QUANTUM PHYSICS AS NUMBER THEORY: PART I

Matti Pitkänen

Rinnekatu 2-4 A 8, Karkkila, 03620, Finland

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0.1 PREFACE

Brief summary of TGD

Towards the end of the year 2023 I became convinced that it would be appropriate to prepare collections about books related to TGD and its applications. The finiteness of human lifetime was my first motivation. My second motivation was the deep conviction that TGD will mean a revolution of the scientific world view and I must do my best to make it easier.

The first collection would relate to the TGD proper and its applications to physics. Second collection would relate to TGD inspired theory of consciousness and the third collection to TGD based quantum biology. The books in these collections would focus on much more precise topics than the earlier books and would be shorter. This would make it much easier for the reader to understand what TGD is, when the time is finally mature for the TGD to be taken seriously. This particular book belongs to a collection of books about TGD proper.

The basic ideas of TGD

TGD can be regarded as a unified theory of fundamental interactions but is not the kind of unified theory as so called GUTs constructed by graduate students in the seventies and eighties using detailed recipes for how to reduce everything to group theory. Nowadays this activity has been completely computerized and it probably takes only a few hours to print out the predictions of this kind of unified theory as an article in the desired format. TGD is something different and I am not ashamed to confess that I have devoted the last 45 years of my life to this enterprise and am still unable to write The Rules.

If I remember correctly, I got the basic idea of Topological Geometrodynamics (TGD) during autumn 1977, perhaps it was October. What I realized was that the representability of physical space-times as 4-dimensional surfaces of some higher-dimensional space-time obtained by replacing the points of Minkowski space with some very small compact internal space could resolve the conceptual difficulties of general relativity related to the definition of the notion of energy. This belief was too optimistic and only with the advent of what I call zero energy ontology the understanding of the notion of Poincare invariance has become satisfactory. This required also the understanding of the relationship to General Relativity.

It soon became clear that the approach leads to a generalization of the notion of space-time with particles being represented by space-time surfaces with finite size so that TGD could be also seen as a generalization of the string model. Much later it became clear that this generalization is consistent with conformal invariance only if space-time is 4-dimensional and the Minkowski space factor of the embedding space is 4-dimensional. During last year it became clear that 4-D Minkowski space and 4-D complex projective space CP_2 are completely unique in the sense that they allow twistor space with Kähler structure.

It took some time to discover that also the geometrization of also gauge interactions and elementary particle quantum numbers could be possible in this framework: it took two years to find the unique internal space (CP_2) providing this geometrization involving also the realization that family replication phenomenon for fermions has a natural topological explanation in TGD framework and that the symmetries of the standard model symmetries are much more profound than pragmatic TOE builders have believed them to be. If TGD is correct, the mainstream particle physics chose the wrong track leading to the recent deep crisis when people decided that quarks and leptons belong to the same multiplet of the gauge group implying instability of the proton.

Instead of trying to describe in detail the path, which led to TGD as it is now with all its side tracks, it is better to summarize the recent view which of course need not be final.

TGD can be said to be a fusion of special and general relativities. The Relativity Principle (Poincare Invariance) of Special Relativity is combined with the General Coordinate Invariance and Equivalence Principle of General Relativity. TGD involves 3 views of physics: physics geometry, physics as number theory and physics as topological physics in some sense.

Physics as geometry

"Geometro-" in TGD refers to the idea about the geometrization of physics. The geometrization program of Einstein is extended to gauge fields allowing realization in terms of the geometry of surfaces so that Einsteinian space-time as abstract Riemann geometry is replaced with sub-manifold geometry. The basic motivation is the loss of classical conservation laws in General Relativity Theory (GRT)(see **Fig. 1**). Also the interpretation as a generalization of string models by replacing string with 3-D surface is natural.

- Standard model symmetries uniquely fix the choice of 8-D space in which space-time surfaces live to $H = M^4 \times CP_2$ [L49]. Also the notion of twistor is geometrized in terms of surface geometry and the existence of twistor lift fixes the choice of H completely so that TGD is unique [L25, L29](see **Fig. 2**). The geometrization applies even to the quantum theory itself and the space of space-time surfaces - "world of classical worlds" (WCW) - becomes the basic object endowed with Kähler geometry (see **Fig. 3**). The mere mathematical existence of WCW geometry requires that it has maximal isometries, which together twistor lift and number theoretic vision fixes it uniquely [L50].

- General Coordinate Invariance (GCI) for space-time surfaces has dramatic implications. A given 3-surface fixes the space-time surface almost completely as analog of Bohr orbit (preferred extremal). This implies holography and leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K114, L34].

- From the beginning it was clear that the theory predicts the presence of long ranged classical electro-weak and color gauge fields and that these fields necessarily accompany classical electromagnetic fields in all scales. It took about 26 years to gain the maturity to admit the obvious: these fields are classical correlates for long range color and weak interactions assignable to the phases of ordinary matter predicted by the number theoretic vision and behaving like dark matter but identifiable as matter explaining the missing baryon problem whereas the galactic dark matter would correspond to the dark energy assignable monopole flux tubes as deformations of cosmic strings. The only possible conclusion is that TGD physics is a fractal consisting of an entire hierarchy of fractal copies of standard model physics. Also the understanding of electro-weak massivation and screening of weak charges has been a long standing problem and p-adic physics solved this problem in terms of p-adic thermodynamics [K23, K51] [L46].

- One of the most recent discoveries of classical TGD is exact general solution of the field equations. Holography can be realized as a generalized holomorphy realized in terms of what I call Hamilton-Jacobi structure [L47]. Space-time surfaces correspond to holomorphic imbeddings of the space-time surface to H with a generalized complex structure defined by the vanishing of 2 analytic functions of 4 generalized complex coordinates of H. These surfaces are automatically minimal surfaces. This is true for any geneneral coordinate invariant action constructed in terms of the induced geometric structures so that the dynamics is universal. Different actions differ only in the sense that singularities at which the minimal surface property fails depend on the action. This affects the scattering amplitudes, which can be constructed in terms of the data related to the singularities [L52].

- Generalized conformal symmetries define an extension of conformal symmetries and one can assign to them Noether charges. Besides this the so called super-symplectic symmetries associated with $\delta M^4_+ \times CP_2$ define isometries of the "world of classical worlds" (WCW), which by holography is essentially the space of Bohr orbits of 3-surfaces as particles so that quantum TGD is expected to reduce to a generalization of wave mechanics.

Physics as number theory

During these years TGD led to a rather profound generalization of the space-time concept. Quite general properties of the theory led to the notion of many-sheeted space-time with sheets representing physical subsystems of various sizes. At the beginning of 90s I became dimly aware of the importance of p-adic number fields and soon ended up with the idea that p-adic thermodynamics for a conformally invariant system allows to understand elementary particle massivation with amazingly few input assumptions. The attempts to understand p-adicity from basic principles led gradually to the vision about physics as a generalized number theory as an approach complementary to the physics as an infinite-dimensional spinor geometry of WCW approach. One of its elements was a generalization of the number concept obtained by fusing real numbers and various p-adic numbers along common rationals. The number theoretical trinity involves besides p-adic number fields also quaternions and octonions and the notion of infinite prime.

Adelic physics [L22, L23] fusing real and various p-adic physics is part of the number theoretic vision, which provides a kind of dual description for the description based on space-time geometry and the geometry of "world of classical words". Adelic physics predicts two fractal length scale hierarchies: p-adic length scale hierarchy and the hierarchy of dark length scales labelled by $h_{eff} = nh_0$, where n is the dimension of extension of rational. The interpretation of the latter hierarchy is as phases of ordinary matter behaving like dark matter. Quantum coherence is possible in arbitarily long scales. These two hierarchies are closely related. p-Adic primes correspond to ramified primes for a polynomial, whose roots define the extension of rationals: for a given extension this polynomial is not unique.

$M^8 - H$ duality

The concrete realization of the number theoretic vision is based on $M^8 - H$ duality (see **Fig. 4**). What the precise form is this duality is, has been far from clear but the recent form is the simplest one and corresponds to the original view [L51]. M^8 corresponds to octonions O but with the number theoretic metric defined by $Re(o^2)$ rather than the standard norm and giving Minkowskian signature.

The physics in M^8 can be said to be algebraic whereas in H field equations are partial differential equations. The dark matter hierarchy corresponds to a hierarchy of algebraic extensions of rationals inducing that for adeles and has interpretation as an evolutionary hierarchy (see **Fig. 5**). p-Adic physics is an essential part of number theoretic vision and the space-time surfaces are such that at least their M^8 counterparts exists also in p-adic sense. This requires that the analytic function defining the space-time surfaces are polynomials with rational coefficients.

$M^8 - H$ duality relates two complementary visions about physics (see **Fig. 6**), and can be seen as a generalization of the momentum-position duality of wave mechanics, which fails to generalize to quantum field theories (QFTs). $M^8 - H$ duality applies to particles which are 3-surfaces instead of point-like particles.

p-Adic physics

The idea about p-adic physics as physics of cognition and intentionality emerged also rather naturally and implies perhaps the most dramatic generalization of the space-time concept in which most points of p-adic space-time sheets are infinite in real sense and the projection to the real imbedding space consists of discrete set of points. One of the most fascinating outcomes was the observation that the entropy based on p-adic norm can be negative. This observation led to the vision that life can be regarded as something in the intersection of real and p-adic worlds. Negentropic entanglement has interpretation as a correlate for various positively colored aspects of conscious experience and means also the possibility of strongly correlated states stable under state function reduction and different from the conventional bound states and perhaps playing key role in the energy metabolism of living matter.

If one requires consistency of Negentropy Mazimization Principle with standard measurement theory, negentropic entanglement defined in terms of number theoretic negentropy is necessarily associated with a density matrix proportional to unit matrix and is maximal and is characterized by the dimension n of the unit matrix. Negentropy is positive and maximal for a p-adic unique prime dividing n.

Hierarchy of Planck constants labelling phases ordinary matter dark matter behaving like dark matter

One of the latest threads in the evolution of ideas is not more than nine years old. Learning about the paper of Laurent Nottale about the possibility to identify planetary orbits as Bohr orbits with a gigantic value of gravitational Planck constant made once again possible to see the obvious. Dynamical quantized Planck constant is strongly suggested by quantum classical correspondence and the fact that space-time sheets identifiable as quantum coherence regions can have arbitrarily large sizes. Second motivation for the hierarchy of Planck constants comes from bio-electromagnetism suggesting that in living systems Planck constant could have large values making macroscopic quantum coherence possible. The interpretation of dark matter as a hierarchy of phases of ordinary matter characterized by the value of Planck constant is very natural.

During summer 2010 several new insights about the mathematical structure and interpretation of TGD emerged. One of these insights was the realization that the postulated hierarchy of Planck constants might follow from the basic structure of quantum TGD. The point is that due to the extreme non-linearity of the classical action principle the correspondence between canonical momentum densities and time derivatives of the imbedding space coordinates is one-to-many and the natural description of the situation is in terms of local singular covering spaces of the imbedding space. One could speak about effective value of Planck constant $h_{eff} = n \times h$ coming as a multiple of minimal value of Planck constant. Quite recently it became clear that the non-determinism of Kähler action is indeed the fundamental justification for the hierarchy: the integer n can be also interpreted as the integer characterizing the dimension of unit matrix characterizing negentropic entanglement made possible by the many-sheeted character of the space-time surface.

Due to conformal invariance acting as gauge symmetry the n degenerate space-time sheets must be replaced with conformal equivalence classes of space-time sheets and conformal transformations correspond to quantum critical deformations leaving the ends of space-time surfaces invariant. Conformal invariance would be broken: only the sub-algebra for which conformal weights are divisible by n act as gauge symmetries. Thus deep connections between conformal invariance related to quantum criticality, hierarchy of Planck constants, negentropic entanglement, effective p-adic topology, and non-determinism of Kähler action perhaps reflecting p-adic non-determinism emerges.

The implications of the hierarchy of Planck constants are extremely far reaching so that the significance of the reduction of this hierarchy to the basic mathematical structure distinguishing between TGD and competing theories cannot be under-estimated.

TGD as an analog of topological QFT

Consider next the attribute "Topological". In condensed matter physical topological physics has become a standard topic. Typically one has fields having values in compact spaces, which are topologically non-trivial. In the TGD framework space-time topology itself is non-trivial as also the topology of $H = M^4 \times CP_2$. Since induced metric is involved with TGD, it is too much to say that TGD is topological QFT but one can for instance say, that space-time surfaces as preferred extremals define representatives for 4-D homological equivalence classes.

The space-time as 4-surface $X^4 \subset H$ has a non-trivial topology in all scales and this together with the notion of many-sheeted space-time brings in something completely new. Topologically trivial Einsteinian space-time emerges only at the QFT limit in which all information about topology is lost (see **Fig. 7**).

Any GCI action satisfying holography=holomorphy principle has the same universal basic extremals: CP_2 type extremals serving basic building bricks of elementary particles, cosmic strings and their thickenings to flux tubes defining a fractal hierarchy of structure extending from CP_2 scale to cosmic scales, and massless extremals (MEs) define space-time correletes for massless particles. World as a set or particles is replaced with a network having particles as nodes and flux tubes as bonds between them serving as correlates of quantum entanglement.

"Topological" could refer also to p-adic number fields obeying p-adic local topology differing radically from the real topology (see **Fig. 8**).

Zero energy ontology

TGD inspired theory of consciousness entered the scheme after 1995 as I started to write a book about consciousness. Gradually it became difficult to say where physics ends and consciousness theory begins since consciousness theory could be seen as a generalization of quantum measurement theory by identifying quantum jump as a moment of consciousness and by replacing the observer with the notion of self identified as a system which is conscious as long as it can avoid entanglement with environment. The somewhat cryptic statement "Everything is conscious and consciousness can be only lost" summarizes the basic philosophy neatly.

General coordinate invariance leads to the identification of space-time surfaces are analogous to Bohr orbits inside causal diamond (CD). CD obtained as intersection of future and past directed light-cones (with CP_2 factor included). By the already described hologamphy, 3-dimensional data replaces the boundary conditions at single 3-surface involving also normal derivatives with conditions involving no derivates.

In zero energy ontology (ZEO), the superpositions of space-time surfaces inside causal diamond (CD) having their ends at the opposite light-like boundaries of CD, define quantum states. CDs form a scale hierarchy (see **Fig. 9** and **Fig. 10**). Quantum states are modes of WCW spinor fields, essentially wave functions in the space WCW consisting of Bohr orbit-like 4-surfaces.

Quantum jumps occur between these and the basic problem of standard quantum measurement theory disappears. Ordinary state function reductions (SFRs) correspond to "big" SFRs (BSFRs) in which the arrow of time changes (see **Fig. 11**). This has profound thermodynamic implications and the question about the scale in which the transition from classical to quantum takes place becomes obsolete. BSFRs can occur in all scales but from the point of view of an observer with an opposite arrow of time they look like smooth time evolutions.

In "small" SFRs (SSFRs) as counterparts of "weak measurements" the arrow of time does not change and the passive boundary of CD and states at it remain unchanged (Zeno effect).

Equivalence Principle in TGD framework

There have been also longstanding problems related to the relationship between inertial mass and gravitational mass, whose identification has been far from obvious.

- Gravitational energy is well-defined in cosmological models but is not conserved. Hence the conservation of the inertial energy does not seem to be consistent with the Equivalence Principle. In this framework the quantum numbers are assigned with zero energy states located at the boundaries of CDs defined as intersections of future and past directed light-cones. The notion of energy-momentum becomes length scale dependent since one has a scale hierarchy for causal diamonds. This allows to understand the non-conservation of energy as apparent.

  Equivalence Principle in the form expressed by Einstein's equations follows from Poincare invariance once it is realized that GRT space-time is obtained from the many-sheeted space-time of TGD by lumping together the space-time sheets to a regionof Minkowski space and endowing it with an effective metric given as a sum of Minkowski metric and deviations of the metrices of space-time sheets from Minkowski metric. Similar description relates classical gauge potentials identified as components of induced spinor connection to Yang-Mills gauge potentials in GRT space-time. Various topological inhomogenities below resolution scale identified as particles are described using energy momentum tensor and gauge currents.

  At quantum level, the Equivalence Principle has a surprisingly strong content. In linear Minkowski coordinates, space-time projection of the M^4 spinor connection representing gravitational gauge potentials the coupling to induced spinor fields vanishes. Also the modified Dirac action for the solutions of the modified Dirac equation seems to vanish identically and in TGD perturbative approach separating interaction terms is not possible.

  The modified Dirac equation however fails at the singularities of the minimal surface representing space-time surface and Dirac action reduces to an integral over singularities for the trace of the second fundamental form slashed between the induced spinor field and its conjugate. Also the M^4 part of the trace is non-vanishing and gives rise to the gravitational coupling. The trace gives both standard model vertices and graviton emission vertices. One could say that at the quantum level gravitational and gauge interactions are eliminated everywhere except at the singularities identifiable as defects of the ordinary smooth structure. The exotic smooth structures [L43], possible only in dimension 4, are ordinary smooth structures apart from these defects serving as vertex representing a creation of a fermion-antifermion pair in the induced gauge potentials. The vertex is universal and essentially the trace of the second fundamental form as an analog of the Higgs field and the gravitational constant is proportional to the square of CP_2 radius.

- There is a delicate difference between inertial and gravitational masses. One can assume that the modes of the imbedding space spinor fields are solutions of massles Dirac equation in either $M^4 \times CP_2$ and therefore eigenstates of inertial momentum or in $CD = cd \times CP_2$: in this case they are only mass eigenstates. The mass spectra are identical for these options. Inertial momenta correspond naturally to the Poincare charges in the space of CDs. For the CD option the spinor modes correspond to mass squared eigenstates for which the mode for H^3 with a given value of light-proper time is a unitary irreducible $SO(1,3)$ representation rather than a representation of translation group. These two eigenmode basis correspond to gravitational basis for spinor modes.

Quantum TGD as a generalization of Einstein's geometrization program

I started the serious attempts to construct quantum TGD after my thesis around 1982. The original optimistic hope was that path integral formalism or canonical quantization might be enough to construct the quantum theory but it turned that this approach fails due to the extreme non-linearity of the theory.

It took some years to discover that the only working approach is based on the generalization of Einstein's program. Quantum physics involves the geometrization of the infinite-dimensional "world of classical worlds" (WCW) identified as the space of 3-dimensional surfaces. Later 3-surfaces where replaced with 4-surfaces satifying holography and therefore as analogs of Bohr orbits.

- If one assumes Bohr orbitology, then strong correlations between the 3-surfaces at the ends of CD follow and mean holography. It is natural to identify the quantum states of the Universe (and sub-Univeverses) as modes of a formally classical spinor field in WCW. WCW gamma matrices are expressible in terms of oscillator operators of free second quantized spinor fields of H. The induced spinor fields identified projections of H spinor fields to the space-time surfaces satisfy modified Dirac equation for the modified Dirac equation. Only quantum jump remains the genuinely quantal aspect of quantum physics.

- Quantum TGD can be seen as a theory for free spinor fields in WCW having maximal isometries and the generalization of the Super Virasoro conditions gives rise to the analog massless Dirac equation at the level of WCW.

The world of classical worlds and its symmetries

The notion of "World of Classical Worlds" (WCW) emerged around 1985 but found its basic form around 1990. Holography forced by the realization of General Coordinate Invariance forced/allowed to give up the attempts to make sense of the path integral.

A more concrete way to express this view is that WCW does not consist of 3-surfaces as particle-like entities but almost deterministic Bohr orbits assignable to them as preferred extremals of Kähler action so that quantum TGD becomes wave mechanics in WCW combined with Bohr orbitology. This view has profound implications, which can be formulated in terms of zero energy ontology (ZEO), solving among other things the basic paradox of quantum measurement theory. ZEO forms also the backbone of TGD inspired theory of consciousness and quantum biology.

WCW geometry exists only if it has maximal isometries: this statement is a generalization of the discovery of Freed for loop space geometries [A30]. I have proposed [K43, K26, K111, K84, L50] that WCW could be regarded as a union of generalized symmetric spaces labelled by zero modes which do not contribute to the metric. The induced Kähler field is invariant under symplectic transformations of CP_2 and would therefore define zero mode degrees of freedom if one assumes that WCW metric has symplectic transformations as isometries. In particular, Kähler magnetic fluxes would define zero modes and are quantized closed 2-surfaces. The induced metric appearing in Kähler action is however not zero mode degree of freedom. If the action contains volume term, the assumption about union of symmetric spaces is not well-motivated.

Symplectic transformations are not the only candidates for the isometries of WCW. The basic picture about what these maximal isometries could be, is partially inspired by string models.

- A weaker proposal is that the symplectomorphisms of H define only symplectomorphisms of WCW. Extended conformal symmetries define also a candidate for isometry group. Remarkably, light-like boundary has an infinite-dimensional group of isometries which are in 1-1 correspondence with conformal symmetries of $S^2 \subset S^2 \times R_+ = \delta M^4_+$.

- Extended Kac Moody symmetries induced by isometries of δM^4_+ are also natural candidates for isometries. The motivation for the proposal comes from physical intuition deriving from string models. Note they do not include Poincare symmetries, which act naturally as isometries in the moduli space of causal diamonds (CDs) forming the "spine" of WCW.

- The light-like orbits of partonic 2-surfaces might allow separate symmetry algebras. One must however notice that there is exchange of charges between interior degrees of freedom and partonic 2-surfaces. The essential point is that one can assign to these surface conserved charges when the dual light-like coordinate defines time coordinate. This picture also assumes a slicing of space-time surface by by the partonic orbits for which partonic orbits associated with wormrhole throats and boundaries of the space-time surface would be special. This slicing would correspond to Hamilton-Jacobi structure.

- Fractal hierarchy of symmetry algebras with conformal weights, which are non-negative integer multiples of fundamental conformal weights, is essential and distinguishes TGD from string models. Gauge conditions are true only the isomorphic subalgebra and its commutator with the entire algebra and the maximal gauge symmetry to a dynamical symmetry with generators having conformal weights below maximal value. This view also conforms with p-adic mass calculations.

- The realization of the symmetries for 3-surfaces at the boundaries of CD and for light-like orbits of partonic 2-surfaces is known. The problem is how to extend the symmetries to the interior of the space-time surface. It is natural to expect that the symmetries at partonic orbits and light-cone boundary extend to the same symmetries.

After the developments towards the end of 2023, it seems that the extension of conformal and Kac-Moody symmetries of string models to the TGD framework is understood. What about symplectic symmetries, which were originally proposed as isometries of WCW? In this article this question is discussed in detail and it will be found that these symmetries act naturally on 3-D holographic data and one can identify conserved charges. By holography this is in principle enough and might imply that the actions of holomorphic and symplectic symmetry algebras are dual. Holography=holomorphy hypothesis is discussed also in the case of the modified Dirac equation.

About the construction of scattering amplitudes

From the point of view of particle physics the ultimate goal is of course a practical construction recipe for the S-matrix of the theory. I have myself regarded this dream as quite too ambitious taking into account how far-reaching re-structuring and generalization of the basic mathematical structure of quantum physics is required. After having made several guesses for what the counterpart of S-matrix could be, it became clear that the dream about explicit formulas is unrealistic before one has understood what happens in quantum jump.

- In ZEO [K114, L34] one must distinguish between "small" state function reductions (SSFRs) and "big" SFRs (BSFRs). BSFR is the TGD counterpart of the ordinary SFRs and the arrow of the geometric time changes in it. SSFR follows the counterpart of a unitary time evolution and the arrow of the geometric time is preserved in SSFR. The sequence of SSFRs is the TGD counterpart for the sequence of repeated quantum measurements of the same observables in which nothing happens to the state. In TGD something happens in SSFRs and this gives rise to the flow of consciousness. When the set of the observables measured in SSFR does not commute with the previous set of measured observables, BSFR occurs.

The evolution by SSFRs means that also the causal diamond changes. At quantum level one has a wave function in the finite-dimensional moduli space of CDs which can be said to form a spine of WCW [L48]. CDs form a scale hierarchy. SSFRs are preceded by a dispersion in the moduli space of CDs and SSFR means localization in this space.

- There are several S-matrix like entities. One can assign an analog of the S-matrix to each analog of unitary time evolution preceding a given SSFR. One can also assign an analog S-matrix between the eigenstate basis of the previous set of observables and the eigenstate basis of new observers: this S-matrix characterizes BSFR. One can also assign to zero energy states an S-matrix like entity between the states assignable to the two boundaries of CD. These S-matrix like objects can be interpreted as a complex square root of the density matrix representable as a diagonal and positive square root of density matrix and unitary S-matrix so that quantum theory in ZEO can be said to define a square root of thermodynamics at least formally.

In standard QFTs Feynman diagrams provide the description of scattering amplitudes. The beauty of Feynman diagrams is that they realize unitarity automatically via the so-called Cutkosky rules. In contrast to Feynman's original beliefs, Feynman diagrams and virtual particles are taken only as a convenient mathematical tool in quantum field theories. The QFT approach is however plagued by UV and IR divergences and one must keep mind open for the possibility that a genuine progress might mean opening of the black box of the virtual particle.

In the TGD framework this generalization of Feynman diagrams indeed emerges unavoidably.

- The counterparts of elementary particles can be identified as closed monopole flux tubes connecting two parallel Minkowskian space-time sheets and have effective ends which are Euclidean wormhole contacts. The 3-D light-like boundaries of wormhole contacts as orbits of partonic 2-surfaces.

  The intuitive picture is that the 3-D light-like partonic orbits replace the lines of Feynman diagrams and vertices are replaced by 2-D partonic 2-surfaces. A stronger condition is that fermion number is carried by light-like fermion lines at the partonic orbits, which can be identified as boundaries string world sheets.

- The localization of the nodes of induced spinor fields to 2-D string world sheets (and possibly also to partonic 2-surfaces) implies a stringy formulation of the theory analogous to stringy variant of twistor formalism with string world sheets having interpretation as 2-braids. In the TGD framework, the fermionic variant of twistor Grassmann formalism combined with the number theoretic vision [L40, L41] led to a stringy variant of the twistor diagrammatics.

- Fundamental fermions are off-mass-shell in the sense that their momentum components are real algebraic integers in an extension of rationals associated with the space-time surfaces inside CD with a momentum unit determined by the CD size scale. Galois confinement states that the momentum components are integer valued for the physical states.

- The twistorial approach suggests also the generalization of the Yangian symmetry to infinite-dimensional super-conformal algebras, which would determine the vertices and scattering amplitudes in terms of poly-local symmetries.

The twistorial approach is however extremely abstract and lacks a concrete physical interpretation. The holography=holomorphy vision led to a breakthough in the construction of the scattering amplitudes by solving the problem of identifying interaction vertices [L52].

1. The basic prediction is that space-time surfaces as analogs of Bohr orbits are holomorphic in a generalized sense and are therefore minimal surfaces. The minimal surface property fails at lower-dimensional singularities and the trace of the second fundamental form (SFF) analogous to acceleration associated with the Bohr orbit of the particle as 3-surface has a delta function like singularity but vanishes elsewhere.

2. The minimal surface property expressess masslessness for both fields and particles as 3-surfaces. At singularities masslessness property fails and singularities can be said to serve as sources which also in QFT define scattering amplitudes.

3. The singularities are analogs of poles and cuts for the 4-D generalization of the ordinary holomorphic functions. Also for the ordinary holomorphic functions the Laplace equation as analog massless field equation and expressing analyticity fails. Complex analysis generalizes to dimension 4.

4. The conditions at the singularity give a generalization of Newton's "F=ma"! I ended up where I started more than 50 years ago!

5. In dimension 4, and only there, there is an infinite number of exotic diff structures [**?**], which differ from ordinary ones at singularities of measure zero analogous to defects. These defects correspond naturally to the singularities of minimal surfaces. One can say that for the exotic diff structure there is no singularity.

6. Group theoretically the trace of the SFF can be regarded as a generalization of the Higgs field, which is non-vanishing only at the vertices and this is enough. Singularities take the role of generalized particle vertices and determine the scattering amplitudes. The second fundamental form contracted with the embedding space gamma matrices and slashed between the second quantized induced spinor field and its conjugate gives the universal vertex involving only fermions (bosons are bound states of fermions in TGD). It contains both gauge and gravitational contributions to the scattering amplitudes and there is a complete symmetry between gravitational and gauge interactions. Gravitational couplings come out correctly as the radius squared of CP_2 as also in the classical picture.

7. The study of the modified Dirac equation leads to the conclusion that vertices as singularities and defects contain the standard electroweak gauge contribution coming from the induced spinor connection and a contribution from the M^4 spinor connection. M^4 part of the generalized Higgs can give rise to a graviton as an $L = 1$ rotational state of the flux tube representing the graviton. It is not clear whether M^4 Kähler gauge potential can give rise to a spin 1 particle. The vielbein part of M^4 spinor connection is pure gauge and could give rise to gravitational topological field theory.

Figures

What I have said above is strongly biased view about the recent situation in quantum TGD. This vision is single man's view and doomed to contain unrealistic elements as I know from experience. My dream is that young critical readers could take this vision seriously enough to try to demonstrate that some of its basic premises are wrong or to develop an alternative based on these or better premises. I must be however honest and tell that 45 years of TGD is a really vast bundle of thoughts and quite a challenge for anyone who is not able to cheat himself by taking the attitude of a blind believer or a light-hearted debunker trusting on the power of easy rhetoric tricks. The books provide a view of how TGD evolved rather than the final theory and there are archeological layers containing mammoth bones, which reflect earlier views not necessarily consistent with the recent view.

Karkkila, April 21, 2024, Finland

Matti Pitkänen

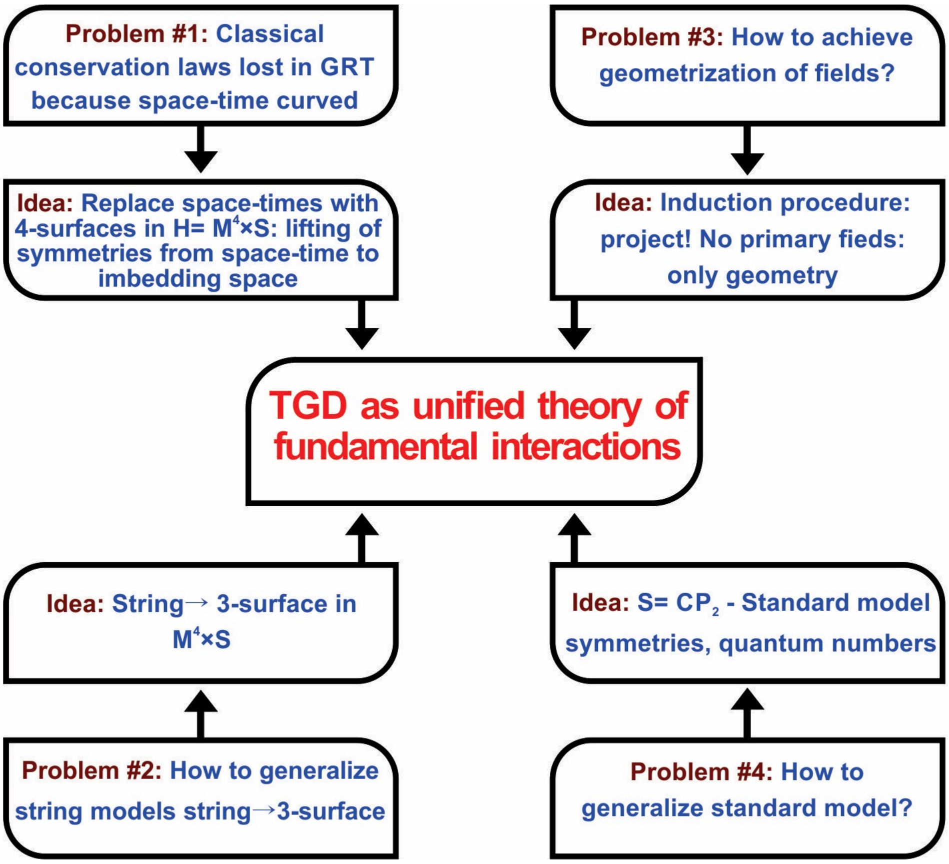

Figure 1: The problems leading to TGD as their solution.

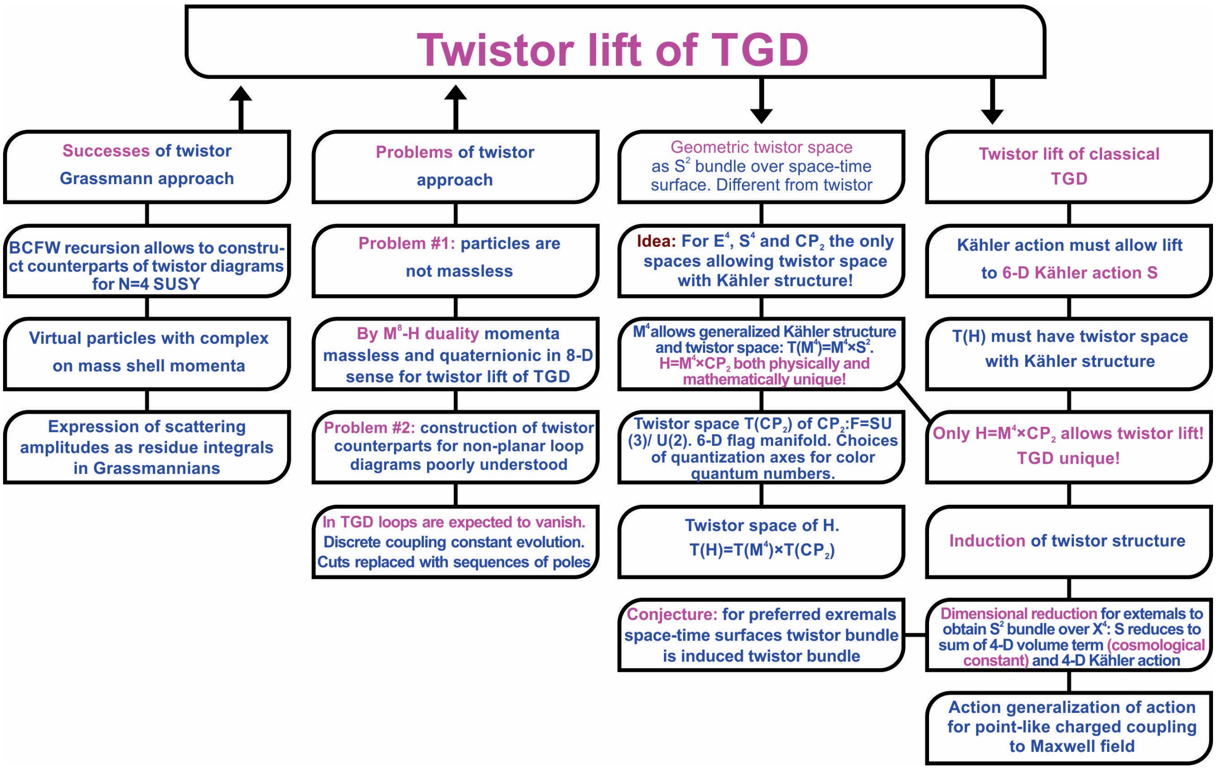

Figure 2: Twistor lift

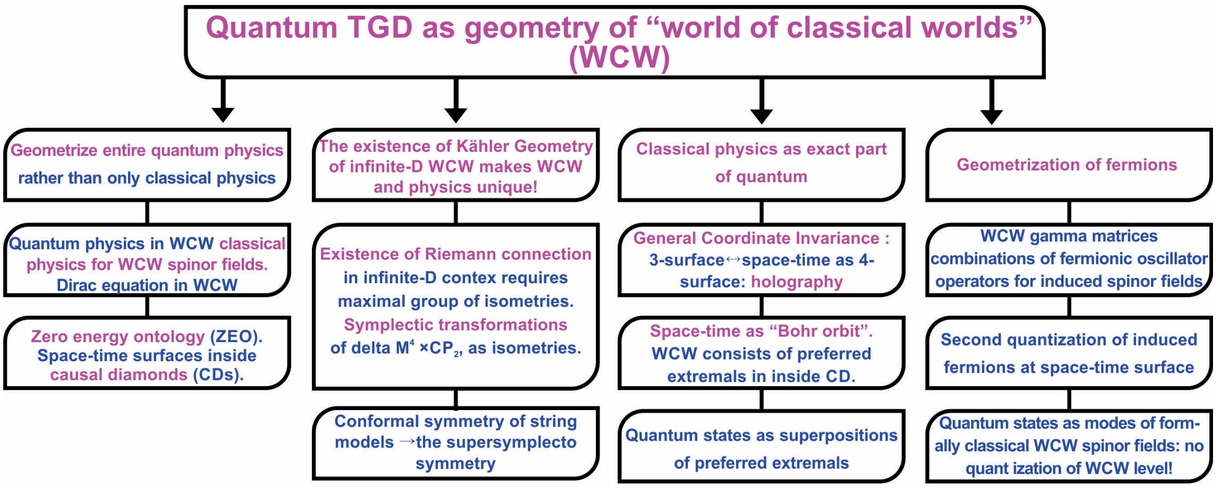

Figure 3: Geometrization of quantum physics in terms of WCW

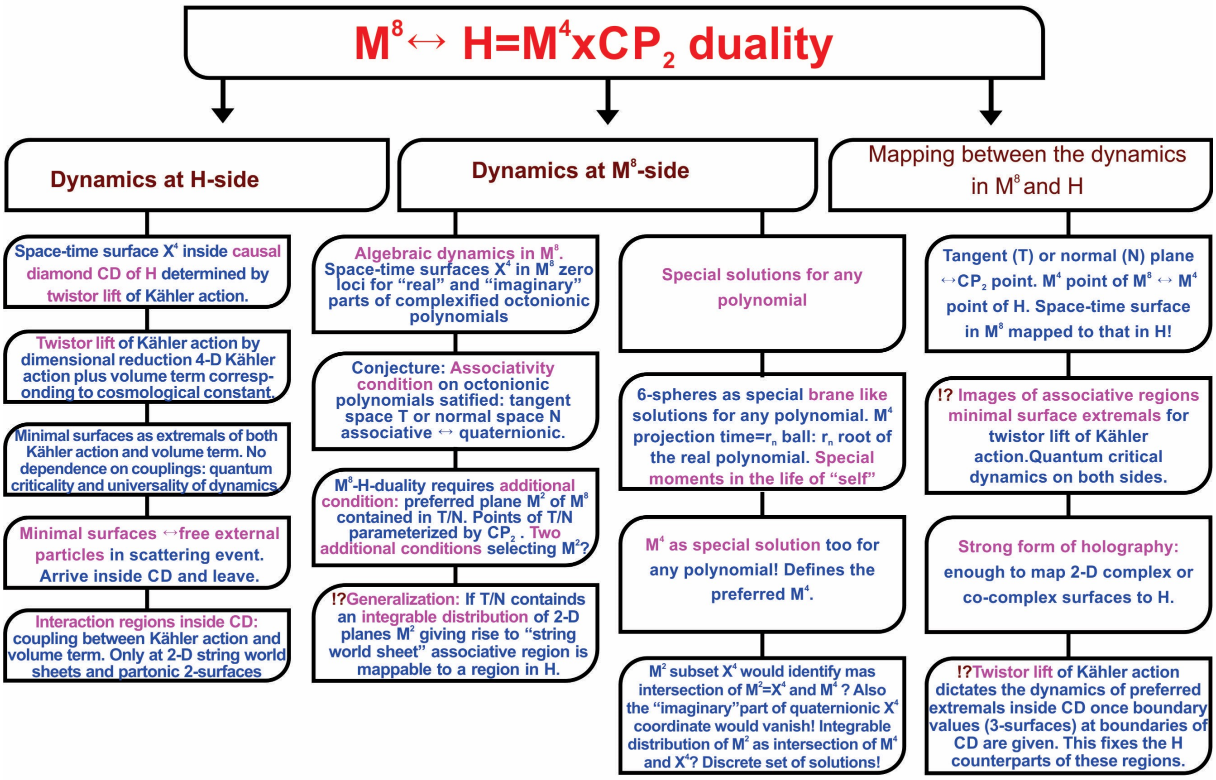

Figure 4: $M^8 - H$ duality

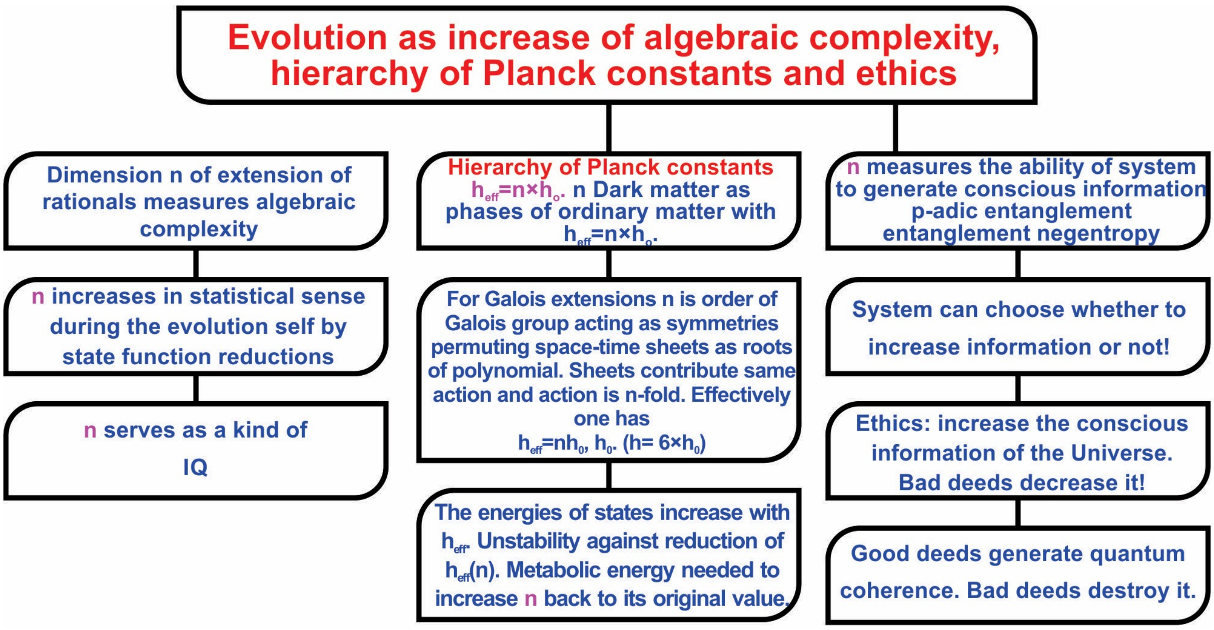

Figure 5: Number theoretic view of evolution

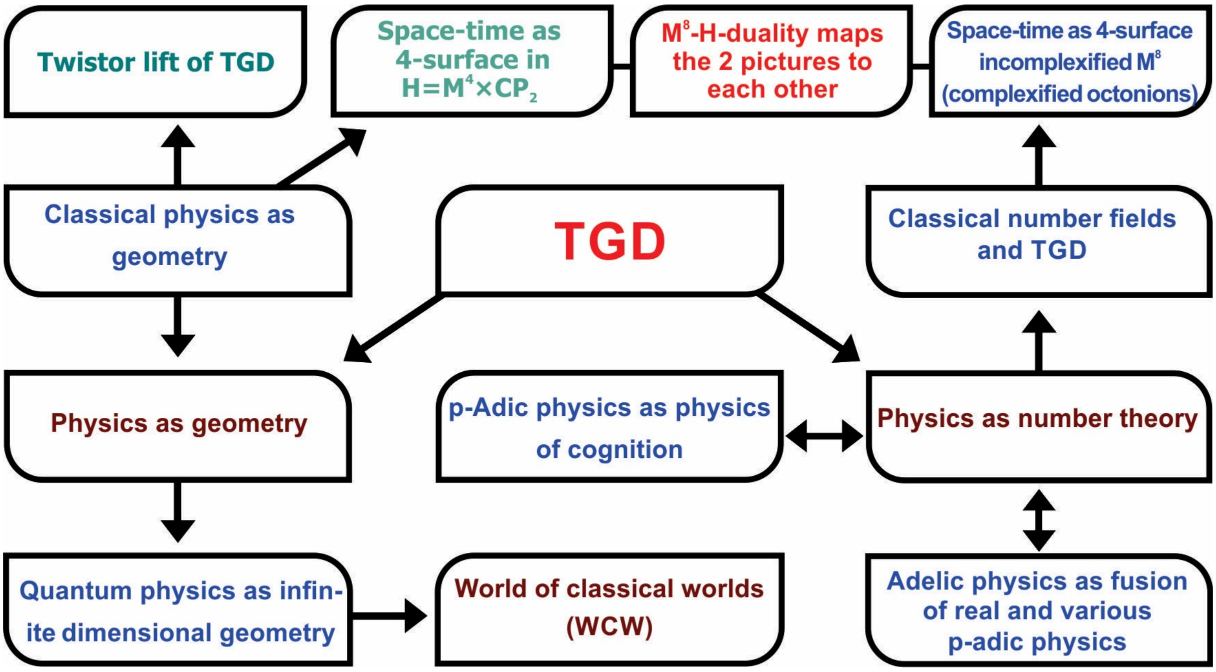

Figure 6: TGD is based on two complementary visions: physics as geometry and physics as number theory.

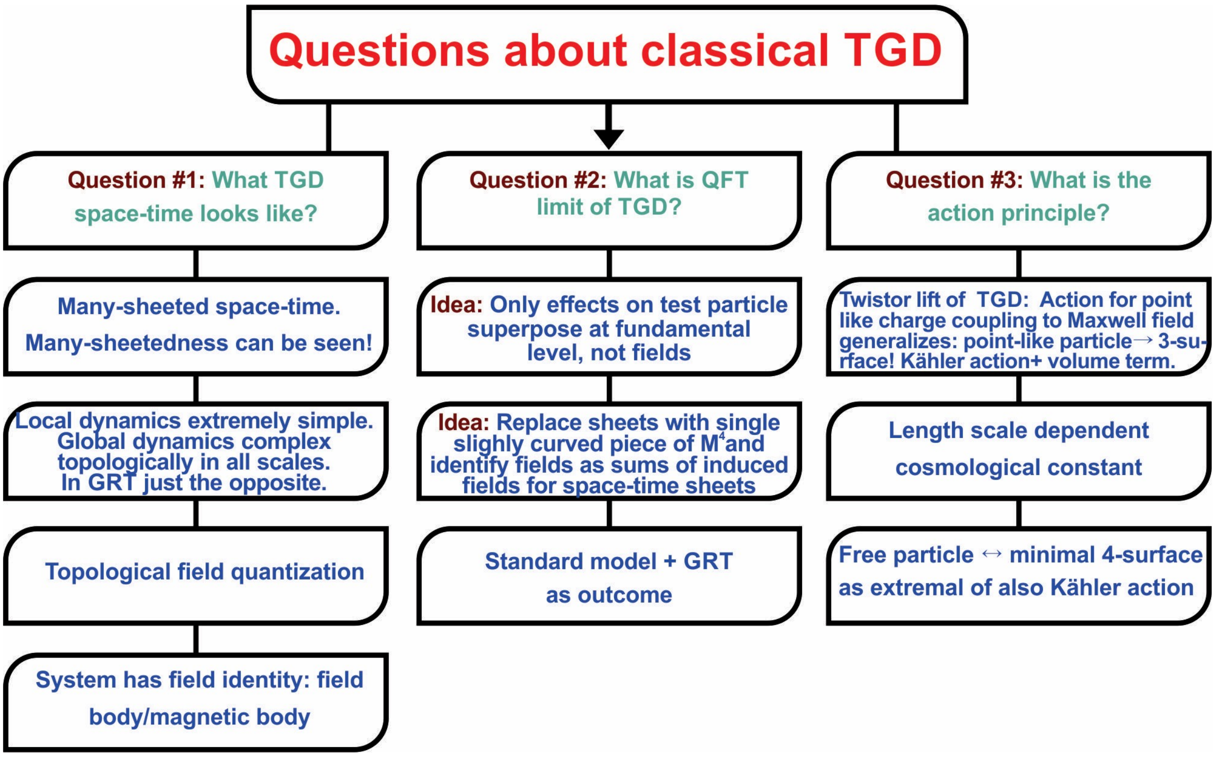

Figure 7: Questions about classical TGD.

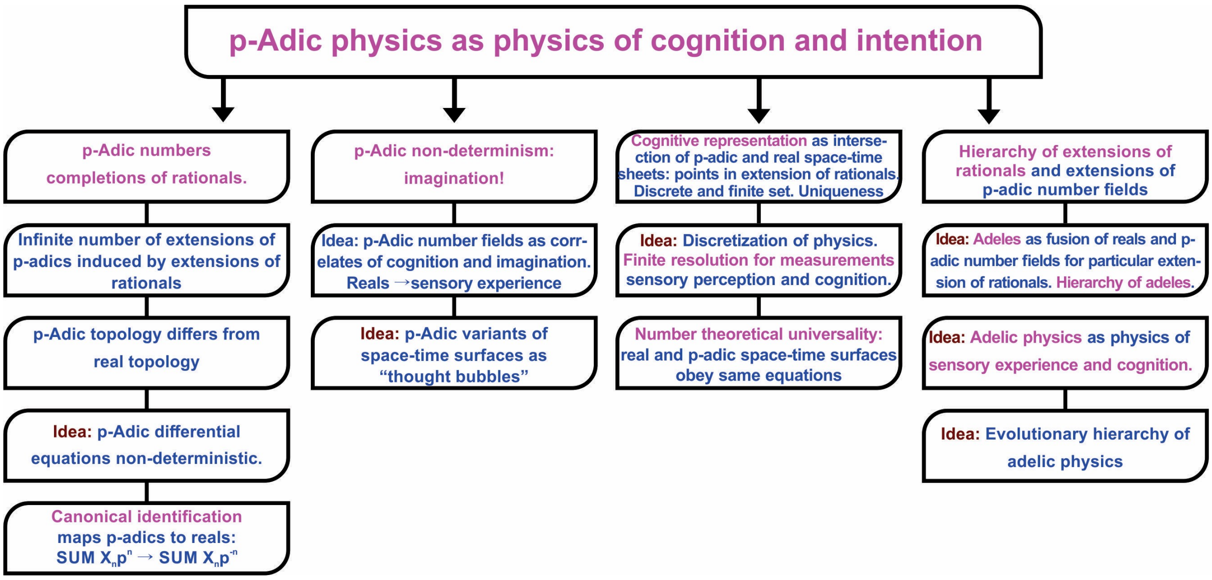

Figure 8: p-Adic physics as physics of cognition and imagination.

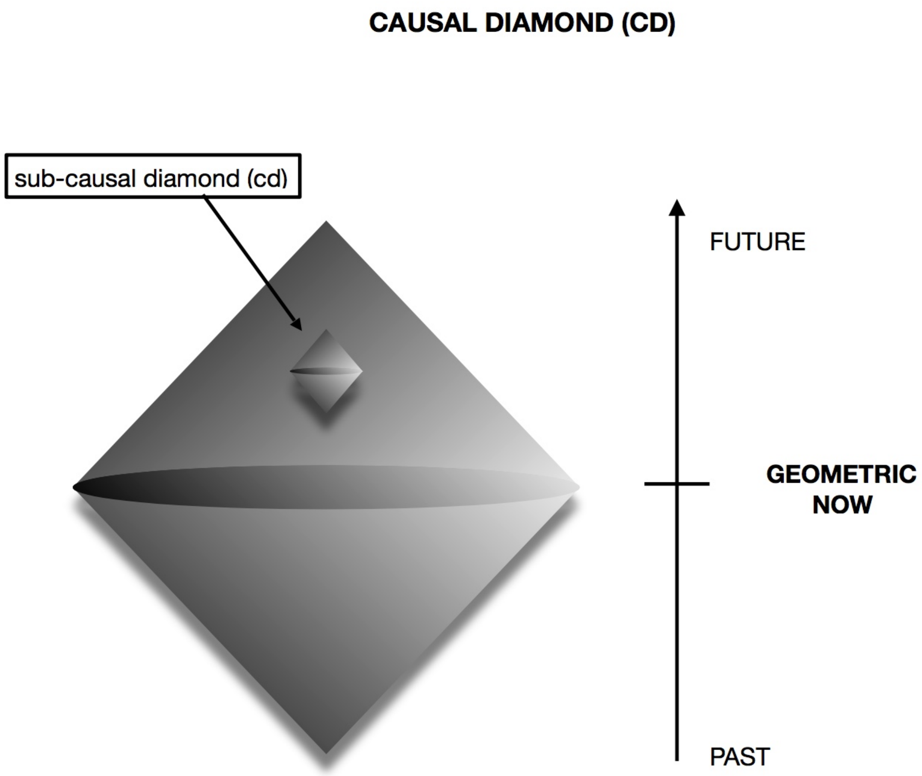

Figure 9: Causal diamond

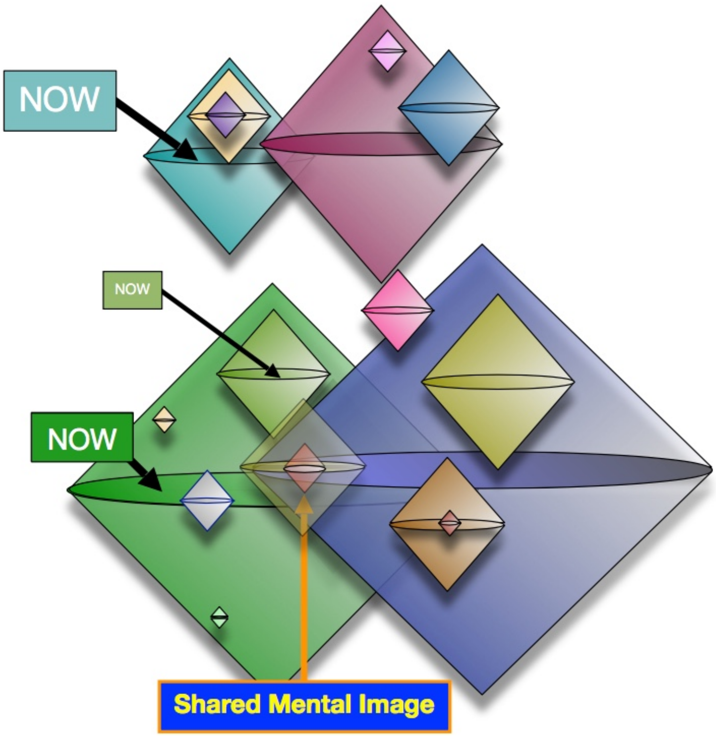

Figure 10: CDs define a fractal "conscious atlas"

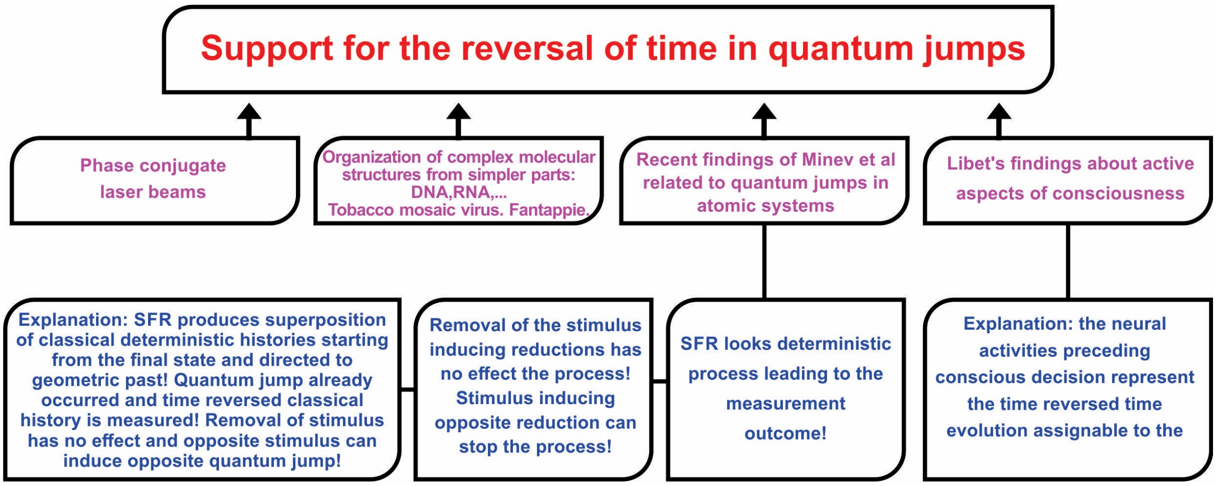

Figure 11: Time reversal occurs in BSFR

ACKNOWLEDGEMENTS

Neither TGD nor these books would exist without the help and encouragement of many people. The friendship with Heikki and Raija Haila and their family and Kalevi and Ritva Tikkanen and their family have been kept me in contact with the everyday world and without this friendship I would not have survived through these lonely 45 lonely years most of which I have remained unemployed as a scientific dissident. I am happy that my children have understood my difficult position and like my friends have believed that what I am doing is something valuable although I have not received any official recognition for it.

