(A_{klm})$, where f is fixed by some constraints. Note that A_{klm} has also interpretations as solid angle and magnetic flux.

8.5.2 What Conditions Could Fix The Symplectic Triangles?

The basic question is how to identify the symplectic triangles. The basic criterion is certainly the symplectic invariance: if one has found N-D symplectic algebra, symplectic transformations of S^2 must provide a new one. This is guaranteed if the areas of the symplectic triangles remain invariant under symplectic transformations. The questions are how to realize this condition and whether it might be replaced with a weaker one. There are two approaches to the problem.

Physics inspired approach

In the first approach inspired by classical physics symplectic invariance for the edges is interpreted in the sense that they correspond to the orbits of a charged particle in a magnetic field defined by the Kähler form. Symplectic transformation induces only a $U(1)$ gauge transformation and leaves the orbit of the charged particle invariant if the vertices are not affected since symplectic transformations are not allowed to act on the orbit directly in this approach. The general functional form of the structure constants C_{klm} as a function $f(A_{klm})$ of the symplectic area should guarantee fusion rules.

If the action of the symplectic transformations does not affect the areas of the symplectic triangles, the construction is invariant under general symplectic transformations. In the case of uncharged particle this is not the case since the edges are pieces of geodesics: in this case however fusion algebra however trivializes so that one cannot conclude anything. In the case of charged particle one might hope that the area remains invariant under general symplectic transformations whose action is induced from the action on vertices. The equations of motion for a charged particle involve a Kähler metric determined by the symplectic structure and one might hope that this is enough to achieve this miracle. If this is not the case - as it might well be - one might hope that although the areas of the triangles are not preserved, the triangles are mapped to each other in such a way that the fusion algebra rules remain intact with a proper choice of the function $f(A_{klm})$. One could also consider the possibility that the function $f(A_{klm})$ is dictated from the condition that the it remains invariant under symplectic transformations. It however turns that this approach does not work as such.

Category theoretical approach

The second realization is guided by the basic idea of category theoretic thinking: the properties of an object are determined its relationships to other objects. Rather than postulating that the symplectic triangle is something which depends solely on the three points involved via some geometric notion like that of geodesic line of orbit of charged particle in magnetic field, one assumes that the symplectic triangle reflects the properties of the fusion algebra, that is the relations of the symplectic triangle to other symplectic triangles. Thus one must assign to each triplet (s_1, s_2, s_3) of points of S^2 a triangle just from the requirement that braided associativity holds true for the fusion algebra.

All symplectic transformations leaving the N points fixed and thus generated by Hamiltonians vanishing at these points would give new gauge equivalent realizations of the fusion algebra and deform the edges of the symplectic triangles without affecting their area. One could even say that symplectic triangulation defines a new kind geometric structure in S^2. The quantum fluctuating degrees of freedom are parameterized by the symplectic group of $S^2 \times CP_2$ in TGD so that symplectic the geometric representation of the triangulation changes but its inherent properties remain invariant.

The elegant feature of category theoretical approach is that one can in principle construct the fusion algebra without any reference to its geometric realization just from the braided associativity and nilpotency conditions and after that search for the geometric realizations. Fusion algebra has also a hierarchy of discrete variants in which the integral over intermediate points in fusion is replaced by a sum over a fixed discrete set of points and this variant is what finite measurement resolution implies. In this case it is relatively easy to see if the geometric realization of a given abstract fusion algebra is possible.

The notion of number theoretical braid

Braids -not necessary number theoretical- provide a realization discretization as a space-time correlate for the finite measurement resolution. The notion of braid was inspired by the idea about quantum TGD as almost topological quantum field theory. Although the original form of this idea has been buried, the notion of braid has survived: in the decomposition of space-time sheets to string world sheets, the ends of strings define representatives for braid strands at light-like 3-surfaces.

The notion of number theoretic universality inspired the much more restrictive notion of number theoretic braid requiring that the points in the intersection of the braid with the partonic 2-surface correspond to rational or at most algebraic points of H in preferred coordinates fixed by symmetry considerations. The challenge has been to find a unique identification of the number theoretic braid or at least of the end points of the braid. The following consideration suggest that the number theoretic braids are not a useful notion in the generic case but make sense and are needed in the intersection of real and p-adic worlds which is in crucial role in TGD based vision about living matter [K45].

It is only the braiding that matters in topological quantum field theories used to classify braids. Hence braid should require only the fixing of the end points of the braids at the intersection of the braid at the light-like boundaries of CDs and the braiding equivalence class of the braid itself. Therefore it is enough is to specify the topology of the braid and the end points of the braid in accordance with the attribute "number theoretic". Of course, the condition that all points of the strand of the number theoretic braid are algebraic is impossible to satisfy.

The situation in which the equations defining X^2 make sense both in real sense and p-adic sense using appropriate algebraic extension of p-adic number field is central in the TGD based vision about living matter [K45]. The reason is that in this case the notion of number entanglement theoretic entropy having negative values makes sense and entanglement becomes information carrying. This motivates the identification of life as something in the intersection of real and p-adic worlds. In this situation the identification of the ends of the number theoretic braid as points belonging to the intersection of real and p-adic worlds is natural. These points -call them briefly algebraic points- belong to the algebraic extension of rationals needed to define the algebraic extension of p-adic numbers. This definition however makes sense also when the equations defining the partonic 2-surfaces fail to make sense in both real and p-adic sense. In the generic case the set of points satisfying the conditions is discrete. For instance, according to Fermat's theorem the set of rational points satisfying $X^n + Y^n = Z^n$ reduces to the point $(0, 0, 0)$ for $n = 3, 4, ...$ Hence the constraint might be quite enough in the intersection of real and p-adic worlds where the choice of the algebraic extension is unique.

One can however criticize this proposal.

1. One must fix the number of points of the braid and outside the intersection and the non-uniquencess of the algebraic extension makes the situation problematic. Physical intuition suggests that the points of braid define carriers of quantum numbers assignable to second quantized induced spinor fields so that the total number of fermions anti-fermions would define the number of braids. In the intersection the highly non-trivial implication is that this number cannot exceed the number of algebraic points.

2. In the generic case one expects that even the smallest deformation of the partonic 2-surface can change the number of algebraic points and also the character of the algebraic extension of rational numbers needed. The restriction to rational points is not expected to help in the generic case. If the notion of number theoretical braid is meant to be practical, must be able to decompose WCW to open sets inside which the numbers of algebraic points of braid at its ends are constant. For real topology this is expected to be impossible and it does not make sense to use p-adic topology for WCW whose points do not allow interpretation as p-adic partonic surfaces.

3. In the intersection of real and p-adic worlds which corresponds to a discrete subset of WCW , the situation is different. Since the coefficients of polynomials involved with the definition of the partonic 2-surface must be rational or at most algebraic, continuous deformations are not possible so that one avoids the problem.

4. This forces to ask the reason why for the number theoretic braids. In the generic case they seem to produce only troubles. In the intersection of real and p-adic worlds they could however allow the construction of the elements of M-matrix describing quantum transitions changing p-adic to real surfaces and vice versa as realizations of intentions and generation of cognitions. In this the case it is natural that only the data from the intersection of the two worlds are used. In [K45] I have sketched the idea about number theoretic quantum field theory as a description of intentional action and cognition.

There is also the problem of fixing the interior points of the braid modulo deformations not affecting the topology of the braid.

1. Infinite number of non-equivalent braidings are possible. Should one allow all possible braidings for a fixed light-like 3-surface and say that their existence is what makes the dynamics essentially three-dimensional even in the topological sense? In this case there would be no problems with the condition that the points at both ends of braid are algebraic.

2. Or should one try to characterize the braiding uniquely for a given partonic 2-surfaces and corresponding 4-D tangent space distributions? The slicing of the space-time sheet by partonic 2-surfaces and string word sheets suggests that the ends of string world sheets could define the braid strands in the generic context when there is no algebraicity condition involved. This could be taken as a very natural manner to fix the topology of braid but leave the freedom to choose the representative for the braid. In the intersection of real and p-adic worlds there is no good reason for the end points of strands in this case to be algebraic at both ends of the string world sheet. One can however start from the braid defined by the end points of string world sheets, restrict the end points to be algebraic at the end with a smaller number of algebraic pointsandthen perform a topologically non-trivial deformation of the braid so that also the points at the other end are algebraic? Non-trivial deformations need not be possible for all possible choices of algebraic braid points at the other end of braid and different choices of the set of algebraic points would give rise to different braidings. A further constraint is that only the algebraic points at which one has assign fermion or anti-fermion are used so that the number of braid points is not always maximal.

3. One can also ask whether one should perform the gauge fixing for the strands of the number theoretic braid using algebraic functions making sense both in real and p-adic context. This question does not seem terribly relevant since since it is only the topology of the braid that matters.

Symplectic triangulations and braids

The identification of the edges of the symplectic triangulation as the end points of the braid is favored by conceptual economy. The nodes of the symplectic triangulation would naturally correspond to the points in the intersection of the braid with the light-like boundaries of CD carrying fermion or anti-fermion number. The number of these points could be arbitrarily large in the generic case but in the intersection of real and p-adic worlds these points correspond to subset of algebraic points belonging to the algebraic extension of rationals associated with the definition of partonic 2-surfaces so that the sum of fermion and anti-fermion numbers would be bounded above. The presence of fermions in the nodes would be the physical prerequisite for measuring the phase factors defined by the magnetic fluxes. This could be understood in terms of gauge invariance forcing to assign to a pair of points of triangulation the non-integrable phase factor defined by the Kähler gauge potential.

The remaining problem is how uniquely the edges of the triangulation can be determined.

1. The allowance of all possible choices for edges would bring in an infinite number of degrees of freedom. These curves would be analogous to freely vibrating strings. This option is not attractive. One should be able to pose conditions on edges and whatever the manner to specify the edges might be, it must make sense also in the intersection of real and p-adic worlds. In this case the total phase factor must be a root of unity in the algebraic extension of rationals involved and this poses quantization rules analogous to those for magnetic flux. The strongest condition is that the edges are such that the non-integrable phase factor is a root of unity for each edge. It will be found that similar quantization is implied also by the associativity conditions and this justifies the interpretation of phase factors defining the fusion algebra in terms of the Kähler magnetic fluxes. This would pose strong constraints on the choice of edges but would not fix completely the phase factors, and it seems that one must allow all possible triangulations consistent with this condition and the associativity conditions so that physical state is a quantum superposition over all possible symplectic triangulations characterized by the fusion algebras.

2. In the real context one would have an infinite hierarchy of symplectic triangulations and fusion algebras satisfying the associativity conditions with the number of edges equal to the total number N of fermions and anti-fermions. Encouragingly, this hierarchy corresponds also to a hierarchy of $\mathcal{N} = N$ SUSY algebras [?] (large values of $\mathcal{N}$ are not a catastrophe in TGD framework since the physical content of SUSY symmetry is not the same as that in the standard approach). In the intersection of real and p-adic worlds the value of $\mathcal{N}$ would be bounded by the total number of algebraic points. Hence the notion of finite measurement resolution, cutoff in $\mathcal{N}$ and bound on the total fermion number would make physics very simple in the intersection of real and p-adic worlds.

Two kinds of symplectic triangulations are possible since one can use the symplectic forms associated with CP_2 and $r_M = constant$ sphere S^2 of light-cone boundary. For a given collection of nodes the choices of edges could be different for these two kinds of triangulations. Physical state would be proportional to the product of the phase factors assigned to these triangulations.

8.5.3 Associativity Conditions And Braiding

The generalized fusion rules follow from the associativity condition for n-point functions modulo phase factor if one requires that the factor assignable to n-point function has interpretation as n-point function. Without this condition associativity would be trivially satisfied by using a product of various bracketing structures for the n fields appearing in the n-point function. In conformal field theories the phase factor defining the associator is expressible in terms of the phase factor associated with permutations represented as braidings and the same is expected to be true also now.

1. Already in the case of 4-point function there are three different choices corresponding to the 4 possibilities to connect the fixed points s_k and the varying point s_r by lines. The options are (1-2, 3-4), (1-3, 2-4), and (1-4, 2-3) and graphically they correspond to s-, t-, and u-channels in string diagrams satisfying also this kind of fusion rules. The basic condition would be that same amplitude results irrespective of the choice made. The duality conditions guarantee associativity in the formation of the n-point amplitudes without any further assumptions. The reason is that the writing explicitly the expression for a particular bracketing of n-point function always leads to some bracketing of one particular 4-point function and if duality conditions hold true, the associativity holds true in general. To be precise, in quantum theory associativity must hold true only in projective sense, that is only modulo a phase factor.

2. This framework encourages category theoretic approach. Besides different bracketing there are different permutations of the vertices of the triangle. These permutations can induce a phase factor to the amplitude so that braid group representations are enough. If one has representation for the basic braiding operation as a quantum phase $q = exp(i2\pi/N)$, the phase factors relating different bracketings reduce to a product of these phase factors since $(AB)C$ is obtained from $A(BC)$ by a cyclic permutation involving to permutations represented as a braiding. Yang-Baxter equations express the reduction of associator to braidings. In the general category theoretical setting associators and braidings correspond to natural isomorphisms leaving category theoretical structure invariant.

3. By combining the duality rules with the condition that 4-point amplitude vanishes, when any two points co-incide, one obtains from $s_k = s_l$ and $s_m = s_n$ the condition stating that the sum (or integral in possibly existing continuum version) of $U^2(A_{klm})|f|^2(x_{kmr})$ over the third point s_r vanishes. This requires that the phase factor U is non-trivial so that Q must be non-vanishing if one accepts the identification of the phase factor as Bohm-Aharonov phase.

4. Braiding operation gives naturally rise to a quantum phase. A good guess is that braiding operation maps triangle to its complement since only in this manner orientation is preserved so that area is A_{klm} is mapped to $A_{klm} - 4\pi$. If the f is proportional to the exponent $exp(-A_{klm}Q)$, braiding operation induces a complex phase factor $q = exp(-i4\pi Q)$.

5. For half-integer values of Q the algebra is commutative. For $Q = M/N$, where M and N have no common factors, only braided commutativity holds true for $N \geq 3$ just as for quantum groups characterizing also Jones inclusions of HFFs. For $N = 4$ anti-commutativity and associativity hold true. Charge fractionization would correspond to non-trivial braiding and presumably to non-standard values of Planck constant and coverings of M^4 or CP_2 depending on whether S^2 corresponds to a sphere of light-cone boundary or homologically trivial geodesic sphere of CP_2.

8.5.4 Finite-Dimensional Version Of The Fusion Algebra

Algebraic discretization due to a finite measurement resolution is an essential part of quantum TGD. In this kind of situation the symplectic fields would be defined in a discrete set of N points of S^2: natural candidates are subsets of points of p-adic variants of S^2. Rational variant of S^2 has as its points points for which trigonometric functions of θ and ϕ have rational values and there exists an entire hierarchy of algebraic extensions. The interpretation for the resulting breaking of the rotational symmetry would be a geometric correlate for the choice of quantization axes in quantum measurement and the book like structure of the embedding space would be direct correlate for this symmetry breaking. This approach gives strong support for the category theory inspired philosophy in which the symplectic triangles are dictated by fusion rules.

General observations about the finite-dimensional fusion algebra

1. In this kind of situation one has an algebraic structure with a finite number of field values with integration over intermediate points in fusion rules replaced with a sum. The most natural option is that the sum is over all points involved. Associativity conditions reduce in this case to conditions for a finite set of structure constants vanishing when two indices are identical. The number $M(N)$ of non-vanishing structure constants is obtained from the recursion formula $M(N) = (N-1)M(N-1)+(N-2)M(N-2)+...+3M(3) = NM(N-1)$, $M(3) = 1$ given $M(4) = 4$, $M(5) = 20$, $M(6) = 120$, ... With a proper choice of the set of points associativity might be achieved. The structure constants are necessarily complex so that also the complex conjugate of the algebra makes sense.

2. These algebras resemble nilpotent algebras ($x^n = 0$ for some n) and Grassmann algebras ($x^2 = 0$ always) in the sense that also the products of the generating elements satisfy $x^2 = 0$ as one can find by using duality conditions on the square of a product $x = yz$ of two generating elements. Also the products of more than N generating elements necessary vanish by braided commutativity so that nilpotency holds true. The interpretation in terms of measurement resolution is that partonic states and vertices can involve at most N fermions

in this measurement resolution. Elements anti-commute for $q=-1$ and commute for $q=1$ and the possibility to express the product of two generating elements as a sum of generating elements distinguishes these algebras from Grassman algebras. For $q=-1$ these algebras resemble Lie-algebras with the difference that associativity holds true in this particular case.

3. I have not been able to find whether this kind of hierarchy of algebras corresponds to some well-known algebraic structure with commutativity and associativity possibly replaced with their braided counterparts. Certainly these algebras would be category theoretical generalization of ordinary algebras for which commutativity and associativity hold true in strict sense.

4. One could forget the representation of structure constants in terms of triangles and think these algebras as abstract algebras. The defining equations are $x_i^2=0$ for generators plus braided commutativity and associativity. Probably there exists solutions to these conditions. One can also hope that one can construct braided algebras from commutative and associative algebras allowing matrix representations. Note that the solution the conditions allow scalings of form $C_{klm} \rightarrow \lambda_k\lambda_l\lambda_m C_{klm}$ as symmetries.

Formulation and explicit solution of duality conditions in terms of inner product

Duality conditions can be formulated in terms of an inner product in the function space associated with N points and this allows to find explicit solutions to the conditions.

1. The idea is to interpret the structure constants C_{klm} as wave functions C_{kl} in a discrete space consisting of N points with the standard inner product

$$\langle C_{kl}, C_{mn}\rangle = \sum_r C_{klr}\overline{C}_{mnr} \quad . \tag{8.5.1}$$

2. The associativity conditions for a trivial braiding can be written in terms of the inner product as

$$\langle C_{kl}, \overline{C}_{mn}\rangle = \langle C_{km}, \overline{C}_{ln}\rangle = \langle C_{kn}, \overline{C}_{ml}\rangle \quad . \tag{8.5.2}$$

3. Irrespective of whether the braiding is trivial or not, one obtains for $k=m$ the orthogonality conditions

$$\langle C_{kl}, \overline{C}_{kn}\rangle = 0 \quad . \tag{8.5.3}$$

For each k one has basis of $N-1$ wave functions labeled by $l \neq k$, and the conditions state that the elements of basis and conjugate basis are orthogonal so that conjugate basis is the dual of the basis. The condition that complex conjugation maps basis to a dual basis is very special and is expected to determine the structure constants highly uniquely.

4. One can also find explicit solutions to the conditions. The most obvious trial is based on orthogonality of function basis of circle providing representation for Z_{N-2} and is following:

$$C_{klm} = E_{klm} \times exp(i\phi_k + \phi_l + \phi_m) \quad , \quad \phi_m = \frac{n(m)2\pi}{N-2} \quad . \tag{8.5.4}$$

Here E_{klm} is non-vanishing only if the indices have different values. The ansatz reduces the conditions to the form

$$\sum_r E_{klr}E_{mnr}exp(i2\phi_r) = \sum_r E_{kmr}E_{lnr}exp(i2\phi_r) = \sum_r E_{knr}E_{mlr}exp(i2\phi_r) \quad . \tag{8.5.5}$$

In the case of braiding one can allow overall phase factors. Orthogonality conditions reduce to

$$\sum_r E_{klr}E_{knr}exp(i2\phi_r) = 0 \quad . \tag{8.5.6}$$

If the integers $n(m)$, $m \neq k, l$ span the range $(0, N-3)$ ortogonality conditions are satisfied if one has $E_{klr} = 1$ when the indices are different. This guarantees also duality conditions since the inner products involving k, l, m, n reduce to the same expression

$$\sum_{r \neq k,l,m,n} exp(i2\phi_r) \quad . \tag{8.5.7}$$

5. For a more general choice of phases the coefficients E_{klm} must have values differing from unity and it is not clear whether the duality conditions can be satisfied in this case.

Do fusion algebras form little disk operad?

The improvement of measurement resolution means that one adds further points to an existing set of points defining a discrete fusion algebra so that a small disk surrounding a point is replaced with a little disk containing several points. Hence the hierarchy of fusion algebras might be regarded as a realization of a little disk operad [A10] and there would be a hierarchy of homomorphisms of fusion algebras induced by the fusion. The inclusion homomorphism should map the algebra elements of the added points to the algebra element at the center of the little disk.

A more precise prescription goes as follows.

1. The replacement of a point with a collection of points in the little disk around it replaces the original algebra element ϕ_{k_0} by a number of new algebra elements ϕ_K besides already existing elements ϕ_k and brings in new structure constants C_{KLM}, C_{KLk} for $k \neq k_0$, and C_{Klm}.

2. The notion of improved measurement resolution allows to conclude

$$C_{KLk} = 0 \quad , \quad k \neq k_0 \quad , \quad C_{Klm} = C_{k_0lm} \quad . \tag{8.5.8}$$

3. In the homomorphism of new algebra to the original one the new algebra elements and their products should be mapped as follows:

$$\begin{aligned} &\phi_K \rightarrow \phi_{k_0} \quad , \\ &\phi_K\phi_L \rightarrow \phi_{k_0}^2 = 0 \quad , \quad \phi_K\phi_l \rightarrow \phi_{k_0}\phi_l \quad . \end{aligned} \tag{8.5.9}$$

Expressing the products in terms of structure constants gives the conditions

$$\sum_M C_{KLM} = 0 \quad , \quad \sum_r C_{Klr} = \sum_r C_{k_0lr} = 0 \quad . \tag{8.5.10}$$

The general ansatz for the structure constants based on roots of unity guarantees that the conditions hold true.

4. Note that the resulting algebra is more general than that given by the basic ansatz since the improvement of the measurement resolution at a given point can correspond to different value of N as that for the original algebra given by the basic ansatz. Therefore the original ansatz gives only the basic building bricks of more general fusion algebras. By repeated local improvements of the measurement resolution one obtains an infinite hierarchy of algebras labeled by trees in which each improvement of measurement resolution means the splitting of the branch with arbitrary number N of branches. The number of improvements of the measurement resolution defining the height of the tree is one invariant of these algebras. The fusion algebra operad has a fractal structure since each point can be replaced by any fusion algebra.

How to construct geometric representation of the discrete fusion algebra?

Assuming that solutions to the fusion conditions are found, one could try to find whether they allow geometric representations. Here the category theoretical philosophy shows its power.

1. Geometric representations for C_{klm} would result as functions $f(A_{klm})$ of the symplectic area for the symplectic triangles assignable to a set of N points of S^2.

2. If the symplectic triangles can be chosen freely apart from the area constraint as the category theoretic philosophy implies, it should be relatively easy to check whether the fusion conditions can be satisfied. The phases of C_{klm} dictate the areas A_{klm} rather uniquely if one uses Bohm-Aharonov ansatz for a fixed the value of Q. The selection of the points s_k would be rather free for phases near unity since the area of the symplectic triangle associated with a given triplet of points can be made arbitrarily small. Only for the phases far from unity the points s_k cannot be too close to each other unless Q is very large. The freedom to chose the points rather freely conforms with the general view about the finite measurement resolution as the origin of discretization.

3. The remaining conditions are on the moduli $|f(A_{klm})|$. In the discrete situation it is rather easy to satisfy the conditions just by fixing the values of f for the particular triangles involved: $|f(A_{klm})| = |C_{klm}|$. For the exact solution to the fusion conditions $|f(A_{klm})| = 1$ holds true.

4. Constraints on the functional form of $|f(A_{klm})|$ for a fixed value of Q can be deduced from the correlation between the modulus and phase of C_{klm} without any reference to geometric representations. For the exact solution of fusion conditions there is no correlation.

5. If the phase of C_{klm} has A_{klm} as its argument, the decomposition of the phase factor to a sum of phase factors means that the A_{klm} is sum of contributions labeled by the vertices. Also the symplectic area defined as a magnetic flux over the triangle is expressible as sum of the quantities $\int A_\mu dx^\mu$ associated with the edges of the triangle. These fluxes should correspond to the fluxes assigned to the vertices deduced from the phase factors of $\Psi(s_k)$. The fact that vertices are ordered suggest that the phase of $\Psi(s_j)$ fixes the value of $\int A_\mu dx^\mu$ for an edge of the triangle starting from s_k and ending to the next vertex in the ordering. One must find edges giving a closed triangle and this should be possible. The option for which edges correspond to geodesics or to solutions of equations of motion for a charged particle in magnetic field is not flexible enough to achieve this purpose.

6. The quantization of the phase angles as multiples of $2\pi/(N-2)$ in the case of N-dimensional fusion algebra has a beautiful geometric correlate as a quantization of symplecto-magnetic fluxes identifiable as symplectic areas of triangles defining solid angles as multiples of $2\pi/(N-2)$. The generalization of the fusion algebra to p-adic case exists if one allows algebraic extensions containing the phase factors involved. This requires the allowance of phase factors $exp(i2\pi/p)$, p a prime dividing $N-2$. Only the exponents $exp(i\int A_\mu dx^\mu) = exp(in2\pi/(N-2))$ exist p-adically. The p-adic counterpart of the curve defining the edge of triangle exists if the curve can be defined purely algebraically (say as a solution of polynomial equations with rational coefficients) so that p-adic variant of the curve satisfies same equations.

Does a generalization to the continuous case exist?

The idea that a continuous fusion algebra could result as a limit of its discrete version does not seem plausible. The reason is that the spatial variation of the phase of the structure constants increases as the spatial resolution increases so that the phases $exp(i\phi(s)$ cannot be continuous at continuum limit. Also the condition $E_{klm} = 1$ for $k \neq l \neq m$ satisfied by the explicit solutions to fusion rules fails to have direct generalization to continuum case.

To see whether the continuous variant of fusion algebra can exist, one can consider an approximate generalization of the explicit construction for the discrete version of the fusion algebra by the effective replacement of points s_k with small disks which are not allowed to intersect. This would mean that the counterpart $E(s_k, s_l, s_m)$ vanishes whenever the distance between two arguments is below a cutoff a small radius d. Puncturing corresponds physically to the cutoff implied by the finite measurement resolution.

1. The ansatz for C_{klm} is obtained by a direct generalization of the finite-dimensional ansatz:

$$C_{klm} = \kappa_{s_k,s_l,s_m}\Psi(s_k)\Psi(s_l)\Psi(s_m) \ . \tag{8.5.11}$$

   where κ_{s_k,s_l,s_m} vanishes whenever the distance of any two arguments is below the cutoff distance and is otherwise equal to 1.

2. Orthogonality conditions read as

$$\Psi(s_k)\Psi(s_l)\int \kappa_{s_k,s_l,s_r}\kappa_{s_k,s_n,s_r}\Psi^2(s_m)d\mu(s_r) = \Psi(s_k)\Psi(s_l)\int_{S^2(s_k,sl,s_n)}\Psi^2(s_r)d\mu(s_r) = 0 \tag{8.5.12}$$

   The resulting condition reads as

$$\int_{S^2(s_k,sl,s_n)}\Psi^2(s_r)d\mu(s_r) = 0 \tag{8.5.13}$$

   This condition holds true for any pair s_k, s_l and this might lead to difficulties.

3. The general duality conditions are formally satisfied since the expression for all fusion products reduces to

$$\begin{aligned}
& \Psi(s_k)\Psi(s_l)\Psi(s_m)\Psi(s_n)X \ , \\
X &= \int_{S^2}\kappa_{s_k,s_l,s_m,s_n}\Psi(s_r)d\mu(s_r) \\
&= \int_{S^2(s_k,s_l,s_m,s_n)}\Psi(s_m)d\mu(s_r) \\
&= -\int_{D^2(s_i)}\Psi^2(s_r)d\mu(s_r) \ , \ \ i=k,l,s,m \ .
\end{aligned} \tag{8.5.14}$$

   These conditions state that the integral of Ψ^2 any disk of fixed radius d is same: this result follows also from the orthogonality condition. This condition might be difficult to satisfy exactly and the notion of finite measurement resolution might be needed. For instance, it might be necessary to restrict the consideration to a discrete lattice of points which would lead back to a discretized version of algebra. Thus it seems that the continuum generalization of the proposed solution to fusion rules does not work.

8.6 Could Operads Allow The Formulation Of The Generalized Feynman Rules?

The previous discussion of symplectic fusion rules leaves open many questions.

1. How to combine symplectic and conformal fields to what might be called symplecto-conformal fields?
2. The previous discussion applies only in super-symplectic degrees of freedom and the question is how to generalize the discussion to super Kac-Moody degrees of freedom. One must of course also try to identify more precisely what Kac-Moody degrees of freedom are!
3. How four-momentum and its conservation in the limits of measurement resolution enters this picture? Could the phase factors assocaited with the symplectic triangulation carry information about four-momentum?
4. At least two operads related to measurement resolution seem to be present: the operads formed by the symplecto-conformal fields and by generalized Feynman diagrams. For generalized Feynman diagrams causal diamond (CD) is the basic object whereas disks of S^2 are the basic objects in the case of symplecto-conformal QFT with a finite measurement resolution. Could these two different views about finite measurement resolution be more or less equivalent and could one understand this equivalence at the level of details.
5. Is it possible to formulate generalized Feynman diagrammatics and improved measurement resolution algebraically?

8.6.1 How To Combine Conformal Fields With Symplectic Fields?

The conformal fields of conformal field theory should be somehow combined with symplectic scalar field to form what might be called symplecto-conformal fields.

1. The simplest thing to do is to multiply ordinary conformal fields by a symplectic scalar field so that the fields would be restricted to a discrete set of points for a given realization of N-dimensional fusion algebra. The products of these symplecto-conformal fields at different points would define a finite-dimensional algebra and the products of these fields at same point could be assumed to vanish.
2. There is a continuum of geometric realizations of the symplectic fusion algebra since the edges of symplectic triangles can be selected rather freely. The integrations over the coordinates z_k (most naturally the complex coordinate of S^2 transforming linearly under rotations around quantization axes of angular momentum) restricted to the circle appearing in the definition of simplest stringy amplitudes would thus correspond to the integration over various geometric realizations of a given N-dimensional symplectic algebra.

Fusion algebra realizes the notion of finite measurement resolution. One implication is that all n-point functions vanish for $n > N$. Second implication could be that the points appearing in the geometric realizations of N-dimensional symplectic fusion algebra have some minimal distance. This would imply a cutoff to the multiple integrals over complex coordinates z_k varying along circle giving the analogs of stringy amplitudes. This cutoff is not absolutely necessary since the integrals defining stringy amplitudes are well-defined despite the singular behavior of n-point functions. One can also ask whether it is wise to introduce a cutoff that is not necessary and whether fusion algebra provides only a justification for the $1 + i\epsilon$ prescription to avoid poles used to obtain finite integrals.

The fixed values for the quantities $\int A_\mu dx^\mu$ along the edges of the symplectic triangles could indeed pose a lower limit on the distance between the vertices of symplectic triangles. Whether this occurs depends on what one precisely means with symplectic triangle.

1. The conformally invariant condition that the angles between the edges at vertices are smaller than π for triangle and larger than π for its conjugate is not enough to exclude loopy edges

and one would obtain ordinary stringy amplitudes multiplied by the symplectic phase factors. The outcome would be an integral over arguments $z_1, z_2, ..z_n$ for standard stringy n-point amplitude multiplied by a symplectic phase factor which is piecewise constant in the integration domain.

2. The condition that the points at different edges of the symplectic triangle can be connected by a geodesic segment belonging to the interior of the triangle is much stronger and would induce a length scale cutoff since loops cannot be used to create large enough value of $\int A_\mu dx^\mu$ for a given side of triangle. Symplectic invariance would be obtained for small enough symplectic transformations. How to realize this cutoff at the level of calculations is not clear. One could argue that this problem need not have any nice solution and since finite measurement resolution requires only finite calculational resolution, the approximation allowing loopy edges is acceptable.

3. The restriction of the edges of the symplectic triangle within a tubular neighborhood of a geodesic -more more generally an orbit of charged particle - with thickness determined by the length scale resolution in S^2 would also introduce the length scale cutoff with symplectic invariance within measurement resolution.

Symplecto-conformal should form an operad. This means that the improvement of measurement resolution should correspond also to an algebra homomorphism in which super-symplectic symplecto-conformal fields in the original resolution are mapped by algebra homomorphism into fields which contain sum over products of conformal fields at different points: for the symplectic parts of field the products reduces always to a sum over the values of field. For instance, if the field at point s is mapped to an average of fields at points s_k, nilpotency condition $x^2 = 0$ is satisfied.

8.6.2 Symplecto-Conformal Fields In Super-Kac-Moody Sector

The picture described above applies only in super-symplectic degrees of freedom. The vertices of generalized Feynman diagrams are absent from the description and CP_2 Kähler form induced to space-time surface which is absolutely essential part of quantum TGD is nowhere visible in the treatment.

How should one bring in Super Kac-Moody (SKM) algebra? The condition that the basic building bricks are same for the treatment of these degrees of freedom is a valuable guideline.

What does SKM algebra mean?

The first thing to consider is what SKM could mean. The recent view is that symplectic algebra corresponds to symplectic transformations for the boundary of causal diamond CD which looks locally like $\delta M^4_\pm \times CP_2$. For this super-algebra fermionic generators would be contractions of covariantly constant right-handed neutrino with the second quantized induced spinor field to which the contraction$j^k_A \Gamma_k$ of symplectic vector field with gamma matrices acts. For SKM algebra corresponding generators would be similar contractions of other spinor modes but involving only Killing vectors fields that is symplectic isometries.

The recent view about quantum criticality strongly suggests that the conformal symmetries act as almost gauge symmetries producing from a given preferred extremal new ones with same action and conserved charges. "Almost" means that sub-algebra of conformal algebra annihilates the physical states. The subalgebras in question form a fractal hierarchy and are isomorphic with the conformal algebra itself. They contain generators for which the conformal weight is multiple of integer n characterizing also the value of Planck constant given by $h_{eff} = n \times h$. n defines the number of conformal equivalence classes of space-time surfaces connecting fixed 3-surfaces at the boundaries of CD (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg` or **Fig. ??** in the appendix of this book).

Since Kähler action reduces for the general ansatz for the preferred extremals to 3-D Chern-Simons terms, the action of the conformal symmetries reduces also to the 3-D space-like surfaces where it is trivial by definition and to non-trivial action to the light-like 3-surfaces at which the signature of the induced metric changes: I have used to call this surface partonic orbits.

It must be however observed that one can consider also the possibility that SKM algebra corresponds to the isometries of $\delta M^4 \pm \times CP_2$ continued to the space-time surface by field equations. These isometries are conformal transformations of S^2 ($\delta M^4_\pm = S^2 \times R_+$) with conformal scaling compensated by the local scaling of the light-like radial coordinate r_M to guarantee that the metric reducing to that for S^2 apart from conformal scaling factor R_M^2 remains invariant. If this is the case the SKM contains also other than symplectic isometries.

Attempt to formulate symplectic triangulation for SKM algebra

The analog of symplectic triangulation for SKM algebra obviously requires that SKM algebra corresponds to symplectic isometries rather than including all $\delta M^4_\pm = S^2 \times R_+$ isometries in one-one correspondence with conformal transformations of S^2.

1. In the transition from super-symplectic to SKM degrees of freedom the light-cone boundary is naturally replaced with the light-like 3-surface X^3 representing the light-like random orbit of parton and serving as the basic dynamical object of quantum TGD. The sphere S^2 of light-cone boundary is in turn replaced with a partonic 2-surface X^2. This suggests how to proceed.

2. In the case of SKM algebra the symplectic fusion algebra is represented geometrically as points of partonic 2-surface X^2 by replacing the symplectic form of S^2 with the induced CP_2 symplectic form at the partonic 2-surface and defining $U(1)$ gauge field. This gives similar hierarchy of symplecto-conformal fields as in the super-symplectic case. This also realizes the crucial aspects of the classical dynamics defined by Kähler action. In particular, for vacuum 2-surfaces symplectic fusion algebra trivializes since Kähler magnetic fluxes vanish identically and 2-surfaces near vacua require a large value of N for the dimension of the fusion algebra since the available Kähler magnetic fluxes are small.

3. In super-symplectic case the projection along light-like ray allows to map the points at the light-cone boundaries of CD to points of same sphere S^2. In the case of light-like 3-surfaces light-like geodesics representing braid strands allow to map the points of the partonic two-surfaces at the future and past light-cone boundaries to the partonic 2-surface representing the vertex. The earlier proposal was that the ends of strands meet at the partonic 2-surface so that braids would replicate at vertices. The properties of symplectic fields would however force identical vanishing of the vertices if this were the case. There is actually no reason to assume this condition and with this assumption vertices involving total number N of incoming and outgoing strands correspond to symplecto-conformal N-point function as is indeed natural. Also now Kähler magnetic flux induces cutoff distance.

4. SKM braids reside at light-like 3-surfaces representing lines of generalized Feynman diagrams. If super-symplectic braids are needed at all, they must be assigned to the two light-like boundaries of CD meeting each other at the sphere S^2 at which future and past directed light-cones meet.

8.6.3 The Treatment Of Four-Momentum

Four-momentum enjoys a special role in super-symplectic and SKM representations in that it does not correspond to a quantum number assignable to the generators of these algebras. It would be nice if the somewhat mysterious phase factors associated with the representation of the symplectic algebra could code for the four-momentum - or rather the analogs of plane waves representing eigenstates of four-momentum at the points associated with the geometric representation of the symplectic fusion algebra.

Also the vision about TGD as almost topological QFT suggests that the symplectic degrees of freedom added to the conformal degrees of freedom defining alone a mere topological QFt somehow code for the physical degrees of freedom should and also four-momentum. If so, the symplectic triangulation might somehow code for four-momentum.

The representation of longitudinal momentum in terms of phase factors

The following argument suggests that S^2 and X^2 triangulations cannot naturally represent four-momentum and that one needs extension into 3-D light-like triangulation to achieve this.

1. The basic question is whether four-momentum could be coded in terms of non-integrable phase factors appearing in the representations of the symplectic fusion algebras.

2. In the symplectic case S^2 triangulation suggests itself as a representation of angular momentum only: it would be kind of spin network. In the SKM case X^2 would suggest representation of color hyper charge and isospin in terms of phases since CP_2 symmetries act non-trivially in Chern-Simons action. Does this mean that symplectic and SKM triangulations must be extended so that they are 3-D and defined for space-like 3-surface and the light-like orbit of partonic 2-surface. This would give additional phase factors assignable to presumably light-like edges. Ligh-like momentum would be natural and the recent twistorial formulation of quantum TGD indeed assigns massless momenta to fermion lines.

Suppose that one has 3-D light-like triangulation eith at δCD or at light-like orbits of partonic 2-surface. Consider first coding of four-momentum assuming only Kähler gauge potential of CP_2 possibly having M^4 part which is pure gauge.

1. Four different phase factors are needed if all components of four-momentum are to be coded. Both number theoretical vision about quantum TGD and the realization of the hierarchy of Planck constants assign to each point of space-time surface the same plane $M^2 \subset M^4$ having as the plane of non-physical polarizations. This condition allows to assign to a given light-like partonic 3-surface unique extremal of Kähler action defining the Kähler function as the value of Kähler action.

   Also p-adic mass calculations support the view that the physical states correspond to eigen states for the components of longitudinal momentum only (also the parton model for hadrons assumes this). This encourages to think that only M^2 part of four-momentum is coded by the phase factors. Transversal momentum squared would be a well defined quantum number and determined from mass shell conditions for the representations of super-symplectic (or equivalently SKM) conformal algebra much like in string model.

2. The phase factors associated with the 3-D symplectic fusion algebra in $S^2 \times R_+$ mean a deviation from conformal n-point functions, and the innocent question is whether these phase factors could be identified as plane-wave phase factors in S^2 could be associated with the transversal part of the four-momentum so that the n-point functions would be strictly analogous with stringy amplitudes. Alternative, and perhaps more natural, interpretation is in terms of spin and angular momentum.

3. Suppose one allows a gauge transformation of Kähler gauge potential inducing a pure gauge M^4 component to the Kähler gauge potential expressible as scalar function of M^4 coordianates. This kind of term might allow to achieve the vanishing of $j^\alpha A_\alpha$ term of at least its integral over space-time surface in Kähler action implying reduction of Kähler action to Chern-Simons terms if weak form of electric magnetic duality holds true. The scalar function can be interpreted as integral of a position dependent momentum along curve defined by $S^2 \times R_+$ triangulation and gives hopes of coding four-momentum in terms of Kähler gauge potential.

   In fact, the identification of the phase factors $exp(i \int A_\mu dx^\mu/\hbar)$ along a path as phase factors $exp(ip_{L,k}\Delta m^k)$ defined by the ends of the path and associated with the longitudinal part of four-momentum would correspond to an integral form of covariant constancy condition $\frac{dx^\mu}{ds}(\partial_\mu - iA_\mu)\Psi = 0$ along the edge of the symplectic triangle of more general path.

4. For the SKM triangulation associated with the light-like orbit X_l^3 of partonic 2-surface analogous phase factor would come from the integral along the (most naturally) light-like curve defining braid strand associated with the point in question. A geometric representation for the two projections of the four-momentum would thus result in SKM degrees of freedom and apart from the non-uniqueness related to the multiples of a 2π the components of M^2

momentum could be deduced from the phase factors. If one is satisfied with the projection of momentum in M^2, this is enough.

5. Neither of these phase factors is able to code all components of four-momentum. One might however hope that together they could give enough information to deduce the four-momentum if it is assumed to correspond to the rest system.

6. The phase factors assignable to the symplectic triangles in S^2 and X^2 have nothing to do with momentum. Because the space-like phase factor $exp(iS_z\Delta\phi/\hbar)$ associated with the edge of the symplectic triangle is completely analogous to that for momentum, one can argue that the symplectic triangulation could define a kind of spin network utilized in discretized approaches to quantum gravity. The interpretation raises the question about the interpretation of the quantum numbers assignable to the Lorentz invariant phase factors defined by the CP_2 Kähler gauge potential.

The quantum numbers associated with phase factors for CP_2 parts of Kähler gauge potentials

Suppose that it is possible to assign two independent and different phase factors to the same geometric representation, in other words have two independent symplectic fields with the same geometric representation. The product of two symplectic fields indeed makes sense and satisfies the defining conditions. One can define prime symplectic algebras and decompose symplectic algebras to prime factors. Since one can allow permutations of elements in the products it becomes possible to detect the presence of product structure experimentally by detecting different combinations for products of phases caused by permutations realized as different combinations of quantum numbers assigned with the factors. The geometric representation for the product of n symplectic fields would correspond to the assignment of n edges to any pair of points. The question concerns the interpretation of the phase factors assignable to the CP_2 parts of Kähler gauge potentials of S^2 and CP_2 Kähler form.

1. The natural interpretation for the two additional phase factors would be in terms of color quantum numbers. Color hyper charge and isospin are mathematically completely analogous to the components of four-momentum so that a possible identification of the phase factors is as a representation of these quantum numbers. The representation of plane waves as phase factors $exp(ip_k\Delta m^k/\hbar)$ generalizes to the representation $exp(iQ_A\Delta\Phi^A/\hbar)$, where Φ_A are the angle variables conjugate to the Hamiltonians representing color hyper charge and isospin. This representation depends on end points only so that the crucial symplectic invariance with respect to the symplectic transformations respecting the end points of the edge is not lost ($U(1)$ gauge transformation is induced by the scalar j^kA_k, where j^k is the symplectic vector field in question).

2. One must be cautious with the interpretation of the phase factors as a representation for color hyper charge and isospin since a breaking of color gauge symmetry would result since the phase factors associated with different values of color isospin and hypercharge would be different and could not correspond to same edge of symplectic triangle. This is questionable since color group itself represents symplectic transformations. The construction of CP_2 as a coset space $SU(3)/U(2)$ identifies $U(2)$ as the holonomy group of spinor connection having interpretation as electro-weak group. Therefore also the interpretation of the phase factors in terms of em charge and weak charge can be considered. In TGD framework electro-weak gauge potential indeed suffer a non-trivial gauge transformation under color rotations so that the correlation between electro-weak quantum numbers and non-integrable phase factors in Cartan algebra of the color group could make sense. Electro-weak symmetry breaking would have a geometric correlate in the sense that different values of weak isospin cannot correspond to paths with same values of phase angles $\Delta\Phi^A$ between end points.

3. If the phase factors associated with the M^4 and CP_2 are assumed to be identical, the existence of geometric representation is guaranteed. This however gives constraints between rest mass, spin, and color (or electro-weak) quantum numbers.

Some general comments

Some further comments about phase factors are in order.

1. By number theoretical universality the plane wave factors associated with four-momentum must have values coming as roots of unity (just as for a particle in box consisting of discrete lattice of points). At light-like boundary the quantization conditions reduce to the condition that the value of light-like coordinate is rational of form m/N, if N: th roots of unity are allowed.

2. In accordance with the finite measurement resolution of four-momentum, four-momentum conservation is replaced by a weaker condition stating that the products of phase factors representing incoming and outgoing four-momenta are identical. This means that positive and negative energy states at opposite boundaries of CD would correspond to complex conjugate representations of the fusion algebra. In particular, the product of phase factors in the decomposition of the conformal field to a product of conformal fields should correspond to the original field value. This would give constraints on the trees physically possible in the operad formed by the fusion algebras. Quite generally, the phases expressible as products of phases $exp(in\pi/p)$, where $p \leq N$ is prime must be allowed in a given resolution and this suggests that the hierarchy of p-adic primes is involved. At the limit of very large N exact momentum conservation should emerge.

3. Super-conformal invariance gives rise to mass shell conditions relating longitudinal and transversal momentum squared. The massivation of massless particles by Higgs mechanism and p-adic thermodynamics pose additional constraints to these phase factors.

8.6.4 What Does The Improvement Of Measurement Resolution Really Mean?

To proceed one must give a more precise meaning for the notion of measurement resolution. Two different views about the improvement of measurement resolution emerge. The first one relies on the replacement of braid strands with braids applies in SKM degrees of freedom and the homomorphism maps symplectic fields into their products. The homomorphism based on the averaging of symplectic fields over added points consistent with the extension of fusion algebra described in previous section is very natural in super-symplectic degrees of freedom. The directions of these two algebra homomorphisms are different. The question is whether both can be involved with both super-symplectic and SKM case. Since the end points of SKM braid strands correspond to both super-symplectic and SKM degrees of freedom, it seems that division of labor is the only reasonable option.

1. Quantum classical correspondence requires that measurement resolution has a purely geometric meaning. A purely geometric manner to interpret the increase of the measurement resolution is as a replacement of a braid strand with a braid in the improved resolution. If one assigns the phase factor assigned with the fusion algebra element with four-momentum, the conservation of the phase factor in the associated homomorphism is a natural constraint. The mapping of a fusion algebra element (strand) to a product of fusion algebra elements (braid) allows to realize this condition. Similar mapping of field value to a product of field values should hold true for conformal parts of the fields. There exists a large number equivalent geometric representations for a given symplectic field value so that one obtains automatically an averaging in conformal degrees of freedom. This interpretation for the improvement of measurement resolution looks especially natural for SKM degrees of freedom for which braids emerge naturally.

2. One can also consider the replacement of symplecto-conformal field with an average over the points becoming visible in the improved resolution. In super-symplectic degrees of freedom this looks especially natural since the assignment of a braid with light-cone boundary is not so natural as with light-like 3-surface. This map does not conserve the phase factor but this could be interpreted as reflecting the fact that the values of the light-like radial coordinate

are different for points involved. The proposed extension of the symplectic algebra proposed in the previous section conforms with this interpretation.

3. In the super-symplectic case the improvement of measurement resolution means improvement of angular resolution at sphere S^2. In SKM sector it means improved resolution for the position at partonic 2-surface. This generalizes also to the 3-D symplectic triangulations. For SKM algebra the increase of the measurement resolution related to the braiding takes place inside light-like 3-surface. This operation corresponds naturally to an addition of sub-CD inside which braid strands are replaced with braids. This is like looking with a microscope a particular part of line of generalized Feynman graph inside CD and corresponds to a genuine physical process inside parton. In super-symplectic case the replacement of a braid strand with braid (at light-cone boundary) is induced by the replacement of the projection of a point of a partonic 2-surface to S^2 with a a collection of points coming from several partonic 2-surfaces. This replaces the point s of S^2 associated with CD with a set of points s_k of S^2 associated with sub-CD. Note that the solid angle spanned by these points can be rather larger so that zoom-up is in question.

4. The improved measurement resolution means that a point of S^2 (X^2) at boundary of CD is replaced with a point set of S^2 (X^2) assignable to sub-CD. The task is to map the point set to a small disk around the point. Light-like geodesics along light-like X^3 defines this map naturally in both cases. In super-symplectic case this map means scaling down of the solid angle spanned by the points of S^2 associated with sub-CD.

8.6.5 How Do The Operads Formed By Generalized Feynman Diagrams And Symplecto-Conformal Fields Relate?

The discussion above leads to following overall view about the situation. The basic operation for both symplectic and Feynman graph operads corresponds to an improvement of measurement resolution. In the case of planar disk operad this means to a replacement of a white region of a map with smaller white regions. In the case of Feynman graph operad this means better space-time resolution leading to a replacement of generalized Feynman graph with a new one containing new sub-CD bringing new vertices into daylight. For braid operad the basic operation means looking a braid strand with a microscope so that it can resolve into a braid: braid becomes a braid of braids. The latter two views are equivalent if sub-CD contains the braid of braids.

The disks D^2 of the planar disk operad has natural counterparts in both super-symplectic and SKM sector.