During the last decade Tapio Tammi has helped me quite concretely by providing the necessary computer facilities and being one of the few persons in Finland with whom to discuss my work. Pertti Kärkkäinen is my old physicist friend and has provided continued economic support for a long time. I have also had stimulating discussions with Samuli Penttinen who has also helped to get through the economical situations in which there seemed to be no hope. The continual updating of fifteen online books means quite a heavy bureaucracy at the level of bits and without a systemization one ends up with endless copying and pasting and internal consistency is soon lost. Tommi Ullgren has provided both economic support and encouragement during years. Pekka Rapinoja has offered his help in this respect and I am especially grateful to him for my Python skills.

During the last five years I have had inspiring discussions with many people in Finland interested in TGD. We have had video discussions with Sini Kunnas and had podcast discussions with Marko Manninen related to the TGD based view of physics and consciousness. Marko has also helped in the practical issues related to computers and quite recently he has done a lot of testing of chatGPT helping me to get an overall view of what it is. The discussions in a Zoom group involving Marko Manninen, Tuomas Sorakivi and Rode Majakka have given me the valuable opportunity to clarify my thoughts.

The collaboration with Lian Sidorov was extremely fruitful and she also helped me to survive economically through the hardest years. The participation in CASYS conferences in Liege has been an important window to the academic world and I am grateful for Daniel Dubois and Peter Marcer for making this participation possible. The discussions and collaboration with Eduardo de Luna and Istvan Dienes stimulated the hope that the communication of new vision might not be a mission impossible after all. Also blog discussions have been very useful. During these years I have received innumerable email contacts from people around the world. I am grateful to Mark McWilliams, Paul Kirsch, Gary Ehlenberg, and Ulla Matfolk and many others for providing links to possibly interesting websites and articles. We have collaborated with Peter Gariaev and Reza Rastmanesh. These contacts have helped me to avoid the depressive feeling of being some kind of Don Quixote of Science and helped me to widen my views: I am grateful for all these people.

In the situation in which the conventional scientific communication channels are strictly closed it is important to have some loop hole through which the information about the work done can at least in principle leak to the public through the iron wall of academic censorship. Without any exaggeration I can say that without the world wide web I would not have survived as a scientist nor as an individual. Homepage and blog are however not enough since only the formally published result is a result in recent day science. Publishing is however impossible without direct support from power holders- even in archives like arXiv.org.

Situation changed as Andrew Adamatsky proposed the writing of a book about TGD when I had already gotten used to the thought that my work would not be published during my lifetime. The Prespacetime Journal and two other journals related to quantum biology and consciousness - all of them founded by Huping Hu - have provided this kind of loophole. In particular, Dainis Zeps, Phil Gibbs, and Arkadiusz Jadczyk deserve my gratitude for their kind help in the preparation of an article series about TGD catalyzing a considerable progress in the understanding of quantum TGD. Also the viXra archive founded by Phil Gibbs and its predecessor Archive Freedom have been of great help: Victor Christianto deserves special thanks for doing the hard work needed to run Archive Freedom. Also the Neuroquantology Journal founded by Sultan Tarlaci deserves a special mention for its publication policy.

And last but not least: there are people who experience as a fascinating intellectual challenge to spoil the practical working conditions of a person working with something which might be called unified theory: I am grateful for the people who have helped me to survive through the virus attacks, an activity which has taken roughly one month per year during the last half decade and given a strong hue of grey to my hair.

For a person approaching his 73th birthday it is somewhat easier to overcome the hard feelings due to the loss of academic human rights than for an inpatient youngster. Unfortunately the economic situation has become increasingly difficult during the twenty years after the economic depression in Finland which in practice meant that Finland ceased to be a constitutional state in the strong sense of the word. It became possible to depose people like me from society without fear about public reactions and the classification as dropout became a convenient tool of ridicule to circumvent the ethical issues. During the period when the right wing held political power this trend was steadily strengthening and the situation is the same as I am writing this. In this kind of situation the concrete help from individuals has been and will be of utmost importance. Against this background it becomes obvious that this kind of work is not possible without the support from outside and I apologize for not being able to mention all the people who have helped me during these years.

Karkkila, August 30, 2023, Finland

Matti Pitkänen

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Chapter 1

Introduction

1.1 Basic Ideas of Topological Geometrodynamics (TGD)

Standard model describes rather successfully both electroweak and strong interactions but sees them as totally separate and contains a large number of parameters which it is not able to predict. For about four decades ago unified theories known as Grand Unified Theories (GUTs) trying to understand electroweak interactions and strong interactions as aspects of the same fundamental gauge interaction assignable to a larger symmetry group emerged. Later superstring models trying to unify even gravitation and strong and weak interactions emerged. The shortcomings of both GUTs and superstring models are now well-known. If TGD - whose basic idea emerged towards the end of 1977 - would emerge now it would be seen as an attempt to solve the difficulties of these approaches to unification.

The basic physical picture behind the geometric vision of TGD corresponds to a fusion of two rather disparate approaches: namely TGD as a Poincare invariant theory of gravitation and TGD as a generalization of the old-fashioned string model. After 1995 number theoretic vision started to develop and was initiated by the success of mass calculations based on p-adic thermodynamics. Number theoretic vision involves all number fields and is complementary to the geometric vision: one can say that this duality is analogous to momentum-position duality of wave mechanics. TGD can be also regarded as topological quantum theory in a very general sense as already the attribute "Topological" in "TGD" makes clear. Space-time surfaces as minimal surfaces can be regarded as representatives of homology equivalence classes and p-adic topologies generalize the notion of local topology and apply to the description of correlates of cognition.

1.1.1 Geometric Vision Very Briefly

T(opological) G(eometro)D(ynamics) is one of the many attempts to find a unified description of basic interactions. The development of the basic ideas of TGD to a relatively stable form took time of about half decade [K2].

The basic vision and its relationship to existing theories is now rather well understood.

1. Space-times are representable as 4-surfaces in the 8-dimensional embedding space $H = M^4 \times CP_2$, where M^4 is 4-dimensional (4-D) Minkowski space and CP_2 is 4-D complex projective space (see Appendix).

2. Induction procedure (a standard procedure in fiber bundle theory, see Appendix) allows to geometrize various fields. Space-time metric characterizing gravitational fields corresponds to the induced metric obtained by projecting the metric tensor of H to the space-time surface. Electroweak gauge potentials are identified as projections of the components of CP_2 spinor connection to the space-time surface, and color gauge potentials as projections of CP_2 Killing vector fields representing color symmetries. Also spinor structure can be induced: induced spinor gamma matrices are projections of gamma matrices of H and induced spinor fields just H spinor fields restricted to space-time surface. Spinor connection is also projected. The interpretation is that distances are measured in embedding space metric and parallel translation using spinor connection of embedding space.

Twistor lift of TGD means that one can lift space-time surfaces in H to 6-D surfaces a analogs of twistor space of space-time surface in the Cartesian product of the twistor spaces of M^4 and CP_2, which are the only 4-manifolds allowing twistor space with Kähler structure [A45]. The twistor structure would be induced in some sense, and should coincide with that associated with the induced metric. Clearly, the 2-spheres defining the fibers of twistor spaces of M^4 and CP_2 must allow identification: this 2-sphere defines the S^2 fiber of the twistor space of the space-time surface. This poses a constraint on the embedding of the twistor space of space-time surfaces as sub-manifold in the Cartesian product of twistor spaces. The existence of Kähler structure allows to lift 4-D Kähler action to its 6-D counterparts and the 6-D counterpart of twistor space is obtained by its dimensional reduction so that one obtains a sphere bundle. This makes possible twistorialization for all space-time surfaces: in general relativity the general metric does not allow this.

3. A geometrization of quantum numbers is achieved. The isometry group of the geometry of CP_2 codes for the color gauge symmetries of strong interactions. Vierbein group codes for electroweak symmetries, and explains their breaking in terms of CP_2 geometry so that standard model gauge group results. There are also important deviations from the standard model: color quantum numbers are not spin-like but analogous to orbital angular momentum: this difference is expected to be seen only in CP_2 scale. In contrast to GUTs, quark and lepton numbers are separately conserved and family replication has a topological explanation in terms of topology of the partonic 2-surface carrying fermionic quantum numbers.

   M^4 and CP_2 are unique choices for many other reasons. For instance, they are the unique 4-D space-times allowing twistor space with Kähler structure. M^4 light-cone boundary allows a huge extension of 2-D conformal symmetries. M^4 and CP_2 allow quaternionic structures. Therefore standard model symmetries have number theoretic meaning.

4. Induced gauge potentials are expressible in terms of embedding space coordinates and their gradients and general coordinate invariance implies that there are only 4 field-like variables locally. Situation is thus extremely simple mathematically. The objection is that one loses linear superposition of fields. The resolution of the problem comes from the generalization of the concepts of particle and space-time.

   Space-time surfaces can be also particle like having thus finite size. In particular, space-time regions with Euclidian signature of the induced metric (temporal and spatial dimensions in the same role) emerge and have interpretation as lines of generalized Feynman diagrams. Particles in space-time can be identified as a topological inhomogeneities in background space-time surface which looks like the space-time of general relativity in long length scales.

   One ends up with a generalization of space-time surface to many-sheeted space-time with space-time sheets having extremely small distances of about 10^4 Planck lengths (CP_2 size). As one adds a particle to this kind of structure, it touches various space-time sheets and thus interacts with the associated classical fields. Their effects superpose linearly in good approximation and linear superposition of fields is replaced with that for their effects.

   This resolves the basic objection. It also leads to the understanding of how the space-time of general relativity and quantum field theories emerges from TGD space-time as effective space-time when the sheets of many-sheeted space-time are lumped together to form a region of Minkowski space with metric replaced with a metric identified as the sum of empty Minkowski metric and deviations of the metrics of sheets from empty Minkowski metric. Gauge potentials are identified as sums of the induced gauge potentials. TGD is therefore a microscopic theory from which the standard model and general relativity follow as a topological simplification, however forcing a dramatic increase of the number of fundamental field variables.

5. A further objection is that classical weak fields identified as induced gauge fields are long ranged and should cause large parity breaking effects due to weak interactions. These effects are indeed observed but only in living matter. The basic problem is that one has long ranged classical electroweak gauge fields. The resolution of the problem is that the quantum averages of induced weak and color gauge fields vanish due to the fact that color rotations affect both space-time surfaces and induced weak and color fields. Only the averages of

electromagnetic fields are nonvanishing. The correlations functions for weak fields are non-vanishing below Compton lengths of weak bosons. In living matter large values of effective Planck constant labelling phases of ordinary matter identified as dark matter make possible long ranged weak fields and color fields.

6. General coordinate invariance requires holography so that space-time surfaces are analogous to Bohr orbits for particles identified as 3-surfaces. Bohr orbit property would be naturally realized by a 4-D generalization of holomorphy of string world sheets and implies that the space-time surfaces are minimal surfaces apart from singularities. This holds true for any action as long as it is general coordinate invariant and constructible in terms of the induced geometry. String world sheets and light-like orbits of partonic 2-surfaces correspond to singularities at which the minimal surface property of the space-time surfaces realizing the preferred extremal property fails. Preferred extremals are not completely deterministic, which implies what I call zero energy ontology (ZEO) meaning that the Bohr orbits are the fundamental objects. This leads to a solution of the basic paradox of quantum measurement theory. Also the mathematically ill-defined path integral disappears and leaves only the well-defined functional integral over the Bohr orbits.

7. A string model-like picture emerges from TGD and one ends up with a rather concrete view about the topological counterpart of Feynman diagrammatics. The natural stringy action would be given by the string world sheet area, which is present only in the space-time regions with Minkowskian signature. Gravitational constant could be present as a fundamental constant in string action and the ratio $\hbar/G/R^2$ would be determined by quantum criticality conditions. The hierarchy of Planck constants $h_{eff}/h = n$ assigned to dark matter in TGD framework would allow to circumvent the objection that only objects of length of order Planck length are possible since string tension given by $T = 1/\hbar_{eff}G$ apart from numerical factor could be arbitrary small. This would make possible gravitational bound states as partonic 2-surfaces as structures connected by strings and solve the basic problem of superstring theories. This option allows the natural interpretation of M^4 type vacuum extremals with CP_2 projection, which is Lagrange manifold as good approximations for space-time sheets at macroscopic length scales. String area does not contribute to the Kähler function at all.

   Whether induced spinor fields associated with Kähler-Dirac action and de-localized inside the entire space-time surface should be allowed remains an open question: super-conformal symmetry strongly suggests their presence. A possible interpretation for the corresponding spinor modes could be in terms of dark matter, sparticles, and hierarchy of Planck constants.

It is perhaps useful to make clear what TGD is not and also what new TGD can give to physics.

1. TGD is *not* just General Relativity made concrete by using embeddings: the 4-surface property is absolutely essential for unifying standard model physics with gravitation and to circumvent the incurable conceptual problems of General Relativity. The many-sheeted space-time of TGD gives rise only at the macroscopic limit to GRT space-time as a slightly curved Minkowski space. TGD is *not* a Kaluza-Klein theory although color gauge potentials are analogous to gauge potentials in these theories.

   TGD space-time is 4-D and its dimension is due to completely unique conformal properties of light-cone boundary and 3-D light-like surfaces implying enormous extension of the ordinary conformal symmetries. Light-like 3-surfaces represent orbits of partonic 2-surfaces and carry fundamental fermions at 1-D boundaries of string world sheets. TGD is *not* obtained by performing Poincare gauging of space-time to introduce gravitation and is plagued by profound conceptual problems.

2. TGD is *not* a particular string model although string world sheets emerge in TGD very naturally as loci for spinor modes: their 2-dimensionality makes among other things possible quantum deformation of quantization known to be physically realized in condensed matter, and conjectured in TGD framework to be crucial for understanding the notion of finite measurement resolution. Hierarchy of objects of dimension up to 4 emerge from TGD: this obviously means analogy with branes of super-string models.

TGD is *not* one more item in the collection of string models of quantum gravitation relying on Planck length mystics. Dark matter becomes an essential element of quantum gravitation and quantum coherence in astrophysical scales is predicted just from the assumption that strings connecting partonic 2-surfaces are responsible for gravitational bound states.

TGD is *not* a particular string model although AdS/CFT duality of super-string models generalizes due to the huge extension of conformal symmetries and by the identification of WCW gamma matrices as Noether super-charges of super-symplectic algebra having a natural conformal structure.

3. TGD is *not* a gauge theory. In TGD framework the counterparts of also ordinary gauge symmetries are assigned to super-symplectic algebra (and its Yangian [A17] [B18, B14, B15]), which is a generalization of Kac-Moody algebras rather than gauge algebra and suffers a fractal hierarchy of symmetry breakings defining hierarchy of criticalities. TGD is *not* one more quantum field theory like structure based on path integral formalism: path integral is replaced with functional integral over 3-surfaces, and the notion of classical space-time becomes an exact part of the theory. Quantum theory becomes formally a purely classical theory of WCW spinor fields: only state function reduction is something genuinely quantal.

4. TGD view about spinor fields is *not* the standard one. Spinor fields appear at three levels. Spinor modes of the embedding space are analogs of spinor modes characterizing incoming and outgoing states in quantum field theories. Induced second quantized spinor fields at space-time level are analogs of stringy spinor fields. Their modes are localized by the well-definedness of electro-magnetic charge and by number theoretic arguments at string world sheets. Kähler-Dirac action is fixed by supersymmetry implying that ordinary gamma matrices are replaced by what I call Kähler-Dirac gamma matrices - this something new. WCW spinor fields, which are classical in the sense that they are not second quantized, serve as analogs of fields of string field theory and imply a geometrization of quantum theory.

5. TGD is in some sense an extremely conservative geometrization of entire quantum physics: *no* additional structures such as gauge fields as independent dynamical degrees of freedom are introduced: Kähler geometry and associated spinor structure are enough. "Topological" in TGD should not be understood as an attempt to reduce physics to torsion (see for instance [B13]) or something similar. Rather, TGD space-time is topologically non-trivial in all scales and even the visible structures of the everyday world represent non-trivial topology of space-time in the TGD Universe.

6. Twistor space - or rather, a generalization of twistor approach replacing masslessness in 4-D sense with masslessness in 8-D sense and thus allowing description of also massive particles - emerged originally as a technical tool, and its Kähler structure is possible only for $H = M^4 \times CP_2$. It however turned out that much more than a technical tool is in question. What is genuinely new is the infinite-dimensional character of the Kähler geometry making it highly unique, and its generalization to p-adic number fields to describe correlates of cognition. Also the hierarchy of Planck constants $h_{eff} = n \times h$ reduces to the quantum criticality of the TGD Universe and p-adic length scales and Zero Energy Ontology represent something genuinely new.

The great challenge is to construct a mathematical theory around these physically very attractive ideas and I have devoted the last 45 years to the realization of this dream and this has resulted in 26 online books about TGD and nine online books about TGD inspired theory of consciousness and of quantum biology.

A collection of 30 online books is now (August 2023) under preparation. The goal is to minimize overlap between the topics of the books and make the focus of a given book sharper.

1.1.2 Two Visions About TGD as Geometrization of Physics and Their Fusion

As already mentioned, TGD as a geometrization of physics can be interpreted both as a modification of general relativity and generalization of string models.

TGD as a Poincare Invariant Theory of Gravitation

The first approach was born as an attempt to construct a Poincare invariant theory of gravitation. Space-time, rather than being an abstract manifold endowed with a pseudo-Riemannian structure, is regarded as a surface in the 8-dimensional space $H = M^4_\times CP_2$, where M^4 denotes Minkowski space and $CP_2 = SU(3)/U(2)$ is the complex projective space of two complex dimensions [A36, A44, A27, A43].

The identification of the space-time as a sub-manifold [A37, A56] of $M^4 \times CP_2$ leads to an exact Poincare invariance and solves the conceptual difficulties related to the definition of the energy-momentum in General Relativity.

It soon however turned out that sub-manifold geometry, being considerably richer in structure than the abstract manifold geometry, leads to a geometrization of all basic interactions. First, the geometrization of the elementary particle quantum numbers is achieved. The geometry of CP_2 explains electro-weak and color quantum numbers. The different H-chiralities of H-spinors correspond to the conserved baryon and lepton numbers. Secondly, the geometrization of the field concept results. The projections of the CP_2 spinor connection, Killing vector fields of CP_2 and of H-metric to four-surface define classical electro-weak, color gauge fields and metric in X^4.

The choice of H is unique from the condition that TGD has standard model symmetries. Also number theoretical vision selects $H = M^4 \times CP_2$ uniquely. M^4 and CP_2 are also unique spaces allowing twistor space with Kähler structure.

TGD as a Generalization of the Hadronic String Model

The second approach was based on the generalization of the mesonic string model describing mesons as strings with quarks attached to the ends of the string. In the 3-dimensional generalization 3-surfaces correspond to free particles and the boundaries of the 3- surface correspond to partons in the sense that the quantum numbers of the elementary particles reside on the boundaries. Various boundary topologies (number of handles) correspond to various fermion families so that one obtains an explanation for the known elementary particle quantum numbers. This approach leads also to a natural topological description of the particle reactions as topology changes: for instance, two-particle decay corresponds to a decay of a 3-surface to two disjoint 3-surfaces.

This decay vertex does not however correspond to a direct generalization of trouser vertex of string models. Indeed, the important difference between TGD and string models is that the analogs of string world sheet diagrams do not describe particle decays but the propagation of particles via different routes. Particle reactions are described by generalized Feynman diagrams for which 3-D light-like surface describing particle propagating join along their ends at vertices. As 4-manifolds the space-time surfaces are therefore singular like Feynman diagrams as 1-manifolds.

Quite recently, it has turned out that fermionic strings inside space-time surfaces define an exact part of quantum TGD and that this is essential for understanding gravitation in long length scales. Also the analog of AdS/CFT duality emerges in that the Kähler metric can be defined either in terms of Kähler function identifiable as Kähler action assignable to Euclidian space-time regions or Kähler action + string action assignable to Minkowskian regions.

The recent view about construction of scattering amplitudes is very "stringy". By strong form of holography string world sheets and partonic 2-surfaces provide the data needed to construct scattering amplitudes. Space-time surfaces are however needed to realize quantum-classical correspondence necessary to understand the classical correlates of quantum measurement. There is a huge generalization of the duality symmetry of hadronic string models.

The proposal is that scattering amplitudes can be regarded as sequences of computational operations for the Yangian of super-symplectic algebra. Product and co-product define the basic vertices and realized geometrically as partonic 2-surfaces and algebraically as multiplication for the elements of Yangian identified as super-symplectic Noether charges assignable to strings. Any computational sequences connecting given collections of algebraic objects at the opposite boundaries of causal diamond (CD) produce identical scattering amplitudes.

Fusion of the Two Approaches via a Generalization of the Space-Time Concept

The problem is that the two approaches to TGD seem to be mutually exclusive since the orbit of a particle like 3-surface defines 4-dimensional surface, which differs drastically from the topologically trivial macroscopic space-time of General Relativity. The unification of these approaches forces a considerable generalization of the conventional space-time concept. First, the topologically trivial 3-space of General Relativity is replaced with a "topological condensate" containing matter as particle like 3-surfaces "glued" to the topologically trivial background 3-space by connected sum operation. Secondly, the assumption about connectedness of the 3-space is given up. Besides the "topological condensate" there could be "vapor phase" that is a "gas" of particle like 3-surfaces and string like objects (counterpart of the "baby universes" of GRT) and the non-conservation of energy in GRT corresponds to the transfer of energy between different sheets of the space-time and possible existence vapour phase.

. What one obtains is what I have christened as many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig. ??** in the appendix of this book). One particular aspect is topological field quantization meaning that various classical fields assignable to a physical system correspond to space-time sheets representing the classical fields to that particular system. One can speak of the field body of a particular physical system. Field body consists of topological light rays, and electric and magnetic flux quanta. In Maxwell's theory the physical system does not possess this kind of field identity. The notion of the magnetic body is one of the key players in TGD inspired theory of consciousness and quantum biology. The existence of monopole flux tubes requiring no current as a source of the magnetic field makes it possible to understand the existence of magnetic fields in cosmological and astrophysical scales.

This picture became more detailed with the advent of zero energy ontology (ZEO). The basic notion of ZEO is causal diamond (CD) identified as the Cartesian product of CP_2 and of the intersection of future and past directed light-cones and having scale coming as an integer multiple of CP_2 size is fundamental. CDs form a fractal hierarchy and zero energy states decompose to products of positive and negative energy parts assignable to the opposite boundaries of CD defining the ends of the space-time surface. The counterpart of zero energy state in positive energy ontology is the pair of initial and final states of a physical event, say particle reaction.

At space-time level ZEO means that 3-surfaces are pairs of space-like 3-surfaces at the opposite light-like boundaries of CD. Since the extremals of Kähler action connect these, one can say that by holography the basic dynamical objects are the space-time surface connecting these 3-surfaces and identifiable as analogs of Bohr orbits. This changes totally the vision about notions like self-organization: self-organization by quantum jumps does not take for a 3-D system but for the entire 4-D field pattern associated with it.

General Coordinate Invariance (GCI) allows to identify the basic dynamical objects as space-like 3-surfaces at the ends of space-time surface at boundaries of CD: this means that space-time surface is analogous to Bohr orbit. An alternative identification of the lines of generalized Feynman diagrams is as light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian . Also the Euclidian 4-D regions can have a similar interpretation. The requirement that the two interpretations are equivalent, leads to a strong form of General Coordinate Invariance. The outcome is effective 2-dimensionality stating that the partonic 2-surfaces identified as intersections of the space-like ends of space-time surface and light-like wormhole throats are the fundamental objects. That only effective 2-dimensionality is in question is due to the effects caused by the failure of strict determinism of Kähler action. In finite length scale resolution these effects can be neglected below UV cutoff and above IR cutoff. One can also speak about a strong form of holography.

The understanding of the super symplectic invariance leads to the proposal that super symplectic algebra and other Kac-Moody type algebras labelled by non-negative multiples of basic conformal weights allow a hierarchy of symmetry breakings in which the analog of gauge symmetry breaks down to a genuine dynamical symmetry. This gives rise to fractal hierarchies of algebras and symmetry breakings. This breaking can occur also for ordinary conformal algebras if one restricts the conformal weights to be non-negative integers.

1.1.3 Basic Objections

Objections are the most powerful tool in theory building. The strongest objection against TGD is the observation that all classical gauge fields are expressible in terms of four embedding space coordinates only- essentially CP_2 coordinates. The linear superposition of classical gauge fields taking place independently for all gauge fields is lost. This would be a catastrophe without many-sheeted space-time. Instead of gauge fields, only the effects such as gauge forces are superposed. Particles topologically condense to several space-time sheets simultaneously and experience the sum of gauge forces. This transforms the weakness to extreme economy: in a typical unified theory the number of primary field variables is countered in hundreds if not thousands, now it is just four.

Second objection is that TGD space-time is quite too simple as compared to GRT space-time due to the embeddability to 8-D embedding space. One can also argue that Poincare invariant theory of gravitation cannot be consistent with General Relativity. The above interpretation makes it possible to understand the relationship to GRT space-time and how the Equivalence Principle (EP) follows from Poincare invariance of TGD. The interpretation of GRT space-time is as effective space-time obtained by replacing many-sheeted space-time with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrics of the space-time sheets from Minkowski metric. Poincare invariance strongly suggests classical EP for the GRT limit in long length scales at least. One can also consider other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define a natural starting point and CP_2 indeed defines a gravitational instanton with a very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of the standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

Topological Field Quantization

Topological field quantization distinguishes between TGD based and more standard - say Maxwellian - notion of field. In Maxwell's fields created by separate systems superpose and one cannot tell which part of field comes from which system except theoretically. In TGD these fields correspond to different space-time sheets and only their effects on test particle superpose. Hence physical systems have well-defined field identifies - field bodies - in particular magnetic bodies.

The notion of magnetic body carrying dark matter with non-standard large value of Planck constant has become central concept in TGD inspired theory of consciousness and living matter, and by starting from various anomalies of biology one ends up to a rather detailed view about the role of magnetic body as intentional agent receiving sensory input from the biological body and controlling it using EEG and its various scaled up variants as a communication tool. Among other thins this leads to models for cell membrane, nerve pulse, and EEG.

1.1.4 Quantum TGD as Spinor Geometry of World of Classical Worlds

A turning point in the attempts to formulate a mathematical theory was reached after seven years from the birth of TGD. The great insight was "Do not quantize". The basic ingredients to the new approach have served as the basic philosophy for the attempt to construct Quantum TGD since then and have been the following ones.

World of Classical Worlds

The notion of WCW reduces the interacting quantum theory to a theory of free WCW spinor fields.

1. Quantum theory for extended particles is free(!), classical(!) field theory for a generalized Schrödinger amplitude identified as WCW spinor in the configuration space CH ("world of classical worlds", WCW) consisting of all possible 3-surfaces in H. "All possible" means that surfaces with arbitrary many disjoint components and with arbitrary internal topology and also singular surfaces topologically intermediate between two different manifold topologies are included.

2. 4-D general coordinate invariance forces holography and replaces the ill-defined path integral over all space-time surfaces with a discrete sum over 4-D analogs of Bohr orbits for particles identified as 3-surfaces. Holography means that basic objects are these analogs of Bohr orbits. Since there is no quantization at the level of WCW, one has an analog of wave mechanics with point-like particles replaced with 4-D Bohr orbits.

3. One must geometrize WCW as the space of Bohr orbits. In an infinite-dimensional situation the existence of geometry requires maximal symmetries already in the case of loop spaces. Physics is unique from its mathematical existence.

   WCW is endowed with metric and spinor structure so that one can define various metric related differential operators, say Dirac operators, appearing in the field equations of the theory [1]

Identification of Kähler function

The evolution of these basic ideas has been rather slow but has gradually led to a rather beautiful vision. One of the key problems has been the definition of Kähler function. Kähler function is Kähler action for a preferred extremal assignable to a given 3-surface but what this preferred extremal is? The obvious first guess was as absolute minimum of Kähler action but could not be proven to be right or wrong. One big step in the progress was boosted by the idea that TGD should reduce to almost topological QFT in which braids would replace 3-surfaces in finite measurement resolution, which could be inherent property of the theory itself and imply discretization at partonic 2-surfaces with discrete points carrying fermion number.

It took long time to realize that there is no discretization in 4-D sense - this would lead to difficulties with basic symmetries. Rather, the discretization occurs for the parameters characterizing co-dimension 2 objects representing the information about space-time surface so that they belong to some algebraic extension of rationals. These 2-surfaces - string world sheets and partonic 2-surfaces - are genuine physical objects rather than a computational approximation. Physics itself approximates itself, one might say! This is of course nothing but strong form of holography.

1. TGD as almost topological QFT vision suggests that Kähler action for preferred extremals reduces to Chern-Simons term assigned with space-like 3-surfaces at the ends of space-time (recall the notion of causal diamond (CD)) and with the light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. Minkowskian and Euclidian regions would give at wormhole throats the same contribution apart from coefficients and in Minkowskian regions the $\sqrt{g_4}$ factorc coming from metric would be imaginary so that one would obtain sum of real term identifiable as Kähler function and imaginary term identifiable as the ordinary Minkowskian action giving rise to interference effects and stationary phase approximation central in both classical and quantum field theory.

   Imaginary contribution - the presence of which I realized only after 33 years of TGD - could also have topological interpretation as a Morse function. On physical side the emergence of Euclidian space-time regions is something completely new and leads to a dramatic modification of the ideas about black hole interior.

2. The way to achieve the reduction to Chern-Simons terms is simple. The vanishing of Coulomb contribution to Kähler action is required and is true for all known extremals if one makes a general ansatz about the form of classical conserved currents. The so called weak form of electric-magnetic duality defines a boundary condition reducing the resulting 3-D terms to Chern-Simons terms. In this way almost topological QFT results. But only "almost" since the Lagrange multiplier term forcing electric-magnetic duality implies that Chern-Simons action for preferred extremals depends on metric.

WCW spinor fields

Classical WCW spinor fields are analogous to Schrödinger amplitudes and the construction of WCW Kähler geometry reduces to the second quantization of free spinor fields of H.

[1] There are four kinds of Dirac operators in TGD. The geometrization of quantum theory requires Kähler metric definable either in terms of Kähler function identified as a the bosonic action for Euclidian space-time regions or as anti-commutators for WCW gamma matrices identified as conformal Noether super-charges associated with the second quantized modified Dirac action consisting of string world sheet term and possibly also modified Dirac action in Minkowskian space-time regions. These two possible definitions reflect a duality analogous to AdS/CFT duality.

1. The WCW metric is given by anticommutators of WCW gamma matrices which also have interpretation as supercharges assignable to the generators of WCW isometries and allowing expression as non-conserved Noether charges. Holography implies zero energy ontology (ZEO) meaning that zero energy states are superpositions of Bohr orbits connecting boundaries of causal diamond (CD). CDs form a fractal hierarchy and their space forming the spine of WCW is finite-dimensional and can be geometrized. The alternative interpretation is as a superposition of pairs of ordinary 3-D fermionic states assignable to the ends of the space-time surfaces.

2. There are several Dirac operators. WCW Dirac operator D_{WCW} appears in Super-symplectic gauge conditions analogous to Super Virasoro conditions. The algebraic variant of the H Dirac operator D_H appears in fermionic correlation functions: this is due to the fact that free fermions appearing as building bricks of WCW gamma matrices are modes of D_H. The modes of $_DH$ define the ground states of super-symplectic representations. There is also the modified Dirac operator D_{X^4} acting on the induced spinors at space-time surfaces and it is dictated by symmetry one the action fixing the space-time surfaces as Bohr orbits is fixed. D_H is needed since it determines the expressions of WCW gamma matrices as Noether charges assignable to 3-surfaces at the ends of WCW.

The role of modified Dirac action

1. By quantum classical correspondence, the construction of WCW spinor structure in sectors assignable to CDs reduces to the second quantization of the induced spinor fields of H. The basic action is so called modified Dirac action in which gamma matrices are replaced with the modified) gamma matrices defined as contractions of the canonical momentum currents of the bosonic action defining the space-time surfaces with the embedding space gamma matrices. In this way one achieves super-conformal symmetry and conservation of fermionic currents among other things and a consistent Dirac equation.

   Modified Dirac action is needed to define WCW gamma matrices as super charges assignable to WCW isometry generators identified as generators of symplectic transformations and by holography are needed only at the 3-surface at the boundaries of WCW. It is important to notice that the modified Dirac equation does not determine propagators since induced spinor fields are obtained from free second quantized spinor fields of H. This means enormous simplification and makes the theory calculable.

2. An important interpretational problem relates to the notion of the induced spinor connection. The presence of classical W boson fields is in conflict with the classical conservation of em charge since the coupling to classical W fields changes em charge.

   One way out of the problem is the fact that the quantum averages of weak and gluon fields vanish unlike the quantum average of the em field. This leads to a rather precise understanding of electroweak symmetry breaking as being due the fact that color symmetries rotate space-time surfaces and also affect the induced weak fields.

   One can also consider a stronger condition. If one requires that the spinor modes have well-defined em charge, one must assume that the modes in the generic situation are localized at 2-D surfaces - string world sheets or perhaps also partonic 2-surfaces - at which classical W boson fields vanish. Covariantly constant right handed neutrinos generating super-symmetries forms an exception. The vanishing of the Z^0 field is possible for Kähler-Dirac action and should hold true at least above weak length scales. This implies that the string model in 4-D space-time becomes part of TGD. Without these conditions classical weak fields can vanish above weak scale only for the GRT limit of TGD for which gauge potentials are sums over those for space-time sheets.

   The localization would simplify the mathematics enormously and one can solve exactly the Kähler-Dirac equation for the modes of the induced spinor field just like in super string models.

   At the light-like 3-surfaces the signature of the induced metric changes from Euclidian to Minkowskian so that $\sqrt{g_4}$ vanishes. One can pose the condition that the algebraic analog of the massless Dirac equation is satisfied by the modes of the modified-Dirac action assignable to the Chern-Simons-Kähler action.

1.1.5 Construction of scattering amplitudes

Reduction of particle reactions to space-time topology

Particle reactions are identified as topology changes [A49, A62, A73]. For instance, the decay of a 3-surface to two 3-surfaces corresponds to the decay $A \rightarrow B + C$. Classically this corresponds to a path of WCW leading from 1-particle sector to 2-particle sector. At quantum level this corresponds to the dispersion of the generalized Schrödinger amplitude localized to 1-particle sector to two-particle sector. All coupling constants should result as predictions of the theory since no nonlinearities are introduced.

During years this naïve and very rough vision has of course developed a lot and is not anymore quite equivalent with the original insight. In particular, the space-time correlates of Feynman graphs have emerged from theory as Euclidian space-time regions and the strong form of General Coordinate Invariance has led to a rather detailed and in many respects un-expected visions. This picture forces to give up the idea about smooth space-time surfaces and replace space-time surface with a generalization of Feynman diagram in which vertices represent the failure of manifold property. I have also introduced the word "world of classical worlds" (WCW) instead of rather formal "configuration space". I hope that "WCW" does not induce despair in the reader having tendency to think about the technicalities involved!

Construction of the counterparts of S-matrices

What does one mean with the counterpart of S-matrix in the TGD framework has been a long standing problem. The development of ZEO based quantum measurement theory has led to a rough overall view of the situation.

1. There are two kinds of state function reductions (SFRs). "Small" SFRs (SSFRs) following the TGD counterpart of a unitary time evolution defines a sequence of SFRs, which is analogous to a sequence of repeated quantum measurements associated with the Zeno effect. In wave mechanics nothing happens in these measurements. In quantum optics these measurements correspond to weak measurements. In TGD SSFR affects the zero energy state but leaves the 3-D state at the passive boundary of CD unaffected.

2. In TGD framework each SSFR is preceded by a counterpart of a unitary time evolution, which means dispersion in the space of CDs and unitary time evolution in fermionic degrees of freedom such that the passive boundary of CDs and 3-D states at it are unaffected but a superposition of CDs with varying active boundaries in the space of CDs is formed. In SSFR a localization in the space of CDs occurs such that the active is fixed. In a statistical sense the size of the CD increases and the increasing distance between the tips of the CD gives rise to the arrow of geometric time.

3. Also "big" SFRS (BSFRs) can occur and they correspond to ordinary SFRs. In BSFR the roles of the active and passive boundary are changed and this means that the arrow of time is changed. Big SFR occurs when the SSFR corresponds to a quantum measurement, which does not commute with the operators, which define the states at the passive boundary of CD as their eigenstates. This means a radical deviation from standard quantum measurement theory and has predictions in all scales.

4. One can assign the counterpart of S-matrix to the unitary time evolution between two subsequent SSFRs and also to the counterpart of S-matrix associated with BSFR. At least in the latter case the dimension of the state space can increase since at least BSFRs lead to the increase of the dimension of algebraic extension of rationals assignable to the space-time surface by $M^8 - H$ duality. Unitarity is therefore replaced with isometry.

5. I have also considered the possibility that unitary S-matrix could be replaced in the fermionic degrees of freedom with Kähler metric of the state space satisfying analogs of unitarity conditions but it seems that this is un-necessary and also too outlandish an idea.

The notion of M-matrix

1. The most ambitious dream is that zero energy states correspond to a complete solution basis for the Dirac operators associated with WCWs associated with the spaces of CDs with fixed passive boundary: this would define an S-matrix assignable to SFR. Also the analog of S-matrix for the localizations of the states to the active boundary assignable to the BSFR changing the state at the passive boundary of CD is needed.

2. If one allows entanglement between positive and energy parts of the zero energy state but assumes that the states at the passive boundary are fixed, one must introduce the counterpart of the density matrix, or rather its square root. This classical free field theory would dictate what I have called M-matrices defined between positive and negative energy parts of zero energy states which form orthonormal rows of what I call U-matrix as a matrix defined between zero energy states. A biven M-matrix in turn would decompose to a product of a hermitian square root of density matrix and unitary S-matrix.

3. M-matrix would define time-like entanglement coefficients between positive and negative energy parts of zero energy states (all net quantum numbers vanish for them) and can be regarded as a hermitian square root of density matrix multiplied by a unitary S-matrix. Quantum theory would be in a well-defined sense a square root of thermodynamics. The orthogonality and hermiticity of the M-matrices commuting with S-matrix means that they span infinite-dimensional Lie algebras acting as symmetries of the S-matrix. Therefore quantum TGD would reduce to group theory in a well-defined sense.

4. In fact the Lie algebra of Hermitian M-matrices extends to Kac-Moody type algebra obtained by multiplying hermitian square roots of density matrices with powers of the S-matrix. Also the analog of Yangian algebra involving only non-negative powers of S-matrix is possible and would correspond to a hierarchy of CDs with the temporal distances between tips coming as integer multiples of the CP_2 time.

   The M-matrices associated with CDs are obtained by a discrete scaling from the minimal CD and characterized by integer n are naturally proportional to a representation matrix of scaling: $S(n) = S^n$, where S is unitary S-matrix associated with the minimal CD [K61]. This conforms with the idea about unitary time evolution as exponent of Hamiltonian discretized to integer power of S and represented as scaling with respect to the logarithm of the proper time distance between the tips of CD.

5. I have also considered the notion of U-matrix. U-matrix elements between M-matrices for various CDs are proportional to the inner products $Tr[S^{-n_1} \circ H^i H^j \circ S^{n_2} \lambda]$, where λ represents unitarily the discrete Lorentz boost relating the moduli of the active boundary of CD and H^i form an orthonormal basis of Hermitian square roots of density matrices. $\circ$ tells that S acts at the active boundary of CD only. I have proposed a general representation for the U-matrix, reducing its construction to that of the S-matrix.

1.1.6 TGD as a generalized number theory

Quantum T(opological)D(ynamics) as a classical spinor geometry for infinite-dimensional configuration space ("world of classical worlds", WCW), p-adic numbers and quantum TGD, and TGD inspired theory of consciousness, have been for last ten years the basic three strongly interacting threads in the tapestry of quantum TGD. The fourth thread deserves the name "TGD as a generalized number theory". It involves three separate threads: the fusion of real and various p-adic physics to a single coherent whole by requiring number theoretic universality discussed already, the formulation of quantum TGD in terms of complexified counterparts of classical number fields, and the notion of infinite prime. Note that one can identify subrings such as hyper-quaternions and hyper-octonions as sub-spaces of complexified classical number fields with Minkowskian signature of the metric defined by the complexified inner product.

The Threads in the Development of Quantum TGD

The development of TGD has involved several strongly interacting threads: physics as infinite-dimensional geometry; TGD as a generalized number theory, the hierarchy of Planck constants interpreted in terms of dark matter hierarchy, and TGD inspired theory of consciousness. In the following these threads are briefly described.

1. Quantum T(opological) G(eometro)D(ynamics) as a classical spinor geometry for infinite-dimensional WCW, p-adic numbers and quantum TGD, and TGD inspired theory of consciousness and of quantum biology have been for last decade of the second millenium the basic three strongly interacting threads in the tapestry of quantum TGD.

2. The discussions with Tony Smith initiated a fourth thread which deserves the name "TGD as a generalized number theory". The basic observation was that classical number fields might allow a deeper formulation of quantum TGD. The work with Riemann hypothesis made time ripe for realization that the notion of infinite primes could provide, not only a reformulation, but a deep generalization of quantum TGD. This led to a thorough and rather fruitful revision of the basic views about what the final form and physical content of quantum TGD might be. Together with the vision about the fusion of p-adic and real physics to a larger coherent structure these sub-threads fused to the "physics as generalized number theory" thread.

3. A further thread emerged from the realization that by quantum classical correspondence TGD predicts an infinite hierarchy of macroscopic quantum systems with increasing sizes, that it is not at all clear whether standard quantum mechanics can accommodate this hierarchy, and that a dynamical quantized Planck constant might be necessary and strongly suggested by the failure of strict determinism for the fundamental variational principle. The identification of hierarchy of Planck constants labelling phases of dark matter would be natural. This also led to a solution of a long standing puzzle: what is the proper interpretation of the predicted fractal hierarchy of long ranged classical electro-weak and color gauge fields. Quantum classical correspondences allows only single answer: there is infinite hierarchy of p-adically scaled up variants of standard model physics and for each of them also dark hierarchy. Thus TGD Universe would be fractal in very abstract and deep sense.

The chronology based identification of the threads is quite natural but not logical and it is much more logical to see p-adic physics, the ideas related to classical number fields, and infinite primes as sub-threads of a thread which might be called "physics as a generalized number theory". In the following I adopt this view. This reduces the number of threads to three corresponding to geometric, number theoretic and topological views of physics.

TGD forces the generalization of physics to a quantum theory of consciousness, and TGD as a generalized number theory vision leads naturally to the emergence of p-adic physics as physics of cognitive representations.

Number theoretic vision very briefly

Number theoretic vision about quantum TGD involves notions like adelic physics, M^8-H duality and number theoretic universality. A short review of the basic ideas that have developed during years is in order.

1. The physical interpretation of M^8 is as an analog of momentum space and M^8-H duality is analogous to momentum-position duality of ordinary wave mechanics.

2. Adelic physics means that all classical number fields, all p-adic number fields and their extensions induced by extensions of rationals and defining adeles, and also finite number fields are basic mathematical building bricks of physics.

   The complexification of M^8, identified as complexified octonions, would provide a realization of this picture and M^8-H duality would map the algebraic physics in M^8 to the ordinary physics in $M^4\times CP_2$ described in terms of partial differential equations.

3. Negentropy Maximization Principle (NMP) states that the conscious information assignable with cognition representable measured in terms of p-adic negentropy increases in statistical sense.

   NMP is mathematically completely analogous to the second law of thermodynamics and number theoretic evolution as an unavoidable statistical increase of the dimension of the algebraic extension of rationals characterizing a given space-time region implies it. There is no paradox involved: the p-adic negentropy measures the conscious information assignable to the entanglement of two systems regarded as a conscious entity whereas ordinary entropy measures the lack of information about the quantums state of either entangled system.

4. Number theoretical universality requires that space-time surfaces or at least their $M^8 - H$ duals in M_c^8 are defined for both reals and various p-adic number fields. This is true if they are defined by polynomials with integer coefficients as surfaces in M^8 obeying number theoretic holography realized as associativity of the normal space of 4-D surface using as holographic data 3-surfaces at mass shells identified in terms of roots of a polynomial. A physically motivated additional condition is that the coefficients of the polynomials are smaller than their degrees.

5. Galois confinement is a key piece of the number theoretic vision. It states that the momenta of physical states are algebraic integers in the extensions of rationals assignable to the space-time region considered. These numbers are in general complex and are not consistent with particle in box quantization. The proposal is that physical states satisfy Galois confinement being thus Galois singlets and having therefore total momenta, whose components are ordinary integers, when momentum unit defined by the scale of causal diamond (CD) is used.

6. The notion of p-adic prime was introduced in p-adic mass calculations that started the developments around 1995. p-Adic length scale hypothesis states that p-adic primes near powers of 2 have a special physical role (as possibly also the powers of other small primes such as $p = 3$).

   The proposal is that p-adic primes correspond to ramified primes assignable to the extension and identified as divisors of the polynomial defined by the products of the root differences for the roots of the polynomial defining space-time space and having interpretation as values of, in general complex, virtual mass squared.

p-Adic TGD and fusion of real and p-adic physics to single coherent whole

The p-adic thread emerged for roughly ten years ago as a dim hunch that p-adic numbers might be important for TGD. Experimentation with p-adic numbers led to the notion of canonical identification mapping reals to p-adics and vice versa. The breakthrough came with the successful p-adic mass calculations using p-adic thermodynamics for Super-Virasoro representations with the super-Kac-Moody algebra associated with a Lie-group containing standard model gauge group. Although the details of the calculations have varied from year to year, it was clear that p-adic physics reduces not only the ratio of proton and Planck mass, the great mystery number of physics, but all elementary particle mass scales, to number theory if one assumes that primes near prime powers of two are in a physically favored position. Why this is the case, became one of the key puzzles and led to a number of arguments with a common gist: evolution is present already at the elementary particle level and the primes allowed by the p-adic length scale hypothesis are the fittest ones.

It became very soon clear that p-adic topology is not something emerging in Planck length scale as often believed, but that there is an infinite hierarchy of p-adic physics characterized by p-adic length scales varying to even cosmological length scales. The idea about the connection of p-adics with cognition motivated already the first attempts to understand the role of the p-adics and inspired "Universe as Computer" vision but time was not ripe to develop this idea to anything concrete (p-adic numbers are however in a central role in TGD inspired theory of consciousness). It became however obvious that the p-adic length scale hierarchy somehow corresponds to a hierarchy of intelligences and that p-adic prime serves as a kind of intelligence quotient. Ironically, the almost obvious idea about p-adic regions as cognitive regions of space-time providing cognitive representations for real regions had to wait for almost a decade for the access into my consciousness.

In string model context one tries to reduces the physics to Planck scale. The price is the inability to say anything about physics in long length scales. In TGD p-adic physics takes care of this shortcoming by predicting the physics also in long length scales.

There were many interpretational and technical questions crying for a definite answer.

1. What is the relationship of p-adic non-determinism to the classical non-determinism of the basic field equations of TGD? Are the p-adic space-time region genuinely p-adic or does p-adic topology only serve as an effective topology? If p-adic physics is direct image of real physics, how the mapping relating them is constructed so that it respects various symmetries? Is the basic physics p-adic or real (also real TGD seems to be free of divergences) or both? If it is both, how should one glue the physics in different number field together to get *the* Physics? Should one perform p-adicization also at the level of the WCW? Certainly the p-adicization at the level of super-conformal representation is necessary for the p-adic mass calculations.

2. Perhaps the most basic and most irritating technical problem was how to precisely define p-adic definite integral which is a crucial element of any variational principle based formulation of the field equations. Here the frustration was not due to the lack of solution but due to the too large number of solutions to the problem, a clear symptom for the sad fact that clever inventions rather than real discoveries might be in question. Quite recently I however learned that the problem of making sense about p-adic integration has been for decades central problem in the frontier of mathematics and a lot of profound work has been done along same intuitive lines as I have proceeded in TGD framework. The basic idea is certainly the notion of algebraic continuation from the world of rationals belonging to the intersection of real world and various p-adic worlds.

Despite various uncertainties, the number of the applications of the poorly defined p-adic physics has grown steadily and the applications turned out to be relatively stable so that it was clear that the solution to these problems must exist. It became only gradually clear that the solution of the problems might require going down to a deeper level than that represented by reals and p-adics.

The key challenge is to fuse various p-adic physics and real physics to single larger structure. This has inspired a proposal for a generalization of the notion of number field by fusing real numbers and various p-adic number fields and their extensions along rationals and possible common algebraic numbers. This leads to a generalization of the notions of embedding space and space-time concept and one can speak about real and p-adic space-time sheets. One can talk about adelic space-time, embedding space, and WCW.

The corresponds of real 4-surfaces with the p-adic ones is induced by number theoretical discretization using points of 4-surfaces $Y^4 \subset M^8_c$ identifiable as 8-momenta, whose components are assumed to be algebraic integers in an extension of rationals defined by the extension of rationals associated with a polynomial P with integer coefficients smaller than the degree of P. These points define a cognitive representation, which is universal in the sense that it exists also in the algebraic extensions of p-adic numbers. The points of the cognitive representations associated with the mass shells with mass squared values identified as roots of P are enough since $M^8 - H$ duality can be used at both M^8 and H sides and also in the p-adic context. The mass shells are special in that they allow for Minkowski coordinates very large cognitive representations unlike the interiors of the 4-surfaces determined by holography by using the data defined by the 3-surfaces at the mass shells. The higher the dimension of the algebraic extension associated with P, the better the accuracy of the cognitive representation.

Adelization providing number theoretical universality reduces to algebraic continuation for the amplitudes from this intersection of reality and various p-adicities - analogous to a back of a book - to various number fields. There are no problems with symmetries but canonical identification is needed: various group invariant of the amplitude are mapped by canonical identification to various p-adic number fields. This is nothing but a generalization of the mapping of the p-adic mass squared to its real counterpart in p-adic mass calculations.

This leads to surprisingly detailed predictions and far reaching conjectures. For instance, the number theoretic generalization of entropy concept allows negentropic entanglement central for the applications to living matter (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book). One can also understand how preferred p-adic primes could emerge as so called ramified primes of algebraic extension of rationals in question and characterizing string world sheets and partonic 2-surfaces. Preferred p-adic primes would be ramified primes for extensions for which the number of p-adic continuations of two-surfaces to space-time surfaces (imaginations) allowing also real continuation (realization of imagination) would be especially large. These ramifications would be winners in the fight for number theoretical survival. Also a generalization of p-adic length scale hypothesis emerges from NMP [K56].

The characteristic non-determinism of the p-adic differential equations suggests strongly that p-adic regions correspond to "mind stuff", the regions of space-time where cognitive representations reside. This interpretation implies that p-adic physics is physics of cognition. Since Nature is probably a brilliant simulator of Nature, the natural idea is to study the p-adic physics of the cognitive representations to derive information about the real physics. This view encouraged by TGD inspired theory of consciousness clarifies difficult interpretational issues and provides a clear interpretation for the predictions of p-adic physics.

Infinite primes

The discovery of the hierarchy of infinite primes and their correspondence with a hierarchy defined by a repeatedly second quantized arithmetic quantum field theory gave a further boost for the speculations about TGD as a generalized number theory.

After the realization that infinite primes can be mapped to polynomials possibly representable as surfaces geometrically, it was clear how TGD might be formulated as a generalized number theory with infinite primes forming the bridge between classical and quantum such that real numbers, p-adic numbers, and various generalizations of p-adics emerge dynamically from algebraic physics as various completions of the algebraic extensions of complexified quaternions and octonions. Complete algebraic, topological and dimensional democracy would characterize the theory.

The infinite primes at the first level of hierarchy, which represent analogs of bound states, can be mapped to irreducible polynomials, which in turn characterize the algebraic extensions of rationals defining a hierarchy of algebraic physics continuable to real and p-adic number fields. The products of infinite primes in turn define more general algebraic extensions of rationals. The interesting question concerns the physical interpretation of the higher levels in the hierarchy of infinite primes and integers mappable to polynomials of $n > 1$ variables.

1.1.7 An explicit formula for $M^8 - H$ duality

$M^8 - H$ duality is a generalization of momentum-position duality relating the number theoretic and geometric views of physics in TGD and, despite that it still involves poorly understood aspects, it has become a fundamental building block of TGD. One has 4-D surfaces $Y^4 \subset M^8_c$, where M^8_c is complexified M^8 having interpretation as an analog of complex momentum space and 4-D spacetime surfaces $X^4 \subset H = M^4 \times CP_2$. M^8_c, equivalently E^8_c, can be regarded as complexified octonions. M^8_c has a subspace M^4_c containing M^4.

Comment: One should be very cautious with the meaning of "complex". Complexified octonions involve a complex imaginary unit i commuting with the octonionic imaginary units I_k. i is assumed to also appear as an imaginary unit also in complex algebraic numbers defined by the roots of polynomials P defining holographic data in M^8_c.

In the following $M^8 - H$ duality and its twistor lift are discussed and an explicit formula for the dualities are deduced. Also possible variants of the duality are discussed.

Holography in H

$X^4 \subset H$ satisfies holography and is analogous to the Bohr orbit of a particle identified as a 3-surface. The proposal is that holography reduces to a 4-D generalization of holomorphy so that X^4 is a simultaneous zero of two functions of complex CP_2 coordinates and of what I have called Hamilton-Jacobi coordinates of M^4 with a generalized Kähler structure.

The simplest choice of the Hamilton-Jacobi coordinates is defined by the decomposition $M^4 = M^2 \times E^2$, where M^2 is endowed with hypercomplex structure defined by light-like coordinates (u, v), which are analogous to z and $\overline{z}$. Any analytic map $u \to f(u)$ defines a new set of light-like coordinates and corresponds to a solution of the massless d'Alembert equation in M^2. E^2 has some complex coordinates with imaginary unit defined by i.

The conjecture is that also more general Hamilton-Jacobi structures for which the tangent space decomposition is local are possible. Therefore one would have $M^4 = M^2(x) \times E^2(x)$. These would correspond to non-equivalent complex and Kähler structures of M^4 analogous to those possessed by 2-D Riemann surfaces and parametrized by moduli space.

Number theoretic holography in M^8_c

$Y^4 \subset M^8_c$ satisfies number theoretic holography defining dynamics, which should reduce to associativity in some sense. The Euclidian complexified normal space $N^4(y)$ at a given point y of Y^4 is required to be associative, i.e. quaternionic. Besides this, $N^4(i)$ contains a preferred complex Euclidian 2-D subspace $Y^2(y)$. Also the spaces $Y^2(x)$ define an integrable distribution. I have assumed that $Y^2(x)$ can depend on the point y of Y^4.

These assumptions imply that the normal space $N(y)$ of Y^4 can be parameterized by a point of $CP_2 = SU(3)/U(2)$. This distribution is always integrable unlike quaternionic tangent space distributions. $M^8 - H$ duality assigns to the normal space $N(y)$ a point of CP_2. M^4_c point y is mapped to a point $x \in M^4 \subset M^4 \times CP_2$ defined by the real part of its inversion (conformal transformation): this formula involves effective Planck constant for dimensional reasons.