1. For the geometric representations of the symplectic algebra the image points vary in continuous regions of S^2 (X^2) since the symplectic area of the symplectic triangle is a highly flexible constraint. Posing the condition that any point at the edges of symplectic triangle can be connected to any another edge excludes symplectic triangles with loopy sides so that constraint becomes non-trivial. In fact, since two different elements of the symplectic algebra cannot correspond to the same point for a given geometric representation, each element must correspond to a connected region of S^2 (X^2). This allows a huge number of representations related by the symplectic transformations S^2 in super-symplectic case and by the symplectic transformations of CP_2 in SKM case. In the case of planar disk operad different representations are related by isotopies of plane.

   This decomposition to disjoint regions naturally correspond to the decomposition of the disk to disjoint regions in the case of planar disk operad and Feynman graph operad (allowing zero energy insertions). Perhaps one might say that N-dimensional elementary symplectic algebra defines an N-coloring of S^2 (S^2) which is however not the same thing as the 2-coloring possible for the planar operad. TGD based view about Higgs mechanism leads to a decomposition of partonic 2-surface X^2 (its light-like orbit X^3) into conformal patches. Since also these decompositions correspond to effective discretizations of X^2 (X^3), these two decompositions would naturally correspond to each other.

2. In SKM sector disk D^2 of the planar disk operad is replaced with the partonic 2-surface X^2 and since measurement resolution is a local notion, the topology of X^2 does not matter. The

improvement of measurement resolution corresponds to the replacement of braid strand with braid and homomorphism is to the direction of improved spatial resolution.

3. In super-symplectic case D^2 is replaced with the sphere S^2 of light-cone boundary. The improvement of measurement resolution corresponds to introducing points near the original point and the homomorphism maps field to its average. For the operad of generalized Feynman diagrams CD defined by future and past directed light-cones is the basic object. Given CD can be indeed mapped to sphere S^2 in a natural manner. The light-like boundaries of CDs are metrically spheres S^2. The points of light-cone boundaries can be projected to any sphere at light-cone boundary. Since the symplectic area of the sphere corresponds to solid angle, the choice of the representative for S^2 does not matter. The sphere defined by the intersection of future and past light-cones of CD however provides a natural identification of points associated with positive and negative energy parts of the state as points of the same sphere. The points of S^2 appearing in n-point function are replaced by point sets in a small disks around the n points.

4. In both super-symplectic and SKM sectors light-like geodesic along X^3 mediate the analog of the map gluing smaller disk to a hole of a disk in the case of planar disk operad defining the decomposition of planar tangles. In super-symplectic sector the set of points at the sphere corresponding to a sub-CD is mapped by SKM braid to the larger CD and for a typical braid corresponds to a larger angular span at sub-CD. This corresponds to the gluing of D^2 along its boundaries to a hole in D^2 in disk operad. A scaling transformation allowed by the conformal invariance is in question. This scaling can have a non-trivial effect if the conformal fields have anomalous scaling dimensions.

5. Homomorphisms between the algebraic structures assignable to the basic structures of the operad (say tangles in the case of planar tangle operad) are an essential part of the power of the operad. These homomorphisms associated with super-symplectic and SKM sector code for two views about improvement of measurement resolution and might lead to a highly unique construction of M-matrix elements.

The operad picture gives good hopes of understanding how M-matrices corresponding to a hierarchy of measurement resolutions can be constructed using only discrete data.

1. In this process the n-point function defining M-matrix element is replaced with a superposition of n-point functions for which the number of points is larger: $n \to \sum_{k=1,...,m} n_k$. The numbers n_k vary in the superposition. The points are also obtained by downwards scaling from those of smaller S^2. Similar scaling accompanies the composition of tangles in the case of planar disk operad. Algebra homomorphism property gives constraints on the compositeness and should govern to a high degree how the improved measurement resolution affects the amplitude. In the lowest order approximation the M-matrix element is just an n-point function for conformal fields of positive and negative energy parts of the state at this sphere and one would obtain ordinary stringy amplitude in this approximation.

2. Zero energy ontology means also that each addition in principle brings in a new zero energy insertion as the resolution is improved. Zero energy insertions describe actual physical processes in shorter scales in principle affecting the outcome of the experiment in longer time scales. Since zero energy states can interact with positive (negative) energy particles, zero energy insertions are not completely analogous to vacuum bubbles and cannot be neglected. In an idealized experiment these zero energy states can be assumed to be absent. The homomorphism property must hold true also in the presence of the zero energy insertions. Note that the Feynman graph operad reduces to planar disk operad in absence of zero energy insertions.

8.7 Possible Other Applications Of Category Theory

It is not difficult to imagine also other applications of category theory in TGD framework.

8.7.1 Categorification And Finite Measurement Resolution

I read a very stimulating article by John Baez with title "Categorification" (see http://tinyurl.com/ych6a8oa) [A86] about the basic ideas behind a process called categorification. The process starts from sets consisting of elements. In the following I describe the basic ideas and propose how categorification could be applied to realize the notion of finite measurement resolution in TGD framework.

What categorification is?

In categorification sets are replaced with categories and elements of sets are replaced with objects. Equations between elements are replaced with isomorphisms between objects: the right and left hand sides of equations are not the same thing but only related by an isomorphism so that they are not tautologies anymore. Functions between sets are replaced with functors between categories taking objects to objects and morphisms to morphisms and respecting the composition of morphisms. Equations between functions are replaced with natural isomorphisms between functors, which must satisfy certain coherence laws representable in terms of commuting diagrams expressing conditions such as commutativity and associativity.

The isomorphism between objects represents equation between elements of set replaces identity. What about isomorphisms themselves? Should also these be defined only up to an isomorphism of isomorphism? And what about functors? Should one continue this replacement ad infinitum to obtain a hierarchy of what might be called n-categories, for which the process stops after n: th level. This rather fuzzy buisiness is what mathematicians like John Baez are actually doing.

Why categorification?

There are good motivations for the categofication. Consider the fact that natural numbers. Mathematically oriented person would think number "3" in terms of an abstract set theoretic axiomatization of natural numbers. One could also identify numbers as a series of digits. In the real life the representations of three-ness are more concrete involving many kinds of associations. For a child "3" could correspond to three fingers. For a mystic it could correspond to holy trinity. For a Christian "faith, hope, love". All these representations are isomorphic representation of threeness but as real life objects three sheeps and three cows are not identical.

We have however performed what might be called decategorification: that is forgitten that the isomorphic objects are not equal. Decatecorification was of course a stroke of mathematical genius with enormous practical implications: our information society represents all kinds of things in terms of numbers and simulates successfully the real world using only bit sequences. The dark side is that treating people as mere numbers can lead to a rather cold society.

Equally brilliant stroke of mathematical genius is the realization that isomorphic objects are not equal. Decategorization means a loss of information. Categorification brings back this information by bringing in consistency conditions known as coherence laws and finding these laws is the hard part of categorization meaning discovery of new mathematics. For instance, for braid groups commutativity modulo isomorphisms defines a highly non-trivial coherence law leading to an extremely powerful notion of quantum group having among other things applications in topological quantum computation.

The so called associahedrons (see http://tinyurl.com/ng2fqro) [A35] emerging in n-category theory could replace space-time and space as fundamental objects. Associahedrons are polygons used to represent geometrically associativity or its weaker form modulo isomorphism for the products of n objects bracketed in all possible ways. The polygon defines a hierarchy containing sub-polygons as its edges containing.... Associativity states the isomorphy of these polygons. Associahedrons and related geometric representations of category theoretical arrow complexes in terms or simplexes allow a beautiful geometric realization of the coherence laws. One could perhaps say that categories as discrete structures are not enough: only by introducing the continuum allowing geometric representations of the coherence laws things become simple.

No-one would have proposed categorification unless it were demanded by practical needs of mathematics. In many mathematical applications it is obvious that isomorphism does not mean identity. For instance, in homotopy theory all paths deformable to each other in continuous manner are homotopy equivalent but not identical. Isomorphism is now homotopy. These paths can be connected and form a groupoid. The outcome of the groupoid operation is determined up to homotopy. The deformations of closed path starting from a given point modulo homotopies form homotopy group and one can interpret the elements of homotopy group as copies of the point which are isomorphic. The replacement of the space with its universal covering makes this distinction explicit. One can form homotopies of homotopies and continue this process ad infinitum and obtain in this manner homotopy groups as characterizes of the topology of the space.

Cateforification as a way to describe finite measurement resolution?

In quantum physics gauge equivalence represents a standard example about equivalence modulo isomorphisms which are now gauge transformations. There is a practical strategy to treat the situation: perform a gauge choice by picking up one representative amongst infinitely many isomorphic objects. At the level of natural numbers a very convenient gauge fixing would correspond the representation of natural number as a sequence of decimal digits rather than image of three cows.

In TGD framework a excellent motivation for categorification is the need to find an elegant mathematical realization for the notion of finite measurement resolution. Finite measurement resolutions (or cognitive resolutions) at various levels of information transfer hierarchy imply accumulation of uncertainties. Consider as a concrete example uncertainty in the determination of basic parameters of a mathematical model. This uncertainty is reflected to final outcome as via a long sequence of mathematical maps and additional uncertainties are produced by the approximations at each step of this process.

How could onbe describe the finite measurement resolution elegantly in TGD Universe? Categorification suggests a natural method. The points equivalent with measurement resolution are isomorphic with each other. A natural guess inspired by gauge theories is that one should perform a gauge choice as an analog of decategorification. This allows also to avoid continuum of objects connected by arrows not n spirit with the discreteness of category theoretical approach.

1. At space-time level gauge choice means discretization of partonic 2-surfaces replacing them with a discrete set points serving as representatives of equivalence classes of points equivalent under finite measurement resolution. An especially interesting choice of points is as rational points or algebraic numbers and emerges naturally in p-adicization process. One can also introduce what I have called symplectic triangulation of partonic 2-surfaces with the nodes of the triangulation representing the discretization and carrying quantum numbers of various kinds.

2. At the level of "world classical worlds" (WCW) this means the replacement of the sub-group if the symplectic group of $\delta M^4 \times CP_2$ -call it G - permuting the points of the symplectic triangulation with its discrete subgroup obtained as a factor group G/H, where H is the normal subgroup of G leaving the points of the symplectic triangulation fixed. One can also consider subgroups of the permutation group for the points of the triangulation. One can also consider flows with these properties to get braided variant of G/H. It would seem that one cannot regard the points of triangulation as isomorphic in the category theoretical sense. This because, one can have quantum superpositions of states located at these points and the factor group acts as the analog of isometry group. One can also have many-particle states with quantum numbers at several points. The possibility to assign quantum numbers to a given point becomes the physical counterpart for the axiom of choice.

   The finite measurement resolution leads to a replacement of the infinite-dimensional world of classical worlds with a discrete structure. Therefore operation like integration over entire "world of classical worlds" is replaced with a discrete sum.

3. What suggests itself strongly is a hierarchy of n-categories as a proper description for the finite measurement resolution. The increase of measurement resolution means increase for the number of braid points. One has also braids of braids of braids structure implied by the possibility to map infinite primes, integers, and rationals to rational functions of several variables and the conjecture possibility to represent the hierarchy of Galois groups involved as symplectic flows. If so the hierarchy of n-categories would correspond to the hierarchy

of infinite primes having also interpretation in terms of repeated second quantization of an arithmetic SUSY such that many particle states of previous level become single particle states of the next level.

The finite measurement resolution has also a representation in terms of inclusions of hyperfinite factors of type II_1 defined by the Clifford algebra generated by the gamma matrices of WCW [K87]

1. The included algebra represents finite measurement resolution in the sense that its action generates states which are cannot be distinguished from each other within measurement resolution used. The natural conjecture is that this indistuinguishability corresponds to a gauge invariance for some gauge group and that TGD Universe is analogous to Turing machine in that almost any gauge group can be represented in terms of finite measurement resolution.

2. Second natural conjecture inspired by the fact that symplectic groups have enormous representabive power is that these gauge symmetries allow representation as subgroups of the symplectic group of $\delta M^4 \times CP_2$. A nice article about universality of symplectic groups is the article "The symplectification of science" (see `http://tinyurl.com/y8us9sgw`) by Mark. J. Gotay [A22].

3. An interesting question is whether there exists a finite-dimensional space, whose symplectomorphisms would allow a representation of any gauge group (or of all possible Galois groups as factor groups) and whether $\delta M^4 \times CP_2$ could be a space of this kind with the smallest possible dimension.

8.7.2 Inclusions Of HFFs And Planar Tangles

Finite index inclusions of HFFs are characterized by non-branched planar algebras for which only an even number of lines can emanate from a given disk. This makes possible a consistent coloring of the k-tangle by black and white by painting the regions separated by a curve using opposite colors. For more general algebras, also for possibly existing branched tangle algebras, the minimum number of colors is four by four-color theorem. For the description of zero energy states the 2-color assumption is not needed so that the necessity to have general branched planar algebras is internally consistent. The idea about the inclusion of positive energy state space into the space of negative energy states might be consistent with branched planar algebras and the requirement of four colors since this inclusion involves also conjugation and is thus not direct.

In [A16] if was proposed that planar operads are associated with conformal field theories at sphere possessing defect lines separating regions with different color. In TGD framework and for branched planar algebras these defect lines would correspond to light-like 3-surfaces. For fermions one has single wormhole throat associated with topologically condensed CP_2 type extremal and the signature of the induced metric changes at the throat. Bosons correspond to pairs of wormhole throats associated with wormhole contacts connecting two space-time sheets modellable as a piece of CP_2 type extremal. Each boson thus corresponds to 2 lines within CP_2 radius so that in purely bosonic case the planar algebra can correspond to that associated with an inclusion of HFFs.

8.7.3 2-Plectic Structures And TGD

Chris Rogers and Alex Hoffnung have demonstrated [A123] that the notion of symplectic structure generalizes to n-plectic structure and in $n = 2$ case leads to a categorification of Lie algebra to 2-Lie-algebra. In this case the generalization replaces the closed symplectic 2-form with a closed 3-form ω and assigns to a subset of one-forms defining generalized Hamiltonians vector fields leaving the 3-form invariant.

There are two equivalent definitions of the Poisson bracket in the sense that these Poisson brackets differ only by a gradient, which does not affect the vector field assignable to the Hamiltonian one-form. The first bracket is simply the Lie-derivate of Hamiltonian one form G with respect to vector field assigned to F. Second bracket is contraction of Hamiltonian one-forms with the three-form ω. For the first variant Jacobi identities hold true but Poisson bracket is antisymmetric only modulo gradient. For the second variant Jacobi identities hold true only modulo gradient but Poisson bracket is antisymmetric. This modulo property is in accordance with category theoretic thinking in which commutativity, associativity, antisymmetry, ... hold true only up to isomorphism.

For 3-dimensional manifolds n=2-plectic structure has the very nice property that *all* one-forms give rise to Hamiltonian vector field. In this case any 3-form is automatically closed so that a large variety of 2-plectic structures exists. In TGD framework the natural choice for the 3-form ω is as Chern-Simons 3-form defined by the projection of the Kähler gauge potential to the light-like 3-surface. Despite the fact the induced metric is degenerate, one can deduce the Hamiltonian vector field associated with the one-form using the general defining conditions

$$i_{v_F}\omega = dF \tag{8.7.1}$$

since the vanishing of the metric determinant appearing in the formal definition cancels out in the expression of the Hamiltonian vector field.

The explicit formula is obtained by writing ω as

$$\begin{aligned} \omega = K\epsilon_{\alpha\beta\gamma} \times \epsilon^{\mu\nu\delta} A_\mu J_{\nu\delta}\sqrt{g} \ &= \epsilon_{\alpha\beta\gamma} \times C-S \ , \\ C-S = KE^{\alpha\beta\gamma} A_\alpha J_{\beta\gamma} \ . \end{aligned} \tag{8.7.2}$$

Here $E^{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$ holds true numerically and metric determinant, which vanishes for light-like 3-surfaces, has disappeared.

The Hamiltonian vector field is the curl of F divided by the Chern-Simons action density $C-S$:

$$v_F^\alpha = \frac{1}{2} \times \frac{\epsilon^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)\sqrt{g}}{C-S\sqrt{g}} = \frac{1}{2} \times \frac{E^{\alpha\beta\gamma}(\partial_\beta F_\gamma - \partial_\gamma F_\beta)}{C-S} \ . \tag{8.7.3}$$

The Hamiltonian vector field multiplied by the dual of 3-form multiplied by the metric determinant has a vanishing divergence and is analogous to a vector field generating volume preserving flow. and the value of Chern Simons 3-form defines the analog of the metric determinant for light-like 3-surfaces. The generalized Poisson bracket for Hamiltonian 1-forms defined in terms of the action of Hamiltonian vector field on Hamiltonian as $J_1^\beta D_\beta F_2\alpha - J_2^\beta D_\beta H_2\alpha$ is Hamiltonian 1-form. Here J_i denotes the Hamiltonian vector field associated with F_i. The bracked unique apart from gradient. The corresponding vector field is the commutator of the Hamiltonian vector fields.

The objection is that gauge invariance is broken since the expression for the vector field assigned to the Hamiltonian one-form depends on gauge. In TGD framework there is no need to worry since Kähler gauge potential has unique natural expression and the $U(1)$ gauge transformations of Kähler gauge potential induced by symplectic transformations of CP_2 are not genuine gauge transformations but dynamical symmetries since the induced metric changes and space-time surface is deformed. Another important point is that Kähler gauge potential for a given CD has M^4 part which is "pure gauge" constant Lorentz invariant vector and proportional to the inverse of gravitational constant G. Its ratio to CP_2 radius squared is determined from electron mass by p-adic mass calculations and mathematically by quantum criticality fixing also the value of Kähler coupling strength.

8.7.4 TGD Variant For The Category Ncob

John Baez has suggested that quantum field theories could be formulated as functors from the category of n-cobordisms to the category of Hilbert spaces [A94, A38]. In TGD framework light-like 3-surfaces containing the number theoretical braids define the analogs of 3-cobordisms and surface property brings in new structure. The motion of topological condensed 3-surfaces along 4-D space-time sheets brings in non-trivial topology analogous to braiding and not present in category nCob.

Intuitively it seems possible to speak about one-dimensional orbits of wormhole throats and -contacts (fermions and bosons) in background space-time (homological dimension). In this case linking or knotting are not possible since knotting is co-dimension 2 phenomenon and only objects whose homological dimensions sum up to $D-1$ can get linked in dimension D. String like objects could topologically condense along wormhole contact which is string like object. The orbits of closed string like objects are homologically co-dimension 2 objects and could get knotted if one does not allow space-time sheets describing un-knotting. The simplest examples are ordinary knots which are not allowed to evolve by forming self intersections. The orbits of point like wormhole contact and closed string like wormhole contact can get linked: a point particle moving through a closed string is basic dynamical example. There is no good reason preventing unknotting and unlinking in absolute sense.

8.7.5 Number Theoretical Universality And Category Theory

Category theory might be also a useful tool to formulate rigorously the idea of number theoretical universality and ideas about cognition. What comes into mind first are functors real to p-adic physics and vice versa. They would be obtained by composition of functors from real to rational physics and back to p-adic physics or vice versa. The functors from real to p-adic physics would provide cognitive representations and the reverse functors would correspond to the realization of intentional action. The functor mapping real 3-surface to p-adic 3-surfaces would be simple: interpret the equations of 3-surface in terms of rational functions with coefficients in some algebraic extension of rationals as equations in arbitrary number field. Whether this description applies or is needed for 4-D space-time surface is not clear.

At the Hilbert space level the realization of these functors would be quantum jump in which quantum state localized to p-adic sector tunnels to real sector or vice versa. In zero energy ontology this process is allowed by conservation laws even in the case that one cannot assign classical conserved quantities to p-adic states (their definition as integrals of conserved currents does not make sense since definite integral is not a well-defined concept in p-adic physics). The interpretation would be in terms of generalized M-matrix applying to cognition and intentionality. This M-matrix would have values in the field of rationals or some algebraic extension of rationals. Again a generalization of Connes tensor product is suggestive.

8.7.6 Category Theory And Fermionic Parts Of Zero Energy States As Logical Deductions

Category theory has natural applications to quantum and classical logic and theory of computation [A38]. In TGD framework these applications are very closely related to quantum TGD itself since it is possible to identify the positive and negative energy pieces of fermionic part of the zero energy state as a pair of Boolean statements connected by a logical deduction, or rather- quantum superposition of them. An alternative interpretation is as rules for the behavior of the Universe coded by the quantum state of Universe itself. A further interpretation is as structures analogous to quantum computation programs with internal lines of Feynman diagram would represent communication and vertices computational steps and replication of classical information coded by number theoretical braids.

8.7.7 Category Theory And Hierarchy Of Planck Constants

Category theory might help to characterize more precisely the proposed geometric realization of the hierarchy of Planck constants explaining dark matter as phases with non-standard value of Planck constant. The situation is topologically very similar to that encountered for generalized Feynman diagrams. Singular coverings and factor spaces of M^4 and CP_2 are glued together along 2-D manifolds playing the role of object and space-time sheets at different vertices could be interpreted as arrows going through this object.

Chapter 9

Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD?

9.1 Introduction

The dynamics of TGD is extremely simple locally: space-times are surfaces of 8-D embedding space so that only four field-like dynamical variables are present and preferred extremals satisfy strong form of holography (SH) meaning that almost 2-D data determine them. TGD Universe looks however also extremely complex. There is a hierarchy of space-times sheets, hierarchy of p-adic length scales, hierarchy of dark matters labelled by the values of Planck constant $h_{eff}/h = n$, hierarchy of extensions of rationals defining hierarchy of adeles in adelic physics view about TGD, hierarchy of infinite primes (and rationals), and also the hierarchy of conscious entities (quantum measurement theory in zero energy ontology can be seen as theory of consciousness [L27]).

During years it has become gradually clear that category theory could be the mathematical language of quantum TGD [K15, K14, K9]. Only category theory gives hopes about unifying various hierarchies making TGD Universe to look so horribly complex. Hierarchy formed by categories, categories of categories, ... could be the mathematics needed to keep book about this complexity and provide also otherwise unexpected constraints.

The arguments developed in the sequel suggest the following overall view.

1. Positive and negative energy parts of zero energy states can be regarded as tensor networks [L10] identifiable as categories. The new element is that one does not have only particles (objects) replaced with partonic 2-surfaces but also strings connecting them (morphisms). Morphisms and functors provide a completely new element not present in the standard model. For instance, S-matrix would be a functor between categories. Various hierarchies of of TGD would in turn translate to hierarchies of categories.

2. The recent view about generalized Feynman diagrams [K29, K8, L26] is inspired by two general ideas. First, the twistor lift of TGD replaces space-time surfaces with their twistor-spaces getting their twistor structure as induced twistor structure from the product of twistor spaces of M^4 and CP_2. Secondly, topological scattering diagrams are analogous to computations and can be reduced to minimal diagrams, which are tree diagrams with braiding. This picture fits very nicely with the picture provided by fusion categories. At fermionic level the basic interaction is 2+2 scattering of fermions occurring at the vertices identifiable as partonic 2-surface and re-distributes the fermion lines between partonic 2-surfaces. This interaction is highly analogous to what happens in braiding interaction defining basic gate in topological quantum computation [K4] but vertices expressed in terms of twistors depend on momenta of fermions.

3. Braiding transformations for fermionic lines identified as boundaries of string world sheets can take place inside the light-like orbits of partonic 2-surfaces defining boundaries of space-time

regions with Minkowskian and Euclidian signature of induced metric respectively. Braiding transformation is essentially a permutation for two braid strands mapping tensor product $A \otimes B$ to $B \otimes A$. R-matrix satisfying Yang-Baxter equation [B33] characterizes this operation algebraically.

4. Reconnections of fermionic strings connecting partonic 2-surfaces are possible and suggest interpretation in terms of 2-braiding generalizing ordinary braiding. I have2-braiding in [K37]: string world sheets get knotted in 4-D space-time forming 2-knots and strings form 1-knots in 3-D space. I do not actually know whether my intuitive believe that 2-braiding reduces to reconnections is correct. Reconnection induces an exchange of braid strands defined by boundaries of the string world sheet and therefore exchange of fermion lines defining boundaries string world sheets. This requires a generalization of quantum algebras to include also algebraic representation for reconnection: this representation could reduce to a representation in terms of an analog of R-matrix.

Yangians [B14] seem to be especially natural quantum algebras from TGD point of view [K77, L26]. Quantum algebras are bi-algebras having co-product Δ, which in well-defined sense is the inverse of the product. This makes the algebra multi-local: this feature is very attractive as far as understanding of bound states is considered. Δ-iterates of single particle system would give many-particle systems with non-trivial interactions reducing to kinematics.

One should assign Yangian to various Super-Kac-Moody algebras (SKMAs) involved and even with super-symplectic algebra (SSA) [K20, K88, K63], which however reduces effectively to SKMA for finite-dimensional Lie group if the proposed gauge conditions meaning vanishing of Noether charges for some sub-algebra H of SSA isomorphic to it and for its commutator $[SSA, H]$ with the entire SSA. Strong form of holography (SH) implying almost 2-dimensionality motivates these gauge conditions. Each SKMA would define a direct summand with its own parameter defining coupling constant for the interaction in question. There is also extended SKMA associated with the light-like orbits of partonic 2-surfaces and it seems natural to identify appropriate sub-algebras of these two algebras as duals in Yangian sense.

There is also partonic super-Kac-Moody algebra (PSKMA) associated with partonic 2-surfaces extending ordinary SKMA. On old conjecture is that SSA and PSKMA are physically dual in the same sense as the conformal algebra and its dual in twistor Grassmannian approach and that this generalizes equivalence principle (EP) to all conserved charges.

The plan of the article is following.

1. The basic notions and ideas about tensor networks as categories and about Yangians as multi-local symmetries and fundamental description of interactions are described.

2. The questions related to the Yangianization in TGD framework are considered. Yangianization of four-momentum and mass squared operator are discussed as examples.

3. The next section is devoted to category theory as tool of TGD: braided categories and fusion categories are briefly described and the notion of category with reconnection is considered.

4. The last section tries to represent the "great vision" in more detail.

9.2 Basic vision

The existing vision about TGD is summarized first and followed by a proposal about tensor networks as categories and Yangians as a multi-local generalization of symmetries with partonic surfaces replacing point like particles.

9.2.1 Very concise summary about basic notions and ideas of TGD

Let us briefly summarize the basic notions and ideas of TGD.

1. Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$, which is fixed uniquely by the condition that the factors of $H = M^4 \times S$ allow twistor space with Kähler structure [A79]. The

twistor spaces of dynamically allowed space-time surfaces are assumed to be representable as 6-D surfaces in twistor space $T(H) = T(M^4) \times T(CP_2)$ getting their twistor structure by induction from that of $T(H)$. $T(M^4)$ is identified as its purely geometric variant $T(M^4) = M^4 \times CP_1$. At the level of momentum space the usual identification is more appropriate. It is also assumed that these space-time surfaces are obtained as extremals of 6-D Kähler action [K77, K8, L26]. At space-time level this gives rise to dimensionally reduced Kähler action equal to the sum of volume term and 4-D Kähler action. Either the entire action or volume term would correspond to vacuum energy parameterized by cosmological constant in standard cosmology. Planck length corresponds to the radius of twistor sphere of M^4.

2. Strong form of holography (SH) implied by strong form of general coordinate invariance (SGCI) stating that light-like 3-surfaces defined by parton orbits and 3-D space-like ends of space-time surface at boundaries of CD separately code 3-D holography. SH states that 2-D data at string world sheets plus condition fixing the points of space-time surface with H-coordinates in extension of rationals fix the real space-time surface.

   (a) SH strongly suggests that the preferred extremals of the dimensionally reduced action satisfy gauge conditions (vanishing Noether charges) for a subalgebra H of super-symplectic algebras (SSA) isomorphic to it and its commutator $[H, SSA]$ with SSA: this effectively reduces SSA to a finite-dimensional Kac-Moody algebra.

   (b) Similar dimensional reduction would take place in fermionic degrees of freedom, where super-conformal symmetry fixes 4-D Dirac action, when bosonic action is known [K88, K63]. This involves the new notion of modified gamma matrices determined in terms of canonical momentum currents associated with the action.

   Quantum classical correspondence (QCC) states that classical Cartan charges for SSA are equal to the eigenvalues of corresponding fermionic charges. This gives a correlation between space-time dynamics and quantum numbers of positive (negative) parts of zero energy states.

   (c) SH implies that fermions are effectively localized at string world sheets: in other words, the induced spinor fields Ψ_{int} in space-time interior are determined their values Ψ_{string} at string world sheets. There are two options: Ψ_{int} is either continuation of Ψ_{string} or Ψ_{string} serves as the source of Ψ_{int} [L13].

3. At space-time level the dynamics is extremely simple locally since by general coordinate invariance (GCI) only 4 field-like variables are dynamical, and one has also SH by SGCI. Topologically the situation is rather complex: one has many-sheeted space-time having hierarchical structure. The GRT limit of TGD [K80] is obtained in long length scales by mapping the many-sheeted structure to a slightly curved piece of M^4 by demanding that the deformation of M^4 metric is sum of the deformation of he induced metrics of space-time surface from M^4 metric. Similar description implies to gauge potentials in terms of induced gauge potentials. The many-sheetedness is visible as anomalies of GRT and plays central role in quantum biology [K58].

4. Zero energy ontology (ZEO) means that one consider space-time surfaces inside causal diamonds (CDs defined as intersections of future and past directed light-cones with points replaced with CP_2) forming a scale hierarchy. Zero energy states are tensor products of positive and negative energy parts at opposite boundaries of CD. Zero energy property means that the total conserved quantum numbers are opposite at the opposite boundaries of CD so that one has consistency with ordinary positive energy ontology. Zero energy states are analogous to physical events in the usual ontology but is much more flexible since given zero energy energy states is in principle creatable from vacuum.

5. The "world of classical worlds" (WCW) [K35, K20, K63] generalizes the superspace of Wheeler. WCW decomposes to sub-WCWs assignable to CDs forming a scale hierarchy. Note that 3-surface in ZEO corresponds to a pair of disjoint collections 3-surfaces at opposite boundaries of CD- initial and final state in standard ontology. Super-symplectic symmetries

(SCA) act as isometries of WCW. Zero energy states correspond to WCW spinor fields and the gamma matrices of WCW are expressible as linear combinations of fermionic oscillator operators for induced spinor fields. Besides SCA there is partonic super-Kac-Moody algebra (PSCA) acting on light-like orbits of partonic 2-surfaces and these algebras are suggested to be dual physically (generalized EP).

6. One ends up with an extension of real physics to adelic physics [L22]. p-Adic physics for various primes are introduced as physical correlates of cognition and imagination: the original motivation come from p-adic mass calculations [K41]. p-Adic non-determinism (pseudo constants) [K50, K73] strongly suggests that one can always assign to 2-D holographic data a p-adic variant of space-time surface as a preferred extremal. In real case this need not be the case so that the space-time surface realized as preferred extremal is imaginable but not necessarily realizable.

   p-Adic physics and real physics are fused to adelic physics: space-time surface isa book-like structure with pages labelled by real number field and p-adic number fields in an extension induced by some extension of rationals. Planck constants $h_{eff} = n \times h$ corresponds to the dimension of the extension dividing the order of its Galois group and favored p-adic primes correspond to ramified primes for favored extensions. Evolution corresponds to increasing complexity of extension of rationals and favored extensions are the survivors in fight for number theoretic survival.

7. Twistor lift of TGD leads to a proposal for the construction of scattering amplitudes assuming Yangian symmetry assignable to Kac-Moody algebras for embedding space isometries, with electroweak gauge group, and for finite-D Lie dynamically generated Lie group selected by conditions on SSA algebra. 2+2 fermion vertex analogous to braiding interaction serves as the basic vertex in the formulation of [L26].

9.2.2 Tensor networks as categories

The challenge has been the identification of relevant categories and physical realization of them. One can imagine endless number of identifications but the identification of absolutely convincing candidate has been difficult. Quite recently an astonishingly simple proposal emerged.

1. The notion of tensor network [B23] has emerged in condensed matter physics to describe strongly entangled systems and complexity associated with them. Holography is in an essential role in this framework. In TGD framework tensor network is realized physically at the level of the topology and geometry of many-sheeted space-time [L10]. Nodes would correspond to objects and links between them to morphisms. This structure would be realized as partonic 2-surfaces - objects - connected by fermionic strings - morphisms - assignable to magnetic flux tubes. Morphisms would be realized as Hilbert space isometries defined by entanglement. Physical state would be category or set of them!

   Functors are morphisms of categories mapping objects to objects and morphisms to morphisms and respecting the composition of morphisms so that the structure of the category is preserved. For instance, in zero energy ontology (ZEO) S-matrix for given space-time surface could be a unitary functor assigning to an initial category final category: they would be represented as quantum states at the opposite boundaries of causal diamond (CD). Also quantum states could be categories of categories of in accordance with various hierarchies.

2. Skeptic could argue as follows. The passive part of zero energy states for which active part evolves by unitary time evolutions following by state function reductions inducing time localization in moduli space of CDs, could be category. But isn't the active path more naturally a quantum superposition of categories? Should one replace time evolution as a functor with its quantum counterpart, which generates a quantum superposition of categories? If so, then state function reduction to opposite boundary of CD would mean localization in the set of categories! This is quite an abstraction from simple localization in 3-space in wave mechanics.

3. Categories form categories with functors between categories acting as morphisms. In principle one obtains an infinite hierarchy of categories identifiable as quantum states. This would fit

nicely with various hierarchies associated with TGD, most of which are induced by the hierarchy of extensions of rationals.

4. The language of categories fits like glove also to TGD inspired theory of consciousness. The fermionic strings and associated magnetic flux tubes would serve as correlates of attention. The associated morphism would define the direction of attention and also define sensory maps as morphisms. Conscious intelligence relies crucially on analogies and functors realize mathematically the notion of analogy. Categorification means basically classification and this is what cognition does all the time.

9.2.3 Yangian as a generalization of symmetries to multilocal symmetries

Mere networks of arrows are not enough. One needs also symmetry algebra associated with them giving flesh around the bones.

1. Various quantum algebras, in particular Yangians are naturally related to physically interesting categories. The article of Jimbo [B33], one of the pioneers of quantum algebras, gives a nice summary of Yang-Baxter equation central in the construction of quantum algebras. R-matrix performs is an endomorphism permuting two tensor factors in quantal matter.

2. One of the nice features of Yangian is that it gives hopes for a proper description of bound states problematic in quantum field theories (one can argue that QCD cannot really describe hadrons and already QED has problems with Bethe-Salpeter equation for hydrogen atom). The idea would be simple. Yangian would provide many-particle generalization of single particle symmetry algebra and give formulas for conserved charges of many-particle states containing also interaction terms. Interactions would reduce to kinematics. This - as I think - is a new idea.

   The iteration of the co-product Δ would map single particle symmetry operator by homomorphism to operator acting in N-parton state space and one would obtain a hierarchy of algebra generators labelled by N and Yangian inariance would dictate the interaction terms completely (as it indeed does in $\mathcal{N}=4$ SUSY in twistor Grassmannian approach [B15]).

3. There is however a delicacy involved. There is a mysterious looking doubling of the symmetry generators. One has besides ordinary local generators T_0^A generators T_1^A: in twistor Grassmann approach the latter correspond to dual conformal symmetries. For T_0^A the co-product is trivial: $\Delta(J_0^A) = J_0^A \otimes 1 + 1 \otimes J_0^A$, just like in non-interacting theory. This is true for all iterates of Δ.

   For J_1^A one has $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \otimes J_1^A + f^A_{BC} J_0^B \otimes J_0^C$. One has two representations and the duality suggests that the eigenvalues J_0^A and J_1^A are same (note that in Witten's approach [B14] $J_1^A = 0$ holds true so that it does not apply as such to TGD). The differences $T_0^A - T_1^A$ would give a precise meaning for "interaction charges" if the duality holds true, and more generally, to the perturbation theory formed by a pair of free and interacting theory. This picture raises hopes about first principle description of bound states: interactions described in wave mechanics in terms of phenomenological interaction Hamiltonians and interaction potentials would be reduced to kinematics.

   For instance, for four-momentum $\Delta(P_1^k)$ would contain besides free particle term $P_0^k \otimes 1 + 1 \otimes P_0^k$ also the interaction term involving generators of - say - conformal group.

4. What about the physical interpretation of the doubling? The most natural interpretation would be in terms of SSA and the extended super-conformal algebra assignable to the light-like orbits of partonic 2-surfaces. An attractive interpretation is in terms of a generalization of Equivalence Principle (EP) stating that inertial and gravitational charges are identical for the physical states.

5. The tensor summands of Kac-Moody algebra would have different coupling constants k_i perhaps assignable to the 4 fundamental interactions and to the dynamical gauge group emerging from the SCA would give further coupling constant. This would give 5 tensor

factors strongly suggested by p-adic mass calculations - p-adic masses depend only on the number of tensor factors [K41].

9.3 Some mathematical background about Yangians

In the following necessary mathematical background about Yangians are summarized.

9.3.1 Yang-Baxter equation (YBE)

Yang-Baxter equation (YBE) has been used for more than four decades in integrable models of statistical mechanics of condensed matter physics and of 2-D quantum field theories (QFTs) [A114]. It appears also in topological quantum field theories (TQFTs) used to classify braids and knots [B14] (see `http://tinyurl.com/mcvvcqp`) and in conformal field theories and models for anyons. Yangian symmetry appears also in twistor Grassmann approach to scattering amplitudes [B15, B20] and thus involves YBE. At the same time new invariants for links were discovered and new braid-type relation was found. YBEs emerged also in 2-D conformal field theories.

Yang-Baxter equation (YBE) has a long history described in the excellent introduction to YBE by Jimbo [B33] (see `http://tinyurl.com/l4z6zyr`, where one can also find a list of references). YBE was first discovered by McGuire (1964) and 3 years later by Yang in quantum mechanical many-body problem involving delta function potential $\sum_{i<j}\delta(x_i-x_j)$. Using Bethe's Ansatz for building wave functions they found that the scattering matrix factorized that it could be constructed using as building brick 2-particle scattering matrix - R-matrix. YBE emerged for R-matrix as a consistency condition for factorization. Baxter discovered 1972 solution of the eight vertex model in terms of YBE. Zamolodchikov pointed ot that the algebraic mechanism behind factorization of 2-D QFTs is same as in condensed matter models.

1978-1979 Faddeev, Sklyanin, and Takhtajan proposed quantum inverse scattering method as a unification of classical and quantum integrable models. Eventually the work with YBE led to the discovery of the notion of quantum group by Drinfeld. Quantum group can be regarded as a deformation $U_q(g)$ of the universal enveloping algebra $U(g)$ of Lie algebra. Drinfeld also introduced the universal R-matrix, which does not depend on the representation of algebra used.

R-matrix satisfying YBE is now the common aspect of all quantum algebras. I am not a specialist in YBE and can only list the basic points of Jimbo's article. Interested reader can look for details and references in the article of Jimbo.

In 2-D quantum field theories R-matrix $R(u)$ depends on one parameter u identifiable as hyperbolic angle characterizing the velocity of the particle. $R(u)$ characterizes the interaction experienced by two particles having delta function potential passing each other (see the figure of `http://tinyurl.com/kyw6xu6`). In 2-D quantum field theories and in models for basic gate in topological quantum computation (for early TGD vision see [K4] were also R-matrix is discussed in more detail) the R-matrix is unitary. One can interpret R-matrix as endomorphism mapping $V_1\otimes V_2$ to $V_2\otimes V_1$ representing permutation of the particles.

YBE

R-matrix satisfies Yang-Baxter equation (YBE)

$$R_{23}(u)R_{13}(u+v)R_{12}(v)=R_{12}(v)R_{13}(u+v)R_{23}(u) \tag{9.3.1}$$

having interpretation as associativity condition for quantum algebras.

At the limit $u,v\rightarrow\infty$ one obtains R-matrix characterizing braiding operation of braid strands. Replacement of permutation of the strands with braid operations replaces permutation group for n strands with its covering group. YBE states that the braided variants of identical permutations (23)(13)(12) and (12)(13)(23) are identical.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving YBE is a difficult challenge. Equations have symmetries, which are obvious on basis of the topological interpretation. Scaling and automorphism induced by linear transformations of

V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

One can pose to the R-matrix some boundary condition. For $V \otimes V$ the condition states that $R(0)$ is proportional to permutation matrix P for the factors.

General results about YBE

The following lists general results about YBE.

1. Belavin and Drinfeld proved that the solutions of YBE can be continued meromorphic functions to complex plane and define with poles forming an Abelian group. R-matrices can be classified to rational, trigonometric, and elliptic R-matrices existing only for $sl(n)$. Rational and trigonometric solutions have pole at origin and elliptic solutions have a lattice of poles. In [B33] (see `http://tinyurl.com/l4z6zyr`) simplest examples about R-matrices for $V_1 = V_2 = C^2$ are discussed, one of each type.

2. In [B33] it is described how the notions of R-matrix can be generalized to apply to a collection of vector spaces, which need not be identical. The interpretation is as commutation relations of abstract algebra with co-product Δ - say quantum algebra or Yangian algebra. YBE guarantees the associativity of the algebra.

3. One can define quasi-classical R-matrices as R-matrices depending on Planck constant like parameter $\hbar$ (which need have anything to do with Planck constant) such that small values of u one has $R = constant \times (I + \hbar r(u) + O(\hbar^2))$. $r(u)$ is called classical r-matrix and satisfies CYBE conditions

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0$$

   obtained by linearizing YBE. $r(u)$ defines a deformation of Lie-algebra respecting Jacobi-identities. There are also non-quasi-classical solutions. The universal solution for r-matrix is formulated in terms of Lie-algebra so that the representation spaces V_i can be any representation spaces of the Lie-algebra.

4. Drinfeld constructed quantum algebras $U_q(g)$ as quantized universal enveloping algebras $U_q(g)$ of Lie algebra g. One starts from a classical r-matrix r and Lie algebra g. The idea is to perform a "quantization" of the Lie-algebra as a deformation of the universal enveloping algebra $U_q(g)$ of $U(g)$ by r. Drinfeld introduces a universal R-matrix independent of the representation used. This construction will not be discussed here since it does not seem to be so interesting as Yangian: in this case co-product Δ does not seem to have a natural interpretation as a description of interaction. The quantum groups are characterized by parameter $q \in C$.

   For a generic value the representation theory of q-groups does not differ from the ordinary one. For roots of unity situation changes due to degeneracy caused by the fact $q^N = 1$ for some N.

5. The article of Jimbo discusses also fusion procedure initiated by Kulish, Restetikhin, and Sklyanin allowing to construct new R-matrices from existing one. Fusion generalizes the method used to construct group representation as powers of fundamental representation. Fusion procedure constructs R-matrix in $W \otimes V^2$, where one has $W = W_1 \otimes W_2 \subset V \otimes V^1$. Picking W is analogous to picking a subspace of tensor product representation $V \otimes V^1$.

9.3.2 Yangian

Yangian algebra $Y(g(u))$ is associative Hopf algebra (see `http://tinyurl.com/qfl8dwu`) that is bi-algebra consisting of associative algebra characterized by product μ: $A \otimes A \rightarrow A$ with unit element 1 satisfying $\mu(1, a) = a$ and co-associative co-algebra consisting of co-product $\Delta A \in A \otimes A$ and co-unit $\epsilon : A \rightarrow C$ satisfying $\epsilon \circ \Delta(a) = a$. Product and co-product are "time reversals" of each other. Besides this one has antipode S as algebra anti-homomorphism $S(ab) = S(b)S(a)$. YBE has interpretation as an associativity condition for co-algebra $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$. Also ϵ satisfies associativity condition $(\epsilon \otimes 1) \circ \Delta = (1 \otimes \epsilon) \circ \Delta$.

There are many alternative formulations for Yangian and twisted Yangian listed in the slides of Vidas Regelskis at `http://tinyurl.com/ms9q8u4`. Drinfeld has given two formulations and there is FRT formulation of Faddeev, Restetikhin and Takhtajan.

Drinfeld's formulation [B33] (see `http://tinyurl.com/qfl8dwu`) involves the notions of Lie bi-algebra and Manin triple, which corresponds to the triplet formed by half-loop algebras with positive and negative conformal weights, and full loop algebra. There is isomorphism mapping the generating elements of positive weight and negative weight loop algebra to the elements of loop algebra with conformal weights 0 and 1. The integer label n for positive half loop algebra corresponds in the formulation based on Manin triple to conformal weight. The alternative interpretation for $n+1$ would be as the number of factors in the tensor power of algebra and would in TGD framework correspond to the number of partonic 2-surfaces. In this interpretation the isomorphism becomes confusing.

In any case, one has two interpretations for $n+1 \geq 1$: either as parton number or as occupation number for harmonic oscillator having interpretation as bosonic occupation number in quantum field theories. The relationship between Fock space description and classical description for n-particle states has remained somewhat mysterious and one can wonder whether these two interpretation improve the understanding of classical correspondence (QCC).

Witten's formulation of Yangian

The following summarizes my understanding about Witten's formulation of Yangian in $\mathcal{N} = 4$ SUSYs [B14], which does not mention explicitly the connection with half loop algebras and loop algebra and considers only the generators of Yangian and the relations between them. This formulation gives the explicit form of Δ and looks natural, when n corresponds to parton number. Also Witten's formulation for Super Yangian will be discussed.

It must be however emphasized that Witten's approach is not general enough for the purposes of TGD. Witten uses the identification $\Delta(J_1^A) = f^A_{BC} J_0^B \times J_0^C$ instead of the general expression $\Delta(J_1^A) = J_1^A \otimes 1 + 1 \times J_1^A + f^A_{BC} J_0^B \times J_0^C$ needed in TGD strongly suggested by the dual roles of the super-symplectic conformal algebra and super-conformal algebra associated with the light-like partonic orbits realizing generalized EP. There is also a nice analogy with the conformal symmetry and its dual twistor Grassmann approach.

The elements of Yangian algebra are labelled by non-negative integers so that there is a close analogy with the algebra spanned by the generators of Virasoro algebra with non-negative conformal weight. The Yangian symmetry algebra is defined by the following relations for the generators labeled by integers $n = 0$ and $n = 1$. The first half of these relations discussed in very clear manner in [B14] follows uniquely from the fact that adjoint representation of the Lie algebra is in question

$$\left[J^A, J^B\right] = f_C^{AB} J^C \quad , \quad \left[J^A, J^{(1)B}\right] = f_C^{AB} J^{(1)C} \quad . \tag{9.3.2}$$

Besides this Serre relations are satisfied. These have more complex form and read as

$$\begin{aligned}
&\left[J^{(1)A}, \left[J^{(1)B}, J^C\right]\right] + \left[J^{(1)B}, \left[J^{(1)C}, J^A\right]\right] + \left[J^{(1)C}, \left[J^{(1)A}, J^B\right]\right] \\
&\qquad = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\} \ , \\
&\left[\left[J^{(1)A}, J^{(1)B}\right], \left[J^C, J^{(1)D}\right]\right] + \left[\left[J^{(1)C}, J^{(1)D}\right], \left[J^A, J^{(1)B}\right]\right] \\
&\qquad = \frac{1}{24} (f^{AGL} f^{BEM} f_K^{CD} \\
&\qquad + f^{CGL} f^{DEM} f_K^{AB}) f^{KFN} f_{LMN} \{J_G, J_E, J_F\} \ .
\end{aligned} \tag{9.3.3}$$

The indices of the Lie algebra generators are raised by invariant, non-degenerate metric tensor g_{AB} or g^{AB}. $\{A, B, C\}$ denotes the symmetrized product of three generators.

The right hand sides have often as a coefficient $\hbar^2$ instead of $1/24$. $\hbar$ need not have anything to do with Planck constant. The Serre relations give constraints on the commutation relations of $J^{(1)A}$. For $J^{(1)A=J^A}$ the first Serre relation reduces to Jacobi identity and second to antisymmetry of Lie bracket. The right hand sided involved completely symmetrized trilinears $\{J_D, J_E, J_F\}$ making sense in the universal covering of the Lie algebra defined by J^A.

Repeated commutators allow to generate the entire algebra whose elements are labeled by non-negative integer n. The generators obtain in this manner are n-local operators arising in $(n-1)$-commutator of $J^{(1)}$: s. For $SU(2)$ the Serre relations are trivial. For other cases the first Serre relation implies the second one so the relations are redundant. Why Witten includes it is for the purposed of demonstrating the conditions for the existence of Yangians associated with discrete one-dimensional lattices (Yangians exists also for continuum one-dimensional index).

Discrete one-dimensional lattice provides under certain consistency conditions a representation for the Yangian algebra. One assumes that each lattice point allows a representation R of J^A so that one has $J^A = \sum_i J_i^A$ acting on the infinite tensor power of the representation considered. The expressions for the generators J^{1A} in Witten's approach are given as

$$J^{(1)A} = f^A_{BC} \sum_{i<j} J_i^B J_j^C \ . \tag{9.3.4}$$

This formula gives the generators in the case of conformal algebra. This representation exists if the adjoint representation of G appears only one in the decomposition of $R \otimes R$. This is the case for SU(N) if R is the fundamental representation or is the representation of by k^{th} rank completely antisymmetric tensors.

This discussion does not apply as such to $\mathcal{N} = 4$ case the number of lattice points is finite and corresponds to the number of external particles so that cyclic boundary conditions are needed guarantee that the number of lattice points reduces effectively to a finite number. Note that the Yangian in color degrees of freedom does not exist for SU(N) SYM.