The 3-D holographic data, which partially fixes 4-surfaces Y^4 is partially determined by a polynomial P with real integer coefficients smaller than the degree of P. The roots define mass squared values which are in general complex algebraic numbers and define complex analogs of mass shells in $M^4_c \subset M^8_c$, which are analogs of hyperbolic spaces H^3. The 3-surfaces at these mass shells define 3-D holographic data continued to a surface Y^4 by requiring that the normal space of Y^4 is associative, i.e. quaternionic. These 3-surfaces are not completely fixed but an interesting conjecture is that they correspond to fundamental domains of tessellations of H^3.

What does the complexity of the mass shells mean? The simplest interpretation is that the space-like M^4 coordinates (3-momentum components) are real whereas the time-like coordinate (energy) is complex and determined by the mass shell condition. One would have $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3 when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as a time coordinate. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to an equation of mass shell when $\sqrt{(Re(E)^2 - Im(E)^2)}$, expressed in terms of $Re(E)$, is taken as new energy coordinate $E_{eff} = \sqrt{(Re(E)^2 - Im(E)^2)}$. Is this deformation of H^3 in imaginary time direction equivalent with a region of the hyperbolic 3-space H^3?

One can look at the formula in more detail. Mass shell condition gives $Re^2(E) - Im(E)^2 - p^2 = Re(m^2)$ and $2Re(E)Im(E) = Im(m^2)$. The condition for the real parts gives H^3, when $\sqrt{Re^2(E) - Im(E)^2}$ is taken as an effective energy. The second condition allows to solve $Im(E)$ in terms of $Re(E)$ so that the first condition reduces to a dispersion relation for $Re(E)^2$.

$$Re(E)^2 = \frac{1}{2}(Re(m^2) - Im(m^2) + p^2)(1 \pm \sqrt{1 + \frac{2Im(m^2)^2}{(Re(m^2) - Im(m^2) + p^2)^2}} . \qquad (1.1.1)$$

Only the positive root gives a non-tachyonic result for $Re(m^2) - Im(m^2) > 0$. For real roots with $Im(m^2) = 0$ and at the high momentum limit the formula coincides with the standard formula. For $Re(m^2) = Im(m^2)$ one obtains $Re(E)^2 \to Im(m^2)/\sqrt{2}$ at the low momentum limit $p^2 \to 0$. Energy does not depend on momentum at all: the situation resembles that for plasma waves.

Can one find an explicit formula for $M^8 - H$ duality?

The dream is an explicit formula for the $M^8 - H$ duality mapping $Y^4 \subset M^8_c$ to $X^4 \subset H$. This formula should be consistent with the assumption that the generalized holomorphy holds true for X^4.

The following proposal is a more detailed variant of the earlier proposal for which Y^4 is determined by a map g of $M^4_c \to SU(3)_c \subset G_{2,c}$, where $G_{2,c}$ is the complexified automorphism group of octonions and $SU(3)_c$ is interpreted as a complexified color group.

This map defines a trivial $SU(3)_c$ gauge field. The real part of g however defines a non-trivial real color gauge field by the non-linearity of the non-abelian gauge field with respect to the gauge potential. The quadratic terms involving the imaginary part of the gauge potential give an additional condition to the real part in the complex situation and cancel it. If only the real part of g contributes, this contribution would be absent and the gauge field is non-vanishing.

How could the automorphism $g(x) \subset SU(3) \subset G_2$ give rise to $M^8 - H$ duality?

1. The interpretation is that $g(y)$ at given point y of Y^4 relates the normal space at y to a fixed quaternionic/associative normal space at point y_0, which corresponds is fixed by some subgroup $U(2)_0 \subset SU(3)$. The automorphism property of g guarantees that the normal space is quaternionic/associative at y. This simplifies the construction dramatically.

2. The quaternionic normal sub-space (which has Euclidian signature) contains a complex sub-space which corresponds to a point of sphere $S^2 = SO(3)/O(2)$, where $SO(3)$ is the quaternionic automorphism group. The interpretation could be in terms of a selection of spin quantization axes. The local choice of the preferred complex plane would not be unique and is analogous to the possibility of having non-trivial Hamilton Jacobi structures in M^4 characterized by the choice of $M^2(x)$ and equivalently its normal subspace $E^2(x)$.

   These two structures are independent apart from dependencies forced by the number theoretic dynamics. Hamilton-Jacobi structure means a selection of the quantization axis of spin and energy by fixing a distribution of light-like tangent vectors of M^4 and the choice of the quaternionic normal sub-space fixes a choice of preferred quaternionic imaginary unit defining a quantization axis of the weak isospin.

3. The real part $Re(g(y))$ defines a point of $SU(3)$ and the bundle projection $SU(3) \rightarrow CP_2$ in turn defines a point of $CP_2 = SU(3)/U(2)$. Hence one can assign to g a point of CP_2 as $M^8 - H$ duality requires and deduce an explicit formula for the point. This means a realization of the dream.

4. The construction requires a fixing of a quaternionic normal space N_0 at y_0 containing a preferred complex subspace at a single point of Y^4 plus a selection of the function g. If M^4 coordinates are possible for Y^4, the first guess is that g as a function of complexified M^4 coordinates obeys generalized holomorphy with respect to complexified M^4 coordinates in the same sense and in the case of X^4. This might guarantee that the $M^8 - H$ image of Y^4 satisfies the generalized holomorphy.

5. Also space-time surfaces X^4 with M^4 projection having a dimension smaller than 4 are allowed. I have proposed that they might correspond to singular cases for the above formula: a kind of blow-up would be involved. One can also consider a more general definition of Y^4 allowing it to have a M^4 projection with dimension smaller than 4 (say cosmic strings). Could one have implicit equations for the surface Y^4 in terms of the complex coordinates of $SU(3)_c$ and M^4? Could this give for instance cosmic strings with a 2-D M^4 projection and CP_2 type extremals with 4-D CP_2 projection and 1-D light-like M^4 projection?

What could the number theoretic holography mean physically?

What could be physical meaning of the number theoretic holography? The condition that has been assumed is that the CP_2 coordinates at the mass shells of $M^4_c \subset M^8_c$ mapped to mass shells H^3 of $M^4 \subset M^4 \times CP_2$ are constant at the H^3. This is true if the $g(y)$ defines the same CP_2 point for a given component X^3_i of the 3-surface at a given mass shell. g is therefore fixed apart from a local $U(2)$ transformation leaving the CP_2 point invariant. A stronger condition would be that the CP_2 point is the same for each component of X^3_i and even at each mass shell but this condition seems to be unnecessarily strong.

Comment: One can o criticize this condition as too strong and one can consider giving up this condition. The motivation for this condition is that the number of algebraic points at the 3-surfaces associated with H^3 explodes since the coordinates associated with normal directions vanish. Kind of cognitive explosion would be in question.

$SU(3)$ corresponds to a subgroup of G_2 and one can wonder what the fixing of this subgroup could mean physically. G_2 is 14-D and the coset space $G_2/SU(3)$ is 6-D and a good guess is that it is just the 6-D twistor space $SU(3)/U(1) \times U(1)$ of CP_2: at least the isometries are the same. The fixing of the $SU(3)$ subgroup means fixing of a CP_2 twistor. Physically this means the fixing of the quantization axis of color isospin and hypercharge.

Twistor lift of the holography

What is interesting is that by replacing $SU(3)$ with G_2, one obtains an explicit formula form the generalization of $M^8 - H$ duality to that for the twistorial lift of TGD!

One can also consider a twistorial generalization of the above proposal for the number theoretic holography by allowing local G_2 automorphisms interpreted as local choices of the color quantization axis. G_2 elements would be fixed apart from a local $SU(3)$ transformation at the components of 3-surfaces at mass shells. The choice of the color quantization axes for a connected 3-surface at a given mass shell would be the same everywhere. This choice is indeed very natural physically since 3-surface corresponds to a particle.

Is this proposal consistent with the boundary condition of the number theoretical holography mean in the case of 4-surfaces in M^8_c and $M^4 \times CP_2$?

1. The selection of $SU(3) \subset G_2$ for ordinary $M^8 - H$ duality means that the $G_{2,c}$ gauge field vanishes everywhere and the choice of color quantization axis is the same at all points of the 4-surface. The fixing of the CP_2 point to be constant at H^3 implies that the color gauge field at $H^3 \subset M^8_c$ and its image $H^3 \subset H$ vanish. One would have color confinement at the mass shells H^3_i, where the observations are made. Is this condition too strong?

2. The constancy of the G_2 element at mass shells makes sense physically and means a fixed color quantization axis. The selection of a fixed $SU(3) \subset G_2$ for entire space-time surface is in conflict with the non-constancy of G_2 element unless G_2 element differs at different points of 4-surface only by a multiplication of a local $SU(3)_0$ element, that is local $SU(3)$ transformation. This kind of variation of the G_2 element would mean a fixed color group but varying choice of color quantization axis.

3. Could one consider the possibility that the local $G_{2,c}$ element is free and defines the twistor lift of $M^8 - H$ duality as something more fundamental than the ordinary $M^8 - H$ duality based on $SU(3)_c$. This duality would make sense only at the mass shells so that only the spaces $H^3 \times CP_2$ assignable to mass shells would make sense physically? In the interior CP_2 would be replaced with the twistor space $SU(3)/U(1) \times U(1)$. Color gauge fields would be non-vanishing at the mass shells but outside the mass shells one would have G_2 gauge fields.

   There is also a physical objection against the G_2 option. The 14-D Lie algebra representation of G_2 acts on the imaginary octonions which decompose with respect to the color group to $1 \oplus 3 \oplus \overline{3}$. The automorphism property requires that 1 can be transformed to 3 or $\overline{3}$ to themselves: this requires that the decomposition contains $3 \oplus \overline{3}$. Furthermore, it must be possible to transform 3 and $\overline{3}$ to themselves, which requires the presence of 8. This leaves only the decomposition $8 \oplus 3 \oplus \overline{3}$. G_2 gluons would both color octet and triplets. In the TDG framework the only conceivable interpretation would be in terms of ordinary gluons and leptoquark-like gluons. This does not fit with the basic vision of TGD.

The choice of twistor as a selection of quantization axes should make sense also in the M^4 degrees of freedom. M^4 twistor corresponds to a choice of light-like direction at a given point of M^4. The spatial component of the light-like vector fixes the spin quantization axis. Its choice together with the light-likeness fixes the time direction and therefore the rest system and energy quantization axis. Light-like vector fixes also the choice of M^2 and of E^2 as its orthogonal complement. Therefore the fixing of M^4 twistor as a point of $SU(4)/SU(3) \times U(1)$ corresponds to a choice of the spin quantization axis and the time-like axis defining the rest system in which the energy is measured. This choice would naturally correspond to the Hamilton-Jacobi structure fixing the decompositions $M^2(x) \times E^2(x)$. At a given mass shell the choice of the quantization axis would be constant for a given X^3_i.

1.1.8 Hierarchy of Planck Constants and Dark Matter Hierarchy

By quantum classical correspondence space-time sheets can be identified as quantum coherence regions. Hence the fact that they have all possible size scales more or less unavoidably implies that Planck constant must be quantized and have arbitrarily large values. If one accepts this then also the idea about dark matter as a macroscopic quantum phase characterized by an arbitrarily large value of Planck constant emerges naturally as does also the interpretation for the long ranged classical electro-weak and color fields predicted by TGD. Rather seldom the evolution of ideas follows simple linear logic, and this was the case also now. In any case, this vision represents the fifth, relatively new thread in the evolution of TGD and the ideas involved are still evolving.

Dark Matter as Large $\hbar$ Phases

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive.

Nottale and Da Rocha believe that their Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems. The space-time sheets in question would carry dark matter.

Nottale's hypothesis would predict a gigantic value of h_{gr}. Equivalence Principle and the independence of gravitational Compton length on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that h_{gr} would be much smaller. Large h_{gr} could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets which is quantum coherent in the required time scale [K89].

It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta) possibly carrying monopole flux and identifiable as remnants of cosmic string phase of primordial cosmology. The magnetic energy of these flux quanta would correspond to dark energy and magnetic tension would give rise to negative "pressure" forcing accelerate cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside magnetic flux tubes identifiable as dark energy.

Certain experimental findings suggest the identification $h_{eff} = n\times = h_{gr}$. The large value of h_{gr} can be seen as a way to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description). The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also Kähler-Dirac action is present, part of the interior degrees of freedom associated with the Kähler-Dirac part of conformal algebra become physical. A possible is that tfermionic oscillator operators generate super-symmetries and sparticles correspond almost by definition to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) of bunch of n low energy gravitons.

Hierarchy of Planck Constants from the Anomalies of Neuroscience and Biology

The quantal ELF effects of ELF em fields on vertebrate brain have been known since seventies. ELF em fields at frequencies identifiable as cyclotron frequencies in magnetic field whose intensity is about 2/5 times that of Earth for biologically important ions have physiological effects and affect also behavior. What is intriguing that the effects are found only in vertebrates (to my best knowledge). The energies for the photons of ELF em fields are extremely low - about 10^{-10} times lower than thermal energy at physiological temperatures- so that quantal effects are impossible in the framework of standard quantum theory. The values of Planck constant would be in these situations large but not gigantic.

This inspired the hypothesis that these photons correspond to so large a value of Planck constant that the energy of photons is above the thermal energy. The proposed interpretation was as dark photons and the general hypothesis was that dark matter corresponds to ordinary matter with non-standard value of Planck constant. If only particles with the same value of Planck constant can appear in the same vertex of Feynman diagram, the phases with different value of Planck constant are dark relative to each other. The phase transitions changing Planck constant can however make possible interactions between phases with different Planck constant but these interactions do not manifest themselves in particle physics. Also the interactions mediated by classical fields should be possible. Dark matter would not be so dark as we have used to believe.

The hypothesis $h_{eff} = h_{gr}$ - at least for microscopic particles - implies that cyclotron energies of charged particles do not depend on the mass of the particle and their spectrum is thus universal although corresponding frequencies depend on mass. In bio-applications this spectrum would correspond to the energy spectrum of bio-photons assumed to result from dark photons by h_{eff} reducing phase transition and the energies of bio-photons would be in visible and UV range associated with the excitations of bio-molecules.

Also the anomalies of biology (see for instance [K78, K79, K75]) support the view that dark matter might be a key player in living matter.

Dark Matter as a Source of Long Ranged Weak and Color Fields

Long ranged classical electro-weak and color gauge fields are unavoidable in TGD framework. The smallness of the parity breaking effects in hadronic, nuclear, and atomic length scales does not however seem to allow long ranged electro-weak gauge fields. The problem disappears if long range classical electro-weak gauge fields are identified as space-time correlates for massless gauge fields created by dark matter. Also scaled up variants of ordinary electro-weak particle spectra are possible. The identification explains chiral selection in living matter and unbroken $U(2)_{ew}$ invariance and free color in bio length scales become characteristics of living matter and of bio-chemistry and bio-nuclear physics.

The recent view about the solutions of Kähler- Dirac action assumes that the modes have a well-defined em charge and this implies that localization of the modes to 2-D surfaces (right-handed neutrino is an exception). Classical W boson fields vanish at these surfaces and also classical Z^0 field can vanish. The latter would guarantee the absence of large parity breaking effects above intermediate boson scale scaling like h_{eff}.

1.1.9 Twistors in TGD and connection with Veneziano duality

The twistorialization of TGD has two aspects. The attempt to generalize twistor Grassmannian approach emerged first. It was however followed by the realization that also the twistor lift of TGD at classical space-time level is needed. It turned out that the progress in the understanding of the classical twistor lift has been much faster - probably this is due to my rather limited technical QFT skills.

Twistor lift at space-time level

8-dimensional generalization of ordinary twistors is highly attractive approach to TGD [K101]. The reason is that M^4 and CP_2 are completely exceptional in the sense that they are the only 4-D manifolds allowing twistor space with Kähler structure [A45]. The twistor space of $M^4 \times CP_2$ is Cartesian product of those of M^4 and CP_2. The obvious idea is that space-time surfaces allowing twistor structure if they are orientable are representable as surfaces in H such that the properly induced twistor structure co-incides with the twistor structure defined by the induced metric.

In fact, it is enough to generalize the induction of spinor structure to that of twistor structure so that the induced twistor structure need not be identical with the ordinary twistor structure possibly assignable to the space-time surface. The induction procedure reduces to a dimensional reduction of 6-D Kähler action giving rise to 6-D surfaces having bundle structure with twistor sphere as fiber and space-time as base. The twistor sphere of this bundle is imbedded as sphere in the product of twistor spheres of twistor spaces of M^4 and CP_2.

This condition would define the dynamics, and the original conjecture was that this dynamics is equivalent with the identification of space-time surfaces as preferred extremals of Kähler action. The dynamics of space-time surfaces would be lifted to the dynamics of twistor spaces, which are sphere bundles over space-time surfaces. What is remarkable that the powerful machinery of complex analysis becomes available.

It however turned out that twistor lift of TGD is much more than a mere technical tool. First of all, the dimensionally reduction of 6-D Kähler action contained besides 4-D Kähler action also a volume term having interpretation in terms of cosmological constant. This need not bring anything new, since all known extremals of Kähler action with non-vanishing induced Kähler form are minimal surfaces. There is however a large number of embeddings of twistor sphere of space-time surface to the product of twistor spheres. Cosmological constant has spectrum and depends on length scale, and the proposal is that coupling constant evolution reduces to that for cosmological constant playing the role of cutoff length. That cosmological constant could transform from a mere nuisance to a key element of fundamental physics was something totally new and unexpected.

1. The twistor lift of TGD at space-time level forces to replace 4-D Kähler action with 6-D dimensionally reduced Kähler action for 6-D surface in the 12-D Cartesian product of 6-D twistor spaces of M^4 and CP_2. The 6-D surface has bundle structure with twistor sphere as fiber and space-time surface as base.

   Twistor structure is obtained by inducing the twistor structure of 12-D twistor space using dimensional reduction. The dimensionally reduced 6-D Kähler action is sum of 4-D Kähler action and volume term having interpretation in terms of a dynamical cosmological constant depending on the size scale of space-time surface (or of causal diamond CD in zero energy ontology (ZEO)) and determined by the representation of twistor sphere of space-time surface in the Cartesian product of the twistor spheres of M^4 and CP_2.

2. The preferred extremal property as a representation of quantum criticality would naturally correspond to minimal surface property meaning that the space-time surface is separately an extremal of both Kähler action and volume term almost everywhere so that there is no coupling between them. This is the case for all known extremals of Kähler action with non-vanishing induced Kähler form.

   Minimal surface property could however fail at 2-D string world sheets, their boundaries and perhaps also at partonic 2-surfaces. The failure is realized in minimal sense if the 3-surface has 1-D edges/folds (strings) and 4-surface 2-D edges/folds (string world sheets) at which some partial derivatives of the embedding space coordinates are discontinuous but canonical momentum densities for the entire action are continuous.

   There would be no flow of canonical momentum between interior and string world sheet and minimal surface equations would be satisfied for the string world sheet, whose 4-D counterpart in twistor bundle is determined by the analog of 4-D Kähler action. These conditions allow the transfer of canonical momenta between Kähler- and volume degrees of freedom at string world sheets. These no-flow conditions could hold true at least asymptotically (near the boundaries of CD).

   $M^8 - H$ duality suggests that string world sheets (partonic 2-surfaces) correspond to images of complex 2-sub-manifolds of M^8 (having tangent (normal) space which is complex 2-plane of octonionic M^8).

3. Cosmological constant would depend on p-adic length scales and one ends up to a concrete model for the evolution of cosmological constant as a function of p-adic length scale and other number theoretic parameters (such as Planck constant as the order of Galois group): this conforms with the earlier picture.

   Inflation is replaced with its TGD counterpart in which the thickening of cosmic strings to flux tubes leads to a transformation of Kähler magnetic energy to ordinary and dark matter. Since the increase of volume increases volume energy, this leads rapidly to energy minimum at some flux tube thickness. The reduction of cosmological constant by a phase transition however leads to a new expansion phase. These jerks would replace smooth cosmic expansion of GRT. The discrete coupling constant evolution predicted by the number theoretical vision could be understood as being induced by that of cosmological constant taking the role of cutoff parameter in QFT picture [L28].

Twistor lift at the level of scattering amplitudes and connection with Veneziano duality

The classical part of twistor lift of TGD is rather well-understood. Concerning the twistorialization at the level of scattering amplitudes the situation is much more difficult conceptually - I already mentioned my limited QFT skills.

1. From the classical picture described above it is clear that one should construct the 8-D twistorial counterpart of theory involving space-time surfaces, string world sheets and their boundaries, plus partonic 2-surfaces and that this should lead to concrete expressions for the scattering amplitudes.

   The light-like boundaries of string world sheets as carriers of fermion numbers would correspond to twistors as they appear in twistor Grassmann approach and define the analog for the massless sector of string theories. The attempts to understand twistorialization have been restricted to this sector.

2. The beautiful basic prediction would be that particles massless in 8-D sense can be massive in 4-D sense. Also the infrared cutoff problematic in twistor approach emerges naturally and reduces basically to the dynamical cosmological constant provided by classical twistor lift.

   One can assign 4-momentum both to the spinor harmonics of the embedding space representing ground states of super-conformal representations and to light-like boundaries of string world sheets at the orbits of partonic 2-surfaces. The two four-momenta should be identical by quantum classical correspondence: this could be seen as a concretization of Equivalence Principle. Also a connection with string model emerges.

3. As far as symmetries are considered, the picture looks rather clear. Ordinary twistor Grassmannian approach boils down to the construction of scattering amplitudes in terms of Yangian invariants for conformal group of M^4. Therefore a generalization of super-symplectic symmetries to their Yangian counterpart seems necessary. These symmetries would be gigantic but how to deduce their implications?

4. The notion of positive Grassmannian is central in the twistor approach to the scattering amplitudes in $calN = 4$ SUSYs. TGD provides a possible generalization and number theoretic interpretation of this notion. TGD generalizes the observation that scattering amplitudes in twistor Grassmann approach correspond to representations for permutations. Since 2-vertex is the only fermionic vertex in TGD, OZI rules for fermions generalizes, and scattering amplitudes are representations for braidings.

   Braid interpretation encourages the conjecture that non-planar diagrams can be reduced to ordinary ones by a procedure analogous to the construction of braid (knot) invariants by gradual un-braiding (un-knotting).

This is however not the only vision about a solution of non-planarity. Quantum criticality provides different view leading to a totally unexpected connection with string models, actually with the Veneziano duality, which was the starting point of dual resonance model in turn leading via dual resonance models to super string models.

1. Quantum criticality in TGD framework means that coupling constant evolution is discrete in the sense that coupling constants are piecewise constant functions of length scale replaced by dynamical cosmological constant. Loop corrections would vanish identically and the recursion formulas for the scattering amplitudes (allowing only planar diagrams) deduced in twistor Grassmann would involve no loop corrections. In particular, cuts would be replaced by sequences of poles mimicking them like sequences of point charge mimic line charges. In momentum discretization this picture follows automatically.

2. This would make sense in finite measurement resolution realized in number theoretical vision by number-theoretic discretization of the space-time surface (cognitive representation) as points with coordinates in the extension of rationals defining the adele [L23]. Similar discretization would take place for momenta. Loops would vanish at the level of discretization but what would happen at the possibly existing continuum limit: does the sequence of poles integrate to cuts? Or is representation as sum of resonances something much deeper?

3. Maybe it is! The basic idea of behind the original Veneziano amplitudes (see `http://tinyurl.com/yyhwvbqb`) was Veneziano duality. This 4-particle amplitude was generalized by Yoshiro Nambu, Holber-Beck Nielsen, and Leonard Susskind to N-particle amplitude (see `http://tinyurl.com/yyvkx7as`) based on string picture, and the resulting model was called dual resonance model. The model was forgotten as QCD emerged. Later came superstring models and led to M-theory. Now it has become clear that something went wrong, and it seems that one must return to the roots. Could the return to the roots mean a careful reconsideration of the dual resonance model?

4. Recall that Veneziano duality (1968) was deduced by assuming that scattering amplitude can be described as sum over s-channel resonances or t-channel Regge exchanges and Veneziano duality stated that hadronic scattering amplitudes have representation as sums over s- or t-channel resonance poles identified as excitations of strings. The sum over exchanges defined by t-channel resonances indeed reduces at larger values of s to Regge form.

   The resonances had zero width, which was not consistent with unitarity. Further, there were no counterparts for the *sum* of s-, t-, and u-channel diagrams with continuous cuts in the kinematical regions encountered in QFT approach. What puts bells ringing is the u-channel diagrams would be non-planar and non-planarity is the problem of twistor Grassmann approach.

5. Veneziano duality is true only for s- and t- channels but not been s- and u-channel. Stringy description makes t-channel and s-channel pictures equivalent. Could it be that in fundamental description u-channels diagrams cannot be distinguished from s-channel diagrams or t-channel diagrams? Could the stringy representation of the scattering diagrams make u-channel twist somehow trivial if handles of string world sheet representing stringy loops in turn representing the analog of non-planarity of Feynman diagrams are absent? The permutation of external momenta for tree diagram in absence of loops in planar representation would be a twist of π in the representation of planar diagram as string world sheet and would not change the topology of the string world sheet and would not involve non-trivial world sheet topology.

   For string world sheets loops would correspond to handles. The presence of handle would give an edge with a loop at the level of 3-surface (self energy correction in QFT). Handles are not allowed if the induced metric for the string world sheet has Minkowskian signature. If the stringy counterparts of loops are absent, also the loops in scattering amplitudes should be absent.

   This argument applies only inside the Minkowskian space-time regions. If string world sheets are present also in Euclidian regions, they might have handles and loop corrections could emerge in this manner. In TGD framework strings (string world sheets) are identified to 1-D edges/folds of 3-surface at which minimal surface property and topological QFT property fails (minimal surfaces as calibrations). Could the interpretation of edge/fold as discontinuity of some partial derivatives exclude loopy edges: perhaps the branching points would be too singular?

A reduction to a sum over s-channel resonances is what the vanishing of loops would suggest. Could the presence of string world sheets make possible the vanishing of continuous cuts even at the continuum limit so that continuum cuts would emerge only in the approximation as the density of resonances is high enough?

The replacement of continuous cut with a sum of *infinitely* narrow resonances is certainly an approximation. Could it be that the stringy representation as a sum of resonances with *finite* width is an essential aspect of quantum physics allowing to get rid of infinities necessarily accompanying loops? Consider now the arguments against this idea.

1. How to get rid of the problems with unitarity caused by the zero width of resonances? Could *finite* resonance widths make unitarity possible? Ordinary twistor Grassmannian approach predicts that the virtual momenta are light-like but complex: obviously, the imaginary part of the energy in rest frame would have interpretation as resonance with.

   In TGD framework this generalizes for 8-D momenta. By quantum-classical correspondence (QCC) the classical Noether charges are equal to the eigenvalues of the fermionic charges in Cartan algebrable (maximal set of mutually commuting observables) and classical TGD indeed predicts complex momenta (Kähler coupling strength is naturally complex). QCC thus supports this proposal.

2. Sum over resonances/exchanges picture is in conflict with QFT picture about scattering of particles. Could *finite* resonance widths due to the complex momenta give rise to the QFT type scattering amplitudes as one develops the amplitudes in Taylor series with respect to the resonance width? Unitarity condition indeed gives the first estimate for the resonance width.

   QFT amplitudes should emerge in an approximation obtained by replacing the discrete set of finite width resonances with a cut as the distance between poles is shorter than the resolution for mass squared.

   In superstring models string tension has single very large value and one cannot obtain QFT type behavior at low energies (for instance, scattering amplitudes in hadronic string model are concentrated in forward direction). TGD however predicts an entire hierarchy of p-adic length scales with varying string tension. The hierarchy of mass scales corresponding roughly to the lengths and thickness of magnetic flux tubes as thickened cosmic strings and characterized by the value of cosmological constant predicted by twistor lift of TGD. Could this give rise to continuous QCT type cuts at the limit when measurement resolution cannot distinguish between resonances?

   The dominating term in the sum over sums of resonances in t-channel gives near forward direction approximately the lowest mass resonance for strings with the smallest string tension. This gives the behavior $1/(t - m_{min}^2)$, where m_{min} corresponds to the longest mass scale involved (the largest space-time sheet involved), approximating the $1/t$-behavior of massless theories. This also brings in IR cutoff, the lack of which is a problem of gauge theories. This should give rise to continuous QFT type cuts at the limit when measurement resolution cannot distinguish between resonances.

1.2 Bird's Eye of View about the Topics of "Quantum Physics as Number theory: Part I"

The focus of this book is the number theoretical vision about physics. This vision involves three loosely related parts.

1. The fusion of real physic and various p-adic physics to a single larger whole by generalizing the number concept by fusing real numbers and various p-adic number fields along common rationals. Extensions of p-adic number fields can be introduced by gluing them along common algebraic numbers to reals. Algebraic continuation of the physics from rationals and their their extensions to various number fields (completion of rational physics to physics in various number fields) is the key idea and the challenge is to understand whether how one could achieve this dream. A very profound implication is that purely local p-adic physics codes for the p-adic fractality of long length length scale real physics and vice versa. As a consequence one can understand the origins of p-adic length scale hypothesis and ends up with a very concrete view about space-time correlates of cognition.

   The fusion of various p-adic physics to single coherent whole leads to what I call adelic physics [L23, L22].

2. Second part of the vision involves what the classical number fields defined as subspaces of their complexifications with Minkowskian signature of the metric. The hypothesis is

that allowed space-time surfaces correspond to quaternionic sub-manifolds of complexified octonionic space. The proposed interpretation of quaternionicity would in terms of being zero for the real or imaginary part of octonionic polynomial with rational or perhaps even algebraic coefficients.

Real/imaginary part refers to a composition of octonion to quaternion and imaginary unit multiplying second quaternion analogous to the decomposition of ordinary complex number to real and imaginary parts. Space-time surface would correspond to imaginary roots (in the sense that they are proportional to the imaginary unit i commuting with the octonionic units). It is argued that this notion of quaternionicity is equivalent with the assumption that the tangent space or normal of space-time surface in M^8 at each point is quaternionic.

Besides this one assumes that one can assign to each point of space-time surface a complex plane M_c^2 as subspace of the quaternionic plane M_c^4. These planes could even depend on point of space-time surface and define an integrable distribution - kind of string world sheet.

Quaternionicity of the tangent plane in this sense allows to map the space-time surface in M^8 to a space-time surface in $H = M^4 \times CP_2$. This involves a projection to M^4 in the decomposition $M^8 = M^4 \times C_2$ and the assignment to the point of space-time surface point of CP_2 labelling its tangent space.

It is not clear whether one can assign also to each point of space-time surface in H a quaternionic tangent or normal in the tangent space M^8 of H. In the case in H this plane could be the tangent/normal plane defined by the modified gamma matrices or induced gamma matrices. These two planes co-incide with each other only for action defined by the metric determinant. Hence the basic variational principle of TGD would have deep number theoretic content. Reduction to a closed form would also mean that classical TGD would define a generalized topological field theory with Noether charges defining topological invariants.

3. The third part of the vision involves infinite primes, which can be identified in terms of an infinite hierarchy of second quantized arithmetic quantum fields theories on one hand, and as having representations as space-time surfaces analogous to zero surfaces of polynomials on the other hand. In this framework space-time surface would represent an infinite number. This vision leads also the conclusion that single point of space-time has an infinitely complex structure since real unity can be represented as a ratio of infinite numbers in infinitely many ways each having its own number theoretic anatomy. Thus single space-time point is in principle able to represent in its structure the quantum state of the entire universe. This number theoretic variant of Brahman=Atman identity also means that Universe is an algebraic hologram.

Besides this holy trinity I will discuss also loosely related topics. Included are the possible applications of the category theory in TGD and in TGD inspired theory of consciousness; various TGD inspired considerations related to Riemann hypothesis - in particular, a strategy for proving Riemann hypothesis using a modification of Hilbert-Polya conjecture replacing quantum states with coherent states of a unique conformally invariant physical system; topological quantum computation in TGD Universe; and TGD inspired approach to Langlands program.

1.2.1 Organization of "Quantum Physics as Number Theory: Part I"

The first of "Quantum Physics as Number Theory: Part I" decomposes into two 2 parts.

1. In the first part of the book the number theoretical vision is introduced. The first chapter describes p-adicization program, second chapter introduces the ideas related to classical number fields and algebraic physics, and the third chapter summarizes the rather speculative notion of infinite primes having amazing structural similarities with a repeatedly second quantized arithmetic super-symmetric quantum field theory with single particle states labelled by primes.

   Although the notion of infinite primes sounds rather academic, it might have deep physical content (infinite primes are finite in p-adic topologies). The hierarchy of infinite primes could relate to the hierarchies of space-time sheets, of extensions of rationals, and of supersymplectic and other alagras appearing in quantum TGD.

2. In the first five chapters of the second part of the book p-adic numbers, and the fusion of various p-adic physics to single coherent whole eventually leading to the notion of adelic physics as a fusion of physics of sensory experience and cognition are discussed.

1.3 Sources

The eight online books about TGD [K107, K102, K82, K67, K20, K62, K42, K92] and nine online books about TGD inspired theory of consciousness and quantum biology [K99, K17, K74, K16, K40, K49, K52, K91, K98] are warmly recommended for the reader willing to get overall view about what is involved.

My homepage (`http://tinyurl.com/ybv8dt4n`) contains a lot of material about TGD. In particular, a TGD glossary at `http://tinyurl.com/yd6jf3o7`).

I have published articles about TGD and its applications to consciousness and living matter in *Journal of Non-Locality* (`http://tinyurl.com/ycyrxj4o` founded by Lian Sidorov and in *Prespacetime Journal* (`http://tinyurl.com/ycvktjhn`), *Journal of Consciousness Research and Exploration* (`http://tinyurl.com/yba4f672`), and *DNA Decipher Journal* (`http://tinyurl.com/y9z52khg`), all of them founded by Huping Hu. One can find the list about the articles published at `http://tinyurl.com/ybv8dt4n`. I am grateful for these far-sighted people for providing a communication channel, whose importance one cannot overestimate.

1.3.1 PART I: NUMBER THEORETICAL VISION

TGD as a Generalized Number Theory I: p-Adicization Program

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation.

The first part of the 3-part chapter is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

1. Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the embedding space coordinates are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics could emerge from the basic equations of the theory. One could interpret the resulting rational-adic or p-adic regions as geometrical correlates for "mind stuff".

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with "mind like" regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of geometric model of "self" and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2. The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

3. Number theoretical Universality and number theoretical criticality

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix (generalization of S-matrix) so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field - number theoretical criticality - becomes a viable concept. This form of principle allows also purely p-adic phenomena such as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough.

4. p-Adicization by algebraic continuation

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. It must be however emphasized that for weaker form of number theoretical universality this restriction applies only at number theoretical quantum criticality. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. Zero energy ontology provides a natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

5. Number theoretic democracy

The interpretation allows all finite-dimensional extensions of p-adic number fields and perhaps even infinite-P p-adics. The notion arithmetic quantum theory generalizes to include Gaussian and Eisenstein variants of infinite primes and corresponding arithmetic quantum field theories. Also the notion of p-adicity generalizes: it seems that one can indeed assign to Gaussian and Eisenstein primes what might be called G-adic and E-adic numbers.

p-Adicization by algebraic continuation gives hopes of continuing quantum TGD from reals to various p-adic number fields. The existence of this continuation poses extremely strong constraints on theory.

TGD as a Generalized Number Theory II: Quaternions, Octonions, and their Hyper Counterparts

This chapter is second one in a multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields in quantum TGD. A central notion is $M^8 - H$ duality which might be also called number theoretic compactification. This duality allows to identify embedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. This duality has been recently extended to a $H-H$ duality making sense if the dualism respects associativity (co-associativity). This would make the space of preferred extremals category with dualism representing the fundamental arrow.

These number theoretical symmetries induce also the symmetries dictating the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D embedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology (ZEO) has become the corner stone of both quantum TGD and number theoretical vision. In ZEO either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW.

The basic number theoretical structures are complex numbers, quaternions and octonions, and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this manner could select 4-D surfaces as surface having as tangent spaces associative

(co-associative) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by space-time tangent vectors spanning associative (co-associative) sub-algebra of complexified octonions generated by embedding space tangent vectors. A more concrete representation of vectors of complexified tangent space as embedding space gamma matrices is not necessary. One can consider also octonionic representation of embedding space gamma matrices but whether it has any physical content, remains an open question. The answer to the question whether octonions could correspond to the Kähler-Dirac gamma matrices associated with Kähler-Dirac action turned out to be "No".

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative (co-commutative) sub-algebra is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations. The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. As has become clear, one can define also the notions of co-associativity and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces.

TGD as a Generalized Number Theory III: Infinite Primes

Infinite primes are besides p-adicization and the representation of space-time region as a associative (co-associative) sub-manifold of hyper-octonionic space the basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

1. Why infinite primes are unavoidable

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. The first speculative view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D complexified octonions containing hyper-octonions as sub-space

would make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic embedding space. This view is addmittedly speculative since the notion of octonionic prime makes sense only for complexified octonions.

2. The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational infinite primes. One can however replace classical associativity with quantum associativity for quantum states assigned with infinite prime. One can also decompose rational infinite primes to hyper-octonionic infinite primes at lower level of the hierarchy. Physically this would mean that the number theoretic 8-momenta have only time-component. This decomposition is completely analogous to the decomposition of hadrons to its colored constituents and might be even interpreted in terms of color confinement. The interpretation of the decomposition of rational primes to primes in the algebraic extensions of rationals, hyper-quaternions, and hyper-octonions would have an interpretation as an increase of number theoretical resolution and the principle of number theoretic confinement could be seen as a fundamental physical principle implied by associativity condition.

4. Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

The representation of space-time surfaces as algebraic surfaces in M^8 is however too naive idea and the attempt to map hyper-octonionic infinite primes to algebraic surfaces seems has not led to any concrete progress.

The endless updating of quantum TGD might be blamed to be a waste of time. The interaction of new ideas with old ones has however again and again turned out to be an extremely fruitful process leading to rather precise view about how infinite hyper-octonionic rationals can be mapped to space-time surfaces without ad hoc assumptions. The progress in quantum TGD during the second half of the first decade of the new millenium led to several new and quite convincing ideas. Mention only zero energy ontology, the generalization of the embedding space concept realizing the hierarchy of Planck constants, hyper-finite factors and their inclusions, and in particular, the realization of quantum classical correspondence in terms of measurement interaction term associated with the Kähler-Dirac action.

The crucial observation is that quantum classical correspondence allows to map quantum numbers of WCW spinor fields to space-time geometry. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries.

5. Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real and also more general units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and embedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the WCW and WCW spinor fields to the number theoretical anatomies of a single point of embedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of WCW spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of embedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

Infinite rationals are in one-one correspondence with quantum states and in zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

1.3.2 PART II: TGD, P-ADIC NUMBERS, AND ADELES

p-Adic Numbers and Generalization of Number Concept

In this chapter the general TGD inspired mathematical ideas related to p-adic numbers are discussed. The extensions of the p-adic numbers including extensions containing transcendentals, the correspondences between p-adic and real numbers, p-adic differential and integral calculus, and p-adic symmetries and Fourier analysis belong the topics of the chapter.

The basic hypothesis is that p-adic space-time regions correspond to cognitive representations for the real physics appearing already at the elementary particle level. The interpretation of the p-adic physics as a physics of cognition is justified by the inherent p-adic non-determinism of the p-adic differential equations making possible the extreme flexibility of imagination.

p-Adic canonical identification and the identification of reals and p-adics by common rationals are the two basic identification maps between p-adics and reals and can be interpreted as two basic types of cognitive maps. The concept of p-adic fractality is defined and p-adic fractality is the basic property of the cognitive maps mapping real world to the p-adic internal world. Canonical identification is not general coordinate invariant and at the fundamental level it is applied only to map p-adic probabilities and predictions of p-adic thermodynamics to real numbers. The correspondence via common rationals is general coordinate invariant correspondence when general coordinate transformations are restricted to rational or extended rational maps: this has interpretation in terms of fundamental length scale unit provided by CP_2 length.

A natural outcome is the generalization of the notion of number. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both embedding space and WCW are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The definition of p-adic manifold is not discussed although it has turned out to be highly non-trivial. The feasible definition of p-adic sub-manifold emerged two decades after the emergence of the notion of of p-adic space-time sheet. The definition relies on the idea that p-adic space-time surfaces serve as p-adic charts - cognitive maps - for real space-time surfaces and vice versa and that both real and p-adic space-time sheets are preferred extremals of Kähler action and defined only modulo finite measurement/cognitive resolution.

p-Adic differential calculus obeys the same rules as real one and an interesting outcome are p-adic fractals involving canonical identification. Perhaps the most crucial ingredient concerning the practical formulation of the p-adic physics is the concept of the p-adic valued definite integral. Quite generally, all general coordinate invariant definitions are based on algebraic continuation by common rationals. Integral functions can be defined using just the rules of ordinary calculus and the ordering of the integration limits is provided by the correspondence via common rationals. Residy calculus generalizes to p-adic context and also free Gaussian functional integral generalizes to p-adic context and is expected to play key role in quantum TGD at WCW level.

The special features of p-adic Lie-groups are briefly discussed: the most important of them being an infinite fractal hierarchy of nested groups. Various versions of the p-adic Fourier analysis are proposed: ordinary Fourier analysis generalizes naturally only if finite-dimensional extensions of p-adic numbers are allowed and this has interpretation in terms of p-adic length scale cutoff. Also p-adic Fourier analysis provides a possible definition of the definite integral in the p-adic context by using algebraic continuation.

p-Adic Physics: Physical Ideas

The most important p-adic concepts and ideas are p-adic fractality, 4-D spin glass analogy, p-adic length scale hypothesis, p-adic realization of the Slaving Principle, p-adic criticality, and the non-determinism of the p-adic differential equations justifying the interpretation of the p-adic space-time regions as cognitive representations. These ideas are discussed in this chapter in a more concrete level than in previous chapters in the hope that this might help the reader to assimilate the material more easily. Some of the considerations might be a little bit out of date since the chapter is written much earlier than the preceding chapters.

1. 2-D thermodunamical criticality is accompanied by conformal invariance. The proposed quantum criticality of quantum TGD motivated the attempt to generalize conformal invariance to the 4-dimensional context providing a motivation of the p-adic approach. After almost two decades after the emergence of the idea about extended conformal invariance

the view about conformal invariance is much more detailed and is indeed associated with quantum criticality, which reflects the non-determinism of Kähler action.

2. In TGD as a generalized number theory approach p-adic space-time regions emerge completely naturally and have interpretation as cognitive representations of the real physics. If this occurs already at the level of elementary particles, one can understand p-adic physics as a model for a cognitive model about physics provided by Nature itself. The basic motivation for this assumption is the p-adic non-determinism of the p-adic field equations making them ideal for the simulation purposes. The p-adic–real phase transitions are the second basic concept allowing to understand how intention is transformed to action and vice versa: the occurrence of this process even at elementary particle level explains why p-adic length scale hypothesis works. This picture is consistent with the idea about evolution occurring already at the level of elementary particles and allowing the survival of the systems with largest cognitive resources.

3. Spin glass analogy, which was the original motivation for p-adicization before the discovery that p-adic regions of space-time emerge automatically from TGD as a generalized number theory approach, is discussed at WCW level. The basic idea is that the maximum (several of them are possible) of the exponential of the Kähler function with respect to the fiber degrees of freedom as function of zero modes is p-adic fractal. This together with spin glass analogy suggest p-adic ultra-metricity of the reduced WCW CH_{red}, the TGD counterpart of the energy landscape.

4. Slaving Principle states that there exists a hierarchy of dynamics with increasing characteristic length (time) scales and the dynamical variables of a given length scale obey dynamics, where the dynamical variables of the longer length (time) scale serve as "masters" that is effectively as external parameters or integration constants. The dynamics of the "slave" corresponds to a rapid adaptation to the conditions posed by the "master". p-Adic length scale hypothesis allows a concrete quantification of this principle predicting a hierarchy of preferred length, time, energy and frequency scales.

5. Critical systems are fractals and the natural guess is that p-adic topology serves also as an effective topology of real space-time sheets in some length scale range and that real non-determinism of Kähler action mimics p-adic non-determinism for some value of prime p. This motivates some qualitative p-adic ideas about criticality.

6. The properties of the CP_2 type extremals providing TGD based model for elementary particles and topological sum contacts, are discussed in detail. CP_2 type extremals could be for TGD what black holes are for General Relativity. Black hole elementary particle analogy is discussed in detail and the generalization of the Hawking-Bekenstein formula is shown to lead to a prediction for the radius of the elementary particle horizon and to a justification for the p-adic length scale hypothesis. A deeper justification for the p-adic length scale hypothesis comes from the assumption that systems with maximal cognitive resources are winners in the fight for survival even in elementary particle length scales.

7. Quantum criticality in its simplest variants states that states Kähler coupling strength α_K is analoggous to critical temperature. In principle allows allows the dependence of the α_K on on zero modes. It would be nice if α_K were RG invariant in strong sense but the expression for gravitational coupling constant implies that it increases rapidly as a function of p-adic length scale in this case. This led to the hypothesis that G is RG invariant. The hypothesis fixes the p-adic evolution of α_K completely and implies logarithmic dependence of α_K on p-adic length scale. It has however turned out that the RG invariance might after all be possible and is actually strongly favored by different physical arguments. The point is that M_{127} is the largest Mersenne prime for which p-adic length scale is non-super-astronomical. If gravitational interaction is mediated by space-time sheets labelled by this Mersenne prime, gravitational constant is effective RG invariant even if α_K is RG invariant in strong sense. This option is also ideal concerning the p-adicization of the theory.

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

The mathematical aspects of p-adicization of quantum TGD are discussed. In a well-defined sense Nature itself performs the p-adicization and p-adic physics can be regarded as physics of cognitive regions of space-time which in turn provide representations of real space-time regions. Cognitive representations presumably involve the p-adicization of the geometry at the level of the space-time and embedding space by a mapping of a real space time region to a p-adic one. One can differentiate between two kinds of maps: the identification induced by the common rationals of real and p-adic space time region and the representations of the external real world to internal p-adic world induced by a canonical identification type maps.

Only the identification by common rationals respects general coordinate invariance, and it leads to a generalization of the number concept. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and WCW are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The definition of p-adic manifold is not discussed although it has turned out to be highly non-trivial. The feasible definition of p-adic sub-manifold emerged two decades after the emergence of the notion of of p-adic space-time sheet. The definition relies on the idea that p-adic space-time surfaces serve as p-adic charts - cognitive maps - for real space-time surfaces and vice versa and that both real and p-adic space-time sheets are preferred extremals of Kähler action and defined only modulo finite measurement/cognitive resolution

The program makes sense only if also extensions containing transcendentals are allowed: the p-dimensional extension containing powers of e is perhaps the most important transcendental extension involved. Entire cognitive hierarchy of extension emerges and the dimension of extension can be regarded as a measure for the cognitive resolution and the higher the dimension the shorter the length scale of resolution. Cognitive resolution provides also number theoretical counterpart for the notion of length scale cutoff unavoidable in quantum field theories: now the length scale cutoffs are part of the physics of cognition rather than reflecting the practical limitations of theory building.

There is a lot of p-adicizing to do.

1. The p-adic variant of classical TGD must be constructed. Field equations make indeed sense also in the p-adic context. The strongest assumption is that real space time sheets have the same functional form as real space-time sheet so that there is non-uniqueness only due to the hierarchy of dimensions of extensions.

2. Probability theory must be generalized. Canonical identification playing central role in p-adic mass calculations using p-adic thermodynamics maps genuinely p-adic probabilities to their real counterparts. p-Adic entropy can be defined and one can distinguish between three kinds of entropies: real entropy, p-adic entropy mapped to its real counterpart by canonical identification, and number theoretical entropies applying when probabilities are in finite-dimensional extension of rationals. Number theoretic entropies can be negative and provide genuine information measures, and it turns that bound states should correspond in TGD framework to entanglement coefficients which belong to a finite-dimensional extension of rationals and have negative number theoretic entanglement entropy. These information measures generalize by quantum-classical correspondence to space-time level.

3. p-Adic quantum mechanics must be constructed. p-Adic unitarity differs in some respects from its real counterpart: in particular, p-adic cohomology allows unitary S-matrices $S = 1 + T$ such that T is hermitian and nilpotent matrix. p-Adic quantum measurement theory based on Negentropy Maximization Principle (NMP) leads to the notion of monitoring, which might have relevance for the physics of cognition.

4. Generalized quantum mechanics results as fusion of quantum mechanics in various number fields using algebraic continuation from the field of rational as a basic guiding principle. It

seems possible to generalize the notion of unitary process in such a manner that unitary matrix leads from rational Hilbert space H_Q to a formal superposition of states in all Hilbert spaces H_F, where F runs over number fields. The basic objection is that p-adic numbers allow non-vanishing zero norm states. If one can avoid this objection, state function reduction could be seen as a number theoretical necessity and involves a reduction to a particular number field followed by state function reduction and state preparation leading ultimately to a state containing only entanglement which is rational or finitely-extended rational and because of its negative number theoretic entanglement entropy identifiable as bound state entanglement stable against NMP.

It has later turned out that negentropic entanglement must correspond to density matrix proportional to a unit matrix in order to achieve consistency with the ordinary quantum measurement theory. Unitary entanglement coefficients characterizing topological quantum computation give rise to negentropic entanglement, which would be stable with respect to NMP.

5. Generalization of the configuration space ("world of classical worlds" (WCW)) and related concepts is also necessary and again gluing along common rationals and algebraic continuation is the basic guide line also now. WCW is a union of symmetric spaces and this allows an algebraic construction of the WCW Kähler metric and spinor structure, whose definition reduces to the supersymplectic algebra defined by the function basis at the light cone boundary. Hence the algebraic continuation is relatively straightforward. Even WCW functional integral could allow algebraic continuation. The reason is that symmetric space structure together with Duistermaat Hecke theorem suggests strongly that WCW integration with the constraints posed by infinite-dimensional symmetries on physical states is effectively equivalent to Gaussian functional integration in free field theory around the unique maximum of Kähler function using contravariant configuration space metric as a propagator. Algebraic continuation is possible for a subset of rational valued zero modes if Kähler action and Kähler function are rational functions of WCW coordinates for rational values of zero modes.

Unified Number Theoretical Vision

An updated view about M^8-H duality is discussed. M^8-H duality allows to deduce $M^4\times CP_2$ via number theoretical compactification. One important correction is that octonionic spinor structure makes sense only for M^8 whereas for $M^4\times CP_2$ complefixied quaternions characterized the spinor structure.

Octonions, quaternions associative and co-associative space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized.

There is a beautiful pattern present suggesting that $H = M^4 \times CP_2$ is completely unique on number theoretical grounds. Consider only the following facts. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. Octonionic projective space OP_2 appears as octonionic twistor space (there are no higher-dimensional octonionic projective spaces). Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing OP_2 to to 12-D $G_2/U(1)\times U(1)$ having same dimension as the the twistor space $CP_3 \times SU(3)/U(1)\times U(1)$ of H assignable to complexified quaternionic representation of gamma matrices.

A further fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically six-sphere. Also the analogy of quaternionicity of preferred extremals in TGD with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program.

Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

A long-standing question has been the origin of preferred p-adic primes characterizing elementary particles. I have proposed several explanations and the most convincing hitherto is related to the algebraic extensions of rationals and p-adic numbers selecting naturally preferred primes as those which are ramified for the extension in question.

Philosophy of Adelic Physics

The p-adic aspects of Topological Geometrodynamics (TGD) will be discussed. Introduction gives a short summary about classical and quantum TGD. This is needed since the p-adic ideas are inspired by TGD based view about physics.

p-Adic mass calculations relying on p-adic generalization of thermodynamics and super-symplectic and super-conformal symmetries are summarized. Number theoretical existence constrains lead to highly non-trivial and successful physical predictions. The notion of canonical identification mapping p-adic mass squared to real mass squared emerges, and is expected to be a key player of adelic physics allowing to map various invariants from p-adics to reals and vice versa.

A view about p-adicization and adelization of real number based physics is proposed. The proposal is a fusion of real physics and various p-adic physics to single coherent whole achieved by a generalization of number concept by fusing reals and extensions of p-adic numbers induced by given extension of rationals to a larger structure and having the extension of rationals as their intersection.

The existence of p-adic variants of definite integral, Fourier analysis, Hilbert space, and Riemann geometry is far from obvious and various constraints lead to the idea of number theoretic universality (NTU) and finite measurement resolution realized in terms of number theory. An attractive manner to overcome the problems in case of symmetric spaces relies on the replacement of angle variables and their hyperbolic analogs with their exponentials identified as roots of unity and roots of e existing in finite-dimensional algebraic extension of p-adic numbers. Only group invariants - typically squares of distances and norms - are mapped by canonical identification from p-adic to real realm and various phases are mapped to themselves as number theoretically universal entities.

Also the understanding of the correspondence between real and p-adic physics at various levels - space-time level, embedding space level, and level of "world of classical worlds" (WCW) - is a challenge. The gigantic isometry group of WCW and the maximal isometry group of embedding space give hopes about a resolution of the problems. Strong form of holography (SH) allows a non-local correspondence between real and p-adic space-time surfaces induced by algebraic continuation from common string world sheets and partonic 2-surfaces. Also local correspondence seems intuitively plausible and is based on number theoretic discretization as intersection of real and p-adic surfaces providing automatically finite "cognitive" resolution. he existence p-adic variants of Kähler geometry of WCW is a challenge, and NTU might allow to realize it.

I will also sum up the role of p-adic physics in TGD inspired theory of consciousness. Negentropic entanglement (NE) characterized by number theoretical entanglement negentropy (NEN) plays a key role. Negentropy Maximization Principle (NMP) forces the generation of NE. The interpretation is in terms of evolution as increase of negentropy resources.

Part I

NUMBER THEORETICAL VISION

Chapter 2

TGD as a Generalized Number Theory I: p-Adicization Program

2.1 Introduction

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential way the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and embedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines sub-algebra of embedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces X^4 correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to X^4 a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved and dual relation between totally different descriptions of the physical world would be in question.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

2.1.1 The Painting Is The Landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios q of infinite integers divided by q. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

2.1.2 Real And P-Adic Regions Of The Space-Time As Geometric Correlates Of Matter And Mind

One could end up with p-adic space-time sheets via field equations. The solutions of the equations determining space-time surfaces are restricted by the requirement that the coordinates are real. When this is not the case, one might apply instead of a real completion with some p-adic completion. It however seems that p-adicity is present at deeper level and automatically present via the generalization of the number concept obtained by fusing reals and p-adics along rationals and common algebraics.

p-Adic non-determinism due to the presence of non-constant functions with vanishing derivative implies extreme flexibility and therefore suggests the identification of the p-adic regions as seats of cognitive representations. Unlike the completion of reals to complex numbers, the completions of p-adic numbers preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with "mind like" regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of "self" and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2.1.3 The Generalization Of The Notion Of Number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

2.1.4 Zero Energy Ontology, Cognition, And Intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts zero energy ontology [K25, K24].

Zero energy ontology classically

In TGD inspired cosmology [K90] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [K104] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

Zero energy ontology at quantum level

Also the construction of S-matrix [K24] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state have interpretation as M-matrix identifiable as a "complex square root" of density matrix expressible as a product of positive diagonal square root of the density matrix and of a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

The collection of M-matrices defines an orthonormal state basis for zero energy states and together they define unitary U-matrix charactering transition amplitudes between zero energy states. This matrix would not be however the counterpart of the usual S-matrix. Rather the unitary matrix phase of a given M-matrix would define the S-matrix measured in laboratory. U-matrix would also characterize the transitions between different number fields possible in the intersection of rel and p-adic worlds and having interpretation in terms of intention and cognition.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by the time scale of the causal diamond (CD) and the rational (perhaps integer) characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also. CDs are indeed important also in TGD inspired cosmology [K90].

Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [K110]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the " of classical worlds", WCW briefly) is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [K110]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same way as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the embedding space this means that p-adic and real space-time sheets intersect only along common rational points of the embedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of embedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [K24]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the embedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

2.1.5 What Number Theoretical Universality Might Mean?

Number theoretic universality has been one of the basic guide lines in the construction of quantum TGD. There are two forms of the principle.