As noticed, Yangian algebra is a Hopf algebra and therefore allows co-product. The co-product Δ is given by

$$\begin{aligned} \Delta(J^A) &= J^A \otimes 1 + 1 \otimes J^A \ , \\ \Delta(J^{(1)A}) &= J^{(1)A} \otimes 1 + 1 \otimes J^{(1)A} + f^A_{BC} J^B \otimes J^C \end{aligned} \tag{9.3.5}$$

Δ allows to imbed Lie algebra to the tensor product in non-trivial manner and the non-triviality comes from the addition of the dual generator to the trivial co-product. In the case that the single spin representation of $J^{(1)A}$ is trivial, the co-product gives just the expression of the dual generator using the ordinary generators as a non-local generator. This is assumed in the recent case and also for the generators of the conformal Yangian.

Super-Yangian

Also the Yangian extensions of Lie super-algebras make sense. From the point of physics especially interesting Lie super-algebras are $SU(m|m)$ and $U(m|m)$. The reason is that $PSU(2,2|4)$ (P refers to "projective") acting as super-conformal symmetries of $\mathcal{N} = 4$ SYM and this super group is a real form of $PSU(4|4)$. The main point of interest is whether this algebra allows Yangian representation and Witten demonstrated that this is indeed the case [B14].

These algebras are Z_2 graded and decompose to bosonic and fermionic parts which in general correspond to n- and m-dimensional representations of $U(n)$. The representation associated with the fermionic part dictates the commutation relations between bosonic and fermionic generators. The anti-commutator of fermionic generators can contain besides identity also bosonic generators if the symmetrized tensor product in question contains adjoint representation. This is the case if fermions are in the fundamental representation and its conjugate. For $SU(3)$ the symmetrize tensor product of adjoint representations contains adjoint (the completely symmetric structure constants d_{abc}) and this might have some relevance for the super $SU(3)$ symmetry.

The elements of these algebras in the matrix representation (no Grassmann parameters involved) can be written in the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad .$$

a and d representing the bosonic part of the algebra are $n \times n$ matrices and $m \times m$ matrices corresponding to the dimensions of bosonic and fermionic representations. b and c are fermionic matrices are $n \times m$ and $m \times n$ matrices, whose anti-commutator is the direct sum of $n \times n$ and $n \times n$ matrices. For $n = m$ bosonic generators transform like Lie algebra generators of $SU(n) \times SU(n)$ whereas fermionic generators transform like $n \otimes \overline{n} \oplus \overline{n} \otimes n$ under $SU(n) \times SU(n)$. Supertrace is defined as $Str(x) = Tr(a) - Tr(b)$. The vanishing of Str defines $SU(n|m)$. For $n \neq m$ the super trace condition removes identity matrix and $PU(n|m)$ and $SU(n|m)$ are same. That this does not happen for $n = m$ is an important delicacy since this case corresponds to $\mathcal{N} = 4$ SYM. If any two matrices differing by an additive scalar are identified (projective scaling as now physical effect) one obtains $PSU(n|n)$ and this is what one is interested in.

Witten shows that the condition that adjoint is contained only once in the tensor product $R \otimes \overline{R}$ holds true for the physically interesting representations of $PSU(2,2|4)$ so that the generalization of the bilinear formula can be used to define the generators of $J^{(1)A}$ of super Yangian of $PU(2,2|4)$. The defining formula for the generators of the Super Yangian reads as

$$\begin{aligned} J_C^{(1)} &= g_{CC'} J^{(1)C'} = g_{CC'} f^{C'}_{AB} \sum_{i<j} J_i^A J_j^B \\ &= g_{CC'} f^{C'}_{AB} g^{AA'} g^{BB'} \sum_{i<j} J^i_{A'} J^j_{B'} \quad . \end{aligned} \tag{9.3.6}$$

Here $g_{AB} = Str(J_A J_B)$ is the metric defined by super trace and distinguishes between $PSU(4|4)$ and $PSU(2,2|4)$. In this formula both generators and super generators appear.

9.4 Yangianization in TGD framework

Yangianization of quantum TGD is quite challenging. Super-conformal algebras are much larger than in say $\mathcal{N} = 4$ SUSY and even in superstring models and reconnection and 2-braiding are new topological elements.

9.4.1 Geometrization of super algebras in TGD framework

Super-conformal algebras allow a geometrization in TGD framework and this should be of considerable help in the Yangianization.

1. The basic generators of various Super-algebras follow from modified Dirac action as Noether charges and their super counterparts obtained by replacing fermion field Ψ (its conjugate $\overline{\Psi}$) by a mode u_m ($\overline{u}_n$) of the induced spinor field [K88, K63]. The anti-commutators of these Noetherian super charges labelled by n define WCW gamma matrices. The replacement of both Ψ and $\overline{\Psi}$ with modes u_m and $\overline{u}_n$ gives a collection of conserved c-number currents and charges labelled by (n,m). These c-number charges define the anti-commutation relations for the induced spinor fields so that quantization reduces to dynamics thanks to the notion of modified gamma matrices forced by super-conformal symmetry.

2. The natural generalization of Sugawara formula to the level of Yangian of SKMA starts from the Dirac operator for WCW defined like ordinary Dirac operator in terms of the contractions of WCW gamma matrices with the isometry generators (SCA) replacing the Super Virasoro generators G_r and WCW d'Alembert operator defined as its square replacing Virasoro generators L_n. Anti-commutators of WCW gamma matrices defined by super charges for super-symplectic generators define WCW Kähler metric [K88] for which action for preferred extremal would define Kähler function for WCW metric [K35].

3. Quarks and leptons give rise to a doubling of WCW metric if associated with same space-time sheet that is with the same sector of WCW. The duplication of the super algebra generators - in particular WCW gamma matrices - does not seem to make sense. Do quarks and leptons therefore correspond to different sectors of WCW and live at different space-time surfaces? But what could distinguish between 3-surfaces associated with quarks and leptons?

   Could quarks be associated with homologically non-trivial partonic 2-surfaces with CP_2 homology charges 2,-1,-1 proportional to color hypercharges $2/3, -1/3, -1/3$ and leptons with partonic 2-surfaces with vanishing homology charges coming as multiples of 3? Vanishing of color hypercharge for color-confined states would topologize to a vanishing of total homology charge. Could spin/isospin half property of fundamental fermions topologize to 2-sheeted structure of the space-time surface representing elementary particle consisting of elementary fermions?

SSA acting as isometries of WCW is not the only super-conformal algebra involved.

1. Partonic 2-surfaces are ends of light-like 3-surfaces- partonic orbits - and give rise to a generalization of SKMA of isometries of H so that they act as local isometries preserving the light-likeness property of the orbits. At the ends of the partonic 2-surface SKMA is associated with complex coordinate of partonic 2-surface. What is the role of this algebra, which is also extended SKMA (already christened PSCA) but with light-like coordinate parameterizing the SKMA generators?

   Is it an additional symmetry combining with string world sheet symmetries to a symmetry involving complex coordinate and complex or hypercomplex coordinate? Or is it dual to the string world sheet symmetry? How do these symmetries relate to SSA? Does SGCI implying SH leave only SKMAs associated with isometries, holonomies of CP_2 (electroweak interactions) and dynamical SKMA remaining as remnant of SCA.

2. I have earlier proposed that Equivalence Principle (EP) as identity of inertial and gravitational charges could reduce to the duality between these SSA assignable to strings and the partonic super-conformal algebra. This picture conforms with the expected form of the generators associated with these algebras. The dual generating elements T_0^A *resp.* T_1^A associated with generic Yangian could naturally correspond to isomorphic sub-algebras of super-conformal algebra associated with orbits of partonic 2-surfaces *resp.* super-symplectic algebra assignable to string world sheets.

9.4.2 Questions

There are many open questions to be answered.

Q1: What Yangianization could mean in TGD framework? The answer is not obvious and one can consider two options.

1. Assuming that SH leads to an effective reduction of super-symplectic algebra to finite-D Kac-Moody algebra, assign to partonic 2-surfaces direct sum of Kac-Moody type algebras $L(g) = g(z, z^{-1})$ assigned with complex coordinate z of partonic 2-surface. One could perform Yangianization for this algebra meaning that these symmetries become multi-local with locus identified as partonic 2-surface.

   In Drinfeld's approach this would mean Yangianization of $L(g)$ rather than g and would involve double loop algebra $L(L(g))$ and its positive and negative energy parts. In Minkowskian space-time regions the generators would be functions of complex coordinate z and hypercomplex coordinate u associated with string world sheet: in Euclidian space-time regions one would have 2 complex coordinates z and w. This would conform with holography. I do not know whether mathematicians have considered this generalization and whether it is possible. In the following this is assumed.

2. Physical states at partonic 2-surfaces consist of pointlike fermions and one can ask whether this actually means that one can consider just the Lie algebra g so that in Drinfeld's approach one would have just string world sheets and $Y(g)$. Already this option requires the

algebraization of reconnection mechanism as a new element. Whether this simpler approach make sense for fermions and by QQC for quantum TGD, is not clear.

Q2: Can one really follow the practice of Grassmannian twistor approach and say that T_1^A and TA^0 are dual?

One has $[T_0^A, T_1^B] = f_C^{AB} T_1^C$. Witten's definition $T_1^A = f_{BC}^A T^B \otimes T^C \equiv T_1^A = f_{BC}^A T^B T^C$ with T_1^A identified as total charges for lattice, identifies T_1^A as 2-particle generators of Yangian. One the other hand, in TGD T_0^A would correspond to partonic super-conformal algebra and T_1^A to bi-local super-symplectic algebra and the general definition to be used regards also T_1^A as single particle generators in Yangian sense and defines the generators at 2-particle level as $\Delta(T_0^A) = T_0^A \otimes 1 + 1 \otimes T_0^A$ and $\Delta(T_1^A) = T_1^A \otimes 1 + 1 \otimes T_1^A + f_{BC}^A T_0^B \otimes T_0^C$.

For the Witten's definition one cannot demand that T_0^A and T_1^A have same eigenvalues for the physical states. For the more general definition of Δ to be followed in the sequel it seems to be possible require that T_0^A and T_1^A obey the same commutation relations for appropriate sub-algebras at least, and that it is possible to diagonalize Cartan algebras simultaneously and even require same total Cartan charges. This issue is not however well-understood.

Q3: What algebras are Yangianized in TGD framework?

The Yangians of SKMAs associated with isometries of $M^4 \times CP_2$ and with the holonomy group $SU(2) \times U(1)$ of CP_2 appear as symmetries. M^4 should give SKMA in transversal degrees of freedom for fermionic string. CP_2 isometries would give SKMA associated with SU(3). $SU(2) \times U(1)$ would be assignable to electroweak symmetries. This gives 4 tensor factors.

Five of them are required by p-adic mass calculations [K41], whose outcome depends only on the number of tensor factors in Virasoro algebra. The estimates for the number of tensor factors has been a chronic head ache: in particular, do M^4 SKMA correspond to single tensor factor or two tensor factors assignable to 2 transversal degrees of freedom.

Supersymplectic algebra (SSA) is assumed to define maximal possible isometry group of WCW guaranteeing the existence of Kähler metric with a well-defined Riemann connection. The Yangian of SSA could be the ultimate symmetry group, which could realize the dream about the reduction of all interactions to mere kinematics. If SSA effectively reduces to a finite-D SKMA for fermionic strings, one would have 5 tensor factors.

Q4: What does SSA mean?

1. SSA is associated with light-cone boundary $\delta M_\pm^4$ with one light-like direction. The generators (to be distinguished from generating elements) are products of Hamiltonians of symplectic transformations of CP_2 assignable to representations of color SU(3) and Hamiltonians for the symplectic transformations of light-cone boundary, which reduce to Hamiltonians for symplectic transformations of sphere S^2 depending parametrically on the light-like radial coordinate r. This algebra is generalized to analog of Kac-Moody algebra defined by finite-dimensional Lie algebra.

2. The radial dependence of Hamiltonians of form r^h. The naïve guess that conformal weights are integers for the bosonic generators of SSA is not correct. One must allow complex conformal weights of form $h = 1/2 + iy$: $1/2$ comes from the scaling invariant inner product for functions at $\delta M_\pm^4$ defined by integration measure dr/r [K20, K63].

3. An attractive guess [L9] is that there is an infinite number of generating elements with radial conformal weights given by zeros of zeta. Conformal confinement must holds true meaning that the total conformal weights are real and thus half-odd integers. The operators creating physical states form a sub-algebra assignable by SH and QCC to fermionic string world sheets connecting partonic 2-surfaces.

4. SH inspires the assumption that preferred extremal property requires that sub-algebra H of SSA isomorphic to itself (conformal weights are integer multiples of SSA) and its commutator SH with SH annihilate physical states and classical Noether charges vanish. This could reduce the symmetry algebra to SKMA for a finite-dimensional Lie group. SSA could be replaced also with the sub-algebra creating physical states having half-odd integer valued radial conformal weights.

Similar conditions could make sense for the generalization of super-conformal KM algebra associated with light-like partonic orbits.

Q5: What is the precise meaning of SH in the fermionic sector?

Are string world sheets with their ends behaving like pointlike particles enough or are also partonic 2-surface needed. For the latter option a generalization of conformal field theory (CFT) would be needed assigning complex coordinate with partonic 2-surfaces and hyper-complex or complex coordinates with string world sheets. Elementary particle vacuum functionals depend on conformal moduli of partonic 2-surface [K17], which supports the latter option.

There could be however duality between partonic 2-surfaces and string world sheets so that either of them could be enough [L26]. There is also uncertainty about the relationship between induced spinor fields at string world sheets and space-time interior. Are 4-D induced spinor fields obtained by process analogous to analytic continuation in 2-complex dimensional space-time or do 2-D induced spinor fields serve as sources for 4-D induced spinor fields?

Quantum algebras are characterized by parameters such as complex parameter q characterizing R-matrices for quantum groups. Adelic physics [L22] demands number theoretical universality and in particular demands that the parameters - say q - of quantum algebraic structures involved are products $q = e^{m/n}xU$, where U is root of unity (note that e^p exists as ordinary p-adic number for Q_p) and x is real number in the extension. This guarantees that the induced extensions of p-adic numbers are finite-dimensional (the hypothesis is that the correlates of cognition are finite-D extensions of p-adic number fields) [K63].

In the recent view about twistorial scattering amplitudes [L26] the fundamental fermionic vertices are $2 \rightarrow 2$ vertices. There is no fermionic contact interaction in the sense of QFT but the fermions coming to the topological vertex defined by partonic 2-surface at which 3 partonic orbits meet (analogy for the 3-vertex for Feynman diagram) are re-distributed between partonic two surfaces. Also in integrable 2-D QFTs in M^2 the vertices are $2 \rightarrow 2$ vertices characterized by R-matrix. The twistorial vertex is however not topological.

9.4.3 Yangianization of four-momentum

The QFT picture about bound states is unsatisfactory. The basic question to be answered is whether one should approach the problem in terms of Lorentz invariant mass squared natural in conformal field theories or in terms of Poincare algebra. It is quite possible that the fundamental formulation allowing to understand binding energies is in terms of SCA and PSCA.

Twistor lift of TGD [L26] however suggests that Poincare and even finite-D conformal transformations associated with M^2 could play important role. These longitudinal degrees of freedom are non-dynamical in string dynamics. Maybe there is kind of sharing of labor between these degrees of freedom. In the following we consider two purely pedagogical examples about Yangianization of four-momentum in M^4 and in 8-D context regarding four-momentum as quaternionic 8-momentum in M^8.

Yangianization of four-momentum in conformal algebra of M^4

Consider as an example what the Yangianization for four-momentum P^k could mean. This is a pedagogical example.

1. The first thing to notice is that the commutation relations between P^k_0 and P^k_1 are inherited from those between P^k_0 and force P^k_1 and P^k_0 to commute. This holds true quite generally for Cartan algebra so that if the correspondence between T^A_0 and T^A_1 respects Cartan algebra property then Cartan algebras of T^A_0 and T^A_1 can be simultaneously diagonalized for the physical states. The Serre relations of Eq. 9.3.3 are identically satisfied for Cartan algebra and its image. This is consistent with the assumption that Cartan algebra is mapped to Cartan algebra but does not prove it.

2. The formula $f^A_{BC}T^A_0 \otimes T^C_0$ for the interaction term appearing in the expresion of Δ should be non-trivial also when T^A corresponds to four-momentum. Already the Poincare algebra gives this kind of term built from Lorentz generators and translation generators.

The extension of Poincare algebra extended to contain dilatation operator D can be considered as also M^4 conformal algebra with generators of special conformal transformations M^A included (see `http://tinyurl.com/nxlmfug`). One has doubling of all algebra generators. The interpretation as gravitational and inertial momenta is one possibility, and EP suggests that the two momenta have same values. In twistor Grassmannian approach the conformal algebras are regarded as dual and suggests the same. Hence one would have $P_0^k = P_1^k$ at the level of eigenvalues.

3. For conformal group the proposed co-product for P_i^k would read as

$$
\begin{aligned}
\Delta(P_0^k) &= P_0^k \otimes 1 + 1 \otimes P_0^k \ , \\
\Delta(P_1^k) &= P_1^k \otimes 1 + 1 \otimes P_1^k + K f_{Al}^k (L_0^A \otimes P_0^l - P_0^k \otimes L_0^A) + K f_{Al}^k (M_0^A \otimes P_0^l - P_0^l \otimes M_0^A) \\
&+ K(D_0 \times P_0^k - P_0^k \times D_0) \ .
\end{aligned}
\tag{9.4.1}
$$

This condition could be combined with the condition for mass squared operator. For $K = 0$ one would have additivity of mass squared requiring that P_1 and P_2 are parallel and light-like. For $K \neq 0$ it might be possible to have a simultaneous solution to the both conditions with massive total momentum.

The Δ-iterates of P_0^k contain no interaction terms. For P_1 one has interaction term. This holds true for all symmetry generators. Assume $P_0 = P_1$: does this mean that the interacting theory associated with P_1 is dual to free theory? The difference $\Delta P_0^k - \Delta(P_1^k)$ defines the analog interaction Hamilton, which would therefore be not due to a somewhat arbitrary decomposition of four-momentum to free and interaction parts. It should be possible to possible to measure this difference and its counterpart for other quantum numbers. One can only make questions about the interpretation for this duality applying to all quantum numbers.

1. In Drinfeld's construction the negative and positive energy parts of loop algebra would be related by the duality. In ZEO it might be possible to relate them to positive and negative energy parts of zero energy states at the opposite boundaries of CD.

2. If n is interpreted as number of partonic surfaces and the generators are interpreted as in Witten's construction then the duality could be seen as a geometric duality in plane mapping edges and vertices (partonic 2-surfaces ordered in sequence and string between them) to each other. In super-conformal algebra of twistor Grassmannian approach the generators T_0^A and T_1^A are associated with vertices and edges of the polygon defining the scattering diagram and this suggests that T_0^A corresponds to partonic 2-surfaces and T_1^A to the strings world sheets.

3. Could the duality be a generalization of for Equivalence Principle identifying inertial and gravitational quantum numbers? This interpretation is encouraged by the presence of SSA action on space-like 3-surfaces at the ends of CDs and extended super-conformal algebra associated with the light-like orbits of partons: SGCI would suggest that these algebras or at least their appropriate sub-algebra are dual. This interpretation conforms also with the above geometric interpretation and twistor Grassmannian interpretation.

Consider for simplicity the situation in which only scaling generator D is present in the extension.

1. Suppose that one has eigenstate of total momentum $\Delta(P_0^k)$ resp. $\Delta(P_1^k)$ with eigenvalue p_0^{tot} resp. p_1^{tot} and that

$$
p_0^{tot} = p_1^{tot} \tag{9.4.2}
$$

holds true.

2. Since D_0 and P_0^k do not commute, the action of D_0 must be realized as differential operator $D_0 = ip_0^k d/dp_0^k$ so that one has following eigenvalue equations

$$\begin{aligned} \Delta(P_0^k)\Psi &= (p_{0,1}^k + p_{0,2}^k)\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_1^k)\Psi &= (p_{1,1}^k + p_{1,2}^k)\Psi + K(ip_{0,1}^k \otimes p_{0,2}^r \frac{d}{dp_{0,2}^r} - ip_{0,1}^r \frac{d}{dp_{0,1}^r} \otimes p_{0,2}^k)\Psi = p_1^{tot}\Psi \end{aligned} \quad (9.4.3)$$

Ψ must be a superposition of states $|p_{0,1}, p_{0,2}\rangle$. One has non-trivial interaction. Analogous interaction terms mixing states with different momenta emerge from the terms involving Lorentz generators and special conformal generators.

Four-momenta as quaternionic 8-momenta in octonionic 8-space

In octonionic approach to twistorial scattering amplitudes particles can be regarded as massless in 8-D sense [L26]. The light-like octonionic momenta are actually quaternionic and one would obtain massive states in 4-D sense. Different 4-D masses would correspond to discrete set of quaternionic momenta for 8-D massless particle. Could the above conditions generalize to this case?

1. Suppose that the symmetries reduce to Poincare symmetry and to a number theoretic color symmetry acting as automorphisms of octonions. In this case the four-momentum for a given $M^4 \subset M^8$ decomposes to a sum of to a direct sum of M^2 invariant under SU(3) and E^2 invariant under $SU(2) \times U(1) \subset SU(3) \subset G_2$. ΔP_1 would be non-trivial for the transversal momentum and of form

$$\begin{aligned} \Delta(P_0^{L,k})\Psi &= (p_{0,1}^{L,k} + p_{0,2}^{L,k})\Psi = p_0^{tot}\Psi \ , \\ \Delta(P_0^{T,k})\Psi &= (P_0^{T,k} \otimes 1 + 1 \otimes P_0^{T,k})\Psi \ , \\ \Delta(P_1^{L,k})\Psi &= (p_{1,1}^{L,k} + p_{1,2}^{L,k})\Psi = P_1^{L,tot}\Psi \ , \\ \Delta(P_1^{T,k})\Psi &= (P_1^{T,k} \otimes 1 + 1 \otimes P_1^{T,k} + Kf_{Al}^k(ip_{0,1}^l \otimes t_{0,2}^A - i(ip_{0,2}^l \otimes t_{0,2}^A)\Psi \ . \end{aligned} \quad (9.4.4)$$

Here P_0^L resp. P_0^T represents longitudinal resp. transversal momentum and T_0^b denotes $SU(2) \subset SU(3)$ generator representable as differential operator acting on complexified momentum and $p_0^T = p_0^{T,x} + ip_0^{T,y}$ and its conjugate.

2. In transversal degrees of freedom the assumption about momentum eigenstates would be probably too strong. String model suggests Gaussian in transversal oscillator degrees of freedom. Hadronic physics suggests an eigenstate of transversal momentum squared. TGD based number theoretic considerations suggest that the transversal state is characterized by color quantum numbers.

   Hence the conditions

$$p_0^{L,tot} = p_1^{L,tot} \quad , \quad (p_0^{T,tot})^2 = (p_1^{T,tot})^2 \quad (9.4.5)$$

   are natural. It would be nice if the momenta p_{01} and p_{02} could be chosen to be on mass shell and satisfy stringy formula for mass squared where transverse momentum squared would correspond to stringy contribution.

One can also add to $\Delta(P)$ the terms coming from conformal group of M^4 or its subgroup. Since octonionic momentum is light-like M^2 momentum for a suitable choice of M^2, one must consider the possibility that the conformal group is that of $M^2 \subset M^4$. Twistorialization supports this view [L26]. The action of conformal generations would be on longitudinal momentum only.

One can wonder how gauge interactions and gravitational interaction do fit to this picture. Is the extension to super-conformal algebra and supersymplectic algebra the only manner to obtain gauge interactions and gravitation into the picture?

9.4.4 Yangianization for mass squared operator

It would be nice to have universal mass formulas as a generalization of mass squared formula for string models in terms of the conformal scaling generator $L_0 = zd/dz$. This operator should have besides single particle contributions also many particle contributions in bound states analogous to interaction Hamiltonian and interaction potential. Yangian as an algebra containing multi-local generators is a natural candidate in this respect.

One can consider Yangianization of Super Virasoro algebra (SVA). The Yangianization of various Super Kac-Moody algebras (SKMA) seems however more elegant if it induces the Yangianization of SVA. Consider first direct Yangianization of SVA. The commutation relations for SVA will be used in the sequel. They can be found in Wikipedia (see `http://tinyurl.com/klsgquz`) so that I do not bother to write them here. It must be emphasized that there might be delicate mathematical constraints on algebras which allow Yangianization as the article of Witten [B14] shows. The considerations here rely on physical intuition with unavoidable grain of wishful thinking.

What about the Yangian variant of mass squared operator m^2in terms of the conformal scaling generator $L_0 = zd/dz$? Consider first the definition of various Super algebras in TGD framework.

1. In standard approach the basic condition at single particle level $L_0\Psi = h_{vac}\Psi$ giving the eigenvalues of m^2. Massless in generalize sense requires $h_{vac} = 0$. One would have $m^2_{op} = L_0^{vib} + h_{vac}Id$, where "vib" refers to vibrational degrees of freedom of Kac-Moody algebra (KMA). Sugawara construction [A75] allows to express the left-hand side of this formula in terms of Kac-Moody generators - one has sum over squares $T^a_n T_a^{-n}$. One can say that mass squared is Casimir operator vibrational degrees of freedom for KMA

2. In absence of interactions - and always for $L_{0,0}$ - mass squared formula gives $m_1^2 + m_2^2 = L_0^{vib,1} + L_0^{vib,2}$ for vanishing vacuum weights. It is important to notice that this does *not* imply the additivity of mass squared since one does not have $(p_1 + p_2)^2 = m_1^2 + m_2^2$, which can hold true only for massless and parallel four-momenta. I have considered the possible additivity of mass mass squared for mesons [K51] but it of course fails for systems like hydrogen atom.

One can look what Yangianization of Super Virasoro algebra could mean.

1. One would have doubling of the generators of SKMA and SVA: one possible explanation is in terms of generalized EP. The difference $\Delta(T_0^A) - \Delta(T_1^A)$ would define the analog of interaction Hamiltonian of the duality holds true.

   One has $L_0 = G_0^2/2$. Quite generally, one has $\{G_r, G_{-r}\} = 2L_0$ apart from the central extension term. Generalization Yangian to Super Algebra suggests that one has

$$\begin{aligned}
\Delta(L_{0,0}) &= L_{0,0} \otimes 1 + 1 \otimes L_{0,0} \ , \\
\Delta(L_{1,0}) &= L_{1,0} \otimes 1 + 1 \otimes L_{1,0} + K \sum_n G_{0,r} \otimes G_{0,-r}
\end{aligned} \tag{9.4.6}$$

   Both operators give the value of h_{vac} expected to vanish when acting on physical states and the eigenvalues of the interaction mass squared $K \sum_n G_2 \otimes G_{-r}/2$ would represent the difference $m^2_{0,1} + m^2_{0,2} - m^2_{2,1} - m^2_{2,2}$. By Lorentz invariance the interaction energy is expected to be proportional to the inner product $P_1 \cdot P_2$ and the interpretation in terms of gravitational interaction energy is attractive. The size scale of K would be determined by $l_P^2/R^2 \simeq 2^{-12}$, where l_P is Planck length and R is CP_2 radius gravitational constant [K8, L26].

2. The action of $k \sum_n G_{0,n} \otimes G_{0,-n}/2$ on state $|p_1, p_2\rangle$ is analogous to the action of a tensor product of Dirac operators on tensor product of spinors. Since Dirac operator changes chirality, this suggests that the states are superpositions of eigenstates of chirality of form

$$\Psi = G_{0,0}\Psi_1 \otimes \Psi_2 + \epsilon \times \Psi_1 \otimes G_{0,0}\Psi_2 \ , \ \ \epsilon = \pm 1 \ .$$

$L_{0,0}\Psi_i = 0$ and $\Delta(L_{0,0})\Psi = 0$ holds true. $\Delta(G_{0,0})$ and $\Delta(G_{1,0})$ are given by

$$\begin{aligned} \Delta(G_{0,0}) &= G_{0,0} \otimes 1 - \epsilon \times 1 \otimes G_{0,0} \ , \\ \Delta(G_1, 0) &= G_{1,0} \otimes 1 - \epsilon \times 1 \otimes G_{1,0} - \tfrac{3K}{2} \textstyle\sum_r r(L_{0,r} \otimes G_{0,-r} - (G_{0,-r} \otimes L_{0,r}) \ , \end{aligned} \tag{9.4.7}$$

and should annihilate Ψ. This is true if $L_{1,r}$ and $L_{0,r}$ annihilate the states.

3. Perhaps the correct approach reduces to the Yangianization of SKMAs (including the dynamically generated SKM two which SSA effectively reduces by gauge conditions) provided that it induces Yangianization of SVA. Momentum components would be associated with KM generators for M^4 excitations of strings such that only transversal excitations are dynamical.

   For fermionic and bosonic generators of SKMA one would have

$$\begin{aligned} \Delta(F_0^a) &= F_0^a \otimes 1 + 1 \times F_0^a \ , \\ (F_1^a) &= F_1^a \otimes 1 + 1 \times F_1^a + K f_a^{Ab}(T_0^A \otimes F_0^b - F_0^b \otimes T_0^A) \ , \\ \\ \Delta(T_0^A) &= T_0^A \otimes 1 + 1 \otimes T_0^A \ , \\ \Delta(T_1^A) &= T_1^A \otimes 1 + 1 \otimes T_1^A + f_{BC}^A (T_0^B \otimes T_0^C \ . \end{aligned} \tag{9.4.8}$$

   Yangianization of SKMA would introduce interaction terms.

9.5 Category theory as a basic tool of TGD

I have already earlier developed ideas about the role of category theory in TGD [K15, K14, K9]. The hierarchy formed by categories, categories of categories, ... could allow to keep book about the complexity due to various hierarchies. WCW geometry with its huge symmetries combined with adelic physics; quantum states identified in ZEO as WCW spinor fields having topological interpretation as braided fusion categories with reconnection; the local symmetry algebras of quantum TGD extended to Yangians realizing elegantly the construction of interacting many-particle states in terms of iterated Δ operation assigning fundamental interactions to tensor summands of SKMAs: these could be the pillars of the basic vision.

9.5.1 Fusion categories

While refreshing my rather primitive physicist's understanding of categories, I found an excellent representation of fusion categories and braided categories [B2] introduced in topological condensed matter physics. The idea about product and co-product as fundamental vertices is not new in TGD [K9, K77, L26] but the physicist's view described in the article provided new insights.

Consider first fusion categories.

1. In TGD framework scattering diagrams generalize Feynman diagrams in the sense that in 3-vertices the 2-D ends for orbits of 3 partonic 2-surfaces are glued together like the ends of lines in 3-vertex of Feynman diagram. One can say that particles fuse or decay. 3-vertex would be fundamental vertex since higher vertices are unstable against splitting to 3-vertices. Braiding and reconnection would bring in additional topological vertices. Note that reconnection represents basic vertex in closed string theory and appears also in open string theory.

   Also fusions and splittings of 3-surfaces analogous to stringy trouser vertex appear as topological vertices but they do not represent particle decays but give rise to two paths along, which particles travel simultaneously: they appear in the TGD based description of double slit experiment. This is a profound departure from string models.

The key idea is that scattering diagrams are analogous to algebraic computations: the simplest computation corresponds to tree diagram apart from possible braiding and reconnections to be discussed below giving rise to purely topological dynamics. One has a generalization of the duality of the hadronic string model: one does not sum over all diagrams but takes only one of them, most naturally the simplest one. This is highly reminiscent to what happens for twistor Grassmann amplitudes.

One can eliminate all loops by moves and modify the tree diagram by moving lines along lines [?] Scattering diagrams would reduce to tree diagrams having in given vertex either product μ or its time reversal Δ plus propagator factors connecting them. The scattering amplitudes associated with tree diagrams related by these moves were earlier assumed to be identical. With better understanding of fusion categories I realized that the amplitudes corresponding to equivalent computations need not be numerically identical but only unitarily related and in this sense physically equivalent in ZEO.

2. Fusion categories indeed realize algebraically in very simple form the idea that all scattering diagrams reduce to tree diagrams with 3-vertices as basic vertices. Fusion categories [B2] (the illustrations `http://tinyurl.com/l2jsrzc` are very helpful) involve typically tensor product $a \otimes b$ of irreducible representations a and b of an algebraic structure decomposed to irreducible representations c. This product is counterpart for the 3-parton vertex generalizing Feynmanian 3-vertex.

   The article gives a graphical representation for various notions involved and these help enormously to concretize the notions. Fusion coefficients in $a \otimes b = N_{ab}^c c$ must satisfy consistency conditions coming from commutativity and associativity forcing the matrices $(N_a)_{bc} = N_{ab}^c$ to commute. One can diagonalize N_a simultaneously and their largest eigenvalues d_a are so called quantum dimensions. Fusion category contains also identity object and its presence leads to the identification of gauge invariants defining also topological invariants.

   The fusion product $a \otimes b$ has decomposition $V_{ab}^{c\alpha}|c, \alpha\rangle$ for each c. Co-product is an analog of the decay of particle to two particles and product and co-product are inverses of each other in a well-defined sense expressed as an algebraic identities. This gives rise to completeness relations from the condition stating that states associated with various c form a complete basis for states for $a \otimes b$ and orthogonality relations for the states of associated with various c coefficients. Square roots of quantum dimensions d_a appear as normalization factors in the equations.

   Diagrammatically the completeness relation means that scattering $ab \rightarrow c \rightarrow cd$ is trivial. This cannot be the case and the completeness relation must be more general. One would expect unitary S-matrix instead of identity matrix. The orthogonality relation says that loop diagram for $c \rightarrow ab \rightarrow c$ gives identity so that one can eliminate loops.

   Further conditions come from the fact that the decay of particle to 3 particles can occur in two ways, which must give the same outcome apart from a unitary transformation denoted by matrix F (see Eq. (106) of `http://tinyurl.com/l2jsrzc`). Similar consistency conditions for decay to 4 particles give so called pentagon equation as a consistency condition (see Eq. (107) and Fig. 9 of `http://tinyurl.com/l2jsrzc`). These equations are all that is needed to get an internally consistent category.

In TGD framework the fusion algebra would be based on Super Yangian with super Variant of Lie-algebra commutator as product and Yangian co-product of form already discussed and determining the basic interaction vertices in amplitudes. Perhaps the scattering amplitude for a given space-time surface transforming two categories at boundaries of CD to each other could be seen as a diagrammatic representation of category defined by zero energy state.

9.5.2 Braided categories

Braided categories [B2] (see `http://tinyurl.com/l2jsrzc`) are fusion categories with braiding relevant in condensed matter physics and also in TGD.

1. Braiding operation means exchange of braid strands defining particle world-lines at 3-D light-like orbits of partonic 2-surfaces (wormhole throats) defining the boundaries between Minkowskian and Euclidian regions of space-time surface. Braid operation is naturally realized in TGD for fermion lines at orbits of partonic 2-surfaces since braiding occurs in codimension 2.

2. For quantum algebras braiding operation is algebraically realized as R-matrix satisfying YBE (see `http://tinyurl.com/l4z6zyr`). R-matrix is a representation for permutation of two objects represented quantally. Group theoretically the braid group for n-braid system is covering group of the ordinary permutation group.

   In 2-D QFTs braiding operation defines the fundamental $2 \rightarrow 2$ scattering defining R-matrix as a building brick of S-matrix. This scattering matrix is trivial in the sense that the scattering involves only a phase lag but no exchange of quantum numbers: particles just pass by each other in the 2-particle scattering. This kind of S-matrix characterizes also topological quantum field theories used to deduce knot invariants as its quantum trace [A59, A23, A68]. I have considered knots from TGD point of view in [K37] [L3].

3. For braided fusion categories one obtains additional conditions known as hexagon conditions since there are two ways to end up from $1 \rightarrow 3$ fusion diagram involving two 3-vertices and 2 braidings to an equivalent diagram using sliding of lines along lines and braiding operation (see Fig. 10 of `http://tinyurl.com/l2jsrzc`).

9.5.3 Categories with reconnections

Fusion and braiding are not enough to satisfy the needs of TGD.

1. In TGD one does not have just objects - point like particles, whose world lines define braid strands in time direction. One has also the morphisms represented by the strings between the particles. Partonic 2-surfaces are connected by strings and these strings have topological interaction: they can reconnect or just go through each other. Reconnection is in key role in TGD inspired theory of consciousness and quantum biology [K58].

   Reconnection is an additional topological reaction besides braiding and one must assign to it a generalization of R-matrix. Reconnection and going through each other are just the basic operations used to unknot ordinary knots in the construction of knot invariants in topological quantum field theories. Now topological time evolution would be a generalization of this process connecting the knotted and linked structures at boundaries of CD and allowing both knotting and un-knotting.

2. Although 2-knots and braids are difficult to construct and visualize, it seems rather obvious (to me at least) that the reconnections correspond in 4-D space-time surface to basic operations giving rise to 2-knots [A48] - a generalization of ordinary knot that is 1-knot. 2-knots could be seen as a cobordism between 1-knots and this suggests a construction of 2-knot invariants as generalization of that for 1-knots [K37]. 2-knot would be the process transforming 1-knot by re-connections and "going through" the second 1-knot. The trace of the topological unitary S-matrix associated with it would give a knot invariant. If this view is correct, a generalization of TQFT for ordinary braids to include reconnection could give a TQFT for 2-braids with invariants as invariants of knot-cobordism. It must be however emphasized that the identification of 2-braids as knot-cobordisms is only an intuitive guess.

3. From the point of view of braid strands at the ends of strings, reconnection means exchange of braid strands. Composite particles consisting of strands would exchange their building bricks - the analogy with a chemical reaction is obvious and various reactions could be interpreted as knot cobordisms. Since exchange is involved also now, one expects that the generalization of R-matrix to algebraically describe this process should obey the analog of YBE stating that the two braided versions of permutation $abc \rightarrow cba$ are identical.

   If the strings are oriented, one could have YBEs separately for left and right ends such that braid operation would correspond to the exchange of braid between braid pairs. The topological interaction for strings AB and CD could correspond to a) trivial operation "going through" (AB + CD $\rightarrow$ AB+CD) visible in in the topological intersection matrix characterizing the union of string world sheets, exchanges of either left (AB+CD$\rightarrow$ CB+AD) or right ends (AB+CD$\rightarrow$ AD+CB), or exchange or right and left ends (AB+CD$\rightarrow$ CD+AB) representable as composition of braid operation for string ends and exchange of right or left ends and giving rise to braiding operation for pairs AB and CD.

The following braiding operations would be involved.

(a) Internal braiding operation $A \otimes B \rightarrow B \otimes A$ for string like object.

(b) Braiding operation $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes D) \otimes (A \otimes B)$ for two string like objects.

(c) Reconnection as braiding operation: $(A \otimes B) \otimes (C \otimes D) \rightarrow (A \otimes D) \otimes (C \otimes B)$ and $(A \otimes B) \otimes (C \otimes D) \rightarrow (C \otimes B) \otimes (A \otimes D)$.

I have not found by web search whether this generalization of YBE exists in mathematics literature or whether it indeed reduces to ordinary braiding for the exchanged braids for different options emerging in reconnection. One can ask whether the fusion procedure for R-matrices as an analog for the formation of tensor products already briefly discussed could allow to construct the R-matrix for the reconnection of 2 strings with braids as boundaries.

4. The intersections of braid strands are stable against small perturbations unless one modifies the space-time surface itself (in TGD 2-braids are 2-surfaces inside 4-surfaces). Also the intersections of world lines in M^2 integrable theories are stable. Hence it would be natural to assign analog of R-matrix also to the intersections.

5. Light-like 3-D partonic orbits can contain several fermion lines identifiable as boundaries of string world sheets so that reconnections could induce also more complex reactions in which partonic 2-surfaces exchange fermions. Quite generally one would have braid of braids able to braid and also exchange their constituent braids. This would give rise to a hierarchy of braids within braids and presumably to a hierarchy of categories. This might provide a first principle topological description of both hadronic, nuclear, and (bio-)chemical reactions. For instance, the mysterious looking ability of bio-molecules to find each other in dense molecular soup could rely on magnetic flux tubes (and associated strings) connecting them [K58].

6. Reconnection requires a generalization of various quantum algebras, in particular Yangian, which seems to be especially relevant to TGD since it generalizes local symmetries to multi-local symmetries with locus identifiable as partonic 2-surface in TGD. Since braid strands are replaced with pairs of them, one might expect that the generalization of R-matrix involves two parameters instead of one.

9.6 Trying to imagine the great vision about categorification of TGD

The following tries to summarize the ideas described. This is mostly free play with the ideas in order to see what objects and arrows might be relevant physically and whether category theory might be of help in understanding poorly understood issues related to various hierarchies of TGD.

9.6.1 Different kind of categories

Category theory could be much more than mere book keeping device in TGD. Morphisms and functors could allow to see deep structural similarities between different levels of TGD remaining otherwise hidden.

Geometric and number theoretic categories

There are three geometric levels involved: space-time, CDs at embedding space level, sectors of WCW assignable with CDs their subsectors characterized by a point for moduli space of CDs with second boundary fixed.

There are also number theoretic categories.

1. Adelic physics would define a hierarchy of categories defined by extensions of rationals and identifiable as an evolutionary hierarchy in TGD inspired theory of consciousness. Inclusion of extensions parameterized by Galois group and ramified primes defining preferred p-adic primes would define a functor. The parameters of quantum algebras should be number theoretically universal and belong to the extension of rationals defining the adele in question. Powers or roots of e, roots of unity, and algebraic numbers would appear as building bricks. The larger the p-adic prime p the higher the dimension of extension containing e and possibly also some of its roots, the better the accuracy of the cognitive representation.

2. These inclusions should relate closely to the inclusions of hyperfinite factors of type II_1 assignable to finite measurement resolution [K87]. The measurement resolution at space-time level would characterize the cognitive representation defined in terms of points with embedding space coordinates in the extension of rationals defining the adele. The larger the extension, the larger the cognitive representation and the higher the accuracy of the representation.

   Should the points of cognitive representation be assigned

   (a) only with partonic 2-surfaces (each point of representation is accompanied by fermion)

   (b) or also with the interior of space-time surface (it is not natural to assign fermion to the point unless the point belongs to string world sheet, even in this case this is questionable)?

   Many-fermion states define naturally a tensor product of quantum Boolean algebras at the opposite boundaries of CD in ZEO and the interpretation of time evolution as morphism of quantum Boolean algebras is natural. If cognition is always Boolean then the first option is more plausible.

3. The hierarchy of Planck constants $h_{eff}/h = n$ with $n \leq ord(G)$ naturally the number of sheets and dividing the order $ord(G)$ of the Galois group G of the extension would relate closely to the hierarchy of extensions. n would be dimension of the covering of space-time surface defined by the action of Galois group to space-time sheet. Ramified primes for extensions are in special position for given extension. The conjecture is that p-adic primes near powers of two or more generally of small primes ramified primes for extensions, which are winners in number theoretic fight for survival [L22].

4. The hierarchy of infinite primes [K72] might characterize many-sheeted space-time and leads to a generalization of number concept with infinitely complex number theoretic anatomy provided by infinite rationals, which correspond to real and p-adic units. The inclusion of lower level primes to the higher level primes would define morphism now. One can assign hierarchy of infinite primes with primes of any extension of rationals.

Consciousness and categories

Categories are especially natural from the point of view of cognition. Classification is the basic cognitive function and category is nothing but classification by defining objects as equivalence classes. Morphisms and functors serve as correlates for analogies and would provide the tool of understanding the power of analogies in conscious intelligence. Also attention could involve morphism and its direction would correlate with the direction of attention. Perhaps isomorphism corresponds to the state of consciousness in which the distinction between observer and observed is reported by meditators to cease. Cognitive representations would be provided by adelic physics at both space-time level, embedding space level, and WCW level (the preferred coordinates for WCW would be in extension of rationals defining the adele).

One would have a hierarchy of increasingly complex cognitive representations with inclusions as arrows and their sub-WCWs labelled by moduli of CDs and arrow of geometric time telling which boundary is affected in the sequence of state function reductions defining self as generalized Zeno effect [L27].

9.6.2 Geometric categories

Geometric categories appear at WCW level, embedding space level, and space-time level.

WCW level

The hierarchies formed by the categories defined by the hierarchies of adeles, space-time sheets and hierarchy of CDs would be mapped also to the level of WCW. The preferred coordinates of WCW points would be in extension of rationals defining the adele and one would form inclusion hierarchy. The extension at the level of WCW would induce that at the level of embedding space and space-time surface. Sub-CDs would correspond to sub-WCWs and the moduli space for given CD would correspond to moduli space for corresponding sub-WCWs. The different arrows of embedding space time would correspond to sub-WCW and its time reflection. By the breaking of CP,T, and P the space-time surfaces within time reversed sub-WCWs would not be mere CP, T and P mirror images of each other [L25, L15].

Embedding space level

ZEO emerges naturally at embedding space level and CDs are key notion at this level. Consider next the categories that might be natural in ZEO [K48].

1. Hierarchy of CDs could allow interpretation as hierarchy of categories. Overlapping CDs would define an analog of covering of manifold by open sets: one might speak of atlas with CDs defining conscious maps. Chart maps would be morphisms between different CDs assignable to common pieces of space-time surfaces. These morphisms would be also realized at the level of conscious experience. The sub-CD associated with CD would correspond to mental image defined by sub-self as image of the morphism.

2. Quantum state of single space-time sheet at boundary of CD would define a geometric and topological representation for categories. States at partonic 2-surfaces would be the objects connected by fermionic strings and the associated flux tubes would serve as space-time correlates of attention in TGD inspired theory of consciousness. The arrows represented by fermionic strings would correspond to some morphisms, at least thre Hilbert space isometries defined by entanglement with coefficients in an extension of rationals. Unitary entanglement gives rise to a density matrix proportional to unitary matrix and maximal entanglement in both real and p-adic sense. Much more general entanglement gives rise to maximal entanglement in p-adic sense for some primes.

3. Zero energy states the states at passive boundary would be naturally identifiable as categories. At active boundary quantum superpositions of categories could be in question. Maybe one should talk about quantum categories defined by the superposition of space-time sheets with category assigned with an equivalence class of space-time sheets satisfying the conditions for preferred extremal.

4. One can imagine a hierarchy of zero energy states corresponding to the hierarchy of space-time sheets. One can build zero energy states also by adding zero energy states associated with smaller sub-CDs near the boundaries of CD to get an infinite hierarchy of zero energy states. The interpretation as a hierarchy of reflective levels of consciousness would be natural.

5. Zero energy states would correspond to generalized Feynman diagrams interpreted as unitary functors between initial and final state categories. Scattering diagram would be seen as algebraic computation in a fusion category defined by Yangian. All diagrams would be

reducible to braided tree diagrams with braidings and reconnections. The time evolution between boundaries could be seen as a topological evolution a of tensor net [L10].

Category theory would provide cognitive representations as morphisms. Morphisms would become the key element of physics completely discarded in the existing billiard ball view about Universe: Universe would be like Universal computer mimicking itself at all hierarchy levels. This extends dramatically the standard view about cognition where brain is seen as an isolated seat of cognition.

Space-time level

Many-sheeted space-time is the most obvious application for categorification.

1. Smaller space-time sheets condensed at large space-time surface regarded as categories become objects at the level of larger space-time sheet. Functors between the categories defined by smaller space-time sheets define morphisms between them. Also now fermion lines and flux tubes connecting the condensed space-time sheets to each other via wormhole contacts with flux going along another space-time sheet could define functors. Closed loops involving larger space-time sheets and smaller space-time sheets are needed if monopole flux in question. The loop could visitat smaller space-time sheets.

2. Interactions would reduce to product and co-product. Interaction term in Δ for generalized Yangian would characterize fundamental interactions with dynamically generated SKMAs assignable to SSA as additional interactions. The coupling parameters with Δ assigned to a direct sum of SKMAs would define coupling constants of fundamental interactions. Iteration of the co-product Δ would give rise to a hierarchy of many-particle states. The fact that morphism is in question would map the structure of single particle states to that of many-particle states.

SH would involve a functor mapping the category of string world sheets (and partonic 2-surfaces) to that of space-time surfaces having same points with coordinates in extension of rationals. In p-adic sectors this morphism presumably exists for all p-adic primes thanks to p-adic pseudo-constants. In real sector this need not be the case: all imaginations are not realizable.

The morphisms would be mediated by either continuation of strings world sheets (and partonic 2-surfaces) to space-time interiors (morphism would be analogous to a continuation of holomorphic functions of two complex coordinates from 2-D data at surfaces, where the functions are real). Possible quaternion analyticity [K77] encourages to consider even continuation of 1-D data to 4-D surfaces and twistor lift gives some support for this idea.

In the fermionic sector one must continue induced spinor fields at string world sheets to those at space-time surfaces. The 2-D induced spinor fields could also serve as sources for 4-D spinor fields.

Chapter 10

Are higher structures needed in the categorification of TGD?

10.1 Introduction

I encountered a very interesting work by Urs Schreiber related to so called higher structures and realized that these structures are part of the mathematical language for formulating quantum TGD in terms of Yangians and quantum algebras in a more general way.

10.1.1 Higher structures and categorification of physics

What theoretical physicist Urs Screiber calls "higher structures" are closely related to the categorification program of physics. Baez, David Corfield and Urs Schreiber founded a group blog n-Category Cafe about higher category theory and its applications. John Baez is a mathematical physicists well-known from is pre-blog "This Week's Finds" (see `http://tinyurl.com/yddcabfl`) explaining notions of mathematical physics.

Higher structures or n-structures involve "higher" variants of various mathematical structures such as groups, algebras, homotopy theory, and also category theory (see `http://tinyurl.com/ydz9mbtp`. One can assign a higher structure to practically anything. Typically one loosens some conditions on the structure such as commutativity or associativity: a good example is the product for octonionic units which is associative only apart from sign factors [K74]. Braid groups and fusion algebras [L16], which seem to play crucial role in TGD can be seen as higher structures.

The key idea is simple: replace "=" with homotopy understood in much more general sense than in topology and identified as the procedure proving $A = B$! Physicist would call this operationalism. I would like a more concrete interpretation: "=" is replaced with "=" in a given measurement resolution. Even homotopies can be defined only modulo homotopies of homotopies - that is within measurement resolution - and one obtains a hierarchy of homotopies and at the highest level coherence conditions state that one has "=" almost in the good old sense. This kind of hierarchical structures are characteristic for TGD: hierarchy of space-time sheet, hierarchy of p-adic length scales, hierarchy of Planck constants and dark matters, hierarchy of inclusions of hyperfinite factors, hierarchy of extensions of rationals defining adels in adelic TGD, hierarchy of infinite primes, self hierarchy, etc...

10.1.2 Evolution of Schreiber's ideas

One of Schreiber's articles in Physics Forum articles has title "*Why higher category theory in physics?*" (see `http://tinyurl.com/ydcylrun`) telling his personal history concerning the notion of higher category theory. Supersymmetric quantum mechanics and string theory/M-theory are strongly involved with his story.

Wheeler's superspace and its deformations as starting point

Schreiber started with super variant of Wheeler's super-space. Intriguingly, also the "world of classical worlds" (WCW) of TGD [K35, K20, K63] emerged as a counterpart of superspace of Wheeler in which the generalization of super-symmetries is geometrized in terms of spinor structure of WCW expressible in terms of fermionic oscillator operators so that there is something common at least.

Screiber consider deformation theory of this structure. Deformations appear also in the construction of various quantum structures such as quantum groups and Yangians. Both quantum groups characterized by quantum phase, which is root of unity, and Yangians ideal for reduction of many-particle states and their interactions to kinematics seem to be the most important from the TGD point of view [L16].

These deformations are often called "quantizations" but this nomenclature is to my opinion misleading. In TGD framework the basic starting point is "*Do not quantize*" meaning the reduction of the entire quantum theory to classical physics at the level of WCW: modes of a formally classical WCW spinor fields correspond to the states of the Universe.

This does not however prevent the appearance of the deformations of basic structures also in TGD framework and they might be the needed mathematical tool to describe the notions of finite measurement resolution and cognitive resolution appearing in the adelic version of TGD. I proposed more than decade ago that inclusions of hyperfinite factors of II_1 (HFFs) [K87, K28] might provide a natural description of finite measurement resolution: the action of included factor would generate states equivalent under the measurement resolution used.

The description of non-point-like objects in terms of higher structures

Schreiber ends up with the notion of higher gauge field by considering the space of closed loops in 4-D target space [B35]. At the level of target space the loop space connection (1-form in loop space) corresponds to 2-form at the level of target space. At space-time level 1- form A defines gauge potentials in ordinary gauge theory and non-abelian 2-form B as its generalization with corresponding higher gauge field identified as 3-form $F = dB$.

The idea is that the values of 2-form B are defined for a string world sheet connecting two string configuration just like the values of 1-form are defined for a world-line connecting two positions of a point-like particle. The new element is that the ordinary curvature form does not anymore satisfy the usual Bianchi identities stating that magnetic monopole currents are vanishing (see `http://tinyurl.com/ya3ur2ad`).

It however turns out that one has $B = DA = F$ (D denotes covariant derivative) so that B is flat by the usual Bianchi-identities implying $dB = 0$ so that higher gauge field vanishes. B also turns out to be Abelian. In the Abelian case the value of 2-form would be magnetic flux depending only on the boundary of string world sheet. By $dB = 0$ gauge fields in loop space would vanish and only topology of field configurations would make itself manifest as for locally trivial gauge potentials in topological quantum field theories (TQFT): a generalization of Aharonov-Bohm effect would be in question. Schreiber calls this "*fake flatness condition*". This could be seen as an unsatisfactory outcome since dynamics would reduce to topological dynamics.

The assumption that loop space gauge fields reduce to those in target space could be argued to be non-realistic in TGD framework . For instance, high mass excitations of theories of extended structures like strings would be lost. In the case of loop spaces there is also problem with general coordinate invariance (GCI): one would like to have 2-D GCI assignable to string world sheets. In TGD the realization that one must have 4-D GCI for 3-D fundamental objects was a breakthrough, which occurred around 1990 about 12 years after the discovery of the basic idea of TGD and led to the discovery of WCW Kähler geometry and to "Do not quantize".

Understanding "fake flatness" condition

Schreiber tells how he encountered the article of John Baez titled "*Higher Yang-Mills Theory*" [B30] (see `http://tinyurl.com/yagkqsut`) based on the notion of 2-category and was surprised to find that also now the "*fake flatness condition*" emerged.

Schreiber concludes that the "*fake flatness condition*" results from "a kind of choice of coordinate composition": non-Abelian higher gauge field would reduce to Abelian gauge field over a background of ordinary non-Abelian gauge fields. Schreiber describes several string theory related examples involving branes and introduces connection with modern mathematics. Since branes in the stringy sense are not relevant to TGD and I do not know much about them, I will not discuss these here.

However, dimensional hierarchies formed by fermions located to points at partonic 2-surfaces, their world lines at 3-D light-like orbits of partons, strings and string world sheets as their orbits, and space-time surfaces as 4-D orbits of 3-surfaces definitely define a TGD analog for the brane hierarchy of string models. It is not yet completely clear whether strong form of holography (SH) implies that string world sheets and strings provide dual descriptions of 4-D physics or whether one could regard all levels of this hierarchy independent to some degree at least [L13].

Since the motion of measurement resolution is fundamental in TGD [K87, K28], it is interesting to see whether n-structures could emerge naturally also in TGD framework. There is also second aspect involved: various hierarchies appearing in TGD have basically the structure of abstraction hierarchy of statements about statements and higher structures seem to define just this kind of hierarchies. Of course, human mind - at least my mind - is in grave difficulties already with few lowest levels but here category theory and its computerization might come into a rescue.

10.1.3 What higher structures are?

Schreiber describes in very elegant and comprehensible way the notion of higher structures (see `http://tinyurl.com/ydfspcld`). This description is a real gem for a physicists frustrated to the impenetrable formula jungle of the usual mathematical prose. Just the basic ideas and the reader can start to think using his/her own brains. The basic ideas ideas are very simple and general. Even if one were not enthusiastic about the notion of higher gauge field, the notion of higher structure is extremely attractive concerning the mathematical realization of the notion of finite measurement resolution.

1. The idea is to reconsider the meaning of "=". Usually it is understood as equivalence: $A = B$ if A and B belong to same equivalence class defined by equivalence relation. The idea is to replace "=" with its operational definition, with the proof of equivalence. This could be seen as operationalism of physics applied to mathematics. Schreiber calls this proof homotopy identified as a generalization of a map f_t: $S \to X$ depending on parameter $t \in [0, 1]$ transforming two objects of a topological space X to each other in continuous way: $f_0(S)$ is the initial object and $f_1(S)$ is the final object. Now homotopy would be much more general.

2. One can also improve the precision of "=" meaning that equivalence classes decompose to smaller ones and equivalent homotopies decompose to subclasses of equivalent homotopies related by homotopies. One might say that "=" is deconstructed to more precise "=". Physicist would see this as a partial opening of a black box by improving the measurement resolution. This gives rise to n-variants of various algebraic structures.

3. This hierarchy would have a finite number of levels. At highest level the accuracy would be maximal and "=" would have almost its usual meaning. This idea is formulated in terms of coherence conditions. Braiding involving R-matrix represents one example: permutations are replaced by braidings and permutation group is lifted to braid group but associativity still holds true for Yang-Baxter equation (YBE). Second example is 2-group for which associativity holds true only modulo homotopy so that $(x \circ y) \circ z$ is related to $x \circ (y \circ z)$ by homotopy $a_{x,y,z}$ depending on x, y, z and called an associator. For 2-group the composite homotopy $((w \circ x) \circ y) \circ z \to w \circ (x \circ (y \circ z))$ is however unique albeit non-trivial.

   This gives rise to the so called pentagon identity encountered also in the theory of quantum groups and Yangians. The outcome is that all homotopies associated with re-bracketings of an algebraic expression are identical. One can define in similar way n-group and formally even infinity-group.

10.1.4 Possible applications of higher structures to TGD

Before listing some of the applications of higher structures imaginable in TGD framework, let us summarize the basic principles.

1. Physics as WCW geometry [K78, K35, K20, K63] having super-symplectic algebra (SSA) and partonic super-conformal algebra (PSCA) as fundamental symmetries involving a generalization of ordinary conformal invariance to that for light-like 3-surfaces defined by the boundary of CD and by the light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian.

2. Physics as generalized number theory [K53] [L22] leading to the notion of adelic physics with a hierarchy of adeles defined by the extensions of rationals.

3. In adelic physics finite resolutions for sensory and cognitive representations (see the glossary of Appendix) could would characterize "=". Hierarchies of resolutions meaning hierarchies of n-structures rather than single n-structure would give inclusion hierarchies for HFFs, SSA, and PSCA, and extensions of rationals characterized by Galois groups with order identifiable as $h_{eff}/h = n$ and ramified primes of extension defining candidates for preferred p-adic primes.

   Finite measurement resolution defined by SSA and its isomorphic sub-algebra acting as pure gauge algebra would reduce SSA to finite-dimensional SKMA. WCW could become effectively a coset space of Kac-Moody group or of even Lie group associated with it. Same would take place for PSCA. This would give rise to n-structures. Quantum groups and Yangians would indeed represent examples of n-structures.

   In TGD the "conformal weight" of Yangian however corresponds to the number of partonic surfaces - parton number - whereas for quantum groups and Kac-Moody algebras it is analogous to harmonic oscillator quantum number n, which however has also interpretation as boson number. Maybe this co-incidence involves something much deeper and relates to quantum classical correspondence (QCC) remaining rather mysterious in quantum field theories (QFTs).

4. An even more radical reduction of degrees of freedom can be imagined. Cognitive representations could replace space-time surfaces with discrete structures and points of WCW could have cognitive representations as disretized WCW coordinates.

5. Categorification requires morphisms and homomorphisms mapping group to sub-group having normal sub-group defining the resolution as kernel would define "resolution morphisms". This normal sub-group principle would apply quite generally. One expects that the representations of the groups involved are those for quantum groups with quantum phase q equal to a root of unity.

Some examples helps to make this more concrete.

Scattering amplitudes as computations

The deterministic time devolution connecting two field patterns could define analog of homotopy in generalized sense. In TGD framework space-time surface (preferred extremals) having 3-D space-like surfaces at the opposite boundaries of causal diamond (CD) could therefore define analog of homotopy.

1. Preferred extremal defines a topological scattering diagram in which 3-vertices of Feynman diagram are replaced with partonic 2-surfaces at which the ends of light-like orbits of partonic 2-surfaces meet and fermions moving along lines defined by string world sheets scatter classically, and are redistributed between partonic orbits [K77, K8, L26]. Also braidings and reconnections of strings are possible. It is important to notice that one does not sum over these topological diagrams. They are more like possible classical backgrounds.

   The conjecture is that scattering diagrams are analogous to algebraic computations so that one can find the shortest computation represented by a tree diagram. Homotopy in the roughest sense could mean identification of topological scattering diagrams connecting two states at boundaries of CD and differing by addition of topological loops. The functional integral in WCW is proposed to trivialize in the sense that loop corrections vanish as a manifestation of quantum criticality of Kähler coupling strength and one obtains an exponent of Kähler function which however cancels in scattering amplitudes if only single maximum of Kähler function contributes.

2. In the optimal situation one could eliminate all loops of these diagrams and also move line ends along the lines of diagrams to get tree diagrams as representations of scattering diagrams. Similar conditions hold for fusion algebras. This might however hold true only in the minimal resolution. In an improved measurement resolution the diagrams could become more complex. For instance, one might obtain genuine topological loops.

3. The diagrams and state spaces with different measurement resolutions could be related by Hilbert space *isometries* but would not be unitarily equivalent: Hilbert space isometries are also defined by entanglement in tensor nets [L10]. This would give an n-levelled hierarchy of higher structures (rather than single n-structure!) and at the highest level with best resolution one would have coherence rules. Generalized fusion algebras would partially realize this vision. In improved measurement resolution the diagrams would not be identical anymore and equivalence class would decompose to smaller equivalence classes. This brings in mind renormalization group equations with cutoff.

4. Intuitively the improvement of the accuracy corresponds to addition of sub-CDs of CDs and smaller space-time sheets glued to the existing space-time sheets.

Zero energy ontology (ZEO)

In ZEO [K48] "=" could mean the equivalence of two zero energy states indistinguishable in given measurement resolution. Could one say that the 3-surfaces at the ends of space-time surface are equivalent in the sense that they are connected by preferred extremal and have thus same total Noether charges, or that entangled many-fermion states at the boundaries of CD correspond to quantal logical equivalences (fermionic oscillator algebra defines a quantum Boolean algebra)?

In the case of zero energy states "=" could tolerate a modification of zero energy state by zero energy state in smaller scale analogous to a quantum fluctuation in quantum field theories (QFTs). One could add to a zero energy state for given CD zero energy states associated with smaller CDs within it.

In TGD inspired theory of consciousness [L27] sub-CDs are correlates for the perceptive fields of conscious entities and the states associated with sub-CDs would correspond to sub-selves of self defining its mental images. Also this could give rise to hierarchies of n-structures with n characterizing the number of CDs with varying sizes. An interesting proposal is the distance between the tips of CD is integer multiple of CP_2 for number theoretic reasons. Primes and primes near powers of 2 are suggested by p-adic length scale hypothesis [K41, K46, K47] [L22].

"World of classical worlds" (WCW)

At the level of "world of classical worlds" (WCW) "=" could have both classical meaning and meaning in terms of quantum state defining the measurement resolution. At the level of WCW geometry n-levelled hierarchies formed by the isomorphic sub-algebras of SSA and PSCA are excellent candidates for n-structures. The sub-SCA or sub-PSCA would define the measurement resolution. The smaller the sub-SSA or sub-PSCA, the better the resolution.

This could correspond to a hierarchy of inclusions of HFFs [K87, K28] to which one can assign ADE SKMA by McKay correspondence or its generalization allowing also other Lie groups suggested by the hierarchy of extensions of rationals with Galois groups that are groups of Lie type. The conjecture generalizing McKay correspondence is that the Galois group Gal is representable as a subgroup of G in the case that it is of Lie type.

An attractive idea is that WCW is effectively reduced to a finite-dimensional coset space of the Kac-Moody group defined by the gauge conditions. Number theoretic universality requires that these parameters belong to the extension of rationals considered so that the Kac-Moody group G is discretized and also homotopies are discretized. SH raises the hope that it is enough to consider string world sheets with parameters (WCW coordinates) in the extension of rationals.

One can define quite concretely the action of elements of homotopy groups of Kac-Moody Lie groups G on space-time surfaces as induced action changing the parameters characterizing the space-time surface. $n+1$-dimensional homotopy would be 1-dimensional homotopy of n-dimensional homotopy. Also the spheres defining homotopies could be discretized so that the coordinates of its points would belong to the extension of rationals.

These kind of homotopy sequences could define analogs of Berry phases (see `http://tinyurl.com/yd4agwnt`) in Kac-Moody group. Could gauge theory for Kac-Moody group give an approximate description of the dynamical degrees of freedom besides the standard model degrees of freedom? This need not be a good idea. It is better to base the considerations of the physical picture provided by TGD. I have however discussed the TGD analog of the *fake flatness condition* in the Appendix.

Adelic physics

Also number theoretical meaning is possible for "=". It is good to start with an objection against adelic physics. The original belief was that adelic physics forces preferred coordinates. Indeed, the property of belonging to an extension of rationals does not conform with general coordinate invariance (GCI). Coordinate choice however matters cognitively as any mathematical physicist knows! One can therefore introduce preferred coordinates at the embedding space level as cognitively optimal coordinates: they are dictated to a high degree by the isometries of H. One can use a sub-set of these coordinates also for space-time surfaces, string world sheets, and partonic 2-surfaces.

1. Space-time surfaces can be regarded as multi-sheeted Galois coverings of a representative sheet [L22]. Minimal resolution means that quantum state is Galois singlet. Improving resolution means requiring that singlet property holds true only for normal sub-group H of Galois group Gal and states belong to the representations of Gal/H. Maximal resolution would mean that states are representations of the entire Gal. The hierarchy of normal sub-groups of Gal would define a resolution hierarchy and perhaps an analog of n-structure. $h_{eff}/h = n$ hypothesis suggests hierarchies of Galois groups with dimensions n_i dividing n_{i+1}. The number of extensions in the hierarchy would characterize n-structure.

2. The increase of the complexity for the extension of rationals would bring new points in the *cognitive representations* defined by the points of the space-time surface with embedding space coordinates in the extension of rationals used (see the glossary in Appendix). Also the size of the Gal would increase and higher-D representations would become possible. The value of $h_{eff}/h = n$ identifiable as dimension of Gal would increase. The cognitive representation would become more precise and the topology of the space-time surface would become more complex.

3. In adelic TGD "=" could have meaning at the level of cognitive representations. One could go really radical and ask whether discrete cognitive representations replacing space-time surfaces with the set of points with H-coordinates in an extension of rationals (see the glossary in Appendix) defining the adele should provide the fundamental data and that all group representations involved should be realized as representations of Gal. This might apply in cognitive sector.

   This would also replace space-time surfaces as points of WCW with their cognitive representations defining their WCW coordinates! All finite groups can appear as Galois groups for some number field. Whether this is case when one restricts the consideration to the extensions of rationals, is not known. Most finite groups are groups of Lie type and thus representable as rational points of some Lie group. Note that rational point can also mean rational point in extension of rationals as ratio of corresponding algebraic integers identifiable as roots of monic polynomials $P_n(x) = x^n + ...$ having rational coefficients.

4. By SH space-time surface would in information theoretic sense effectively reduce to string world sheets and even discrete set of points with H-coordinates in extension of rationals. These points could even belong to the partonic 2-surface at the ends of strings at ends of CD carrying fermions and the partonic 2-surfaces defining topological vertices. If only this data is available, the WCW coordinates of space-time surface would reduce to these points of $H = M^4 \times CP_2$ and to the direction angles of strings emerging from these points

and connecting them to the corresponding points at other partonic 2-surfaces besides Gal identifiable as sub-group of Lie group G of some Kac-Moody group! Not all pairs $Gal - G$ are possible.

5. Could these data be enough to describe mathematically what one knows about space-time surface as point of WCW and the physics? One could indeed deduce $h_{eff}/h = n$ as the order of Gal and preferred p-adic primes as ramified primes of extension. The Galois representations acting on the covering defining space-time surface or string world sheets should be identifiable as representations of physical states. There is even number theoretical vision about coupling constant evolution relying on zeros of Riemann zeta [L9],

6. This sounds fine but one must notice that there is also the global information about the conformal moduli of partonic 2-surfaces and the elementary particle vacuum functionals defined in this moduli space [K17] explain family replication phenomenon. There is also information about moduli of CDs. Also the excitations of SKMA representations with higher conformal weights are present and play a crucial role in p-adic thermodynamics predicting particle masses [K41]. It is far from clear whether the approach involving only cognitive representation is able to describe them.

To help the reader I have included a vocabulary at the end of the article and include here a list of the abbreviations used in the text.

General abbreviations: Quantum field theory (QFT); Topological quantum field theory (TQFT); Hyper-finite factor of type II_1 (HFF); General coordinate invariance (GCI); Equivalence Principle (EP).

TGD related abbreviations: Topological Geometrodynamics (TGD); General Relativity Theory (GRT); Zero energy ontology (ZEO); Strong form of holography (SH); Strong form of general coordinate invariance (SGCI); Quantum classical correspondence (QCC); Negentropy Maximization Principle (NMP); Negentropic entanglement (NE); Causal diamond (CD); Super-symplectic algebra (SSA); Partonic superconformal algebra (PSCA); Super Virasoro algebra (SVA); Kac-Moody algebra (KMA); Super-Kac-Moody algebra (SKMA);

10.2 TGD very briefly

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K78] and physics as generalized number theory [K53]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

10.2.1 World of classical worlds (WCW)

TGD is a fusion of two approaches to physics. Physics as infinite-dimensional geometry based on the notion of "(" []WCW) [K78] and physics as generalized number theory [K53]. Here some aspects of the vision about physics as WCW geometry are discussed very briefly.

Construction of WCW geometry briefly

In the following the vision about physics in terms of classical physics of spinor fields of WCW is briefly summarized.

1. The idea is to geometrize not only the classical physics in terms of geometry of space-time surfaces but also quantum physics in terms of WCW [K63]. Quantum states of the Universe would be modes of classical spinor fields in WCW and there would be no quantization. One must construct Kähler metric and Kähler form of WCW: in complex coordinates they differ by a multiplicative imaginary unit. Kähler geometry makes possible to geometrize hermitian conjugation fundamental for quantum theory.

2. One manner to build WCW metric this is via the construction of gamma matrices of WCW in terms of second quantized oscillator operators for fermions described by induced spinor fields at space-time surfaces. By strong form of holography this would reduce to the construction

of second quantized induced spinor fields at string world sheets. The anti-commutators of of WCW gamma matrices expressible in terms of oscillator operators would define WCW metric with maximal isometry group (SCA) [K88, K63].

3. Second manner to achieve the geometrization is to construct Kähler metric and Kähler form directly [K35, K20, K63]. The idea is to induce WCW geometry from the Kähler form J of the embedding space $H = M^4 \times CP_2$. The mere existence of the Riemann connection forces a maximal group of isometries. In fact, already in the case of loop space the Kähler geometry is essentially unique.

   The original construction used only the Kähler form of CP_2. The twistor lift of TGD [L26] forces to endow also M^4 with the Minkowskian analog of Kähler form involving complex and hypercomplex part and the sum of the two Kähler forms can be used to define what might be called flux Hamiltonians. They would define the isometries of WCW as symplectic transformations. What was surprising and also somewhat frustrating was that what I called almost 2-dimensionality of 3-surfaces emerges from the condition of general coordinate invariance and absence of dimensional parameters apart from the size scale of CP_2.

   In the recent formulation this corresponds to SH: 2-D string world sheets and 2-D partonic 2-surfaces would contain data allowing to construct space-time surfaces as preferred extremals. In adelic physics also the specification of points of space-time surface belonging to extension of rationals defining the adele would be needed. There are several options to consider but the general idea is clear.

   SH is analogous to a construction of analytic function of 2-complex from its real values at 2-D surface and the analogy at the level of twistor lift is holomorphy as generalization of holomorphy of solutions gauge fields in the twistor approach of Penrose. Also quaternionic analyticity [K77] is suggestive and might mean even stronger form of holography in which 1-D data allow to construct space-time surfaces as preferred extremals and quantum states.

   I have proposed formulas for the Kähler form of WCW in terms of flux Hamiltonians but the construction as anti-commutators of gamma matrices is the more convincing definition. Fermions and second quantize induced spinor fields could be an absolutely essential part of WCW geometry.

4. WCW allows as infinitesimal isometries huge super-symplectic algebra (SSA) [K35, K20] acting on space-like 3-surfaces at the ends of space-time surfaces inside causal diamond (CD) and also generalization of Kac-Moody and conformal symmetries acting on the 3-D light-like orbits of partonic 2-surfaces (partonic super-conformal algebra (PSCA)). These symmetry algebras have a fractal structure containing a hierarchy of sub-algebras isomorphic to the full algebra. Even ordinary conformal algebra with non-negative conformal weights has similar fractal structure as also Yangian. In fact, quantum algebras are formulated in terms of these half algebras.

   The proposal is that sub-algebra of SSA (with non-negative conformal weights) and isomorphic to entire SSA and its commutator with the full algebra annihilate the physical states. What remains seems to be finite-D Kac-Moody algebra as an effective "coset" algebra obtained. Note that the resulting normal sub-group is actually quantum group.

   There is direct analogy with the decomposition of a group Gal to a product of sub-group and normal sub-group H. If the normal sub-group H acts trivially on the representation the representation of Gal reduces to that of the group Gal/H. Now one works at Lie algebra level: Gal is replaced with SSA and H with its sub-algebra with conformal weights multiples of those for SSA.

Super-symplectic conformal weights, zeros of Riemann zeta, and quantum phases?

In [L9] I have considered the possibility that the generators of super-symplectic algebra could correspond to zeros $h = 1/2 + iy$ of zeta. The hypothesis has several variants.

1. The simplest variant is that the non-trivial zeros of zeta are labelling the generators of SSA associated with Hamiltonians proportional to the functions $f(r_M)$ of the light-like radial

coordinate of light-cone boundary as $f(r_M) = (r_M/_0)^h \equiv exp(hu)$, $u = log(r_M/r_0)$, $h = -1/2 + iy$. For infinitely large size of CD the plane waves are orthogonal but for finite-sized CD orthogonality is lost. Orthogonality requires periodic boundary conditions and these are simultanwously possible only for a finite number of zeros of zeta.

2. One could modify the hypothesis by allowing superpositions of zeros of zeta but with a subtraction of half integer to make the real part of ih equal to 1/2 so that one obtains an analog of plane-wave when using $u = log(r_M/r_0)$ as a radial coordinate. Equivalently, one can take dr_M/r_M out as integration measure and assume $h = iy$ plus the condition that the Riemannian plane waves are orthogonal and satisfy periodic boundary conditions for the allowed zeros $z = 1/2 + iy$.

3. Periodic boundary conditions can be satisfied for given zero of zeta if the condition $r_{max}/r_{min} = p^n$ holds true and the additional conjecture that given non-trivial zeros of zeta correspond to prime $p(y)$ and p^{iy} is a root of unity. Given basis of $f(r_M)$ would correspond to n-ary p-adic length scales and also the size scales of CDs would correspond to powers of p-adic primes. This conjecture is rather attractive physically and I have not been able to prove it wrong.

   One can associate to given zero $z = 1/2 + iy$ single and only single prime $p(y)$ by demanding that $p^{iy} = exp(i2\pi q)$, $q = m/n$ rational, implying $log(p)y = 2\pi q$. If there were two primes p_1 and p_2 of this kind, one one ends up with contradiction $p_1^m = p_2^n$ for some integers m and n.

   One could however associate several zeros $y_i(p)$ to the same prime p as discussed in [L9]. If $N = \prod_i n_i$ is the smallest common denominator of q_i allowed conformal weights would be superpositions $ih = iN \sum n_i y_i(p)$ and conformal weights would form higher dimensional lattice rather than 1-D lattice as usually. If only single prime $p(y)$ can be associated to given y, then the original hypothesis identifying $h = 1/2 + iy$ as conformal weight would be natural.

4. The understanding of the p-adic length scale hypothesis is far from complete and one can ask whether preferred p-adic primes near powers of 2 and possibly also other small primes could be primes for which there are several roots $y_i(p)$.

10.2.2 Strong form of holography (SH)

There are several reasons why string world sheets and partonic 2-surfaces should code for physics. One reason for SH comes from $M^8 - H$ correspondence [K86]. Second motivation comes from the condition that spinor modes at string world sheets are eigenstates of em charge [K88]. The third reason could come the requirement that the notion of commutative quantum sub-manifold [A29] is equivalent with its number theoretic variant.

SH and $M^8 - H$ correspondence

The strongest form of $M^8 - H$ correspondence [K74, K86, L26] assumes that the 4-surfaces $X^4 \subset M^8$ have fixed $M^2 \subset M^4 \subset M^8$ as part of tangent space. A weaker form states that these 2-D subspaces M^2 define an integrable distribution and therefore 2-D surface in M^4. This condition guarantees that the quaternionic (associative) tangent space of X^4 is parameterized by a point of CP_2 so that the map of X^4 to a 4-surface in $M^4 \times CP_2$ is possible. One can consider also co-associative space-time surfaces having associative normal spaces. m Note that $M^8 - H$ [K74, K86] correspondence respects commutativity and quaternionic property by definition since it maps space-time surfaces having quaternionic tangent space having fixed M^2 as sub-set of tangent space.

What could be the relationship between SH and $M^8 - H$ correspondence? Number theoretic vision suggests rather obvious conjectures.

1. Could the tangent spaces of string world sheets in H be commutative in the sense of complexified octonions and therefore be hyper-complex in Minkowskian regions. By $M^8 - H$ duality the commutative sub-manifolds would correspond to those of octonionic M^8 and finding of these could be the first challenge. The co-commutative manifolds in quaternionic X^4 would have commutative normal spaces. Could they correspond to partonic 2-surfaces?

2. There is however a delicacy involved. Could world sheets and partonic 2-surfaces correspond to hyper-complex and co-hyper-complex sub-manifolds of space-time surface X^4 identifiable as quaternionic surface in octonionic M^8 mappable to similar surfaces in H. Or could their M^4 (CP_2) projections define hypercomplex (co-hypercomplex) 2-manifolds?

3. Could co-commutativity condition for a foliation by partonic 2-surfaces select preferred string world sheets as normal spaces integrable to 2-surfaces identifiable as string world sheets? Note that induced gauge field on 2-surface is always Abelian so that QFT and number theory based views about commutativity co-incide.

   Preferred choices for these 2-surfaces would serve as natural representatives for the equivalence classes of string world sheets and partonic 2-surfaces with fermions at the boundaries of string world sheets serving as markers for the representatives? The end points of the string orbits would belong to extension of rationals or even correspond to singular points at which the different sheets co-incide and have rational coordinates: this possibility was considered in [L28].

Real curves correspond to the lowest level of the dimensional hierarchy of continuous surfaces. Could string world lines along light-like partonic orbits correspond to real sub-manifolds of octonionic M^8 mapped to $M^4 \times CP_2$ by $M^8 - H$ correspondence and carrying fermion number?

What about the set of points with coordinates in the extension of rationals? Do all these points carry fermion number? If so they must correspond to the edges of the boundaries of string world sheets at partonic 2-surfaces at the boundaries of CD or edges at the partonic 2-surfaces defining generalized vertices to which sub-CDs could be assigned.

Well-definedness of em charge forces 2-D fundamental objects

The proposal has been that the representative string world sheets should have vanishing induced W fields so that induced spinors could have well-defined em and Z^0 charges and partonic 2-surfaces would correspond to the ends of 3-D boundaries between Euclidian and Minkowskian space-time regions [K88, K63].

As a matter of fact, the projections of electroweak gauge fields to 2-D surfaces are always Abelian and by using a suitable $SU(2)_L \times U(1)$ rotation one can always find a gauge in which the induced W fields and even Z^0 field vanish. The highly non-trivial conclusion is that string world sheets as fundamental dynamical objects coding 4-D physics by SH would guarantee well-definedness of em charge as fermionic quantum number. Also the projections of all classical color gauge fields, whose components are proportional to $H^A J$, where H^A is color Hamiltonian and J is Kähler form of CP_2, are Abelian and in suitable gauge correspond to hypercharge and isospin.

One can imagine a foliation of space-time surfaces by string world sheets and partonic 2-surfaces. Could there be a $U(1)$ gauge invariance allowing to chose partonic 2-surfaces and string world sheets arbitrarily? If so, the assignment of the partonic 2-surfaces to the light-like boundaries between Minkowskian and Euclidian space-time regions would be only one - albeit very convenient - choice. I have proposed that this choice is equivalent with the choice of complex coordinates of WCW. The change of complex coordinates would introduce a $U(1)$ transformation of Kähler function of WCW adding to it a real part of holomorphic function and of Kähler gauge potential leaving the Kähler form and Kähler metric of WCW invariant.

String world sheets as sub-manifolds of quantum spaces for which commuting sub-set of coordinates are diagonalized?

The third notion of commutativity relates to the notion of non-commutative geometry. Unfortunately, I do not know much about non-commutative geometry.

1. Should one follow Connes [A29] and replace string world sheets with non-commutative geometries with quantum dimension identifiable as fractal dimension. I must admit that I have felt aversion towards non-commutative geometries. For linear structures such as spinors the quantum Clifford algebra looks natural as a "coset space" obtained by taking the orbits of included factor as elements of quantum Clifford algebra. The application of this idea to string world sheets does not look attractive to me.

2. The basic reason for my aversion is that non-commutative quantum coordinates lead to problems with general coordinate invariance (GCI). There is however a possible loophole here. One can approach the situation from two angles: number theoretically and from the point view of non-commutative space. Commutativity could mean two things: number theoretic commutativity and commutativity of quantum coordinates for H seen as observables. Could these two meanings be equivalent as quantum classical correspondence (QCC) encourages to think?

   Could the discreteness for cognitive representations correspond to a discretization of the eigenvalue spectrum of the coordinates as quantum operators? The choice of the coefficient number field for Hilbert space as extension of rationals would automatically imply this and resolve the problems related to continuous spectra.

   Quantum variant of string world sheet could correspond to a quantization using a sub-set of embedding space coordinates as quantum commutative coordinates as coordinates for string world sheet. H-coordinates for string world sheet would correspond to eigenvalues of commuting quantum coordinates.

The above three views about SH suggests that Abelianity at the fundamental level is unavoidable because basic observable objects are 2-dimensional. This would correspond $A = J = -B = 0$ for non-Abelian gauge fields reducing to Abelian ones in Schreiber's approach. Also Schreiber finds that with suitable choice of coordinates this holds true always. In TGD this choice would correspond to gauge choice in which all induced gauge fields are Abelian (see Appendix).

Ordinary twistorialization maps points of M^4 to bi-spinors allowing quantum variants. Could twistorialization of M^4 and CP_2 allow something analogous?

10.3 The notion of finite measurement resolution

Finite measurement resolution [K87, K28] is central in TGD. It has several interpretations and the challenge is to unify the mutually consistent views.

10.3.1 Inclusions of HFFs, finite measurement resolution and quantum dimensions

Concerning measurement resolution the first proposal was that the inclusions of HFFs chacterize it.

1. The key idea is simple. Yangians and/or quantum algebras associated with the dynamical SKMAs defined by pairs of SSA and its isomorphic sub-algebra acting as pure gauge transformations are characterized by quantum phases [L16] characterizing also inclusions of HFFs [K87, K28]. Quantum parameter would characterize the measurement resolution.

   The Lie group characterizing SKMA would be replaced by its quantum counterpart. Quantum groups involve quantum parameter $q \in C$ involved also with n-structures. This parameter - in particular its phase- should belong to the extension of rationals considered. Notions like braiding making sense for 2-D structures are crucial. Remarkably, the representation theory for quantum groups with q different from a root of unity does not differ from that for ordinary groups. For the roots of unity the situation is different.

2. The levels in the hierarchy of inclusions for HFFs [K87] are labelled by integer $n \in [3, \infty)$ or equivalenly by quantum phases $q = exp(i\pi/n)$ and quantum dimension is given by $d_q = 4cos^2(\pi/n)$. $n = 3$ gives $d = 2$ that is ideal SH with minimal measurement resolution. For instance, in extension of rationals only phases, which are powers of $exp(i\pi/3)$ are represented p-adically so that angle measurement is very imprecise. The hierarchy would correspond to an increasing measurement resolution and at the level $n \to \infty$ one would have $d_q \to 4$. Could the interpretation be that one sees space-time as 4-dimensional? This strongly suggests that the hierarchy of Lie groups characterizing SKMAs are characterized by the same quantum phase as inclusions of HFFs.

How does quantal dimension show itself at space-time level?

1. Could SH reduce the 4-surfaces to effectively fractal objects with quantum dimension d_q? Could one speak of quantum variant of SH perhaps describe finite measurement resolution. In adelic picture this limit could correspond to an extension of rational consists of algebraic numbers extended by all rational powers of e. How much does this limit deviate from real numbers?

2. McKay correspondence (see `http://tinyurl.com/z48d92t`) states that the hierarchy of finite sub-groups of $SU(2)$ corresponds to the hierarchy ADE Kac-Moody algebras in the following sense. The so called McKay graph codes for the information about the multiplicities of the tensor products of given representation of finite group (spin 1/2 doublet) - obviously one can assign McKay graph to any Galois group. McKay correspondence says that the McKay graph for the so called canonical representation of finite sub-group of $SU(2)$ co-incides with the Dynkin diagram for ADE type Kac-Moody algebra.

3. A physically attractive idea is that these algebras correspond to a hierarchy of reduced SSAs and PSCAs defined by the gauge conditions of SSA and PSCA. The breaking of maximal effective gauge symmetry characterizing measurement resolution to isomorphic sub-algebra would bring in additional degrees of freedom increasing the quantum dimension of string world sheets from the minimal value $d_q = 2$.

   My naïve physical intuition suggests that McKay correspondence generalizes to a much wider class of Galois groups identifiable as finite groups of Lie type identifiable as sub-groups of Lie groups (for the periodic table of finite groups see (see `http://tinyurl.com/y75r68hp`)). In general, the irreducible representation (irrep) of group is reducible representation of subgroup. The rule could be that the representations of the quantum Lie groups *allowed* as ground states of SKMA representations are *irreducible* also as representations of Galois group in case that it is Lie-type subgroup.

What about the concrete geometric interpretation of d_q? Two interpretations, which do not exclude each other, suggest themselves.

1. A very naïve idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. The idea about fractal string world sheets does not however conform with the fact that preferred extremals must be rather smooth.

   String world sheets could be however locally smooth if they define an analog of discretization for the space-time surface. At the limit $d_q \rightarrow 4$ string world sheets would fill space-time surface. Analogously, strings (string orbits) would fill the space-like 3-surfaces at the boundaries of CD (the light-like 3-surfaces connecting the partonic 2-surfaces at boundaries of CD). The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution at the level of physics. For anyonic systems [K55] one indeed would have have large number of fermions at 2-D surfaces.

2. An alternative idea is that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. A precise mathematical representation of ignorance can be extremely successful trick as ordinary thermodynamics and also p-adic thermodynamics for particle masses [K41] demonstrate!

10.3.2 Three options for the identification of quantum dimension

The quantum dimension would increase as the measurement accuracy increases but what quantum dimension of string world sheets could mean at space-time level? Identification of quantum dimension as fractal dimension could be the answer but how could one concretely define this notion? Could one find an elegant formulation for the fractality at space-time level.

Option I

One could argue that quantum dimension is temperature like parameter coding for the ignorance about the details of space-time surface and string world sheet due to finite cognitive resolution. Cognitive representation consists of a discrete set of points of H in an extension of rationals defining the adele and quantum dimension would represent this ignorance. One would give up the attempts to represent quantum superposition of space-time surfaces with single classical surface. This option would use only the discrete cognitive representations (see the glossary in Appendix).

1. This would mean a radical simplification and could make sense for cognitive representations. String world sheet would be replaced by this discrete cognitive representation and one should be able to deduce its quantum dimension. Gal acts on this representation.

2. Could one imagine q-variants of the representations of Gal defining also representations of the Lie group defining KMA? If one can imbed Gal to Lie-group as discrete sub-group then the q-representation of the Lie-group would define a q-representation of discrete group and one might be able to talk about q-Galois groups.

3. On the other hand, the condition that these representations restricted to representations of Galois group remain irreducible poses similar condition. Are these two criteria equivalent? Could this allow to identify the value of root of unity associated with given Galois group and corresponding Lie group defining SKMA in case that it contains representations that remain irreps of Galois group? If so, the notion of quantum group would follow from adelic physics in a natural manner.

   This would allow to assign quantum dimension to the discretized string world sheet without clumsy fractal constructions at space-time level involving a lot of redundant information. The really nice thing would be that one would use only the information defining the cognitive representations and the fact that one does not know about the rest. Just as in thermodynamics, things would become extremely simple!

4. One might argue that giving just discrete points at partonic 2-surfaces gives very little information. If one however assumes that also the functions characterizing space-time surfaces as points of sub-WCW involved are constructed from rational polynomials with roots in the extension of rationals used, the situation improves dramatically.

Option II

A very naïve idea is that string world sheets effectively fill the space-time surface as the measurement accuracy increases. Smooth strings would fill the space-like 3-surfaces at the boundaries of CD and light-like 3-surface connecting the partonic 2-surfaces at boundaries of CD. The number of fermions at partonic 2-surfaces would increase and lead to an increased measurement resolution. For anyonic systems one indeed would have have large number of fermions at 2-D surfaces.

This option would be based on fractal dimension of some kind. Most naturally the fractal dimension would be that of space-time surface discretized using string world sheets and possibly also partonic 2-surface instead of points. It is however difficult to imagine a practical realization for fractal dimension in this sense.

1. Assume reference string world sheets in the minimal resolution defined by an extension of rationals with total area S_0. Study the total area S associated with string world sheets as function of the extension of rationals.

2. As the size of the extension grows, new points of extension emerge at partonic 2-surfaces and therefore also new string world sheets and the total area of string worlds sheets increases. Twistor lift suggests that one can take the area S_1 defined by Planck length squared and the area S_2 of CP_2 geodesic sphere as units. Suppose that one has $S/S_0 = (S_1/S_2)^d$, where d depends on the extension and equals to $d = 0$ for rationals, holds true. Could $d+2$ define the fractal dimension equal to d_q for Jones inclusions in the range $[2, 4)$? If the proposed notion of quantum Galois group makes sense this could be the case.

One must admit that the hopes of proving this picture works in practice are rather meager. Too much redundant information is involved.

Option III

One can also imagine an approach quantum dimension identifying quantum dimension as fractal dimension for space-time surface. If SH makes sense, one can consider the possibility that this dimension determined by the geometry of space-time surface as Riemann manifold has fractal dimension equal to the fractal dimension of string world sheets as sub-manifold.

1. The spectral dimension of classical geometry is discussed in http://tinyurl.com/yadcmjd6). One considers heat equation describing essentially random walk in a given metric and constructs so called heat kernel as a solution of the heat equation. The Laplacian depends on metric only - now the induced metric. The trace of heat kernel characterizes the probability to return to the original position. The derivative of the logarithm of the heat trace with respect to the logarithm of fictive time coordinate gives time dependent spectral dimension, which for short times approaches to topological dimension and for flat space equals to it always. For long times the dimension is smaller than the topological dimension due to curvature effects and SH raises the hope that this dimension corresponds to the fractal dimension of string world sheets identified as quantum dimension.

2. This approach can be criticized for the introduction of fictive time coordinate. Furthermore, Laplacian would be replaced with d'Alembertian in Minkowskian regions so that one cannot speak about diffusion anymore. Could one replace the heat equation with 4-D spinor d'Alembertian or modified Dirac operator so that also the induced gauge fields would appear in the equation? Artificial time coordinate would be replaced with some time coordinate for M^4 - light-cone proper time is the most natural choice. The probability would be defined as modulus squared for the fermionic propagator integrated over space-time surface.

   The problem is that this approach is rather formal and might be of little practical value.

10.3.3 n-structures and adelic physics

TGD involves several concepts, which could relate to n-structures. The notion of finite measurement resolution realized in terms of HFFs is the oldest notion [K87, K28]. Adelic physics suggests that the measurement resolution could be realized in terms of a hierarchy of extensions of rationals [L22]. The parameters characterizing space-time surfaces and by SH the string world sheets would belong to the extension. Also the points of space-time surface in the extension would be data coding for the preferred extremals. The reconnection points and intersection points would belong to the extension [L16]. n-structures relate closely to the notion of non-commutative space and strings world sheets could be such. Also the role of classical number fields - in particular $M^8 - H$ correspondence suggest the same. The challenge is to develop a coherent view about all these structures.

1. There should be also a connection with the adelic view. In this picture string world sheets and points of space-time surface with coordinates in the extension of rationals defining the adele code for the data for preferred extremals and quantum states. What these points are - could they correspond to points of partonic 2-surfaces carrying fermions or could the correspond also to the points in the interior of space-time surface is not clear. The larger the extension of rationals, the larger the number of these points, and the better the resolution and the larger the deviation of SH from ideal. The hierarchy of Galois groups of extension of rationals should relate closely to the inclusion hierarchies.

2. Galois extension with given Galois group Gal allows hierarchy of intermediate extensions defining inclusion sequence for Galois groups. Besides inclusion homomorphisms there exists homomorphisms from Galois group Gal with order $h_{eff}/h = n$ to its sub-groups $H \subset Gal$ with order $h_{eff}/h = m < n$ dividing n. If it exists the sub-group mapped to identity element is normal sub-group H for which right and left cosets gH and Hg are identical. These homomorphisms to sub-groups identify the sheets of Galois covering of the space-time surface transformed to each other by H and thus define different number theoretical resolutions: measurement resolution would have precise geometric meaning. This would mean looking states with $h_{eff}/h = n$ in poorer resolution defined by $h_{eff}/h = m < n$.

These arrows would define "resolution morphisms" in category theoretic description. Also the analogy with the homotopies of n-structures is obvious. There would be a finite number of normal sub-groups with order dividing n for given higher structure. Quantum phase equal to root of unity ($q = exp(i2\pi/k)$) could appear in these representations and distinguish them from ordinary group representations.

10.3.4 Could normal sub-groups of symplectic group and of Galois groups correspond to each other?

Measurement resolution realized in terms of various inclusion is the key principle of quantum TGD. There is an analogy between the hierarchies of Galois groups, of fractal sub-algebras of SSA, and of inclusions of HFFs. The inclusion hierarchies of isomorphic sub-algebras of SSA and of Galois groups for sequences of extensions of extensions should define hierarchies for measurement resolution. Also the inclusion hierarchies of HFFs are proposed to define hierarcies of measurement resolutions. How closely are these hierarchies related and could the notion of measurement resolution allow to gain new insights about these hierarchies and even about the mathematics needed to realize them?

1. As noticed, SSA and its isomorphic sub-algebras are in a relation analogous to the between normal sub-group H of group Gal (analog of isomorphic sub-algebra) and the group G/H. One can assign to given Galois extension a hierarchy of intermediate extensions such that one proceeds from given number field (say rationals) to its extension step by step. The Galois groups H for given extension is normal sub-group of the Galois group of its extension. Hence Gal/H is a group. The physical interpretation is following. Finite measurement resolution defined by the condition that H acts trivially on the representations of Gal implies that they are representations of Gal/H. Thus Gal/H is completely analogous to the Kac-Moody type algebra conjecture to result from the analogous pair for SSA.

2. How does this relate to McKay correspondence stating that inclusions of HFFs correspond to finite discrete sub-groups of $SU(2)$ acting as isometries of regular n-polygons and Platonic solids correspond to Dynkin diagrams of ADE type SKMAs determined by ADE Lie group G. Could one identify the discrete groups as Galois groups represented geometrically as sub-groups of $SU(2)$ and perhaps also those of corresponding Lie group? Could the representations of Galois group correspond to a sub-set of representations of G defining ground states of Kac-Moody representations. This might be possible. The sub-groups of $SU(2)$ can however correspond only to a very small fraction of Galois groups.

Can one imagine a generalization of ADE correspondence? What would be required that the representations of Galois groups relate in some natural manner to the representations as Kac-Moody groups.

Some basic facts about Galois groups and finite groups

Some basic facts about Galois groups mus be listed before continuing. Any finite group can appear as a Galois group for an extension of some number field. It is known whether this is true for rationals (see `http://tinyurl.com/hus4zso`).

Simple groups appear as building bricks of finite groups and are rather well understood. One can even speak about periodic table for simple finite groups (see `http://tinyurl.com/y75r68hp`). Finite groups can be regarded as a sub-group of permutation group S_n for some n. They can be classified to cyclic, alternating , and Lie type groups. Note that alternating group A_n is the subgroup of permutation group S_n that consists of even permutations. There are also 26 sporadic groups and Tits group.

Most simple finite groups are groups of Lie type that is rational sub-groups of Lie groups. Rational means ordinary rational numbers or their extension. The groups of Lie type (see `http://tinyurl.com/k4hrqr6`) can be characterized by the analogs of Dynkin diagrams characterizing Lie algebras. For finite groups of Lie type the McKay correspondence could generalize.

Representations of Lie groups defining Kac-Moody ground states as irreps of Galois group?

The goal is to generalize the McKay correspondence. Consider extension of rationals with Galois group Gal. The ground staes of KMA representations are irreps of the Lie group G defining KMA. Could the allow ground states for given Gal be irreps of also Gal?

This constraint would determine which group representations are possible as ground states of SKMA representations for a given Gal. The better the resolution the larger the dimensions of the allowed representations would be for given G. This would apply both to the representations of the SKMA associated with dynamical symmetries and maybe also those associated with the standard model symmetries. The idea would be quantum classical correspondence (QCC) space-time sheets as coverings would realize the ground states of SKMA representations assignable to the various SKMAs.

This option could also generalize the McKay correspondence since one can assign to finite groups of Lie type an analog of Dynkin diagram (see `http://tinyurl.com/k4hrqr6`). For Galois groups, which are discrete finite groups of $SU(2)$ the hypothesis would state that the Kac-Moody algebra has same Dynkin diagram as the finite group in question.

To get some perspective one can ask what kind of algebraic extensions one can assign to ADE groups appearing in the McKay correspondence? One can get some idea about this by studying the geometry of Platonic solids (see `http://tinyurl.com/p4rwc76`). Also the geometry of Dynkin diagrams telling about the geometry of root system gives some idea about the extension involved.

1. Platonic solids have p vertices and q faces. One has $\{p,q\} \in \{\{3,3\},\{4,3\},\{3,4\},\{5,3\},\{3,5\}\}$. Tetrahedron is self-dual (see `http://tinyurl.com/qdl4sss` object whereas cube and octahedron and also dodecahedron and icosahedron are duals of each other. From the table of `http://tinyurl.com/p4rwc76` one finds that the cosines and sines for the angles between the vectors for the vertices of tetrahedron, cube, and octahedron are rational numbers. For icosahedron and dodecahedron the coordinates of vertices and the angle between these vectors involve Golden Mean $\phi = (1+\sqrt{5})/2$ so that algebraic extension must involve $\sqrt{5}$ at least.