1. The strong form of number theoretical universality states that physics for any system should effectively reduce to a physics in algebraic extension of rational numbers at the level of M-matrix so that an interpretation in both real and p-adic sense (allowing a suitable algebraic extension of p-adics) is possible. One can however worry whether this principle only means that physics is algebraic so that there would be no need to talk about real and p-adic physics at the level of M-matrix elements. It is not possible to get rid of real and p-adic numbers at the level of classical physics since calculus is a prerequisite for the basic variational principles used to formulate the theory. For this option the possibility of completion is what poses conditions on M-matrix.

2. The weak form of principle requires only that both real and p-adic variants of physics make sense and that the intersection of these physics consist of physics associated with various algebraic extensions of rational numbers. In this rational physics would be like rational numbers allowing infinite number of algebraic extensions and real numbers and p-adic number fields as its completions. Real and p-adic physics would be completions of rational physics. In this framework criticality with respect to phase transitions changing number field becomes a viable concept. This form of principle allows also purely p-adic phenomena such

as p-adic pseudo non-determinism assigned to imagination and cognition. Genuinely p-adic physics does not however allow definition of notions like conserved quantities since the notion of definite integral is lacking and only the purely local form of real physics allows p-adic counterpart.

Experience has taught that it is better to avoid too strong statements and perhaps the weak form of the principle is enough. It is however clear that number theoretical criticality could provide important insights to quantum TGd. p-Adic thermodynamics [K62] is an excellent example of this. In consciousness theory the transitions transforming intentions to actions and actions to cognitions would be key applications. Needless to say, zero energy ontology is absolutely essential: otherwise this kind of transitions would not make sense.

2.1.6 P-Adicization By Algebraic Continuation

The basic challenges of the p-adicization program are following.

1. The first problem -the conceptual one- is the identification of preferred coordinates in which functions are algebraic and for which algebraic values of coordinates are in preferred position. This problem is encountered both at the level of space-time, embedding space, and configuration space. Here the group theoretical considerations play decisive role and the selection of preferred coordinates relates closely to the selection of quantization axes. This selection has direct physical correlates at the level of embedding space and the hierarchy of Planck constants has interpretation as a correlate for the selection of quantization axes [K36].

   Algebraization does not necessarily mean discretization at space-time level: for instance, the coordinates characterizing partonic 2-surface can be algebraic so that algebraic point of the configuration space results and surface is not discretized. If this kind of function spaces are finite-dimensional, it is possible to fix X^2 completely data for a finite number of points only.

2. Local physics generalizes as such to p-adic context (field equations, etc...). The basic stumbling block of this program is integration already at space-time (Kähler action etc..). The problem becomes really horrible looking at configuration space level (functional integral). Algebraic continuation could allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics. Also the existence of the Kähler geometry does this and the solution to the constraint is that WCW is a union of symmetric spaces. In the case of symmetric spaces Fourier analysis generalizes to harmonics analysis and one can reduces integration to summation for functions allowing Fourier decomposition. In p-adic context the existence of plane waves requires an algebraic extension allowing roots of unity characterizing the measurement accuracy of angle like variables. This leads in the case of symmetric spaces to a general p-adicization recipe. One starts from a discrete variantnof the symmetric space defined for which points correspond to roots of unity and replaces each discrete point with is p-adic completion representing the p-adic variant of the symmetric space. There is infinite hierarchy of p-adicizations corresponding to measurement resolutions and to the choice of preferred coordinates and the interpretation is in terms of cognitive representations and refined view about General Coordinate Invariance taking into account the fact that cognition is also part of the quantum state.

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane.

1. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is also such a function.

2. For instance, residue calculus essential in the construction of N-point functions of conformal field theory might be generalized so that the value of an integral along the real axis could be

calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

3. Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition.

4. For instance, the idea that number theoretically critical partonic 2-surfaces are expressible in terms of rational functions with rational or algebraic coefficients so that also p-adic variants of these surfaces make sense, is very attractive.

5. Finite sums and products respect algebraic number property and the condition of finiteness is coded naturally by the notion of finite measurement resolution in terms of the notion of (number theoretic) braid. This simplifies dramatically the algebraic continuation since configuration space reduces to a finite-dimensional space and the space of configuration space spinor fields reduces to finite-dimensional function space.

The real configuration space can well contain sectors for which p-adicization does not make sense. For instance, if the exponent of Kähler function and Kähler function are not expressible in terms of algebraic functions with rational or at most algebraic functions or more general functions making sense p-adically, the continuation is not possible. p-Adic non-determinism in p-adic sectors makes also impossible the continuation to real sector. All this is consistent with vision about rational and algebraic physics as as analog of rational and algebraic numbers allowing completion to various continuous number fields.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. For instance, most points p-adic space-time sheets reside at infinity in real sense and p-adically infinitesimal is infinite in real sense. Two types of cutoffs are predicted p-adic length scale cutoff and a cutoff due to phase resolution related to the hierarchy of Planck constants. Zero energy ontology provides natural realization for the p-adic length scale cutoff. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and quantum phases $exp(i2\pi/n)$, $n \geq 3$, coming as roots of unity and defining extensions of rationals and p-adics allowing to define p-adically sensible trigonometric functions. These phases relate closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra.

2.1.7 For The Reader

Most of this chapter has been written for about decade before the above discussion of number theoretical universality and criticality. Therefore the chapter in its original form reflects the first violent burst of ideas of an innocent novice rather than the recent more balanced vision about the role of number theory in quantum TGD. For instance, in the original view about number theoretic universality is the strong one and is un-necessarily restricting. Although I have done my best to update the sections, the details of the representation may still reflect in many aspects quantum TGD as I understood it for a decade ago and the recent vision differs dramatically from this view.

The plan of the chapter is following. In the first one half I describe general ideas as they emerged years ago in a rather free flowing "Alice in the Wonderland" mood. I also describe phenomenological applications, such as conjectures about number theoretic anatomy of coupling constants which are now at rather firm basis. The chapter titled "The recent view about Quantum

TGD" represents kind of turning point and introduces quantum TGD in its recent formulation in the real context. The remaining chapters are devoted to the challenge of understanding p-adic counterpart of this general theory.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

2.2 How P-Adic Numbers Emerge From Algebraic Physics?

The new algebraic vision leads to several generalization of the p-adic philosophy. Besides p-adic topologies more general rational-adic topologies are possible. Topology is purely dynamically determined and ..-adic topologies are quite "real". There is a physics oriented review article by Brekke and Freund [A26]. The books of Gouvêa [A40] and Khrennikov [A19] give a more mathematics-oriented views about p-adics.

This section is written before the discovery that it is possible to generalize the notion of the number field by the fusion reals and various p-adic numbers fields and their extensions together along common rationals (and also common algebraic numbers) to form a book like structure. The interpretation of p-adic physics as physics of intention and cognition removes interpretational problems. This vision provides immediately an answer to many questions raised in the text. In particular, it leads naturally to a complete algebraic democracy. The introduction of infinite primes, which are discussed in next chapter, extends the algebraic democracy even further and gives hopes of describing mathematically also mathematical cognition.

2.2.1 Basic Ideas And Questions

It is good to list the basic ideas and pose the basic question before more detailed considerations.

Topology is dynamical

The dynamical emergence of p-adicity is strongly supported both by the applications of p-adic and algebraic physics. The solutions of polynomial equations involving more than one variable involve roots of polynomials. Only roots in the real algebraic extensions of rationals are allowed since the components of quaternions must be real numbers. When the root is complex in real topology, one can however introduce p-adic topology such that the root exists as a number in a real extension of p-adics. In p-adic context only a finite-dimensional algebraic extension of rational numbers is needed. The solutions of the derivative conditions guaranteeing Lagrange manifold property involve p-adic pseudo constants so that the p-adic solutions are non-deterministic. The interpretation is that real roots of polynomials correspond to geometric correlates of matter whereas p-adic regions are geometric correlates of mind in consistency with the p-adic non-determinism.

Does this picture imply the physically attractive working hypothesis stating that the decomposition of infinite prime into primes of lower level corresponds to a decomposition of the space-time surface to various p-adic regions appearing in the definition of the infinite prime? Generating infinite primes correspond to quaternionic rationals and these rationals contain powers of quaternionic primes defining the infinite prime. The convergence of the power series solution of the polynomial equations defining space-time surface might depend crucially on the norms of these rationals in the p-adic topology used. This could actually force in a given space-time region p-adic topology associated with some prime involved in the expansion. This is in complete accordance with the idea that p-adic topologies are topologies of sensory experience and real topology is the topology of reality.

Various generalizations of p-adic topologies

p-Adicized quaternions is not a number field anymore. One could allow also rational-adic extensions [A19] for which pinary expansions are replaced by expansions in powers of rational. These extensions give rise to rings with unit but not to number fields. In this approach p-adic, or more generally rational-adic, topology determined by the algebraic number field on a given space-time sheet would be absolutely "real" rather than mere effective topology. Space-time surface decomposes into regions which look like fractal dust when seen by an observer characterized by different number field unless the observer uses some resolution.

This approach suggests even further generalizations. The original observation stimulated by the work with Riemann hypothesis was that the primes associated with the algebraic extensions of rationals, in particular Gaussian primes and Eisenstein primes, have very attractive physical interpretation. Quaternionic primes and rationals might in turn define what might be regarded as non-commutative generalization of the p-adic and rational-adic topology.

...-Adic topology measures the complexity of the quantum state

The higher the degree of the polynomial, and thus the number of particles in the physical state and its complexity, the higher the algebraic dimension of the rational quaternions. A complete algebraic and quaternion and octonion-dimensional democracy would prevail. Accordingly, space-time topology would be completely dynamical in the sense that space-time contains both rational-adic, p-adic regions, and real regions. Physical evolution could be seen as evolution of mathematical structures in this framework: p-adic topologies would be obviously winners over rational-adic topologies and p-adic length scale hypothesis would select the surviving p-adic topologies. For instance, Gaussian-adic and Eisenstein-adic topologies would in turn be higher level survivors possibly associated with biological systems.

Dimensional democracy would be realized in the sense that one can regard the space-time sheets defining n-sheeted topological condensate also as a $4n$-dimensional surface in H^n. This hypothesis fixes the interactions associated with the topological condensation, and the hierarchical structure of the topological condensate conforms with the hierarchical ordering of the quaternionic arguments of the polynomials to which infinite primes are mapped. Polynomials (infinite integers) at a given level of hierarchy in turn can be interpreted in terms of formation of bound states by the formation of join along boundaries bonds/flux tubes.

Is adelic principle consistent with the dynamical topology?

There is competing, and as it seems, almost diametrically opposite view. Just like adelic formula allows to express the norm of a rational number as product of its p-adic norms, various algebraic number fields and even more general structures such as quaternions allowing the notion of prime, provide a collection of incomplete but hopefully calculable views about physics. The net description gives rise to quantum TGD formulated using real numbers. These descriptions would be like summary over all experiences about world of conscious experiencers characterized by p-adic completions of various four-dimensional algebraic number rationals. What is important is that the descriptions using algebraic number fields or their generalization might be calculable. This view need not be conflict with the dynamical view and one could indeed claim that the p-adic physics associated with various algebraic extensions of rational quaternions provide a model about physics constructed by various conscious observers. For a given quantum state there would be however minimal algebraic extension containing all points of the space-time surface in it.

2.2.2 Are More General Adics Indeed Needed?

The considerations related to Riemann hypothesis inspired the notion of G- and E-adic numbers in which rational prime p is replaced with Gaussian or Eisenstein prime. The notion of Eisenstein prime is so attractive because it makes possible to circumvent the complexification of p-adic numbers for $p\ mod\ 4 = 1$ for which $\sqrt{-1}$ exists as a p-adic number. What forces to take the notion of G-adics very seriously is that Gaussian Mersennes correspond to the p-adic length scale of atomic nucleus and to important biological length scales in the range between 10 nanometers and few micrometers. Also the key role of Golden Mean τ in biology and self-organizing systems could be understood if $Q(\tau, i)$ defines D-adic topology. Thus there is great temptation to believe that the notion of p-adic number generalizes in these sense that any irreducible associated with real or complex algebraic extension defines generalization of p-adic numbers and that these extensions appear in the algebraic extensions of quaternions.

Thus one must consider seriously also generalized p-adic numbers, D-adics as they were called in [K85]. D-adics would correspond to powers series of a prime belonging to a complex algebraic extension of rationals. Quaternions decompose naturally in longitudinal and transversal part and transversal part can be interpreted as a complex algebraic extension of rationals in the case of both M^4 and CP_2. Thus some irreducibles of this complex extension could define a generalization of p-adic numbers used to define the algebraic extension of rational quaternions reduced to a pair of complex coordinates.

Perhaps one could go even further: quaternion-adics defined as power series of quaternionic primes of norm p suggest themselves. What would be nice that this prime could perhaps be interpreted as a representation for the momentum of corresponding space-time sheets. The components of the prime belong to algebraic extension of rationals and would even code information about external world if the proposed interpretations are correct. One can also ask whether quaternionic primes could define what might be called quaternion-adic algebras and whether these algebras might be a basic element of algebraic physics.

This would mean that space-time topology would code information about the quantum numbers of a physical state. Rings with unit rather than number fields are in question since the p-adic counterparts of quaternionic integers in general fail to have inverse. It must be emphasized that the field property might not be absolutely essential. For instance "rational-adics" [A19], for which prime p is replaced with a rational q such that norm comes as a power of q, exists as rings with unit and define topology. Rational-adic topologies could have also quaternionic counterparts.

The idea of q-rational topologies is supported by the physical picture about the correspondence between Fock states and space-time sheets. Single 3-surface can in principle carry arbitrarily high fermion and boson numbers but is unstable to a topological decay to 3-surfaces carrying single fermion and boson states. The translation of this statement to...-adic context would be that the Fock states associated with infinite primes which correspond to rational-adic quaternionic topologies are unstable against decay to states described by polynomial primes in which each factor corresponds to prime (bosons) or its inverse (fermions) in algebraic extension of quaternions. This tendency to evolve to prime-adic topologies could be seen also as a manifestation of p-adic evolution and self-organization. Rational-adic topologies would be simply losers in the fight for survival against topologies defining number fields. Since also quaternion-adic topologies fail to define number fields they are expected to be losers in the fight for survival. Winners would be...-adic topologies defining number fields. At the level of Fock states this would mean the instability of states which contain more than one prime: that this is indeed the case, is one of the basic assumptions of quantum TGD forced by the experimental fact that elementary particles correspond to simplest Fock states associated with WCW spinor s.

2.2.3 Why Completion To P-Adics Necessarily Occurs?

There is rather convincing argument in favor of...-adic physics. Typically one must find zeros of rational functions of several variables. Simplifying somewhat, at the first level one must find zeros of polynomials $P(x_1, x_2)$. Newton's theorem states that the monic polynomial $P_n(y, x) = y^n + a_{n-1}x^{n-1} + ..$ allows a factorization in an algebraically closed number field

$$P(y, x^m) = \prod_k (y - f_k(x)) \ . \tag{2.2.1}$$

Here f_k are polynomials and m is integer which divides n and equals to n for an irreducible polynomial P. Since the multiplication of x by m: th root of unity (ζ_m) leaves left hand side invariant it must permute the factors on right hand side. Thus one can express the formula also as

$$P(y, x) = \prod_{k=1,..m} (y - f_k(\zeta_m^k x^{1/m})) \ . \tag{2.2.2}$$

When number field is not algebraically closed this means that one must introduce an algebraic extension by m: th roots of all rationals.

The problem is that these roots are not real in general and one cannot solve the problem by using a completion to complex numbers since only real extensions for the components of quaternion are possible. Only in the region where some of the roots of the polynomial are real, this is possible. The only manner to achieve consistency with the reality requirement is to allow p-adic topology or possibly rational-adic topology: in this case also the algebraic extension allowing m: th roots is always finite-dimensional. For instance, for $m = 2$ p-adic extension of rationals would be 4-dimensional for $p > 2$. The situation is similar for rational-adic topology.

If this argument is correct, one can conclude that real topology is possible only in the regions where real roots of the polynomial equation are possible: in the regions where all roots are complex, p-adicization gives rise to roots in the algebraic extension of p-adics and p-adic topology emerges naturally. This picture provides a precise view about how the space-time surface defined by the polynomial of quaternions decomposes to real and p-adic regions. Also a connection with catastrophe theory [A35] emerges: the boundaries of the catastrophe regions where some roots coincide, serve also as boundaries between...-adic and real regions.

2.2.4 Decomposition Of Space-Time To...-Adic Regions

Number-theoretic constraints are important in determining which...-adic topologies are possible in a given space-time region. There is no hope of building any unique vision unless one poses some general principles. Complete algebraic and topological democracy and the generalization of the notion of p-adic evolution to what might be called rational-adic evolution allow to build plausible and sufficiently general working hypothesis not requiring too much ad hoc assumptions and allowing at least mathematical testing. A further natural principle states that the topology for a given region is such that complex extension of rationals is not needed and that the series defining the normal quaternionic coordinate as function of the space-time quaternionic coordinate converges and gives rise to a smooth surface.

The power series defining solutions of polynomial equations must converge in some topology

The roots of polynomials of several variables can be expressed as Taylor series. When the root is complex, real topology is not possible and some p-adic topology must be considered. This suggests a very attractive dynamical mechanism of p-adicization. In the regions where the root belongs to a complex extension of rationals in the real topology, one could find those values of p for which the series converges p-adically. The rational numbers characterizing the polynomials associated with the generating infinite primes certainly determine the convergence and the primes for which p-adic convergence occurs are certainly functions of these rationals. Hence it could occur that the p-adic topologies for which convergence occurs correspond to the primes appearing as factors in these rationals.

In this approach topology is a result of dynamics. Note that also the notion of symmetry depends on the region of space-time. Contrary to the basic working hypothesis, ...-adic topology of a given space-time sheet is its "real" topology rather than being only an effective topology and the topology of space-time is completely dynamical being dictated by algebraic physics and smoothness requirement.

It is also possible that convergence does not occur with respect to any ...-adic topology and in this case the topology would be discrete. This situation would correspond to primordial chaos but still the algebraic formulation and Fock space description of the theory would make sense.

Space-time surfaces must be smooth in the completion

The completion must give rise to a smooth or at least continuous ....-adic or real surface defining a critical extremal of Kähler action in the sense of having an infinite number of deformations for which the second variation of Kähler action vanishes. This requirement might allow only finite number of...-adic topologies for a given space-time region. If the completion involves functions expandable in powers of a (possibly quaternionic) rational $q = m/n$, then the prime factors of m define natural p-adic number fields for which completion is possible. Also q itself could define rational-adic topology. Since the space-time surface decomposes into regions labeled by rationals in an algebraic extension of rationals q_1, there is interesting possibility that q_1 as such defines the rational-adic topology so that there would be no need to understand why the space-time region labeled by q decomposes into space-time sheets labeled by the prime factors of q.

Whatever the details of the coding are, the coding would mean that the quantum numbers associated with the space-time sheet would determine the generalized...-adic topology associated with it. The information about quantum systems would be mapped to space-time physics and the coding of quantum numbers to...-adic topology would solve at a general level the problem how the information about quantum state is coded into the structure of space-time.

2.2.5 Universe As An Algebraic Hologram?

Quaternionic primes have a natural indentification as four-momenta. If the Minkowski norm for the quaternion is defined using the algebraic norm of the real extension of rationals involved with the state, mass squared is integer-valued as in super-conformal theories. The use of the algebraic norm means a loss of information carried by the units of the real algebraic extension $K(\theta)$ (see the appendix of this chapter). Hence one can say that besides ordinary elementary particle quantum numbers there are algebraic quantum numbers which presumably carry algebraic information. Very effective coding of information about quantum numbers becomes possible and these quantum numbers commute with ordinary quantum numbers. This information does not become manifest for matter-like regions where a real completion of rationals are used. In p-adic regions representing geometric correlates of mind the situation is different since p-adic number field in question is a finite algebraic extension of rationals.

Almost every calculation is approximation and completion to reals or p-adics makes possible to measure how good the approximation is. Real numbers are extremely practical in this respect but the failure of the real number based physics is that it reduces number to a mere quantity having a definite size but no number-theoretical properties. This is practical from the point of view of numerics but means huge loss of capacity for information storage and representation. In algebraic number theory number contains representation for its construction recipe. It seems that the correct manner to see numbers is as elements of the state space provided by the algebraic extension. p-Adic physics using p-adic versions of the algebraic extensions does not lead to a loss of this information unlike real physics. Thus the basic topology of the space-time sheet could code the quantum numbers associated with it.

Since the algebraic extension of rationals, and hence also of p-adics, depends on the number of particles present in the Fock state coded by the infinite prime, the only possible interpretation is that the additional quantum numbers code information about the many-particle state. Hence the idea about "cognitive representation" of the fractal quantum numbers of particles of the external world suggests itself naturally. In particular, the degree of the minimal polynomial for the real extension $Q(\theta)$ is n, where n is the number of particles in the Fock state in the case the resulting state represents infinite prime. This means that there are $n-1$ quantum numbers represented by fractal scalings (see Appendix for Dirichlet's unit theorem). The interpretation as a representation for the fractal quantum numbers representing information about states of other particles in the system suggests itself. One cannot exclude the possibility that the fractal quantum numbers represent momenta or some other quantum numbers of other particles.

If this rather un-orthodox interpretation is correct, then cognitive representations are present already at the elementary particle level in p-adic regions associated with particles and are realized as algebraic holograms. Universe as a Computer consisting of sub-computers mimicking each other would be realized already at the elementary particle level. This view is consistent with the TGD inspired theory of consciousness. Algebraic physics would also make possible kind of a Gödelian loop by providing a representation for how the information about the structure of a physical system is coded into its properties.

This view has also immediate implications for complexity theory. The dimension of the minimal algebraic extension containing the algebraic number is a unique measure for its complexity. More concretely: the degree of the minimal polynomial measures the complexity. Everyone can solve second order polynomial but very few of us remembers formulas for the roots of fourth order polynomials. For higher orders quadratures do not even exist. Of course, numbers represent typically coordinates and this is consistent with the general coordinate invariance only if some preferred coordinates exist. In TGD based physics these coordinates exist: embedding space allows (apart from isometries) unique coordinates in which the components of the metric tensor are rational functions of the coordinates.

Similar realization is fundamental in the second almost-proof of Riemann hypothesis described in [K85]. In this case ζ is interpreted as an element in an infinite-dimensional algebraic extension of rationals allowing all roots of rationals. The vanishing of ζ requires that all components of this infinite-dimensional vector contain a common rational factor which vanishes. This is possible only if an infinite number of partition functions in the product representation of the modulus squared of ζ are rational and their product vanishes. This implies Riemann hypothesis. The assumption that only square roots of rationals are needed is very probably wrong and must be replaced with the assumption that p^{iy} is algebraic numbers when $z = 1/2 + iy$ is zero of ζ for any prime p. It is quite possible that the almost-proof survives this generalization.

The notion of Platonia discussed already in the introduction adds cognition to this picture and allows to understand where all those mathematical structures continually invented by mathematicians but not realized physically in the conventional sense of the word reside. This notion takes also the notion of algebraic hologram to its extreme by making space-time points infinitely structured.

2.2.6 How To Assign A P-Adic Prime To A Given Real Space-Time Sheet?

p-Adic mass calculations force to assign p-adic prime also to the real space-time sheets and the longstanding problem is how this p-adic prime, or possibly many of them, are determined. Number theoretic view about information concept provides a possible solution of this long-standing problem.

Number theoretic information concept

The notion of information in TGD framework differs in some respects from the standard notion.

1. The definition of the entropy in p-adic context is based on the notion p-adic logarithm depending on the p-adic norm of the argument only ($Log_p(x) = Log_p(|x|_p) = n$) [K56]. For rational- and even algebraic number valued probabilities this entropy can be regarded as a real number. The entanglement entropy defined in this manner can be negative so that the entanglement can carry genuine positive information. Rationally/algebraically entangled p-adic system has a positive information content only if the number of the entangled state pairs is proportional to a positive power of the p-adic prime p.

2. This kind of definition of entropy works also in the real-rational/algebraic case and makes always sense for finite ensembles. This would have deep implications. For ordinary definition of the entropy NMP [K56] states that entanglement is minimized in the state preparation process. For the number theoretic definition of entropy entanglement could be generated during state preparation for both p-adic and real sub-systems, and NMP forces the emergence of p-adicity (say the number of entangled state is power of prime). The fragility of quantum coherence is the basic problem of quantum computations and the good news would be that Nature itself (according to TGD) tends to stabilize quantum coherence both in the real and p-adic contexts.

3. Quantum-classical correspondence suggests that the notion of information is well defined also at the space-time level. In the presence of the classical non-determinism of Kähler action and p-adic non-determinism one can indeed define ensembles, and therefore also probability distributions and entropies. For a given space-time sheet the natural ensemble consists of the deterministic pieces of the space-time sheet regarded as different states of the same system.

Are living systems in the intersection of real and p-adic world?

NMP combined with number theoretic entropies leads to an important exception to the rule that the generation of bound state entanglement between system and its environment during U process leads to a loss of consciousness. When entanglement probabilities are rational (or even algebraic) numbers, the entanglement entropy defined as a number theoretic variant of Shannon entropy can be negative so that entanglement carries information. NMP favors the generation of algebraic entanglement. The attractive interpretation is that the generation of algebraic entanglement leads to an expansion of consciousness ("fusion into the ocean of consciousness") instead of its loss. Rational and even algebraic entanglement coefficients make sense in the intersection of real and p-adic words, which suggests that life and conscious intelligence reside in the intersection of the real and p-adic worlds. Life would represent number theoretically criticality so that the quantum criticality of TGD Universe would allow to understand also life.

1. To be in the intersection of real and p-adic worlds means that partonic 2-surfaces and their 4-D tangent planes representing the information about space-time sheet (holography) have a mathematical representation allowing an interpretation either as a real or p-adic surface (just like rationals can be regarded as being common to reals and p-adic numbers). Number theoretical criticality makes also possible the transformation of intentions to actions as transformations of a p-adic 2-surfaces to a real 2-surfaces via leakage through this common intersection. This process makes sense only in zero energy ontology. This would generalize the observation that rationals and algebraics in a well-defined sense represent islands of order in the seas of chaos defined by real and p-adic continua.

2. A more concrete interpretation for the intersection of real and p-adic worlds would be as the intersection of real and p-adic variants of space-time surface allowing interpretation in both number fields. This intersection is discrete set containing besides rational points also algebraic points common to reals and algebraic extension of p-adics involved.

3. These two interpretations for the intersection of real and p-adic worlds need not be independent. The absence of definite integral in p-adic number fields suggests that the transition amplitudes between p-adic and real sectors must be expressible using only the data associated with rational and common algebraic points (in the algebraic extension of p-adic numbers used) of embedding space. This intersection is discrete and could even consist of a finite number of points. For instance, Fermat's last theorem tells that the surface $x^n+y^n = z^n$ contains only origin as rational point for $n = 3, 4, ...$ whereas for $n = 2$ it contains all rational multiples of integer valued points defining Pythagorean triangles: this is due to the homogenuity of the polynomial in question. Therefore p-adic-to real transition amplitudes would have a purely number theoretical interpretation. One could speak of number theoretical field theory as an analogy for topological field theory.

Does space-time sheet represent integer and its prime factorization?

A long-standing problem of quantum TGD is how to associate to a given real space-time sheet a (not necessarily) unique p-adic prime as required by the p-adic length scale hypothesis. One could achieve this by requiring that for this prime the negentropy associated with the ensemble is maximal. The simplest hypothesis is that a real space-time sheet consisting of N deterministic pieces corresponds to p-adic prime defining the largest factor of N. One could also consider a more general possibility. If N contains p^n as a factor, then the real fractality above n-ary p-adic length scale $L_p(n) = p^{(n-1)/2}L_p$ corresponds to smoothness in the p-adic topology. This option is more attractive since it predicts that the fundamental p-adic length scale L_p for a given p can be effectively replaced by any integer multiple NL_p, such that N is not divisible by p. There is indeed a considerable evidence for small p p-adicity in long length scales. For instance, genetic code and the appearance of binary pairs like cell membrane consisting of liquid layers suggests 2-adicity in nano length scales. This view means that the fractal structure of a given real space-time sheet represents both an integer N and its decomposition to prime factors physically. This obviously conforms with the physics as a generalized number theory vision.

Quantum-classical correspondence suggests that quantum computation processes might have counterparts at the level of space-time. An especially interesting process of this kind is the factorization of integers to prime factors. The classical cryptography relies on the fact that the factorization of large integers to prime factors is a very slow process using classical computation: the time needed to factor 100 digit number using modern computer would take more than the recent age of the universe. For quantum computers the factorization is achieved very rapidly using the famous Shor's algorithm. Does the factorization process indeed have a space-time counterpart?

Suppose that one can map the integer N to be factored to a real space-time sheet with N deterministic pieces. If one can measure the powers $p_i^{n_i}$ of primes p_i for which the fractality above the appropriate p-adic length scale looks smoothness in the p-adic topology, it is possible to deduce the factorization of N by direct physical measurements of the p-adic length scales characterizing the representative space-time sheet (say from the resonance frequencies of the radiation associated with the space-time sheet). If only the p-adic topology corresponding to the largest prime p_1 is realized in this manner, one can deduce first it, and repeat the process for N/p_1^n, and so on, until the full factorization is achieved. A possible test is to generate resonant radiation in a wave guide of having length which is an integer multiple of the fundamental p-adic length scale and to see whether frequencies which correspond to the factors of N appear spontaneously.

2.2.7 Gaussian And Eistenstein Primes And Physics

Gaussian and Eisenstein primes could give rise to what might be called G- and E-adicities and also these -adicities might be of physical interest.

Gaussian and Eisenstein primes and elementary particle quantum numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

1. The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales L_p: this suggests a direct connection with quantum TGD.

2. Each non-degenerate (purely real or imaginary) Gaussian prime of given norm p corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the embedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spin like quantum number but is analogous to orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states. The 4-fold degeneracy associated with the $p\ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electro-weak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

3. For Eisenstein prime E_{p_1} the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Eisenstein primes $r + isw$ and $s + irw$ have however the same norm squared so that also now the 8-fold degeneracy is present. When p_1^{iy} is of the general form $r + i\sqrt{k}s$ this degeneracy is not present.

4. The basic character of the quark color is triality realized as phases w which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, w or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor Z_{p_1} with $p_1\ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor Z_{p_1} satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime p_i associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

   There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. ζ decomposes into a product $\zeta_1 \times \zeta_3$, such that ζ_1 is the product of $p\ mod\ 4 = 1$ partition functions and ζ_3 the product of $p\ mod\ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p\ mod\ 4 = 3$.

5. What about the most general factors $r+is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of n ($r^2+s^2k=n^2$). The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

6. What about interpretation of the components of the complex integers? For Super Virasoro representations basic quantum numbers of this kind correspond to energy and longitudinal momentum. This suggests the interpretation of r^2+s^2k as energy, r^2-s^2k as mass, and $2rs\sqrt{k}$ as momentum. For the squares $r^2-s^2+(2rs-s^2)w$ of Eisenstein primes $r^2-s^2/2-rs$ corresponds to mass, r^2+s^2-rs to energy, and $(2rs-s^2)\sqrt{3}/2$ to momentum. Note that the sign of mass changes for Gaussian primes in the interchange $r \leftrightarrow s$. The fact that the hexagonal lattice defined by Eisenstein integers correspond to the root lattice of $SU(3)$ group means that energy, momentum and mass corresponds to the sides of the triangles in the root lattice of color group.

The following argument suggests that finite Gaussian and Eistenstein primes might be forced by zero energy ontology (ZEO)

1. In ZEO M-matrix is in a well-defined sense "complex" square root of density matrix reducing to a product of Hermitian square root of density matrix multiplied by unitary S-matrix. A natural guess is that p-adic thermodynamics possesses this kind of square root or better to say: is modulus squared for it.

2. For fermions the value of p-adic temperature is however $T=1$ and thus maximal. It is not possible to construct real square root by simply taking the square root of thermodynamical probabilities for various conformal weights. One manner to solve the problem is to assume that one has quadratic algebraic extension of p-adic numbers in which the p-adic prime splits as $p=\pi\overline{\pi}$, $\pi = m+\sqrt{-k}n$. For $k=1$ primes $p \mod 4 = 1$ indeed allow a representation as product of Gaussian prime and its conjugate.

3. For primes $p \mod 4 = 3$ this is not the case and Mersenne primes are important examples of these primes. Eisenstein primes provide the simplest extension of rationals splitting Mersenne primes. For Eisenstein primes one has $k=3$ and all ordinary primes satisfying either $p=3$ or $p \mod 3 = 1$ (true for Mersenne primes) allows this splitting. For the square root of p-adic thermodynamics the complex square roots of probabilities would be given by $\pi^{L_0/T}/\sqrt{Z}$, and the moduli squared would give thermodynamical probabilities as $p^{L_0/T}/Z$. Here Z is the partition function.

4. An interesting question is whether $T=1$ for fermions means that complex square of thermodynamics is indeed complex and whether $T=1/2$ for bosons means that the square root is actually real valued.

G-adic, E-adic and even more general fractals?

Still one line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Eisenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers. In fact, the argument generalizes to the case of all nine $\sqrt{-d}$ type extensions of rationals allowing a unique prime decomposition so that one might perhaps speak about D-adics.

1. Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the representation of the algebraic extension involved (complex integers in the case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \quad . \tag{2.2.3}$$

Here z_n is Gaussian integer with norm smaller than $|G_p|$, which equals to p for $p\ mod\ 4 = 3$ and $\sqrt{p}$ for $p\ mod\ 4 = 1$.

2. If any Gaussian integer z has a unique expansion in powers of G_p such that coefficients have norm squared smaller than p, modulo G arithmetics makes sense and one can construct the inverse of G and number field results. This is the case if Gaussian integers behave with respect to modulo G_p arithmetics like finite field $G(p, 2)$. For $p\ mod\ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in the case$p\ mod\ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p\ mod\ 4 = 1$, perhaps some kind of square root of R_p, and if they indeed form a number field, do they reduce to some known algebraic extension of R_p?

3. In the case of Eisenstein numbers one can identify the coefficients z_n in the formal power series $E = \sum z_n E_p^n$ as Eisenstein numbers having modulus square smaller than p associated with E_p and similar argument works also in this case.

4. As already noticed, in the case of complex extensions of form $r + \sqrt{-d}s$ a unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [A38]. The poor man's argument above does not distinguish between G- and E-adics ($d = 1, 3$) and these extensions.One might perhaps call this extensions generally D-adics. This suggests that generalized p-adics could exist also in this case. In fact, the generalization p-adics could make sense also for higher-dimensional algebraic extensions allowing unique prime decomposition. For $d = 2$ complex algebraic primes are of form $r + s\sqrt{-2}$ satisfying the condition $r^2 + 2s^2 = p$. For $d > 2$ complex algebraic primes are of form $(r + s\sqrt{-d})/2$ such that both r and s are even or odd. Quite generally, the condition $p\ mod\ d = k^2$ must be satisfied. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$ so that the powers of a complex primes in this case have a finite number of possible phase angles: this might make G- and E-adics physically special.

TGD suggests rather interesting physical applications of D-adics.

1. What is interesting from the physics point of view is that for $p\ mod\ 4 = 1$ the points D_p^n are on the logarithmic spiral $z_n = p^{n/2} exp(in\phi_0/2)$, where ϕ is the phase associated with D_p^2. The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: D-adics might provide the mathematics for the modelling of these structures.

2. p-Adic length scale hypothesis should hold true also for Gaussian primes, in particular, Gaussian Mersennes of form $(1 \pm i)^k - 1$ should be especially interesting from TGD point of view.

   (a) The integers k associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79, 113$. $k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon. Thus all known charged leptons, rather than only e and τ, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

   (b) The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 μm. This strongly suggests that cellular organisms have evolved to their present form through four basic stages.

   (c) $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 min. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about 2.6×10^6 light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely super-astronomical time and length scale.

3. Eisenstein integers form a hexagonal lattice equivalent with the root lattice of the color group $SU(3)$. Microtubular surface defines a hexagonal lattice on the surface of a cylinder which suggests an interpretation in terms of E-adicity. Also the patterns of neural activity form often hexagonal lattices.

Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates a further line of thoughts.

1. As has been found, the zeros of ζ code for the physical states of a super-symmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labeled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or ± 1-factor which could be interpreted as a phase factor associated with any $p \ mod \ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

2. The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure of the previous chapter allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD.

2.2.8 P-Adic Length Scale Hypothesis And Quaternionic Primality

p-Adic length scale hypothesis states that fundamental length scales correspond to the so called p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives strong support for this hypothesis. Elementary particles correspond to the so called CP_2 type extremals in TGD. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [A70] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one can identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime. The crucial point from the TGD point of view is the appearance of the *square* of the norm instead of the norm. One can even define the product of spheres $R^2 = n_1$ and $R^2 = n_2$ by defining the product sphere with norm squared $R^2 = n_1 n_2$ to consist of the quaternions, which are products of quaternions with norms squared n_1 and n_2 respectively. Prime spheres correspond to $n = p$. The powers of sphere p correspond to a multiplicatively closed structure consisting of powers p^n of the sphere p. It is also possible to speak about the multiplication of balls and prime balls in the case of integer quaternions.

p-Adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the p-adic QFT limit of TGD as also in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k: th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface. This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the criticality of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

2.3 Scaling Hierarchies And Physics As A Generalized Number Theory

The scaling hierarchies defined by powers of Φ and primes p probably reflect something very profound. Mueller has proposed also a scaling law in powers of e [B1]. This scaling law can be however questioned since $\Phi^2 = 2.6180..$ is rather near to $e = 2.7183...$ Note that powers of e define p-dimensional extension of R_p since e^p exists as a p-adic number in this case.

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a way analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of WCW spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type II_1 (not that WCW corresponds here to the "world of classical worlds"). The emergence of powers of e has been discussed in [K85] and will not be discussed here.

2.3.1 P-Adic Physics And The Construction Of Solutions Of Field Equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, WCW spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field R_p as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers n decomposing into products of powers of primes p_i. One can expect that for p_i-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the Kähler-Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results. The value of the dynamical Planck constant characterizes in this approach the scale factor of the M^4 metric in various number theoretical variants of the embedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of H. The values of $\hbar$ are determined from the requirement of quantum criticality [K110] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the embedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete embedding spaces.

p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [K14]. This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases “short distance” and “long distance”. The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement “Short length scale physics codes for long length scale physics” the attribute “short”/“long” could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational embedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

1. Select a rational point of the embedding space and solve field equations in the real sense in an arbitrary small neighborhood U of this point. This can be done with an arbitrary accuracy by chosing U to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

2. Select a subset of rational points in U and interpret them as points of p-adic embedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

3. Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to 1) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

1. Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

2. The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

3. Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

4. It is also possible to generalize the procedure by changing the value of p at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

5. One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labeled by

$p_1, ..., p_n$ belong to the real surface. This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ..., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

p-Adic length scale hypothesis

Approximate p_1-adicity implies also approximate p_2-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor λ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

1. The mass calculations based on p-adic thermodynamics for Virasoro generator L_0 predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that L_p is proportional to $\sqrt{p}$ rather than p, which looks more natural if common rationals define the correspondence between real and p-adic physics.

2. It would seem that length $d_p \simeq pR$, R or order CP$_2$ length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in M^4. This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with M^4 distance r_{M^4} has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP$_2$ degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in CP_2 degrees of freedom so that r_{X^4} increases and can depend non-linearly on r_{M^4}.

3. If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding M^4 size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

4. The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to L_p. At longer length scales the space-time distance d_p associated with L_p becomes the unit of space-time distance and geodesic distance r_{X^4} is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \quad , \tag{2.3.1}$$

and is thus linear in M^4 distance r_{M^4}.

Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers m and n in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

2.3.2 A More Detailed View About How Local P-Adic Physics Codes For P-Adic Fractal Long Range Correlations Of The Real Physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of M^4 and CP_2 coordinates and to the mapping of p-adic H-coordinates to their real counterparts and vice versa.

How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, m and n have finite power expansions in powers of p and one can always write $q = p^k \times r/s$ such that r and s are not divisible by p and thus have pinary expansion of in powers of p as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where y has p-adic norm 1 and has expansion in non-negative powers of p. When x is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number x can be written as $x = p^n y$, where y is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of p. As a p-adic number y is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book). In the following I shall be somewhat sloppy and treat the rational points of the embedding space as if they were points of real axis in order to avoid clumsy formulas.

1. The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of M^4. The possibility of p-adic pseudo constants means that for rational points of M^4 having sufficiently large p-adic norm, the values of CP_2 coordinates or induced spinor fields can be chosen more or less freely.

2. One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point q of M^4, where y has a unit p-adic norm and select the values of fields at different points q_1 and q_2 freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

The points in the p-adic continuum part of these solutions are at an infinite distance from q in M^4. The points which are well-defined in real sense form a discrete subset of rational points of M^4. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

3. All rational points q of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q + p^n y\}$ corresponding to real numbers in the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

p-Adic scalings act only in M^4 degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the M^4 coordinates having origin at $x = 0$ by p^n of the solution defined in a neighborhood of x and leaving CP_2 coordinates as such. The known extremals of Kähler action indeed allow M^4 scalings as dynamical symmetries.

One can understand why no scaling should appear in CP_2 degrees of freedom. CP_2 is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space S in $H = M^4 \times S$ must be a projective space.

What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real M^4 coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic CP_2 coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of CP_2 coordinates as function of M^4 coordinates for solutions representable as maps $M^4 \rightarrow CP_2$. Indeed, if the rational valued p-adic CP_2 coordinates are mapped as such to real coordinates, the addition of large power p^n to CP_2 coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of CP_2 at p-adically scaled M^4 points would be completely lost.

The situation changes if the map of p-adic CP_2 coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

1. Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

2. The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{2.3.2}$$

is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

3. The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of CP_2 maximally.

Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of CP_2 are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary 2×2-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

4. The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of p in r and s $(q = p^n r/s)$ are not higher than p^N. Write $x = \sum_{n\geq 0} x_n p^n = x^{N)} + p^{N+1}y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1}I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

5. There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric 2×2-matrix ϵ_{ij} satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \ mod \ 4 = 3$ imaginary unit should be introduced as an algebraic extension and CP_2 in this sense would be an algebraic extension of RP_2. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of CP_2. For $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

Preferred CP_2 coordinates as a space-time correlate for the selection of quantization axis

Complex CP_2 coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the embedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

Relationship between real and p-adic induced spinor fields

Besides embedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as CP_2 coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [K90]. Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

1. Realization of intention as a scattering process

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real M^4) to a real finite sized surface (infinitely distant p-adic surface in p-adic M^4) would be in question.

2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [K25], the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

1. Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

2. The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of embedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed. Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in the latter case. The inherent nature of cognition would be that it favors localization in the position space.

3. Objection and its resolution

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the Kähler-Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [K111, K14]. The idea is that Dirac determinant can be defined as a product of eigenvalues of the Kähler-Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Planck constant. The elegance of this approach is that no discretization at space-time level would be neededeverything reduces to the generalized eigenvalue spectrum of the Kähler-Dirac operator.

4. A more detailed view

Consider the proposed approach in more detail.

1. Fermionic oscillator operators are assigned with the generalized eigenvectors of the Kähler-Dirac operator defined at the light-like causal determinants:

$$\begin{aligned} \Psi &= \sum_n \Psi_n b_n \ , \\ D\Psi_n &= \Gamma^\alpha D_\alpha \Psi_n = \lambda_n O \Psi_n \ , \ \ O \equiv n_\alpha \Gamma^\alpha \ . \end{aligned} \quad (2.3.3)$$

   Here $\Gamma^\alpha = T^{\alpha k}\Gamma_k$ denote so called Kähler-Dirac gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [K111]. The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the super-symmetries of the Kähler-Dirac action are consistent with the property of being an extremal of Kähler action. n_α is a light like vector assignable to the light-like causal determinant and $O = n_\alpha \Gamma^\alpha$ must be rational and have the same value at real and p-adic side at rational points. The integer n labels the eigenvalues λ_n of the Kähler-Dirac operator, and b_n corresponds to the corresponding fermionic oscillator operator.

2. The condition that the p-adic and real variants Ψ if the Ψ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

3. Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators b_n is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret WCW spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

4. Dirac determinant can be defined as the product of the eigenvalues λ_n restricted to a given algebraic extension of rationals. The solutions of the Kähler-Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

5. Only those operators b_n for which λ_n belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that WCW Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type II_1 for which finite truncations by definition allow excellent approximations [K110]. One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type II_1 would provide an extremely powerful tool.

2.3.3 Cognition, Logic, And P-Adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with n "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space M^4 in terms of generalized Boolean functions emerges naturally so that M^4 projections of p-adic space-time would represent Boolean functions for a logic with n taboos.

2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients f_{nk} in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of N bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in N: th bit of $f(x)$ is introduced, B^M-valued function in B^N results, where B denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by p truth values representable as $0, ..., p-1$.

p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of p^k, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, k integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

$p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectivesandOR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms B^k with n Boolean statements dropped out so that one obtains what might be called $\hat{b^k}$. Since n is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functionsandOR with the constraint that taboos are respected in other words, both the inputs and values of functions belong to $\hat{b^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

The projections of p-adic space-time sheets to real embedding space as representations of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space M^4. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \rightarrow B^{4N}$. Here the number of lattice points in a given coordinate direction of M^4 is M and N is the number of bits allowed by binary cutoff for CP_2 coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{b}^{4M} \rightarrow \hat{b}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of M^4 the lowest bits define a time evolution of a generalized Boolean function: $B \rightarrow B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to B^k with n axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of embedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

1. Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of CP_2 coordinates as a function of 2-adic M^4 coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of N^4 pinary digits of M^4 coordinates and N^4 pinary digits of CP_2 coordinates? Is this time non-polynomial for M^d and S_d, S_d d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

2. The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$. Therefore the question would be whether p-adic length scale hypothesis and dimensions of embedding space and space-time are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of B^k is in question.

Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

1. One-dimensional case

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n \geq n_0} x_n p^n$.

1. p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N + 1$: th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0 \leq n \geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than N in $x = \sum_{n \geq n_0} x_n p^n$ can be chosen arbitrarily.

2. Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

   i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius p^{-n} consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume $N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M \leq n \leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ q is always integer valued and the scaling of x by a suitable power of p always allows to get a finite integer lattice at x-axis.

iii) The lowest pinary digit in the expansion of $f(q)$ in powers of p in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

2. Generalization to the space-time level

This picture generalizes to space-time level in a rather straight forward manner. y is replaced with CP_2 coordinates, x is replaced with M^4 coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of M^4 coordinates is allowed. The restriction of 4 CP_2 coordinates to a finite integer lattice of M^4 defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

2.3.4 Fibonacci Numbers, Golden Mean, And Jones Inclusions

The picture discussed above does not apply in the case of Golden Mean since powers of Φ do not have any special role for the algebraic extension of rationals by $\sqrt{5}$. It is however possible to understand the emergence of Fibonacci numbers and Golden Mean using quantum classical correspondence and the fact that the Clifford algebra and its sub-algebras associated with configuration space spinors corresponds to the so called hyper-finite factor of type II_1 (WCW refers to the "world of classical worlds").

Infinite braids as representations of Jones inclusions

The appearance of hyper-finite factor of type II_1 at the level of basic quantum tGD justifies the expectation that Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of these factors play a key role in TGD Universe. For instance, subsystem system inclusions could induce Jones inclusions.

For the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ $\mathcal{M}$ can be regarded as an $\mathcal{N}$-module with fractal dimension given by Beraha number $B_n = 4cos^2(\pi/n)$, $n \geq 3$ or equivalently by the quantum group phases $exp(i\pi/n)$. B_5 satisfies $B_5 = 4cos^2(\pi/5) = \Phi^2 = \Phi + 1$ so that the special role of $n = 5$ inclusion could explain the special role of Golden Mean in Nature.

Hecke algebras H_n, which are also characterized by quantum phase $q = exp(i\pi/n)$ or the corresponding Beraha number $B_n = 4cos^2(\pi/n)$, characterize the anyonic quantum statistics of n-braid system. Braids are understood as threads which can get linked and define in this manner braiding. Braid group describes these braidings. Like any algebra, Hecke algebra H_n can be decomposed into a direct sum of matrix algebras. Fibonacci numbers characterize the dimensions of these matrix algebras for $n = 5$. Interestingly, topological quantum computation is based on the idea that computer programs can be coded into braidings. What is remarkable is that $n = 5$ characterizes the simplest universal quantum computer so that Golden Mean could indeed have very deep roots to quantum information processing.

The so called Bratteli diagrams characterize the inclusions of various direct summands of H_k to direct summands H_{k+1} in the sequence $H_3 \subset H_4 \subset ... \subset H_k \subset ...$ of Hecke algebras. Essentially the reduction of the representations of H_{k+1} to those of H_k is in question. The same Bratteli diagrams characterize also the Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors of type II_1 with index n as a limit of a finite-dimensional inclusion. Thus Jones inclusion can be visualized as a system consisting of infinite number of braids. In TGD framework the braids could be represented by magnetic flux lines or flux tubes.

Logarithmic spirals as representations of Jones inclusions

The inclusion sequence for Hecke algebras has a representations as a logarithmic spiral. The angle $\pi/5$ can be identified as a limit for angles ϕ_n with $cos(\phi_n) = F_{n+1}/2F_n$ assignable to orthogonal triangle with hypothenuse $2F_n$ and short side F_{n+1} and $\sqrt{4F_n^2 - F_{n+1}^2}$. Fibonacci sequence defines via this prescription a logarithmic spiral as a symbolic representation of the $n = 5$ Jones inclusion representable also in terms of infinite number of braids.

DNA as a topological quantum computer?

Quantum classical correspondence encourages to think that space-time geometry could define a correlate for Jones inclusions of hyper-finite factors of Clifford sub-algebras associated with Clifford algebra of WCW spinor s. The appearance of Fibonacci series in living systems could represent one example of this correspondence. The angle $\pi/10$ closely related to Golden Mean characterizes the winding of DNA double strand. Could this mean that DNA allows to realize topological quantum computer programs as braidings? A possible realization would be based on the notion of super-genes [K50], which are like pages of a book identified as magnetic flux sheets containing genomes of sequences of cell nuclei as text lines. These text lines would represent line through which magnetic flux lines traverse.

The braiding of magnetic flux lines (or possibly flux sheets regarded as flattened tubes) would define the braiding and the particles involved would be anyons obeying dynamics having quantum group $SU(2)_q$, $q = exp(i\pi/5)$, as its symmetries. The anyons could be assigned with DNA nucleotides or triplets.

TGD predicts also different kind of new physics to DNA double strand. So called H_N-atoms consist of ordinary proton an N dark electrons at space-time sheet which is λ-fold covering of space-time sheet of ordinary hydrogen atom. The effective charge of H_N-atom is $1 - N/\lambda$ since the fine structure constant for dark electrons is scaled down by $1/\lambda$. H_λ-atoms have full electron shell and are therefore exceptionally stable. The proposal is that H_λ-atoms could replace ordinary hydrogen atoms in hydrogen bonds [K50, ?]. Single base pair corresponds to 2 or 3 hydrogen bonds. The question is whether λ-hydrogen atom might somehow relate to the anyons involved with topological quantum computation.

Anyons could be dark protons resulting in the formation dark hydrogen bond in the fusion of H_N atom and its conjugate H_{N_c}, $N_c = \lambda - N$. Neutron scattering and electron diffraction suggest

2.4 The Recent View About Quantum TGD

Before detailed discussion of what p-adicization of quantum TGD could mean, it is good to have an overall view about what quantum TGD in real context is.

2.4.1 Basic Notions

The notions of embedding space, 3-surface (and 4-surface), and WCW (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably. Therefore it seems better to begin directly from the recent picture.

The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K96, K94].

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The

generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K111, K25] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized power of 2 multiples of CP_2 length, p-adic length scale hypothesis [K66] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K36] led to a further generalization of the notion of embedding space. Generalized embedding space is obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW .

The notions of 3-surface and space-time surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to Equivalence implied by General Coordinate Invariance. There was a problem related to the realization of GCI since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory. On basis of these symmetries light-like 3-surfaces can be regarded as orbits of partonic 2-surfaces so that the theory is locally 2-dimensional. It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed. The mathematical formulation of this vision is however highly nontrivial challenge: is it due to analogs of gauge symmetries or should effective 2-dimensionality formulated explicitly as assumed until 2014 when stringy formulation of WCW geometry emerged.

3. Rather recently came the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. This means that the light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams. The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals actually has however remained open.

1. The obvious guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This choice had some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

   The identification of $X^3(X^3)$ as preferred extremal is however not consistent with quantum criticality suggesting in zero energy ontology (ZEO) a large number of space-time sheets associated with same 3-surface at the ends of causal diamond CD, and having same value of Kähler function. More technically, the Kähler action would have degenerate Hessian as a functional of X^4 with fixed ends X^3).

2. Much later number theoretical vision led to the conclusion that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes a connected component of the light-like 3-surfaces X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW . This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

3. The next step [K111] was the realization that the construction of WCW geometry in terms of modified Dirac action strengthens the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X^3_l. One could say that at X^3_l situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. The weakest form of number theoretic compactification [K96] states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, , where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$: in fact that 2-surface would have identical induced metric and Kähler form so that this conjecture would follow trivial. $M^8 - H$ duality would in this sense be Kähler isometry.

The notion of WCW ("world of classical worlds")

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW .

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW . The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD.

   The original long-held belief was that the second conformal symmetry corresponds to local embedding space isometries for light-like 3-surfaces X^3_l, which are either boundaries of X^4 (probably not: it seems that boundary conditions cannot be satisfied so that space-time surfaces must consists of regions defining at least double coverings of M^4) or light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry would be identifiable as the counterpart of the Kac Moody symmetry of string models.

   It has turned out that one can assume Kac-Moody algebra to be sub-algebra of symplectic algebra consisting of the symplectic isometries of $\delta M^4_\pm \times CP_2$. This super Kac-Moody algebra is generated by super-currents assignable to the modes of induced spinor fields other than right-handed neutrino and localized at string world sheets. The symplectic algebra would correspond to right-handed neutrino and one would have direct sum of these two. The beauty of this option is that localization would be inherent property of both algebras and with respect to the light-like coordinate of light-cone boundary rather than forced by hand. The issues related to diffeo-invariance would be avoided in this manner.

   Strong form of holography implied by strong form of GCI suggests the duality between space-like 3-surfaces at the end of CD and light-like 3-surfaces. By parallel translating the boundary of CD one can indeed define the action of symplectic algebra at the light-like 3-surfaces. Therefore also the symplectic and Kac-Moody algebras associated with these surfaces could be used to generate zero energy states, and one would have effective 2-dimensionality in the sense that only the partonic 2-surfaces defined by the intersections of space-like and light-like 3-surfaces and their 4-D tangent space data would code for quantum physics.

A rather plausible conclusion is that WCW (WCW) is a union of sub- WCW s associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

A further piece of understanding emerged from the following observations.

1. The induced Kähler form at the partonic 2-surface X^2 - the basic dynamical object if holography is accepted- can be seen as a fundamental symplectic invariant so that the values of $\epsilon^{\alpha\beta} J_{\alpha\beta}$ at X^2 define local symplectic invariants not subject to quantum fluctuations in the sense that they would contribute to the WCW metric. Hence only induced metric corresponds to quantum fluctuating degrees of freedom at WCW level and TGD is a genuine theory of gravitation at this level.

2. WCW can be divided into slices for which the induced Kähler forms of CP_2 and $\delta M^4_\pm$ at the partonic 2-surfaces X^2 at the light-like boundaries of CDs are fixed. The symplectic group of $\delta M^4_\pm \times CP_2$ parameterizes quantum fluctuating degrees of freedom in given scale (recall the presence of hierarchy of CDs).

3. This leads to the identification of the coset space structure of the sub- WCW associated with given CD in terms of the generalized coset construction for super-symplectic and super Kac-Moody type algebras. WCW in quantum fluctuating degrees of freedom for given values of zero modes can be regarded as being obtained by dividing symplectic group with subgroup leaving the 3-surface invariant for a preferred 3-surface which could be chosen maximum/minimum of Kähler function. Equivalently, the local coset space associated with $S^2 \times CP_2$ is in question: this was one of the first ideas about WCW which I gave up as too naïve!