   The dihedral angle θ between the faces of Platonic solid $\{p,q\}$ is given by $sin(\theta/2) = cos(\pi/q)/sin(\pi/p)$. For tetrahedron, cube and octahedron $sin(\theta)$ and $cos(\theta)$ involve $\sqrt{3}$. For icosahedron dihedral angle is $tan(\theta/2) = \phi$. For instance, the geometry of tetrahedron involves both $\sqrt{2}$ and $\sqrt{3}$. For dodecahedron more complex algebraic numbers are involved.

2. The rotation matrices for for the triangles of tetrahedron and icosahedron involve $cos(2\pi/3)$ and $sin(2\pi/3)$ associated with the quantum phase $q = exp(i2\pi/3)$ associated with it. The rotation matrices performing rotation for a pentagonal face of dodecahedron involves $cos(2\pi/5)$ and $sin(2\pi/5)$ and thus $q = exp(i2\pi/5)$ characterizing the extension. Both $q = exp(i2\pi/3)$ and $q = exp(i2\pi/5)$ are thus involved with icosahedral and dodecahedral rotation matrices. The rotation matrices for cube and for octahedron have rational matrix elements.

3. The Dynkin diagrams characterize both the finite discrete groups of $SU(2)$ and those of ADE groups. The Dynkin diagrams of Lie groups reflecting the structure of corresponding Weyl groups involve only the angles $\pi/2, 2\pi/3, \pi-\pi/6, 2\pi-\pi/6$ between the roots. They would naturally relate to quadratic extensions.

   For ADE Lie groups the diagram tells that the roots associated with the adjoint representation are either orthogonal or have mutual angle of $2\pi/3$ and have same length so that length ratios are equal to 1. One has $sin(2\pi/3) = \sqrt{3}/2$. This suggests that $\sqrt{3}$ belongs to the algebraic extension associated with ADE group always. For the non-simply laced Lie groups of type B, C, F, G the ratios of some root lengths can be $\sqrt{2}$ or $\sqrt{3}$.

For ADE groups assignable to n-polygons ($n > 5$) Galois group must involve the cyclic extension defined by $exp(i2\pi/n)$. The simplest option is that the extension corresponds to the roots of the polynomial $x^n = 1$.

10.3.5 A possible connection with number theoretic Langlands correspondence

I have discussed number theoretic version of Langlands correspondence in [K38, L11] trying to understand it using physical intuition provided by TGD (the only possible approach in my case). Concerning my unashamed intrusion to the territory of real mathematicians I have only one excuse: the number theoretic vision forces me to do this.

Number theoretic Langlands correspondence relates finite-dimensional representations of Galois groups and so called automorphic representations of reductive algebraic groups defined also for adeles, which are analogous to representations of Poincare group by fields. This is kind of relationship can exist follows from the fact that Galois group has natural action in algebraic reductive group defined by the extension in question.

The "Resiprocity conjecture" of Langlands states that so called Artin L-functions assignable to finite-dimensional representations of Galois group Gal are equal to L-functions arising from so called automorphic cuspidal representations of the algebraic reductive group G. One would have correspondence between finite number of representations of Galois group and finite number of cuspidal representations of G.

This is not far from what I am naïvely conjecturing on physical grounds: finite-D representations of Galois group are reductions of certain representations of G or of its subgroup defining the analog of spin for the automorphic forms in G (analogous to classical fields in Minkowski space). These representations could be seen as induced representations familiar for particle physicists dealing with Poincare invariance. McKay correspondence encourages the conjecture that the allowed spin representations are irreducible also with respect to Gal. For a childishly naïve physicist knowing nothing about the complexities of the real mathematics this looks like an attractive starting point hypothesis.

In TGD framework Galois group could provide a geometric representation of "spin" (maybe even spin 1/2 property) as transformations permuting the sheets of the space-time surface identifiable as Galois covering. This geometrization of number theory in terms of cognitive representations analogous to the use of algebraic groups in Galois correspondence might provide a totally new geometric insights to Langlands correpondence. One could also think that Galois group represented in this manner could combine with the dynamical Kac-Moody group emerging from SSA to form its Langlands dual.

Skeptic physicist taking mathematics as high school arithmetics might argue that algebraic counterparts of reductive Lie groups are rather academic entities. In adelic physics the situation however changes completely. Evolution corresponds to a hierarchy of extensions of rationals reflected directly in the physics of dark matter in TGD sense: that is as phases of ordinary matter with $h_{eff}/h = n$ identifiable as divisor of the order of Galois group for an extension of rationals. Algebraic groups and their representations get physical meaning and also the huge generalization of their representation to adelic representations makes sense if TGD view about consciousness and cognition is accepted.

In attempts to understand what Langlands conjecture says one should understand first the rough meaning of many concepts. Consider first the Artin L-functions appearing at the number theoretic side. Consider first the Artin L-functions appearing at the number theoretic side.

1. L-functions (see `http://tinyurl.com/y8dc4zv9`) are meromorphic functions on complex plane that can be assigned to number fields and are analogs of Riemann zeta function factorizing into products of contributions labelled by primes of the number field. The definition of L-function involves Direchlet characters: character is very general invariant of group representation defined as trace of the representation matrix invariant under conjugation of argument.

2. In particular, there are Artin L-functions (see `http://tinyurl.com/y7thhodk`) assignable to the representations of *non-Abelian* Galois groups. One considers finite extension L/K of fields with Galois group G. The factors of Artin L-function are labelled by primes p of K. There are two cases: p is un-ramified or ramified depending on whether the number of primes of L to which p decomposes is maximal or not. The number of ramified primes is finite and in TGD framework they are excellent candidates for physical preferred p-adic primes for given extension of rationals.

These factors labelled by p analogous to the factors of Riemann zeta are identified as characteristic polynomials for a representation matrix associated with any element in a preferred conjugacy class of G. This preferred conjugacy class is known as Frobenius element $Frob(p)$ for a given prime ideal p , whose action on given algebraic integer in O_L is represented as its p:th power. For un-ramified p the characteristic polynomial is explicitly given as determinant $det[I - t\rho(Frob(p))]^{-1}$, where one has $t = N(p)^{-s}$ and $N(p)$ is the field norm of p in the extension L (see `http://tinyurl.com/o4saw2l`).

In the ramified case one must restrict the representation space to a sub-space invariant under inertia subgroup, which by definition leaves invariant integers of O_L/p that is the lowest part of integers in expansion of powers of p.

At the other side of the conjecture appear representations of algebraic counterparts of reductive Lie groups and their L-functions and the two number theoretic and automorphic L-functions would be identical.

1. Automorphic form F generalizes the notion of plane wave invariant under discrete subgroup of the group of translations and satisfying Laplace equation defining Casimir operator for translation group. Automorphic representations can be seen as analogs for the modes of classical fields with given mass having spin characterized by a representation of subgroup of Lie group G ($SO(3)$ in case of Poincare group).

   Automorphic functions as field modes are eigen modes of some Casimir operators assignable to G. Algebraic groups would in TGD framework relate to adeles defined by the hierarchy of extensions of rationals (also roots of e can be considered in extensions). Galois groups have natural action in algebraic groups.

2. Automorphic form (see `http://tinyurl.com/create.php`) is a complex vector valued function F from topological group to some vector space V. F is an eigen function of certain Casimir operators of G. In the simplest situation these function are invariant under a discrete subgroup $\Gamma \subset G$ identifiable as the analog of the subgroup defining spin in the case of induced representations.

   In general situation the automorphic form F transforms by a factor j of automorphy under Γ. The factor can also act in a finite-dimensional representation of group Γ, which would suggest that it reduces to a subgroup of Γ obtained by dividing with a normal subgroup. j satisfies 1-cocycle condition $j(g_1, g_2g_3) = j(g_1g_2, g_3)$ in group cohomology guaranteeing associativity (see `http://tinyurl.com/on7ffy9`). Cuspidality relates to the conditions on the growth of F at infinity.

3. Elliptic functions in complex plane characterized by two complex periods are meromorphic functions of this kind. A less trivial situation corresponds to non-compact group $G = SL(2, R)$ and $\Gamma \subset SL(2, Q)$.

There are more groups involved: Langlands group L_F and Langlands dual group ${}^LG$. A more technical formulation says that the automorphic representations of a reductive Lie group G correspond to homomorphisms from so called Langlands group L_F (see `http://tinyurl.com/ycnhkvm2`) at the number theoretic side to L-group ${}^LG$ or Langlands dual of algebraic G at group theory side (see `http://tinyurl.com/ycnk9ga5`). It is important to notice that ${}^LG$ is a complex Lie group. Note also that homomorphism is a representation of Langlands group L_F in L-group ${}^LG$. In TGD this would be analogous to a homomorphism of Galois group defining it as subgroup of the group G defining Kac-Moody algebra.

1. Langlands group L_F of number field is a speculative notion conjectured to be a extension of the Weil group of extension, which in turn is a modification of the absolute Galois group. Unfortunately, I was not able to really understand the Wikipedia definition of Weil group (`http://tinyurl.com/hk74sw7`). If E/F is finite extension as it is now, the Weil group would be $W_{E/F} = W_F/W_E^c$, W_E^c refers to the commutator subgroup W_E defining a normal subgroup, and the factor group is expected to be finite. This is not Galois group but should be closely related to it.

Only finite-D representations of Langlands group are allowed, which suggests that the representations are always trivial for some normal subgroup of L_F For Archimedean local fields L_F is Weil group, non-Archimedean local fields L_F is the product of Weil group of L and of SU(2). The first guess is that SU(2) relates to quaternions. For global fields the existence of L_F is still conjectural.

2. I also failed to understand the formal Wikipedia definition of the L-group ${}^LG$ appearing at the group theory side. For a reductive Lie group one can construct its root datum $(X^*, \Delta, X_*, \Delta^c)$, where X^* is the lattice of characters of a maximal torus, X_* its dual, Δ the roots, and Δ^c the co-roots. Dual root datum is obtained by switching X^* and X_* and Δ and Δ^c. The root datum for G and ${}^LG$ are related by this switch.

   For a reductive G the Dynkin diagram of ${}^LG$ is obtained from that of G by exchanging the components of type B_n with components of type C_n. For simple groups one has $B_n \leftrightarrow C_n$. Note that for ADE groups the root data are same for G and its dual and it is the Kac-Moody counterparts of ADE groups, which appear in McKay correspondence. Could this mean that only these are allowed physically?

3. Consider now a reductive group over some field with a separable closure K (say k for rationals and K for algebraic numbers). Over K G as root datum with an action of Galois group of K/k. The full group ${}^LG$ is the semi-direct product ${}^LG^0 \rtimes Gal(K/k)$ of connected component as Galois group and Galois group. $Gal(K/k)$ is infinite (absolute group for rationals). This looks hopelessly complicated but it turns it that one can use the Galois group of a finite extension over which G is split. This is what gives the action of Galois group of extension (l/k) in ${}^LG$ having now finitely many components. The Galois group permutes the components. The action is easy to understand as automorphism on Gal elements of G.

Could TGD picture provide additional insights to Langlands duality or vice versa?

1. In TGD framework the action of Gal on algebraic group G is analogous to the action of Gal on cognitive representation at space-time level permuting the sheets of the Galois covering, whose number in the general case is the order of Gal identifiable as $h_{eff}/h = n$. The connected component ${}^LG0$ would correspond to one sheet of the covering.

2. What I do not understand is whether ${}^LG = G$ condition is actually forced by physical contraints for the dynamical Kac-Moody algebra and whether it relates to the notion of measurement resolution and inclusions of HFFs.

3. The electric-magnetic duality in gauge theories suggests that gauge group action of G on electric charges corresponds in the dual phase to the action of ${}^LG$ on magnetic charges. In self-dual situation one would have $G =^L G$. Intriguingly, CP_2 geometry is self-dual (Kähler form is self-dual so that electric and magnetic fluxes are identical) but induced K̈ahler form is self-dual only at the orbits of partonic 2-surfaces if weak form of electric-magnetic duality holds true. Does this condition leads to ${}^LG = G$ for dynamical gauge groups? Or is it possible to distinguish between the two dynamical descriptions so that Langlands duality would correspond to electric-magnetic duality. Could this duality correspond to the proposed duality of two variants of SH: namely, the electric description provided by string world sheets and magnetic description provided by partonic 2-surfaces carrying monopole fluxes?

10.3.6 A formulation of adelic TGD in terms of cognitive representations?

The vision about p-adic physics as cognitive representations of real physics [L22] encourages to consider an amazingly simple formulation of TGD diametrically opposite to but perhaps consistent with the vision based on the notion of WCW and WCW spinor fields. Finiteness of cognitive and measurement resolutions would not be enemies of the theoretician but could make possible to deduce highly non-trivial predictions from the theory by getting rid of all irrelevant information and using only the most significant bits. Number theoretic physics need not of course cover the entire quantum physics and could be analogous to topological quantum field theories: even this might provide huge amounts of precise information about the quantum physics of TGD Universe.

Could the discrete variant of WCW geometry make sense?

The first thing that one can imagine is number theoretic discretization of WCW by assuming that WCW coordinates belong to an extension of rationals. Integration would reduce to a summation but the problem is that there are too many points in the extension so that sums do not make sense in real sense. In the case of space-time surfaces the problems are solved by the fact that space-time surfaces have dimension lower than the embedding space and the number of points with coordinates in the extension is in typical case finite: exceptions are surfaces such as canonically imbedded M^4 or CP_2. This option does not work at the level of WCW.

Cognitive representations however carry information about the points with coordinates in the extension of rationals defining the adele and possibly about the directions of strings emanating from these points. The effective WCW is kind of coset space with most of degrees of freedom not visible in the cognitive representation. Cognitive representations would specify the points in the extension of rationals for space-time surface, string world sheets, or even for their intersection with partonic surfaces at the ends of CD carrying fermion number plus those at the ends of sub-CDs forming a hierarchy.

Could one use the points of cognitive representation as coordinates for this effective WCW so that everything including WCW integration would reduce to well-defined summations? This would solve the problem of too many points in sub-WCW associated with the extension. Could one formulate everything that one can know at given level of cognitive hierarchy defined by extensions?

This idea was already suggested by the interpretation of p-adic mass calculations.

1. p-Adic mass calculations would correspond to cognitive representation of real physics [K17, K41]. For large p-adic primes p-adic thermodynamics converges extremely rapidly as powers $p^{-n/2}$ and the results from two lowest orders are practically exact.

2. What is however required is a justification for the map of p-adic mass squared values to real numbers by canonical identification. Quite generally this map makes sense for group invariants - say Lorentz invariants defined by inner products of momenta. As a matter of fact, the construction of quantum algebras and Yangians demands p-adic topology for the antipode to exist mathematically so that this approach could be forced by mathematical consistency [B1].

Could scattering amplitudes be constructed in terms of cognitive representations?

The crazy looking idea that cognitive representations defined by common points of real and p-adic variants of space-time surfaces or even partonic 2-surfaces is at least worth of showing to be wrong. If the idea works, cognitive representations could code what can be known about classical and even quantum dynamics and reduce physics to number theory. Also WCW would be discretized with points of discretized space-time surface defining WCW coordinates. Functional integral over WCW would reduce to a converging sum over cognitive representations.

It is interesting to look what this could mean if scattering amplitudes correspond in some sense to algebraic computations in bi-algebra besides product also co-product as its time reversal and interpreted as 3-vertex physically.

1. For the simplest option fermions would reside at the intersection points of partonic 2-surfaces and string world sheets. One possibility considered earlier is that at these points the Galois coverings are singular meaning that all sheets co-incide. This might be too strong condition and might be replacable by a weaker condition that Galois group at these points reduces to its sub-group and normal subgroup leaves amplitudes invariant. A reduction of measurement resolution would be in question.

2. If the basic computational operation involves a fusion of representations of Galois group, fusion algebra could describe the situation [L16]. The Galois groups assignable to the incoming lines of 3-vertex must correspond to Galois groups, which define groups of 3-levelled hierarchy of extension of rationals allowing inclusion homomorphism. Therefore the values of Planck constant would be of from $h_{eff}/h \in \{n_1, n_1n_2, n_1n_2n_3\}$. The tensor product decomposition would tell the outcome of tensor product. One can consider also 2-vertices corresponding to a phase transition $n_1 \leftrightarrow n_1n_2$ changing the value of h_{eff}/h.

McKay graphs (see http://tinyurl.com/z48d92t) for Galois groups describe the decomposition of the tensor products of representations of Galois groups. In general the tensor products for corresponding KMAs restricted to Galois group are not irreducible. What could this mean? Are they allowed to occur? Are there general results allowing to conclude how do the analogs of McKay graphs for the tensor products of the irreps of the group defining Kac-Moody group relate to the McKay graphs for its finite discrete sub-groups?

Possible problems relate to the description of momenta and higher excitations of SKMAs. In topological QFTs one loses information about metric properties such as mass but what happens in number theoretic QFT? Could the Galois approach expanded to include also discrete variants of quaternions and octonions assigna ble to extensions of rationals allow also the number theoretic description of also momenta?

1. Octonions and quaternions have $G(2)$ and $SO(3)$ as automorphisms groups (analogs of Galois groups). The octonionic automorphisms respecting chosen imaginary consist of $SU(3)$ rotations. These groups would be replaced with their dicrete variants with matrix elements in an extension of rationals.

   The automorphism group Gal for the extension of rationals and automorphism group $Aut \in \{G_2, SU(3), SO(3)\}$ for octonions/for octonions with fixed unit/for quaternions form a semi-direct product $Gal \rtimes Aut$ with multiplication rule $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_2g_1(g_b))$, where $g_1(g_b)$ represents the element of Aut obtained by performing Gal automorphism g_1 for g_b. For rational elements g_b one has $(g_1, g_a) \circ (g_2, g_b) = (g_1g_2, g_ag_b)$ so that Gal Aut_Q commute. An interesting possibility is that the automorphisms of $Aut \in \{SU(3), SO(3)\}$ can be interpreted in terms of standard model symmetries whereas Gal would relate to the dynamical symmetries.

   In M^8 picture one has naturally wave functions in the space of quaternionic light-like 8-momenta and it is natural to decompose quaternionic momenta to longitudinal M^2 piece and transversal E^2 piece. The physical interpretation of this condition has been discussed thoroughly in [L26]. One has thus more than mere analog of TQFT.

2. If fermions propagate along the lines of the TGD analogs twistor graphs, one must have an analog of propagator. Twistor approach [L26] implies that the propagator is replaced with the inverse of the fermion propagator for quaternionic 8-momentum as a residue with sigma matrices representing the quaternionic units. This is non-vanishing only if the fermion chirality is "wrong". This has co-homological interpretation: for external lines the inverse of the propagator would annihilate the state (co-closedness) unlike for internal lines.

3. Triality holds true for the octonionic vector representation assignable to momenta and octonionic spinors and their conjugates. All these should be quaternionic, in other words belong to some complexified quaternionic $M^4 \subset M^8$. The components of these spinors should belong to an extension of rational used with imaginary unit commuting with octonionic imaginary units.

4. The condition that the amplitudes belong to an extension of rationals could be extremely powerful when combined with category theoretic view implying the Hilbert space isometries allowing to relate amplitudes at different levels of the hierarchy. This conditions should be true also for the twistors in terms which momenta can be expressed. Also the space $SU(3)/U(1) \times U(1)$ of CP_2 twistors would be replaced with a sub-space with points in an extension of rationals.

10.4 Could McKay correspondence generalize in TGD framework?

McKay correspondence is rather mysterious looking correspondence appearing in several fields. This correspondence is extremely interesting from point of view of adelic TGD [L24] [L22].

1. McKay graphs code for the fusion algebra of irreducible representations (irreps) of finite groups (see http://tinyurl.com/z48d92t). For finite subgroups of $G \subset SU(2)$ McKay graphs are extended Dynkin diagrams for affine (Kac-Moody) algebras of ADE type coding the structure of the root diagram for these algebras. The correspondence looks mysterious since Dynkin diagrams have quite different geometric interpretation.

2. McKay graphs for finite subgroups of $G \subset SU(2)$ characterize also the fusion rules of minimal conformal field theories (CFTs) having Kac-Moody algebra (KMA) of $SU(2)$ as symmetries (see http://tinyurl.com/y7doftpe). Fusion rules characterize the decomposition of the tensor products of primary fields in CFT. For minimal CFTs the primary fields belonging to the irreps of $SU(2)$ are in 1-1 correspondence with irreps of G, and the fusion rules for primary fields are same as for the irreps of G. The irreps of $SU(2)$ are also irreps of G.

   Could the ADE type affine algebra appear as dynamical symmetry algebra too? Could the primary fields for ADE defining extended ADE Cartan algebra be constructed as G-invariants formed from the irreps of G and be exponentiated using the standard free field construction using the roots of the ADE KMA a give ADE KMA acting as dynamical symmetries?

3. McKay graphs for $G \subset SU(2)$ characterize also the double point singularities of algebraic surfaces of real dimension 4 in C^3 (or CP^3 , one variant of twistor space!) with real dimension 6 (see http://tinyurl.com/ydz93hle). The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This partially explains the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of spheres is the number of irreps for G is not obvious to me.

   The double point singularities of $C^2 \subset C^3$ allow thus ADE classification. The number of added points corresponds to the dimension of Cartan algebra for ADE type affine algebra, whose Dynkin diagram codes for the finite subgroup $G \subset SU(2)$ leaving the algebraic surface looking locally like C^2 invariant and acting as isotropy group of the singularity.

These results are highly inspiring concerning adelic TGD.

1. The appearance of Dynkin diagrams in the classification of minimal CFTs inspires the conjecture that in adelic physics Galois groups Gal or semi-direct products $G \triangleleft Gal$ of Gal with a discrete subgroup G of automorphism group $SO(3)$ (having $SU(2)$ as double covering!) classifies TGD generalizations of minimal CFTs. Also discrete subgroups of octonionic automorphism group can be considered. The fusion algebra of irreps of Gal would define also the fusion algebra for KMA for the counterparts of minimal fields. This would provide deep insights to the general structure of adelic physics.

2. One cannot avoid the question whether the extended ADE diagram could code for a dynamical symmetry of a minimal CFT or its modification? If the Gal singlets formed from the primary fields of minimal model define primary fields in Cartan algebra of ADE type KMA, then standard free field construction would give the charged KMA generators. In TGD framework this conjecture generalizes.

3. A further conjecture is that the singularities of space-time surface imbedded as 4-surface in its 6-D twistor bundle with twistor sphere as fiber could be classified by McKay graph of Gal. The singular intersection of the Euclidian and Minkowskian regions of space-time surface is especially interesting: the twistor spheres at the common points defining light-like partonic orbits need not be same but have intersections with intersection matrix given by McKay graph for Gal. The basic information about adelic CFT would be coded by the general character of singularities for the twistor bundle.

4. In TGD also singularities in which the group Gal is reduced to its subgroup Gal/H, where H is normal group are possible and would correspond to phase transition reducing the value of Planck constant. What happens in these phase transitions to single particle states would be dictated by the decomposition of representations of Gal to those of Gal/H and transition matrix elements could be evaluated.

One can find from web excellent articles about the topics to be discussed in this article.

1. The article "*Cartan matrices, finite groups of quaternions, and Kleinian singularities*" of John McKay [A90] (see `http://tinyurl.com/ydygjgge`) summarizes McKay correspondence.

2. Miles Reid has written an article " *The Du Val singularities* A_n, D_n, E_6, E_7, E_8" [A107] (see `http://tinyurl.com/ydz93hle`). Also the article " *Chapters on algebraic surfaces*" [A109](see `http://tinyurl.com/yaty9rzy`) of Reid should be helpful. There is also an article "*Resolution of Singularities in Algebraic Varieties*" [A58] (see `http://tinyurl.com/yb7cuwkf`) of Emma Whitten about resolution of singularities.

3. Andrea Cappelli and Jean-Benard Zuber have written an article "*A-D-E Classification of Conformal Field Theories*" [B12] about ADE classification of minimal CFT models (see `http://tinyurl.com/y7doftpe`).

4. McKay correspondence appears also in M-theory, and the thesis "*On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective*" [B27] (see `http://tinyurl.com/ycmyjukn`) of Yang-Hui He might be helful for the reader. In this work the possible generalization of McKay correspondence so that it would apply form finite subgroups of $SU(n)$ is discussed. $SU(3)$ acting as subgroup of automorphism group G_2 of octonions is especially interesting in this respect. The idea is rather obvious: the fusion diagram for the theory in question would be the McKay graph for the finite group in question.

10.4.1 McKay graphs in mathematics and physics

McKay graphs for subgroups of $SU(2)$ reducing to Dynkin diagrams for affine Lie algebras of ADE type appear in several ways in mathematics and physics.

McKay graphs

McKay graphs [A90] (see `http://tinyurl.com/ydygjgge`) code for the fusion algebra of irrpes of finite groups G (for Wikipedia article see `http://tinyurl.com/z48d92t`). One considers the tensor products of irreps with the canonical representation (doublet representation for the finite sub-groups of $SU(2)$), call it V. The irreps V_i correspond to nodes and their number is equal to the number of irreps G.

Two nodes i and j are no connected if the decomposition of $V \otimes V_i$ to irreps does not contain V_j. There is arrow pointing from $i \rightarrow j$ in this case. The number $n_{ij} > 0$ or number of arrows tells how many times j is contained in $V \otimes V_j$. For $n_{ij} = n_{ji}$ there is no arrow.

One can characterize the fusion rules by matrix $A = d\delta_{ij} - n_{ij}$, where d is the dimension of the canonical representation. The eigenvalues of this matrix turn out to be given by $d - \xi_V(g)$, where $\xi_V(g)$ is the character of the canonical representation, which depends on the conjugacy class of g only. The number of eigenvalues is therefore equal to the number $n(class, G)$ of conjugacy classes. The components of eigenvectors in turn are given by the values $\chi_i(g)$ of characters of irreps.

MacKay graphs and Dynkin diagrams

The nodes of the Dynkin diagram (see `http://tinyurl.com/hpm5y9s`) are positive simple root vectors identified as vectors formed by the eigenvalues of the Cartan sub-algebra generators under adjoint action on Lie algebra. In the case of affine Lie algebra the Cartan algebra contains besides the Cartan algebra of the Lie group also scaling generator $L_0 = td/dt$ and the number of nodes increases by one.

The number of positive simple roots equals to the dimension of the root space. The number n_{ij} codes now for the angle between positive simple roots. The number of edges connecting root vectors is $n = 0, 1, 2, 3$ depending on whether the angle between root vectors is $\pi/2, 2\pi/3, 3\pi/4$, or $5\pi/6$. The ratios of lengths of connected roots can have values $\sqrt{n}$, $n \in \{1, 2, 3\}$, and the number n of edges corresponds to this ratio. The arrow is directed to the shorter root if present. For simply laced Lie groups (ADE groups) the roots have unit length so that only single undirected edge can connect the roots. Weyl group acts as symmetries of the root diagram as reflections in hyperplanes orthogonal to the roots.

The Dynkin diagrams of affine algebras are obtained by adding to the Cartan algebra a generator which corresponds to the scaling generator $L_0 = td/dt$ of affine algebra assumed to act via adjoint action to the Lie algebra. Depending on the position of the added node one obtains also twisted versions of the KMA.

For the finite subgroups of $SU(2)$ the McKay graphs reduce to Dynkin diagrams of affine Lie algebras of ADE type [A90] (see `http://tinyurl.com/ydygjgge`) so that one has either $n_{ij} = 0$ or $n_{ij} = 1$ for $i \neq j$. There are no self-loops ($n_{ii} \neq 0$). The result looks mysterious since the two diagrams describe quite different things. One can also raise the question whether ADE type affine algebra might somehow emerge in minimal CFT involving $SU(2)$ KMA for which ADE classification emerges.

In TGD framework the interpretation of finite groups $G \subset SU(2)$ in terms of quaternions is an attractive possibility since rotation group $SO(3)$ acts as automorphisms of quaternions and has $SU(2)$ as its covering group.

ADE diagrams and subfactors

ADE classification emerges also naturally for the inclusions of hyper-finite factors of type II_1 [K87, K28]. Subfactors with index smaller than four have so called principal graphs characterizing the sequence of inclusions equal to one of the A, D or E Coxeter-Dynkin diagrams: see the article "*In and around the origin of quantum groups*" of Vaughan Jones [A131] (see `http://tinyurl.com/ycbbbvpq`). As a matter of fact, only the D_{2n} and E_6 and E_8 do occur. It is also possible to construct $M : N = 4$ sub-factor such that the principle graph is that for any subgroup $G \subset SU(2)$. This suggests that the subfactors $M : N = 4cos^2(\pi/n) < 4$ correspond to quantum groups. The basic objects can be seen as quantum spinors so that again the appearance of subgroups of $SU(2)$ looks natural. One can still wonder whether ADE KMAs might be involved.

ADE classification for minimal CFTs

.

CFTs on torus [B12] are characterized by modular invariant partition functions, which can be expressed in terms of characters of the scaling generator L_0 of Virasoro algebra (VA) given by

$$Z(\tau) = Tr(X) \ , \quad X = exp\{i2\pi \left[\tau(L_0 - c/24) - \overline{\tau}(\overline{L}_0 - c/24)\right]\} \ . \tag{10.4.1}$$

Modular invariance requires that $Z(\tau)$ is invariant under modular transformations leaving the conformal equivalence class of torus invariant. Modular group equals to $SL(2, Z)$ has as generators the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The partition function can be expressed as

$$Z(\tau) = \sum N_{j\overline{j}} \chi_j(q) \chi_{\overline{j}}(\overline{q}) \ , \quad q = exp(i2\pi\tau) \ , \quad \overline{q} = exp(-i2\pi\overline{\tau}) \ . \tag{10.4.2}$$

Here χ_j corresponds to the trace of $L_0 - c/24$ for a representation of KMA inducing the VA representation. Modular invariance of the partition function requires $SNS^\dagger = N$ and $TNT^\dagger = N$.

The ADE classification for minimal conformal models summarized in [B12] (see `http://tinyurl.com/y7doftpe`) involves $SU(2)$ affine algebra with central extension parameter k. The central extension parameter for the VA is $c < 1$. The fusion algebra for primary fields in representations of $SU(2)$ KMA characterizes the CFT to a high degree.

The fusion rules characterized the decomposition of the tensor product of representation D_i with representation D_j as $i \otimes j = N_{ij}^k D_k$. Due to the properties of the tensor product the matrices $\mathcal{N}_i = N_{ij}^k$ form and associative and commutative algebra and one can diagonalize these matrices simultaneously. This algebra is known as Verlinde algebra and its elements can be expressed in terms of unitary modular matrix S_{ij} representing the transformation of characters in the modular transformation $\tau \to -1/\tau$.

The generator of the Verlinde algebra is fusion algebra for the 2-D representation of $SU(2)$ generating the fusion algebra (this corresponds to the fact that tensor powers of this representations give rise to all representations of $SU(2)$). It turns out that for minimal models with a finite number of primary fields (KMA representations) the fusion algebra of KMA reduces to that for a finite subgroup of $SU(2)$ and thus corresponds to ADE KMA. The natural interpretation is that the condition that the number of primary fields is finite is realized if the primary fields correspond also to the irreps of finite subgroup of SU(2).

Could the ADE type KMA actually correspond to a genuine dynamical symmetry of minimal CFT? For this conjecture makes sense, the roots of ADE type KMA should be in 1-1 correspondence with the irreps of $G \subset SU(2)$ assignable to primary fields. How could this be possible? In the free field construction of ADE type KMA generators one constructs charged KMA generators from free fields in Cartan algebra by exponentiating the quantities $\alpha\cdot\phi$, where α is the root and ϕ is a primary field corresponding to the element of Cartan algebra of KMA. Could $SU(2)$ invariants formed from the primary fields defined by each G- (equivalently $SU(2)$-) multiplet give rise to $SU(2)$ neutral multiplet of primary fields of ADE type Cartan algebra and could their exponentiation give rise to ADE type KMA acting as dynamical symmetries of a minimal CFT?

The resolution of singularities of algebraic surfaces and extended Dynkin diagrams of ADE type

The classification of singularities of algebraic surfaces leads also to extended Dynkin diagrams of ADE type.

1. Classification of singularities

In algebraic geometry the classification of singularities of algebraic varieties [A58] is a central task. The singularities of curves in plane represent simplest singularities (see `http://tinyurl.com/y8ub2c4s`). The resolution of singularities of complex curves in C^3 is less trivial task.

The resolution of singularity (`http://tinyurl.com/y8veht3p`) is a central concept and means elimination of singularity by modifying it locally. There is extremely general theorem by Hiroka stating that the resolution of singularities of algebraic varieties is always possible for fields with characteristic zero (reals and p-adic number fields included) using a sequence of birational transformations. For finite groups the situation is unclear for dimensions $d > 3$.

The articles of Reid [A107] and Whitten [A58] describe the resolution for algebraic surfaces (2-D surfaces with real dimension equal to four). The article of Reid describes how the resolutions of double-point singularities of $m = d_c = 2$-D surfaces in $n = d_c = 3$-D C^3 or CP_3 (d_c refers to complex dimension) are classified by ADE type extended Dynkin diagrams. Subgroups $G \subset SU(2)$ appear naturally because the surface has dimension $d_c = 2$. This is the simplest non-trivial situation since for Riemann surface with $(m, n) = (1, 2)$ the group would be discrete subgroup of $U(1)$.

2. Singularity and Jacobians

What does one mean with singularity and its resolution? Reid [A107] (see `http://tinyurl.com/ydz93hle`) discusses several examples. The first example is the singularity of the surface $P(x_1, x_2, x_3) = x_1^2 - x_2x_3 = 0$.

1. One can look the situation from the point of view of embedding of the 2-surface to C^3: one considers map from tangent space of the surface to the embedding space C^3. The Jacobian of the embedding map $(x_2, x_3) \to (x_1, x_2, x_3) = \pm\sqrt{x_2x_3}, x_2, x_3)$ becomes ill-defined at origin since the partial derivatives $\partial x_1/\partial x_2 = (\sqrt{x_3/x_2})/2$ and $\partial x_1/\partial x_3 = (\sqrt{x_2/x_3})/2$ have all possible limiting values at singularity. The resolution of singularity must as a coordinate transformation singular at the origin should make the Jacobian well-defined. Obviously this must mean addition of points corresponding to the directions of various lines of the surface through origin.

2. A more elegant dual approach replaces parametric representation with representation in terms of conditions requiring function to be constant on the surface. Now the Jacobian of a map from C^3 to the 1-D normal space of the singularity having polynomial $P(x_1, x_2, x_3)$ as coordinate is considered. Singularity corresponds to the situation when the rank of the Jacobian defined by partial derivatives is less than maximal so that one has $\partial P/\partial x_i = 0$. The resolution of singularity means that the rank becomes maximal. Quite generally, for

co-dimension m algebraic surface the vanishing of polynomials P_i, $i = 1, ..., m$ defines the surface. At the singularity the reduction of the rank for the matrix $\partial P_i/\partial x_n$ from its maximal value takes place.

3. Blowing up of singularity

Codimension one algebraic surface is defined by the condition $P(x_1, x_2, ..., x_n) = 0$, where $P(x_1, ..., x_n)$ is polynomial. For higher codimensions one needs more polynomials and the situation is not so neat anymore since so called complete intersection property need not hold anymore. Reid [A107] gives an easy-to- understand introduction to the blowing up of double-point singularities. Also the article "*Resolution of Singularities in Algebraic Varieties*" of Emma Whitten [A58] (see `http://tinyurl.com/yb7cuwkf`) is very helpful.

1. Coordinates are chosen such that the singularity is at the origin $(x, y, z) = (0, 0, 0)$ of complex coordinates. The polynomial has vanishing linear terms at singularity and the first non-vanishing term is second power of some coordinate, say x_1, so that one has $x_1 = \pm\sqrt{P_1(x_1, x_2, x_3}$, where x_1 in P_1 appears in powers higher than 2. At the singularity the two roots co-incide. One can of course have also more complex singularities such as triple-points.
2. The simplest example $P(x_1, x_2, x_3) = x_1^2 - x_2 x_3 = 0$ has been already mentioned. This singularity has the structure of double cone since one as $x_1 = \pm\sqrt{x_2 x_3}$. At $(0, 0, 0)$ the vertices of the two cones meet.
3. One can look this particular situation from the perspective of projective geometry. Homogenous polynomials define a surface invariant under scalings of coordinates so that modulo scalings the surface can be regarded also as complex curve in CP_2. The conical surface can be indeed seen as a union of lines $(x_1 = k^2 x_3, x_2 = k x_3)$, where k is complex number. The ratio $x_1 : x_2 : x_3$ for the coordinates at given line is determined by $x_1 : x_2 = k$ and $x_2 : x_3 = k$ so that the surface can be parameterized by k and the coordinate along given line.

   In this perspective the singularity decomposes to the directions of the lines going through it and the situation becomes non-singular. The replacement of the original view with this gives a geometric view idea about the resolution of singularity: the 2-surface is replaced by a bundle lines of surfaces going through the singularity and singularity is replaced with a union of directions for these lines.

Quite generally, in the resolution of singularity, origin is replaced by a set of points (x_1, x_2, x_3) with a well-defined ratio $(x_1 : x_2 : x_3)$. This interpretation applies also to more general singularities. One can say that origin is replaced with a projective sub-manifold of 2-D projective space CP_2 (very familiar to me)! This procedure is known as blowing up. Strictly speaking, one only replaces origin with the directions of lines in C^3.

Remark: In TGD the wormhole contacts connecting space-time sheets of many-sheeted space-time could be seen as outcomes of blowing up procedure.

Blowing up replaces the singular point with projective space CP_1 for which points with same value of $(x_1 : x_2 : x_3)$ are identified. Blowing up can be also seen as a process analogous to seeing the singularity such as self-intersection of curve as an illusion: the curve is actually a projection of a curve in higher dimensional space to which it is lifted so that the intersection disappears [A58] (see `http://tinyurl.com/yb7cuwkf`). Physicist can of course protest by saying that in space-time physics is is not allowed to introduce additional dimensions in this manner!

There is an analytic description for what happens at the singular point in blowing up process [A58] (see `http://tinyurl.com/yb7cuwkf`).

1. In blowing up one lifts the surface in higher-dimensional space $C^3 \times CP_2$ (C^3 can be replaced by any affine space). The blowing up of the singularity would be the set of lines $\overline{q}$ of the surface S going through the singularity that is the set $B = \{(q, \overline{q}) | q \in S\}$. This set can be seen as a subset of $C^3 \times CP_2$ and one can represent it explicitly by using projective coordinates (y_1, y_2, y_3) for CP_2. Consider points of C^3 and CP_2 with coordinates $z = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The coordinate vectors must be parallel x is to be at line y. This requires that all 2×2 sub-determinants of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{10.4.3}$$

vanish: that is $x_iy_j - x_jy_i = 0$ for all pairs $i < j$. This description generalizes to the higher-dimensional case. The added CP_1s defined what is called exceptional divisor in the blown up surface. Recall that divisors (see `http://tinyurl.com/yc7x3ohx`) are by definition formal combinations of points of algebraic surface with integer coefficients. The principal divisors defined by functions are sums over their zeros and poles with integer weight equal to the order of zero (negative for pole).

The above example considers a surface $x_1^2 - x_2x_3 = 0$ which allows interpretation as a projective surface. The method however works also for more general case since the idea about replacing point with directions is applied only at origin.

2. One can consider a more practical resolution of singularity by performing a bi-rational coordinate transformation becoming singular at the singular point. This can improve the singularity by blowing it up or make it worse by inducing blowing down. The idea is to perform a sequence of this kind of coordinate changes inducing blowing ups so that final outcome is free of singularities.

   Since one considers polynomial equations both blowing up and its reversal must map polynomials to polynomials. Hence a bi-rational transformation b acting as a surjection from the modified surface to the original one must be in question (for bi-rational geometry see `http://tinyurl.com/yadoo3ot`). At the singularity b is many-to-one y so that at this point inverse image is multivalued and gives rise to the blowing up.

   The equation $P(x_1, x_2, x_3) = 0$ combined with the equations $x_iy_j - x_jy_i = 0$ by putting $y_3 = 1$ (the coordinates are projective) leads to a parametric representation of S using y_1 and y_2 as coordinates instead of x_1 and x_2. Origin is replaced with CP_1. This representation is actually much more general. Whitten [A58] gives a systematic description of resolution of singularities using this representation. For instance, cusp singularity $P(x_1, x_2) = x_1^2 - x_2^3 = 0$ is discussed as a special case.

3. Topologically the blow up process corresponds to the gluing of CP_2 to the algebraic surface $A : A \rightarrow A\#CP_2$ and clearly makes it more complex. One can say that gluing occurs along sphere CP_1 and since the process involves several steps several spheres are involved with the resolution of singularities.

4. ADE classification for resolutions of double point singularities of algebraic surfaces

ADE classification emerges for co-dimension one double point singularities of complex surfaces in C^3 known as Du Val singularities. The surface itself can be seen locally as C^2. These surfaces are 4-D in real sense can have self-intersections with real dimension 2. In the singular point the dimension of the intersection is reduced and the dimension of tangent space is reduced (the rank of Jacobian is not maximal). The vertices of cone and cusp are good examples of singularities.

The subgroup $G \subset SU(2)$ has a natural action in C^2 and it appears in the canonical representation of the singularity as orbifold C^2/G. This helps to understand the appearance of the McKay graph of G. The resolved singularities are characterized by a set of projective lines CP_1 with intersection matrix in CP_2 characterized by McKay graph of G. Why the number of projective lines equals to the number of irrepss of G appearing as nodes in McKay graph looks to me rather mysterious. Reid's article [A107] gives the characterization of groups G and canonical forms of the polynomials defining the singular surfaces.

The reason why Du Val singularities are so interesting from TGD point of view is that complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces as preferred extremals have real dimension 4 and assumed possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure [K80].

10.4.2 Do McKay graphs of Galois groups give overall view about classical and quantum dynamics of quantum TGD?

McKay graphs for Galois groups are interesting from TGD view point for several reasons. Galois groups are conjectured to be the number theoretical symmetries for the hierarchy of extensions of rationals defining hierarchy of adelic physics [L24] [L22] and the notion of CFT is expected to generalize in TGD framework so that ADE classification for minimal CFTs might generalize to a classification of minimal number theoretic CFTS by Galois groups.

1. Vision

The arguments leading to the vision are roughly following.

1. Adelic physics postulates a hierarchy of quantum physics with adeles at given level associated with extension of rationals characterized partially by Galois group and ramified primes of extension. The dimension of the extensions dividing the order of Galois group is excellent candidate for defining the value of Planck constant $h_{eff}/h = n$ and ramified primes could correspond to preferred p-adic primes. The discrete sets of points of space-time surface for which embedding space coordinates are in the extension define what I have interpreted as cognitive representations and can be said to be in the intersection of all number fields involved forming kind of book like structure with pages intersecting at the points with coordinates in extension.

   Galois groups would define a hierarchy of theories and the natural first guess is that Galois groups take the role of subgroups of $SU(2)$ in CFTs with $SU(2)$ KMA as symmetry. Could the MacKay graphs defining the fusion algebra of Galois group define the fusion algebra of corresponding minimal number theoretic QFTs in analogy with minimal conformal models? This would fix the primary fields of theories assignable to given level of adele hierarchy to be minimal representations of *Gal* perhaps having also interpretation as representations of KMAs or their generalization to TGD framework.

2. The analogies between TGD and the theory of Du Val singularities is intriguing. Complex surfaces in Du Val theory have real dimension 4 and are surfaces in space of real dimension 6. The intersections of the branches of the 4-surfaces have real dimension $D = 2$ in the generic case. In TGD space-time surfaces have real dimension 4 and possess complex structure or its Minkowskian generalization that I have called Hamilton-Jacobi structure.

   The twistor bundle of space-time surface has 2-sphere CP_1 as a fiber and space-time surface as base [K8, L26]. Space-time surfaces can be realized as sections in their own 6-D twistor bundle obtained by inducing twistor structure from the product $T(M^4) \times T(CP_2)$ of twistor bundles of M^4 and CP_2. Section is fixed only modulo gauge choice, which could correspond to the choice of the Kähler form defining twistor structure from quaternionic units represented as points of S^2. Even if this choice is made, $U(1)$ gauge transformations remain and could correspond to gauge transformations of WCW changing its Kähler gauge potential by gradient and adding to Kähler function a real part of holomorphic function of WCW coordinates.

   If the embedding of 4-D space-time surface as section can become singular in given gauge, it will have self-intersections with dimension 2 possibly assignable to partonic 2-surfaces and maybe also string world sheets playing a key role in strong form of holography (SH). Could SH mean that information about classical and quantum theory is coded by singularities of the embedding of space-time surface to twistor bundle. This would be highly analogous to what happens in the case of complex functions and also in twistor Grassmann theory whether the amplitudes are determined by the data at singularities.

3. Where would the intersections take place? Space-time regions with Minkowskian and Euclidian signature of metric have light-like orbits of partonic 2-surfaces as intersections. These surfaces are singular in the sense that the metric determinant vanishes and tangent space of space-time surface becomes effectively 3-D: this would correspond to the reduction of tangent space dimension of algebraic surface at singularity. It is attractive to think that the lifts of Minkowskian and Euclidian space-time sheets have twistor spheres, which only intersect and have intersection matrix represented by McKay graph of *Gal*.

What about string world sheets? Does it make sense to regard them as intersections of 4-D surfaces? This does not look plausible idea but there are also other characterizations of string world sheets. One can also ask about the interpretation of the boundaries of string world sheets, in particular the points at the partonic 2-surfaces. How could they relate to singularities? The points of cognitive representation at partonic 2-surfaces carrying fermion number should belong to cognitive representation with embedding space coordinates belonging to an extension of rationals.

4. In Du Val theory the resolution of singularity means that one adds additional points to a double singularity: the added points form projective sphere CP_1. The blowing up process is like lifting self-intersecting curve to a non-singular curve by embedding it into 3-D space so that the original curve is its projection. Could singularity disappear as one looks at 6-D objects instead of 4-D object? Could the blowing up correspond in TGD to a transition to a new gauge in which the self intersection disappears or is shifted on new place? The intersections of 4-surfaces in 6-space analogous to roots of polynomial are topologically stable suggesting that they can be only shifted by a new choice of gauge.

   Self-intersection be a genuine singularity if the spheres CP_1 defining the fibers of the twistor bundles of branches of the space-time surface do not co-incide in the 2-D intersection. In the generic case they would only intersect in the intersection. Could the McKay diagram of Galois group characterize the intersection matrix?

5. The big vision could be following. Galois groups characterize the singularities at given level of the adelic hierarchy and code for the multiplets of primary fields and for the analogs of their fusion rules for TGD counterparts of minimal CFTs. Note that singularities themselves identified as partonic 2-surfaces and possibly also light partonic orbits and possibly even string world sheets are not restricted in any manner.

This idea need not be so far-fetched as it might look at first.

1. One considers twistor lift and self-intersections indeed occur also in twistor theory. When the M^4 projections of two spheres of twistor space CP_3 (to which the geometric twistor space $T(M^4) = M^4 \times S^2$ has a projection) have light-like separation, they intersect. In twistor diagrams the intersection corresponds to an emission of massless particle.

2. The physical expectation is that this kind of intersections could occur also for the twistor bundle associated with the space-time surface. Most naturally, they could occur along the light-like boundary of causal diamond (CD) for points with light-like separation. They could also occur along the partonic orbits which are light-like 3-surfaces defining the boundaries between Minkowskian and Euclidian space-time regions. The twistor spheres at the ends of light-like curve could intersect.

Why the number of intersecting twistor spheres should reduce to the number $n(irred, Gal)$ of irreducible representations (irreps) of Gal, which equals to $n(Gal)$ in Abelian case but is otherwise smaller? This question could be seen as a serious objection.

1. Does it make sense to think that although there are $n(Gal)$ in the local fiber of twistor bundle, the part of Galois fiber associated with the twistor fiber CP_1 has only $n(irrep, Gal)$ CP_1:s and even that the spheres could correspond to irreps of Gal. I cannot invent any obvious objection against this. What would happen that Could this mean realization of quantum classical correspondence at space-time level.

2. There are $n(irrep, G)$ irreps and $\sum_i n_i^2 = n(G)$. n_i^2 points at corresponding sheet labelled by irrep. The number of twistor spheres collapsing to single one would be n_i for n_i-D irrep so that instead of states of representations the twistor spheres would correspond to irrep. One would have analogy with the fractionization of quantum numbers. The points assignable to n_i-D representations would become effectively $1/n_i$-fractionized. At the level of base space this would not happen.

Phase transitions reducing h_{eff}/h

In TGD framework one can imagine also other kinds of singularities. The reduction of Gal to its subgroup Gal/H, where H is normal subgroup defining Galois group for the Gal as extension of Gal/H is one such singularity meaning that the H orbits of space-time sheets become trivial.

1. The action of Gal could reduce locally to a normal subgroup H so that Gal would be replaced with Gal/H. In TGD framework this would correspond to a phase transition reducing the value of Planck constant $h_{eff}/h = n(Gal)$ labelling dark matter phases to $h_{eff}/h = n(Gal/H) = n(Gal)/n(H)$. The reduction to Gal/H would occur automatically for the points of cognitive representation belonging to a lower dimensional extension having Gal/H as Galois group. The singularity would occur for the cognitive points of both space-time surface and twistor sphere and would be analogous to $n(H)$-point singularity.

2. A singularity of the discrete bundle defined by Galois group would be in question and is assumed to induce similar singularity of $n(Gal)$ -sheeted space-time surface and its twistor lift. Although the singularity would occur for the ends of strings it would induce reduction of the extension of rationals to Gal/H, which should also mean that string world sheets have representation with WCW coordinates in smaller extension of rationals.

3. This would be visible as a reduction in the spectrum of primary fields of number theoretic variant of minimal model. I have considered the possibility that the points at partonic 2-surfaces carrying fermions located at the ends of string world sheets could correspond to singularities of this kind. Could string world sheets could correspond to this kind of bundle singularities? This singularity would not have anything to do with the above described self-interactions of the twistor spheres associated with the Minkowskian and Euclidian regions meeting at light-like orbits of partonic 2-surfaces.

4. This provides a systematic procedure for constructing amplitudes for the phase transitions reducing $h_{eff}/h = n(Gal)$ to $h_{eff}/h = n(Gal/H)$. The representations of Gal would be simply decomposed to the representations of $Gal(G/H)$ in the vertex describing the phase transition. In the simplest 2-particle vertex the representation of Gal remains irreducible as representation of Gal/H. Transition amplitudes are given by overlap integrals of representation functions of group algebra representations of Gal restricted to Gal/H with those of Gal/H.

   The description of transitions in which particles with different Galois groups arrive in same diagram would look like follows. The Galois groups must form an increasing sequence $... \subset Gal_i = Gal_{i+1}/H_{i+1} \subset ....$ The representations of the largest Galois group would be decomposed to the representations of smallest Galois group so that the scattering amplitudes could be constructed using the fusion algebra of the smallest Galois group. The decomposition to should be associative and commutative and could be carried in many ways giving the same outcome at the final step.

Also quaternionic and octonionic automorphisms might be important

What about the role of subgroups of $SU(2)$? What roles they could have? Could also they classify singularities in TGD framework?

1. SU(2) is indeed realize as multiplication of quaternions. $M^8 - H$ correspondence suggests that space-time surfaces in M^8 can be regarded as associative or co-associative (normal space-is associative. Associative translates to quaternionic. Associativity makes sense also at the level of H although it is not necessary. This would mean that the tangent space of space-time surface has quaternionic structure and the multiplication by quaternions is makes sense.

2. The Galois group of quaternions is $SO(3)$ and has discrete subgroups having discrete subgroups of $SU(2)$ as covering groups. Quaternions have action on the spinors from which twistors are formed as pairs of spinors. Could quaternionic automorphisms be lifted to a an $SU(2)$ action on these spinors by quaternion multiplication? Could one imagine that the representations formed as tensor powers of these representations give finite irreps of discrete

subgroups of $SU(2)$ defining ground states of $SU(2)$ KMA a representations and define the primary fields of minimal models in this manner?

3. Galois groups for extensions of rationals have automorphic action on $SO(3)$ and its algebraic subgroups replacing matrix elements with their automorphs: for subgroups represented by rational matrices the action is trivial. One would have analogs of representations of Lorentz group $SL(2, C)$ induced from spin representations of finite subgroups $G \subset SU(2)$ by Lorentz transformations realizing the representation in Lobatchevski space. Lorentz group would be replaced by Gal and the Lobatchevski spaces as orbit with the representation of Gal in its group algebra. An interesting question is whether the hierarchy of discrete subgroups of $SU(2)$ in McKay correspondence relates to quaternionicity.

G_2 acts as octonionic automorphisms and SU(3) appears as its subgroup leaving on octonionic imaginary unit invariant. Could these semi-direct products of Gal with these automorphism groups have some role in adelic physics?

About TGD variant of ADE classification for minimal models

I already considered the ADE classification of minimal models. The first question is whether the finite subgroups $G \subset SU(2)$ are replaced in TGD context with Galois groups or with their semi-direct products $G \triangleleft Gal$. Second question concerns the interpretation of the Dynkin diagram of affine ADE type Lie algebra. Does it correspond to a real dynamical symmetries.

1. Could the MacKay correspondence and ADE classification generalize? Could fusion algebras of minimal models for KMA associated with general compact Lie group G be classified by the fusion algebras of the finite subgroups of G. This generalization seems to be discussed in [B27] (see `http://tinyurl.com/ycmyjukn`).

2. Could the fusion algebra of Galois group Gal give rise to a generalization of the minimal model associated with a KMA of Lie group $G \supset Gal$. The fusion algebra of Gal would be identical with that for the primary fields of KMA for G. Galois groups could be also grouped to classes consisting of Galois groups appearing as a subgroup of a given Lie group G.

3. In TGD one has a fractal hierarchy of isomorphic supersymplectic algebras (SSAs) (the conformal weights of sub-algebra are integer multiples of those of algebra) with gauge conditions stating that given sub-algebra of SSA and its commutator with the entire algebra annihilates the physical states. The remnant of the full SSA symmetry algebra would be naturally KMA.

   The pair formed by full SSA and sub-SSA would correspond to pair formed by group G and normal subgroup H and the dynamical KMA would correspond to the factor group G/H. This conjecture generalizes: one can replace G with Galois group and $SU(2)$ KMA with a KMA continuing Gal as subgroup. One the other hand, one has also hierarchies of extensions of rationals such that $i+1$:th extension of rationals is extension of i:th extension. G_i is a normal subgroup of G_{i+1} so that the group $Gal_{i+1,i} = Gal_{i+1}/Gal_i$ acts as the relative Galois group for $i+1$:th extension as extensions of i:th extension.

   This suggest the conjecture that the Galois groups Gal_i for extension hierarchies correspond to the inclusion hierarchies $SSA_i \supset SSA_{i+1}$ of fractal sub-algebras of SSA such that the gauge conditions for SSA_i define a hierarchy KMA_i of dynamical KMAs acting as dynamical symmetries of the theory. The fusion algebra of KMA_i theory would be characterized by Galois group Gal_i.

4. I have considered the possibility that the McKay graphs for finite subgroups $G \subset SU(2)$ indeed code for root diagrams of ADE type KMAs acting as dynamical symmetries to be distinguished from $SU(2)$ KMA symmetry and from fundamental KMA symmetries assignable to the isometries and holonomies of $M^4 \times CP_2$.

   One can of course ask whether also the fundamental symmetries could have a representation in terms of Gal or its semi-direct product $G \triangleleft Gal$ with a finite sub-group automorphism group $SO(3)$ of quaternions lifting to finite subgroup $G \subset SU(2)$ acting on spinors. This is not necessary since Gal can form semidirect products with the algebraic subgroups of Lie groups of fundamental symmetries (Langlands program relies on this). In the generic case the algebraic subgroups spanned by given extension of rationals are infinite. When the finite subgroup $G \subset SU(2)$ is closed under Gal automorphism, the situation changes, and these extensions are expected to be in a special role physically.

The number theoretic generalization of the idea that affine ADE group acts as symmetries would be roughly like following. The nodes of the McKay graph of $G \triangleleft Gal$ label its irreps, which should be in 1-1 correspondence with the Cartan algebra of the KMA. The KMA counterparts of the local bilinear Gal invariants associated with Gal irreps would give currents of dynamical KMA having unit conformal weight. The convolution of primary fields with respect to conformal weight would be completely analogous to that occurring in the expression of energy momentum tensor as local bilinears of KMA currents.

If the free field construction using the local invariants as Cartan algebra defined by the irreps of $G \triangleleft Gal$ works, it gives rise to charged primary fields for the dynamical KMA labelled by roots of the corresponding Lie algebra. For trivial Gal one would have ADE group acting as dynamical symmetries of minimal model associated with $G \subset SU(2)$.

5. Number theoretic Langlands conjecture [L14] [L11] generalizes this to the semidirect product $G_0 \triangleleft Gal$ algebraic subgroup G_0 of the original KMA Lie group (p-adicization allows also powers of roots of e in extension). One can imagine a hierarchy of KMA type algebras KMA_n obtained by repeating the procedure for the $G_1 \triangleleft Gal$, where G_1 is discrete subgroup of the new KMA Lie group.

6. In CFTs are also other ways to extend VA or SVA (Super-Virasoro algebra) to a larger algebra by discovering new dynamical symmetries. The hope is that symmetries would allow to solve the CFT in question. The general constraint is that the conformal weights of symmetry generators are integer or half-integer valued. For the energy momentum tensor defining VA the conformal weight is $h = 2$ whereas the conformal weights of primary fields for minimal models are rational numbers.

   The simplest extension is SVA involving super generators with $h = 3/2$. Extension of (S)VA by (S)KMA so that (S)VA acts by semidirect product on (S)KMA means adding (S)KMA generators with with $h = 1$ (and 1/2). The generators of W_n-algebras (see `http://tinyurl.com/y93f6eoo`) have either integer or half integer conformal weights and the algebraic operations are defined as ordered products (an associative operation). These extensions are different from the proposed number theoretic extension for which the restriction to a discrete subgroup of KMA Lie group is essential.

10.5 Appendix

I have left the TGD counterpart of *fake flatness condition* in Appendix. Also a little TGD glossary is included.

10.5.1 What could be the counterpart of the fake flatness in TGD framework?

Schreiber considers the n-variant of gauge field concept with gauge potential A and gauge field $F = DA$ replaced with a hierarchy of gauge potential like entities $A^{k)}$, $k = 1, .., n$ with $DA^{n)} = 0$ and ends up in $n = 2$ case to what he calls *fake flatness condition* $DA^{1)} = A^{2)}$. This raises a chain of questions.

Could higher gauge fields of Schreiber and Baez [B35, B30] provide a proper description of the situation in finite measurement resolution? Could induction procedure make sense for them? Should one define the projections of the classical fields by replacing ordinary H-coordinates with their quantum counterparts? Could these reduce to c-numbers for number-theoretically commutative 2-surfaces with commutative tangent space? What about second fundamental form orthogonal to the string world sheet? Must its trace vanish so that one would have minimal 2-surface?

The proposal of Schreiber is a generalization of a massless gauge theory. My gut feeling is that the non-commutative counterpart of space-time surface is not promising in TGD framework. My feelings are however mixed.

1. The effective reduction of SSA and PSCA to quantal variants of Kac-Moody algebras gives rise to a theory much more complex than gauge theory. On the other hand, the reduction to Galois groups by finiteness of measurement resolution would paradoxically reduce TGD to extremely simple theory.

2. Analog of Yang Mills theory is not enough since it describes massless particles. Massless states in 4-D sense are only a very small portion of the spectrum of states in TGD. Stringy mass squared spectrum characterizes these theories rather than massless spectrum. On the other hand, in TGD particles are massless in 8-D sense and this is crucial for the success of generalized twistor approach.

3. To define a generalization of gauge theory in WCW one needs homology and cohomology for differential forms and their duals. For infinite-dimensional WCW the notion of dual is difficult to define. The effective reduction of SSA and PSCA to SKMAs could however effectively replace WCW with a coset space of the Lie-group associated with SKMA and finite dimension would allow tod define dual.

4. The idea about non-Abelian counterparts of Kähler gauge potential A and J in WCW does not look promising and the TGD counterpart of the *fake flatness condition* does not however encourage this.

Just for curiosity one could however ask whether one could generalize the Kähler structure of WCW to n-Kähler structure to describe finite measurement resolution at the level of WCW and whether also now something analogous to the *fake flatness condition* emerges. The "fake flatness" condition has interesting analogy in TGD framework starting from totally different geometric vision.

1. SSA acts as dynamical symmetries on fermions at string world sheets. Gauge condition would make the situation effectively finite-dimensional and allow to define if the effectively finite-D variant of WCW n-structures using ordinary homotopies and homology and cohomology. Also n-gauge fields could be defined in this effectively finite-D WCW and they would allow a description in terms of string world sheets. The interpretation could be in terms of generalization of Bohm-Aharonov phase from space-time level to Berry phase in abstract configuration space defined now in reduced WCW.

2. The Kähler form of $H = M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) can be induced to space-time level. When limited to the string world sheet is both the curvature form of Kähler potential and the analog of flat 2-connection defining the 1-connection in the approaches of Schreiber's and Baez so that one would have $B = J$ and $dB = 0$.

3. 2-form J as it is interpreted in Screiber's approach is hwoever not enough to construct WCW geometry. The generalization of the geometry of $M^4 \times CP_2$ (involving also the analog of Kähler form for M^4) to involve higher forms and its induction to the space-time level and level of WCW looks rather awkward idea and does not bring in anything new.

10.5.2 A little glossary

Topological Geometrodynamics (TGD): TGD can be regarded as a unified theory of fundamental interactions. In General Relativity space-time time is abstract pseudo-Riemannian manifold and the dynamical metric of the space-time describes gravitational interactions. In TGD space-time is a 4-dimensional surface of certain 8-dimensional space, which is non-dynamical and fixed by either physical or purely mathematical requirements. Hence space-time has shape besides metric properties. This identification solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity. Even more, sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry behind General Relativity, leads to a geometrization of known fundamental interactions and elementary particle quantum numbers.

Many-sheeted space-time, topological quantization, quantum classical correspondence (QCC): TGD forces the notion of many-sheeted space-time (see http://tinyurl.com/mf99gpv) with space-time sheets identified as geometric correlates of various physical objects (elementary particles, atoms, molecules, cells, ..., galaxies, ...). Quantum-classical correspondence (QCC) states that all quantum notions have topological correlates at the level of many-sheeted space-time.

Topological quantization: Topological field quantization is one of the basic distinctions between TGD and Maxwell's electrodynamics and GRT and means that various fields decompose to topological field quanta: radiation fields to "topological light rays"; magnetic fields to flux tube structures; and electric fields to electric flux quanta (electrets). Topological field quantization means that one can assign to every material system a field (magnetic) body, usually much larger than the material system itself, and providing a representation for various quantum aspects of the system.

Strong form of holography (SH): SH states that space-time surfaces as preferred extremals can be constructed from the data given at 2-D string world sheets and by a discrete set of points defining the cognitive representation of the space-time surface as points common to real and various p-adic variants of the space-time surface (intersection of realities and various p-adicities). Points of the cognitive representation have embedding space coordinates in the extension of rationals defining the adele in question. Effective 2-dimensionality is a direct analogy for the continuation of 2-D data to analytic function of two complex variables.

Zero energy ontology (ZEO): In ZEO quantum states are replaced by pairs of positive and negative energy states having opposite total quantum numbers. The pair corresponds to the pair of initial and final state for a physical event in classical sense. The members of the pair are at opposite boundaries of causal diamond (CD) (see http://tinyurl.com/mh9pbay), which is intersection of future and past directed light-cones with each point replaced with CP_2. Given CD can be regarded as a correlate for the perceptive field of conscious entity.

p-Adic physics, adelic physics, hierarchy of Planck constants, p-adic length scale hypothesis: p-Adic physics is a generalization of real number based physics to p-adic number fields and interpreted as a correlate for cognitive representations and imagination. Adelic physics fuses real physics with various p-adic physics ($p = 2, 3, 5, ...$) to adelic physics. Adele is structure formed by reals and extensions of various p-adic number fields induced by extensions of rationals forming an evolutionary hierarchy. Hierarchy of Planck constants corresponds to the hierarchy of orders of Galois groups for these extensions. Preferred p-adic primes satisfying p-adic length scale hypothesis $p \simeq 2^k$, are so called ramified primes for certain extension of rationals appearing as winners in algebraic evolution.

Cognitive representation: Cognitive representation corresponds to the intersection of the sensory and cognitive worlds - realities and p-adicities - defined by real and p-adic space-time surfaces. The points of the cognitive representation have H-coordinates which belong to an extension of rationals defining the adele. The choice of H-coordinates is in principle free but symmetries of H define preferred coordinates especially suitable for cognitive representations. The Galois group of the extension of rationals has natural action in the cognitive representation, and one can decompose it into orbits, whose points correspond the sheets of space-time surface as Galois covering. The number n of sheets equals to the dimension of the Galois group in the general case and is identified as the value $h_{eff}/h = n$ of effective Planck constant characterizing levels in the dark matter hierarchy. One can also consider replacing space-time surfaces as points of WCW with their cognitive representations defined by the cognitive representation of the space-time surface and defining the natural coordinates of WCW point.

Quantum entanglement, negentropic entanglement (NE), Negentropy maximization principle (NMP): Quantum entanglement does not allow any concretization in terms of everyday concepts. Schrödinger cat is the classical popularization of the notion (see http://tinyurl.com/lpjcjm9): the quantum state, which is a superposition of the living cat + the open bottle of poison and the dead cat + the closed bottle of poison represents quantum entangled state. Schrödinger cat has clearly no self identity in this state.

In adelic physics one can assign to the same entanglement both real entropy and various p-adic negentropies identified as measures of conscious information. p-Adic negentropy - unlike real - can be positive, and one can speak of negentropic entanglement (NE). Negentropy Maximization Principle (NMP) states that it tends to increase. In the adelic formulation NMP holding true only in statistical sense is a consequence rather than separate postulate.

Self, subself, self hierarchy: In ZEO self is generalized Zeno effect. At the passive boundary nothing happens to the members of state pairs and the boundary remains unaffected. At active boundary members of state pairs change and boundary itself moves farther away from the passive boundary reduction by reduction inducing localization of the active boundary in the moduli space of CDs after unitary evolution inducing delocalization in it. Self dies as the first reduction takes place at opposite boundary. A self hierarchy extending from elementary particle level to the level of the entire Universe is predicted. Selves can have sub-selves which they experience as mental images. Sub-selves of two separate selves can quantum entangle and this gives rise to fusion of the mental images and the fused mental image is shared by both selves.

Sensory representations: The separation of data processing and its representation is highly desirable. In computers processing of the data is performed inside CPU and representation is realized outside it (monitor screen, printer,...). In standard neuroscience it is however believed that both data processing and representations are realized inside brain. TGD leads the separation of data processing and representations: the "manual" of the material body provided by field (or magnetic) body serves as the counterpart of the computer screen at which the sensory and other representations of the data processed in brain are realized. Various attributes of the objects of the perceptive field processed by brain are quantum entangled with simple "something exists" mental images at the MB. The topological rays of EEG serve are the electromagnetic bridges serving as the topological correlates for this entanglement.

Chapter 11

Is Non-associative Physics and Language Possible only in Many-Sheeted Space-time?

11.1 Introduction

In Thinking Allowed Original (see `https://www.facebook.com/groups/thinkallowed/`) there was very interesting link added by Ulla about the possibility of non-associative quantum mechanics (see `http://phys.org/news/2015-12-physicists-unusual-quantum-mechanics.html#jCp`).

Also I have been forced to consider this possibility.

1. The 8-D embedding space of TGD has octonionic tangent space structure and octonions are non-associative. Octonionic quantum theory however has serious mathematical difficulties since the operators of Hilbert space are by definition associative. The representation of say octonionic multiplication table by matrices is possible but is not faithful since it misses the associativity. More concretely, so called associators associated with triplets of representation matrices vanish. One should somehow transcend the standard quantum theory if one wants non-associative physics.

2. Associativity seems to be fundamental in quantum theory as we understand it recently. Associativity is a fundamental and highly non-trivial constraint on the correlation functions of conformal field theories. It could be however broken in weak sense: as a matter of fact, Drinfeld's associator emerges in conformal field theory context. In TGD framework classical physics is an exact part of quantum theory so that quantum classical correspondence suggests that associativity could play a highly non-trivial role in classical TGD. The conjecture is that associativity requirement fixes the dynamics of space-time sheets - preferred extremals of Kähler action - more or less uniquely. One can endow the tangent space of 8-D imbedding $H = M^4 \times CP_2$ space at given point with octonionic structure: the 8 tangent vectors of the tangent space basis obey octonionic multiplication table.

   Space-time realized as n-D surface in 8-D H must be either associative or co-associative: this depending on whether the tangent space basis or normal space basis is associative. The maximal dimension of space-time surface is predicted to be the observed dimension $D = 4$ and tangent space or normal space allows a quaternionic basis.

3. There are also other conjectures [K77] about what the preferred extremals of Kähler action defining space-time surfaces are.

   (a) A very general conjecture states that strong form of holography allows to determine space-time surfaces from the knowledge of partonic 2-surfaces and 2-D string world sheets.

(b) Second conjecture involves quaternion analyticity and generalization of complex structure to quaternionic structure involving generalization of Cauchy-Riemann conditions.

(c) $M^8 - M^4 \times CP_2$ duality stating that space-time surfaces can be regarded as surfaces in either M^8 or $M^4 \times CP_2$ is a further conjecture.

(d) Twistorial considerations select $M^4 \times CP_2$ as a completely unique choice since M^4 and CP_2 are the only spaces allowing twistor space with Kähler structure. The conjecture is that preferred extremals can be identified as base spaces of 6-D sub-manifolds of the product $CP_3 \times SU(3)/U(1) \times U(1)$ of twistor spaces associated with M^4 and CP_2 having the property that it makes sense to speak about induced twistor structure.

The "super(optimistic)" conjecture is that all these conjectures are equivalent.

The motivation for what follows emerged from the observation that language is an essentially non-associative structure as the necessity to parse linguistic expressions essential also for computation using the hierarchy of brackets makes obvious. Hilbert space operators are however associative so that non-associative quantum physics does not seem plausible without an extension of what one means with physics. Associativity of the classical physics at the level of *single* space-time sheet in the sense that tangent or normal spaces of space-time sheets are associative as sub-spaces of the octonionic tangent space of 8-D embedding space $M^4 \times CP_2$ is one of the key conjectures of TGD.

But what about many-sheeted space-time? The sheets of the many-sheeted space-time form hierarchies labelled by p-adic primes and values of Planck constants $h_{eff} = n \times h$. Could these hierarchies provide space-time correlates for the parsing hierarchies of language and music, which in TGD framework can be seen as kind of dual for the spoken language? For instance, could the braided flux tubes inside larger braided flux tubes inside... realize the parsing hierarchies of language, in particular topological quantum computer programs? And could the great differences between organisms at very different levels of evolution but having very similar genomes be understood in terms of widely different numbers of levels in the parsing hierarchy of braided flux tubes-that is in terms of magnetic bodies as indeed proposed. If the intronic portions of DNA connected by magnetic flux tubes to the lipids of lipid layers of nuclear and cellular membranes make them topological quantum computers, the parsing hierarchy could be realized at the level of braided magnetic bodies of DNA.

Fortunately the mathematics needed to describe the breaking of associativity at fundamental level seems to exist. The hierarchy of braid group algebras forming an operad combined with the notions of quasi-bialgebra and quasi-Hopf algebra discovered by Drinfeld are highly suggestive concerning the realization of weak breaking of associativity. With good luck this breaking of associativity is all that is needed. With not so good luck this breaking of associativity takes place already at the level of single space-time sheets and something else is needed in many-sheeted space-time.

11.2 Is Non-associative Physics Possible In Many-sheeted Space-time?

The key question in the sequel is whether non-associative physics could emerge in TGD via *many-sheeted* space-time as an outcome of many-sheetedness and therefore distinguishing TGD from GRT and various QFTs.

11.2.1 What Does Non-associativity Mean?

To answer this question one must first understand what non-associativity could mean.

1. In non-associative situation brackets matter. $A(BC)$ is different from $(AB)C$. Here AB need not be restricted to a product or sum: it can be anything depending on A and B. From schooldays or at least from the first year calculus course one recalls the algorithm: when calculating the expression involving brackets one first finds the innermost brackets and calculates what is inside them, then proceed to the next innermost brackets, etc... In

computer programs the realization of the command sequences involving brackets is called parsing and compilers perform it. Parsing involves decomposition of program to modules calling modules calling.... Quite generally, the analysis of linguistic expressions involves parsing. Bells start to ring as one realizes that parsings form a hierarchy as also do the space-time sheets!

2. More concretely, there is hierarchy of brackets and there is also a hierarchy of space-time sheets labelled by p-adic primes and perhaps also by Planck constants $h_{eff} = n \times h$. B and C inside brackets form (BC), something analogous to a bound state or chemical compound. In TGD this something could correspond to a "glueing" space-time sheets B and C at the same larger space-time sheet. More concretely, (BC) could correspond to braided pair of flux tubes B and C inside larger flux tube, whose presence is expressed as brackets (..). As one forms $A(BC)$ one puts flux tube A and flux tube (BC) containing braided flux tubes B and C inside larger flux tube. For $(AB)C$ flux one puts tube (AB) containing braided flux tubes A and B and tube C inside larger flux tube. The outcomes are obviously different.

3. Non-associativity in this sense would be a key signature of many-sheeted space-time. It could show itself in say molecular chemistry, where putting on same sheet could mean formation of chemical compound AB from A and B. Another highly interesting possibility is hierarchy of braids formed from flux tubes: braids can form braids, which in turn can form braids,... Flux tubes inside flux tubes inside... Maybe this more refined breaking of associativity could underly the possible non-associativity of biochemistry: biomolecules looking exactly the same would differ in subtle manner.

4. What about quantum theory level? Non-associativity at the level of quantum theory could correspond to the breaking of associativity for the correlation functions of n fields if the fields are not associated with the same space-time sheet but to space-time sheets labelled by different p-adic primes. At QFT limit of TGD giving standard model and GRT the sheets are lumped together to single piece of Minkowski space and all physical effects making possible non-associativity in the proposed sense are lost. Language would be thus possible only in TGD Universe!

11.2.2 Language And Many-sheeted Physics?

Non-associativity is an essentially linguistic phenomenon and relates therefore to cognition. p-Adic physics labelled by p-adic primes fusing with real physics to form adelic physics are identified as the physics of cognition in TGD framework.

1. Could many-sheeted space-time of TGD provides the geometric realization of language like structures? Could sentences and more complex structures have many-sheeted space-time structures as geometrical correlates? p-Adic physics as physics of cognition would suggest that p-adic primes label the sheets in the parsing hierarchy. Could bio-chemistry with the hierarchy of magnetic flux tubes added, realize the parsing hierarchies?

2. DNA is a language and might provide a key example about parsing hierarchy. The mystery is that human DNA and DNAs of most simplest creatures do not differ much. Our cousins have almost identical DNA with us. Why do we differ so much? Could the number of parsing levels be the reason- p-adic primes labelling space-time sheets? Could our DNA language be much more structured than that of our cousins. At the level of concrete language the linguistic expressions of our cousin are indeed simple signals rather than extremely complex sentences of old-fashioned German professor forming a single lecture each. Could these parsing hierarchies realize themselves as braiding hierarchies of magnetic flux tubes physically and - more abstractly - as analos of parsing hierarchies for social structures. Indeed, I have proposed that the presence of collective levels of consciousness having the hierarchy of magnetic bodies as a space-time correlates distinguishes us from our cousins so that this explanation is consistent with more quantitative one relying on language.

3. I have also proposed that intronic portion of DNA is crucial for understanding why we differ so much from our cousins [K3, K81]. How does this view relate to the above proposal? In the

simplest model for DNA as topological quantum computer introns would be connected by flux tubes to the lipids of nuclear and cell membranes. This would make possible topological quantum computations with the braiding of flux tubes defining the topological quantum computer program.

Ordinary computer programs rely on computer language. Same should be true about quantum computer programs realized as braidings. Now the hierarchical structure of parsings would correspond to that of braidings: one would have braids, braids of braids, etc... This kind of structure is also directly visible as the multiply coiled structure of DNA. The braids beginning from the intronic portion of DNA would form braided flux tubes inside larger braided flux tubes inside.... defining the parsing of the topological quantum computer program. The higher the number of parsing levels, the higher the position in the evolutionary hierarchy. Each braiding would define one particular fundamental program module and taking this kind of braided flux tubes and braiding them would give a program calling these programs as sub-programs.

4. The phonemes of language have no meaning to us (at our level of self hierarchy) but the words formed by phonemes and involving at basic level the braiding of "phoneme flux tubes" would have. Sentences and their substructures would in turn involve braiding of "word flux tubes". Spoken language would correspond to temporal sequences of braidings of flux tubes at various hierarchy levels.

5. The difference between us and our cousins (or other organisms) would not be at the level of visible DNA but at the level of magnetic body. Magnetic bodies would serve as correlates also for social structures and associated collective levels of consciousness. The degree of braiding would define the level in the evolutionary hierarchy. This is of course the basic vision of TGD inspired quantum biology and quantum bio-chemistry in which the double formed by organism and environment is completed to a triple by adding the magnetic body.

11.2.3 What About The Hierarchy Of Planck Constants?

p-Adic hierarchy is not the only hierarchy in TGD Universe: there is also the hierarchy of Planck constants $h_{eff} = n \times h$ giving rise to a hierarchy of intelligences. What is the relationship between these hierarchies?

1. I have proposed that speech and music are fundamental aspects of conscious intelligence and that DNA realizes what I call bio-harmonies in quite concrete sense [L7] [K61]: DNA codons would correspond to 3-chords. DNA would both talk and sing. Both language and music are highly structured. Could the relation of h_{eff} hierarchy to language be same as the relation of music to speech?

2. Are both musical and linguistic parsing hierarchies present? Are they somehow dual? What does parsing mean for music? How musical heard sounds could give rise to the analog of braided strands? Depending on the situation we hear music both as separate notes and as chords as separate notes fuse in our mind to a larger unit like phonemes fuse to a word. Could chords played by single instrument correspond to braidings of flux tubes at the same level? Could the duality between linguistic and musical intelligence (analogous to that between function and its Fourier transform) be very concrete and detailed and reflect itself also as the possibility to interpret DNA codons both as three letter words and as 3-chords [L7]?

11.3 Braiding Hierarchy Mathematically

More precise formulation of the braided flux tube hierarchy leads naturally to the notions of braid group and operad that I have considered earlier. They have a close relationship with quantum groups - more precisely, bialgebras and Hopf algebras and their generalizations quasi-bialgebras and quasi-Hopf algebras, which in turn allow to characterize what might be called minimal breaking of associativity in terms of Drinfeld associator. These notions are already familiar from conformal field theories and string theories them so that there are good hopes that no completely new mathematics is not needed.

It must be made clear that I am not a mathematician and the following is just a modest attempt to understand what the problem is. I try to identify the algebraic structure possibly allowing to realize the big vision and gather some results about these structures from Wikipedia: I confess that I do not understand the formulas at the deeper level and my goal is to find their physical interpretation in TGD framework.

11.3.1 How To Represent The Hierarchy Of Braids?

Before going to web to see how modern mathematics could help in the problem, try first to formulate the situation more concretely. One must consider a more detailed representation for braids and for their hierarchy.

Consider first rough physical geometric view about braids of braids represented in terms of flux tubes.

1. Braid strands have two ends: one can label them as "lower" and "upper". Flux tubes can be labelled by p-adic prime p and $h_{eff} = n\times h$. Magnetic flux tubes can carry monopole flux and this could be crucial for the breaking of associativity - at least it is so in the proposed model (see `http://tinyurl.com/y7oom5kh`). The possibility of apparent magnetic monopoles in TGD framework indeed involves many-sheetedness in an essential manner: monopole flux flows from space-time sheet to another one through wormhole contact. This can be taken as one possible hint about the concrete physics involved.

2. One can get more precise picture by using formulas. One has labelling of flux tubes by primes p and Planck constants h_{eff}: to be short call this label $a, b, c, ...$ Since the values of p and h_{eff} are graded one could also speak of grading. The states for given value of a assignable to braid strands are labelled by the quantum states $A, B, ...$ associated with them and analogous to algebra elements. One must however consider all possible situations so that has operators $A_a, B_a, ...$ analogous to algebra elements of a graded algebra about which Clifford algebras and super-algebras are familiar examples.

3. Consider now the physical interpretation for the breaking of associativity. For ordinary associative algebra one considers $A(BC) = (AB)C$. This condition as such make sense if $A(BC)$ and $(AB)C$ are inside same flux tube and perhaps also that the strands A, B, C are not braids. In the general case one must must add the labels a, b, c, d and a, b_1, c_1, d_1 and one obtains $((A_dB_d)_cC_b)_a$ and $(A_{b_1}(B_{d_1}C_{d_1}))_{c_1})_a$. Obviously, these two states need not identical unless one has $a = b = c = d = b_1 = c_1 = d_1$, which is also possible and means that all strands are at the same flux tube labelled by a. The challenge is to combine various almost copies of algebraic structure defined by braidings and labelled by $a, b, ..$ to larger algebraic structure and formulate the breaking of associativity for this structure.

11.3.2 Braid Groups As Coverings Of Permutation Groups

Consider next the definition of braid group.

1. The notion of braiding can be algebraized using the notion of braid group B_n of n strands, which is covering of the permutation group S_n. For ordinary permutations generating permutations are exchanges of P_i two neighboring elements in the ordered set $(a_1, ..., a_n)$: $(a_i, a_{i+1}) \rightarrow (a_{i+1}, a_i)$. Obviously one has P_i^2 so that permutation is analogous to reflection. For braid group permutation is replaced to twisting of neighboring braid strand. It looks like permutation if one looks at the ends of strands only. If one looks entire strands, there is no reason to have $P_i^2 = 1$ except possibly for the representation of braid group. For arbitrarily large n that one has $P_i^n \neq 1$. 2-D braid group B_n can be represented as a homotopies of 2-D plane with n punctures identifiable as ends of braid strands defined by their non-intersecting orbits.

2. At the level of quantum description one must allow quantum superpositions of different braidings and must describe the quantum state of braid as wave function in braid group: one has element of group algebra of braid group. To each element of braid group one can assign unitary matrix representing the braiding and this unitary matrix would define a "topological

time evolution" defined by braiding transforming the initial state at the lower end of braid to the state at upper end of braid. Hence it seems that braid group algebra is the proper mathematical notion. One has quantum superposition of topological time evolutions: something rather abstract.

11.3.3 Braid Having Braids As Strands

Many-sheeted space-time makes possible fractal hierarchy of braids. Braid group in above sense would act on flux tubes at the same space-time sheets or space-time of QFT and GRT. Braids can have as strands braids so that there is hierarchy of braiding levels. The hierarchy of coilings of DNA provides a simple example (very simple having not much to do with the hierarchy of braidings for flux tubes).

1. Suppose that one has only two levels in the hierarchy. One has n braid strands/flux tubes altogether and there are k larger flux tubes containing n_i, $i = 1,..,k$ flux tubes so that one has $\sum_{i=1}^{k} n_i = n$. One can imagine a coloring of the braid strands inside given flux tube characterizing it. Only braid strands inside same flux tube - with the same color - can be braided. The full braid group B_n braiding freely all n braid strands is restricted to a subgbroup $B_{n_1} \times .... \times B_{n_2}$. This group can be regarded as subgroup of B_n so that permutations of B_{n_i} have a well-defined outcome, which seems however to be trivial classically. In quantum situation the exchange of the factors B_{n_i} however corresponds to braiding and for non-trivial quantum deformations its action is non-trivial. One has braided commutativity instead of commutativity.

2. Besides this there are braidings for the k braids of braids and this gives braid group B_k acting at upper level of hierarchy. Clearly the higher level braids b_i, $i = 1,...,k$ and lower level braids b_{ij}, $j = 1,...,n_i$ form a two-levelled entity. The braid groups B_k and B_{n_i} form an algebraic entity such that B_k acts by permuting the entities. Same holds true for the braid group algebras. This structure generalizes to an entire hierarchy of braid groups and their group algebras.

The hierarchy of braid group algebras seems to closely relate to a very general notion known as operad (see `http://tinyurl.com/yavyhcsk`). The key motivation of the operad theory is to model the computational trees resulting from parsing. The action of permutations/braidings on the basic objects is central notion and one indeed has hierarchy of symmetric groups/braid groups such that the symmetric/braid group at $n+1$:th level permutes/braids the objects at n:th level. Now the objects would be braids whose strands are braided. The braids can be strands of higher level braids and these strands can be braided. The action of braidings extends to that on braid group algebras defining candidates for wave functions.

11.4 General Formulation For The Breaking Of Associativity In The Case Of Operads

The formulas characterizing weak form of associativity by Drinfeld and others look rather mysterious without understanding of their origins. This understanding emerges from very simple but general basic arguments. Instead of studying given algebra one transcends to a higher abstraction level and studies - not the results of algebraic expressions - but the very process how the algebraic expression is evaluated and what kind of rules one can pose on it. The rules can be abstracted to what is called algebraic coherence.

The evaluation process - parsing - starts from inner most brackets and proceeds outwards so that eventually all brackets have disappeared and one has the value for the expression. This process can be regarded as a tree which starts from n inputs which are algebra elements, in the recent case they could be braid group algebra elements.

For instance, $(AB)C$ corresponds to an tree in which A, B, C are the branches. As one comes downwards, A and B fuse in the upper node and AB and C in the lower node. One manner to see this is as particle reaction proceeding backwards in time. For $A(BC)$ B and C fuse to BC in the upper node and A and BC at the lower node. Associativity says that the two trees give the same result. "Braided associativity" would say that these trees give results differing by an isomorphism just as braided commutativity says that AB and BA give results differing by isomorphism.

One can formulate this more concretely by denoting algebra decomposition $A \otimes B \in V \otimes V \rightarrow AB \in V$ by θ. In associativity condition one has 3 inputs so that 3-linear map $V \otimes V \otimes V \rightarrow V$ is in question. $(AB)C$ corresponds to $\theta \circ (\theta, 1)$ applied to $(A \otimes B \otimes C)$. Indeed, $(\theta, 1)$ gives $(AB, C) \in V \otimes V$. Second step $\theta \circ$ applied to this gives $(AB)C$. In the same manner, $A(BC)$ corresponds to $(\theta \circ (1, \theta)$ and associativity condition can be expressed as

$$\theta \circ (\theta, 1) = \theta \circ (1, \theta) \ .$$

An important delicacy should be mentioned. Although operations can be non-associative, the composition of operations is assumed to be associative. One can imagine obtaining $((ab)c)d$ either by $\theta \circ (\theta, 1) \circ (\theta, 1, 1))$ or by $(\theta \circ (\theta, 1)) \circ (\theta, 1, 1))$. The condition that these expressions are identical is completely analogous to the associativity for the composition of functions $f \circ (g \circ h) = (f \circ g) \circ h$ and this axiom looks obvious becomes one is used to *define* $f \circ g$ using this formula (starting from rightmost brackets). One could however imagine starting the evaluation of the composition of operators also from leftmost brackets. This makes sense if the composition can be done without the substitution of the value of argument.

11.4.1 How Associativity Could Be Broken?

How to obtain the breaking of associativity? The first thing is to get some idea about what (weak) breaking of associativity could mean.

Breaking of associativity at the level of algebras

Basic examples about breaking of associativity might help in the attempts to understand how many-sheetedness could induce the breaking of associativity. The intuitive feeling is that the effect is not large and disappears at QFT limit of TGD.

In the case of algebras one has bilinear map $V \otimes V \rightarrow V$. Now this map is from $V \otimes V \rightarrow V \otimes V$ so that the two situations need not have much common. Despite this one can look the situation in the case of algebras.

Lie-algebras and Jordan algebras represent key examples about non-associative algebras. Associative algebras, Lie-algebras, and Jordan algebras can be unified by weakning the associativity condition $A(BC) = (AB)C$ to a condition obtained by cyclically symmetrizing this condition to get the condition

$$A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$$

plus the condition

$$(A^2 B)A = A^2(BA)$$

defining together with commutativity condition $AB = BA$ Jordan algebra (`http://tinyurl.com/y8n9ol9p`). Note that Jordan algebra with multiplication $A \cdot B$ is realized in terms of associative algebra product as $A \cdot B = (AB + BA)/2$. A good guess is that the non-associative Malcev algebra formed by imaginary octonions with product $xy - yx$ satisfies these conditions.

Could the analog of the condition $A(BC) + B(CA) + C(AB) = (AB)C + (BC)A + (CA)B$ make sense also for the braiding group algebra assignable to quantum states of braids? The condition would say that cyclic symmetrization by superposing different braiding topologies gives a quantum state, which is in well-defined sense associative. Cyclic symmetry looks attractive because it plays also a key role in twistor Grassmannian approach.

Bi-algebras and Hopf algebras

One must start from bi-algebra $(B, \nabla, \eta, \Delta, \epsilon)$. One has product ∇ and co-product Δ analogous to replication of algebra element: particle physicists has tendency to see it as "time reversal" of product analogous to particle decay as reversal of particle fusion. The key idea is that co-multiplication is algebra homomorphism for multiplication and multiplication algebra homomorphism for co-multiplication. This leads to four commutative diagrams essentially expressing this property (see `http://tinyurl.com/y897z3es`).

Instead of giving the general definitions it is easier to consider concrete example of bi-algebra defined by group algebra. Bi-algebra has product $\nabla : H\otimes H \rightarrow H$ and co-product $\Delta : H \rightarrow H\otimes H$, which intuitively corresponds to inverse or time reversal of product. In the case of group algebra this holds true in very precise sense since one has $\Delta(g) = g\otimes g$: Δ is clearly analogous to replication. Besides this one has map $\epsilon : H \rightarrow K$ assigning to the algebra element a scalar and inverse map taking the unit 1 of the field to unit element of H, called also 1 in the following. For group algebra one has $\epsilon(g) = 1$. Bi-algebras are associative and co-associative. Commutativity is however only braided commutativity.

Hopf algebra $(H, \nabla, \eta, \Delta, \epsilon, S)$ is special case of bi-algebra and often loosely called quantum group. The additional building brick is algebra anti-homomorphism $S : H \rightarrow H$ known as antipode. S is analogous to mapping element of h to its inverse (it need not exist always). For group algebra one indeed has $S(g) = g^{-1}$. Besides the four commuting diagrams for bi-algebra one has commutative diagrams $\nabla(S, 1)\Delta = \eta\epsilon$ and $\nabla(1, S)\Delta = \eta\epsilon$, where ϵ is co-unit. The right hand side gives a scalar depending on h multiplied by unit element of H. For group algebra this gives unit at both sides. In the general case the situation $\Delta(h) = h \otimes h$ is true for group like element only and one has more complex formula $\Delta(h) = \sum_i a_i \otimes b_i$. One also defines primitive elements as elements satisfying $\Delta(h) = h \otimes 1 + 1 \otimes h$. Also Hopf algebras are associative and co-associative.

Quasi-bialgebras and quasi-Hopf algebras

Quasi-bi-algebras giving as special case quasi-Hopf algebras were discovered by Russian mathematician Drinfeld (for technical definition, which does not say much to non-specialist see `http://tinyurl.com/y7b6lpop` and `http://tinyurl.com/y89cs5oy`). They are non-associative or associative modulo isomoprhism.

Consider first quasi-bi-algebra $(B, \Delta, \epsilon, \Phi, l, r)$. Δ and ϵ are as for bi-algebra. Besides this one has invertible elements Φ (Drinfeld associator) and r, l called right and lef unit constraints. The conditions satisfied are following

- $$(1 \otimes \Delta) \circ \Delta(a) = \Phi[((\Delta \otimes 1) \circ \Delta(a)]\Phi^{-1} \quad .$$

  For $\Phi = 1 \otimes 1 \otimes 1$ one obtains associativity.

- $$[(1 \otimes 1 \times \Delta)(\Phi)][(\Delta \otimes 1 \otimes 1)(\Phi)] = (1 \otimes \Phi)[1 \otimes \Delta \otimes 1)(\Phi)(\Phi \otimes 1) \quad .$$

- $$(\epsilon \otimes 1)(\Delta(a)) = l^{-1}al \quad , \quad (1 \otimes \epsilon)(\Delta(a)) = r^{-1}ar \quad .$$

- $$1 \otimes \epsilon \otimes 1)(\Phi) = 1 \otimes 1 \quad .$$

These mysterious looking conditions express the fact that Drinfeld associator is a bialgebra co-cycle.

Quasi-bialgebra is braided if it has universal R-matrix which is invertible element in $B \otimes B$ such that the following conditions hold true.

$$(\Delta^{op})(a) = R\Delta(a)R^{-1} \quad . \tag{11.4.1}$$

Note that for group algebra with $\Delta g = g \otimes g$ one has $\Delta^{op} = \Delta$ so that R must commute with Δ. Whether this forces R to be trivial is unclear to me. Certainly there are also other homomorphisms. A good candidate for a non-symmetric co-product is $\Delta g = g \times h(g)$ where h is a homomorpism of the braid group. This requires the replacement $S(g) \rightarrow S(h^{-1}g)$ in order to obtain unitarity for $\nabla(1, S)\Delta$ loop removing the braiding.

$$(1 \otimes \Delta)(R) = \Phi^{-1}_{231}R_{13}\Phi_{213}R_{12}\Phi^{-1}_{213} \quad . \tag{11.4.2}$$

$$(\Delta \otimes 1)(R) = \Phi_{321}^{-1} R_{13} \Phi_{213}^{-1} R_{23} \Phi_{123} \quad . \tag{11.4.3}$$

This and second condition imply for trivial R that also Φ is trivial.
For $\Phi = 1 \otimes 1 \otimes 1$ the conditions reduces to those for ordinary braiding. The universal R-matrix satisfies the non-associative version of Yang-Baxter equation

$$R_{12}\Phi_{321}R_{13}(\Phi_{132})^{-1}R_{23}\Phi_{123} = \Phi_{321}R_{23}(\Phi_{231})^{-1}R_{13}\Phi_{213}R_{12} \quad . \tag{11.4.4}$$

Quasi-Hopf algebra is a special case of quasi-bialgebra. Also now one has product ∇, co-product Δ, antipode S not present in bialgebra, and maps ϵ and η. Besides this one has two special elements α and β of H such that the conditions $\nabla(S,\alpha)\cdot\Delta = \alpha$ and $\nabla(1,\beta S)\cdot\Delta = \alpha$. To my understanding these conditions generalize the conditions $\nabla(S,1)\Delta = \eta\epsilon$ and $\nabla(1,S)\Delta = \eta\epsilon$.

Associativity holds but only modulo a morphism in the same way as commutativity becomes braided commutativity in the case of quantum groups. The braided commutativity is characterized by R-matrix. The morphism defining "braided associativity" is characterized by the product $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ acting on triple tensor product $V \otimes V \otimes V$ and satisfying certain algebraic conditions. Φ has "inverse" $\Phi^{-1} = \sum_i P_i \otimes Q_i \otimes R_i$ The conditions $(1,\beta S,\alpha)\Phi = 1$ and $(S,\alpha,\beta S)\Phi = 1$. Here the action of S is that of algebra anti-homomorphism rather than algebra multiplication.

Drinfeld associator, which is a non-abelian bi-algebra 3-cocycle satisfying conditions analogous to the condition for weakened associativity holding true for Lie and Jordan algebras. These quasi-Hopf algebras are known in conformal field theory context and appear in Knizhnik-Zamolodchikov equations so that a lot of mathematical knowhow exists. According to Wikipedia, quasi-Hopf algebras are associated with finite-D irreps of quantum affine algebras in terms of F-matrices used to factorize R-matrix. The representations give rise to solutions of Quantum Yang-Baxter equation. The generalization of conformal invariance in TGD framework strongly suggests the relevance of Quasi-Hopf algebras in the realization of non-associativity in TGD framework.

Drinfeld double

Drinfeld double provides a concrete example about breaking of associativity. It can be formulated for finite groups as well as discrete groups. Drinfeld's approach is essentially algebraic: one works at the level of group algebra. In TGD framework the approach is geometric: algebraic constructs should emerge naturally from geometry. Braiding operations should induce algebras.

The basic notions involved are following.

1. One begins from a trivial tensor product of Hopf algebras and modified. In trivial case algebra product is tensor product of products, co-product is tensor product of co-products, antipode is tensor product of antipodes, map ϵ is product of the maps from the factors of the tensor product and delta maps unit element of field K to a product of unit elements. Drinfeld double represents a non-trivial tensor product of Hopf algebras.

2. One application of Drinfeld double construction is tensor product of group algebra and its dual. One can also interpret it as tensor product of braids as non-closed paths and closed braids (knots) as closed paths: in TGD framework this interpretation is suggestive and will be discussed later.

3. Drinfeld double allows breaking of associativity. It can be broken by introducing 3-cocycle (see `http://tinyurl.com/y9vcsmyg`) of group cohomology (see `http://tinyurl.com/y755gd36`). In the recent case group cohomology relies on homomorphism of group braid G to abelian group $U(1)$. n-cocycle is a map $G^n \rightarrow U(1)$ satisfying the condition that its derivation vanishes $d_n f = 0$. $d_n \circ d_{n-1} = 0$ holds true identically.

   The explicit definition of n-cocycle is in additive notion for U(1) product (usually multiplicative notation is used is) given by to illustrate that d_n acts like exterior derivative.

$$(d_nf)(g_1,g_2,g_n,g_{n+1}) = g_1f(g_1,...g_n) - f(g_1g_2,g_2,...,g_{n+1}) + f(g_1,g_2g_3,...,g_{n+1}) \\ -... + (-1)^n f(g_1,g_2...g_ng_{n+1}) + (-1)^{n+1}f(g_1,g_2...g_n) \ . \tag{11.4.5}$$

This formula is easy to translate to multiplicative notion. The fact that group cohomology is universal concept strongly suggests that 3 co-cycle can be introduced quite generally to break associativity in the sense that different associations differ only by isomorphism.

The construction of quantum double of Hopf algebras is discussed in detail at `http://tinyurl.com/ybbvjaw5`. Here however non-associative option is not discussed. In `http://tinyurl.com/ya8n98o5` one finds explicit formula for Drinfeld double for the Drinfeld double formed by group algebra and its dual. Just to give some idea what is involved the following gives the formula for the product:

$$(h,y)\circ(g,x) = \frac{\omega(h,g,x)\omega(hgx((hg)^{-1},h,g)}{\omega(h,gx(g)^{-1},h,g)}(hg,x) \ . \tag{11.4.6}$$

Without background it does not tell much. What is essential however that the starting point is algebraic. The product is non-vanishing only between (g,x) and (h,gxg^{-1}). For gauge group like structure one would have x instead of $g^{-1}xg^{-1}$. ω is 3-cocycle: it it is non-trivial one as associativity modulo isomorphism.