4. The original construction of WCW metric was in terms of flow Hamiltonians induced by those of $\delta M^4_\pm \times CP_2$. Matrix elements of WCW metric were constructed as anti-commutators of super-Hamiltonians having interpretation also as WCW Hamiltonians.

   The construction had problematic aspects. Flux Hamiltonians were strictly 2-D objects and also the fact that they contained very little explicit information about the dynamics of the Kähler-Dirac action. The realization of super-Hamiltonians in terms of conserved symplectic super-charges of Kähler-Dirac action labelled by the modes of the Kähler-Dirac operator cures the situation and the construction becomes 3-dimensional although effective 2-dimensionality still holds true. Anti-commutations are fixed completely and the construction works for dimension $D = 8$ of embedding space only. The stringy picture forced by the solutions of the Kähler-Dirac operator becomes very explicit at the level of WCW . 8-D embedding space only [K84].

5. Generalized coset construction and coset space structure have very deep physical meaning. Symmetric space structure requires involution and it corresponds to inversion in light-like radial coordinate r_M of $\delta M^4_\pm$ (determined only up to Lorentz transformation). Super Virasoro algebra realizes quantum criticality, and one obtains hierarchy of criticalities represented by the hierarchy of sub-algebras of Super Virasoro algebra.

Head aches from Equivalent Principle

Equivalence Princple (EP) has been continual source of headaches during years. It is not even clear whether the uncritical assumption that there gravitational and inertial masses exist at separate notions creates the problem as a pseudoproblem. Stringy description of graviton mediated scattering predicted also by TGD indeed suggests this.

1. A longstanding conjecture has been that coset representations could Equivalence Principle (EP) at quantum level: the identity of Super Virasoro generators for super-symplectic and super Kac-Moody algebras was proposed to imply that inertial and gravitational four-momenta are identical. This conjecture is probably wrong.

2. The equivalence of classical Noether momentum associated with Kähler action with eigenvalues of the corresponding quantal momentum for Kähler-Dirac action certainly realizes quantum classical correspondence. It could also realized EP. Zero energy ontology suggests an alternative formulation for the same idea.

3. A further option is that EP reduces to the identification of the four momenta assignable to Super Virasoro representations assignable to space-like and light-like 3-surfaces and therefore become part of strong form of holography and quantum classical correspondence (QCC).

So it seems that EP might reduce to holography, GCI, or QCC and it might well be that it is trivially true! At classical level the understanding of the relationship between TGD and GRT led to the final break through in the understanding of EP. The recent view is that EP at quantum level reduces to Quantum Classical Correpondence (QCC) in the sense that Cartan algebra Noether charges assignable to 3-surface in case of Kähler action (inertial charges) are identical with eigenvalues of the quantal variants of Noether charges for Kähler-Dirac action (gravitational charges). The well-definedness of the latter charges is due to the conformal invariance assignable to 2-D surfaces (string world sheets and possibly partonic 2-surfaces) at which the spinor modes are localized in generic case. This localization follows from the condition that em charge has well defined value for the spinor modes. The localization is possibly only for the Kähler-Dirac action and key role is played by the modification of gamma matrices to Kähler-Dirac gamma matrices. The gravitational four-momentum is thus completely analogous to stringy four-momentum.

At classical level EP follows from the interpretation of GRT space-time as effective space-time obtained by replacing many-sheeted space-time (see **Fig.** `http://tgdtheory.fi/appfigures/manysheeted.jpg` or **Fig.** 9 in the appendix of this book) with Minkowski space with effective metric determined as a sum of Minkowski metric and sum over the deviations of the induced metrices of space-time sheets from Minkowski metric. Poincare invariance suggests strongly classical EP for the GRT limit in long length scales at least. One can consider also other kinds of limits such as the analog of GRT limit for Euclidian space-time regions assignable to elementary particles. In this case deformations of CP_2 metric define anatural starting point and CP_2 indeed defines a gravitational instanton with very large cosmological constant in Einstein-Maxwell theory. Also gauge potentials of standard model correspond classically to superpositions of induced gauge potentials over space-time sheets.

2.4.2 The Most Recent Vision About Zero Energy Ontology

The generalization of the number concept obtained by fusing real and p-adics along rationals and common algebraics is the basic philosophy behind p-adicization. This however requires that it is possible to speak about rational points of the embedding space and the basic objection against the notion of rational points of embedding space common to real and various p-adic variants of the embedding space is the necessity to fix some special coordinates in turn implying the loss of a manifest general coordinate invariance. The isometries of the embedding space could save the situation provided one can identify some special coordinate system in which isometry group reduces to its discrete subgroup. The loss of the full isometry group could be compensated by assuming that WCW is union over sub- WCW : s obtained by applying isometries on basic sub- WCW with discrete subgroup of isometries.

The combination of zero energy ontology realized in terms of a hierarchy causal diamonds and hierarchy of Planck constants providing a description of dark matter and leading to a generalization of the notion of embedding space suggests that it is possible to realize this dream. The article [L4] provides a brief summary about recent state of quantum TGD helping to understand the big picture behind the following considerations.

Zero energy ontology briefly

1. The basic construct in the zero energy ontology is the space $CD \times CP_2$, where the causal diamond CD is defined as an intersection of future and past directed light-cones with time-like separation between their tips regarded as points of the underlying universal Minkowski space M^4. In zero energy ontology physical states correspond to pairs of positive and negative energy states located at the boundaries of the future and past directed light-cones of a particular CD. CD: s form a fractal hierarchy and one can glue smaller CD: s within larger CD along the upper light-cone boundary along a radial light-like ray: this construction recipe allows to understand the asymmetry between positive and negative energies and why the arrow of experienced time corresponds to the arrow of geometric time and also why the contents of sensory experience is located to so narrow interval of geometric time. One can imagine evolution to occur as quantum leaps in which the size of the largest CD in the hierarchy of personal CD: s increases in such a way that it becomes sub-CD of a larger CD. p-Adic length scale hypothesis follows if the values of temporal distance T between tips of CD come in powers of 2^n: a weaker condition would be $T_p = pT_0$, p prime, and would assign all p-adic time scales to the size scale hierarchy of CDs. All conserved quantum numbers for zero energy states have vanishing net values. The interpretation of zero energy states in the framework of positive energy ontology is as physical events, say scattering events with positive and negative energy parts of the state interpreted as initial and final states of the event.

2. In the realization of the hierarchy of Planck constants $CD \times CP_2$ is replaced with a Cartesian product of book like structures formed by almost copies of CD: s and CP_2: s defined by singular coverings and factors spaces of CD and CP_2 with singularities corresponding

to intersection $M^2 \cap CD$ and homologically trivial geodesic sphere S^2 of CP_2 for which the induced Kähler form vanishes. The coverings and factor spaces of CD: s are glued together along common $M^2 \cap CD$. The coverings and factors spaces of CP_2 are glued together along common homologically non-trivial geodesic sphere S^2. The choice of preferred M^2 as subspace of tangent space of X^4 at all its points and having interpretation as space of non-physical polarizations, brings M^2 into the theory also in different manner. S^2 in turn defines a subspace of the much larger space of vacuum extremals as surfaces inside $M^4 \times S^2$.

3. WCW (the world of classical worlds, WCW) decomposes into a union of sub-WCW:s corresponding to different choices of M^2 and S^2 and also to different choices of the quantization axes of spin and energyand color isospin and hyper-charge for each choice of this kind. This means breaking down of the isometries to a subgroup. This can be compensated by the fact that the union can be taken over the different choices of this subgroup.

4. p-Adicization requires a further breakdown to discrete subgroups of the resulting sub-groups of the isometry groups but again a union over sub- WCW : s corresponding to different choices of the discrete subgroup can be assumed. Discretization relates also naturally to the notion of number theoretic braid.

Consider now the critical questions.

1. Very naïvely one could think that center of mass wave functions in the union of sectors could give rise to representations of Poincare group. This does not conform with zero energy ontology, where energy-momentum should be assignable to say positive energy part of the state and where these degrees of freedom are expected to be pure gauge degrees of freedom. If zero energy ontology makes sense, then the states in the union over the various copies corresponding to different choices of M^2 and S^2 would give rise to wave functions having no dynamical meaning. This would bring in nothing new so that one could fix the gauge by choosing preferred M^2 and S^2 without losing anything. This picture is favored by the interpretation of M^2 as the space of longitudinal polarizations.

2. The crucial question is whether it is really possible to speak about zero energy states for a given sector defined by generalized embedding space with fixed M^2 and S^2. Classically this is possible and conserved quantities are well defined. In quantal situation the presence of the light-cone boundaries breaks full Poincare invariance although the infinitesimal version of this invariance is preserved. Note that the basic dynamical objects are 3-D light-like "legs" of the generalized Feynman diagrams.

Definition of energy in zero energy ontology

Can one then define the notion of energy for positive and negative energy parts of the state? There are two alternative approaches depending on whether one allows or does not allow wave-functions for the positions of tips of light-cones.

Consider first the naïve option for which four momenta are assigned to the wave functions assigned to the tips of CD: s.

1. The condition that the tips are at time-like distance does not allow separation to a product but only following kind of wave functions

$$\Psi = exp[ip \cdot (m_+ - m_-)]\Theta(T^2)\Theta(m_+^0 - m_-^0)\Phi(p) \ , \ T^2 = (m_+ - m_-)^2 \ . \tag{2.4.1}$$

Here m_+ and m_- denote the positions of the light-cones and Θ denotes step function. Φ denotes WCW spinor field in internal degrees of freedom of 3-surface. One can introduce also the decomposition into particles by introducing sub-CD: s glued to the upper light-cone boundary of CD.

2. The first criticism is that only a local eigen state of 4-momentum operators $p_\pm = \hbar\nabla/i$ is in question everywhere except at boundaries and at the tips of the CD with exact translational invariance broken by the two step functions having a natural classical interpretation. The second criticism is that the quantization of the temporal distance between the tips to $T = 2^kT_0$ is in conflict with translational invariance and reduces it to a discrete scaling invariance.

The less naïve approach relying of super conformal structures of quantum TGD assumes fixed value of T and therefore allows the crucial quantization condition $T = 2^kT_0$.

1. Since light-like 3-surfaces assignable to incoming and outgoing legs of the generalized Feynman diagrams are the basic objects, can hope of having enough translational invariance to define the notion of energy. If translations are restricted to time-like translations acting in the direction of the future (past) then one has local translation invariance of dynamics for classical field equations inside $\delta M^4_\pm$ as a kind of semigroup. Also the M^4 translations leading to interior of X^4 from the light-like 2-surfaces surfaces act as translations. Classically these restrictions correspond to non-tachyonic momenta defining the allowed directions of translations realizable as particle motions. These two kinds of translations have been assigned to super-symplectic conformal symmetries at $\delta M^4_\pm \times CP_2$ and and super Kac-Moody type conformal symmetries at light-like 3-surfaces.

2. The condition selecting preferred extremals of Kähler action is induced by a global selection of M^2 as a plane belonging to the tangent space of X^4 at all its points [K25]. The M^4 translations of X^4 as a whole in general respect the form of this condition in the interior. Furthermore, if M^4 translations are restricted to M^2, also the condition itself - rather than only its general form - is respected. This observation, the earlier experience with the p-adic mass calculations, and also the treatment of quarks and gluons in QCD encourage to consider the possibility that translational invariance should be restricted to M^2 translations so that mass squared, longitudinal momentum and transversal mass squared would be well defined quantum numbers. This would be enough to realize zero energy ontology. Encouragingly, M^2 appears also in the generalization of the causal diamond to a book-like structure forced by the realization of the hierarchy of Planck constant at the level of the embedding space.

3. That the cm degrees of freedom for CD would be gauge like degrees of freedom sounds strange. The paradoxical feeling disappears as one realizes that this is not the case for sub-CD: s, which indeed can have non-trivial correlation functions with either upper or lower tip of the CD playing a role analogous to that of an argument of n-point function in QFT description. One can also say that largest CD in the hierarchy defines infrared cutoff.

2.5 P-Adicization At The Level Of Embedding Space And Space-time

In this section p-adicization program at the level if embedding space and space-time is discussed. The general problems of p-adicization, namely the selection of preferred coordinates and the problems caused by the non-existence of p-adic definite integral and algebraic continuation a solution of these problems has been discussed in the introduction.

2.5.1 P-Adic Variants Of The Embedding Space

Consider now the construction of p-adic variants of the embedding space.

1. Rational values of p-adic coordinates are non-negative so that light-cone proper time $a_{4,+} = \sqrt{t^2 - z^2 - x^2 - y^2}$ is the unique Lorentz invariant choice for the p-adic time coordinate near the lower tip of CD. For the upper tip the identification of a_4 would be $a_{4,-} = \sqrt{(t-T)^2 - z^2 - x^2 - y^2}$. In the p-adic context the simultaneous existence of both square roots would pose additional conditions on T. For 2-adic numbers $T = 2^nT_0$, $n \geq 0$ (or more generally $T = \sum_{k \geq n_0} b_k 2^k$), would allow to satisfy these conditions and this would be one additional reason for $T = 2^nT_0$ implying p-adic length scale hypothesis. Note however that

also $T_p = pT_0$, p prime, can be considered. The remaining coordinates of CD are naturally hyperbolic cosines and sines of the hyperbolic angle $\eta_{\pm,4}$ and cosines and sines of the spherical coordinates θ and ϕ.

2. The existence of the preferred plane M^2 of un-physical polarizations would suggest that the 2-D light-cone proper times $a_{2,+} = \sqrt{t^2 - z^2}$ $a_{2,-} = \sqrt{(t-T)^2 - z^2}$ can be also considered. The remaining coordinates would be naturally $\eta_{\pm,2}$ and cylindrical coordinates (ρ, ϕ).

3. The transcendental values of a_4 and a_2 are literally infinite as real numbers and could be visualized as points in infinitely distant geometric future so that the arrow of time might be said to emerge number theoretically. For M^2 option p-adic transcendental values of ρ are infinite as real numbers so that also spatial infinity could be said to emerge p-adically.

4. The selection of the preferred quantization axes of energy and angular momentum unique apart from a Lorentz transformation of M^2 would have purely number theoretic meaning in both cases. One must allow a union over sub-WCWs labeled by points of $SO(1,1)$. This suggests a deep connection between number theory, quantum theory, quantum measurement theory, and even quantum theory of mathematical consciousness.

5. In the case of CP_2 there are three real coordinate patches involved [?]. The compactness of CP_2 allows to use cosines and sines of the preferred angle variable for a given coordinate patch.

$$\begin{aligned} \xi^1 &= tan(u)exp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= tan(u)exp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{2.5.1}$$

The ranges of the variables u, Θ, Φ, Ψ are $[0, \pi/2], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively. Note that u has naturally only the positive values in the allowed range. S^2 corresponds to the values $\Phi = \Psi = 0$ of the angle coordinates.

6. The rational values of the (hyperbolic) cosine and sine correspond to Pythagorean triangles having sides of integer length and thus satisfying $m^2 = n^2 + r^2$ ($m^2 = n^2 - r^2$). These conditions are equivalent and allow the well-known explicit solution [A10]. One can construct a p-adic completion for the set of Pythagorean triangles by allowing p-adic integers which are infinite as real integers as solutions of the conditions $m^2 = r^2 \pm s^2$. These angles correspond to genuinely p-adic directions having no real counterpart. Hence one obtains p-adic continuum also in the angle degrees of freedom. Algebraic extensions of the p-adic numbers bringing in cosines and sines of the angles π/n lead to a hierarchy increasingly refined algebraic extensions of the generalized embedding space. Since the different sectors of WCW directly correspond to correlates of selves this means direct correlation with the evolution of the mathematical consciousness. Trigonometric identities allow to construct points which in the real context correspond to sums and differences of angles.

7. Negative rational values of the cosines and sines correspond as p-adic integers to infinite real numbers and it seems that one use several coordinate patches obtained as copies of the octant $(x \geq 0, y \geq 0, z \geq 0,)$. An analogous picture applies in CP_2 degrees of freedom.

8. The expression of the metric tensor and spinor connection of the embedding in the proposed coordinates makes sense as a p-adic numbers in the algebraic extension considered. The induction of the metric and spinor connection and curvature makes sense provided that the gradients of coordinates with respect to the internal coordinates of the space-time surface belong to the extensions. The most natural choice of the space-time coordinates is as subset of embedding space-coordinates in a given coordinate patch. If the remaining embedding space coordinates can be chosen to be rational functions of these preferred coordinates with coefficients in the algebraic extension of p-adic numbers considered for the preferred extremals of Kähler action, then also the gradients satisfy this condition. This is highly non-trivial

condition on the extremals and if it works might fix completely the space of exact solutions of field equations. Space-time surfaces are also conjectured to be hyper-quaternionic [K96], this condition might relate to the simultaneous hyper-quaternionicity and Kähler extremal property. Note also that this picture would provide a partial explanation for the decomposition of the embedding space to sectors dictated also by quantum measurement theory and hierarchy of Planck constants.

2.5.2 P-Adicization At The Level Of Space-Time

Number theoretical Universality in weak sense does not seem to pose problems. The field equations defining the preferred extremals of Kähler action make sense also p-adically if the preferred extremals correspond to critical space-time sheets for which the second variation of Kähler action vanishes for some deformations [K111]: this guarantees that the Noether currents associated with the Kähler-Dirac action are conserved. In this case the matrix determined by second variations has rank which is not maximal. The interpretation is in terms of a generalized catastrophe theory: space-time surfaces are critical with respect to the variation of Kähler action. These conditions are algebraic and make sense also p-adically. Also the conditions implied by number theoretical compactification make sense p-adically. Therefore one can construct the p-adic variants of preferred extremals of Kähler action. The new element is the possibility of p-adic pseudo constants depending on finite number of pinary digits only.

The natural expectation is that the number of critical deformations is infinite and corresponds to conformal symmetries naturally assignable to criticality. The number n of conformal equivalence classes of the deformations can be finite and n would naturally relate to the hierarchy of Planck constants $h_{eff} = n \times h$ (see **Fig.** `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`, or **Fig. ??** in the appendix of this book).

At number theoretical criticality it should be possible to assign to the real partonic 2-surface a unique p-adic counterpart. This might be true also for X^3_l and even for the space-time sheet $X^4(X^3_l)$. This is possible if the objects in question are defined by algebraic equations making sense also p-adically. Also trigonometric functions and exponential functions can be considered. Obviously p-adic pseudo constants are genuine constants for the geometric objects being shared in algebraic sense by the worlds defined by different number fields.

1. The starting point are the algebraic equations defining light-like partonic 3-surfaces X^3_l via the condition that the determinant of the induced metric vanishes. If the coordinate functions appearing in the determinant are algebraic functions with algebraic coefficients, p-adicization should make sense.

2. General Coordinate Invariance would suggest that this true also for the light-like 3-surfaces parallel to X^3_l appearing in the slicing of $X^4(X^3_l)$ assumed in the quantization of induced spinor fields and suggested by the properties of known extremals.

3. If the 4-dimensional real space-time sheet is expressible as a hyper-quaternionic surface of hyper-octonionic variant M^8 of the embedding space as number-theoretic vision suggests [K96], it might be possible to construct also the p-adic variant of the space-time sheet by algebraic continuation in the case that the functions appearing in the definition of the space-time sheet are algebraic.

Some preferred space-time coordinates are necessary.

1. Standard Minkowski coordinates associated with $M^2 \times E^2$ decomposition are implied by the selection of quantization axes also also preferred CP_2 coordinates and preferred coordinates for geodesic sphere S^2_i, $i = I$ or II. These coordinates could be used to define coordinates also for X^4. Which combination of coordinate variables is good would be determined by the dimensions of projections to M^4 and CP_2.

2. The construction of solutions of field equations leads to the so called Hamilton-Jacobi coordinates for M^4, when the induced metric has Minkowski signature [K14]. These coordinates define a slicing of $X^4(X^3_l)$ by string world sheets and their partonic duals required also by the number theoretic compactification. For 4-D M^4 projection these coordinates could be used

also as X^4 coordinates. The light-like coordinates u, v assigned with the string world sheets *resp.* complex coordinate w associated with the partonic 2-surface would give a candidate for preferred coordinates fixed apart from hyper-conformal *resp.* conformal transformations.

3. A good candidate for preferred coordinates for $X^2(v)$ is defined by the fluxes $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ and their canonical conjugates assignable to partonic 2-surfaces X^2 and their translates $X^2(v)$ along $X^3_l(X^2)$. Here J could correspond to either S^2 or CP_2 Kähler form. These coordinates are discussed in detail in the section about number theoretic braids.

4. For u, v coordinates the basic condition is that v varies along $X^3_l(u)$ and u labels these slices. This condition allows only scalings as hyper-complex analytic transformations and one might hope of fixing this scaling uniquely.

2.5.3 P-Adicization Of Second Quantized Induced Spinor Fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at light-like 3-surfaces and satisfy Kähler-Dirac action associated with Kähler action possibly complexified by addition imaginary CP breaking instanton term. The Kähler-Dirac equation makes sense also p-adically as also the anti-commutation relations of the induced spinor fields at different points of the (number theoretic) braid. Here discreteness is essential since delta functions are not easy to define in p-adic context.

Possible difficulties relate to the definition of p-adic variants of plane wave factors appearing in the construction and being defined with respect to the variable u labeling the slices in the slicing of $X^4(X^3_l)$ by light-like 3-surfaces $X^3_l(v)$ "parallel" to X^3_l. Exponent function as such is well-defined in p-adic context if the argument has p-adic norm smaller than one. It however fails to have the basic properties of its real variant failing to be periodic and having fixed unit p-adic norm for all values of its argument. Periodicity does not however seem to be essential for the formulation of quantum TGD in its recent form. The exponential functions involved are of form $exp(i\sqrt{n}u)$, and are not periodic even in real sense. The p-adic existence requires $u \ mod \ p = 0$ unless one introduces e and possibly also some roots of e to the extension of p-adics used (e^p exists so that the extension would be finite-dimensional).

These observations raise the hope that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure at the level of principle.

2.6 P-Adicization At The Level Of WCW

This section is not a distilled final answer to the challenges involved with the p-adicization of WCW geometry and spinor structure. There are several questions. What is the precise meaning of concepts like number theoretical universality and criticality? What does p-adicization mean and is it needed/possible? Is algebraic continuation the manner to achieve it?

The notion of reduced WCW implied by the notion of finite measurement resolution is what gives hopes about performing this continuation in practice.

1. The weaker notion of reduced WCW emerges from finite measurement resolution and for given induced Kähler form at partonic 2-surfaces reduces WCW to a finite-dimensional space $(\delta M^4_\pm \times CP_2)^n/S_n$ for given number of points of number theoretic braid. The metric and Kähler structure of this space is determined dynamically in terms of the spectrum of the Kähler-Dirac operator.

2. The stronger notion of reduced WCW identified as the space of the maxima of Kähler function in quantum fluctuating degrees of freedom labeled by symplectic group is second key notion and suggests strongly discretization. The points of reduced configuration space with rational of algebraic coordinates would correspond to those 3-surfaces through which leakage between different sectors of WCW is possible. Reduced configuration space in this sense is

the direct counterpart of the spin glass landscape known to obey ultra-metric topology naturally. This approach is reasonably concrete and relies heavily on the most recent, admittedly still speculative, view about quantum TGD.

2.6.1 Generalizing The Construction Of WCW Geometry To The P-Adic Context

A problematics analogous to that related with the entanglement between real and p-adic number fields is encountered also in the construction of WCW geometry. The original construction was performed in the real context. What is needed are Kähler geometry and spinor structure for the WCW , and a construction of the WCW spinor fields. What might solve these immense architectural challenges are the equally immense symmetries of WCW and algebraic continuation as the method of p-adicization.

What one can hope that everything of physical interest reduces to the level of algebra (rational or algebraic numbers) and that topology (be it real or p-adic) disappears totally at the level of the matrix elements of the metric and of U-matrix mediating transitions between sectors of WCW corresponding to different number fields. It is not necessary to require this to happen for M-matrix identified as time-like entanglement coefficients between positive and negative energy parts of zero energy states.

The notions of number theoretical universality and number theoretical criticality

An essential question is however what one means with the notions of number theoretical universality and criticality.

1. The weak form of the number theoretical universality means that there are sub- WCW s which can be regarded as real, those which are genuinely p-adic, and those which are algebraic in the sense that the representation of partonic 2-surface, perhaps also 3-surface, and perhaps even space-time surface is in terms of rational/algebraic functions allows the interpretation in terms of both real and p-adic numbers. These surfaces would be like rational and algebraic numbers common for the continua formed by reals and p-adics. This poses conditions on the representations of surfaces and typically rational functions with rational coefficients would represent these surfaces.

   For these surfaces - and only for these- physics should be expressible in terms of algebraic numbers and define as a completion the physics in real and p-adic number fields. This would allow p-adic non-determinism. Book analogy is convenient here: the physics corresponding to various number fields would be like pages of books glued together along rational and algebraic physics. If the transitions between states in different number field taking place via a leakage between different pages of the book are allowed, one can regard the algebraic sectors of the WCW as critical. This number theoretic criticality could be interpreted in terms of intentionality and cognition, and living matter would represent a school example about number theoretically critical phase. For this option it is not at all obvious whether it makes sense to speak about WCW geometry. The construction of WCW spinor structure reducing exponent of Kähler function to determinant is what gives some hopes.

2. A much stronger condition - which I adopted originally - is that all 3-surfaces allow interpretation as as both real and p-adic surfaces: in this case p-adic non-determinism would be excluded. The objection is that this kind of number theoretical universality might reduce to a purely algebraic physics. This condition has interpretation in terms of number theoretical criticality if the weaker notion of universality is adopted.

Generalizing the construction for WCW metric

It is not enough to generalize this construction to the p-adic context. 3-surfaces contain both real and p-adic regions and should be able to perform the construction for this kind of objects.

1. Very naïvely, one could start from the Riemannian construction of the line element which tells the length squared between infinitesimally close points at each point of the Riemann

manifold. The notion of line element involves the notion of nearness and one obviously cannot do without topology here. The line element makes formally sense sense for real and p-adic contexts but since p-adic definite integral does not exist, the notions of p-adic length and volume do not exist naturally. Of course, p-adic norm defines very rough measure of distance in number theoretic sense. The notion of line-element is not needed in the quantum theory at WCW level since only the matrix elements of the WCW metric matter.

2. WCW metric can be constructed in terms if Dirac determinant identified as exponent of Kähler function and the formula for matrix elements is expressible in terms of derivatives of logarithms of the eigen values of the Kähler-Dirac operator with respect to complex coordinates of WCW . This means enormous simplification if the number of eigenvalues is finite as implied by finite measurement resolution realized in terms of braids defined by physical conditions. If eigenvalues are algebraic functions of complex coordinates of WCW then also the exponent of Kähler function and WCW covariant metric defining as its inverse as propagator in WCW degrees of freedom are algebraic functions.

I have also proposed a formula for the matrix elements of configuration space metric and Kähler form between the Killing vector fields of isometry generators. Isometries are identified as X^2 local symplectic symmetries. These expressions can be given also in terms of WCW Hamiltonians as "half Poisson brackets" in complex coordinates. Also the construction of quantum states involves WCW Hamiltonians and their super counterparts.

1. The definition of WCW s Hamiltonians involves definite integrals of corresponding complexified Hamiltonians of $(\delta M^4_\pm \times CP_2)^n$ over X^2. Definite integrals are problematic in the p-adic context, as is clear from the fact that in-numerable number of definitions of definite integral have been proposed.

2. Finite measurement resolution would reduce integrals to sums since WCW reduces to $(\delta M^4_\pm \times CP_2)^n/S_n$ for given CD. Furthermore, only the Hamiltonians corresponding to triplet *resp.* octet representations of $SO(3)$ *resp.* $SU(3)$ would be needed to coordinatize $S^2 \times CP_2$ part of the reduced WCW .

3. Without number theoretic braids the definition of these integrals seems really difficult in p-adic context. Residue calculus might give some hopes but One might however hope that one could reduce the construction in the real case to that for the representations of superconformal and symplectic symmetries, and analytically continue the construction from the real context to the p-adic contexts by *defining* the matrix elements of the metric to be what the symmetry respecting analytical continuation gives.

WCW integration should be also continued algebraically to the p-adic context. Quantum criticality realized as the vanishing of loop corrections associated with the WCW integral, would reduce WCW integration to purely algebraic process much like in free field theory and this would give could hopes about p-adicization. Matrix elements would be proportional to the exponent of Kähler function at its maximum plus matrix elements expressible as correlation functions of conformal field theory: the recent state of construction is considered in [K24]. This encourages further the hopes about complete algebraization of the theory so that the independence of the basic formulation on number field could be raised to a principle analogous to general coordinate invariance.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in WCW inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the WCW would be possible in the entire WCW . Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also WCW spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X_l^3 to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the WCW . This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub- WCW s induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [K111]. The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_K are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics as physics of cognition also supportthis conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} . \quad (2.6.1)$$

An expression of same form but with sum over eigenvalues of the Kähler-Dirac operator with F replaced with eigenvalue results if exponent of Kähler function is expressible as Dirac determinant:

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} \lambda_k}{\lambda_k} - \frac{\partial_K \lambda \partial_{\overline{L}} \lambda_k}{\lambda_k^2} . \quad (2.6.2)$$

What is important that this formula in principles relates WCW geometry directly to quantum physics as represented by the Kähler-Dirac operator.

Generalizing the notion of WCW spinor field

One must also construct spinor structure. Also this construction relies crucially super Kac-Moody and super-symplectic symmetries. Spinors at a given point of WCW correspond to the Fock space spanned by fermionic oscillator operators and again one might hope that super-symmetries would allow algebraization of the whole procedure.

The identification of WCW gamma matrices as super Hamiltonians of WCW . The generators of various super-algebras are also needed in order to construction configuration space spinors at given point of WCW . In ideal measurement resolution these algebra elements are expressible as integrals of Hamiltonians and super-Hamiltonians of $\delta M^4_\pm \times CP_2$ and this leads to difficulties in p-adic context. It might be that finite measurement resolution which seems to be coded by the classical dynamics provides the only possible solution of these difficulties. In the case of reduced WCW the construction of orthonormalized based of WCW spinor fields looks a rather reasonable challenge and the continuation of this procedure to p-adic context might make sense.

2.6.2 WCW Functional Integral

One can make some general statements about WCW functional integral.

1. If only braid points are specified, there is a functional integral over a huge number of 2-surfaces meaning sum of perturbative contributions from very large number of partonic 2-surfaces selected as maxima of Kähler function or by stationary phase approximation. This kind of non-perturbative contribution makes it very difficult to understand what is involved so that it seems that some restrictions must be posed. Also all information about crucial vacuum degeneracy of Kähler action would be lost as a non-local information.

2. Induced Kähler form represents perhaps the most fundamental zero modes since it remains invariant under symplectic transformations acting as isometries of WCW . Therefore it seems natural organize WCW integral in such a way that each choice of the induced Kähler form represents its own quantized theory and functional integral is only over deformations leaving induced Kähler form invariant. The deformations of the partonic 2-surfaces would leave invariant both the induced areas and magnetic fluxes. The symplectic orbits of the partonic 2-surfaces (and 3-surfaces) would therefore define a slicing of WCW with separate quantization for each slice.

3. The functional integral would be over the symplectic group of CP_2 and over M^4 degrees of freedom -perhaps also in this case over the symplectic group of δM^4_+ - a rather well-defined mathematical structure. Symplectic transformations of CP_2 affect only the CP_2 part of the induced metric so that a nice separation of degrees of freedom results and the functional integral can be assigned solely to the gravitational degrees of freedom in accordance with the idea that fundamental quantum fluctuating bosonic degrees of freedom are gravitational.

4. WCW integration around a partonic 2-surface for which the Kähler function is maximum with respect to quantum fluctuating degrees of freedom should give only tree diagrams with propagator factors proportional to g_K^2 if loop corrections to the WCW integral vanish. One could hope that there exist preferred S^2 and CP_2 coordinates such that vertex factors involving finite polynomials of S^2 and CP_2 coordinates reduce to a finite number of diagrams just as in free field theory.

If WCW functional integral algebraizes by the vanishing of loop corrections, one has hopes that even p-adic variant of WCW functional integral might make sense. The exponent of Kähler function appears and if given by the Dirac determinant it would reduce to a finite product of eigenvalues of Kähler-Dirac operator which makes sense also p-adically.

Algebraization of WCW functional integral

WCW is a union of infinite-dimensional symmetric spaces labeled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case.

1. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

2. If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each WCW spinor field would define a vertex from which lines representing the propagators defined by the contravariant WCW metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic WCW integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the WCW integrals would exist also in the p-adic sense.

Should one p-adicize only the reduced configuration space?

An attractive approach to p-adicization might be characterized as minimalism and would involve geometrization of only the reduced WCW consisting of the maxima of Kähler function in quantum fluctuating degrees of freedom. A further reduction results from the finite measurement resolution replacing WCW effectively with $(\delta M^4_{\pm} \times CP_2)^n/S_n$. In zero modes discretization realizing quantum classical correspondence is attractive possibility.

1. If Duistermaat-Heckman theorem [A31] holds true in TGD context, one could express real WCW functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function in quantum fluctuating degrees of freedom defining what might be called reduced WCW CH_{red}. The exponent of Kähler function and propagator identified as contravariant metric of WCW could be deduced from the spectrum of the modified Dirac operator.

2. The huge super-conformal symmetries raise the hope that the rest of M-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of M-matrix.

3. A possible problem of this reduction is that the number of degrees of freedom in functional integral is still infinite, which might mean problems in terms of algebraization. For instance, the inverse of covariant metric identified as algebraic function need not represent algebraic object. Finite measurement resolution improves the situation in this respect. Finite measurement resolution realized in terms of number theoretic braids would reduce WCW to $(\delta M^4_{\pm} \times CP_2)^n/S_n$ for given CD and this would reduce the situation to a finite dimensional one and maxima of Kähler function would form a discrete set, possibly only single point of $(\delta M^4_{\pm} \times CP_2)^n/S_n$. Also in this case exponent of Kähler function and the spectrum of Kähler-Dirac operator are needed. Also the values of $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic braids labeled by $\delta M^4_{\pm} \times CP_2/S_n$ are needed.

Zero modes pose a further problem.

1. The absence of functional integral measure in zero modes would suggest that states depend on finite number of zero modes only and that there is localization in this degrees of freedom. Finite measurement resolution suggests the same. The extrema of the quantity $J = \epsilon^{\alpha\beta} J_{\alpha\beta}\sqrt{g_2}$ at the points of number theoretic represent finite set of values of fundamental zero modes assignable to X^2 forming a finite-dimensional space naturally. Non-local isometry invariants can be defined as Kähler magnetic fluxes if it is possible to define symplectic triangulation of X^2 with vertices identifiable naturally as points of number theoretic braid corresponding to the extrema of J. The notion of symplectic fusion algebra based on this kind of triangulation is discussed in [K19].

2. Kac-Moody group parameterizes zero modes assignable to X^3_l and a correlation between these zero modes and the quantum numbers of quantum state is natural and could result by stationary phase approximation if finite-dimensional variant of functional integral can be defined. If there is localization in zero modes, this correspondence could be discrete and implied by classical equations of motion for braid points. A unique selection of preferred quantization axis would be made possible by the hierarchy of Planck constants selecting $M^2 \subset M^4$ and $S^2_i \subset CP_2$ as critical manifolds with respect to the change of Planck constant.

What other difficulties can one imagine?

1. The optimal situation would be that M-matrix elements in real case are algebraic functions or at least functions continuable to the p-adic context in a form having sensible physical interpretation.

2. If one starts directly from Fourier transforms in p-adic context, difficulties are caused by trigonometric functions and exponent function whose p-adic counterparts do not behave in physically acceptable manner. It seems that it is phase factors defined by plane waves which should should restricted to roots of unity and continued to the p-adic realm as such. In p-adic context either momentum or position makes sense as p-adic number unless one introduces infinite-dimensional extension containing logarithms and π. Maybe the only manner to avoid problems is to accept discretization and algebraization of the phase factors.

Concerning number field changing transitions at number theoretical criticality possibly relevant for U-matrix some comments are in order. For real$\leftrightarrow$ p-adic transitions only the algebraic points of number theoretic braid common to both real and p-adic variant of partonic 2-surface are relevant and situation reduces to algebraic braid points in $(\delta M^4_\pm \times CP_2)/S_n$. Algebraic points in a given extension of rationals would be common to real and p-adic surfaces. It could happen that there are very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$. The integral over reduced WCW should reduce to a sum over possible values of coordinates for these points. If only maxima of Kähler function an analytic continuation of real M-matrix to p-adic-real M-matrix could make sense.

If this picture is correct, the p-adicization of WCW would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. Finite measurement resolution simplifies the situation dramatically. If CH_{red} is a discrete subset of CH or its finite-dimensional variant, ultra-metric topology induced from finite-p p-adic norm is indeed natural for it. "Discrete set in CH" need not mean a discrete set in the usual sense and the reduced WCW could be even finite-dimensional continuum. p-Adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

2.6.3 Number Theoretic Constraints On M-Matrix

Assume that U-matrix assignable to quantum jump between zero energy states exists simultaneously in all number fields and perhaps even between different number fields at number theoretical quantum criticality (allowing finite-dimensional extensions of p-adics). If so the immediate question is whether also the construction procedure of the M-matrix defined as time-like entanglement coefficients between positive and negative energy parts of zero energy state could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. The identification of M-matrices as building blocks of U-matrix in the manner discussed in [K24] supports affirmative answer to the first question. Not only the WCW but also Kähler function and its exponent, Kähler metric, and WCW functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Number theoretical Universality and M-matrix

Number theoretic constraints on M-matrix are non-trivial even for the weaker notion of number theoretical universality. Number theoretical criticality (or number theoretical universality in strong sense) requires that M-matrix elements are algebraic numbers. This is achieved naturally if the definition of M-matrix elements involves only the data associated with the number theoretic braid. Note that this data is non-local since it involves information about tangent space of X^4 at the point so that discretization happens in geometric sense but not in information theoretic sense. Note also that for algebraic surfaces finite number of points of surface allows to deduce the parameters of the polynomials involved and thus to deduce the entire surface.

If quantum version of WCW is adopted one must perform quantization for $E^2 \subset M^4$ coordinates of points S^2_i braid and CP_2 coordinates of M^2 braid. In this kind of situation it becomes unclear whether one can speak about braiding anymore. This might make sense if each braid topology corresponds to its own quantization containing information about the fact that deformations of X^3_l respect the braiding topology.

The partonic vertices appearing in M-matrix elements should be expressible in terms of N-point functions of some rational super-conformal field theory but with the p-adically questionable N-fold integrals over string appearing in the definition of n-point functions. The most elegant manner to proceed is to replace them with their explicit expressions if they are algebraic functions- quite generally or at number theoretical criticality. Spin chain type string discretization is an alternative, not so elegant option.

Propagators, that is correlations between partonic 2-surfaces, would be due to the interior dynamics of space-time sheets which means a deviation from super string theory. Another function of interior degrees of freedom is to provide zero modes of metric of WCW identifiable as classical degrees of freedom of quantum measurement theory entangling with quantal degrees of freedom at partonic 3-surfaces.

Number theoretical criticality and M-matrix

Number theoretical criticality poses very strong conditions on the theory.

1. The p-adic variants of 4-D field equations associated with Kähler action make sense. Also the notion of preferred extremal makes sense in p-adic context if it corresponds to quantum criticality in the sense that second variation of Kähler action vanishes for dynamical symmetries. A natural further condition is that the surface is representable in terms of algebraic equations involving only rational or algebraic coefficients and thus making sense both in real and p-adic sense. In this case also Kähler action and classical charges could exist in some algebraic extension of p-adic numbers.

2. Also Kähler-Dirac equation makes sense p-adically. The exponent of Kähler function defining vacuum functional is well-defined notion p-adically if the identification as product of finite number of eigenvalues of the Kähler-Dirac operator is accepted and eigenvalues are algebraic. Also the notion of WCW metric expressible in terms of derivatives of the eigenvalues with respect to complex coordinates of WCW makes sense.

3. The functional integral over WCW can be defined only as an algebraic extension of real functional integral around maximum of Kähler function if the theory is integrable and gives as a result an algebraic number. One might hope that algebraic p-adicization makes sense for the vacuum function at points corresponding to the maxima of Kähler function with respect to quantum fluctuating degrees of freedom (assuming they exist) and with respect to zero modes. As discussed already earlier, in the case of zero modes quantum classical correspondence allows to select preferred value of zero modes even if functional integral in zero modes does not make sense. The basic requirement is that the inverse of the matrix defined by the Kähler metric defining propagator is algebraic function of the complex coordinate of WCW . If the eigen-values of the modified Dirac operator satisfy this condition this is indeed the case.

4. Ordinary perturbation series based on Feynman diagrams makes sense also in p-adic sense since the presence of cutoff for the size of CD implies that the number of terms if finite. One must be however cautious with momentum integrations which should reduce to finite sum due to the presence of both IR and UV cutoff implied by the finite size of CD. The formulation in terms of number theoretic braids whose intersections with partonic 2-surfaces consist of finite number of points supports the possibility of number theoretic universality.

There are hopes that M-matrix make sense p-adically. As far M-matrix is considered, The most plausible interpretation relies on the weaker form of number theoretic universality so that genuinely p-adic M-matrices should exist.

1. Dirac determinant exists for any p-adic 3-surfaces since the eigenvalues of Kähler-Dirac operator represent a purely local notion sensible also in p-adic context. The reason is that finite measurement resolution - now deducible from the vacuum degeneracy of Kähler action-implies that the number of eigenvalues is finite. Preferred extremals of Kähler action obey quantum criticality condition meaning that the second variation of Kähler action vanishes. This condition makes sense also p-adically.

2. If loops vanish, WCW integration gives only contractions with propagator expressible as the contravariant WCW Kähler metric expressible in terms of derivatives of the Kähler function with respect to the preferred complex coordinates of WCW . If this function is algebraic function, it allows algebraic continuation to p-adic context and all that is needed for calculation of M-matrix elements makes sense p-adically. The crucial question is whether the Kähler metric is algebraic function in preferred coordinates.

3. N-point functions involve also symplectically invariant multiplicative factors discussed in [K19] in terms of symplectic fusion algebras. For them algebraic universality holds true. N-point functions of conformal field theory associated with the generalized vertices should also be algebraic functions.

4. Finite measurement resolution realized in terms of braids for given $J = \epsilon^{\alpha\beta} J_{\alpha\beta}$ means a reduction of a given sector of WCW in quantum fluctuating degrees of freedom to finite-dimensional space $\delta M^4_\pm \times CP_2/S_n$ associated with the boundaries of CD. For instance, configuration space Hamiltonians reduce apart from J factor to those assignable naturally to the reduced WCW . Finite-dimensionality gives hopes of algebraic continuation of M-matrix defined in terms of general Feynman diagrams in real context using finite purely algebraic operations due to the cutoff in the size of CDs. In zero modes the simplest option would be that quantum states correspond to sums over different values of zero modes, in particular J as function in X^2.

Also number theoretical criticality is consistent with this picture.

1. If partonic 2-surface X^2 is determined by algebraic equations involving only rational coefficients, same equations define real and p-adic variants of X^2.

2. Number theoretic criticality for braids means that their points are algebraic and common to real and p-adic partonic 2-surfaces. The extrema of J -determined by algebraic conditions-must be algebraic numbers.

3. At quantum criticality Dirac determinant is algebraic number if the number of eigenvalues is finite (implied by finite measurement resolution) and if they are algebraic numbers. If the p-adic counterpart of X^3_l exists, this allows to assign to the p-adic counterpart of X^3_l the exponent of Kähler function as Dirac determinant although Kähler action remains ill-defined p-adically.

The relationship between U-matrix and M-matrix

The following represents the latest result concerning the relationship between the notions of U-matrix and M-matrix and probably provides answer to some of the questions posed in the chapter. What is highly satisfactory that U-matrix dictates M-matrix completely via unitarity conditions. A more detailed discussion can be [K56] discussing Negentropy Maximization Principle, which is the basic dynamical principle of TGD inspired theory of consciousness and states that the information content of conscious experience is maximal.

If state function reduction associated with time-like entanglement leads always to a product of positive and negative energy states (so that there is no counterpart of bound state entanglement and negentropic entanglement possible for zero energy states: these notions are discussed below) U-matrix and can be regarded as a collection of M-matrices

$$U_{m_+n_-,r_+,s_-} = M(m_+,n_-)_{r_+,s_-} \tag{2.6.3}$$

labeled by the pairs (m_+, n_-) labelling zero energy states assumed to reduced to pairs of positive and negative energy states. M-matrix element is the counterpart of S-matrix element $S_{r,s}$ in positive energy ontology. Unitarity conditions for U-matrix read as

$$\begin{aligned}(UU^\dagger)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} M(m_+,n_-)_{k_+,l_-}\overline{M}(r_+,s_-)_{k_+,l_-} = \delta_{m+r_+,n_-s_-} \ , \\ (U^\dagger U)_{m_+n_-,r_+s_-} &= \sum_{k_+,l_-} \overline{M}(k_+,l_-)_{m_+,n_-}M(k_+,l_-)_{r_+,s_-} = \delta_{m+r_+,n_-s_-} \ .\end{aligned} \quad (2.6.4)$$

The conditions state that the zero energy states associated with different labels are orthogonal as zero energy states and also that the zero energy states defined by the dual M-matrix

$$M^\dagger(m_+,n_-)_{k_+,l_-} \equiv \overline{M}(k_+l_-)_{m_+,n_-} \quad (2.6.5)$$

-perhaps identifiable as phase conjugate states- define an orthonormal basis of zero energy states.

When time-like binding and negentropic entanglement are allowed also zero energy states with a label not implying a decomposition to a product state are involved with the unitarity condition but this does not affect the situation dramatically. As a matter fact, the situation is mathematically the same as for ordinary S-matrix in the presence of bound states. Here time-like bound states are analogous to space-like bound states and by definition are unable to decay to product states (free states). Negentropic entanglement makes sense only for entanglement probabilities, which are rationals or belong to their algebraic extensions. This is possible in what might be called the intersection of real and p-adic worlds (partonic surfaces in question have representation making sense for both real and p-adic numbers). Number theoretic entropy is obtained by replacing in the Shannon entropy the logarithms of probabilities with the logarithms of their p-adic norms. They satisfy the same defining conditions as ordinary Shannon entropy but can be also negative. One can always find prime p for which the entropy is maximally negative. The interpretation of negentropic entanglement is in terms of formations of rule or association. Schrödinger cat knows that it is better to not open the bottle: open bottle-dead cat, closed bottle-living cat and negentropic entanglement measures this information.

2.7 How To Define Generalized Feynman Diagrams?

S-matrix codes to a high degree the predictions of quantum theories. The longstanding challenge of TGD has been to construct or at least demonstrate the mathematical existence of S-matrix- or actually M-matrix which generalizes this notion in ZEO (ZEO) [K82] . This work has led to the notion of generalized Feynman diagram and the challenge is to give a precise mathematical meaning for this object. The attempt to understand the counterpart of twistors in TGD framework [K101] has inspired several key ideas in this respect but it turned out that twistors themselves need not be absolutely necessary in TGD framework.

1. The notion of generalized Feyman diagram defined by replacing lines of ordinary Feynman diagram with light-like 3-surfaces (elementary particle sized wormhole contacts with throats carrying quantum numbers) and vertices identified as their 2-D ends - I call them partonic 2-surfaces is central. Speaking somewhat loosely, generalized Feynman diagrams (plus background space-time sheets) define the "world of classical worlds" (WCW). These diagrams involve the analogs of stringy diagrams but the interpretation is different: the analogs of stringy loop diagrams have interpretation in terms of particle propagating via two different routes simultaneously (as in the classical double slit experiment) rather than as a decay of particle to two particles. For stringy diagrams the counterparts of vertices are singular as manifolds whereas the entire diagrams are smooth. For generalized Feynman diagrams vertices are smooth but entire diagrams represent singular manifolds just like ordinary Feynman diagrams do. String like objects however emerge in TGD and even ordinary elementary particles are predicted to be magnetic flux tubes of length of order weak gauge boson Compton

length with monopoles at their ends as shown in accompanying article. This stringy character should become visible at LHC energies.

2. ZEO (ZEO) and causal diamonds (intersections of future and past directed light-cones) define second key ingredient. The crucial observation is that in ZEO it is possible to identify off mass shell particles as pairs of on mass shell fermions at throats of wormhole contact since both positive and negative signs of energy are possible and one obtains also space-like total momenta for wormhole contact behaving as a boson. The localization of fermions to string world sheets and the fact that super-conformal generator G carries fermion number combined with twistorial consideration support the view that the propagators at fermionic lines are of form $(1/G)ip^k\gamma_k(1/G^\dagger + h.c.$ and thus hermitian. In strong models $1/G$ would serve as a propagator and this requires Majorana condition fixing the dimension of the target space to 10 or 11.

3. A powerful constraint is number theoretic universality requiring the existence of Feynman amplitudes in all number fields when one allows suitable algebraic extensions: roots of unity are certainly required in order to realize p-adic counterparts of plane waves. Also embedding space, partonic 2-surfaces and WCW must exist in all number fields and their extensions. These constraints are enormously powerful and the attempts to realize this vision have dominated quantum TGD for last two decades.

4. Representation of 8-D gamma matrices in terms of octonionic units and 2-D sigma matrices is a further important element as far as twistors are considered [K101] . Kähler-Dirac gamma matrices at space-time surfaces are quaternionic/associative and allow a genuine matrix representation. As a matter fact, TGD and WCW could be formulated as study of associative local sub-algebras of the local Clifford algebra of 8-D embedding space parameterized by quaternionic space-time surfaces.

5. A central conjecture has been that associative (co-associative) 4-surfaces correspond to preferred extremals of Kähler action [K111]. It took long time to realize that in ZEO the notion of preferred extremal might be un-necessary! The reason is that 3-surfaces are now pairs of 3-surfaces at boundaries of causal diamonds and for deterministic dynamics the space-time surface connecting them is expected to be more or less unique. Now the action principle is non-deterministic but the non-determinism would give rise to additional discrete dynamical degrees of freedom naturally assignable to the hierarchy of Planck constants $h_{eff} = n \times h$, n the number of space-time surface with same fixed ends at boundaries of CD and with same values of Kähler action and of conserved quantities. One must be however cautions: this leaves the possibility that there is a gauge symmetry present so that the n sheets correspond to gauge equivalence classes of sheets. Conformal invariance is associated with criticality and is expected to be present also now.

   One can of course also ask whether one can assume that the pairs of 3-surfaces at the ends of CD are totally un-correlated. If this assumption is not made then preferred extremal property would make sense also in ZEO and imply additional correlation between the members of these pairs. This kind of correlations would correspond to the Bohr orbit property, which is very attractive space-time correlate for quantum states. This kind of correlates are also expected as space-time counterpart for the correlations between initial and final state in quantum dynamics.

6. A further conjecture has been that preferred extremals are in some sense critical (second variation of Kähler action could vanish for infinite number of deformations defining a super-conformal algebra). The non-determinism of Kähler action implies this property for $n > 0$ in $h_{eff} = nh$. If the criticality is present, it could correspond to conformal gauge invariance defined by sub-algebras of conformal algebra with conformal weights coming as multiples of n and isomorphic to the conformal algebra itself.

7. As far as twistors are considered, the first key element is the reduction of the octonionic twistor structure to quaternionic one at space-time surfaces and giving effectively 4-D spinor and twistor structure for quaternionic surfaces.

Quite recently quite a dramatic progress took place in this approach [K111, K101] .

1. The progress was stimulated by the simple observation that on mass shell property puts enormously strong kinematic restrictions on the loop integrations. With mild restrictions on the number of parallel fermion lines appearing in vertices (there can be several since fermionic oscillator operator algebra defining SUSY algebra generates the parton states)- all loops are manifestly finite and if particles has always mass -say small p-adic thermal mass also in case of massless particles and due to IR cutoff due to the presence largest CD- the number of diagrams is finite. Unitarity reduces to Cutkosky rules [B29] automatically satisfied as in the case of ordinary Feynman diagrams.

2. Ironically, twistors which stimulated all these development do not seem to be absolutely necessary in this approach although they are of course possible. Situation changes if one does not assume small p-adically thermal mass due to the presence of massless particles and one must sum infinite number of diagrams. Here a potential problem is whether the infinite sum respects the algebraic extension in question.

This is about fermionic and momentum space aspects of Feynman diagrams but not yet about the functional (not path-) integral over small deformations of the partonic 2-surfaces. The basic challenges are following.

1. One should perform the functional integral over WCW degrees of freedom for fixed values of on mass shell momenta appearing in the internal lines. After this one must perform integral or summation over loop momenta. Note that the order is important since the space-time surface assigned to the line carries information about the quantum numbers associated with the line by quantum classical correspondence realized in terms of Kähler-Dirac operator.

2. One must define the functional integral also in the p-adic context. p-Adic Fourier analysis relying on algebraic continuation raises hopes in this respect. p-Adicity suggests strongly that the loop momenta are discretized and ZEO predicts this kind of discretization naturally.

It indeed seems that the functional integrals over WCW could be carried out at general level both in real and p-adic context. This is due to the symmetric space property (maximal number of isometries) of WCW required by the mere mathematical existence of Kähler geometry [K43] in infinite-dimensional context already in the case of much simpler loop spaces [A30] .

1. The p-adic generalization of Fourier analysis allows to algebraize integration- the horrible looking technical challenge of p-adic physics- for symmetric spaces for functions allowing the analog of discrete Fourier decomposition. Symmetric space property is indeed essential also for the existence of Kähler geometry for infinite-D spaces as was learned already from the case of loop spaces. Plane waves and exponential functions expressible as roots of unity and powers of p multiplied by the direct analogs of corresponding exponent functions are the basic building bricks and key functions in harmonic analysis in symmetric spaces. The physically unavoidable finite measurement resolution corresponds to algebraically unavoidable finite algebraic dimension of algebraic extension of p-adics (at least some roots of unity are needed). The cutoff in roots of unity is very reminiscent to that occurring for the representations of quantum groups and is certainly very closely related to these as also to the inclusions of hyper-finite factors of type II_1 defining the finite measurement resolution.

2. WCW geometrization reduces to that for a single line of the generalized Feynman diagram defining the basic building brick for WCW. Kähler function decomposes to a sum of "kinetic" terms associated with its ends and interaction term associated with the line itself. p-Adicization boils down to the condition that Kähler function, matrix elements of Kähler form, WCW Hamiltonians and their super counterparts, are rational functions of complex WCW coordinates just as they are for those symmetric spaces that I know of. This would allow a continuation to p-adic context.

In the following this vision about generalized Feynman diagrams is discussed in more detail.

2.7.1 Questions

The goal is a proposal for how to perform the integral over WCW for generalized Feynman digrams and the best manner to proceed to this goal is by making questions.

What does finite measurement resolution mean?

The first question is what finite measurement resolution means.

1. One expects that the algebraic continuation makes sense only for a finite measurement resolution in which case one obtains only finite sums of what one might hope to be algebraic functions. The finiteness of the algebraic extension would be in fact equivalent with the finite measurement resolution.

2. Finite measurement resolution means a discretization in terms of number theoretic braids. p-Adicization condition suggests that one must allow only the number theoretic braids. For these the ends of braid at boundary of CD are algebraic points of the embedding space. This would be true at least in the intersection of real and p-adic worlds.

3. The question is whether one can localize the points of the braid. The necessity to use momentum eigenstates to achieve quantum classical correspondence in the Kähler-Dirac action [K111] suggests however a de-localization of braid points, that is wave function in space of braid points. In real context one could allow all possible choices for braid points but in p-adic context only algebraic points are possible if one wants to replace integrals with sums. This implies finite measurement resolution analogous to that in lattice. This is also the only possibility in the intersection of real and p-adic worlds.

   A non-trivial prediction giving a strong correlation between the geometry of the partonic 2-surface and quantum numbers is that the total number $n_F + n_{\overline{F}}$ of fermions and anti-fermions is bounded above by the number n_{alg} of algebraic points for a given partonic 2-surface: $n_F + n_{\overline{F}} \leq n_{alg}$. Outside the intersection of real and p-adic worlds the problematic aspect of this definition is that small deformations of the partonic 2-surface can radically change the number of algebraic points unless one assumes that the finite measurement resolution means restriction of WCW to a sub-space of algebraic partonic surfaces.