I do not have any detailed understanding of quasi-Hopf algebras but to me they seem to provide a very promising approach in attempts to understand the character of non-associativity associated with the braiding hierarchy. The algebraic construction of Drinfeld double does not seem interesting from TGD point of view but the idea that group cocycle is behind the breaking of associativity is attractive. Also the generalization of construction of Drinfeld double to code what happens in braiding geometrically is attractive. One of the many difficult challenges is to understand the role of the varying parameters p,h_{eff},q at the level of braid group algebras and their projective representations characterized by quantum phase q.

11.4.2 Construction Of Quantum Braid Algebra In TGD Framework

It seems that there is no hope that naïve application of existing formulas makes sense. The variety of different variants of quantum algebras is huge and one should have huge mathematical knowledge and understanding in order to find the correct option if it exists at all. Therefore I bravely take the approach of physicists. I try to identify the physical picture and then look whether I can identify the algebraic structure satisfying the axioms of Hopf algebra. In the following I first list various inputs which help to identify constraints on the algebraic structure, which should be simple if it is to be fundamental.

Trying to map out the situation

Usually physicists have enough trouble when dealing with single algebraic structure: say group and its representations. Unfortunately, this does not seem to be possible now. It seems that one must deal with entire collection of algebraic structures defined by braid groups B_n with varying value of n forming a hierarchy in which braid groups act on lower level braid groups.

1. What is clear that the algebraic operation $(A\otimes B)\rightarrow AB$ is somehow related to the braiding of flux tubes or fermionic strings connecting partonic 2-surfaces. One can also consider strings connecting the ends of light-like 3-surfaces so that one has both space-like and time-like braiding. One has flux tubes inside flux tubes.

   The challenge is to identify the natural algebra. It seems best to work with the braiding operations themselves - analogs of linguistic expressions - than the states to which they act. Braiding operations form discrete group, braid group. One must deal with the quantum superpositions of braidings so that one has wave functions in braid group identifiable as elements of discrete group algebra of braid group B_n. One can multiply group algebra elements and include the group algebra of B_m to that of B_n m a factor of n so that the desired product structure is obtained. The group algebras associated with various braid numbers can be organized to operad.

The operad formed by the braid group algebras has the desired hierarchical structure, and braid group algebra is one of the basic structures and quantum groups can be assigned with its projective representations.

2. For a given flux tube (and perhaps also for the fermionic string(s) assigned with it) one has degrees of freedom due different values of the quantum deformation parameter q for which roots of unity define preferred values in TGD framework. In TGD framework also hierarchy $h_{eff}/h = n$ of Planck constants brings in additional complexity. Also the p-adic prime p is expected to characterize the situation: preferred p-adic primes can be interpreted as so called ramified primes in the adelic vision about quantum TGD [K86] unifying real and various p-adic physics to a coherent whole. This brings in new elements. It is still unclear how closely n and $q = exp(i2\pi/m)$ are related and whether one might have $m = n$. Also the relationship of p to n is not well-understood. For instance, could p divide n.

3. Geometrically the association of braid strands means that they belong to the same flux tube. Moving the brackets in expression to transform say $(A(BC))$ to $((AB)C)$ means that strands are transferred from flux tube another one. Hence the breaking of associativity should take place at all hierarchy levels except the lowest one for which flux tube contains single irreducible braid strand - fermion line.

   The general mechanism for a weak breaking of associativity is describable in terms of Drinfeld's associator for quasi-bialgebras and known in some cases explicitly - in particular, shown by Drinfeld to exists when the number field used is rational numbers - is the first guess for the mechanism of the breaking of associativity. Drinfeld's associator is determined completely by group cohomology, which encourages to think that it can be used as such as as a multipler in the definition of product in suitable tensor product algebra. How the Drinfeld's associator depends on the p,n, and q is the basic question.

4. Besides the geometric action of braidings it is important to understand how the braidings act on the fundamental fermions. An attractive idea is that the representation is as holonomies defined by the induced weak gauge potentials as non-integrable phase factors at the boundaries of string world sheets defining fermion lines. The vanishing of electroweak gauge fields at them implies that the non-Abelian part of holonomy is pure gauge as in topological gauge field theories for which the classical solutions have vanishing gauge field. The em part of the induce spinor curvature is however non-vanishing unless one poses the vanishing of electromagnetic field at the boundaries of string world sheets as boundary condition. This seems un-necessary. The outcome would be non-trivial holonomy and restriction to a particular representation of quantum group with quantum phase q coming as root of unity means conditions on the boundaries of string world sheets. Quantum phase would make itself visible also classically as properties of string world sheets which together with partonic 2-surfaces determined space-time surface by strong form of holography. An interesting question relates to the possibility of non-commutative statistics: it should come from the weak part of induced connection which is pure gauge and seems possible as it is possible also in topological QFTs based on Chern-Simons action.

Hints about the details of the braid structure

Concerning the details of the braid structure one has also strong hints.

1. There two are two basic types of braids: I have called them time-like and space-like braids. Time-like (or rather light-like) braids are associated with the 3-D light-like orbits of partonic 2-surfaces at which the signature of the induced metric changes signature from Minkowskian to Euclidian. Braid strands correspond to fermionic lines identifiable as parts of boundaries of string world sheets. Space-like braids are associated with the space-like 3-surfaces at the

ends of causal diamond (CD). Also they consist of fermionic lines. These braids could be called fundamental.

If these braids are associated with magnetic flux tubes carrying monopole flux, the flux tubes are closed. Typically they connect wormhole throats at first space-time sheet, go to the second space-time sheet and return. Hence two-sheeted objects are in question. The braids in question can closed to knots and could correspond to closed loops assigned with the Drinfeld quantum double. The tensor product of the groupoid algebra associated with time-like braids and group algebra associated with space-like braids is highly suggestive as the analog of Drinfeld double.

Also magnetic flux tubes and light-like orbits of partonic 2-surfaces can become braided and one obtains the hierarchies of braids.

2. Since strong world sheets and partonic 2-surfaces have co-dimension 2 as sub-manifolds of space-time surface they can also get braided and knotted and give rise to 2-braids and 2-knots. This is something totally new. The unknotting of ordinary knots would take place via reconnections and the reconnections could correspond to the basic vertices for 2-knots analogous to the crossing of the plane projections of ordinary knot. Reconnections actually correspond to string vertices. A fascinating mathematical challenge is to generalize existing theories so that they apply to 2-braids and 2-knots.

3. Dance metaphor emerged in the model for DNA-lipid membrane system as topological quantum computer [K3, K81]. Dancers whose feet are connected to wall by threads define time-like braiding and also space-like braiding through the resulting entanglement of threads. The assumption was that DNA codons or nucleotides are connected by space-like flux tubes to the lipids of lipid layer of cell membrane or nuclear membrane.

   If they carry monopolo flux they make closed loops at the structure formed by two space-time sheets. The lipid layer of cell membrane is 2-dimensional and can be in liquid crystal state. The 2-D liquid flow of lipids induces braiding of both space-like braids if the DNA end is fixed and of time-like braids. This leads to the dance metaphor: the liquid flow is stored at space-time level to the topology of space-time as a space-like braiding of flux tubes induced by it. Space-like braiding would be like written text. Time-like braiding would be like spoken language.

4. If the space-like braids are closed, they form knots and the flow caused at the second end of braid by liquid flow must be compensated at the parallel flux tube by its reversal since braid strands cannot be cut. The isotopy equivalence class of knot remains unchanged since knots get gg^{-1} piece which can be deformed away. Second interpretation is that the braid X transforms to gXg^{-1}. This kind of transformation appears also in Drinfeld construction. This suggests that the purely algebraic tensor product of braid algebra and its dual corresponds in TGD framework semi-direct tensor product of the groupoid of time-like braids and space-like braids associated with closed knots. The semi-direct tensor product would define the fundamental topological interaction between braids.

5. One can also consider sequence of n tensor factors each consisting of time-like and space-like braids. This require a generalization of the product of two tensor factors to $2n$ tensor factors. Dance metaphor suggests that a kind of chain reaction occurs.

What the structure of the algebra could be?

With this background one can try to guess what the structure of the algebra in question is. Certainly the algebra is semi-direct product of above defined braid group algebras. The multiplication rule would have purely geometric interpretation.

1. The multiplication rule inspired by dance metaphor for 2 tensor factors would be

$$(a_1, a_2) \circ (b_1, b_2) = (a_1 a_2 b_1 a_2^{-1}, a_2 b_2) \ . \tag{11.4.7}$$

Here a_1, b_1 correspond label elements of time-like braid groupoid and a_2, b_2 the elements of braid group associated with the space-like braid. This would replace the trivial product rule $(a_1, a_2)(b_1 g_) = (a_1 b_1, a_2 b_2)$ for the trivial tensor product. The structure is same as for Poincare group as semi-direct product of Lorentz group and translation group: $(\Lambda_1, T_1)(\Lambda_2, T_2) = (\Lambda_1 \Lambda_2, T_1 + \Lambda_1(T_2))$.

It is easy to check that this product is associative. One can however add exactly the same 3-cocycle factor

$$(h, y) \circ (g, x) = \frac{\omega(h, g, x)\omega(hgx((hg)^{-1}, h, g)}{\omega(h, gx(g)^{-1}, h, g)}(hg, x) \quad . \tag{11.4.8}$$

Here (h, y) corresponds to (a_1, a_2) and (g, x) to (b_1, b_2). This should give breaking of non-associativity and third group cohomology of braid group B_n would characterize the non-equivalent associators.

2. The product rule generalizes to n factors. This generalization could be relevant for the understanding of braid hierarchy.

$$(a_1, a_2, ...a_n) \circ (b_1, b_2, ...b_n) \equiv (c_1, ..., c_n) \quad , \tag{11.4.9}$$

where one has

$$\begin{array}{lll} c_n = a_n b_n \ , & c_{n-1} = a_{n-1} Ad_{a_n}(b_{n-1}) \ , & c_{n-2} = a_{n-2} Ad_{a_{n-1}a_n}(b_{n-2}) \ , \\ c_{n-3} = a_{n-3} Ad_{a_{n-2}a_{n-1}a_n}(b_{n-3}) & ,.... & c_1 = a_1 Ad_{a_2.....a_n}(b_1) \ . \\ Ad_x(y) = xyx^{-1} \ . & & \end{array} \tag{11.4.10}$$

In this case a good guess for the breaking of associativity is that the associator is defined in terms of n-cocyle in group cohomology.

What is remarkable that this formula guarantees without any further assumptions the condition

$$\begin{gathered} \nabla_{1\otimes 2}(\Delta_1(a), \Delta_2(b)) = \nabla_1(\Delta_1(a))\nabla_2(\Delta_2(b)) = \sum_{(a)} a_1 a_2 \sum_{(b)} b_1 b_2 \ , \\ \Delta_1(a) = \sum_{(a)} a_1 \otimes a_2 \quad , \quad \Delta_2(b) = \sum_{(b)} b_1 \otimes b_2 \end{gathered} \tag{11.4.11}$$

as a little calculation shows. For group algebra one has $\Delta(a) = g \otimes g$. $\nabla_{1\otimes 2}$ refers to the product defined above.

3. The formula for $\Delta_{1\otimes 2}$ is also needed. The simplest guess is that it corresponds to replication for both factors. This would mean $\Delta^{op} = \Delta$: non-symmetric form guaranteeing non-trivial braiding is however desirable. A candidate satisfying this condition in $n = 2$ case is asymmetric replication:

$$\begin{array}{l} \Delta_{1\otimes 2}(bab^{-1}, b) \otimes (a, b) \\ \Delta^{op}_{1\otimes 2}(a, b) \otimes (bab^{-1}, b) \ . \end{array} \tag{11.4.12}$$

4. In $n = 2$ case the formula for antipode would read as

$$S(a_1, a_2) = (a_2^{-1} a_1^{-1} a_2, a_2^{-1}) \tag{11.4.13}$$

instead of $S(a_1, a_2) = (a_1^{-1}, a_2^{-1})$. Again the semi-direct structure would be involved. One can check that the formula

$$\nabla_{1\otimes 2}(1, S)\Delta_{1\otimes 2} = 1 \otimes 1 \tag{11.4.14}$$

holds true.

11.4.3 Should One Quantize Complex Numbers?

The TGD inspired proposal for the concrete realization of quantum groups might help in attempts to understand the situation. The approach relies on what might be regarded as quantization of complex numbers appearing as matrix elements of ordinary matrices.

1. Quantum matrices are obtained by replacing complex number valued of matrix elements of ordinary matrices with operators. They are are products of hermitian non-negative matrix P analogous to modulus of complex number and unitary matrix S analogous to its phase. One can also consider the condition $[P, S] = iS$ inspired by the idea that radial momentum and phase angle define analog of phase space.

2. The notions of eigenvalue and eigenstate are generalized. Hermitian operator or equivalently the spectrum of its eigenvalues replaces real number. The condition that eigenvalue problem generalizes, demands that the symmetric functions formed from the elements of quantum matrix commute and can be diagonalized simultaneously. The commutativity of symmetric functions holds also for unitary matrices. These conditions is highly non-trivial, and consistent with quantum group conditions if quantum phases are roots of unity. In this framework also Planck constant is replaced by a hermitian operator having $h_{eff} = n \times h$ as its spectrum. Also $q = exp(in2\pi/m)$ generalizes to a unitary operator with these eigenvalues.

3. This leads to a possible concrete representation of quantum group in TGD framework allowing to realize the hierarchy of inclusions of hyperfinite factors obtained by repeatedly replacing the operators appearing as matrix elements with quantum matrices.

4. This procedure can be repeated. One might speak of a fractal quantization. At the first step one obtains what might be called 1-hermitian operators with eigenvalues replaced with hermitian operators. For 1-unitary matrices eigenvalues, which are phases are replaced with unitary operators. At the next step one considers what might be called 2-hermitian and 2-unitary operators. An abstraction hierarchy in which instance (localization to a point as member of class) is replaced with wave function in the class. This hierarchy is analogous to that formed by infinite primes and by the sheets of the many-sheeted space-time. Also braids of braids of ... form this kind of abstraction hierarchy as also the parsing hierarchy for linguistic expressions.

I have proposed that generalized Feynman diagrams or rather - TGD analogs of twistor diagrams - should have interpretation as sequences of arithmetic operators with each vertex representing product or co-product and having interpretation as time reversal of the product operation.

1. The arithmetic operations could be induced by the algebraic operations for Yangian algebra [A26] [B20, B14, B15] assignable to the super-symplectic algebra. I have also proposed that there TGD allows a very powerful symmetry generalizing the duality symmetry of old-fashioned string models relating s- and t-channel exchanges. This symmetry would state

that one can freely move the ends of the propagator lines around the diagrams and that one can remove loops by transforming the loop to tadpole and snipping it away. This symmetry would allow to consider only tree diagrams as shortest representations for computations: this would reduce enormously the calculational complexity. The TGD view about coupling constant evolution allows still to have discrete coupling constant evolution induced by the spectrum of critical values of Kähler coupling strength: an attractive conjecture is that the critical values can be expressed in terms of zeros of Riemann zeta [L9].

2. One can represent the tree representing a sequence of computations in algebra as an analog of twistor diagram and the proposed symmetry implies associativity since moving the line ends induces motion of brackets. If co-algebra operations are allowed also loops become possible and can be eliminated by this symmetry provided the loop acts as identity transformation. This would suggest strong form of associativity at the level of single sheet and weaker form at the level of many-sheeted space-time. One could however still hope that loops can be cancelled so that one would still have only tree diagrams in the simplest description. One would have however sum over amplitudes with different association structures.

3. Co-product could be associated with the basic vertices of TGD, which correspond to a fusion of light-like parton orbits along their ends having no counterpart in super-string models (tensor product vertex) or the decay of light-like parton orbit analogous to a splitting of closed string (direct sum vertex). For the direct sum vertex one has direct sum (unlike string models): one can say that the particle propagates along two path in the sense of superposition as photons in double slit experiment. For the tensor product vertex $D(g) = \Delta(g) = g \times g$ is the first guess. $D(g) = (1, S)\Delta(g) = g \otimes Sg$ or $D(g) = Sg \otimes g$ or their sum suitably normalized is natural second guess. Unitarity allows only the latter option since $\nabla\Delta$ does not conserve probability for probability amplitudes unlike $\nabla(1, S)\Delta$ although it does so for probability distributions. For the direct sum vertex $\Delta(g) = 1 \otimes g \oplus g \otimes 1$ suitably normalized is the natural first guess.

4. Co-product Δ might allow interpretation as annihilation vertex in particle physics context. Co-product might also allow interpretation in terms of replication - at least at the level of topological dynamics of braiding. The possible application of co-product to the replication occurring biology assumed to be induce by replication of magnetic flux tubes in TGD based vision is highly suggestive idea. Is the identification of co-product as replication consistent with its identification as particle annihilation?

   Second question relates to the antipode S, which is anti-homomorphism and brings in mind time reversal. Could one interpret also S as an operation, which should be included to the braid group algebra in the same way as the inclusion of complex conjugation to the algebra of complex numbers produces quaternions? Could one interpret the identity $\nabla(1 \otimes S)\Delta(g) = \eta\epsilon(g) = 1$ by saying that the annihilation to $g \otimes S(g)$ followed by fusion produces braid wave function concentrated on trivial braiding and destroying the information associated with braiding completely. The fusion would produce non-braided particle rather than destroying particles altogether.

5. The condition that loop involving product and annihilation does not affect braid group wave function would require that it takes g to g. For the standard realization of co-product Δ of group algebra $g \to g \otimes g \to g^2$ so that this is not the case. The condition defining Δ is not easy to modify since one loses homomorphism property of Δ. The repetitions of loops would give sequence of powers g^{2n}. For wave function $\sum D(g)g$ this would give the sequence $\sum D(g)g \to \sum D(g)g^2 \to .... \to \sum D(g)g^{2n}$: since given group element has typically several roots one expects that eventually the wave function becomes concentrated to unity with coefficient $\sum D(g)$! For wave functions one has $\sum D(g) = 0$ if they are orthogonal to $D(g) = constant$ as is natural to require. Almost all wave functions would approach to zero so that unitary would be lost. For probability distributions the evolution would make sense since the normalization condition would be respected.

   Also the irreversible behaviour looks strange from particle physics perspective unless $D(g)$ is concentrated on identity so that braiding is trivial. Topological dissipation might take care that this is the case. For elementary particles partonic 2-surfaces carry in the first approximation only single fermion so that braid group would be trivial. Braiding effects become interesting only for strand number larger than 2. The situations in which partonic surface carries large number of fermion lines would be more interesting. Anyonic systems to which TGD based model assigns large h_{eff} and parton surfaces of nanoscopic size could represent a condensed matter example of this situation.

6. Does the behavior of Δ force to regard generalized Feynman diagrams representing computations with different numbers of self-energy loops non-equivalent and to sum over self-energy loops in the construction of scattering amplitudes? The time evolution implied by topological self energy loops is not unitary which suggest that one must perform the sum. There are hopes that the sum converges since the contributions approaches to $\sum D(g) = 0$. This does not however look elegant and is in conflict with the general vision.

   Particle physics intuition tells that in pair annihilation second line has opposite time direction. Should one therefore identify annihilation $g \rightarrow g \otimes S(g)$. Antiparticles would differ from particles by conjugation in braid group. The self energy loop would give trivial braiding with coefficient $\sum D(g)D(g^{-1}) = \sum D(g)D(g)^* = 1$ so that unitarity would be respected and higher self energy loops would be trivial. The conservation of fermion number at fundamental level could also prevent the decays $g \rightarrow g \otimes g$.

One could also take biological replication as a guide line.

1. In biological scales replication by $g \rightarrow g \otimes g$ vertex might not be prevented by fermion number conservation but probability conservation favors $g \rightarrow g \otimes Sg$. Braid replication might be perhaps said to provide replicas of information: whether this conforms with no-cloning theorem remains to be seen. Braid replication followed by fusion means topological dissipation by a loss of braiding and loss of information. Could the fusion of reproduction cells corresponds to product and that replication to co-product possibly involving the action of S one the second line. Fusion followed by replication would lead to a loss of braiding: for $g \rightarrow g \otimes g$ perhaps making sense in probabilistic description gradually and for $g \rightarrow g \otimes Sg$ instantaneously: a reset for memory? Could these mechanisms serve as basic mechanisms of evolution?

2. There might be also a connection with the p-adic length scale hypothesis. The naïve expectation is that $g \rightarrow g^2$ in fusion followed by Δ means the increase of the length of braid by factor 2 - kind of ageing? Could the appearance of powers of two for the length of braid relate to the p-adic length scale hypothesis stating that primes p near powers of 2 are of special importance?

To summarize, the proposed framework gives hopes about description of braids of braids of .... Abstraction would mean transition from classical to quantum: from localized state to a de-localized one: from configuration space to the space of complex valued wave functions in configuration space. Now the configuration space would involve different braidings and corresponding evolutions, and various values of p, h_{eff} and q. If this general framework is to be useful it should be able to tell how the braiding matrices depend on p and h_{eff}: note that p and h_{eff} would be fixed only at the highest abstraction level - the largest flux tubes. This indeterminacy could be interpreted in terms of finite measurement resolution and inclusions of HFFs should help to describe the situation. Indeterminacy could also be interpreted in terms of abstraction in a way similar to the interpretation of negentropically entangled state as a rule for which the state pairs in the superposition represent instances of the rule.

Chapter i

Appendix

A-1 Introduction

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of embedding space and related spaces are discussed and the relationship of CP_2 to the standard model is summarized. The basic vision is simple: the geometry of the embedding space $H = M^4 \times CP_2$ geometrizes standard model symmetries and quantum numbers. The assumption that space-time surfaces are basic objects, brings in dynamics as dynamics of 3-D surfaces based on the induced geometry. Second quantization of free spinor fields of H induces quantization at the level of H, which means a dramatic simplification.

The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Also the relationship to string models is discussed briefly.

Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure analogous to adele [L23, L24]. In the recent view of quantum TGD [L72], both notions reduce to physics as number theory vision, which relies on $M^8 - H$ duality [L45, L46] and is complementary to the physics as geometry vision.

Zero energy ontology (ZEO) [L38] [K89] has become a central part of quantum TGD and leads to a TGD inspired theory of consciousness as a generalization of quantum measurement theory having quantum biology as an application. Also these aspects of TGD are briefly discussed.

A-2 Embedding space $M^4 \times CP_2$

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that embedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Embedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) [L38, L57] [K89] causal diamond

(CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure [A79] so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2.1 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrate that it codes for the symmetries of standard models via its isometries and holonomies.

CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A61] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.2}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi + \Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi - \Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.3}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://tgdtheory.fi/appfigures/cp2.jpg`

Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i \bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.4}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-2.5}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.6}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r, \phi) = (tan(\theta/2), \phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-2.7}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-2.8}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-2.9}$$

are given by

$$\begin{aligned} e^0 &= \tfrac{dr}{F} \ , & e^1 &= \tfrac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \tfrac{r\sigma_2}{\sqrt{F}} \ , & e^3 &= \tfrac{r\sigma_3}{F} \ . \end{aligned} \tag{A-2.10}$$

The explicit representations of vierbein vectors are given by

$$
\begin{array}{llllll}
e^0 & = & \frac{dr}{F} \ , & e^1 & = & \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\
e^2 & = & \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 & = & \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ .
\end{array}
\tag{A-2.11}
$$

The explicit representation of the line element is given by the expression

$$
ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{A-2.12}
$$

From this expression one finds that at coordinate infinity $r = \infty$ line element reduces to $\frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2)$ of S^2 meaning that 3-sphere degenerates metrically to 2-sphere and one can say that CP_2 is obtained by adding to R^4 a 2-sphere at infinity.

The vierbein connection satisfying the defining relation

$$
de^A = -V^A_B \wedge e^B \ , \tag{A-2.13}
$$

is given by

$$
\begin{array}{llllll}
V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\
V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\
V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ .
\end{array}
\tag{A-2.14}
$$

The representation of the covariantly constant curvature tensor is given by

$$
\begin{array}{llllll}
R_{01} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , & R_{23} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\
R_{02} & = & e^0 \wedge e^2 - e^3 \wedge e^1 \ , & R_{31} & = & -e^0 \wedge e^2 + e^3 \wedge e^1 \ , \\
R_{03} & = & 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , & R_{12} & = & 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ .
\end{array}
\tag{A-2.15}
$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$
J = -is_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.16}
$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$
J^k_{\ r} J^{rl} = -s^{kl} \ . \tag{A-2.17}
$$

The condition states that J and g give representations of real unit and imaginary units related by the formula $i^2 = -1$.

Kähler form is expressible locally in terms of Kähler gauge potential

$$
J = dB \ , \tag{A-2.18}
$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

$dJ = ddB = 0$ gives the topological half of Maxwell equations (vanishing of magnetic charges and Faraday's induction law) and self-duality $*J = J$ reduces the remaining equations to $dJ = 0$. Hence the Kähler form can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling).

The magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{aligned} B &= 2re^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2}dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F}sin\Theta d\Theta \wedge d\Phi \ . \end{aligned} \tag{A-2.19}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1, 1).

Useful coordinates for CP_2 are the so called canonical (or symplectic or Darboux) coordinates in which the Kähler potential and Kähler form have very simple expressions

$$\begin{aligned} B &= \sum_{k=1,2} P_k dQ_k \ , \\ J &= \sum_{k=1,2} dP_k \wedge dQ_k \ . \end{aligned} \tag{A-2.20}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned} P_1 &= -\frac{1}{1+r^2} \ , \\ P_2 &= -\frac{r^2 cos\Theta}{2(1+r^2)} \ , \\ Q_1 &= \Psi \ , \\ Q_2 &= \Phi \ . \end{aligned} \tag{A-2.21}$$

Spinors In CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A45]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0,1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1-factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential

$exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the embedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A125] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \ \text{ for } \ X,Y,Z \in t \ . \tag{A-2.22}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2.2 CP_2 geometry and Standard Model symmetries

Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B31] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-2.23}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \otimes \gamma_5$, $1 \otimes \gamma_5$ and $\gamma_5 \otimes 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4)$ having as its covering group $SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \tag{A-2.24}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{lllllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r-\frac{1}{r})e^3 \ , & V_{12} & = & (2r+\frac{1}{r})e^3 \ , \end{array} \tag{A-2.25}$$

and

$$B = 2re^3 \ , \tag{A-2.26}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \tag{A-2.27}$$

where one have defined

$$\begin{aligned} I_L^1 &= \frac{(\Sigma_{01}-\Sigma_{23})}{2} \ , \\ I_L^2 &= \frac{(\Sigma_{02}-\Sigma_{13})}{2} \ . \end{aligned} \tag{A-2.28}$$

A_{ch} is clearly left handed so that one can perform the identification of the gauge potential as

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} \ , \tag{A-2.29}$$

where $W^\pm$ denotes the charged intermediate vector boson.

The covariantly constant curvature tensor is given by

$$\begin{array}{l} R_{01} = -R_{23} = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} = -R_{31} = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \\ R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , \\ R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \tag{A-2.30}$$

The charged part of the curvature tensor is left handed.

This is to be compared with the Weyl tensor, which defines a representation of quaternionic imaginary units.

$$\begin{array}{l} W_{03} = W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ W_{01} = W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ W_{02} = W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \end{array} \quad \text{(A-2.31)}$$

The charged part of the Weyl tensor is right-handed and that the relative sign of the two terms in the curvature tensor and Weyl tensor are opposite.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$\begin{aligned} X &= re^3 \ , \\ Y &= \frac{e^3}{r} \ , \end{aligned} \quad \text{(A-2.32)}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\begin{aligned} \bar{\gamma} &= aX + bY \ , \\ \bar{Z}^0 &= cX + dY \ , \end{aligned} \quad \text{(A-2.33)}$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned} A_{nc} &= [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\ &+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \end{aligned} \quad \text{(A-2.34)}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5\Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \ . \quad \text{(A-2.35)}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \quad \text{(A-2.36)}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$\begin{aligned} Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ , \\ I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \end{aligned} \quad \text{(A-2.37)}$$

The fields γ and Z^0 are defined via the relations

$$\begin{aligned} \gamma &= 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ , \\ Z^0 &= 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \end{aligned} \quad \text{(A-2.38)}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} , \tag{A-2.39}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of the Weinberg angle is a dynamical problem. The original approach was based on the assumption that it makes sense to talk about electroweak action defined at fundamental level and introduce a symmetry breaking by adding an additional term proportional to Kähler action. The recent view is that Kähler action plus volume term defines the fundamental action.

The Weinberg angle is completely fixed if one requires that the electroweak action contains no cross term of type γZ^0. This leads to a definite value for the Weinberg angle.

One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle. As a matter fact, color gauge action identifying color gauge field as proportional to $H^A J_{\alpha\beta}$ is proportional to Kähler action. A possible interpretation would be as a sum of electroweak and color gauge interactions.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) , \tag{A-2.40}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) , \end{aligned} \tag{A-2.41}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) . \tag{A-2.42}$$

Evaluating the expressions above, one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} , \\ Z^0 &= 2R_{03} . \end{aligned} \tag{A-2.43}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) . \tag{A-2.44}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) , \end{aligned} \tag{A-2.45}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$
\begin{aligned}
X &= -\frac{K}{2g^2} + \frac{fp}{18} \ , \\
K &= Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ ,
\end{aligned}
\tag{A-2.46}
$$

This parameter can be calculated by substituting the values of quark and lepton charges and weak isospins.

In the general case the value of the coefficient K is given by

$$
K = \sum_i \left[-\frac{(18 + 2n_i^2)sin^2\theta_W}{9} \right] \ , \tag{A-2.47}
$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18 + n_i^2))} \ . \tag{A-2.48}
$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$
sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \tag{A-2.49}
$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is not far from the typical value 9/24 of GUTs at high energies [B6]. The experimental value at the scale length scale of the electron can be deduced from the ratio of W and Z boson masses as $sin^2\theta_W = 1 - (m_W/m_Z)^2 \simeq .22290$. This ratio and also the weak boson masses depend on the length scale.

If one interprets the additional term proportional to J as color action, one could perhaps interpret the value of Weinberg angle as expressing a connection between strong and weak coupling constant evolution. The limit $f \to 0$ should correspond to an infinite value of color coupling strength and at this limit one would have $sin^2\theta_W = \frac{9}{28}$ for $f/g^2 \to 0$. This does not make sense since the Weinberg angle is in the standard model much smaller in QCD scale Λ corresponding roughly to pion mass scale. The Weinberg angle is in principle predicted by the p-adic coupling constant evolution fixed by the number theoretical vision of TGD.

One could however have a sum of electroweak action, correction terms changing the value of Weinberg angle, and color action and coupling constant evolution could be understood in terms of the coupling parameters involved.

Electroweak symmetry breaking

One of the hardest challenges in the development of the TGD based view of weak symmetry breaking was the fact that classical field equations allow space-time surfaces with finite but arbitrarily large size. For a fixed space-time surface, the induced gauge fields, including classical weak fields, are long ranged. On the other hand, the large mass for weak bosons would require a short correlation length. How can one understand this together with the fact that a photon has a long correlation length?

In zero energy ontology quantum states are superpositions of space-time surfaces as analogs of almost unique Bohr orbits of particles identified as 3-D surfaces. For some reason the superposition should be such that the quantum averages of weak gauge boson fields vanish below the weak scale whereas the quantum average of electromagnetic fields is non-vanishing.

This is indeed the case.

1. The supersymplectic symmetries form isometries of the world of classical worlds (WCW) and they act in CP_2 degrees of freedom as symplectic transformations leaving the CP_2 symplectic form J invariant and therefore also its contribution to the electromagnetic field since this part is the same for all space-time surfaces in the superposition of space-time surfaces as a representation of supersymplectic isometry group (as a special case a representation of color group).

2. In TGD, color and electroweak symmetries acting as holonomies are not independent and for the $SU(2)_L$ part of induced spinor connection the symplectic transformations induces $SU(2)_L \times U(1)_R$ gauge transformation. This suggests that the quantum expectations of the induced weak fields over the space-time surfaces vanish above the quantum coherence scale. The averages of W and of the left handed part of Z^0 should therefore vanish.

3. $\langle Z^0 \rangle$ should vanish. For $U(1)_R$ part of Z^0, the action of gauge transformation is trivial in gauge theory. Now however the space-time surface changes under symplectic transformations and this could make the average of the right-handed part of Z^0 vanishing. The vanishing of the average of the axial part of the Z^0 is suggested by the partially conserved axial current hypothesis.

One can formulate this picture quantitatively.

1. The electromagnetic field [L77] contains, besides the induced Kähler form, also the induced curvature form R_{12}, which couples vectorially. Conserved vector current hypothesis suggests that the average of R_{12} is non-vanishing. One can express the neutral part of the induced gauge field in terms of induced spinor curvature and Kähler form J as

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) = J + 2e^0 \wedge e^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) = 3J - 2e^0 \wedge e^3 \quad , \end{aligned} \tag{A-2.50}$$

2. The induced fields γ and Z^0 (photon and Z- boson) can be expressed as

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} = 2(J + 2e^0 \wedge e^3) \qquad \text{(A-2.51)} \\ per. & \qquad \text{(A-2.52)} \end{aligned}$$

The condition $\langle Z^0 \rangle = 0$ gives $2\langle e^0 \wedge e^3 \rangle = -2J$ and this in turn gives $\langle R_{12} \rangle = 4J$. The average over γ would be

$$\langle \gamma \rangle = (3 - 4sin^2\theta_W)J \ .$$

For $sin^2\theta_W = 3/4\ langle\gamma\rangle$ would vanish.

The quantum averages of classical weak fields quite generally vanish. What about correlation functions?

1. One expects that the correlators of classical weak fields as color invariants, and perhaps even symplectic invariants, are non-vanishing below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as hadron.

2. The intuitive picture is that in longer length scales one has disjoint 3-surfaces with a size scale of Compton length. If the states associated with two disjoint 3-surfaces are separately color invariant there are no correlations in color degrees of freedom and correlators reduce to the products of expectations of classical weak fields and vanish. This could also hold when the 3-surfaces are connected by flux tube bonds.

   Below the Compton length weak bosons would thus behave as correlated massless fields. The Compton lengths of weak bosons are proportional to the value of effective Planck constant h_{eff} and in living systems the Compton lengths are proposed to be even of the order of cell size. This would explain the mysterious chiral selection in living systems requiring large parity violation.

3. What about the averages and correlators of color gauge fields? Classical color gauge fields are proportional to the products of Hamiltonians of color isometries induced Kähler form and the expectations of color Hamiltonians give vanishing average above Compton length and therefore vanishing average. Correlators are non-vanishing below the hadron scale. Gluons do not propagate in long scales for the same reason as weak bosons. This is implied by color confinement, which has also classical description in the sense that 3-surfaces have necessarily a finite size.

   A large value of h_{eff} allows colored states even in biological scales below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as dark hadron.

Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

1. Symmetries must be realized as purely geometric transformations.

2. Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B9] .

   The action of the reflection P on spinors of is given by

$$\Psi \quad \rightarrow \quad P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.53}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.54}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.55}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

A-3.1 Induction procedure for gauge fields and spinor connection

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has embedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as a base space the imbedded manifold, whose points have as fiber the fiber if embedding space at their image points. In the recent case the embedding of space-time surface to embedding space defines the induction procedure. The induced gauge potentials and gauge fields are projections of the spinor connection of the embedding space to the space-time surface (see `http://tgdtheory.fi/appfigures/induct.jpg`).

Induction procedure makes sense also for the spinor fields of embedding space and one obtains geometrization of both electroweak gauge potentials and of spinors. The new element is induction of gamma matrices which gives their projections at space-time surface.

As a matter fact, the induced gamma matrices cannot appear in the counterpart of massless Dirac equation. To achieve super-symmetry, Dirac action must be replaced with Kähler-Dirac action for which gamma matrices are contractions of the canonical momentum currents of Kähler action with embedding space gamma matrices. Induced gamma matrices in Dirac action would correspond to 4-volume as action.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://tgdtheory.fi/appfigures/induct.jpg`.

A-3.2 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP$_2$ projection, only vacuum extremals and space-time surfaces for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

A-3.3 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four embedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. `http://tgdtheory.fi/appfigures/manysheeted.jpg`

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. `http://tgdtheory.fi/appfigures/wormholecontact.jpg`

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. `http://tgdtheory.fi/appfigures/fieldsuperpose.jpg`

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of embedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)") as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-3.4 Embedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

1. Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

2. Spinor harmonics of embedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of embedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

1. Although the embedding space spinor connection carries W gauge potentials one can say that the embedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

2. The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the

spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

3. Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

4. One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

5. This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-3.5 About induced gauge fields

In the following the induced gauge fields are studied for general space-time surface without assuming the preferred extremal property (Bohr orbit property). Therefore the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta) d\Phi) + \frac{r^2}{2F} sin(\Theta) d\Theta \wedge d\Phi \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-3.1}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta) d\Phi) + (3+p)\frac{r^2}{2F} sin(\Theta) d\Theta \wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.2}$$

where Θ_W denotes Weinberg angle.

1. The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p) sin(\Theta) = 0 \ , \end{aligned} \tag{A-3.3}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$
\begin{aligned}
r &= \sqrt{\frac{X}{1-X}} \ , \\
X &= D\left[|\frac{k+u}{C}|\right]^{\epsilon} \ , \\
u &\equiv cos(\Theta) \ , \ C = k + cos(\Theta_0) \ , \ D = \frac{r_0^2}{1+r_0^2} \ , \ \epsilon = \frac{3+p}{3+2p} \ ,
\end{aligned}
\tag{A-3.4}
$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u + k| = [(1 + r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$
sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k+1 \ ,
$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$
\begin{aligned}
J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\
Z^0 &= -\frac{6}{p} J \ .
\end{aligned}
\tag{A-3.5}
$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

2. The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

3. The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$
\begin{aligned}
Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\
r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\
\gamma &= -\frac{p}{2} Z^0 \ .
\end{aligned}
\tag{A-3.6}
$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$
\begin{aligned}
ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\
s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\
s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ ,
\end{aligned} \tag{A-3.7}
$$

and is useful in the construction of vacuum embedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$
\begin{aligned}
\Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\
\Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ .
\end{aligned} \tag{A-3.8}
$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global embedding for, say a constant magnetic field. Although global embedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$
\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \tag{A-3.9}
$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-4 The relationship of TGD to QFT and string models

The recent view of the relationship of TGD to QFT and string models has developed slowly during years and it seems that in a certain sense TGD means a return to roots: instead of QFT like description involving path integral one would have wave mechanics for 3-surfaces.

A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces

The first vision of TGD was as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

The later work has revealed that TGD could be seen as a generalization of the wave mechanism based on the replacement of a point-like particle with 3-D surface. This is due to holography implied by general coordinate invariance. The definition of the metric of the "world of classical worlds" (WCW) must assign a unique or at least almost unique space-time surface to a given 3-surface. This 4-surface is analogous to Bohr orbit so that also Bohr orbitology becomes an exact part of quantum physics. The failure of strict determinism forces to replace 3-surfaces with 4-surfaces and this leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K35, K20, K63] [L60, L72].

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://tgdtheory.fi/appfigures/particletgd.jpg`

A-4.2 Extension of superconformal invariance

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in a completely unique position as far as symmetries are considered.

In fact, this leads to a generalization of the Kac-Moody type symmetries of string models. $\delta M^4_+ \times CP_2$ allows huge supersymplectic symmetries for which the radial light-like coordinate of δM^4_+ plays the role of complex string coordinate in string models. These symmetries are assumed to act as isometries of WCW.

A-4.3 String-like objects and strings

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of the TGD Universe and the inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string-like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situations in Minkowskian regions of space-time surface. `http://tgdtheory.fi/appfigures/fermistring.jpg`

A-4.4 TGD view of elementary particles

The TGD based view about elementary particles has two key aspects.

1. The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidean signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

2. At the level of H Fermion number is carried by the modes of the induced spinor field. In space-time regions with Minkowski signature the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle orbit corresponds to a 4-D generalization of a world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidean signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacts carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompanied by fermionic strings connecting the throats at the same sheet: the strings do not extend inside the wormhole contacts. `http://tgdtheory.fi/appfigures/elparticletgd.jpg`

Particle interactions involve both stringy and QFT aspects.

1. The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having a hadronic string as a physical counterpart. Long strings should be distinguished from wormhole contacts which due to their super-conformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

2. Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats behave like virtual bosons so that one has BFF type vertices typically.

3. In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particles along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. `http://tgdtheory.fi/appfigures/tgdgraphs.jpg`

A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K35, K63].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

A-5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [L26] [L60, L62, L63] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and $T(CP_2)$

of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A79] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

1. Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

2. Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

3. The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

   For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

1. The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

2. Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

   For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

A-5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

1. If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

2. WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L72]. There are also other important algebras but these will not be discussed now.

1. The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

   The super symplectic algebra A has an infinite hierarchy of sub-algebras [L72] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

   This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

2. The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

   The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on $M^8 - H$ duality [L72] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

1. The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by $M^8 - H$ duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

2. This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L72] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P) = n(braid) = n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

3. The actions related to different values of $n(P) = n(braid) = n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

   WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

4. Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

   A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L66] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i = 2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r = 1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

   A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

5. A subset of p-adic primes allowed by the p-adic length scale hypothesis $p \simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K46, K47]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

1. Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

   P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L66], and the forward scattering amplitude corresponds to the propagator.

   This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

2. According to [L66], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

   A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L66].

3. p-Adic length scale hypothesis [L73] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

   For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

   Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

1. The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

   In [L72] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

2. The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K35, K20]. As isometries they would naturally permute the maxima with each other.

A-6 Number theoretic vision of TGD

Physics as number theory vision is complementary to the physics as geometry vision and has developed gradually since 1993. Langlands program is the counterpart of this vision in mathematics [L70].

The notion of p-adic number fields emerged with the motivation coming from the observation that elementary particle mass scales and mass ratios could be understood in terms of the so-called p-adic length scale hypothesis [K49, K41, K17]. The fusion of the various p-adic physics leads to what I call adelic physics [L23, L24]. Later the hypothesis about hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter emerged [K22, K23, K24, K24].

Eventually this led to that the values of effective Planck constant could be identified as the dimension of an algebraic extension of rationals assignable to polynomials with rational coefficients. This led to the number theoretic vision in which so-called M^8-H duality [L45, L46] plays a key role. M^8 (actually a complexification of real M^8) is analogous to momentum space so that the duality generalizes momentum position duality for point-like particles. M^8 has an interpretation as complexified octonions.

The dynamics of 4-surfaces in M^8 is coded by polynomials with rational coefficients, whose roots define mass shells H^3 of $M^4 \subset M^8$. It has turned out that the polynomials satisfy stringent additional conditions and one can speak of number theoretic holography [L66, L70]. Also the ordinary 3$\rightarrow$ 4 holography is needed to assign 4-surfaces with these 3-D mass shells. The number theoretic dynamics is based on the condition that the normal space of the 4-surface in M^8 is associative (quaternionic) and contains a commutative complex sub-space. This makes it possible to assign to this surface space-time surface in $H = M^4 \times CP_2$.

At the level of H the space-time surfaces are by holography preferred extremals and are assumed to be determined by the twistor lift of TGD [L26] giving rise to an action which is sum of the Kähler action and volume term. The preferred extremals would be minimal surfaces analogous to soap films spanned by frames. Outside frames they would be simultaneous extremals of the Kähler action, which requires a generalization of the holomorphy characterizing string world sheets.

In the following only p-adic numbers and hierarchy of Planck constants will be discussed.

A-6.1 p-Adic numbers and TGD

p-Adic number fields

p-Adic numbers (p is prime: 2, 3, 5, ...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A40]. p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k,\ x(k) = 0,....,p-1 \ . \qquad \text{(A-6.1)}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} . \quad \text{(A-6.2)}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) , \quad \text{(A-6.3)}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x,y) = |x-y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} . \quad \text{(A-6.4)}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D . \quad \text{(A-6.5)}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

2. Distances of points x and y inside single class are smaller than distances between different classes.

3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B25]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

1. Basic form of the canonical identification

There exists a natural continuous map $I : R_p \to R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \to x = \sum_{k<N} y_k p^{-k} , \\ y_k &\in \{0, 1, .., p-1\} . \end{aligned} \quad \text{(A-6.6)}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$
\begin{aligned}
x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\
x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ .
\end{aligned}
\tag{A-6.7}
$$

The p-adic images associated with these expansions are different

$$
\begin{aligned}
y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\
y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\
&= y_1 + (x_N - 1)p^N - p^{N+1} \ ,
\end{aligned}
\tag{A-6.8}
$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

2. The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p |y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{A-6.9}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p |y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{A-6.10}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.11}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3. Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.12}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4. Generalization of number concept and notion of embedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic embedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real embedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \ mod \ p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real embedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles".

Fig. 16. The basic idea between p-adic manifold. `http://tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

1. Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

2. Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

3. Canonical identification violates general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic embedding space with chart maps to real embedding space and assuming preferred coordinates made possible by isometries of embedding space: one however obtains several in-equivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-6.2 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the embedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of embedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of embedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-6.3 $M^8 - H$ duality as it is towards the end of 2021

The view of M^8-H duality (see Appendix **??**) has changed considerably towards the end 2021 [L60] after the realization that this duality is the TGD counterpart of momentum position duality of wave mechanics, which is lost in QFTs. Therefore M^8 and also space-time surface is analogous to momentum space. This forced us to give up the original simple identification of the points $M^4 \subset M^4 \times E^4 = M^8$ and of $M^4 \times CP_2$ so that it respects Uncertainty Principle (UP).

The first improved guess for the duality map was the replacement with the inversion $p^k \rightarrow m^k = \hbar_{eff} p^k/p^2$ conforming in spirit with UP but turned out to be too naive.

The improved form [L60] of the $M^8 - H$ duality map takes mass shells $p^2 = m^2$ of $M^4 \subset M^8$ to cds with size $L(m) = \hbar_{eff}/m$ with a common center. The slicing by mass shells is mapped to a Russian doll like slicing by cds. Therefore would be no CDs in M^8 contrary to what I believed first.

Quantum classical correspondence (QCC) inspires the proposal that the point $p^k \in M^8$ is mapped to a geodesic line corresponding to momentum p^k starting from the common center of cds. Its intersection with the opposite boundary of cd with size $L(m)$ defines the image point. This is not yet quite enough to satisfy UP but the additional details [L60] are not needed in the sequel.

The 6-D brane-like special solutions in M^8 are of special interest in the TGD inspired theory of consciousness. They have an M^4 projection which is $E = E_n$ 3-ball. Here E_n is a root of the real polynomial P defining $X^4 \subset M^8_c$ (M^8 is complexified to M^8_c) as a "root" of its octonionic continuation [L45, L46]. E_n has an interpretation as energy, which can be complex. The original interpretation was as moment of time. For this interpretation, $M^8 - H$ duality would be a linear identification and these hyper planes would be mapped to hyperplanes in $M^4 \subset H$.

This motivated the term "very special moment in the life of self" for the image of the $E = E_n$ section of $X^4 \subset M^8$ [L34]. This notion does not make sense at the level M^8 anymore.

The modified $M^8 - H$ duality forces us to modify the original interpretation [L60]. The point $(E_n, p = 0)$ is mapped $(t_n = \hbar_{eff}/E_n, 0)$. The momenta (E_n, p) in $E = E_n$ plane are mapped to the boundary of cd and correspond to a continuous time interval at the boundary of CD: "very special moment" becomes a "very special time interval".

The quantum state however corresponds to a set of points corresponding to quark momenta, which belong to a cognitive representation and are therefore algebraic integers in the extension determined by the polynomial. These active points in E_n are mapped to a discrete set at the boundary of cd(m). A "very special moment" is replaced with a sequence of "very special moments".

So called Galois confinement [L53] forces the total momenta for bound states of quarks and antiquarks to be rational integers invariant under Galois group of extension of rationals determined by the polynomial P [L60]. These states correspond to states at boundaries of sub-CDs so that one obtains a hierarchy. Galois confinement provides a universal number theoretic mechanism for the formation of bound states.

A-7 Zero energy ontology (ZEO)

ZEO is implied by the holography forced in the TGD framework by general coordinate invariance.

A-7.1 Basic motivations and ideas of ZEO

The following gives a brief summary of ZEO [L38] [K89].

1. In ZEO quantum states are not 3-dimensional but superpositions of 4-dimensional deterministic time evolutions connecting ordinary initial 3-dimensional states. By holography they are equivalent to pairs of ordinary 3-D states identified as initial and final states of time evolution. One can say that in the TGD framework general coordinate invariance implies holography and the slight failure of its determinism in turn forces ZEO.

   Quantum jumps replace this state with a new one: a superposition of deterministic time evolutions is replaced with a new superposition. Classical determinism of individual time evolution is not violated and this solves the basic paradox of quantum measurement theory. There are two kinds of quantum jumps: ordinary ("big") state function reductions (BSFRs) changing the arrow of time and "small" state function reductions (SSFRs) (weak measurements) preserving it and giving rise to the analog of Zeno effect [L38].

2. To avoid getting totally confused it is good to emphasize some aspects of ZEO.

   (a) ZEO does not mean that physical states in the usual 3-D sense as snapshots of time evolution would have zero energy state pairs defining zero energy states as initial and final states have same conserved quantities such as energy. Conservation implies that one can adopt the conventions that the values of conserved quantities are opposite for these states so that their sum vanishes: one can think that incoming and outgoing particles come from geometric past and future is the picture used in quantum field theories.

   (b) ZEO means two times: subjective time as sequence of quantum jumps and geometric time as space-time coordinate. These times are identifiable but are strongly correlated.

3. In BSFRs the arrow of time is changed and the time evolution in the final state occurs backwards with respect to the time of the external observer. BSFRs can occur in all scales since TGD predicts a hierarchy of effective Planck constants with arbitrarily large values. There is empirical support for BSFRs.