4. Braids defining propagator lines for fundamental fermions (to be distinguished from observer particles) emerges naturally. Braid strands correspond to the boundaries of string world sheets at which the modes of induced spinor fields are localized from the condition that em charge is well-defined: induced W field and above weak scale also Z^0 field vanish at them.

   In order to obtain non-trivial fermion propagator one must add to Dirac action 1-D Dirac action in induced metric with the boundaries of string world sheets at the light-like parton orbits. Its bosonic counterpart is line-length in induced metric. Field equations imply that the boundaries are light-like geodesics and fermion has light-like 8-momentum. This suggests strongly a connection with quantum field theory and an 8-D generalization of twistor Grassmannian approach. By field equations the bosonic part of this action does not contribute to the Kähler action. The light-like 8-momenta p^k have same M^4 and CP_2 mass squared and latter correspond to the the eigenvalues of the CP_2 spinor d'Alembertian by quantum-classical correspondence.

5. One has also discretization of the relative position of the second tip of CD at the hyperboloid isometric with mass shell. Only the number of braid points and their momenta would matter, not their positions.

6. The quantum numbers characterizing positive and negative energy parts of zero energy states couple directly to space-time geometry via the measurement interaction terms in Kähler action expressing the equality of classical conserved charges in Cartan algebra with their quantal counterparts for space-time surfaces in quantum superposition. This makes sense if classical charges parametrize zero modes. The localization in zero modes in state function reduction would be the WCW counterpart of state function collapse.

How to define integration in WCW degrees of freedom?

The basic question is how to define the integration over WCW degrees of freedom.

1. What comes mind first is Gaussian perturbation theory around the maxima of Kähler function. Gaussian and metric determinants cancel each other and only algebraic expressions remain. Finiteness is not a problem since the Kähler function is non-local functional of 3-surface so that no local interaction vertices are present. One should however assume the vanishing of loops required also by algebraic universality and this assumption look unrealistic when one considers more general functional integrals than that of vacuum functional since free field theory is not in question. The construction of the inverse of the WCW metric defining the propagator is also a very difficult challenge. Duistermaat-Hecke theorem states that something like this known as localization might be possible and one can also argue that something analogous to localization results from a generalization of mean value theorem.

2. Symmetric space property is more promising since it might reduce the integrations to group theory using the generalization of Fourier analysis for group representations so that there would be no need for perturbation theory in the proposed sense. In finite measurement resolution the symmetric spaces involved would be finite-dimensional. Symmetric space structure of WCW could also allow to define p-adic integration in terms of p-adic Fourier analysis for symmetric spaces. Essentially algebraic continuation of the integration from the real case would be in question with additional constraints coming from the fact that only phase factors corresponding to finite algebraic extensions of rationals are used. Cutoff would emerge automatically from the cutoff for the dimension of the algebraic extension.

How to define generalized Feynman diagrams?

Integration in symmetric spaces could serve as a model at the level of WCW and allow both the understanding of WCW integration and p-adicization as algebraic continuation. In order to get a more realistic view about the problem one must define more precisely what the calculation of the generalized Feynman diagrams means.

1. WCW integration must be carried out separately for all values of the momenta associated with the internal lines. The reason is that the spectrum of eigenvalues λ_i of the Kähler-Dirac operator D depends on the momentum of line and momentum conservation in vertices translates to a correlation of the spectra of D at internal lines.

2. For tree diagrams algebraic continuation to the p-adic context if the expression involves only the replacement of the generalized eigenvalues of D as functions of momenta with their p-adic counterparts besides vertices. If these functions are algebraically universal and expressible in terms of harmonics of symmetric space , there should be no problems.

3. If loops are involved, one must integrate/sum over loop momenta. In p-adic context difficulties are encountered if the spectrum of the momenta is continuous. The integration over on mass shell loop momenta is analogous to the integration over sub-CDs, which suggests that internal line corresponds to a $sub - CD$ in which it is at rest. There are excellent reasons to believe that the moduli space for the positions of the upper tip is a discrete subset of hyperboloid of future light-cone. If this is the case, the loop integration indeed reduces to a sum over discrete positions of the tip. p-Adizication would thus give a further good reason why for ZEO.

4. Propagator is expressible in terms of the inverse of generalized eigenvalue and there is a sum over these for each propagator line. At vertices one has products of WCW harmonics assignable to the incoming lines. The product must have vanishing quantum numbers associated with the phase angle variables of WCW. Non-trivial quantum numbers of the WCW harmonic correspond to WCW quantum numbers assignable to excitations of ordinary elementary particles. WCW harmonics are products of functions depending on the "radial" coordinates and phase factors and the integral over the angles leaves the product of the first ones analogous to Legendre polynomials $P_{l,m}$, These functions are expected to be rational functions or at least algebraic functions involving only square roots.

5. In ordinary QFT incoming and outgoing lines correspond to propagator poles. In the recent case this would mean that incoming stringy lines at the ends of CD correspond to fermions satisfying the stringy mass formula serving as a generalization of masslessness condition.

2.7.2 Generalized Feynman Diagrams At Fermionic And Momentum SpaceLevel

Negative energy ontology has already led to the idea of interpreting the virtual particles as pairs of positive and negative energy wormhole throats. Hitherto I have taken it as granted that ordinary Feynman diagrammatics generalizes more or less as such. It is however far from clear what really happens in the verties of the generalized Feynman diagrams. The safest approach relies on the requirement that unitarity realized in terms of Cutkosky rules in ordinary Feynman diagrammatics allows a generalization. This requires loop diagrams. In particular, photon-photon scattering can take place only via a fermionic square loop so that it seems that loops must be present at least in the topological sense.

One must be however ready for the possibility that something unexpectedly simple might emerge. For instance, the vision about algebraic physics allows naturally only finite sums for diagrams and does not favor infinite perturbative expansions. Hence the true believer on algebraic physics might dream about finite number of diagrams for a given reaction type. For simplicity generalized Feynman diagrams without the complications brought by the magnetic confinement since by the previous arguments the generalization need not bring in anything essentially new.

The basic idea of duality in early hadronic models was that the lines of the dual diagram representing particles are only re-arranged in the vertices. This however does not allow to get rid of off mass shell momenta. ZEO encourages to consider a stronger form of this principle in the sense that the virtual momenta of particles could correspond to pairs of on mass shell momenta of particles. If also interacting fermions are pairs of positive and negative energy throats in the interaction region the idea about reducing the construction of Feynman diagrams to some kind of lego rules might work.

Virtual particles as pairs of on mass shell particles in ZEO

The first thing is to try to define more precisely what generalized Feynman diagrams are. The direct generalization of Feynman diagrams implies that both wormhole throats and wormhole contacts join at vertices.

1. A simple intuitive picture about what happens is provided by diagrams obtained by replacing the points of Feynman diagrams (wormhole contacts) with short lines and imagining that the throats correspond to the ends of the line. At vertices where the lines meet the incoming on mass shell quantum numbers would sum up to zero. This approach leads to a straightforward generalization of Feynman diagrams with virtual particles replaced with pairs of on mass shell throat states of type $++$, $--$, and $+-$. Incoming lines correspond to $++$ type lines and outgoing ones to $--$ type lines. The first two line pairs allow only time like net momenta whereas $+-$ line pairs allow also space-like virtual momenta. The sign assigned to a given throat is dictated by the sign of the on mass shell momentum on the line. The condition that Cutkosky rules generalize as such requires $++$ and $--$ type virtual lines since the cut of the diagram in Cutkosky rules corresponds to on mass shell outgoing or incoming states and must therefore correspond to $++$ or $--$ type lines.

2. The basic difference as compared to the ordinary Feynman diagrammatics is that loop integrals are integrals over mass shell momenta and that all throats carry on mass shell momenta. In each vertex of the loop mass incoming on mass shell momenta must sum up to on mass shell momentum. These constraints improve the behavior of loop integrals dramatically and give excellent hopes about finiteness. It does not however seem that only a finite number of diagrams contribute to the scattering amplitude besides tree diagrams. The point is that if a the reactions $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$,, where N_i denote particle numbers, are possible in a common kinematical region for N_2-particle states then also the diagrams $N_1 \rightarrow N_2 \rightarrow N_2 \rightarrow N_3$ are possible. The virtual states N_2 include all all states in the intersection of kinematically allow regions for $N_1 \rightarrow N_2$ and $N_2 \rightarrow N_3$. Hence the dream about finite number possible

diagrams is not fulfilled if one allows massless particles. If all particles are massive then the particle number N_2 for given N_1 is limited from above and the dream is realized.

3. For instance, loops are not possible in the massless case or are highly singular (bringing in mind twistor diagrams) since the conservation laws at vertices imply that the momenta are parallel. In the massive case and allowing mass spectrum the situation is not so simple. As a first example one can consider a loop with three vertices and thus three internal lines. Three on mass shell conditions are present so that the four-momentum can vary in 1-D subspace only. For a loop involving four vertices there are four internal lines and four mass shell conditions so that loop integrals would reduce to discrete sums. Loops involving more than four vertices are expected to be impossible.

4. The proposed replacement of the elementary fermions with bound states of elementary fermions and monopoles $X_\pm$ brings in the analog of stringy diagrammatics. The 2-particle wave functions in the momentum degrees of freedom of fermion and $X_\pm$ might allow more flexibility and allow more loops. Note however that there are excellent hopes about the finiteness of the theory also in this case.

Loop integrals are manifestly finite

One can make also more detailed observations about loops.

1. The simplest situation is obtained if only 3-vertices are allowed. In this case conservation of momentum however allows only collinear momenta although the signs of energy need not be the same. Particle creation and annihilation is possible and momentum exchange is possible but is always light-like in the massless case. The scattering matrices of supersymmetric YM theories would suggest something less trivial and this raises the question whether something is missing. Magnetic monopoles are an essential element of also these theories as also massivation and symmetry breaking and this encourages to think that the formation of massive states as fermion $X_\pm$ pairs is needed. Of course, in TGD framework one has also high mass excitations of the massless states making the scattering matrix non-trivial.

2. In YM theories on mass shell lines would be singular. In TGD framework this is not the case since the propagator is defined as the inverse of the 3-D dimensional reduction of the Kähler-Dirac operator D containing also coupling to four-momentum (this is required by quantum classical correspondence and guarantees stringy propagators),

$$\begin{aligned} D &= i\hat{\Gamma}^\alpha p_\alpha + \hat{\Gamma}^\alpha D_\alpha \ , \\ p_\alpha &= p_k \partial_\alpha h^k \ . \end{aligned} \tag{2.7.1}$$

The propagator does not diverge for on mass shell massless momenta and the propagator lines are well-defined. This is of course of essential importance also in general case. Only for the incoming lines one can consider the possibility that 3-D Dirac operator annihilates the induced spinor fields. All lines correspond to generalized eigenstates of the propagator in the sense that one has $D_3\Psi = \lambda\gamma\Psi$, where γ is Kähler-Dirac gamma matrix in the direction of the stringy coordinate emanating from light-like surface and D_3 is the 3-dimensional dimensional reduction of the 4-D Kähler-Dirac operator. The eigenvalue λ is analogous to energy. Note that the eigenvalue spectrum depends on 4-momentum as a parameter.

3. Massless incoming momenta can decay to massless momenta with both signs of energy. The integration measure $d^2k/2E$ reduces to dx/x where $x \geq 0$ is the scaling factor of massless momentum. Only light-like momentum exchanges are however possible and scattering matrix is essentially trivial. The loop integrals are finite apart from the possible delicacies related to poles since the loop integrands for given massless wormhole contact are proportional to dx/x^3 for large values of x.

4. Irrespective of whether the particles are massless or not, the divergences are obtained only if one allows too high vertices as self energy loops for which the number of momentum degrees of freedom is $3N-4$ for N-vertex. The construction of SUSY limit of TGD in [?] led to the conclusion that the parallelly propagating N fermions for given wormhole throat correspond to a product of N fermion propagators with same four-momentum so that for fermions and ordinary bosons one has the standard behavior but for $N>2$ non-standard so that these excitations are not seen as ordinary particles. Higher vertices are finite only if the total number N_F of fermions propagating in the loop satisfies $N_F > 3N-4$. For instance, a 4-vertex from which $N=2$ states emanate is finite.

Taking into account magnetic confinement

What has been said above is not quite enough. The weak form of electric-magnetic duality [B4] leads to the picture about elementary particles as pairs of magnetic monopoles inspiring the notions of weak confinement based on magnetic monopole force. Also color confinement would have magnetic counterpart. This means that elementary particles would behave like string like objects in weak boson length scale. Therefore one must also consider the stringy case with wormhole throats replaced with fermion-$X_\pm$ pairs ($X_\pm$ is electromagnetically neutral and $\pm$ refers to the sign of the weak isospin opposite to that of fermion) and their super partners.

1. The simplest assumption in the stringy case is that fermion-$X_\pm$ pairs behave as coherent objects, that is scatter elastically. In more general case only their higher excitations identifiable in terms of stringy degrees of freedom would be created in vertices. The massivation of these states makes possible non-collinear vertices. An open question is how the massivation fermion-$X_\pm$ pairs relates to the existing TGD based description of massivation in terms of Higgs mechanism and Kähler-Dirac operator.

2. Mass renormalization could come from self energy loops with negative energy lines as also vertex normalization. By very general arguments supersymmetry implies the cancellation of the self energy loops but would allow non-trivial vertex renormalization [?] .

3. If only 3-vertices are allowed, the loops containing only positive energy lines are possible if on mass shell fermion-$X_\pm$ pair (or its superpartner) can decay to a pair of positive energy pair particles of same kind. Whether this is possible depends on the masses involved. For ordinary particles these decays are not kinematically possible below intermediate boson mass scale (the decays $F_1 \rightarrow F_2 + \gamma$ are forbidden kinematically or by the absence of flavor changing neutral currents whereas intermediate gauge bosons can decay to on mass shell fermion-anti-fermion pair).

4. The introduction of IR cutoff for 3-momentum in the rest system associated with the largest CD (causal diamond) looks natural as scale parameter of coupling constant evolution and p-adic length scale hypothesis favors the inverse of the size scale of CD coming in powers of two. This parameter would define the momentum resolution as a discrete parameter of the p-adic coupling constant evolution. This scale does not have any counterpart in standard physics. For electron, d quark, and u quark the proper time distance between the tips of CD corresponds to frequency of 10 Hz, 1280 Hz, and 160 Hz: all these frequencies define fundamental bio-rhythms [K33] .

These considerations have left completely untouched one important aspect of generalized Feynman diagrams: the necessity to perform a functional integral over the deformations of the partonic 2-surfaces at the ends of the lines- that is integration over WCW. Number theoretical universality requires that WCW and these integrals make sense also p-adically and in the following these aspects of generalized Feynman diagrams are discussed.

2.7.3 Harmonic Analysis In WCW As a way To Calculate WCWFunctional Integrals

Previous examples suggest that symmetric space property, Kähler and symplectic structure and the use of symplectic coordinates consisting of canonically conjugate pairs of phase angles and corresponding "radial" coordinates are essential for WCW integration and p-adicization. Kähler function, the components of the metric, and therefore also metric determinant and Kähler function depend on the "radial" coordinates only and the possible generalization involves the identification the counterparts of the "radial" coordinates in the case of WCW.

Conditions guaranteeing the reduction to harmonic analysis

The basic idea is that harmonic analysis in symmetric space allows to calculate the functional integral over WCW.

1. Each propagator line corresponds to a symmetric space defined as a coset space G/H of the symplectic group and Kac-Moody group and one might hope that the proposed p-adicization works for it- at least when one considers the hierarchy of measurement resolutions forced by the finiteness of algebraic extensions. This coset space is as a manifold Cartesian product $(G/H) \times (G/H)$ of symmetric spaces G/H associated with ends of the line. Kähler metric contains also an interaction term between the factors of the Cartesian product so that Kähler function can be said to reduce to a sum of "kinetic" terms and interaction term.

2. Effective 2-dimensionality and ZEO allow to treat the ends of the propagator line independently. This means an enormous simplification. Each line contributes besides propagator a piece to the exponent of Kähler action identifiable as interaction term in action and depending on the propagator momentum. This contribution should be expressible in terms of generalized spherical harmonics. Essentially a sum over the products of pairs of harmonics associated with the ends of the line multiplied by coefficients analogous to $1/(p^2 - m^2)$ in the case of the ordinary propagator would be in question. The optimal situation is that the pairs are harmonics and their conjugates appear so that one has invariance under G analogous to momentum conservation for the lines of ordinary Feynman diagrams.

3. Momentum conservation correlates the eigenvalue spectra of the Kähler-Dirac operator D at propagator lines [K111] . G-invariance at vertex dictates the vertex as the singlet part of the product of WCW harmonics associated with the vertex and one sums over the harmonics for each internal line. p-Adicization means only the algebraic continuation to real formulas to p-adic context.

4. The exponent of Kähler function depends on both ends of the line and this means that the geometries at the ends are correlated in the sense that Kähler form contains interaction terms between the line ends. It is however not quite clear whether it contains separate "kinetic" or self interaction terms assignable to the line ends. For Kähler function the kinetic and interaction terms should have the following general expressions as functions of complex WCW coordinates:

$$\begin{aligned} K_{kin,i} &= \sum_n f_{i,n}(Z_i)\overline{f_{i,n}(Z_i)} + c.c \ , \\ K_{int} &= \sum_n g_{1,n}(Z_1)\overline{g_{2,n}(Z_2)} + c.c \ , i = 1,2 \ . \end{aligned} \tag{2.7.2}$$

Here $K_{kin,i}$ define "kinetic" terms and K_{int} defines interaction term. One would have what might be called holomorphic factorization suggesting a connection with conformal field theories.

Symmetric space property -that is isometry invariance- suggests that one has

$$f_{i,n} = f_{2,n} \equiv f_n \ , \ g_{1,n} = g_{2,n} \equiv g_n \tag{2.7.3}$$

such that the products are invariant under the group H appearing in G/H and therefore have opposite H quantum numbers. The exponent of Kähler function does not factorize although the terms in its Taylor expansion factorize to products whose factors are products of holomorphic and antiholomorphic functions.

5. If one assumes that the exponent of Kähler function reduces to a product of eigenvalues of the Kähler-Dirac operator eigenvalues must have the decomposition

$$\lambda_k = \prod_{i=1,2} exp\left[\sum_n c_{k,n} g_n(Z_i)\overline{g_n(Z_i)} + c.c\right] \times exp\left[\sum_n d_{k,n} g_n(Z_1)\overline{g_n(Z_2)} + c.c\right] \quad (2.7.4)$$

Hence also the eigenvalues coming from the Dirac propagators have also expansion in terms of G/H harmonics so that in principle WCW integration would reduce to Fourier analysis in symmetric space.

Generalization of WCW Hamiltonians

This picture requires a generalization of the view about configuration space Hamiltonians since also the interaction term between the ends of the line is present not taken into account in the previous approach.

1. The proposed representation of WCW Hamiltonians as flux Hamiltonians [K26, K111]

$$\begin{aligned} Q(H_A) &= \int H_A(1+K)Jd^2x \ , \\ J &= \epsilon^{\alpha\beta}J_{\alpha\beta} \ , \ J^{03}\sqrt{g_4} = KJ_{12} \ . \end{aligned} \quad (2.7.5)$$

works for the kinetic terms only since J cannot be the same at the ends of the line. The formula defining K assumes weak form of self-duality (^{03} refers to the coordinates in the complement of X^2 tangent plane in the 4-D tangent plane). K is assumed to be symplectic invariant and constant for given X^2. The condition that the flux of $F^{03} = (\hbar/g_K)J^{03}$ defining the counterpart of Kähler electric field equals to the Kähler charge g_K gives the condition $K = g_K^2/\hbar$, where g_K is Kähler coupling constant. Within experimental uncertainties one has $\alpha_K = g_K^/4\pi\hbar_0 = \alpha_{em} \simeq 1/137$, where α_{em} is finite structure constant in electron length scale and $\hbar_0$ is the standard value of Planck constant.

The assumption that Poisson bracket of WCW Hamiltonians reduces to the level of embedding space - in other words $\{Q(H_A), Q(H_B\} = Q(\{H_A, H_B\})$ - can be justified. One starts from the representation in terms of say flux Hamiltonians $Q(H_A)$ and *defines* $J_{A,B}$ as $J_{A,B} \equiv Q(\{H_A, H_B\})$. One has $\partial H_A/\partial t_B = \{H_B, H_A\}$, where t_B is the parameter associated with the exponentiation of H_B. The inverse J^{AB} of $J_{A,B} = \partial H_B/\partial t_A$ is expressible as $J^{A,B} = \partial t_A/\partial H_B$. From these formulas one can deduce by using chain rule that the bracket $\{Q(H_A), Q(H_B\} = \partial t_C Q(H_A) J^{CD} \partial t_D Q(H_B)$ of flux Hamiltonians equals to the flux Hamiltonian $Q(\{H_A, H_B\})$.

2. One should be able to assign to WCW Hamiltonians also a part corresponding to the interaction term. The symplectic conjugation associated with the interaction term permutes the WCW coordinates assignable to the ends of the line. One should reduce this apparently non-local symplectic conjugation (if one thinks the ends of line as separate objects) to a non-local symplectic conjugation for $\delta CD \times CP_2$ by identifying the points of lower and upper end of CD related by time reflection and assuming that conjugation corresponds to time reflection. Formally this gives a well defined generalization of the local Poisson brackets between time reflected points at the boundaries of CD. The connection of Hermitian conjugation and time reflection in quantum field theories is is in accordance with this picture.

3. The only manner to proceed is to assign to the flux Hamiltonian also a part obtained by the replacement of the flux integral over X^2 with an integral over the projection of X^2 to a sphere S^2 assignable to the light-cone boundary or to a geodesic sphere of CP_2, which come as two varieties corresponding to homologically trivial and non-trivial spheres. The projection is defined as by the geodesic line orthogonal to S^2 and going through the point of X^2. The hierarchy of Planck constants assigns to CD a preferred geodesic sphere of CP_2 as well as a unique sphere S^2 as a sphere for which the radial coordinate r_M or the light-cone boundary defined uniquely is constant: this radial coordinate corresponds to spherical coordinate in the rest system defined by the time-like vector connecting the tips of CD. Either spheres or possibly both of them could be relevant.

   Recall that also the construction of number theoretic braids and symplectic QFT [K24] led to the proposal that braid diagrams and symplectic triangulations could be defined in terms of projections of braid strands to one of these spheres. One could also consider a weakening for the condition that the points of the number theoretic braid are algebraic by requiring only that the S^2 coordinates of the projection are algebraic and that these coordinates correspond to the discretization of S^2 in terms of the phase angles associated with θ and ϕ.

   This gives for the corresponding contribution of the WCW Hamiltonian the expression

$$Q(H_A)_{int} = \int_{S^2_\pm} H_A X \delta^2(s_+,s_-) d^2 s_\pm = \int_{P(X^2_+)\cap P(X^2_-)} \frac{\partial(s^1,s^2)}{\partial(x^1_\pm,x^2_\pm)} d^2 x_\pm \ . \qquad (2.7.6)$$

   Here the Poisson brackets between ends of the line using the rules involve delta function $\delta^2(s_+,s_-)$ at S^2 and the resulting Hamiltonians can be expressed as a similar integral of $H_{[A,B]}$ over the upper or lower end since the integral is over the intersection of S^2 projections.

   The expression must vanish when the induced Kähler form vanishes for either end. This is achieved by identifying the scalar X in the following manner:

$$\begin{aligned} X &= J_+^{kl} J^-_{kl} \ , \\ J_\pm^{kl} &= (1+K_\pm)\partial_\alpha s^k \partial_\beta s^l J_\pm^{\alpha\beta} \ . \end{aligned} \qquad (2.7.7)$$

   The tensors are lifts of the induced Kähler form of $X^2_\pm$ to S^2 (not CP_2).

4. One could of course ask why these Hamiltonians could not contribute also to the kinetic terms and why the brackets with flux Hamiltonians should vanish. This relate to how one *defines* the Kähler form. It was shown above that in case of flux Hamiltonians the definition of Kähler form as brackets gives the basic formula $\{Q(H_A),Q(H_B)\} = Q(\{H_A,H_B\}$ and same should hold true now. In the recent case $J_{A,B}$ would contain an interaction term defined in terms of flux Hamiltonians and the previous argument should go through also now by identifying Hamiltonians as sums of two contributions and by introducing the doubling of the coordinates t_A.

5. The quantization of the Kähler-Dirac operator must be reconsidered. It would seem that one must add to the super-Hamiltonian completely analogous term obtained by replacing $(1+K)J$ with $X\partial(s^1,s^2)/\partial(x^1_\pm,x^2_\pm)$. Besides the anti-commutation relations defining correct anti-commutators to flux Hamiltonians, one should pose anti-commutation relations consistent with the anti-commutation relations of super Hamiltonians. In these anti-commutation relations $(1+K)J\delta^2(x,y)$ would be replaced with $X\delta^2(s^+,s^-)$. This would guarantee that the oscillator operators at the ends of the line are not independent and that the resulting Hamiltonian reduces to integral over either end for $H_{[A,B]}$.

6. In the case of CP_2 the Hamiltonians generating isometries are rational functions. This should hold true also now so that p-adic variants of Hamiltonians as functions in WCW would make sense. This in turn would imply that the components of the WCW Kähler form are rational functions. Also the exponentiation of Hamiltonians make sense p-adically if one allows the exponents of group parameters to be functions $Exp_p(t)$.

Does the expansion in terms of partial harmonics converge?

The individual terms in the partial wave expansion seem to be finite but it is not at all clear whether the expansion in powers of K actually converges.

1. In the proposed scenario one performs the expansion of the vacuum functional $exp(K)$ in powers of K and therefore in negative powers of α_K. In principle an infinite number of terms can be present. This is analogous to the perturbative expansion based on using magnetic monopoles as basic objects whereas the expansion using the contravariant Kähler metric as a propagator would be in positive powers of α_K and analogous to the expansion in terms of magnetically bound states of wormhole throats with vanishing net value of magnetic charge. At this moment one can only suggest various approaches to how one could understand the situation.

2. Weak form of self-duality and magnetic confinement could change the situation. Performing the perturbation around magnetic flux tubes together with the assumed slicing of the space-time sheet by stringy world sheets and partonic 2-surfaces could mean that the perturbation corresponds to the action assignable to the electric part of Kähler form proportional to α_K by the weak self-duality. Hence by $K = 4\pi\alpha_K$ relating Kähler electric field to Kähler magnetic field the expansion would come in powers of a term containing sum of terms proportional to α_K^0 and α_K. This would leave to the scattering amplitudes the exponents of Kähler function at the maximum of Kähler function so that the non-analytic dependence on α_K would not disappear.

A further reason to be worried about is that the expansion containing infinite number of terms proportional to α_K^0 could fail to converge.

1. This could be also seen as a reason for why magnetic singlets are unavoidable except perhaps for $\hbar < \hbar_0$. By the holomorphic factorization the powers of the interaction part of Kähler action in powers of $1/\alpha_K$ would naturally correspond to increasing and opposite net values of the quantum numbers assignable to the WCW phase coordinates at the ends of the propagator line. The magnetic bound states could have similar expansion in powers of α_K as pairs of states with arbitrarily high but opposite values of quantum numbers. In the functional integral these quantum numbers would compensate each other. The functional integral would leave only an expansion containing powers of α_K starting from some finite possibly negative (unless one assumes the weak form of self-duality) power. Various gauge coupling strengths are expected to be proportional to α_K and these expansions should reduce to those in powers of α_K.

2. Since the number of terms in the fermionic propagator expansion is finite, one might hope on basis of super-symmetry that the same is true in the case of the functional integral expansion. By the holomorphic factorization the expansion in powers of K means the appearance of terms with increasingly higher quantum numbers. Quantum number conservation at vertices would leave only a finite number of terms to tree diagrams. In the case of loop diagrams pairs of particles with opposite and arbitrarily high values of quantum numbers could be generated at the vertex and magnetic confinement might be necessary to guarantee the convergence. Also super-symmetry could imply cancellations in loops.

Could one do without flux Hamiltonians?

The fact that the Kähler functions associated with the propagator lines can be regarded as interaction terms inspires the question whether the Kähler function could contain only the interaction terms so that Kähler form and Kähler metric would have components only between the ends of the lines.

1. The basic objection is that flux Hamiltonians too beautiful objects to be left without any role in the theory. One could also argue that the WCW metric would not be positive definite if only the non-diagonal interaction term is present. The simplest example is Hermitian 2×2-matrix with vanishing diagonal for which eigenvalues are real but of opposite sign.

2. One could of course argue that the expansions of $exp(K)$ and λ_k give in the general powers $(f_n\overline{f_n})^m$ analogous to diverging tadpole diagrams of quantum field theories due to local interaction vertices. These terms do not produce divergences now but the possibility that the exponential series of this kind of terms could diverge cannot be excluded. The absence of the kinetic terms would allow to get rid of these terms and might be argued to be the symmetric space counterpart for the vanishing of loops in WCW integral.

3. In ZEO this idea does not look completely non-sensical since physical states are pairs of positive and negative energy states. Note also that in quantum theory only creation operators are used to create positive energy states. The manifest non-locality of the interaction terms and absence of the counterparts of kinetic terms would provide a trivial manner to get rid of infinities due to the presence of local interactions. The safest option is however to keep both terms.

Summary

The discussion suggests that one must treat the entire Feynman graph as single geometric object with Kähler geometry in which the symmetric space is defined as product of what could be regarded as analogs of symmetric spaces with interaction terms of the metric coming from the propagator lines. The exponent of Kähler function would be the product of exponents associated with all lines and contributions to lines depend on quantum numbers (momentum and color quantum numbers) propagating in line via the coupling to the Kähler-Dirac operator. The conformal factorization would allow the reduction of integrations to Fourier analysis in symmetric space. What is of decisive importance is that the entire Feynman diagrammatics at WCW level would reduce to the construction of WCW geometry for a single propagator line as a function of quantum numbers propagating on the line.

2.8 Appendix: Basic Facts About Algebraic Numbers, Quaternions And Octonions

To understand the detailed connection between infinite primes, polynomial primes and Fock states, some basic concepts of algebraic number theory related to the generalization of prime and prime factorization [A38, A53, A68] (the first reference is warmly recommended for a physicist because it teaches the basic facts through exercises; also second book is highly enjoyable reading because of its non-Bourbakian style of representation).

2.8.1 Generalizing The Notion Of Prime

Algebraic numbers are defined as roots of polynomial equations with rational coefficients. Algebraic integers are identified as roots of monic polynomials (highest coefficient equals to one) with integer coefficients. Algebraic number fields correspond to algebraic extensions of rationals and can have any dimension as linear spaces over rationals. The notion of prime is extremely general and involves rather abctract mathematics in general case.

Quite generally, commutative ring R called integral domain, if the product ab vanishes only if a or b vanishes. To a given integral domain one can assign a number field by essentially the same construction by which one assigns the field of rationals to ordinary integers. The integer valued function $a \rightarrow N(a)$ in R is called norm if it has the properties $N(ab) = N(a)N(b)$ and $N(1) = 1$. For instance, for the algebraic extension $Q(\sqrt{-D})$ of rationals consisting of points $z = r + \sqrt{-D}s$, the function $N(z) = r^2 + Ds^2$ defines norm. More generally, the determinant of the linear map defined by the action of z in algebraic number field defines norm function. This determinant reduces to the product of all conjugates of z in K and is n: th order polynomial with respect to the components of z when K is n-dimensional.

Irreducible elements (almost the counterparts of primes) can be defined as elements P of integral domain having the property that if one has $P = bc$, then either b or c has unit norm. Elements with unit norm are called units and elements differing by a multiplication with unit are called associates. Note that in the case of p-adics all p-adic numbers with unit norm are units.

2.8.2 Ufds, Pids And Eds

If the elements of R allow a unique factorization to irreducible elements, R is said to be unique factorization domain (UFD). Ordinary integers are obviously UFD. The field $Z(\sqrt{-5})$ is not UFd for instance, one has $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. The fact that prime factorization is not unique forces to generalize the notion of primeness such that ideals in the ring of algebraic integers take the role of integers. The counterparts of primes can be identified as irreducible elements, which generate prime ideals containing one and only one rational prime. Irreducible elements, such as $1 \pm \sqrt{-5}$ in $Z(\sqrt{-5})$, are not primes in this sense.

Principal ideal domain (PID) is defined as an integral domain for which all ideals are principal, that is are generated as powers of single element. In the case of ordinary integers powers of integers define PID.

Euclidian domain (ED) is integral domain with the property that for any pair a and b one can find pair (q, r) such that $a = bq + r$ with $N(r) < N(a)$. This guarantees that the Euclidian algorithm used in the division of rationals converges. Integers form an Euclidian domain but polynomials with integer coefficients do not (elements 2 and x do not allow decomposition $2 = q(x)x + r$). It can be shown that EDs are PIDs in turn are UFDs. For instance, for complex quadratic extensions of integers $Z(\sqrt{-d})$ there are only 9 UFDs and they correspond to $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For extensions of type $Z(\sqrt{d})$ the number of UFD: s is infinite. There are not too many quadratic extensions which are ED: s and the possible values of d are $d = -1, \pm 2, \pm 3, 5, 6, \pm 7, \pm 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$.

Any algebraic number field K is representable always as a polynomial ring $Q[\theta]$ obtained from the polynomial ring $Q[x]$ by replacing x with an algebraic number θ, which is a root of an irreducible polynomial with rational coefficients. This field has dimension n over rationals, where n is the degree of the polynomial in question.

2.8.3 The Notion Of Prime Ideal

As already noticed, a general algebraic number field K does not allow a unique factorization into irreducibles and one must generalize the notion of prime number and integer in order to achieve a unique factorization. The ideals of the ring O_K of algebraic integers in K take the role of integers whereas prime ideals take the role of primes. The factorization of an ideal to a product of prime ideals is unique and each prime ideal contains single rational prime characterizing it. One can assign to an ideal norm which orders the ideals: $N(a) < N(b) \leftrightarrow b \subset a$. The smaller the integer generating ideal, the larger the ideal is and the ideals generated by primes are maximal ones in PID. The equivalence classes of the ideals of O_K under equivalence defined by integer multiplication form a group. The number of classes is a characteristic of an algebraic number field. For class-one algebraic number fields prime factorization of ideals is equivalent with the factorization to irreducibles in K. $Z(\sqrt{-5})$, which is not UFD, allows two classes of prime ideals. Cyclotomic number fields $Q(\zeta_m)$, where ζ_m is m: th root of unity have class number one for $3 \leq m \leq 10$. In particular, the four-dimensional algebraic number fields $Q(\zeta_8)$ and $Q(\zeta_5) = Q(\zeta_{10})$ are ED and thus UFD.

Basic facts about primality for polynomial rings

The notion of primality can be abstracted to the level of polynomial algebras in field K and these polynomial algebras seem to be more or less identical with the algebra formed by infinite integers. The following two results are crucial for the argument demonstrating that this is indeed the case.

Polynomial ring associated with any number field is UFD

The elements in the ring $K[x_1, ..., x_n]$ formed by the polynomials having coefficients in *any* field K and x_i having values in K, allow a unique decomposition into prime factors. This means that things are much simpler at the next abstraction level, since there is no need for refined class theories needed in the case of algebraic number fields.

The number field K appearing as a coefficient field of polynomials could correspond to finite fields (Galois fields), rationals, any algebraic number field obtained as an extension of rational, p-adic numbers, reals or complex numbers. For $Q[x]$, where Q denotes rationals, the simplest prime factors are monomials of form $x - q$, q rational number. More complicated prime factors correspond to minimal polynomials having algebraic number α and its conjugates as their roots. In the case of complex number field only monomomials $x - z$, z complex number are the only prime polynomials. Clearly, the primes at the higher level of abstraction are generalized rationals of previous level plus numbers which are algebraic with respect to the generalized rationals.

The polynomial rings associated with any UFD are UFD

If R is a unique factorization domain (UFD), then also $R[x]$ is UFD: this holds also for $R[x_1, ..., x_n]$. Hence one obtains an infinite hierarchy of UFDs by a repeated abstraction process by starting from a given algebraic number field K. At the first step one obtains the ring $K[x]$ of polynomials in K. At the next step one obtains the ring of polynomials $K^{2)}[y]$ having as coefficient ring the ring $K[x] \equiv K^{1)}[x]$ of polynomials. At the next step one obtains $K^{2)}[z]$, etc.. Note that $O_K[x]$ is not ED in general and need not be UFD neither unless O_K is UFD. $O_K[x]$ is not however interesting from the viewpoint of TGD.

An element of $K^{2)}(y)$ corresponds to a polynomial $P(y, x)$ of y such that its coefficients are K-rational functions of x. A polynomial in $K^{3)}(z)$ corresponds to a polynomial of $P(z, y, z)$ such that the coefficients of z are K-rational functions of functions of y with coefficients which are K-rational functions of z. Note that as a special case, polynomials of all n variables result. Note also the hierarchical ordering of the variables. Thus the hierarchy of polynomials gives rise to a hierarchy of functions having increasingly number of independent variables.

2.8.4 Examples Of Two-Dimensional Algebraic Number Fields

The general two-dimensional (in algebraic sense) algebraic extension of rationals corresponds to $K(\theta)$, where $\theta = (-b \pm \sqrt{b^2 - 4c})/2$ is root of second order irreducible polynomial $x^2 + bx + c$. Depending on whether the discriminant $D = b^2 - 4c$ is positive or negative, one obtains real and complex extensions. θ and its conjugate generate equivalent extensions and all extensions can be obtained as extensions of form $Q(\sqrt{\pm d})$.

For $Q(\sqrt{d})$, d square-free integer, units correspond to powers of $x = \pm(p_{n-1} + q_{n-1}\sqrt{d})$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k: th convergent in the continued fraction expansion. For $Q(\sqrt{-d})$, $d > 1$ units form group Z_2. For $d = 1$ the group is Z_2^2 and for $Q(w)$ where $w = -1/2 + \sqrt{3}/2$ is the third root of unity ($w^3 = 1$), this group is $Z_2 \times Z^3$ (note that in this case the minimal polynomial is $(x^3 - 1)/(x - 1)$.

$Z(w)$ and $Z(i)$ are exceptional in the sense that the group of the roots of unity is exceptionally large. $Z(i)$ and $Z(w)$ allow a unique factorization of their elements into products of irreducibles. The primes π of $Z(w)$ consist of rational primes p, $p \ mod \ 4 = 3$ and complex Gaussian primes satisfying $N(\pi) = \pi\overline{\pi} = p$, $p \ mod \ 4 = 1$. Squares of the Gaussian primes generate as their product complex numbers giving rise to Pythagorean phases. The primes π of $Z(w)$ consist of rational primes p, $p \ mod \ 3 = 2$ and complex Eisenstein primes satisfing $N(\pi) = \pi\overline{\pi} = p$, $p \ mod \ 3 = 1$.

2.8.5 Cyclotomic Number Fields As Examples Of Four-Dimensional Algebraic Number Fields

By the "theorem of primitive element" all algebraic number fields are obtained by replacing the polynomial algebra $Q[x]$, by $Q[\theta]$, where θ is a root of an irreducible minimal polynomial which is of fourth order. One can readily calculate the extensions associated with a given irreducible polynomial by using quadratures for 4: th order polynomials. These polynomials are of general form $P_4(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and by a substitution $x = y - a_3/4)$ which does not change the nature of algbebraic number field, they can be reduced to a canonical form $P_4(x) = x^4 + a_2x^2 + a_1x + a_0$. Thus a very rough view is that three rationals parametrize the 4-dimensional algebraic number fields.

A second manner to represent extensions is in form $K(\theta_1, \theta, ..)$ such that the units θ_i have no common factors different from one. In this case the dimension of the extension is 2^n, where n is the number of units. Examples of four-dimensional extensions are the algebraic extensions $Q(\sqrt{\pm d_1}, \sqrt{\pm d_2})$ of rationals, where d_i are square-free integers, reduce to form $Q(\theta)$. The cyclic extension of rationals by the powers of the m: th root of unity with $m = 5, 8, 12$ are four-dimensional extensions called cyclotomic number fields. Also the extensions $Q((\pm)d)^{1/4})$ are simple four-dimensional extensions. These extensions allow completion to a corresponding p-adic algebraic extension for some p-adic primes.

Quite generally, cyclotomic number fields $Q(\zeta_m)$ are obtained from polynomial algebra $Q[x]$ by replacing x with the m: th primitive root of unity denoted by ζ_m and thus satisfying $\zeta_m^m = 1$. There are three cyclic extensions of dimension 4 and they correspond to $Q(\zeta_5) = Q(\zeta_{10})$, $Q(\zeta_8)$ and $Q(\zeta_{12})$. Cyclotomic extensions are higly symmetric since the roots of unity act as symmetries of the norm.

The units of cyclotomic field $Q(\zeta_m)$ form group $Z_2 \times Z_m \times Z$. Z corresponds to the powers of units for $Q(\zeta_m + 1/\zeta_m)$. These powers have unit norm only with respect to the norm of $Q(\zeta_m)$ whereas with respect to the ordinary complex norm they correspond to fractal scalings. What looks fractal obtained by repeated scalings of the same structure with respect to the real norm looks like a lattice when algebraic norm is used.

1. $Q(\zeta_8)$

The cyclotomic number field $Q(\zeta_8)$, $\zeta_8 = exp(i\pi/4)$ satisfying $\zeta_8^8 = 1$, consists of numbers of form $k = m + in + \sqrt{i}(r + is)$. All roots ($\pm i^{1/2}$ and $\pm i^{3/2}$) are complex. The group of units is $Z_2^4 \times Z$. Z corresponds in real topology to the fractal scalings generated by $L = 1 + \sqrt{2}$. The integer multiples of $log(L)$ could be interpreted as a quantized momentum. $Q(\zeta_8)$ can be generated by $\pm\zeta_8$ and $\pm i\zeta_8$. This means additional Z_2^2 Galois symmetry which does not define multiplicative quantum number.

2. $Q(\zeta_{12})$

The extension $Q(\sqrt{-1}, w)$, $w = \zeta_3$, can be regarded as a cyclic extension $Q(iw) = Q(\zeta_{12})$ as is clear from the fact that the six lowest powers of iw come as $iw, -w^2, -i, w = -1 - w^2, iw^2 = -iw - i, -1$. $Z(iw)$ is especially interesting because it contains $Q(i)$ and $Q(w)$ for which primes correspond to Gaussian and Eisenstein primes. A unique factorization to a product of irreducibles is possible only for $Q(\zeta_m)$ $m \leq 10$: thus the algebraic integers in $Z(iw)$ do not always allow a unique decomposition into irreducibles. The most obvious candidates for primes not allowing unique factorization are primes satisfying simultaneously the conditions $p \ mod\ 4 = 3 = 1$ implying decomposition into a product of Gaussian prime and its conjugate and $p\ mod\ 3 = 1$ guaranteeing the decomposition into a product of Eisenstein prime and its conjugate.

The group of units reduces to $Z_2^2 \times Z_3 \times Z$ might have something to do with the group of discrete quantum numbers C, P and $SU(3)$ triality telling the number of quarks modulo 3 in the state. For the extensions $Q(\sqrt{-1}, \sqrt{d})$ the roots of unity form the group Z_2^2: these extensions could correspond to gauge bosons and the quantum numbers would correspond to C and P. For real extensions the group of the roots of unity reduces to Z_2: in this case the interpretation inters of parity suggests itself.

The lattice defined by Z corresponds to the scalings by powers of $\sqrt{3} + 2$. It could be also interpreted also as the lattice of longitudinal momenta for hadronic quarks which move collinearly inside space-time sheet which might be identified as a massless extremal (ME) for which longitudinal direction is a preferred spatial direction.

$Q(\zeta_{12})$ can be generated by $\pm iw, \pm iw^2$ and the replacement of iw with these alternatives generates Z_2^2 symmetry not realizable as a multiplication with units.

3. $Q(\zeta_5)$ and biology

$Q(\zeta_5)$ indeed gives 4-dimensional extension of rationals since one has $1 + \zeta_5 + ...\zeta_5^4 = 0$ implying that $\zeta_5^4 = 1/\zeta_5$ is expressible as rational combination of other units. Both $Q(\zeta_5)$ and $Q(\zeta_8)$ allows a unique decomposition of rational integers into prime factors. The primes in $Q(\zeta_5)$ allow decomposition to a product of $r = 1, 2$ or 4 primes of $Q(\zeta_5)$ [A53]. The value of r for a given p is fixed by the requirement that $f = 4/r$ is the smallest natural number for which $p^f - 1\ mod\ p = 0$ holds true. For instance, $p = 2, 3$ correspond to $f = 4$ and are primes of $Q(\zeta_5)$, $p = 11$ has decomposition into a product of four primes of $Q(\zeta_5)$, and $p = 19$ has decomposition into two primes of $Q(\zeta_5)$).

What makes this extension interesting is that the phase angle associated with ζ_5 corresponds to the angle of 72 degrees closely related with Golden Mean $\tau = (1+\sqrt{5})/2$ satisfying the equation $\tau^2 - \tau - 1 = 0$. The phase of the fifth root is given by $\zeta_5 = (\tau - 1 + i\sqrt{2+\tau})2$. The group of units is $Z_2 \times Z_5 \times Z$. Z corresponds to the fractal scalings by $\tau = (1+\sqrt{5})/2$. The conjugations $\zeta_5 \to \zeta_5^k$, $k = 1,2,3,4$ leave the norm invariant and generate group Z_2^5.

Fractal scalings by Golden Mean and the closely related Fibonacci numbers are closely related with the fractal structures associated with living systems (botany is full of logarithmic spirals involving Golden Mean and the phase angle 36 is involved even with DNA). Of course, the very fact that Golden Mean emerges in biological length scales provides strongest evidence for its dynamical origin in algebraic framework.

$Q(\zeta_5)$ cannot be realized as an algebraic extension $K(\theta, i)$ naturally associated with the transversal part of quaternionic primes but can appear only as a subfield of the 8-dimensional extension $K(i, cos(2\pi/5), sin(2\pi/5)$ containing also 20: th root of unity as $\zeta_{20} = i\zeta_5$. In [K4] it is indeed found that Golden Mean plays a fundamental role in topological quantum computation and is indeed a fundamental constant in TGD Universe.

Fractal scalings

By Dirichlet's unit theorem the group of units quite generally reduces to $Z_m \times Z^r$, where Z_m is cyclic group of roots of unity and Z^r can be regarded as an r-dimensional lattice with latticed units determined by the extension. For real extensions Z_m reduces to Z_2 since the only real roots of unity are $\{\pm 1\}$. All components of four-momentum represented by a quaternionic prime can be multiplied by separate real units of $Q(\theta)$. For a given quaternionic prime, one can always factor out the common factor of the units of $Q(\theta)$ or $Q(\theta, i)$.

The units generate nontrivial transformations at the level of single quaternionic prime. If the dimension of the real extension is n, the transformations form an $n-1$-dimensional lattice of scalings. Alternative but less plausible interpretation is that the logarithms of the scalings represent $n-1$-dimensional momentum lattice. Particle would be like a part of an algebraic hologram carrying information about external world in accordance with the ideas about fractality. Of course, units represent fractal scalings only with respect to ordinary real norm, with respect to number theoretical norm they act like phase factors.

For instance, in the case of $Q(\sqrt{5})$ the units correspond to scalings by powers of Golden Mean $\tau = (1+\sqrt{5})/2$ having number theoretic norm equal to one. Bio-systems are indeed full of fractals with scaling symmetry. For $K = Q(\sqrt{3})$ the scalings correspond to powers of $L = 2+\sqrt{3}$. An interesting possibility is that hadron physics might reveal fractality in powers of L. More generally, for $Q(\sqrt{d})$, d square-free integer, the basic fractal scaling is $L = p_{n-1} + q_{n-1}\sqrt{d}$, where n defines the period of the continued fraction expansion of $\sqrt{d}$ and p_k/q_k defines k: th convergent in the continued fraction expansion.

Four-dimensional algebraic extensions are very interesting for several reasons. First, algebraic dimension four is a borderline in complexity in the sense that for higher-dimensional irreducible algebraic extensions there is no general quadratures analogous to the formulas associated with second order polynomials giving the roots of the polynomial. Secondly, in transversal degrees of freedom the minimal dimension for $K(\theta, i)$ is four. The units of K which are algebraic integers having a unit norm in K. Quite generally, the group of units is a product $Z_{2k} \times Z_r$ of two groups. $Z_{2k} = Z_2 \times Z_k$ is the cyclic group generated by k: th root of unity. For real extensions one has $k = 1$. In transversal degrees of freedom one can have $k > 1$ since extension is $Q(\theta, i)$. The roots of unity possible in four-dimensional case correspond to $k = 2,4,6,8,10,12$. Corresponding cyclic groups are products of Z_2^i, Z_3 and Z_5. Z_2, Z_2 and Z_3 and act as symmetries of the root lattices of Cartan algebras.

Z_3 gives rise to the Cartan algebra of $SU(3)$ and an interesting question is whether color symmetry is generated dynamically or whether it can be regarded as a basic symmetry with the lattice of integer quaternions providing scaled-up version for the root lattice of color group. Note that in TGD quark color is not spin like quantum number but corresponds to CP_2 partial waves for quark like spinors.

Permutations of the real roots of the minimal polynomial of θ

The replacements of the primitive element θ of $K(\theta)$ with a new one obtained by acting in it with the elements of Galois group of the minimal polynomial of θ generate different internal states of number theoretic fermions and bosons. The subgroup G_1 of Galois group permuting the real roots of the minimal polynomial with each other acts also as a symmetry. The number of equivalent primitive elements is $n_1 = n - 2r_1$, where r_2 is the number of complex root pairs. For instance, for 2-dimensional extensions these symmetries permute the real roots of a second order polynomial irreducible in the set of rationals. Since the entire polynomial has rational coefficients, kind of G_1-confinement is realized. One could say that kind of algebraically confined n-color is in question.

2.8.6 Quaternionic Primes

Primeness makes sense for quaternions and octonions. The following considerations are however restricted to quaternionic primes but can be easily generalized to the octonionic case. Quaternionic primes have Euclidian norm squared equal to a rational prime. The number $N(p)$ of primes associated with a given rational p depends on p and each p allows at least two primes. Quaternionic primes correspond to points of 3-sphere with prime-valued radius squared. Prime-valued radius squared is consistent with p-adic length scale hypothesis, and one can indeed reduce p-adic length scale hypothesis to the assumption that the Euclidian region associated with CP_2 type extremal has prime-valued radius squared.

It is interesting to count the number of quaternionic primes with same prime valued length squared.

1. In the case of algebraic extensions the first definition of quaternionic norm is by using number theoretic norm either for entire quaternion squared or for each component of quaternion separately. The construction of infinite primes suggests that the first definition is more appropriate. Both definitions of norm are natural for four-momentum squared since they give integer valued mass squared spectrum associated with super-conformally invariant systems. One could also decompose quaternion to two parts as $q = (q_0 + Iq_1) + J(q_2 + Iq_3)$ and define number theoretic norm with respect to the algebraic extension $Q(\theta, I)$.

2. Quaternionic primes with the same norm are related by $SO(4)$ rotation plus a change of sign of the real component of quaternion. The components of integer quaternion are analogous to components of four-momentum.

3. There are 2^4 quaternionic $\pm E_i$ and multiplication by these units defines symmetries. Non-commutativity of the quaternionic multiplication makes the interpretation of units as parity like quantum numbers somewhat problematic since the net parity associated with a product of primes representing physical particles associated with the infinite primes depends on the order of quaternionic primes. For real algebraic extensions $K = Q(\theta)$ there is also the units defining a "momentum" lattice with dimension $n - 1$, where n is the degree of the minimal polynomial $P(\theta)$.

4. Quaternionic primes cannot be real so that a given quaternionic prime with $k \geq 2$ components has 2^k conjugates obtained by changing the signs of the components of quaternion. Basic conjugation changes the signs of imagy components of quaternion. This corresponds to group $Z_2^k \subset Z_2^4$, $2 \leq k \leq 4$.

5. The group S_4 of $4! = 24$ permutations of four objects preserves the norm of a prime quaternion: these permutations are representable as a multiplication with non-prime quaternion and thus identifiable as subgroup of $SO(4)$ and also as a subgroup of $SO(3)$ (invariance group of tetrahedron). In degenerate cases (say when some components of q are identical), some subgroup of S_4 leaves quaternionic prime invariant and the rotational degeneracy reduces from $D = 24$ to some smaller number which is some factor of 24 and equals to $4, 6$ or 12 as is easy to see. There are 16 quaternionic conjugations corresponding to change of sign of any quaternion unit but all these conjugations are obtained from single quaternionic conjugation changing the sign of the imaginary part of quaternion by combining them with a multiplication with unit and its inverse. Thus the restricted group of symmetries is $S_4 \times Z_2$.

6. It is possible to find for every prime p at least two quaternionic (primes with norm squared equal to p. For a given prime p there are in general several quaternionic primes not obtainable from each other by transformations of S_4. There must exist some discrete subgroup of $SO(4)$ relating these quaternionic primes to each other.

7. The maximal number of quaternionic primes generated by $S_4 \times Z_2$ is 24×2. In non-commutative situation it is not clear whether units can be regarded as parity type quantum numbers. In any case, one can divide the entire group with Z_2^4 to obtain Z_3. This group corresponds to cyclic permutations of imaginary quaternion units.

$D = 24$ is the number of physical dimensions in bosonic string model. In TGD framework a possible interpretation is based on the observation that infinite primes constructed from rational primes the product of all primes contains the first power of each prime having interpretation as a representation for a single filled state of the fermionic sea. In the case of quaternions the Fock vacuum defined as a product of all quaternionic primes gives rise to a vacuum state

$$X = \prod_p p^{N(p)/2} \quad ,$$

since each prime and its quaternionic conjugate contribute one power of p.

2.8.7 Embedding Space Metric And Vielbein Must Involve Only Rational Functions

Algebraization requires that embedding space exists in the algebraic sense containing only points for which preferred coordinate variables have values in some algebraic extension of rationals. Embedding space metric at the algebraic level can be defined as a quadratic form without any reference to metric concepts like line element or distance. The metric tensors of both M^4_+ and CP_2 are indeed represented by algebraic functions in the preferred coordinates dictated by the symmetries of these spaces.

One should also construct spinor structure and this requires the introduction of an algebraic extension containing square roots since vielbein vectors appearing in the definition of the gamma matrices involve square roots of the components of the metric. In CP_2 degrees of freedom this forces the introduction of square root function, and thus all square roots, unless one restricts the values of the radial CP_2 coordinate appearing in the vielbein in such a way that rationals result. What is interesting is that all components of spinor curvature and Kähler form of CP_2 are quadratic with respect to vierbein and algebraic functions of CP_2 complex coordinates. Also the square root of the determinant of the induce metric appears only as a multiplicative factor in the Euler-Lagrange equations so that one can get rid of the square roots.

Induced spinor structure and Dirac equation relies on the notion of the induced gamma matrices and here the projections of the vierbein of CP_2 containing square roots are unavoidable. In complex coordinates the components of CP_2 vielbein in complex coordinates ξ_1, ξ_2, in which the action of $U(2)$ is linear holomorphic transformation, involve the square roots $r = \sqrt{|\xi|^2 + |\xi_2|^2}$ and $\sqrt{1+r^2}$ (for detailed formulas see Appendix at the end of the book). If one has $r = m/n$, the requirement that $\sqrt{1+r^2}$ is rational, implies $m^2 + n^2 = k^2$ so that (m,n) defines Pythagorean square. Thus induced Dirac equation is rationalized if the allowed values of r correspond to Pythagorean phases. The notion of the phase preserving canonical identification [K39], crucial for the earlier formulation of TGD, is consistent with this assumption. The metric of $S^2 = CP_1$ is a simplified example of what happens. One can write the metric as $g_{z\bar{z}=r^2} = \frac{1}{1+r^2}$ and vielbein component is proportional to $1/\sqrt{1+r^2}$, this exists for $r = m/n$ as rational number if one has $m^2 + n^2 = k^2$, which indeed defines Pythagorean triangle.