   (a) The findings of Minev et al [L32] in atomic scale can be explained by the same mechanism [L32]. In BSFR a final zero energy state as a superposition of classical deterministic time evolutions emerges and for an observer with a standard arrow of time looks like a superposition of deterministic smooth time evolutions leading to the final state. Interestingly, once this evolution has started, it cannot be stopped unless one changes the stimulus signal inducing the evolution in which case the process does not lead to anywhere: the interpretation would be that BSFR back to the initial state occurs!

(b) Libets' experiments about active aspects of consciousness [J3] can be understood. Subject person raises his finger and neural activity starts before the conscious decision to do so. In the physicalistic framework it is thought to lead to raising of the finger. The problem with the explanation is that the activity beginning .5 seconds earlier seems to be dissipation with a reversed arrow of time: from chaotic and disordered to ordered at around .15 seconds. ZEO explanation is that macroscopic quantum jump occurred and generated a signal proceeding backwards in time and generated neural activity and dissipated to randomness.

(c) Earthquakes involve a strange anomaly: they are preceded by ELF radiation. One would expect that they generate ELF radiation. The identification as BSFR would explain the anomaly [L33]. In biology the reversal of the arrow of time would occur routinely and be a central element of biological self-organization, in particular self-organized quantum criticality (see [L36, L81].

A-7.2 Some implications of ZEO

ZEO has profound implications for understanding self-organization and self-organized quantum criticality in terms of dissipation with non-standard arrow of time looking like generation of structures [L36, L81]. ZEO could also allow understanding of what planned actions - like realizing the experiment under consideration - could be.

1. Second law in the standard sense does not favor - perhaps even not allow - realization of planned actions. ZEO forces a generalization of thermodynamics: dissipation with a non-standard arrow of time for a subsystem would look like self-organization and planned action and its realization.

   Could most if not all planned action be like this - induced by BSFR in the geometric future and only apparently planned? There would be however the experience of planning and realizing induced by the signals from geometric future by a higher level in the hierarchy of conscious entities predicted by TGD! In long time scales we would be realizing our fates or wishes of higher level conscious entities rather than agents with completely free will.

2. The notion of magnetic body (MB) serving as a boss of ordinary matter would be central. MB carries dark matter as $h_{eff} = nh_0$ phases of ordinary matter with n serving as a measure for algebraic complexity of extension of rationals as its dimension and defining a kind of universal IQ. There is a hierarchy of these phases and MBs labelled by extension of rationals and the value of n.

   MBs would form a hierarchy of bosses - a realization for master slave hierarchy. Ordinary matter would be at the bottom and its coherent behavior would be induced from quantum coherence at higher levels. BSFR for higher level MB would give rise to what looks like planned actions and experienced as planned action at the lower levels of hierarchy. One could speak of planned actions inducing a cascade of planned actions in shorter time scales and eventually proceeding to atomic level.

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred embedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes. There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. `http://tgdtheory.fi/appfigures/padictoreal.jpg`

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. `http://tgdtheory.fi/appfigures/sharing.jpg`

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see **Fig.** `http://tgdtheory.fi/appfigures/timemirror.jpg` or **Fig.** 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. `http://tgdtheory.fi/appfigures/timemirror.jpg`

Acknowledgements: I am grateful for Dainis Zeps for enlightnening discussions concerning CP_2 geometry.

REFERENCES

Mathematics

[A1] A quantum octonion algebra. Available at: https://arxiv.org/abs/math/9801141.

[A2] Atyiah-Singer index-theorem. Available at: https://en.wikipedia.org/wiki/Atiyah-Singer_index_theorem.

[A3] Category theory. Available at: https://en.wikipedia.org/wiki/Category_theory.

[A4] Farey sequence. Available at: https://en.wikipedia.org/wiki/Farey_sequence.

[A5] Frames and locales. Available at: https://en.wikipedia.org/wiki/Frames_and_locales.

[A6] Groupoid. Available at: https://en.wikipedia.org/wiki/Groupoid.

[A7] Hyperfinite type II factor. Available at: https://en.wikipedia.org/wiki/Hyperfinite_type_II-1_factor.

[A8] KMS state. Available at: https://en.wikipedia.org/wiki/KMS_state.

[A9] Langlands program. Available at: https://en.wikipedia.org/wiki/Langlands_program.

[A10] Little disk operad. Available at: https://en.wikipedia.org/wiki/Little_disks_operad#Examples.

[A11] Multicategories. Available at: https://en.wikipedia.org/wiki/Multicategory.

[A12] Operad theory. Available at: https://en.wikipedia.org/wiki/Operad.

[A13] Particular point topology. Available at: https://en.wikipedia.org/wiki/Particular_point_topology.

[A14] Planar algebra. Available at: https://en.wikipedia.org/wiki/Planar_algebra.

[A15] Planar algebra. Available at: https://en.wikipedia.org/wiki/Planar_algebra.

[A16] Planar Algebras, TFTs with Defects. Available at: https://golem.ph.utexas.edu/category/2008/09/planar_algebras_tfts_with_defe.html.

[A17] Pointless topology. Available at: https://en.wikipedia.org/wiki/Pointless_topology.

[A18] Quadratic resiprocity theorem. Available at: https://en.wikipedia.org/wiki/Quadratic_reciprocity.

[A19] Quasicrystals. Available at: https://en.wikipedia.org/wiki/Quasicrystal.

[A20] Set theory. Available at: https://en.wikipedia.org/wiki/Set_theory.

[A21] Sierpinski space. Available at: https://en.wikipedia.org/wiki/Sierpinski_space.

[A22] Symplectification of science. Available at: https://www.pims.math.ca/~gotay/Symplectization(E).pdf.

[A23] Video about Edward Witten's talk relating to his work with knot invariants. Available at: https://video.ias.edu/webfm_send/1787.

[A24] Von Neumann algebra. Available at: https://en.wikipedia.org/wiki/Von_Neumann_algebra.

[A25] Weekly Finds (Week 102). Available at: https://math.ucr.edu/home/baez/week202.html.

[A26] Yangian symmetry. Available at: https://en.wikipedia.org/wiki/Yangian.

[A27] *Objects of categories as complex numbers*, 2007. Available at: https://www.arxiv.org/abs/math.CT/0212377.

[A28] Connes A. Une classification des facteurs de type III. *Ann Sci Ecole Norm Sup*, 6, 1973.

[A29] Connes A. *Non-commutative Geometry.* Academic Press, San Diego, 1994.

[A30] Connes A. Quantum Fields and Motives, 2005. Available at: https://arxiv.org/abs/hep-th/0504085.

[A31] Connes A. Noncommutative Geometry Year 2000, 2018. Available at: https://arxiv.org/abs/math/0011193.

[A32] Wassermann A. Operator algebras and conformal field theory. III. Fusion of positive energy representations of LSU(N) using bounded operators. *Invent Math*, 133(3), 1998.

[A33] Beardon AF. Composition factors of polynomials. *Complex Variables, Theory and Application: An International Journal*, 43(3-4):225–239, 2001. Available at: https://doi.org/10.1080/17476930108815314.

[A34] Haken W Appel K. Every Planar Map is Four Colorable. *AMS*, 98, 1989.

[A35] Casselman B. Strange associations, 1998. Montly Essays on Mathematical Topics (AMS). https://www.ams.org/samplings/feature-column/fcarc-associahedra.

[A36] Mitchell B. *Theory of Categories.* Academic Press, 1965.

[A37] Schroer B. Lectures on Algebraic Quantum Field Theory and Operator Algebras, 2001. Available at: https://xxx.lanl.gov/abs/math-ph/0102018.

[A38] Stay M Baez J. Physics, topology, logic and computation: a Rosetta Stone, 2008. Available at: https://math.ucr.edu/home/baez/rosetta.pdf.

[A39] Gosh SK Bisch D, Das P. Planar algebra of group-type subfactors, 2008.Available at: https://arxiv4.library.cornell.edu/abs/0807.4134.

[A40] Shafarevich IR Borevich ZI. *Number Theory.* Academic Press, 1966.

[A41] Robinson DW Bratteli O. *Operator Algebras and Quantum Statistical Mechanics.* Springer Verlag, New York, 1979.

[A42] Kassel C. *Quantum Groups.* Springer Verlag, 1995.

[A43] Ershov M Carbone L. Abstract simplicity of complete Kac-Moody groups over finite fields, 2010. Available at: https://arxiv.org/abs/math/0612772.

[A44] Pressley A Chari V. *A Guide to Quantum Groups.* Cambridge University Press, Cambridge, 1994.

[A45] Pope CN. Eigenfunctions and $Spin^c$ Structures on CP_2, 1980.

[A46] Kreimer D Connes A. *Hofp algebras, renormalization, and non-commutative geometry*, volume 1999. Kluwer, 1998. Available at: https://arxiv.org/abs/hep-th/9912092.

[A47] Rovelli C Connes A. Von Neumann algebra automorphisms and time-thermodynamics relation in general covariant quantum theories, 1994. Available at: https://arxiv.org/PS_cache/gr-qc/pdf/9406/9406019v1.pdf.

[A48] Martellini M Cotta-Rasmussino P. BF theories and 2-knots, 1994. Available at: https://arxiv.org/pdf/hep-th/9407097.

[A49] Bisch D. Subfactors and Planar Algebras, 2003. Available at: https://arxiv.org/abs/math/0304340.

[A50] Ruberman D. Comment in discussion about unitary cobordisms. Available at: https://math.ucr.edu/home/baez/quantum/ruberman.html.

[A51] Evans DE and Pugh M. Spectral measures for G_2, 2014. Available at: https://www.researchgate.net/publication/261437422_Spectral_Measures_for_G_2_II_finite_subgroups.

[A52] Roberts JE Doplicher S, Haag R. Local Observables and Particle Statistics I. *Ann Math*, 23(1974):75–119, 1971.

[A53] Hofstadter DR. *Gödel, Escher, Bach: an Eternal Braid.* Penguin Books, 1980.

[A54] Freed DS. The Geometry of Loop Groups, 1985.

[A55] Frenkel E. Free Field Realizations in Representation Theory and Conformal Field Theory. Available at: https://www.mathunion.org/ICM/ICM1994.2/Main/icm1994.2.1256.1269.ocr.pdf.

[A56] Frenkel E. Recent Advances in Langlands program. *AMS*, 41(2):151–184, 2004.

[A57] Frenkel E. Lectures on Langlands program and conformal field theory, 2005. Available at: https://arxiv.org/abs/hep-th/0512172.

[A58] Whitten E. Resolution of singularities in algebraic varieties, 2008. Available at:https://tinyurl.com/yb7cuwkf.

[A59] Witten E. Quantum field theory and the Jones polynomial. *Comm Math Phys*, 121:351–399, 1989.

[A60] Titchmarch EC. *The Theory of Riemann Zeta Function. 2nd ed. revised by R. D. Heath-Brown.* Oxford Univ. Press, 1986.

[A61] Hanson J Eguchi T, Gilkey B. *Phys Rep*, 66, 1980.

[A62] Eisenhart. *Riemannian Geometry.* Princeton University Press, 1964.

[A63] Kawahigashi Y Evans D. *Quantum symmetries on operator algebras.* Oxford University Press, New York, 1998.Available at: https://www.ams.org/bull/2001-38-03/S0273-0979-01-00906-5/S0273-0979-01-00906-5.pdf.

[A64] Butin F. Branching Law for the Finite Subgroups of $SL(4,C)$. *Ukrainian Mathematical Journal*, 67:1484–1497, 2016. Available at: https://link.springer.com/article/10.1007/s11253-016-1167-8.

[A65] Butin F and Perets GS. Branching law for finite subgroups of $SL(3,C)$ and McKay correspondence. *Journal of Group Theory, De Gruyter*, 17(2):191–251, 2013. Available at: https://hal.archives-ouvertes.fr/hal-00412643/document.

[A66] Jones FR. *Braid groups, Hecke algebras and type II_1 factors.* 1983.

[A67] Kato G. Sheaf Cohomology of Conscious Entity. *Int J Comp Antic Systems*, 2002.

[A68] Witten E Gaiotto D. Knot Invariants from Four-Dimensional Gauge Theory, 2011. Available at: https://arxiv.org/abs/1106.4789v1.

[A69] Shapiro ZYa Gelfand IM, Minklos RA. *Representations of the rotation and Lorentz groups and their applications.* Pergamon Press, 1963.

[A70] Pope CN Gibbons GW. CP_2 as gravitational instanton. *Comm Math Phys*, 55, 1977.

[A71] Jones VFR Goodman FM, Harpe la de P. *Coxeter graphs and towers of algebras.* Springer Verlag, 1989.

[A72] Browne H. Connectivity Properties of McKay Quivers, 2020. Available at: `https://arxiv.org/pdf/2003.09502.pdf`.

[A73] Saleur H. Zeroes of chromatic polynomials: a new approach to the Beraha conjecture using quantum groups. *Comm Math Phys*, 132, 1990.

[A74] Schubert H. *Categories.* Springer Verlag, New York, 1972.

[A75] Sugawara H. A field theory of currents. *Phys Rev*, 176, 1968.

[A76] Wenzl H. Hecke algebra of type A_n and subfactors. *Invent Math*, 92, 1988.

[A77] Kastler D Haag R. An Algebraic Approach to Quantum Field Theory. *J Math Phys*, 5:848–861, 1964.

[A78] Pope CN Hawking SW. Generalized Spin Structures in Quantum Gravity. *Phys Lett*, (1), 1978.

[A79] N. Hitchin. Kählerian twistor spaces. *Proc London Math Soc*, 8(43):133–151, 1981.. Available at: `https://tinyurl.com/pb8zpqo`.

[A80] Borchers HJ. On Revolutionizing QFT with Tomita's Modular Theory. *J Math Phys*, pages 3604–3673, 2000. Available at: `https://www.lqp.uni-goettingen.de/papers/99/04/99042900.html`.

[A81] Halvorson HP. Locality, Localization, and the Particle Concept: Topics in the Foundations of Quantum Field Theory, 2001.Available at: `https://philsci-archive.pitt.edu/archive/00000346/00/main-new.pdf`.

[A82] Nakamura I. McKay Correspondence, 2002. Available at: `https://www.math.sci.hokudai.ac.jp/~nakamura/McKay071002.pdf`.

[A83] Butterfield J Isham CJ. Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity, 1999. Available at: `https://arxiv.org/abs/gr-gc/9910005`.

[A84] Zuber J-B Iztykson C. *Quantum Field Theory*, volume 549. Mc Graw-Hill, New York, 1980.

[A85] Baez J. Higher-dimensional algebra II: 2-Hilbert spaces, 1997. Available at: `https://arxiv.org/abs/q-alg/9609018`.

[A86] Baez J. Categorification, 1998. Available at: `https://arxiv.org/pdf/math/9802029v1`.

[A87] Baez J. Quantum Quandaries, 2007. Available at: `https://math.ucr.edu/home/baez/quantum/node1.html`.

[A88] Dixmier J. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981. First published in French in 1957: Les Algebres d'Operateurs dans l'Espace Hilbertien, Paris: Gauthier-Villars.

[A89] Dixmier J. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981. First published in French in 1957: Les Algebres d'Operateurs dans l'Espace Hilbertien, Paris: Gauthier-Villars.

[A90] McKay J. Cartan matrices, finite groups of quaternions, and kleinian singularities. *Proc AMS*, 1981. Available at: `https://tinyurl.com/ydygjgge`.

[A91] Mickelson J. Gerbes, (Twisted) K-Theory, and the Supersymmetric WZW Model, 2002. Available at: `https://arxiv.org/abs/hep-th/0206139`.

[A92] Milnor J. *Topology form Differential Point of View.* The University Press of Virginia, Virginia, 1965.

[A93] Yngvason J. The role of Type III Factors in Quantum Field Theory, 2004. Available at: https://arxiv.org/abs/math-ph/0411058.

[A94] Baez JC. The Octonions. *Bull Amer Math Soc*, 39(2002), 2001. Available at: https://math.ucr.edu/home/baez/Octonions/octonions.html.

[A95] Grabowski JE and Launois S. Graded quantum cluster algebras and an application to quantum Grassmannians, 2018. Available at: https://arxiv.org/pdf/1301.2133.pdf.

[A96] Ritt JF. Prime and Composite Polynomials. *Transactions of the American Mathematical Society*, 23(1):51–66, 1922. Available at: https://doi.org/10.2307/1988911.

[A97] Vidick T Wright J Ji Z, Natarajan A and Yuen H. MIP*=RE, 2020. Available at: https://arxiv.org/abs/2001.04383.

[A98] Struppa DC Kato G. Category Theory and Consciousness. 2001.

[A99] Lambropoulou S Kauffman LH. Hard Unknots and Collapsing Tangles, 2006. Available at: https://arxiv.org/abs/math/0601525.

[A100] Schwartz L. Generalisation de la Notion de Fonction, de Derivation, de Transformation de Fourier et Applications Mathematiques et Physiques. *Publications de l'Institut de Mathematique de l'Universite de Strasbourg, Vols 9-10, Paris: Hermann*, 1945.

[A101] Abdel-Khalek K Leo De S. Octonionic quantum mechanics and complex geometry. *Prog. Theor. Phys.*, 96:823–832, 1996. Available at: https://arxiv.org/abs/hep-th/9609032.

[A102] Horwitz LP. Hypercomplex quantum mechanics, 1996. Available at: https://arxiv.org/abs/quant-ph/9602001.

[A103] Brenner M. Quantum octonions. *Communications in Algebra*, 27, 1999. Available at: https://math.usask.ca/~bremner/research/publications/qo.pdf.

[A104] Choda M. Conjugate but non inner conjugate subfactors. *Proc AMS*, 124(1), 1996. Available at: https://tinyurl.com/nle4p7b.

[A105] Khovanov M. MacKay correspondence, Available at: https://www.math.columbia.edu/~khovanov/finite/QiYou1.pdf.

[A106] Rainer M. Algebraic Quantum Field Theory on Manifolds: A Haag-Kastler Setting for Quantum Geometry, 2000. Available at: https://arxiv.org/abs/gr-qc/9911076.

[A107] Reid M. The du val singularities an, dn, e6, e7, e8. Available at:https://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf.

[A108] Reid M. McKay correspondence. Available at: https://arxiv.org/abs/alg-geom/9702016.

[A109] Reid M. Chapters on algebraic surfaces, 1996. Available at:https://arxiv.org/abs/alg-geom/9602006.

[A110] Spivak M. *Differential Geometry I,II,III,IV.* Publish or Perish, Boston, 1970.

[A111] Takesaki M. *Tomita's Theory of Modular Hilbert Algebras and Its Applications*, volume 128. Springer, Berlin, 1970.

[A112] Sheppeard MD. Gluon Phenomenology and a Linear Topos, 2007. Available at: https://arcadianfunctor.files.wordpress.com/2007/11/thesismds.pdf.

[A113] Neumann von J Murray FJ. On Rings of Operators. *Ann Math*, pages 37116–229, 1936.

[A114] MacKay NJ. Introduction to Yangian symmetry in integrable field theory, 2009. Available at: https://wenku.baidu.com/view/01e4ebdbad51f01dc281f180.html.

[A115] Dirac PAM. A New Notation for Quantum Mechanics. *Proc Cambridge Phil Soc*, 35:416–418, 1939.

[A116] Goldblatt R. *The Categorial Analysis of Logic*. North-Holland, Amsterdam, 1984.

[A117] Haag R. *Local Quantum Physics*. Springer, Berlin, 1992.

[A118] Longo R. Operators algebras and Index Theorems in Quantum Field Theory, 2004. Andrejevski lectures in Göttingen 2004 (lecture notes).

[A119] Stefanik R. *Structuralism, Category Theory and Philosophy of Mathematics*. MSG Press, Washington, 1994.

[A120] Thom R. *Comm Math Helvet*, 28, 1954.

[A121] Stöltzner M Redei M. *John von Neumann and the Foundations of Quantum Physics. Vol. 8, Dordrecht*. Kluwer, 2001.

[A122] Rivers RJ. *Path Integral Methods in Quantum Field Theory*. Cambridge University Press, Cambridge, 1987.

[A123] Hoffnung A Rogers C. Categorified symplectic geometry and the classical string, 2008. Available at: https://math.ucr.edu/home/baez/2plectic_1.pdf.

[A124] Feynman RP. Space-Time Approach to Non-Relativistic Quantum Mechanics. *Rev Mod Phys*, 20:367–387, 1948.

[A125] Helgason S. *Differential Geometry and Symmetric Spaces*. Academic Press, New York, 1962.

[A126] Okubo S. Angular momentum, quaternion, octonion, and Lie-super algebra $Osp(1, 2)$, 1997. Available at: https://arxiv.org/pdf/physics/9710038.

[A127] Sawin S. Links, Quantum Groups, and TQFT's, 1995.Available at: https://arxiv.org/abs/q-alg/9506002.

[A128] Weber M. (Editors) Stammeier M, Voiculescu D-V. EMS Press, 2016.

[A129] Lieb EH Temperley NH. Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices:some exact results for the percolation problem. *Proc R Soc London*, 322, 1971.

[A130] Meyer U. Quantum determinants, 1994. Available at: https://arxiv.org/pdf/hep-th/9406172.pdf.

[A131] Jones V. In and around the origin of quantum groups, 2003. Available at: https://arxiv.org/abs/math/0309199.

[A132] Jones VF. Hecke algebra representations of braid groups and link polynomial. *Ann Math*, 126, 1987.

[A133] Jones VF. *The planar algebra of a bipartite graph*, pages 94–117. World Science Publishing, 2000.

[A134] Neumann von J. Quantum Mechanics of Infinite Systems, 1937.

[A135] Wallace. *Differential Topology*. W. A. Benjamin, New York, 1968.

[A136] Yen Y-H and Song JS. Of McKay Correspondence, Non-linear Sigma-model and Conformal Field Theory. *Advances in Theoretical and Mathematical Physics*, 4(4), 1999. Available at: https://cutt.ly/jLQPgkQ.

[A137] Ge ML Yang CN. *Braid Group, Knot Theory, and Statistical Mechanics*. World Scientific, 1989.

Theoretical Physics

[B1] The quantum Super Yangian and Casimir operators of $U_q(gl(m|n))$, 1994. Available at:https://tinyurl.com/l8asm33.

[B2] Bernewig A and Neupert T. Topological Superconductors and Category Theory, 2015. Available at:https://tinyurl.com/l2jsrzc.

[B3] Jadczyk A. Conformally compactified minkowski space: myths and facts, 2011. Available at: https://arxiv.org/abs/1803.00545.

[B4] Kitaev A. Quantum computations: algorithms and error correction. *Russ Math Survey*, pages 52–61, 1997.

[B5] Kitaev A. Fault tolerant quantum computation by anyons, 1997. Available at: https://arxiv.org/abs/quant-ph/9707021.

[B6] Zee A. *The Unity of Forces in the Universe*. World Sci Press, Singapore, 1982.

[B7] Trnka Y Arkani-Hamed N. The Amplituhedron, 2013. Available at: https://arxiv.org/abs/1312.2007.

[B8] Schroer B. Lectures on Algebraic Quantum Field Theory and Operator Algebras, 2001. Available at: https://arxiv.org/abs/math-ph/0102018.

[B9] Drell S Björken J. *Relativistic Quantum Fields*. Mc Graw-Hill, New York, 1965.

[B10] Feng B Witten E Britto R, Cachazo F. Direct Proof of Tree-Level Recursion Relation in Yang- Mills Theory. *Phys Rev*, 94:181602, 2005. Available at: https://arxiv.org/abs/hep-th/0501052.

[B11] Witten E Cachazo F, Svrcek P. MHV Vertices and Tree Amplitudes In Gauge Theory, 2004. Available at: https://arxiv.org/abs/hep-th/0403047.

[B12] Zuber J-B Cappelli A. A-D-E Classification of Conformal Field Theories, 2010. Available at:https://tinyurl.com/h9val5g.

[B13] Rapoport D. Stochastic processes in conformal Riemann-Cartan-Weyl gravitation, 1991. Available at: https://link.springer.com/article/10.1007/BF00675614.

[B14] Witten E Dolan L, Nappi CR. Yangian Symmetry in $D = 4$ superconformal Yang-Mills theory, 2004. Available at: https://arxiv.org/abs/hep-th/0401243.

[B15] Plefka J Drummond J, Henn J. Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, 2009. Available at: https://cdsweb.cern.ch/record/1162372/files/jhep052009046.pdf.

[B16] Witten E. Perturbative Gauge Theory As a String Theory In Twistor Space, 2003. Available at: https://arxiv.org/abs/hep-th/0312171.

[B17] Huang Y-T Elvang H. Scattering amplitudes, 2013. Available at: https://arxiv.org/pdf/1308.1697v1.pdf.

[B18] Arkani-Hamed N et al. Scattering amplitides and the positive Grassmannian. Available at: https://arxiv.org/pdf/1212.5605v1.pdf.

[B19] Arkani-Hamed N et al. A duality for the S-matrix, 2009. Available at: https://arxiv.org/abs/0907.5418.

[B20] Arkani-Hamed N et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM, 2010. Available at: https://arxiv.org/abs/1008.2958.

[B21] Arkani-Hamed N et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM, 2011. Available at: https://arxiv.org/abs/1008.2958..

[B22] Minev ZK et al. To catch and reverse a quantum jump mid-flight, 2019. Available at: https://arxiv.org/abs/1803.00545.

[B23] Preskill J et al. Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence, 2015. Available at: https://arxiv.org/pdf/1503.06237.pdf.

[B24] Wang Z Freedman M, Larsen H. A modular functor which is universal for quantum computation. *Comm Math Phys*, 1(2):605–622, 2002. Available at: https://arxiv.org/abs/quant-ph/0001108.

[B25] Parisi G. *Field Theory, Disorder and Simulations*. World Scientific, 1992.

[B26] Kastler D Haag R. An Algebraic Approach to Quantum Field Theory. *J Math Phys*, 5, 1964.

[B27] Yang-Hui He. On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective, 2002. Available at: https://arxiv.org/pdf/hep-th/0209230.pdf..

[B28] Borchers HJ. On Revolutionizing QFT with Tomita's Modular Theory. *J Math Phys*, 41:3604–3673, 2000. Available at: https://www.lqp.uni-goettingen.de/papers/99/04/99042900.html.

[B29] Nakamura I Ito Y. Hilbert schemes and simple singularities. *Proc. of EuroConference on Algebraic Geometry, Warwick*, pages 151–233, 1996. Available at: https://www.math.sci.hokudai.ac.jp/~nakamura/ADEHilb.pdf.

[B30] Baez JC. Higher Yang-Mills Theory, 2002. Available at: https://xxx.lanl.gov/abs/hep-th/0206130.

[B31] Huang K. *Quarks,Leptons* & *Gauge Fields*. World Scientific, 1982.

[B32] Horwitz LP. Hypercomplex quantum mechanics, 1996. Available at: https://arxiv.org/abs/quant-ph/9602001.

[B33] Jimbo M. Introduction to the Yang-Baxter equation. *Int J Mod Phys. A*, 4(15):3759–3777, 1989. Available at: https://tinyurl.com/l4z6zyr.

[B34] Adamo T. Twistor actions for gauge theory and gravity, 2013. Available at: https://arxiv.org/pdf/1308.2820.pdf.

[B35] Schreiber U. Nonabelian 2-forms and loop space connections from SCFT deformations, 2004. Available at:https://arxiv.org/abs/hep-th/0407122.

[B36] Trnka Y. Grassmannian Origin of Scattering Amplitudes, 2013. Available at: https://www.princeton.edu/physics/graduate-program/theses/theses-from-2013/Trnka-Thesis.pdf.

Particle and Nuclear Physics

[C1] Egede U. A theoretical limit on Higgs mass, 1998. Available at: `https://www.hep.lu.se/atlas//thesis/egede/thesis-node20.html`.

Condensed Matter Physics

[D1] Wilczek F Mackenzie R. *Rev Mod Phys . A*, 3:2827, 1988.

[D2] Laughlin RB. *Phys Rev*, 50, 1990.

Cosmology and Astro-Physics

[E1] Nottale L Da Rocha D. Gravitational Structure Formation in Scale Relativity, 2003. Available at: `https://arxiv.org/abs/astro-ph/0310036`.

Physics of Earth

Fringe Physics

Biology

Neuroscience and Consciousness

[J1] Khrennikov A. Bell's inequality for conditional probabilities as a test for quantum like behaviour of mind, 2004. Available at: `https://arxiv.org/abs/quant-ph/0402169`.

[J2] Khrennikov A Adenier G. Is the Fair Sampling Assumption Supported by EPR Experiments?, 2006. Available at: `https://arxiv.org/abs/quantum-ph/0606122`.

[J3] Libet B. Readiness potentials preceding unrestricted spontaneous and preplanned voluntary acts, 1982. Available at: `https://tinyurl.com/jqp1`. See also the article *Libet's Research on Timing of Conscious Intention to Act: A Commentary* of Stanley Klein at `https://tinyurl.com/jqp1`.

[J4] Weihs G et al. *Phys Rev*, 81:5039, 1998.

[J5] Sacks O. *The man who mistook his wife for a hat.* 1998. Available at: First edition 1985.

[J6] Suzuki S. *Zen Mind, Beginner's Mind.* Water hill,, New York, 1988.

Books related to TGD

[K1] Pitkänen M. TGD Inspired Theory of Consciousness. In *Topological Geometrodynamics: Overview: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*.

[K2] Pitkänen M. *Topological Geometrodynamics.* 1983. Thesis in Helsinki University 1983.

[K3] Pitkänen M. DNA as Topological Quantum Computer. In *Quantum - and Classical Computation in TGD Universe.* *https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: https://tgdtheory.fi/pdfpool/dnatqc.pdf, 2015.

[K4] Pitkänen M. Topological Quantum Computation in TGD Universe. In *Quantum - and Classical Computation in TGD Universe.* *https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: https://tgdtheory.fi/pdfpool/tqc.pdf, 2015.

[K5] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/adelephysics.pdf, 2017.

[K6] Pitkänen M. About Nature of Time. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timenature.pdf, 2023.

[K7] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/prext.pdf, 2023.

[K8] Pitkänen M. About twistor lift of TGD? In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/hgrtwistor.pdf, 2023.

[K9] Pitkänen M. Appendix A: Quantum Groups and Related Structures. In *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Available at: https://tgdtheory.fi/pdfpool/bialgebra.pdf, 2023.

[K10] Pitkänen M. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/class.pdf, 2023.

[K11] Pitkänen M. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://www.tgdtheory.fi/public_html/pdfpool/append.pdf, 2023.

[K12] Pitkänen M. *Bio-Systems as Conscious Holograms.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/holography.html., 2023.

[K13] Pitkänen M. *Bio-Systems as Self-Organizing Quantum Systems.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioselforg.html., 2023.

[K14] Pitkänen M. Category Theory and Quantum TGD. In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/categorynew.pdf, 2023.

[K15] Pitkänen M. Category Theory, Quantum TGD, and TGD Inspired Theory of Consciousness. In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/categoryc.pdf, 2023.

[K16] Pitkänen M. Classical TGD. In *Topological Geometrodynamics: Overview: Part I:https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdclass.pdf, 2023.

[K17] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K18] Pitkänen M. Construction of Quantum Theory: M-matrix. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/towards.pdf, 2023.

[K19] Pitkänen M. Construction of Quantum Theory: Symmetries. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/quthe.pdf, 2023.

[K20] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K21] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/Yangianagain.pdf, 2023.

[K22] Pitkänen M. Criticality and dark matter: part I. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark1.pdf, 2023.

[K23] Pitkänen M. Criticality and dark matter: part II. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark2.pdf, 2023.

[K24] Pitkänen M. Criticality and dark matter: part III. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark3.pdf, 2023.

[K25] Pitkänen M. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/eegdark.pdf, 2023.

[K26] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/fermizeta.pdf, 2023.

[K27] Pitkänen M. Does TGD Predict a Spectrum of Planck Constants? In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/Planck, 2023.

[K28] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Topological Geometrodynamics: Overview: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdoverview2*. Available at: https://tgdtheory.fi/pdfpool/vNeumannnew, 2023.

[K29] Pitkänen M. From Principles to Diagrams. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/diagrams.pdf, 2023.

[K30] Pitkänen M. *Genes and Memes.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/genememe.html., 2023.

[K31] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/tensornet.pdf, 2023.

[K32] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck.html., 2023.

[K33] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck1.html., 2023.

[K34] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy: Part II.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck2.html., 2023.

[K35] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/kahler.pdf., 2023.

[K36] Pitkänen M. Infinite Primes and Consciousness. In *Mathematical Aspect of Consciousness. https://tgdtheory.fi/tgdhtml/mathconsc.html*. Available at: https://tgdtheory.fi/pdfpool/infpc.pdf, 2023.

[K37] Pitkänen M. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/knotstgd.pdf, 2023.

[K38] Pitkänen M. Langlands Program and TGD. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/Langlands.pdf, 2023.

[K39] Pitkänen M. Langlands Program and TGD: Years Later. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/langlandsnew.pdf, 2023.

[K40] Pitkänen M. *Magnetospheric Consciousness.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/magnconsc.html., 2023.

[K41] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K42] Pitkänen M. *Mathematical Aspects of Consciousness Theory.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/mathconsc.html, 2023.

[K43] Pitkänen M. Matter, Mind, Quantum. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/conscic.pdf, 2023.

[K44] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory: Part III. https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infmotives.pdf, 2023.

[K45] Pitkänen M. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/nmpc.pdf, 2023.

[K46] Pitkänen M. New Physics Predicted by TGD: Part I. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys1.pdf, 2023.

[K47] Pitkänen M. New Physics Predicted by TGD: Part II. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys2.pdf, 2023.

[K48] Pitkänen M. Number theoretic vision, Hyper-finite Factors and S-matrix. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/UandM.pdf, 2023.

[K49] Pitkänen M. *p-Adic length Scale Hypothesis*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/padphys.html., 2023.

[K50] Pitkänen M. p-Adic Numbers and Generalization of Number Concept. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/padmat.pdf, 2023.

[K51] Pitkänen M. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mass3.pdf, 2023.

[K52] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K53] Pitkänen M. Physics as a Generalized Number Theory. In *Topological Geometrodynamics: Overview: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdnumber.pdf, 2023.

[K54] Pitkänen M. Quantum Adeles. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/galois.pdf, 2023.

[K55] Pitkänen M. Quantum Hall effect and Hierarchy of Planck Constants. In *TGD and Condensed Matter.* *https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/anyontgd.pdf, 2023.

[K56] Pitkänen M. *Quantum Hardware of Living Matter*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioware.html., 2023.

[K57] Pitkänen M. Quantum Mind and Neuroscience. In *TGD and EEG: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPN.pdf, 2023.

[K58] Pitkänen M. Quantum Mind, Magnetic Body, and Biological Body. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPB.pdf, 2023.

[K59] Pitkänen M. Quantum Model for Bio-Superconductivity: I. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondI.pdf, 2023.

[K60] Pitkänen M. Quantum Model for Bio-Superconductivity: II. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondII.pdf, 2023.

[K61] Pitkänen M. Quantum Model for Hearing. In *TGD and EEG: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdeeg2.html*. Available at: https://tgdtheory.fi/pdfpool/hearing.pdf, 2023.

[K62] Pitkänen M. *Quantum TGD*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdquantum.html., 2023.

[K63] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K64] Pitkänen M. Riemann Hypothesis and Physics. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/riema.pdf, 2023.

[K65] Pitkänen M. Self and Binding: Part I. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/selfbindc.pdf, 2023.

[K66] Pitkänen M. Some questions related to the twistor lift of TGD. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistquestions.pdf, 2023.

[K67] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds". In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwsymm.pdf, 2023.

[K68] Pitkänen M. TGD and Astrophysics. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/astro.pdf, 2023.

[K69] Pitkänen M. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cosmo.pdf, 2023.

[K70] Pitkänen M. *TGD and EEG.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdeeg.html, 2023.

[K71] Pitkänen M. *TGD and Fringe Physics.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/freenergy.html., 2023.

[K72] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K73] Pitkänen M. TGD as a Generalized Number Theory: p-Adicization Program. In *Quantum Physics as Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visiona.pdf, 2023.

[K74] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionb.pdf, 2023.

[K75] Pitkänen M. *TGD Based View About Living Matter and Remote Mental Interactions.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdlian.html, 2023.

[K76] Pitkänen M. *TGD Inspired Theory of Consciousness.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdconsc.html., 2023.

[K77] Pitkänen M. The classical part of the twistor story. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistorstory.pdf, 2023.

[K78] Pitkänen M. The Geometry of the World of the Classical Worlds. In *Topological Geometrodynamics: Overview: Part I:**https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgeom.pdf, 2023.

[K79] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

[K80] Pitkänen M. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgrt.pdf., 2023.

[K81] Pitkänen M. Three new physics realizations of the genetic code and the role of dark matter in bio-systems. In *Genes and Memes: Part II.* *https://tgdtheory.fi/tgdhtml/Bgenememe2.html*. Available at: https://tgdtheory.fi/pdfpool/dnatqccodes.pdf, 2023.

[K82] Pitkänen M. Time and Consciousness. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timesc.pdf, 2023.

[K83] Pitkänen M. *Topological Geometrodynamics: an Overview.* Online book.Available at: https://www.tgdtheory.fi/tgdhtml/tgdview.html., 2023.

[K84] Pitkänen M. Topological Geometrodynamics: Basic Visions. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPTGD.pdf, 2023.

[K85] Pitkänen M. Topological Geometrodynamics: Three Visions. In *Topological Geometrodynamics: Overview: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgd2010.pdf, 2023.

[K86] Pitkänen M. Unified Number Theoretical Vision. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/numbervision.pdf, 2023.

[K87] Pitkänen M. Was von Neumann Right After All? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/vNeumann.pdf, 2023.

[K88] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

[K89] Pitkänen M. Zero Energy Ontology. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/ZEO.pdf, 2023.

Articles about TGD

[L1] Pitkänen M. TGD Inspired Theory of Consciousness. Available at: https://tgdtheory.fi/public_html/articles/tgdconsc.cpdf., 2008.

[L2] Pitkänen M. Topological Geometrodynamics: What Might Be The Basic Principles. Available at: https://tgdtheory.fi/public_html/articles/tgd2008.pdf., 2008.

[L3] Pitkänen M. Algebraic braids, sub-manifold braid theory, and generalized Feynman diagrams. Available at: https://tgdtheory.fi/public_html/articles/algknots.pdf., 2011.

[L4] Pitkänen M. Langlands conjectures in TGD framework. Available at: https://tgdtheory.fi/public_html/articles/Langlandsagain.pdf., 2011.

[L5] Pitkänen M. CMAP representations about TGD. Available at: https://www.tgdtheory.fi/cmaphtml.html, 2014.

[L6] Pitkänen M. CMAP representations about TGD, and TGD inspired theory of consciousness and quantum biology. Available at: https://www.tgdtheory.fi/tgdglossary.pdf., 2014.

[L7] Pitkänen M. Geometric theory of harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[L8] Pitkänen M. Could one realize number theoretical universality for functional integral? Available at: https://tgdtheory.fi/public_html/articles/ntu.pdf., 2015.

[L9] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? . Available at: https://tgdtheory.fi/public_html/articles/fermizeta.pdf., 2015.

[L10] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. Available at: https://tgdtheory.fi/public_html/articles/tensornet.pdf., 2016.

[L11] Pitkänen M. Langlands Program and TGD: Years Later. Available at: https://tgdtheory.fi/public_html/articles/langlandsnew.pdf., 2016.

[L12] Pitkänen M. About number theoretic aspects of NMP. Available at: https://tgdtheory.fi/public_html/articles/nmpagain.pdf., 2017.

[L13] Pitkänen M. About some unclear issues of TGD. Available at: https://tgdtheory.fi/public_html/articles/fermionsector.pdf., 2017.

[L14] Pitkänen M. Are higher structures needed in the categorification of TGD? Available at: https://tgdtheory.fi/public_html/articles/nstructures.pdf., 2017.

[L15] Pitkänen M. Breaking of CP, P, and T in cosmological scales in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/rotuniverse.pdf., 2017.

[L16] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? Available at: https://tgdtheory.fi/public_html/articles/Yangianagain.pdf., 2017.

[L17] Pitkänen M. Could McKay correspondence generalize in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2017.

[L18] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry? Available at: https://tgdtheory.fi/public_html/articles/ratpoints.pdf., 2017.

[L19] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part I. Available at: https://tgdtheory.fi/public_html/articles/ratpoints1.pdf., 2017.

[L20] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part II. Available at: https://tgdtheory.fi/public_html/articles/ratpoints2.pdf., 2017.

[L21] Pitkänen M. Does $M^8 - H$ duality reduce classical TGD to octonionic algebraic geometry?: part III. Available at: https://tgdtheory.fi/public_html/articles/ratpoints3.pdf., 2017.

[L22] Pitkänen M. p-Adicization and adelic physics. Available at: https://tgdtheory.fi/public_html/articles/adelicphysics.pdf., 2017.

[L23] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L24] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L25] Pitkänen M. Quantum computations without definite causal structure: TGD view. Available at: https://tgdtheory.fi/public_html/articles/causalstructure.pdf., 2017.

[L26] Pitkänen M. Questions about twistor lift of TGD. Available at: https://tgdtheory.fi/public_html/articles/twistquestions.pdf., 2017.

[L27] Pitkänen M. Re-examination of the basic notions of TGD inspired theory of consciousness. Available at: https://tgdtheory.fi/public_html/articles/conscrit.pdf., 2017.

[L28] Pitkänen M. What could be the role of complexity theory in TGD? Available at: https://tgdtheory.fi/public_html/articles/numberchaos.pdf., 2017.

[L29] Pitkänen M. TGD view about coupling constant evolution. Available at: https://tgdtheory.fi/public_html/articles/ccevolution.pdf., 2018.

[L30] Pitkänen M. The Recent View about Twistorialization in TGD Framework. Available at: https://tgdtheory.fi/public_html/articles/smatrix.pdf., 2018.

[L31] Pitkänen M. An overall view about models of genetic code and bio-harmony. Available at: https://tgdtheory.fi/public_html/articles/gcharm.pdf., 2019.

[L32] Pitkänen M. Copenhagen interpretation dead: long live ZEO based quantum measurement theory! Available at: https://tgdtheory.fi/public_html/articles/Bohrdead.pdf., 2019.

[L33] Pitkänen M. Earhquakes and volcanic eruptions as macroscopic quantum jumps in zero energy ontology. Available at: https://tgdtheory.fi/public_html/articles/earthquakes.pdf., 2019.

[L34] Pitkänen M. $M^8 - H$ duality and consciousness. Available at: https://tgdtheory.fi/public_html/articles/M8Hconsc.pdf., 2019.

[L35] Pitkänen M. Minimal surfaces: comparison of the perspectives of mathematician and physicist. Available at: https://tgdtheory.fi/public_html/articles/minimalsurfaces.pdf., 2019.

[L36] Pitkänen M. Quantum self-organization by h_{eff} changing phase transitions. Available at: https://tgdtheory.fi/public_html/articles/heffselforg.pdf., 2019.

[L37] Pitkänen M. Scattering amplitudes and orbits of cognitive representations under subgroup of symplectic group respecting the extension of rationals . Available at: https://tgdtheory.fi/public_html/articles/symplorbsm.pdf., 2019.

[L38] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L39] Pitkänen M. SUSY in TGD Universe. Available at: https://tgdtheory.fi/public_html/articles/susyTGD.pdf., 2019.

[L40] Pitkänen M. Tesla still inspires. Available at: https://tgdtheory.fi/public_html/articles/teslastill.pdf., 2019.

[L41] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, and Inclusions of Hyperfinite Factors. Available at: https://tgdtheory.fi/public_html/articles/McKay.pdf., 2019.

[L42] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors. Available at: https://tgdtheory.fi/public_html/articles/McKaygeneral.pdf., 2019.

[L43] Pitkänen M. Twistors in TGD. Available at: https://tgdtheory.fi/public_html/articles/twistorTGD.pdf., 2019.

[L44] Pitkänen M. Two anomalies of hadron physics from TGD perspective. Available at: https://tgdtheory.fi/public_html/articles/twoanomalies.pdf., 2019.

[L45] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L46] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L47] Pitkänen M. Could ZEO provide a new approach to the quantization of fermions? Available at: https://tgdtheory.fi/public_html/articles/secondquant.pdf., 2020.

[L48] Pitkänen M. Fermionic variant of $M^8 - H$ duality. Available at: https://tgdtheory.fi/public_html/articles/M8Hfermion.pdf., 2020.

[L49] Pitkänen M. Summary of Topological Geometrodynamics. https://tgdtheory.fi/public_html/articles/tgdarticle.pdf., 2020.

[L50] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: https://tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[L51] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory. Available at: https://tgdtheory.fi/public_html/articles/kahlersmhort.pdf., 2020.

[L52] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory (short version). Available at: https://tgdtheory.fi/public_html/articles/kahlersm.pdf., 2020.

[L53] Pitkänen M. About the role of Galois groups in TGD framework. https://tgdtheory.fi/public_html/articles/GaloisTGD.pdf., 2021.

[L54] Pitkänen M. Is genetic code part of fundamental physics in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/TIH.pdf., 2021.

[L55] Pitkänen M. Is $M^8 - H$ duality consistent with Fourier analysis at the level of $M^4 \times CP_2$? https://tgdtheory.fi/public_html/articles/M8Hperiodic.pdf., 2021.

[L56] Pitkänen M. Negentropy Maximization Principle and Second Law. Available at: https://tgdtheory.fi/public_html/articles/nmpsecondlaw.pdf., 2021.

[L57] Pitkänen M. Some questions concerning zero energy ontology. https://tgdtheory.fi/public_html/articles/zeonew.pdf., 2021.

[L58] Pitkänen M. Spin Glasses, Complexity, and TGD. https://tgdtheory.fi/public_html/articles/sg.pdf., 2021.

[L59] Pitkänen M. TGD and Condensed Matter. https://tgdtheory.fi/public_html/articles/TGDcondmatshort.pdf., 2021.

[L60] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L61] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.

[L62] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[L63] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[L64] Pitkänen M. About the number theoretic aspects of zero energy ontology. https://tgdtheory.fi/public_html/articles/ZEOnumber.pdf., 2022.

[L65] Pitkänen M. Can quasi-time crystal be created by a Fibonacci process?: TGD point of view. https://tgdtheory.fi/public_html/articles/qtc.pdf., 2022.

[L66] Pitkänen M. Finite Fields and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L67] Pitkänen M. Intersection form for 4-manifolds, knots and 2-knots, smooth exotics, and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L68] Pitkänen M. McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product? . https://tgdtheory.fi/public_html/articles/McKayGal.pdf., 2022.

[L69] Pitkänen M. Quantum Gravitation and Topological Quantum Computation. https://tgdtheory.fi/public_html/articles/TQCTGD.pdf., 2022.

[L70] Pitkänen M. Some New Ideas Related to Langlands Program *viz.* TGD. https://tgdtheory.fi/public_html/articles/Langlands2022.pdf., 2022.

[L71] Pitkänen M. TGD inspired model for freezing in nano scales. https://tgdtheory.fi/public_html/articles/freezing.pdf., 2022.

[L72] Pitkänen M. Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole. https://tgdtheory.fi/public_html/articles/fusionTGD.pdf., 2022.

[L73] Pitkänen M. Two objections against p-adic thermodynamics and their resolution. https://tgdtheory.fi/public_html/articles/padmass2022.pdf., 2022.

[L74] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[L75] Pitkänen M. New findings related to the number theoretical view of TGD. https://tgdtheory.fi/public_html/articles/M8Hagain.pdf., 2023.

[L76] Pitkänen M. New result about causal diamonds from the TGD view point of view. https://tgdtheory.fi/public_html/articles/CDconformal.pdf., 2023.

[L77] Pitkänen M. Reduction of standard model structure to CP_2 geometry and other key ideas of TGD. https://tgdtheory.fi/public_html/articles/cp2etc.pdf., 2023.

[L78] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[L79] Pitkänen M. A fresh look at M^8H duality and Poincare invariance. https://tgdtheory.fi/public_html/articles/TGDcritics.pdf., 2024.

[L80] Pitkänen M. What gravitons are and could one detect them in TGD Universe? https://tgdtheory.fi/public_html/articles/whatgravitons.pdf., 2024.

[L81] Pitkänen M and Rastmanesh R. Homeostasis as self-organized quantum criticality. Available at: https://tgdtheory.fi/public_html/articles/SP.pdf., 2020.

Index

M^4, 76

, 411, 412

associativity, 102, 302

category theory, 265, 302
Clifford algebra, 76, 102
cognition, 266
commutant, 77
conformal field, 302
coset space, 102
crossed product, 77

density matrix, 39

embedding space, 266

factor of type II_1, 26, 102
factors of type I, 102
factors of type III, 40
finite measurement resolution, 78, 102, 302

gamma matrices, 77, 102

Hilbert space, 40
holomorphic function, 77

Kähler magnetic flux, 78
Kähler metric, 77

Lobatchevski space, 77

M-matrix, 40
measurement resolution, 26, 78, 102, 302
Minkowski space, 40

observable, 102

path integral, 78
presheaf, 266
propagator, 78
psychological time, 266

quantum computation, 26, 40
quantum spinors, 79

space-time correlate, 266
space-time sheet, 40

tensor product, 26, 77
TGD inspired theory of consciousness, 26
trace, 39
translation, 77

von Neumann algebra, 26, 76, 102

WCW, 76
world of classical worlds, 76

zero energy ontology, 78, 266, 302
zero mode, 77