The restriction of the phases associated with the CP_2 coordinates to Pythagorean ones has deeper coordinate-invariant meaning. Rational CP_2 can be defined as a coset space $SU_Q(3)/U_Q(2)$ of rational groups $SU_Q(3)$ and $U_Q(2)$: rationality is required in the linear matrix representation of these groups.

Chapter 3

TGD as a Generalized Number Theory II: Quaternions, Octonions, and their Hyper Counterparts

3.1 Introduction

This chapter is second one in a multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields in quantum TGD. A central notion is $M^8 - H$ duality which might be also called number theoretic compactification. This duality allows to identify embedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. These number theoretical symmetries induce also the symmetries dictating the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D embedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology has become the corner stone of both quantum TGD and number theoretical vision. In zero energy ontology either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW.

The basic number theoretical structures are complex numbers, quaternions and octonions, and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this way could select 4-D surfaces as surface having as tangent spaces associative (co-associative) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by space-time tangent vectors spanning associative (co-associative) sub-algebra of complexified octonions generated by embedding space tangent vectors. A more concrete representation of vectors of complexified tangent

space as embedding space gamma matrices is not necessary. One can consider also octonionic representation of embedding space gamma matrices but whether it has any physical content, remains an open question. The answer to the question whether octonions could correspond to the modified (Kähler Dirac) gamma matrices associated with Kähler-Dirac action turned out to be "No".

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative and thus hyper-complex sub-algebra is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations [K14]. The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. Even this is not yet the whole story: one can define also the notions of co-associativity and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces.

3.1.1 Hyper-Octonions And Hyper-Quaternions

The discussions for years ago with Tony Smith [A70] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D embedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

At the end of the chapter it will be found that it might be possible to do without the hyper-variants of classical number fields (not of course number fields!). The idea is obvious already from string model context.

1. For strings in Minkowskian target space the target space coordinates as function of string world sheet coordinates are analytic with respect to hyper-complex coordinate. Quantum theory is however constructed by performing first a Wick rotation to Euclidian target space, calculating the n-point functions using ordinary Euclidian theory, and performing the reverse of Wick rotation.

2. One could generalize the procedure in TGD framework so that octonionic variant of conformal field theory results by algebraic continuation from complex number field to octonionic realm. Octonionic real-analytic functions $f(o)$ are expressible as $f(o) = q_1 + Iq_2$, where q_i are quaternion valued functions and I is octonionic imaginary unit anti-commuting with quaternionic imaginary units. They map the Euclidian variant of $H = M^4 \times CP_2$ to itself. Space-time surfaces can be identified as quaternionic (co-quaternionic) 4-surfaces defined as surfaces for which the imaginary (real) part of an octonion real-analytic function vanishes.The

reversal of Wick rotation maps these Euclidian surfaces to space-time surfaces. One could also see the this process as a complexification in of octonions in which real-analytic functions of complexified octonions are restricted to octonionic and hyper-octonionic sectors. Therefore the two views should be more or less equivalent.

Note that hyper-variants of number fields make also sense p-adically unlike the notions of number fields themselves unless restricted to be algebraic extensions of rational variants of number fields. What deserves separate emphasis is that the basic structure of the standard model would reduce to number theory.

3.1.2 Number Theoretical Compactification And $M^8 - H$ Duality

The notions of associative and hyper-octonionic manifold make sense and one could endow the tangent space of $H = M^4 \times CP_2$ with hyper-octonionic manifold structure. Situation becomes very simple if H is replaced with hyper-octonionic M^8. Suppose that $X^4 \subset M^8$ consists of associative and co-associative regions. The basic observation is that the associative sub-spaces of M^8 with a fixed hyper-complex structure (containing in their tangent space a fixed hyper-complex subspace M^2 or at least one of the light-like lines of M^2) are labeled by points of CP_2. Hence each associative and co-associative four-surface of M^8 defines a 4-surface of $M^4 \times CP_2$. One can loosely say that the number-theoretic analog of spontaneous compactification occurs: this of course has nothing to do with dynamics.

This picture was still too naive and it became clear that not all known extremals of Kähler action contain fixed $M^2 \subset M^4$ or light-like line of M^2 in their tangent space.

1. The first option represents the minimal form of number theoretical compactification. M^8 is interpreted as the tangent space of H. Only the 4-D tangent spaces of light-like 3-surfaces X^3_l (wormhole throats or boundaries) are assumed to be associative or co-associative and contain fixed M^2 or its light-like line in their tangent space. Hyper-quaternionic regions would naturally correspond to space-time regions with Minkowskian signature of the induced metric and their co-counterparts to the regions for which the signature is Euclidian. What is of special importance is that this assumption solves the problem of identifying the boundary conditions fixing the preferred extremals of Kähler action since in the generic case the intersection of M^2 with the 3-D tangent space of X^3_l is 1-dimensional. The surfaces $X^4(X^3_l) \subset M^8$ would be associative or co-associative but would not allow a local mapping between the 4-surfaces of M^8 and H.

2. One can also consider a more local map of $X^4(X^3_l) \subset H$ to $X^4(X^3_l) \subset M^8$. The idea is to allow $M^2 \subset M^4 \subset M^8$ to vary from point to point so that $S^2 = SO(3)/SO(2)$ characterizes the local choice of M^2 in the interior of X^4. This leads to a quite nice view about strong geometric form of $M^8 - H$ duality in which M^8 is interpreted as tangent space of H and $X^4(X^3_l) \subset M^8$ has interpretation as tangent for a curve defined by light-like 3-surfaces at X^3_l and represented by $X^4(X^3_l) \subset H$. Space-time surfaces $X^4(X^3_l) \subset M^8$ consisting of associative and co-associative regions would naturally represent a preferred extremal of E^4 Kähler action. The value of the action would be same as CP_2 Kähler action. $M^8 - H$ duality would apply also at the induced spinor field and at the level of configuration space.

3. Strong form of $M^8 - H$ duality satisfies all the needed constraints if it represents Kähler isometry between $X^4(X^3_l) \subset M^8$ and $X^4(X^3_l) \subset H$. This implies that light-like 3-surface is mapped to light-like 3-surface and induced metrics and Kähler forms are identical so that also Kähler action and field equations are identical. The only differences appear at the level of induced spinor fields at the light-like boundaries since due to the fact that gauge potentials are not identical.

4. The map of $X^3_l \subset H \rightarrow X^3_l \subset M^8$ would be crucial for the realization of the number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as those preferred coordinates in which the points of embedding space are rational/algebraic. Thus the point of $X^4 \subset H$ is algebraic if it is mapped to algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not

have practical meaning. Number theoretical compactication could thus be motivated by the number theoretical universality.

5. The possibility to use either M^8 or H picture might be extremely useful for calculational purposes. In particular, M^8 picture based on $SO(4)$ gluons rather than $SU(3)$ gluons could perturbative description of low energy hadron physics. The strong $SO(4)$ symmetry of low energy hadron physics can be indeed seen direct experimental support for the M^8-H duality.

One can imagine an interesting generalization of the $M^8 - H$ duality to $H - H$ duality. One can assign to an associative (co-associative) 4-surface of H a surface of H by the same rule as in the case of $M^8 - H$ duality. If the outcome is also associative (co-associative) surface one can iterate this map and get infinite number of associative (co-associative) surfaces serving as candidates for preferred extremals and obviously forming a category.

3.1.3 Romantic Stuff

Octonions and quaternions have generated a lot of romantic speculations and my only defence is that I did not know! Combined with free speculation about dualities this generated a lot of non-sense which has been dropped from this version of the chapter.

1. A long standing romantic speculation was that conformal invariance could somehow extend to 4-D context. Conformal invariance indeed extends to 3-D situation in the case of light-like 3-surfaces and they indeed are the basic dynamical objects of quantum TGD. It seems however un-necessary to extend the conformal invariance to 4-D context except by slicing $X^4(X^3_l)$ by 3-D light-like slices possessing the 3-D conformal invariance.

2. The triality between 8-D spinors, their conjugates, and vectors has generated a lot of speculative literature and this triality is indeed important in super string models. If M^8 has hyper-octonionic structure, one can ask whether also the spinors of M^8 could be regarded as complexified octonions. Complexified octonions provide also a representation of 8-D gamma matrices which is not a matrix representation. In this framework the Clifford algebra defined by gamma matrices degenerates to algebra of complexified octonions identifiable as the algebra of octonionic spinors and coordinates of M^8_c. One can make all kinds of questions. For instance, could it be that hyper-octonionic triality for hyper-octonionic spinor fields could allow construction of N-point functions in interaction vertices? One cannot exclude the possibility that trialities are important but the recent formulation of M-matrix elements does quite well without them.

3. The $1 + \overline{1} + 3 + \overline{3}$ decomposition of complexified octonion units with respect to group $SU(3) \subset G_2$ acting as automorphisms of octonions inspired the idea that hyper-octonion spinor field could represent leptons, antileptons, quarks and antiquarks. This proposal is problematic. Hyper-octonionic coordinates would carry color and generic hyper-octonionic spinor is superposition of spinor components which correspond to quarks, leptons and and their anti-fermions and a lot of super-selection rules would be needed. The motivations behind these speculations was that in H picture color would correspond to CP_2 partial waves and spin and ew quantum numbers to spin like quantum numbers whereas in M^8 picture color would correspond to spin like quantum number and spin and electro-weak quantum numbers to E^4 partial waves.

3.1.4 About Literature

The reader not familiar with the basic algebra of quaternions and octonions is encouraged to study some background material: the home page of Tony Smith provides among other things an excellent introduction to quaternions and octonions [A70]. String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [A29, A66, A61] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [A68]. which is not so elementary as the name would suggest, introduces in enjoyable way the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [A38] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

3.1.5 Notations

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the associative and -octonionic sub-spaces can be consideredthese algebras have a representation in the space of spinors of embedding space $H = M^4 \times CP_2$.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

3.2 Quaternion And Octonion Structures And Their Hyper Counterparts

In this introductory section the notions of quaternion and octonion structures and their hyper counterparts are introduced with strong emphasis on the physical interpretation. Literature contains several variants of these structures (Hyper Kähler structure and quaternion Kähler structure possed also by CP_2 [A46]). The notion introduced here is inspired by the physical motivations coming from TGD. As usual the first proposal based on the notions of (hyper-)quaternion and (hyper-)octonion analyticity was not the correct one. Much later a local variant of the notion based on tangent space emerged.

3.2.1 Octonions And Quaternions

In the following only the basic definitions relating to octonions and quaterions are given. There is an excellent article by John Baez [A50] describing octonions and their relations to the rest of mathematics and physics. For the octonionic multiplication table see Fig. ??.

Octonions can be expressed as real linear combinations $\sum_k x^k I_k$ of the octonionic real unit $I_0 = 1$ (counterpart of the unit matrix) and imaginary units I_a, $a = 1, ..., 7$ satisfying

$$\begin{aligned} I_0^2 &= I_0 \equiv 1 \ , \\ I_a^2 &= -I_0 = -1 \ , \\ I_0 I_a &= I_a \ . \end{aligned} \tag{3.2.1}$$

Octonions are closed with respect to the ordinary sum of the 8-dimensional vector space and with respect to the octonionic multiplication, which is neither commutative ($ab \neq ba$ in general) nor associative ($a(bc) \neq (ab)c$ in general).

A concise manner to summarize octonionic multiplication is by using octonionic triangle. Each line (6 altogether) containing 3 octonionic imaginary units forms an associative triple which

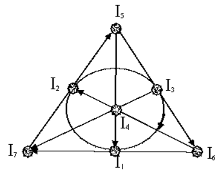

Figure 3.1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

together with $I_0 = 1$ generate a division algebra of quaternions. Also the circle spanned by the 3 imaginary units at the middle of the sides of the triangle is associative triple. The multiplication rules for each associative triple are simple:

$$I_a I_b = \epsilon_{abc} I_c \ , \tag{3.2.2}$$

where ϵ_{abc} is 3-dimensional permutation symbol. $\epsilon_{abc} = 1$ for the clockwise sequence of vertices (the direction of the arrow along the circumference of the triangle and circle). As a special case this rule gives the multiplication table of quaternions. A crucial observation for what follows is that any pair of imaginary units belongs to one associative triple.

The non-vanishing structure constants ${d_{ab}}^c$ of the octonionic algebra can be read directly from the octonionic triangle. For a given pair I_a, I_b one has

$$\begin{aligned} I_a I_b &= {d_{ab}}^c I_c \ , \\ d_{ab} \quad c &= {\epsilon_{ab}}^c \ , \\ I_a^2 &= {d_{aa}}^0 I_0 = -I_0 \ , \\ I_0^2 &= {d_{00}}^0 I_0 \ , \\ I_0 I_a &= {d_{0a}}^a I_a = I_a \ . \end{aligned} \tag{3.2.3}$$

For ϵ_{abc} c belongs to the same associative triple as ab.

Non-associativity means that is not possible to represent octonions as matrices since matrix product is associative. Quaternions can be represented and the structure constants provide the defining representation as $I_a \rightarrow d_{abc}$, where b and c are regarded as matrix indices of 4×4 matrix. The algebra automorphisms of octonions form 14-dimensional group G_2, one of the so called exceptional Lie-groups. The isotropy group of imaginary octonion unit is the group $SU(3)$. The Euclidian inner product of the two octonions is defined as the real part of the product $\overline{x}y$

$$(x,y) = Re(\overline{x}y) = \sum_{k=0,..7} x_k y_k \ ,$$
$$\overline{x} = x^0 I_0 - \sum_{i=1,..,7} x^k I_k \ , \qquad (3.2.4)$$

and is just the Euclidian norm of the 8-dimensional space.

3.2.2 Hyper-Octonions And Hyper-Quaternions

The Euclidicity of the quaternion norm suggests that octonions are not a sensible concept in TGD context. One can imagine two ways to circumvent this conclusion.

1. Minkowskian metric for octonions and quaternions is obtained by identifying Minkowski inner product xy as the real counterpart of the product

$$x \cdot y \equiv Re(xy) = x^0 y^0 - \sum_k x^k y^k \ . \qquad (3.2.5)$$

   $SO(1,7)$ ($SO(1,3)$ in quaternionic case) Lorentz invariance appears completely naturally as the symmetry of the real part of the octonion (quaternion) product and hence of octonions/quaternions and there is no need to perform the complexification of the octonion algebra. Furthermore, only the signature $(1,7)$ ($(1,3)$ in the quaternionic case) is possible and this would raise $M^4_+ \times CP_2$ in a preferred position.

   This norm does not give rise to a number theoretic norm defining a homomorphism to real numbers. Indeed, the number theoretic norm defined by the determinant of the linear mapping defined by the multiplication with quaternion or octonion, is inherently Euclidian. This is in conflict with the idea that quaternionic and octonionic primes and their infinite variants should have key role in TGD [K94].

2. Hyper-octonions and hyper-quaternions provide a possible solution to these problems. These are obtained by multiplying imaginary units by commutative and associative $\sqrt{-1}$. These numbers form a sub-space of complexified octonions/quaternions and the cross product of imaginary parts leads out from this sub-space. In this case number theoretic norm induced from Q_C/O_C gives the fourth/eighth power of Minkowski length and Lorentz group acts as its symmetries. Light-like hyper-quaternions and -octonions causing the failure of the number field property have also a clear physical interpretation.

   A criticism against the notion of hyper-quaternionic and octonionic primeness is that the tangent space as an algebra property is lost and the notion of primeness is inherited from Q_C/O_C. Also non-commutativity and non-associativity could cause difficulties.

Zero energy ontology leads to a possible physical interpretation of complexified octonions. The moduli space for causal diamonds corresponds to a Cartesian product of $M^4 \times CP_2$ whose points label the position of either tip of $CD \times CP_2$ and space I whose points label the relative positive of the second tip with respect to the first one. p-Adic length scale hypothesis results if one assumes that the proper time distance between the tips comes in powers of two so that one has union of hyperboloids $H_n \times CP_2$, $H_n = \{m \in M^4_+ | a = 2^n a_0)\}$. A further quantization of hyperboloids H_n is obtained by replacing it with a lattice like structure is highly suggestive and would correspond to an orbit of a point of H_n under a subgroup of $SL(2,Q_C)$ or $SL(2,Z_C)$ acting as Lorentz transformations in standard manner. Also algebraic extensions of Q_C and Z_C can be considered. Also in the case of CP_2 discretization is highly suggestive so that one would have an orbit of a point of CP_2 under a discrete subgroup of $SU(3,Q)$.

The outcome could be interpreted by saying that the moduli space in question is $H \times I$ such that H corresponds to hyper-octonions and I to a discretized version of $\sqrt{-1}H$ and thus a subspace of complexified octonions. An open question whether the quantization has some deeper mathematical meaning.

3.2.3 Basic Constraints

Before going to details it is useful to make clear the constraints on the concept of the hyper-octonionic structure implied by TGD view about physics.

$M^4 \times CP_2$ cannot certainly be regarded as having any global octonionic structure (for instance in the sense that it could be regarded as a coset space associated with some exceptional group). There are however clear indications for the importance of the hyper-quaternionic and -octonionic structures.

1. $SU(3)$ is the only simple 8-dimensional Lie-group and acts as the group of isometries of CP_2: if $SU(3)$ had some kind of octonionic structure, CP_2 would become unique candidate for the space S. The decomposition $SU(3) = h + t$ to $U(2)$ subalgebra and its complement corresponds rather closely to the decomposition of (hyper-)octonions to (hyper-)quaternionic sub-space and its complement. The electro-weak $U(2)$ algebra has a natural 1+3 decomposition and generators allow natural hyper-quaternionic structure. Hyper Kähler structure with three covariantly constant quaternionic imaginary units represented Kähler forms is however not possible. The components of the Weyl tensor of CP_2 behave with respect to multiplication like quaternionic imaginary units but only one of them is covariantly constant so that hyper-Kähler structure is not possible. These tensors and metric tensor however define quaternionic structure.

2. M^4_+ has a natural 1+3 decomposition and a unique cosmic time coordinate defined as the light cone proper time. Hyper-quaternionic structure is consistent with the Minkowskian signature of the inner product and hyper quaternion units have a natural representation in terms of covariantly constant self-dual symplectic forms and their contractions with sigma matrices. It is not however clear whether this representation is physically intereting.

3.2.4 How To Define Hyper-Quaternionic And Hyper-Octonionic Structures?

I have considered several proposals for how to define quaternionic and octonionic structures and their hyper-counterparts.

1. (Hyper-)octonionic manifolds would obtained by gluing together coordinate patches using (hyper-)octonion analytic functions with real Laurent coefficients (this guarantees associativity and commutativity). This definition does not yet involve metric or any other structures (such as Kähler structure). This approach does not seem to be physically realistic.

2. Second option is based on the idea of representing quaternionic and octonionic imaginary units as antisymmetric tensors. This option makes sense for quaternionic manifolds and CP_2 indeed represents an example represents of this kind of manifold. The problem with the octonionic structure is that antisymmetric tensors cannot define non-associative product.

3. If the manifold is endowed with metric, octonionic structure should be defined as a local tangent space structure analogous to eight-bein structure and local gauge algebra structures. This can be achieved by contracting octo-bein vectors with the standard octonionic basis to get octonion form I_k. Each vector field a^k defines naturally octonion field $A = a^k I_k$. The product of two vector fields can be defined by the octonionic multiplication and this leads to the introduction of a tensor field d_{klm} of these structure constants obtained as the contraction of the octobein vectors with the octonionic structure constants d_{abc}. Hyper-octonion structure can defined in a completely analogous manner.

   It is possible to induce octonionic structure to any 4-dimensional space-time surface by forming the projection of I_k to the space-time surface and redefining the products of I_k: s by dropping away that part of the product, which is orthogonal to the space-time surface. This means that the structure constants of the new 4-dimensional algebra are the projections of d_{klm} to the space-time surface. One can also define similar induced algebra in the 4-dimensional normal space of the space-time surface. The hypothesis would be that the induced tangential is associative or hyper-quaternionic algebra. Also co-associativity defined as associativity of the normal space algebra is possible. This property would give for the 4-dimensionality of the space-time surface quite special algebraic meaning. The problem is now that there is no direct connection with quantum TGD proper- in particular the connection with the classical dynamics defined by Kähler action is lacking.

4. 8-dimensional gamma matrices allow a representation in terms of tensor products of octonions and 2×2 matrices. Genuine matrices are of course not in question since the product of the gamma matrices fails to be associative. An associative representation is obtained by restricting the matrices to a quaternionic plane of complex octonions. If the space-time surface is hyper-quaternionic in the sense that induced gamma matrices define a quaternionic plane of complexified octonions at each point of space-time surface the resulting local Clifford algebra is associative and structure constants define a matrix representation for the induced gamma matrices.

   A more general definition allows gamma matrices to be Kähler-Dirac gamma matrices defined by Kähler action appearing in the Kähler-Dirac action and forced both by internal consistency and super-conformal symmetry [K111]. The Kähler-Dirac gamma matrices associated with Kähler action do not in general define tangent space of the space-time surface as the induced gamma matrices do. Also co-associativity can be considered if one can identify a preferred imaginary unit such that the multiplication of the Kähler-Dirac gamma matrices with this unit gives a quaternionic basis. This condition makes sense only if the preferred extremals of the action are hyper-quaternionic surfaces in the sense defined by the action. That this is true for Kähler action at least is an is an unproven conjecture.

In the sequel only the fourh option will be considered.

3.2.5 How To End Up To Quantum TGD From Number Theory?

An interesting possibility is that quantum TGD could emerge from a condition that a local version of hyper-finite factor of type II_1 represented as a local version of infinite-dimensional Clifford algebra exists. The conditions are that "center or mass" degrees of freedom characterizing the position of CD separate uniquely from the "vibrational" degrees of freedom being represented in terms of octonions and that for physical states associativity holds true. The resulting local Clifford algebra would be identifiable as the local Clifford algebra of WCW (being an analog of local gauge groups and conformal fields).

The uniqueness of M^8 and $M^4 \times CP_2$ as well as the role of hyper-quaternionic space-time surfaces as fundamental dynamical objects indeed follow from rather weak conditions if one restricts the consideration to gamma matrices and spinors instead of assuming that M^8 coordinates are hyper-octonionic as was done in the first attempts.

1. The unique feature of M^8 and any 8-dimensional space with Minkowski signature of metric is that it is possible to have an octonionic representation of the complexified gamma matrices [K111, K25] and of spinors. This does not require octonionic coordinates for M^8. The restriction to a quaternionic plane for both gamma matrices and spinors guarantees the associativity.

2. One can also consider a local variant of the octonionic Clifford algebra in M^8. This algebra contains associative subalgebras for which one can assign to each point of M^8 a hyper-quaternionic plane. It is natural to assume that this plane is either a tangent plane of 4-D manifold defined naturally by the induced gamma matrices defining a basis of tangent space or more generally, by Kähler-Dirac gamma matrices defined by a variational principle (these gamma matrices do not define tangent space in general). Kähler action defines a unique candidate for the variational principle in question. Associativity condition would automatically select sub-algebras associated with 4-D hyper-quaternionic space-time surfaces.

3. This vision bears a very concrete connection to quantum TGD. In [K25] the octonionic formulation of the Kähler-Dirac equation is studied and shown to lead to a highly unique general solution ansatz for the equation working also for the matrix representation of the Clifford algebra. An open question is whether the resulting solution as such defined also

solutions of the Kähler-Dirac equation for the matrix representation of gammas. Also a possible identification for 8-dimensional counterparts of twistors as octo-twistors follows: associativity implies that these twistors are very closely related to the ordinary twistors. In TGD framework octo-twistors provide an attractive manner to get rid of the difficulties posed by massive particles for the ordinary twistor formalism.

4. Associativity implies hyperquaternionic space-time surfaces (in a more general sense as usual) and this leads naturally to the notion of WCW and local Clifford algebra in this space. Number theoretic arguments imply $M^8 - H$ duality. The resulting infinite-dimensional Clifford algebra would differ from von Neumann algebras in that the Clifford algebra and spinors assignable to the center of mass degrees of freedom of causal diamond CD would be expressed in terms of octonionic units although they are associative at space-time surfaces. One can therefore say that quantum TGD follows by assuming that the tangent space of the embedding space corresponds to a classical number field with maximal dimension.

5. The slicing of the Minkowskian space-time surface inside CD by stringy world sheets and by partonic 2-surfaces inspires the question whether the Kähler-Dirac gamma matrices associated with the stringy world sheets *resp.* partonic 2-surfaces could be could commutative *resp.* co-commutative. Commutativity would also be seen as the justification for why the fundamental objects are effectively 2-dimensional.

This formulation is undeniably the most convincing one found hitherto since the notion of hyper-quaternionic structure is local and has elegant formulation in terms of Kähler-Dirac gamma matrices.

3.2.6 P-Adic Length Scale Hypothesis And Quaternionic And Hyper-Quaternionic Primes

p-Adic length scale hypothesis [K66] states that fundamental length scales correspond to the p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives a partial theoretical justification for this hypothesis [K66]. A strong empirical support for the hypothesis comes from p-adic mass calculations [K51, K51, K64, K57].

Elementary particles correspond to the so called CP_2 type extremals in TGD Universe [K14, K66]. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [A70] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one could identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime, with integer value E^4 coordinates. The worries are of course raised by the Euclidian signature of the number theoretical norm of quaternions.

Hyper-quaternionic and -octonionic primes and effective 2-dimensionality

The notion of prime generalizes to hyper-quaternionic and -octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p - 1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. The interpretation of hyper-quaternionic primes (or integers) as four-momenta suggests itself. Note that it is not possible to find a rest system for a massive particle unless the energy is allowed to be a square root of integer.

The notion of "irreducible" (see Appendix of [K95]) is defined as the equivalence class of primes related by a multiplication with a unit (integer with unit norm) and is more fundamental than that of prime. All Lorentz boosts of a hyper prime obtained by multiplication with units labeling $SO(D-1)$ cosets of $SO(D-1,1)$, $D=4,8$ to a hyper prime, combine to form a hyper irreducible. Note that the units cannot correspond to real particles in the arithmetic quantum field theory in which primes correspond to D-momenta labeling the physical states.

If the situation for $p>2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the hyper-complex case. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality.

Hyper-complex numbers H_2 define the maximal sub-algebra of HQ and HO. In the case of H_2 the failure of the number field property is due to the existence of light-like hyper-complex numbers with vanishing norm. The light-likeness of hyper-quaternions and -octonions is expected to have a deep physical significance and could define a number theoretic analog of propagator pole and light-like 3-D and 7-D causal determinants.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes. Note that the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

The situation becomes more complex if also space-like hyper primes with negative norm squared $n_0^2-n_1^2-...=-p$ are allowed. Gaussian primes with $p \ mod 4 \ =1$ would be representable as primes of form $(0,n_1,n_2,0)$: $n_1^2+n_2^2=p$. If all quaternionic primes allow a representation as a quaternionic integer with three non-vanishing components, they can be identified as space-like hyper-quaternionic primes. Space-like primes with $p \ mod \ 4=3$ have at least 3 non-vanishing components which are odd integers. By their tachyonity space-like primes are not physically favored.

Hyper-quaternionic hyperboloids and p-adic length scale hypothesis

In the hyper-quaternionic case the 3-dimensional sphere $R^2=p$ is replaced with Lobatchevski space (hyperboloid of M^4 with points having integer valued M^4 coordinates. Hence integer valued hyper-quaternions allow interpretation as quantized four-momenta.

Prime mass hyperboloids correspond to $n=p$. It is not possible to multiply hyperboloids since the cross product leads out of hyper sub-space. It is however possible to multiply the 2-dimensional hyperboloids and act on these by units to get full 3-D hyperboloids. The powers of hyperboloid p correspond to a multiplicatively closed structure consisting of powers p^n of the hyperboloid p. At space-time level the hyper-quaternionic lattice gives rise to a one-dimensional lattices of hyperboloidal lattices labeled by powers p^n, and the values of light-cone proper time $a\propto\sqrt{p}$ are expected to define fundamental p-adic time scales.

Also the space-like hyperboloids $R^2=-n$ are possible and the notion of primeness makes sense also in this case. The space-like hyperboloids define one-dimensional lattice of space-like hyper-quaternionic lattices and an explanation for the spatial variant of the p-adic length scale hypothesis stating that p-adic length scales are proportional to $\sqrt{p}$ emerges in this manner naturally.

Euclidian version of the p-adic length scale hypothesis

Hyper-octonionic integers have a decomposition into hyper-quaternion and a product of $\sqrt{-1}K$ with quaternion so that quaternionic primes can be identified as hyper-octonionic space-like primes. The Euclidian version of the p-adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2=k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2

type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k: th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface.

This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

3.3 Quantum TGD In Nutshell

This section provides a very brief summary about quantum TGD. The discussions are based on the general vision that quantum states of the Universe correspond to the modes of *classical* spinor fields in the "world of the classical worlds" identified as the infinite-dimensional WCW of light-like 3-surfaces of $H = M^4 \times CP_2$ (more or less-equivalently, the corresponding 4-surfaces defining generalized Bohr orbits). This implies a radical deviation from path integral formalism, in which one integrates over all space-time surfaces. A second important deviation is due to Zero Energy Ontology. The properties of Kähler action imply a further crucial deviation, which in fact forced the introduction of WCW , and is behind the hierarchy of Planck constants, hierarchy of quantum criticalities, and hierarchy of inclusions of hyper-finite factors.

I include also an excerpt from [K101] representing the most recent view about how scattering amplitudes could be constructed in TGD using the notion of super-symplectic Yangian and generalization of the notion of twistor structure so that it applies at the level of 8-D embedding space.

3.3.1 Basic Physical And Geometric Ideas

TGD relies heavily on geometric ideas, which have gradually generalized during the years. Symmetries play a key role as one might expect on basis of general definition of geometry as a structure characterized by a given symmetry.

Physics as infinite-dimensional Kähler geometry

1. The basic idea is that it is possible to reduce quantum theory to WCW geometry and spinor structure. The geometrization of loop spaces inspires the idea that the mere existence of Riemann connection fixes WCW Kähler geometry uniquely. Accordingly, WCW can be regarded as a union of infinite-dimensional symmetric spaces labeled by zero modes labeling classical non-quantum fluctuating degrees of freedom.

   The huge symmetries of WCW geometry deriving from the light-likeness of 3-surfaces and from the special conformal properties of the boundary of 4-D light-cone would guarantee the maximal isometry group necessary for the symmetric space property. Quantum criticality is the fundamental hypothesis allowing to fix the Kähler function and thus dynamics of TGD uniquely. Quantum criticality leads to surprisingly strong predictions about the evolution of coupling constants.

2. WCW spinors correspond to Fock states and anti-commutation relations for fermionic oscillator operators correspond to anti-commutation relations for the gamma matrices of the WCW . WCW gamma matrices contracted with Killing vector fields give rise to a super-algebra which together with Hamiltonians of WCW forms what I have used to called super-symplectic algebra.

   WCW metric can be expressed in two ways. Either as anti-commutators of WCW gamma matrices identified as super-symplectic Noether super charges (this is highly non-trivial!) or in terms of the second derivatives of Kähler function expressible as Kähler action for the space-time regions with 4-D CP_2 projection and Euclidian signature of the induced metric (wormhole contacts).

   This leads to a generalization of AdS/CFT duality if one assumes that spinor modes are localized at string world sheets to guarantee well-definedness of em charge for the spinor modes following from the assumption that induced classical W fields vanish at string world sheets. Also number theoretic argument requiring that octonionic spinor structure for the embedding space is equivalent with ordinary spinor structure implies the localization. String model in space-time becomes part of TGD.

3. Super-symplectic degrees of freedom represent completely new degrees of freedom and have no electroweak couplings. In the case of hadrons super-symplectic quanta correspond to what has been identified as non-perturbative sector of QCD they define TGD correlate for the degrees of freedom assignable to hadronic strings. They could be responsible for the most of the mass of hadron and resolve spin puzzle of proton.

   It has turned out that super-symplectic quanta would naturally give rise to a hierarchy of dark matters labelled by the value of effective Planck constant $h_{eff} = n \times h$. n would characterize the breaking of super-symplectic symmetry as gauge symmetry and for $n = 1$ (ordinary matter) there would be no breaking.

   Besides super-symplectic symmetries there extended conformal symmetries associated with light-cone boundary and light-like orbits of partonic 2-surfaces and Super-Kac Moody symmetries assignable to light-like 3-surfaces. A further super-conformal symmetry is associated with the spinor modes at string world sheets and it corresponds to the ordinary super-conformal symmetry. The existence of quaternion conformal generalization of these symmetries is suggestive and the notion of quaternion holomorphy [A74] indeed makes sense [K84]. Together these algebras mean a gigantic extension of the conformal symmetries of string models [L12]. Some of these symmetries act as dynamical symmetries instead of mere gauge symmetries. The construction of the representations of these symmetries is one of the main challenges of quantum TGD.

   The original proposal was that the commutator algebras of super-symplectic and super Kac-Moody algebra annihilate physical states. Recently the possibility that a sub-algebra of super-symplectic algebra (at least this algebra) with conformal weights coming as multiples of integer some integer n annihilates physical states at both boundaries of CD. This would correspond to broken gauge symmetry and would predict fractal hierarchies of quantum criticalities defined by sequences of integers $n_{i+1} = \prod_{k<i+1} m_k$. The conformal algebra of string world sheet could always correspond to $n = 1$. Super Virasoro conditions could be regarded as analogs of WCW Dirac equation. These sequences would define hierarchies of inclusions of hyper finite factors of type II_1 and the identification $n = h_{eff}/h$ would relate this hierarchy to the hierarchy of Planck constants. n would also characterize the non-determinism of Kähler action: there would be n conformal gauge equivalence classes connecting members of a pair of 3-surfaces at the boundaries of CD and defining the ends of space-time.

   An intriguing possibility consistent with this picture is that the conformal weights of the super-symplectic algebra charactering the exponent $h$of the power r_M^h of the light-like radial coordinate r_M appearing in the Hamiltonian of the symplectic transformation of $\delta M^4_\pm \times CP_2$ is not an integer but a linear combination of zeros of Riemann Zeta with integer coefficients. For physical states the weights would be real integers (if mass squared corresponds to conformal weight): one would have conformal confinement in the sense that the sum of imaginary parts of conformal weights would be zero. This is an old idea that I already gave up but seems rather attractive in the recent framework.

Modular invariance is one aspect of conformal symmetries and plays a key role in the understanding of elementary particle vacuum functionals and the description of family replication phenomenon in terms of the topology of partonic 2-surfaces.

4. WCW spinors define a von Neumann algebra known as hyper-finite factor of type II_1 (HFFs). This realization has led also to a profound generalization of quantum TGD through a generalization of the notion of embedding space to characterize quantum criticality. The resulting space has a book like structure with various almost-copies of embedding space representing the pages of the book meeting at quantum critical sub-manifolds. The outcome of this approach is that the exponents of Kähler function and Chern-Simons action are not fundamental objects but reduce to the Dirac determinant associated with the Kähler-Dirac operator assigned to the light-like 3-surfaces.

3.3.2 The Notions Of Embedding Space, 3-Surface, And Configuration Space

The notions of embedding space, 3-surface (and 4-surface), and WCW (world of classical worlds (WCW)) are central to quantum TGD. The original idea was that 3-surfaces are space-like 3-surfaces of $H = M^4 \times CP_2$ or $H = M^4_+ \times CP_2$, and WCW consists of all possible 3-surfaces in H. The basic idea was that the definition of Kähler metric of WCW assigns to each X^3 a unique space-time surface $X^4(X^3)$ allowing in this manner to realize general coordinate invariance. During years these notions have however evolved considerably.

The notion of embedding space

Two generalizations of the notion of embedding space were forced by number theoretical vision [K95, K96, K94].

1. p-Adicization forced to generalize the notion of embedding space by gluing real and p-adic variants of embedding space together along rationals and common algebraic numbers. The generalized embedding space has a book like structure with reals and various p-adic number fields (including their algebraic extensions) representing the pages of the book.

2. With the discovery of zero energy ontology [K111, K25] it became clear that the so called causal diamonds (CDs) interpreted as intersections $M^4_+ \cap M^4_-$ of future and past directed light-cones of $M^4 \times CP_2$ define correlates for the quantum states. The position of the "lower" tip of CD characterizes the position of CD in H. If the temporal distance between upper and lower tip of CD is quantized in power-of-two multiples of CP_2 length, p-adic length scale hypothesis [K66] follows as a consequence. The upper *resp.* lower light-like boundary $\delta M^4_+ \times CP_2$ *resp.* $\delta M^4_- \times CP_2$ of CD can be regarded as the carrier of positive *resp.* negative energy part of the state. All net quantum numbers of states vanish so that everything is creatable from vacuum. Space-time surfaces assignable to zero energy states would would reside inside $CD \times CP_2$s and have their 3-D ends at the light-like boundaries of $CD \times CP_2$. Fractal structure is present in the sense that CDs can contains CDs within CDs, and measurement resolution dictates the length scale below which the sub-CDs are not visible.

3. The realization of the hierarchy of Planck constants [K36] suggests a further generalization of the notion of embedding space, which has however turned out to be an auxiliary tool only.

   Generalized embedding space would be obtained by gluing together Cartesian products of singular coverings and factor spaces of CD and CP_2 to form a book like structure. The particles at different pages of this book behave like dark matter relative to each other. This generalization also brings in the geometric correlate for the selection of quantization axes in the sense that the geometry of the sectors of the generalized embedding space with non-standard value of Planck constant involves symmetry breaking reducing the isometries to Cartan subalgebra. Roughly speaking, each CD and CP_2 is replaced with a union of CDs and CP_2s corresponding to different choices of quantization axes so that no breaking of Poincare and color symmetries occurs at the level of entire WCW .

It is now clear that this generealization only provides a description for the non-determinism realized in terms of n conformal equivalences of preferred extremals connecting 3-surfaces at the opposite boundaries of CD.

4. The construction of quantum theory at partonic level brings in very important delicacies related to the Kähler gauge potential of CP_2. Kähler gauge potential must have what one might call pure gauge parts in M^4 in order that the theory does not reduce to mere topological quantum field theory. Hence the strict Cartesian product structure $M^4 \times CP_2$ breaks down in a delicate manner. These additional gauge components -present also in CP_2- play key role in the model of anyons, charge fractionization, and quantum Hall effect [K73].

The notion of 3-surface

The question what one exactly means with 3-surface turned out to be non-trivial.

1. The original identification of 3-surfaces was as arbitrary space-like 3-surfaces subject to equivalence believed to be implied by General Coordinate Invariance. There was a problem related to the realization of equivalence since it was not at all obvious why the preferred extremal $X^4(Y^3)$ for Y^3 at $X^4(X^3)$ and Diff^4 related X^3 should satisfy $X^4(Y^3) = X^4(X^3)$.

2. Much later it became clear that light-like 3-surfaces identified as boundaries between regions of Minkowskian and Euclidian signature (wormhole contacts and exterior) have unique properties for serving as basic dynamical objects, in particular for realizing the General Coordinate Invariance in 4-D sense (obviously the identification resolves the above mentioned problem) and understanding the conformal symmetries of the theory.

   The condition that light-like parton orbits and space-like 3-surfaces at the ends of CD are physically equivalent allows to conclude that partonic 2-surfaces and their tangent space data should be enough for physics. One would have strong form of General Coordinate Invariance (GCI) and strong form of holography. The condition that the symplectic Noether charges for the above mentioned sub-algebra of the symplectic algebra vanish for space-like 3-surfaces at the ends of CD would be natural in this framework.

   It is however important to emphasize that this indeed holds true only locally. At the level of WCW metric this means that the components of the Kähler form and metric can be expressed in terms of data assignable to 2-D partonic surfaces. It is however essential that information about normal space of the 2-surface is needed.

3. An important step of progress was the realization that light-like 3-surfaces can have singular topology in the sense that they are analogous to Feynman diagrams. The light-like 3-surfaces representing lines of Feynman diagram can be glued along their 2-D ends playing the role of vertices to form what I call generalized Feynman diagrams ("Feynman" could be replaced with twistor, or braid, or something else). The ends of lines are located at boundaries of sub-CDs. This brings in also a hierarchy of time scales: the increase of the measurement resolution means introduction of sub-CDs containing sub-Feynman diagrams. As the resolution is improved, new sub-Feynman diagrams emerge so that effective 2-D character holds true in discretized sense and in given resolution scale only.

The notion of space-time surface

The basic vision has been that space-time surfaces correspond to preferred extremals $X^4(X^3)$ of Kähler action. Kähler function $K(X^3)$ defining the Kähler geometry of the world of classical worlds would correspond to the Kähler action for the preferred extremal. The precise identification of the preferred extremals turned out to be far from trivial. The recent discussion of this topic can be found at [K8].

1. The obvious first guess motivated by physical intuition was that preferred extremals correspond to the absolute minima of Kähler action for space-time surfaces containing X^3. This

choice would have some nice implications. For instance, one can develop an argument for the existence of an infinite number of conserved charges. If X^3 is light-like surface- either light-like boundary of X^4 or light-like 3-surface assignable to a wormhole throat at which the induced metric of X^4 changes its signature- this identification circumvents the obvious objections.

This choice might well be correct for (non-negative) Kähler function identifiable as Kähler action in Euclidian space-time regions (wormhole contacts). In Minkowskian regions Kähler action is imaginary ($\sqrt{g}$ factor is imaginary) and gives a complex phase to vacuum functional and clearly serves as the analog of action in quantum field theories. The identification as preferred extremal does not look natural now.

2. The recent identification has been already described: the vanishing of symplectic Noether charges in a sub-algebra isomorphic to the entire algebra would define the conformal gauge and fix the preferred extremals in ZEO highly uniquely. For a generic pair of 3-surfaces at the boundaries of CD it is not clear whether any preferred extremal exists. The non-determinism of Kähler action makes it difficult to make any conclusions in this respect.

3. I have consider many other identifications of preferred extremals during years. In Minkowskian regions the contraction $j \cdot A$ of Kähler current and Kähler gauge potential vanishes for the known extremals. Together with the weak form of electric-magnetic duality stating $\epsilon_{ijnt}J^{nt} = kJ_{ij}$, k proportionality constant, this condition would reduce Kähler action to 3-D Chern-Simons terms. This would realize TGD as almost topological QFT. Whether this condition makes sense in Euclidian regions and whether it is strong enough remains an open question.

   The construction of WCW geometry suggests also the strengthening the boundary conditions to the condition that there exists space-time coordinates in which the induced CP_2 Kähler form and induced metric satisfy the conditions $J_{ni} = 0$, $g_{ni} = 0$ hold at X_l^3 (n denote normal direction). One could say that at X_l^3 situation is static both metrically and for the Maxwell field defined by the induced Kähler form. There are reasons to hope that this is the final step in a long process.

4. One possible identification of preferred extremals would be as quaternionic sub-manifolds of embedding space with the property that quaternionic tangent space at given point contains a preferred M^2 identifiable as a commutative sub-space of quaternionic tangent spaces. One can also consider the possibility that M^2 depends on the point of space-time surface but that one has an integrable distribution defining string world sheet in M^4: this leads to the notion of Hamilton-Jacobi structure [K8]. $M^8 - H$ duality allowing to map surfaces of M^8 with this property to surfaces in M^8 by mapping the local tangent space to a point of CP_2 relates closely to this proposal.

5. The localization of the modes of Kähler-Dirac equation to string world sheets with vanishing W fields (to guarantee well-defined em charge for the modes) requires that Frobenius integrability conditions are satisfied for the 2-D tangent spaces and that the energy momentum currents as vectors of X^4 have no components normal to the string world sheet. I remains to be proven that these conditions can be satisfied.

   This suggests that one should construct preferred extremals as a concrete realization of holography. One would start from data given by string world sheets and partonic 2-surfaces and possibly also space-like 3-surface and the light-like orbits of partonic 2-surfaces by posing the conditions that sub-algebra of symplectic algebra acts as gauge algera. The reason for fixing of 3-surfaces apart from symplectic gauge transformation in an appropriate sub-algebra is that otherwise the possibility of strings and their orbits to get knotted and linked becomes impossible to describe. One clearly would have effective 2-dimensionality.

   According to the recent view about Kähler-Dirac action the boundaries of string world sheets are embedding space geodesics characterizing by light-like 8-momentum. This suggests that the braiding along partonic orbits is probably possible only if one allows intermediate partonic 2-surfaces in which the direction of four-momentum changes. The particle physics interpretation would be that braiding must respect conservation of momentum and thus occurs by exchange of say bosonic quanta. So that braiding diagram would be replaced by the analog of Feynman diagram.

6. One bundle of ideas relates is inpired by basic thinking about massless fields and relies on the observation that the known extremals seems to decompose in Minkowskian regions to pieces having interpretation as classical analogs of massless field quanta allowing local polarization vector and light-like 4-momentum vector orthogonal to each other. The simplest example is provided by massless extremals for which one has linear superposition of modes in the direction of four-momentum. One has therefore very quantal behavior already classically. In particular, linear superposition fails and can be realized only for effects experienced by a particle like 3-surface topologically condensed to several space-time sheets. At GRT-QFT limit superposition of effects becomes superposition of fields when the many-sheeted space-time is approximated with slightly curved M^4.

   Also number theoretical vision led to a related proposal that $X^4(X^3_{l,i})$, where $X^3_{l,i}$ denotes i^th connected component of the light-like 3-surface X^3_l, contain in their 4-D tangent space $T(X^4(X^3_{l,i}))$ a subspace $M^2_i \subset M^4$ having interpretation as the plane of non-physical polarizations. This means a close connection with super string models. Geometrically this would mean that the deformations of 3-surface in the plane of non-physical polarizations would not contribute to the line element of WCW . This is as it must be since complexification does not make sense in M^2 degrees of freedom.

   In number theoretical framework M^2_i has interpretation as a preferred hyper-complex subspace of hyper-octonions defined as 8-D subspace of complexified octonions with the property that the metric defined by the octonionic inner product has signature of M^8. A stronger condition would be that the condition holds true at all points of $X^4(X^3)$ for a global choice M^2 but this is un-necessary and leads to strong un-proven conjectures. The condition $M^2_i \subset T(X^4(X^3_{l,i}))$ in principle fixes the tangent space at $X^3_{l,i}$, and one has good hopes that the boundary value problem is well-defined and fixes $X^4(X^3)$ uniquely as a preferred extremal of Kähler action. This picture is rather convincing since the choice $M^2_i \subset M^3$ plays also other important roles.

7. The weakest form of number theoretic compactification states that light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^8$, where $X^4(X^3)$ hyper-quaternionic surface in hyper-octonionic M^8 can be mapped to light-like 3-surfaces $X^3 \subset X^4(X^3) \subset M^4 \times CP_2$, where $X^4(X^3)$ is now preferred extremum of Kähler action. The natural guess is that $X^4(X^3) \subset M^8$ is a preferred extremal of Kähler action associated with Kähler form of E^4 in the decomposition $M^8 = M^4 \times E^4$, where M^4 corresponds to hyper-quaternions. The conjecture would be that the value of the Kähler action in M^8 is same as in $M^4 \times CP_2$.

The notion of WCW

From the beginning there was a problem related to the precise definition of WCW ("world of classical worlds" (WCW)). Should one regard CH as the space of 3-surfaces of $M^4 \times CP_2$ or $M^4_+ \times CP_2$ or perhaps something more delicate.

1. For a long time I believed that the question "M^4_+ or M^4?" had been settled in favor of M^4_+ by the fact that M^4_+ has interpretation as empty Roberson-Walker cosmology. The huge conformal symmetries assignable to $\delta M^4_+ \times CP_2$ were interpreted as cosmological rather than laboratory symmetries. The work with the conceptual problems related to the notions of energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering M^4 instead of M^4_+.

2. With the discovery of zero energy ontology it became clear that the so called causal diamonds (CDs) define excellent candidates for the fundamental building blocks of WCW or "world of classical worlds" (WCW). The spaces $CD \times CP_2$ regarded as subsets of H defined the sectors of WCW .

3. This framework allows to realize the huge symmetries of $\delta M^4_\pm \times CP_2$ as isometries of WCW . The gigantic symmetries associated with the $\delta M^4_\pm \times CP_2$ are also laboratory symmetries.

Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $\delta M^4_\pm \times CP_2$ of the embedding space representing the upper and lower boundaries of CD. Second conformal symmetry corresponds to light-like 3-surface X^3_l, which can be boundaries of X^4 and light-like surfaces separating space-time regions with different signatures of the induced metric. This symmetry is identifiable as the counterpart of the Kac Moody symmetry of string models.

A rather plausible conclusion is that WCW is a union of sub- WCW s associated with the spaces $CD \times CP_2$. CDs can contain CDs within CDs so that a fractal like hierarchy having interpretation in terms of measurement resolution results. Since the complications due to p-adic sectors and hierarchy of Planck constants are not relevant for the basic construction, it reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

3.3.3 Could The Universe Be Doing Yangian Arithmetics?

One of the old TGD inspired really crazy ideas about scattering amplitudes is that Universe is doing some sort of arithmetics so that scattering amplitude are representations for computational sequences of minimum length. The idea is so crazy that I have even given up its original form, which led to an attempt to assimilate the basic ideas about bi-algebras, quantum groups [K11], Yangians [K101], and related exotic things. The work with twistor Grassmannian approach inspired a reconsideration of the original idea seriously with the idea that super-symplectic Yangian could define the arithmetics. I try to describe the background, motivation, and the ensuing reckless speculations in the following.

Do scattering amplitudes represent quantal algebraic manipulations?

I seems that tensor product $\otimes$ and direct sum $\oplus$ - very much analogous to product and sum but defined between Hilbert spaces rather than numbers - are naturally associated with the basic vertices of TGD. I have written about this a highly speculative chapter - both mathematically and physically [K68]. The chapter [K11] is a remnant of earlier similar speculations.

1. In $\otimes$ vertex 3-surface splits to two 3-surfaces meaning that the 2 "incoming" 4-surfaces meet at single common 3-surface and become the outgoing 3-surface: 3 lines of Feynman diagram meeting at their ends. This has a lower-dimensional shadow realized for partonic 2-surfaces. This topological 3-particle vertex would be higher-D variant of 3-vertex for Feynman diagrams.

2. The second vertex is trouser vertex for strings generalized so that it applies to 3-surfaces. It does not represent particle decay as in string models but the branching of the particle wave function so that particle can be said to propagate along two different paths simultaneously. In double slit experiment this would occur for the photon space-time sheets.

3. The idea is that Universe is doing arithmetics of some kind in the sense that particle 3-vertex in the above topological sense represents either multiplication or its time-reversal co-multiplication.

The product, call it $\circ$, can be something very general, say algebraic operation assignable to some algebraic structure. The algebraic structure could be almost anything: a random list of structures popping into mind consists of group, Lie-algebra, super-conformal algebra quantum algebra, Yangian, etc.... The algebraic operation $\circ$ can be group multiplication, Lie-bracket, its generalization to super-algebra level, etc...). Tensor product and thus linear (Hilbert) spaces are involved always, and in product operation tensor product $\otimes$ is replaced with $\circ$.

1. The product $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is analogous to a particle reaction in which particles A_k and A_l fuse to particle $A_k \otimes A_l \rightarrow C = A_k \circ A_l$. One can say that $\otimes$ between reactants is transformed to $\circ$ in the particle reaction: kind of bound state is formed.

2. There are very many pairs A_k, A_l giving the same product C just as given integer can be divided in many ways to a product of two integers if it is not prime. This of course suggests

that elementary particles are primes of the algebra if this notion is defined for it! One can use some basis for the algebra and in this basis one has $C = A_k \circ A_l = f_{klm} A_m$, f_{klm} are the structure constants of the algebra and satisfy constraints. For instance, associativity $A(BC) = (AB)C$ is a constraint making the life of algebraist more tolerable and is almost routinely assumed.

For instance, in the number theoretic approach to TGD associativity is proposed to serve as fundamental law of physics and allows to identify space-time surfaces as 4-surfaces with associative (quaternionic) tangent space or normal space at each point of octonionic embedding space $M^4 \times CP_2$. Lie algebras are not associative but Jacobi-identities following from the associativity of Lie group product replace associativity.

3. Co-product can be said to be time reversal of the algebraic operation $\circ$. Co-product can be defined as $C = A_k \rightarrow \sum_{lm} f_k^{lm} A_l \otimes A_m$, where f_k^{lm} are the structure constants of the algebra. The outcome is quantum superposition of final states, which can fuse to C (the "reaction" $A_k \otimes A_l \rightarrow C = A_k \circ A_l$ is possible). One can say that $\circ$ is replaced with $\otimes$: bound state decays to a superposition of all pairs, which can form the bound states by product vertex.

There are motivations for representing scattering amplitudes as sequences of algebraic operations performed for the incoming set of particles leading to an outgoing set of particles with particles identified as algebraic objects acting on vacuum state. The outcome would be analogous to Feynman diagrams but only the diagram with minimal length to which a preferred extremal can be assigned is needed. Larger ones must be equivalent with it.

The question is whether it could be indeed possible to characterize particle reactions as computations involving transformation of tensor products to products in vertices and co-products to tensor products in co-vertices (time reversals of the vertices). A couple of examples gives some idea about what is involved.

1. The simplest operations would preserve particle number and to just permute the particles: the permutation generalizes to a braiding and the scattering matrix would be basically unitary braiding matrix utilized in topological quantum computation.

2. A more complex situation occurs, when the number of particles is preserved but quantum numbers for the final state are not same as for the initial state so that particles must interact. This requires both product and co-product vertices. For instance, $A_k \otimes A_l \rightarrow f_{kl}^m A_m$ followed by $A_m \rightarrow f_m^{rs} A_r \otimes A_s$ giving $A_k \rightarrow f_{kl}^m f_m^{rs} A_r \otimes A_s$ representing 2-particle scattering. State function reduction in the final state can select any pair $A_r \otimes A_s$ in the final state. This reaction is characterized by the ordinary tree diagram in which two lines fuse to single line and defuse back to two lines. Note also that there is a non-deterministic element involved. A given final state can be achieved from a given initial state after large enough number of trials. The analogy with problem solving and mathematical theorem proving is obvious. If the interpretation is correct, Universe would be problem solver and theorem prover!

3. More complex reactions affect also the particle number. 3-vertex and its co-vertex are the simplest examples and generate more complex particle number changing vertices. For instance, on twistor Grassmann approach on can construct all diagrams using two 3-vertices. This encourages the restriction to 3-vertice (recall that fermions have only 2-vertices)

4. Intuitively it is clear that the final collection of algebraic objects can be reached by a large - maybe infinite - number of ways. It seems also clear that there is the shortest manner to end up to the final state from a given initial state. Of course, it can happen that there is no way to achieve it! For instance, if $\circ$ corresponds to group multiplication the co-vertex can lead only to a pair of particles for which the product of final state group elements equals to the initial state group element.

5. Quantum theorists of course worry about unitarity. How can avoid the situation in which the product gives zero if the outcome is element of linear space. Somehow the product should be such that this can be avoided. For instance, if product is Lie-algebra commutator, Cartan algebra would give zero as outcome.

Generalized Feynman diagram as shortest possible algebraic manipulation connecting initial and final algebraic objects

There is a strong motivation for the interpretation of generalized Feynman diagrams as shortest possible algebraic operations connecting initial and final states. The reason is that in TGD one does not have path integral over all possible space-time surfaces connecting the 3-surfaces at the ends of CD. Rather, one has in the optimal situation a space-time surface unique apart from conformal gauge degeneracy connecting the 3-surfaces at the ends of CD (they can have disjoint components).

Path integral is replaced with integral over 3-surfaces. There is therefore only single minimal generalized Feynman diagram (or twistor diagram, or whatever is the appropriate term). It would be nice if this diagram had interpretation as the shortest possible computation leading from the initial state to the final state specified by 3-surfaces and basically fermionic states at them. This would of course simplify enormously the theory and the connection to the twistor Grassmann approach is very suggestive. A further motivation comes from the observation that the state basis created by the fermionic Clifford algebra has an interpretation in terms of Boolean quantum logic and that in ZEO the fermionic states would have interpretation as analogs of Boolean statements $A \rightarrow B$.

To see whether and how this idea could be realized in TGD framework, let us try to find counterparts for the basic operations $\otimes$ and $\circ$ and identify the algebra involved. Consider first the basic geometric objects.

1. Tensor product could correspond geometrically to two disjoint 3-surfaces representing 3-particles. Partonic 2-surfaces associated with a given 3-surface represent second possibility. The splitting of a partonic 2-surface to two could be the geometric counterpart for co-product.

2. Partonic 2-surfaces are however connected to each other and possibly even to themselves by strings. It seems that partonic 2-surface cannot be the basic unit. Indeed, elementary particles are identified as pairs of wormhole throats (partonic 2-surfaces) with magnetic monopole flux flowing from throat to another at first space-time sheet, then through throat to another sheet, then back along second sheet to the lower throat of the first contact and then back to the thirst throat. This unit seems to be the natural basic object to consider. The flux tubes at both sheets are accompanied by fermionic strings. Whether also wormhole throats contain strings so that one would have single closed string rather than two open ones, is an open question.

3. The connecting strings give rise to the formation of gravitationally bound states and the hierarchy of Planck constants is crucially involved. For elementary particle there are just two wormhole contacts each involving two wormhole throats connected by wormhole contact. Wormhole throats are connected by one or more strings, which define space-like boundaries of corresponding string world sheets at the boundaries of CD. These strings are responsible for the formation of bound states, even macroscopic gravitational bound states.

Does super-symplectic Yangian define the arithmetics?

Super-symplectic Yangian would be a reasonable guess for the algebra involved.

1. The 2-local generators of Yangian would be of form $T_1^A = f_{BC}^A T^B \otimes T^C$, where f_{BC}^A are the structure constants of the super-symplectic algebra. n-local generators would be obtained by iterating this rule. Note that the generator T_1^A creates an entangled state of T^B and T^C with f_{BC}^A the entanglement coefficients. T_n^A is entangled state of T^B and T_{n-1}^C with the same coefficients. A kind replication of T_{n-1}^A is clearly involved, and the fundamental replication is that of T^A. Note that one can start from any irreducible representation with well defined symplectic quantum numbers and form similar hierarchy by using T^A and the representation as a starting point.

   That the hierarchy T_n^A and hierarchies irreducible representations would define a hierarchy of states associated with the partonic 2-surface is a highly non-trivial and powerful hypothesis about the formation of many-fermion bound states inside partonic 2-surfaces.

2. The charges T^A correspond to fermionic and bosonic super-symplectic generators. The geometric counterpart for the replication at the lowest level could correspond to a fermionic/bosonic string carrying super-symplectic generator splitting to fermionic/bosonic string and a string carrying bosonic symplectic generator T^A. This splitting of string brings in mind the basic gauge boson-gauge boson or gauge boson-fermion vertex.

   The vision about emission of virtual particle suggests that the entire wormhole contact pair replicates. Second wormhole throat would carry the string corresponding to T^A assignable to gauge boson naturally. T^A should involve pairs of fermionic creation and annihilation operators as well as fermionic and anti-fermionic creation operator (and annihilation operators) as in quantum field theory.

3. Bosonic emergence suggests that bosonic generators are constructed from fermion pairs with fermion and anti-fermion at opposite wormhole throats: this would allow to avoid the problems with the singular character of purely local fermion current. Fermionic and anti-fermionic string would reside at opposite space-time sheets and the whole structure would correspond to a closed magnetic tube carrying monopole flux. Fermions would correspond to superpositions of states in which string is located at either half of the closed flux tube.

4. The basic arithmetic operation in co-vertex would be co-multiplication transforming T^A_n to $T^A_{n+1} = f^A_{BC} T^B_n \otimes T^C$. In vertex the transformation of T^A_{n+1} to T^A_n would take place. The interpretations would be as emission/absorption of gauge boson. One must include also emission of fermion and this means replacement of T^A with corresponding fermionic generators F^A, so that the fermion number of the second part of the state is reduced by one unit. Particle reactions would be more than mere braidings and re-grouping of fermions and anti-fermions inside partonic 2-surfaces, which can split.

5. Inside the light-like orbits of the partonic 2-surfaces there is also a braiding affecting the M-matrix. The arithmetics involved would be therefore essentially that of measuring and "co-measuring" symplectic charges.

   Generalized Feynman diagrams (preferred extremals) connecting given 3-surfaces and many-fermion states (bosons are counted as fermion-anti-fermion states) would have a minimum number of vertices and co-vertices. The splitting of string lines implies creation of pairs of fermion lines. Whether regroupings are part of the story is not quite clear. In any case, without the replication of 3-surfaces it would not be possible to understand processes like e-e scattering by photon exchange in the proposed picture.

It is easy to hear the comments of the skeptic listener in the back row.

1. The attribute "minimal" - , which could translate to minimal value of Kähler function - is dangerous. It might be very difficult to determine what the minimal diagram is - consider only travelling salesman problem or the task of finding the shortest proof of theorem. It would be much nicer to have simple calculational rules.

   The original proposal might help here. The generalization of string model duality was in question. It stated that it is possible to move the positions of the vertices of the diagrams just as one does to transform s-channel resonances to t-channel exchange. All loops of generalized diagrams could be be eliminated by transforming the to tadpoles and snipped away so that only tree diagrams would be left. The variants of the diagram were identified as different continuation paths between different paths connecting sectors of WCW corresponding to different 3-topologies. Each step in the continuation procedure would involve product or co-product defining what continuation between two sectors means for WCW spinors. The continuations between two states require some minimal number of steps. If this is true, all computations connecting identical states are also physically equivalent. The value of the vacuum functional be same for all of them. This looks very natural.

   That the Kähler action should be same for all computational sequences connecting the same initial and final states looks strange but might be understood in terms of the vacuum degeneracy of Kähler action closed related to quantum criticality, which means infinite gauge degeneracy associated with the Yangian of a sub-algebra of super-symplectic algebra.

2. QFT perturbation theory requires that should have superposition of computations/continuations. What could the superposition of QFT diagrams correspond to in TGD framework?

   Could it correspond to a superposition of generators of the Yangian creating the physical state? After all, already quantum computer perform superpositions of computations. The fermionic state would not be the simplest one that one can imagine. Could AdS/CFT analogy allow to identify the vacuum state as a superposition of multi-string states so that single super-symplectic generator would be replaced with a superposition of its Yangian counterparts with same total quantum numbers but with a varying number of strings? The weight of a given superposition would be given by the total effective string world sheet area. The sum of diagrams would emerge from this superposition and would basically correspond to functional integration in WCW using exponent of Kähler action as weight. The stringy functional integral ("functional" if also wormhole contacts contain string portion, otherwise path integral) would give the perturbation theory around given string world sheet. One would have effective reduction of string theory.

How does this relate to the ordinary perturbation theory?

One can of course worry about how to understand the basic results of the usual perturbation theory in this picture. How does one obtain a perturbation theory in powers of coupling constant, what does running coupling constant mean, etc...? I have already discussed how the superposition of diagrams could be understood in the new picture.

1. The QFT picture with running coupling constant is expected at QFT limit, when many-sheeted space-time is replaced with a slightly curved region of M^4 and gravitational field and gauge potentials are identified as sums of the deviations of induced metric from M^4 metric and classical induced gauge potentials associated with the sheets of the many-sheeted space-time. The running coupling constant would be due to the dependence of the size scale of CD, and p-adic coupling constant evolution would be behind the continuous one.

2. The notion of running coupling constant is very physical concept and should have a description also at the fundamental level and be due to a finite computational resolution, which indeed has very concrete description in terms of Noether charges of super-symplectic Yangian creating the states at the ends of space-time surface at the boundaries of CD. The space-time surface and the diagram associated with a given pair of 3-surfaces and stringy Noether charges associated with them can be characterized by a complexity measured in terms of the number of vertices (3-surface at which three 3-surfaces meet).

   For instance, 3-particle scattering can be possible only by using the simplest 3-vertex defined by product or co-product for pairs of 3-surfaces. In the generic case one has more complex diagram and what looks first 3-particle vertex has complex substructure rather than being simple product or co-product.

3. Complexity seems to have two separate aspects: the complexities of the positive and negative parts of zero energy state as many-fermion states and the complexity of associated 3-surfaces. The generalization of AdS/CFT however suggests that once the string world sheets and partonic 2-surfaces appearing in the diagram have been fixed, the space-time surface itself is fixed. The principle also suggests that the fixing partonic 2-surface and the strings connecting them at the boundaries of CD fixes the 3-surface apart from the action of sub-algebra of Yangian acting as gauge algebra (vanishing classical Noether charges). If one can determine the minimal sequence of allowed algebraic operation of Yangian connecting initial and final fermion states, one knows the minimum number of vertices and therefore the topological structure of the connecting minimal space-time surface.

4. In QFT spirit one could describe the finite measurement resolution by introducing effective 3-point vertex, which is need not be product/co-produce anymore. 3-point scattering amplitudes in general involve microscopic algebraic structure involving several vertices. One can however give up the nice algebraic interpretation and just talk about effective 3-vertex for practical purposes. Just as the QFT vertex described by running coupling constant decomposes to sum of diagrams, product/co-product in TGD could be replaced with effective

product/co-product expressible as a longer computation. This would imply coupling constant evolution.

Fermion lines could however remain as such since they are massless in 8-D sense and mass renormalization does not make sense.

Similar practical simplification could be done the initial and final states to get rid of superposition of the Yangian generators with different numbers of strings ("cloud of virtual particles"). This would correspond to wave function renormalization.

The number of vertices and wormhole contact orbits serves as a measure for the complexity of the diagram.

1. Since fermion lines are associated with wormhole throats assignable with wormhole contacts identifiable as deformations CP_2 type vacuum extremals, one expects that the exponent of the Kähler function defining vacuum functional is in the first approximation the total CP_2 volume of wormhole contacts giving a measure for the importance of the contribution in functional integral. If it converges very rapidly only Gaussian approximation around maximum is needed.

2. Convergence depends on how large the fraction of volume of CP_2 is associated with a given wormhole contact. The volume is proportional to the length of the wormhole contact orbit. One expects exponential convergence with the number of fermion lines and their lengths for long lines. For short distances the exponential damping is small so that diagrams with microscopic structure of diagrams are needed and are possible. This looks like adding small scale details to the algebraic manipulations.

3. One must be of course be very cautious in making conclusions. The presence of $1/\alpha_K \propto h_{eff}$ in the exponent of Kähler function would suggest that for large values of h_{eff} only the 3-surfaces with smallest possible number of wormhole contact orbits contribute. On the other hand, the generalization of AdS/CFT duality suggests that Kähler action reducible to area of string world sheet in the effective metric defined by canonical momentum currents of Kähler action behaves as $\alpha_K^2 \propto 1/h_{eff}^2$. How $1/h_{eff}^2$ proportionality might be understood is discussed in [?] in terms electric-magnetic duality.

To sum up, the identification of vertex as a product or co-product in Yangian looks highly promising approach. The Nother charges of the super-symplectic Yangian are associated with strings and are either linear or bilinear in the fermion field. The fermion fields associated with the partonic 2-surface defining the vertex are contracted with fermion fields associated with other partonic 2-surfaces using the same rule as in Wick expansion in quantum field theories. The contraction gives fermion propagator for each leg pair associated with two vertices. Vertex factor is proportional to the contraction of spinor modes with the operators defining the Noether charge or super charge and essentially Kähler-Dirac gamma matrix and the representation of the action of the symplectic generator on fermion realizable in terms of sigma matrices. This is very much like the corresponding expression in gauge theories but with gauge algebra replaced with symplectic algebra. The possibility of contractions of creation and annihilation operator for fermion lines associated with opposite wormhole throats at the same partonic 2-surface (for Noether charge bilinear in fermion field) gives bosonic exchanges as lines in which the fermion lines turns in time direction: otherwise only regroupings of fermions would take place.

This was not the whole story yet

The proposed amplitude represents only the value of WCW spinor field for single pair of 3-surfaces at the opposite boundaries of given CD. Hence Yangian construction does not tell the whole story.

1. Yangian algebra would give only the vertices of the scattering amplitudes. On basis of previous considerations, one expects that each fermion line carries propagator defined by 8-momentum. The structure would resemble that of super-symmetric YM theory. Fermionic propagators should emerge from summing over intermediate fermion states in various vertices and one would have integrations over virtual momenta which are carried as residue integrations in twistor Grassmann approach. 8-D counterpart of twistorialization would apply.

2. Super-symplectic Yangian would give the scattering amplitudes for single space-time surface and the purely group theoretical form of these amplitudes gives hopes about the independence of the scattering amplitude on the pair of 3-surfaces at the ends of CD near the maximum of Kähler function. This is perhaps too much to hope except approximately but if true, the integration over WCW would give only exponent of Kähler action since metric and poorly defined Gaussian and determinants would cancel by the basic properties of Kähler metric. Exponent would give a non-analytic dependence on α_K.

   The Yangian supercharges are proportional to $1/\alpha_K$ since covariant Kähler-Dirac gamma matrices are proportional to canonical momentum currents of Kähler action and thus to $1/\alpha_K$. Perturbation theory in powers of $\alpha_K = g_K^2/4\pi\hbar_{eff}$ is possible after factorizing out the exponent of vacuum functional at the maximum of Kähler function and the factors $1/\alpha_K$ multiplying super-symplectic charges.

   The additional complication is that the characteristics of preferred extremals contributing significantly to the scattering amplitudes are expected to depend on the value of α_K by quantum interference effects. Kähler action is proportional to $1/\alpha_K$. The analogy of AdS/CFT correspondence states the expressibility of Kähler function in terms of string area in the effective metric defined by the anti-commutators of K-D matrices. Interference effects eliminate string length for which the area action has a value considerably larger than one so that the string length and thus also the minimal size of CD containing it scales as h_{eff}. Quantum interference effects therefore give an additional dependence of Yangian super-charges on h_{eff} leading to a perturbative expansion in powers of α_K although the basic expression for scattering amplitude would not suggest this.

3.3.4 The Construction Of U, M-, And S-Matrices

The general architecture of matrices is now rather well-understood and described in chapter [K61]. A brief summary is also given in the introduction. The key matrix is U-matrix acting in the space of zero states but leaving the states at the second boundary of CD invariant. M-matrix acts between positive and negative energy parts of given zero energy state being the product of a hermitian square root of density matrix and of a unitary S-matrix. The hermitian matrices involved would naturally form a representation of super-symplectic algebra or its sub-algebra and their "moduli squared" define a density matrix characterizing the second part of zero energy state. An open question is whether this density matrix relates to thermodynamics only formally or whether there is a deeper connection.

The recipe reduces the decisive step to a construction of S-matrix for a given CD and of a unitary time evolution operator in the moduli space of CDs providing unitary representation for a discrete subgroup of Lorentz group. The S-matrix for a given CD is n:th power of fundamental S-matrix S^n for CD whose size is n times the minimal size of CD characterized by the CP_2 time scale.

The construction of S-matrix involves several ideas that have emerged during last years and involve symmetries in an essential manner.

Emergence of particles as bound state of fundamental fermions, extended space-time supersymmetry, and generalized twistors

During year 2009 several new ideas emerged and give hopes about a concrete construction of M-matrix.

1. The notion of bosonic emergence follows from the fact that gauge bosons are identifiable as pairs of fermion and anti-fermion at opposite light-like throats of wormhole contact. As a consequence, bosonic propagators and vertices are generated radiatively from a fundamental action for fermions and their super partners. At QFT limit without super-symmetry this means that Dirac action coupled to gauge bosons is the fundamental action and the counterpart of YM action is generated radiatively. All coupling constants follow as predictions as they indeed must do on basis of the general structure of quantum TGD.

2. Whether the counterparts of space-time supersymmetries are possible in TGD Universe has remained a long-standing open question and my cautious belief has been that the super

partners do not exist. The resolution of the problem came with the increased understanading of the dynamics of the Kähler-Dirac action [?]. In particular, the localization of the electroweakly charged modes at 2-D surfaces - string world sheets and possibly also partonic 2-surfaces- meant an enormous simplification since the solutions of the Kähler-Dirac equation are conformal spinor modes.

The oscillator operators associated with the modes of the induced spinor field satisfy the anti-commutation relations defining the generalization of space-time super-symmetry algebra and these oscillator operators serve as the building blocks of various super-conformal algebras. The number of super-symmetry generators is very large, perhaps even infinite. This forces a generalization of the standard super field concept. The action for chiral super-fields emerges as a generalization of the Dirac action to include all possible super-partners. The huge super-symmetry gives excellent hopes about cancelation of UV divergences. The counterpart of super-symmetric YM action emerges radiatively. This formalism works at the QFT limit. The generalization of the formalism to quantum TGD proper is yet to be carried out.

3. Twistor program has become one of the most promising approaches to gauge theories. This inspired the question whether TGD could allow twistorialization [K101, K10]. Massive states -both real and virtual- are the basic problem of twistor approach. In TGD framework the obvious idea is that massive on mass shell states can be interpreted as massless states in 8-D sense. Massive off-mass shell states in turn could be regarded as pairs of positive and negative on mass shell states. This means opening of the black box of virtual state attempted already in the model for bosonic propagators inspired by the bosonic emergence , and one can even hope that individual loop integrals are finite and that Wick rotation is not needed. The third observation is that 8-dimensional gamma matrices allow a representation in terms of octonions (matrices are not in question anymore). If the Kähler-Dirac gamma "matrices" associated with space-time surface define a quaternionic sub-algebra of the complexified octonion algebra, they allow a matrix representation defined by octonionic structure constants. This holds true for are hyper-quaternionic space-time surfaces so that a connection with number theoretic vision emerges. This would more or less reduce the notion of twistor to its 4-dimensional counterpart.

Generalization of Feynman diagrams

An essential difference between TGD and string models is the replacement of stringy diagrams with generalized Feynman diagrams obtained by gluing 3-D light-like surfaces (instead of lines) together at their ends represented as partonic 2-surfaces. This makes the construction of vertices very simple. The notion of number theoretic braid in turn implies discretization having also interpretation in terms of non-commutativity due to finite measurement resolution replacing anti-commutativity along stringy curves with anti-commutativity at points of braids. Braids can replicate at vertices which suggests an interpretation in terms of topological quantum computation combined with non-faithful copying and communication of information. The analogs of stringy diagrams have quite different interpretation in TGD for instance, photons traveling via two different paths in double slit experiment are represented in terms of stringy branching of the photonic 2-surface.

Scattering amplitudes as computations in Yangian arithmetics?

One of the old TGD inspired really crazy ideas about scattering amplitudes is that Universe is doing some sort of arithmetics so that scattering amplitude are representations for computational sequences of minimum length and that all diagrams connecting the same states at the boundaries of CD produce the same scattering amplitude. This would mean enormous calculational simplification.

The idea is so crazy that I have even given up its original form, which led to an attempt to assimilate the basic ideas about bi-algebras, quantum groups [K11], Yangians [K101], and related exotic things. The work with twistor Grassmannian approach inspired a reconsideration of the original idea seriously with the idea that super-symplectic Yangian could define the arithmetics.

The identification of universal 3-vertex as a product or co-product in Yangian looks highly promising approach to the construction of the scattering amplitude. The Nother charges of the super-symplectic Yangian are associated with strings and are either linear or bilinear in the fermion field. The fermion fields associated with the partonic 2-surface defining the vertex are contracted with fermion fields associated with other partonic 2-surfaces using the same rule as in Wick expansion in quantum field theories. The contraction gives fermion propagator for each leg pair associated with two vertices. Vertex factor is proportional to the contraction of spinor modes with the operators defining the Noether charge or super charge and essentially Kähler-Dirac gamma matrix and the representation of the action of the symplectic generator on fermion realizable in terms of sigma matrices.

This resembles strongly the corresponding expression in gauge theories but with gauge algebra replaced with symplectic algebra. The possibility of contractions of creation and annihilation operator for fermion lines associated with opposite wormhole throats at the same partonic 2-surface (for Noether charge bilinear in fermion field) gives bosonic exchanges as lines in which the fermion lines turns in time direction: otherwise only regroupings of fermions would take place.

Could correlation functions, S-matrix, and coupling constant evolution be coded the statistical properties of preferred extremals?

How to calculate the correlation functions and coupling constant evolution has remained a basic unresolved challenge. Generalized Feynman diagrams provide a powerful vision which however does not help in practical calculations. Some big idea has been lacking.

Quantum classical correspondence states that all aspects of quantum states should have correlates in the geometry of preferred extremals. In particular, various elementary particle propagators should have a representation as properties of preferred extremals. This would allow to realize the old dream about being able to say something interesting about coupling constant evolution although it is not yet possible to calculate the M-matrices and U-matrix. The general structure of U-matrix is however understood [K61]. Hitherto everything that has been said about coupling constant evolution has been rather speculative arguments except for the general vision that it reduces to a discrete evolution defined by p-adic length scales. General first principle definitions are however much more valuable than ad hoc guesses even if the latter give rise to explicit formulas.

In quantum TGD and also at its QFT limit various correlation functions in given quantum state should code for its properties. By quantum classical correspondence these correlation functions should have counterparts in the geometry of preferred extremals. Even more: these classical counterparts for a given preferred extremal ought to be identical with the quantum correlation functions for the superposition of preferred extremals. This correspondence could be called quantum ergodicity by its analogy with ordinary ergodicity stating that the member of ensemble becomes representative of ensemble.

This principle would be a quantum generalization of ergodic theorem stating that the time evolution of a single member of ensemble represents the ensemble statistically. This symmetry principle analogous to holography might allow to fix S-matrix uniquely even in the case that the hermitian square root of the density matrix appearing in the M-matrix would lead to a breaking of quantum ergodicity as also 4-D spin glass degeneracy suggests.

This principle would allow to deduce correlation functions from the statistical properties of single preferred extremal alone using just classical intuition. Also coupling constant evolution would be coded by the statistical properties of preferred extremals. Quantum ergodicity would mean an enormous simplification since one could avoid the horrible conceptual complexities involved with the functional integrals over WCW .

This might of course be too optimistic guess. If a sub-algebra of symplectic algebra acts as gauge symmmetries of the preferred extremals in the sense that corresponding Noether charges vanish, it can quite well be that correlations functions correspond to averages for extremals belonging to single conformal equivalence class.

1. The marvellous implication of quantum ergodicity would be that one could calculate everything solely classically using the classical intuition - the only intuition that we have. Quantum ergodicity would also solve the paradox raised by the quantum classical correspondence for momentum eigenstates. Any preferred extremal in their superposition defining momentum eigenstate should code for the momentum characterizing the superposition itself. This is indeed possible if every extremal in the superposition codes the momentum to the properties of classical correlation functions which are identical for all of them.

2. The only manner to possibly achieve quantum ergodicity is in terms of the statistical properties of the preferred extremals. It should be possible to generalize the ergodic theorem stating that the properties of statistical ensemble are represented by single space-time evolution in the ensemble of time evolutions. Quantum superposition of classical worlds would effectively reduce to single classical world as far as classical correlation functions are considered. The notion of finite measurement resolution suggests that one must state this more precisely by adding that classical correlation functions are calculated in a given UV and IR resolutions meaning UV cutoff defined by the smallest CD and IR cutoff defined by the largest CD present.

3. The skeptic inside me immediately argues that TGD Universe is 4-D spin glass so that this quantum ergodic theorem must be broken. In the case of the ordinary spin classes one has not only statistical average for a fixed Hamiltonian but a statistical average over Hamiltonians. There is a probability distribution over the coupling parameters appearing in the Hamiltonian. Maybe the quantum counterpart of this is needed to predict the physically measurable correlation functions.

   Could this average be an ordinary classical statistical average over quantum states with different classical correlation functions? This kind of average is indeed taken in density matrix formalism. Or could it be that the square root of thermodynamics defined by ZEO actually gives automatically rise to this average? The eigenvalues of the "hermitian square root" of the density matrix would code for components of the state characterized by different classical correlation functions. One could assign these contributions to different "phases".

4. Quantum classical correspondence in statistical sense would be very much like holography (now individual classical state represents the entire quantum state). Quantum ergodicity would pose a rather strong constraint on quantum states. This symmetry principle could actually fix the spectrum of zero energy states to a high degree and fix therefore the M-matrices given by the product of hermitian square root of density matrix and unitary S-matrix and unitary U-matrix constructible as inner products of M-matrices associated with CDs with various size scales [K61].

5. In TGD inspired theory of consciousness the counterpart of quantum ergodicity is the postulate that the space-time geometry provides a symbolic representation for the quantum states and also for the contents of consciousness assignable to quantum jumps between quantum states. Quantum ergodicity would realize this strongly self-referential looking condition. The positive and negative energy parts of zero energy state would be analogous to the initial and final states of quantum jump and the classical correlation functions would code for the contents of consciousness like written formulas code for the thoughts of mathematician and provide a sensory feedback.

How classical correlation functions should be defined?

1. General Coordinate Invariance and Lorentz invariance are the basic constraints on the definition. These are achieved for the space-time regions with Minkowskian signature and 4-D M^4 projection if linear Minkowski coordinates are used. This is equivalent with the contraction of the indices of tensor fields with the space-time projections of M^4 Killing vector fields representing translations. Accepting ths generalization, there is no need to restrict oneself to 4-D M^4 projection and one can also consider also Euclidian regions identifiable as lines of generalized Feynman diagrams.

   Quantum ergodicity very probably however forces to restrict the consideration to Minkowskian and Euclidian space-time regions and various phases associated with them. Also CP_2 Killing vector fields can be projected to space-time surface and give a representation for classical gluon fields. These in turn can be contracted with M^4 Killing vectors giving rise to gluon fields as analogs of graviton fields but with second polarization index replaced with color index.

2. The standard definition for the correlation functions associated with classical time evolution is the appropriate starting point. The correlation function $G_{XY}(\tau)$ for two dynamical variables

$X(t)$ and $Y(t)$ is defined as the average $G_{XY}(\tau) = \int_T X(t)Y(t+\tau)dt/T$ over an interval of length T, and one can also consider the limit $T \rightarrow \infty$. In the recent case one would replace τ with the difference $m_1 - m_2 = m$ of M^4 coordinates of two points at the preferred extremal and integrate over the points of the extremal to get the average. The finite time interval T is replaced with the volume of causal diamond in a given length scale. Zero energy state with given quantum numbers for positive and negative energy parts of the state defines the initial and final states between which the fields appearing in the correlation functions are defined.

3. What correlation functions should be considered? Certainly one could calculate correlation functions for the induced spinor connection given electro-weak propagators and correlation functions for CP_2 Killing vector fields giving correlation functions for gluon fields using the description in terms of Killing vector fields. If one can uniquely separate from the Fourier transform uniquely a term of form $Z/(p^2 - m^2)$ by its momentum dependence, the coefficient Z can be identified as coupling constant squared for the corresponding gauge potential component and one can in principle deduce coupling constant evolution purely classically. One can imagine of calculating spinorial propagators for string world sheets in the same manner. Note that also the dependence on color quantum numbers would be present so that in principle all that is needed could be calculated for a single preferred extremal without the need to construct QFT limit and to introduce color quantum numbers of fermions as spin like quantum numbers (color quantum numbers corresponds to CP_2 partial wave for the tip of the CD assigned with the particle).

Many detailed speculations about coupling constant evolution to be discussed in the sections below must be taken as innovative guesses doomed to have the eventual fate of guesses. The notion of quantum ergodicity could however be one of the really deep ideas about coupling constant evolution comparable to the notion of p-adic coupling constant evolution. Quantum Ergodicity (briefly QE) would also state something extremely non-trivial also about the construction of correlation functions and S-matrix. Because this principle is so new, the rest of the chapter does not yet contain any applications of QE. This should not lead the reader to under-estimate the potential power of QE.

3.3.5 Are Both Symplectic And Conformal Field Theories Needed In TGD Framework?

Before one can say anything quantitative about coupling constant evolution, one must have a formulation for its TGD counterpart and thus also a more detailed formulation for how to calculate M-matrix elements. There is also the question about infinities. By very general arguments infinities of quantum field theories are predicted to cancel in TGD Universe - basically by the non-locality of Kähler function as a functional of 3-surface and by the general properties of the vacuum functional identified as the exponent of Kähler function. The precise mechanism leading to the cancellation of infinities of local quantum field theories has remained unspecified. Only the realization that the symplectic invariance of quantum TGD provides a mechanism regulating the short distance behavior of N-point functions changed the situation in this respect. This also leads to one possible concrete view about the generalized Feynman diagrams giving M-matrix elements and at least a resemblance with ordinary Feynman diagrammatics.

It must be of course admitted that there are several apparentely competing visions. Twistorial vision [K101] and the vision about scattering amplitudes as representations for sequences of algebraic operations in super-symplectic Yangian [A17] [B18, B14, B15] seem to be consistent views. Symplectic approach seems to be suitable to understand the integration over WCW zero mode degrees of freedom not included in the other approaches.

Symplectic invariance

Symplectic symmetries of $\delta M^4_+ \times CP_2$ (light-cone boundary briefly) act as isometries of the "world of classical worlds". One can see these symmetries as analogs of Kac-Moody type symmetries with symplectic transformations of $S^2 \times CP_2$, where S^2 is $r_M = constant$ sphere of light-cone boundary, made local with respect to the light-like radial coordinate r_M taking the role of complex coordinate. Thus finite-dimensional Lie group G is replaced with infinite-dimensional group of symplectic transformations. This inspires the question whether a symplectic analog of conformal field theory at $\delta M^4_+ \times CP_2$ could be relevant for the construction of n-point functions in quantum TGD and what general properties these n-point functions would have. This section appears already in the previous chapter about symmetries of quantum TGD [K25] but because the results of the section provide the first concrete construction recipe of M-matrix in zero energy ontology, it is included also in this chapter.

Symplectic QFT at sphere

Actually the notion of symplectic QFT emerged as I tried to understand the properties of cosmic microwave background which comes from the sphere of last scattering which corresponds roughly to the age of 5×10^5 years [K71]. In this situation vacuum extremals of Kähler action around almost unique critical Robertson-Walker cosmology imbeddable in $M^4 \times S^2$, where there is homologically trivial geodesic sphere of CP_2. Vacuum extremal property is satisfied for any space-time surface which is surface in $M^4 \times Y^2$, Y^2 a Lagrangian sub-manifold of CP_2 with vanishing induced Kähler form. Symplectic transformations of CP_2 and general coordinate transformations of M^4 are dynamical symmetries of the vacuum extremals so that the idea of symplectic QFT emerges natural. Therefore I shall consider first symplectic QFT at the sphere S^2 of last scattering with temperature fluctution $\Delta T/T$ proportional to the fluctuation of the metric component g_{aa} in Robertson-Walker coordinates.

1. In quantum TGD the symplectic transformation of the light-cone boundary would induce action in the "world of classical worlds" (light-like 3-surfaces). In the recent situation it is convenient to regard perturbations of CP_2 coordinates as fields at the sphere of last scattering (call it S^2) so that symplectic transformations of CP_2 would act in the field space whereas those of S^2 would act in the coordinate space just like conformal transformations. The deformation of the metric would be a symplectic field in S^2. The symplectic dimension would be induced by the tensor properties of R-W metric in R-W coordinates: every S^2 coordinate index would correspond to one unit of symplectic dimension. The symplectic invariance in CP_2 degrees of freedom is guaranteed if the integration measure over the vacuum deformations is symplectic invariant. This symmetry does not play any role in the sequel.

2. For a symplectic scalar field $n \geq 3$-point functions with a vanishing anomalous dimension would be functions of the symplectic invariants defined by the areas of geodesic polygons defined by subsets of the arguments as points of S^2. Since n-polygon can be constructed from 3-polygons these invariants can be expressed as sums of the areas of 3-polygons expressible in terms of symplectic form. n-point functions would be constant if arguments are along geodesic circle since the areas of all sub-polygons would vanish in this case. The decomposition of n-polygon to 3-polygons brings in mind the decomposition of the n-point function of conformal field theory to products of 2-point functions by using the fusion algebra of conformal fields (very symbolically $\Phi_k\Phi_l = c^m_{kl}\Phi_m$). This intuition seems to be correct.

3. Fusion rules stating the associativity of the products of fields at different points should generalize. In the recent case it is natural to assume a non-local form of fusion rules given in the case of symplectic scalars by the equation

$$\Phi_k(s_1)\Phi_l(s_2) = \int c^m_{kl} f(A(s_1,s_2,s_3))\Phi_m(s) d\mu_s \quad . \tag{3.3.1}$$

   Here the coefficients c^m_{kl} are constants and $A(s_1, s_2, s_3)$ is the area of the geodesic triangle of S^2 defined by the sympletic measure and integration is over S^2 with symplectically invariant measure $d\mu_s$ defined by symplectic form of S^2. Fusion rules pose powerful conditions on n-point functions and one can hope that the coefficients are fixed completely.

4. The application of fusion rules gives at the last step an expectation value of 1-point function of the product of the fields involves unit operator term $\int c_{kl} f(A(s_1, s_2, s))Idd\mu_s$ so that one has

$$\langle \Phi_k(s_1)\Phi_l(s_2)\rangle = \int c_{kl} f(A(s_1,s_2,s))d\mu_s \quad . \tag{3.3.2}$$

Hence 2-point function is average of a 3-point function over the third argument. The absence of non-trivial symplectic invariants for 1-point function means that $n = 1$- an are constant, most naturally vanishing, unless some kind of spontaneous symmetry breaking occurs. Since the function $f(A(s_1, s_2, s_3))$ is arbitrary, 2-point correlation function can have both signs. 2-point correlation function is invariant under rotations and reflections.

Symplectic QFT with spontaneous breaking of rotational and reflection symmetries

CMB data suggest breaking of rotational and reflection symmetries of S^2. A possible mechanism of spontaneous symmetry breaking is based on the observation that in TGD framework the hierarchy of Planck constants assigns to each sector of the generalized embedding space a preferred quantization axes. The selection of the quantization axis is coded also to the geometry of "world of classical worlds", and to the quantum fluctuations of the metric in particular. Clearly, symplectic QFT with spontaneous symmetry breaking would provide the sought-for really deep reason for the quantization of Planck constant in the proposed manner.

1. The coding of angular momentum quantization axis to the generalized embedding space geometry allows to select South and North poles as preferred points of S^2. To the three arguments s_1, s_2, s_3 of the 3-point function one can assign two squares with the added point being either North or South pole. The difference

$$\Delta A(s_1,s_2,s_3) \equiv A(s_1,s_2,s_3,N) - A(s_1,s_2,s_3,S) \tag{3.3.3}$$

of the corresponding areas defines a simple symplectic invariant breaking the reflection symmetry with respect to the equatorial plane. Note that ΔA vanishes if arguments lie along a geodesic line or if any two arguments co-incide. Quite generally, symplectic QFT differs from conformal QFT in that correlation functions do not possess singularities.

2. The reduction to 2-point correlation function gives a consistency conditions on the 3-point functions

$$\begin{aligned}\langle(\Phi_k(s_1)\Phi_l(s_2))\Phi_m(s_3)\rangle &= c^r_{kl}\int f(\Delta A(s_1,s_2,s))\langle\Phi_r(s)\Phi_m(s_3)\rangle d\mu_s \\ &= \end{aligned} \tag{3.3.4}$$

$$c^r_{kl}c_{rm}\int f(\Delta A(s_1,s_2,s))f(\Delta A(s,s_3,t))d\mu_s d\mu_t \quad . \tag{3.3.5}$$

Associativity requires that this expression equals to $\langle\Phi_k(s_1)(\Phi_l(s_2)\Phi_m(s_3))\rangle$ and this gives additional conditions. Associativity conditions apply to $f(\Delta A)$ and could fix it highly uniquely.

3. 2-point correlation function would be given by

$$\langle \Phi_k(s_1)\Phi_l(s_2)\rangle = c_{kl}\int f(\Delta A(s_1,s_2,s))d\mu_s \tag{3.3.6}$$

4. There is a clear difference between $n > 3$ and $n = 3$ cases: for $n > 3$ also non-convex polygons are possible: this means that the interior angle associated with some vertices of the polygon is larger than π. $n = 4$ theory is certainly well-defined, but one can argue that so are also $n > 4$ theories and skeptic would argue that this leads to an inflation of theories. TGD however allows only finite number of preferred points and fusion rules could eliminate the hierarchy of theories.

5. To sum up, the general predictions are following. Quite generally, for $f(0) = 0$ n-point correlation functions vanish if any two arguments co-incide which conforms with the spectrum of temperature fluctuations. It also implies that symplectic QFT is free of the usual singularities. For symmetry breaking scenario 3-point functions and thus also 2-point functions vanish also if s_1 and s_2 are at equator. All these are testable predictions using ensemble of CMB spectra.

Generalization to quantum TGD

(Number theoretic) braids are identifiable as boundaries of string world sheets at which the modes of induced spinor fields are localized in the generic case in Minkowskian space-time regions. Fundamental fermions can be assigned to these lines. Braids are the basic objects of quantum TGD, one can hope that the n-point functions assignable to them could code the properties of ground states and that one could separate from n-point functions the parts which correspond to the symplectic degrees of freedom acting as symmetries of vacuum extremals and isometries of the "world of classical worlds".

1. This approach indeed seems to generalize also to quantum TGD proper and the n-point functions associated with partonic 2-surfaces can be decomposed in such a way that one obtains coefficients which are symplectic invariants associated with both S^2 and CP_2 Kähler form.

2. Fusion rules imply that the gauge fluxes of respective Kähler forms over geodesic triangles associated with the S^2 and CP_2 projections of the arguments of 3-point function serve basic building blocks of the correlation functions. The North and South poles of S^2 and three poles of CP_2 can be used to construct symmetry breaking n-point functions as symplectic invariants. Non-trivial 1-point functions vanish also now.

3. The important implication is that n-point functions vanish when some of the arguments co-incide. This might play a crucial role in taming of the singularities: the basic general prediction of TGD is that standard infinities of local field theories should be absent and this mechanism might realize this expectation.

Next some more technical but elementary first guesses about what might be involved.

1. It is natural to introduce the moduli space for n-tuples of points of the symplectic manifold as the space of symplectic equivalence classes of n-tuples. In the case of sphere S^2 convex n-polygon allows $n + 1$ 3-sub-polygons and the areas of these provide symplectically invariant coordinates for the moduli space of symplectic equivalence classes of n-polygons (2^n-D space of polygons is reduced to $n + 1$-D space). For non-convex polygons the number of 3-sub-polygons is reduced so that they seem to correspond to lower-dimensional sub-space. In the case of CP_2 n-polygon allows besides the areas of 3-polygons also 4-volumes of 5-polygons as fundamental symplectic invariants. The number of independent 5-polygons for n-polygon can be obtained by using induction: once the numbers $N(k, n)$ of independent $k \leq n$-simplices are known for n-simplex, the numbers of $k \leq n + 1$-simplices for $n + 1$-polygon are obtained by adding one vertex so that by little visual gymnastics the numbers $N(k, n+1)$ are given by $N(k, n+1) = N(k-1, n)+N(k, n)$. In the case of CP_2 the allowance of 3 analogs $\{N, S, T\}$ of North and South poles of S^2 means that besides the areas of polygons (s_1, s_2, s_3), (s_1, s_2, s_3, X), (s_1, s_2, s_3, X, Y), and (s_1, s_2, s_3, N, S, T) also the 4-volumes of 5-polygons (s_1, s_2, s_3, X, Y), and of 6-polygon (s_1, s_2, s_3, N, S, T), $X, Y \in \{N, S, T\}$ can appear as additional arguments in the definition of 3-point function.

2. What one really means with symplectic tensor is not clear since the naïve first guess for the n-point function of tensor fields is not manifestly general coordinate invariant. For instance, in the model of CMB, the components of the metric deformation involving S^2 indices would be symplectic tensors. Tensorial n-point functions could be reduced to those for scalars obtained as inner products of tensors with Killing vector fields of $SO(3)$ at S^2. Again a preferred choice of quantization axis would be introduced and special points would correspond to the singularities of the Killing vector fields.

The decomposition of Hamiltonians of the "world of classical worlds" expressible in terms of Hamiltonians of $S^2 \times CP_2$ to irreps of $SO(3)$ and $SU(3)$ could define the notion of symplectic tensor as the analog of spherical harmonic at the level of WCW . Spin and gluon color would have natural interpretation as symplectic spin and color. The infinitesimal action of various Hamiltonians on n-point functions defined by Hamiltonians and their super counterparts is well-defined and group theoretical arguments allow to deduce general form of n-point functions in terms of symplectic invariants.

3. The need to unify p-adic and real physics by requiring them to be completions of rational physics, and the notion of finite measurement resolution suggest that discretization of also fusion algebra is necessary. The set of points appearing as arguments of n-point functions could be finite in a given resolution so that the p-adically troublesome integrals in the formulas for the fusion rules would be replaced with sums. Perhaps rational/algebraic variants of $S^2 \times CP_2 = SO(3)/SO(2) \times SU(3)/U(2)$ obtained by replacing these groups with their rational/algebraic variants are involved. Tedrahedra, octahedra, and dodecahedra suggest themselves as simplest candidates for these discretized spaces. Also the symplectic moduli space would be discretized to contain only n-tuples for which the symplectic invariants are numbers in the allowed algebraic extension of rationals. This would provide an abstract looking but actually very concrete operational approach to the discretization involving only areas of n-tuples as internal coordinates of symplectic equivalence classes of n-tuples. The best that one could achieve would be a formulation involving nothing below measurement resolution.

4. This picture based on elementary geometry might make sense also in the case of conformal symmetries. The angles associated with the vertices of the S^2 projection of n-polygon could define conformal invariants appearing in n-point functions and the algebraization of the corresponding phases would be an operational manner to introduce the space-time correlates for the roots of unity introduced at quantum level. In CP_2 degrees of freedom the projections of n-tuples to the homologically trivial geodesic sphere S^2 associated with the particular sector of CH would allow to define similar conformal invariants. This framework gives dimensionless areas (unit sphere is considered). p-Adic length scale hypothesis and hierarchy of Planck constants would bring in the fundamental units of length and time in terms of CP_2 length.

The recent view about M-matrix described in [K24] is something almost unique determined by Connes tensor product providing a formal realization for the statement that complex rays of state space are replaced with $\mathcal{N}$ rays where $\mathcal{N}$ defines the hyper-finite sub-factor of type II_1 defining the measurement resolution. M-matrix defines time-like entanglement coefficients between positive and negative energy parts of the zero energy state and need not be unitary. It is identified as square root of density matrix with real expressible as product of of real and positive square root and unitary S-matrix. This S-matrix is what is measured in laboratory. There is also a general vision about how vertices are realized: they correspond to light-like partonic 3-surfaces obtained by gluing incoming and outgoing partonic 3-surfaces along their ends together just like lines of Feynman diagrams. Note that in string models string world sheets are non-singular as 2-manifolds whereas 1-dimensional vertices are singular as 1-manifolds. These ingredients we should be able to fuse together. So we try once again!

1. *Iteration* starting from vertices and propagators is the basic approach in the construction of n-point function in standard QFT. This approach does not work in quantum TGD. Symplectic and conformal field theories suggest that *recursion* replaces iteration in the construction. One starts from an n-point function and reduces it step by step to a vacuum expectation value of a 2-point function using fusion rules. Associativity becomes the fundamental dynamical principle in this process. Associativity in the sense of classical number fields has already shown its power and led to a hyper-octoninic formulation of quantum TGD promising a unification of various visions about quantum TGD [K96].

2. Let us start from the representation of a zero energy state in terms of a causal diamond defined by future and past directed light-cones. Zero energy state corresponds to a quantum

superposition of light-like partonic 3-surfaces each of them representing possible particle reaction. These 3-surfaces are very much like generalized Feynman diagrams with lines replaced by light-like 3-surfaces coming from the upper and lower light-cone boundaries and glued together along their ends at smooth 2-dimensional surfaces defining the generalized vertices.

3. It must be emphasized that the generalization of ordinary Feynman diagrammatics arises and conformal and symplectic QFTs appear only in the calculation of single generalized Feynman diagram. Therefore one could still worry about loop corrections. The fact that no integration over loop momenta is involved and there is always finite cutoff due to discretization together with recursive instead of iterative approach gives however good hopes that everything works. Note that this picture is in conflict with one of the earlier approaches based on positive energy ontology in which the hope was that only single generalized Feynman diagram could define the U-matrix thought to correspond directly to physical S-matrix at that time.

4. One can actually simplify things by identifying generalized Feynman diagrams as maxima of Kähler function with functional integration carried over perturbations around it. Thus one would have conformal field theory in both fermionic and WCW degrees of freedom. The light-like time coordinate along light-like 3-surface is analogous to the complex coordinate of conformal field theories restricted to some curve. If it is possible continue the light-like time coordinate to a hyper-complex coordinate in the interior of 4-D space-time sheet, the correspondence with conformal field theories becomes rather concrete. Same applies to the light-like radial coordinates associated with the light-cone boundaries. At light-cone boundaries one can apply fusion rules of a symplectic QFT to the remaining coordinates. Conformal fusion rules are applied only to point pairs which are at different ends of the partonic surface and there are no conformal singularities since arguments of n-point functions do not co-incide. By applying the conformal and symplectic fusion rules one can eventually reduce the n-point function defined by the various fermionic and bosonic operators appearing at the ends of the generalized Feynman diagram to something calculable.

5. Finite measurement resolution defining the Connes tensor product is realized by the discretization applied to the choice of the arguments of n-point functions so that discretion is not only a space-time correlate of finite resolution but actually defines it. No explicit realization of the measurement resolution algebra $\mathcal{N}$ seems to be needed. Everything should boil down to the fusion rules and integration measure over different 3-surfaces defined by exponent of Kähler function and by imaginary exponent of Chern-Simons action. The continuation of WCW Clifford algebra for 3-surfaces with cm degrees of freedom fixed to a hyper-octonionic variant of gamma matrix field of super-string models defined in M^8 (hyper-octonionic space) and $M^8 \leftrightarrow M^4 \times CP_2$ duality leads to a unique choice of the points, which can contribute to n-point functions as intersection of M^4 subspace of M^8 with the counterparts of partonic 2-surfaces at the boundaries of light-cones of M^8. Therefore there are hopes that the resulting theory is highly unique. Symplectic fusion algebra reduces to a finite algebra for each space-time surface if this picture is correct.

6. Consider next some of the details of how the light-like 3-surface codes for the fusion rules associated with it. The intermediate partonic 2- surfaces must be involved since otherwise the construction would carry no information about the properties of the light-like 3-surface, and one would not obtain perturbation series in terms of the relevant coupling constants. The natural assumption is that partonic 2-surfaces belong to future/past directed light-cone boundary depending on whether they are on lower/upper half of the causal diamond. Hyper-octonionic conformal field approach fixes the n_{int} points at intermediate partonic two-sphere for a given light-like 3-surface representing generalized Feynman diagram, and this means that the contribution is just N-point function with $N = n_{out} + n_{int} + n_{in}$ calculable by the basic fusion rules. Coupling constant strengths would emerge through the fusion coefficients, and at least in the case of gauge interactions they must be proportional to Kähler coupling strength since n-point functions are obtained by averaging over small deformations with vacuum functional given by the exponent of Kähler function. The first guess is that one can identify the spheres $S^2 \subset \delta M^4_{\pm}$ associated with initial, final and, and intermediate states so that symplectic n-points functions could be calculated using single sphere.

These findings raise the hope that quantum TGD is indeed a solvable theory. The coupling constant evolution is based on the same mechanism as in QFT and symplectic invariance replaces ad hoc UV cutoff with a genuine dynamical regulation mechanism. Causal diamond itself defines the physical IR cutoff. p-Adic and real coupling constant evolutions reflect the underlying evolution in powers of two for the temporal distance between the tips of the light-cones of the causal diamond and the association of macroscopic time scale as secondary p-adic time scale to elementary particles (.1 seconds for electron) serves as a first test for the picture. Even if one is not willing to swallow any bit of TGD, the classification of the symplectic QFTs remains a fascinating mathematical challenge in itself. A further challenge is the fusion of conformal QFT and symplectic QFT in the construction of n-point functions. One might hope that conformal and symplectic fusion rules could be treated independently.

More detailed view about the construction of M-matrix elements

After three decades there are excellent hopes of building an explicit recipe for constructing M-matrix elements but the devil is in the details.

1. Elimination of infinities and coupling constant evolution

The elimination of infinities could follow from the symplectic QFT part of the theory. The symplectic contribution to n-point functions vanishes when two arguments co-incide. The UV cancellation mechanism has nothing to do with the finite measurement resolution which corresponds to the size of the causal diamonds inside which the space-time sheets representing radiative corrections are. There is also IR cutoff due to the presence of largest causal diamond.

On can decompose the radiative corrections two two types. First kind of corrections appear both at the level of positive/and negative energy parts of zero energy states. Second kind of corrections appear at the level of interactions between them. This decomposition is standard in quantum field theories and corresponds to the renormalization constants of fields *resp.* renormalization of coupling constants. The corrections due to the increase of measurement resolution in time comes as very specific corrections to positive and negative energy states involving gluing of smaller causal diamonds to the upper and lower boundaries of causal diamonds along any radial light-like ray. The radiative corresponds to the interactions correspond to the addition of smaller causal diamonds in the interior of the larger causal diamond. Scales for the corrections come as scalings in powers of 2 rather than as continuous scaling of measurement resolution.

UV finiteness is suggested also by the generalized Feynman rules providing a phenomenological view about what TGD predicts. According to these rules fundamental fermions propagate like massless particles. In twistor Grassmann approach residue integration is expected to reduce internal fermion lines to on mass shell propagation with non-physical helicity. The fundamental 4-fermion interaction is assignable to wormhole contact and corresponds to stringy exchange of four-momentum with propagator being defined by the inverse of super-conformal scaling generator $1/L_0$. Wormhole contacts carrying fermion and antifermion at their throats behave like fundamental bosons. Stringy propagators at wormhole contacts make TGD rules a hybrid of Feynmann and stringy rules. Stringy propagators are necessary in order to avoid logarithmic divergences. Higher mass excitations crucial for finiteness belong to the representations of super-conformal algebra and can be regarded as bound states of massless fermions. Massivation of external particles allows to avoid infrared divergences. Not only physical bosons but also physical fermions emerge from fundamental massless fermions.

2. Conformal symmetries

The basic questions are the following ones. How hyper-octonionic/-quaternionic/-complex super-conformal symmetry relates to the super-symplectic conformal symmetry at the embedding space level and the super Kac-Moody symmetry associated with the light-like 3-surfaces? How do the dual $HO = M^8$ and $H = M^4 \times CP_2$ descriptions (number theoretic compactifcation) relate?

Concerning the understanding of these issues, the earlier construction of physical states poses strong constraints [K25].

1. The state construction utilizes both super-symplectic and super Kac-Moody algebras. super-symplectic algebra has negative conformal weights and creates tachyonic ground states from

which Super Kac-Moody algebra generates states with non-negative conformal weight determining the mass squared value of the state. The commutator of these two algebras annihilates the physical states. This requires that both super conformal algebras must allow continuation to hyper-octonionic algebras, which are independent.

2. The light-like radial coordinate at $\delta M^4_\pm$ can be continued to a hyper-complex coordinate in $M^2_\pm$ defined the preferred commutative plane of non-physical polarizations, and also to a hyper-quaternionic coordinate in $M^4_\pm$. Hence it would seem that super-symplectic algebra can be continued to an algebra in $M^2_\pm$ or perhaps in the entire $M^4_\pm$. This would allow to continue also the operators G, L and other super-symplectic operators to operators in hyper-quaternionic $M^4_\pm$ needed in stringy perturbation theory.

3. Also the super KM algebra associated with the light-like 3-surfaces should be continueable to hyper-quaternionic $M^4_\pm$. Here $HO-H$ duality comes in rescue. It requires that the preferred hyper-complex plane M^2 is contained in the tangent plane of the space-time sheet at each point, in particular at light-like 3-surfaces. We already know that this allows to assign a unique space-time surface to a given collection of light-like 3-surfaces as hyper-quaternionic 4-surface of HO hypothesized to correspond to (an obviously preferred) extremal of Kähler action. An equally important implication is that the light-like coordinate of X^3 can be continued to hyper-complex coordinate M^2 coordinate and thus also to hyperquaternionic M^4 coordinate.

4. The four-momentum appears in super generators G_n and L_n. It seems that the formal Fourier transform of four-momentum components to gradient operators to $M^4_\pm$ is needed and defines these operators as particular elements of the WCW Clifford algebra elements extended to fields in embedding space.

3. What about stringy perturbation theory?

The analog of stringy perturbation theory does not seems only a highly attractive but also an unavoidable outcome since a generalization of massless fermionic propagator is needed. The inverse for the sum of super Kac-Moody and super-symplectic super-Virasoro generators G (L) extended to an operator acting on the difference of the M^4 coordinates of the end points of the propagator line connecting two partonic 2-surfaces should appear as fermionic (bosonic) propagator in stringy perturbation theory. Virasoro conditions imply that only G_0 and L_0 appear as propagators. Momentum eigenstates are not strictly speaking possible since since discretization is present due to the finite measurement resolution. One can however represent these states using Fourier transform as a superposition of momentum eigenstates so that standard formalism can be applied.

Symplectic QFT gives an additional multiplicative contribution to n-point functions and there would be also braiding S-matrices involved with the propagator lines in the case that partonic 2-surface carriers more than 1 point. This leaves still modular degrees of freedom of the partonic 2-surfaces describable in terms of elementary particle vacuum functionals and the proper treatment of these degrees of freedom remains a challenge.

4. What about non-hermiticity of the WCW super-generators carrying fermion number?

TGD represents also a rather special challenge, which actually represents the fundamental difference between quantum TGD and super string models. The assignment of fermion number to WCW gamma matrices and thus also to the super-generator G is unavoidable. Also M^4 and H gamma matrices carry fermion number. This has been a long-standing interpretational problem in quantum TGD and I have been even ready to give up the interpretation of four-momentum operator appearing in G_n and L_n as actual four-momenta. The manner to get rid of this problem would be the assumption of Majorana property but this would force to give up the interpretation of different embedding space chiralities in terms of conserved lepton and quark numbers and would also lead to super-string theory with critical dimension 10 or 11. A further problem is how to obtain amplitudes which respect fermion number conservation using string perturbation theory if $1/G = G^\dagger/L_0$ carries fermion number.

The recent picture does not leave many choices so that I was forced to face the truth and see how everything falls down to this single nasty detail! It became as a total surprise that gamma matrices carrying fermion number do not cause any difficulties in zero energy ontology and make sense even in the ordinary Feynman diagrammatics.

1. Non-hermiticity of G means that the center of mass terms CH gamma matrices must be distinguished from their Hermitian conjugates. In particular, one has $\gamma_0 \neq \gamma_0^dagger$. One can interpret the fermion number carrying M^4 gamma matrices of the complexified quaternion space.

2. One might think that $M^4 \times CP_2$ gamma matrices carrying fermion number is a catastrophe but this is not the case in massless theory. Massless momentum eigen states can be created by the operator $p^k\gamma_k^\dagger$ from a vacuum annihilated by gamma matrices and satisfying massless Dirac equation. The conserved fermion number defined by the integral of $\overline{\Psi}\gamma^0\Psi$ over 3-space gives just its standard value. A further experimentation shows that Feynman diagrams with non-hermitian gamma matrices give just the standard results since ordinary fermionic propagator and boson-emission vertices at the ends of the line containing WCW gamma matrix and its conjugate give compensating fermion numbers [K101].

3. If the theory would contain massive fermions or a coupling to a scalar Higgs, a catastrophe would result. Hence ordinary Higgs mechanism is not possible in this framework. Of course, also the quantization of fermions is totally different. In TGD fermion mass is not a scalar in H. Part of it is given by CP_2 Dirac operator, part by p-adic thermodynamics for L_0, and part by Higgs field which behaves like vector field in CP_2 degrees of freedom, so that the catastrophe is avoided.

4. In zero energy ontology zero energy states are characterized by M-matrix elements constructed by applying the combination of stringy and symplectic Feynman rules and fermionic propagator is replaced with its super-conformal generalization reducing to an ordinary fermionic propagator for massless states. The norm of a single fermion state is given by a propagator connecting positive energy state and its conjugate with the propagator G_0/L_0 and the standard value of the norm is obtained by using Dirac equation and the fact that Dirac operator appears also in G_0.

5. The hermiticity of super-generators G would require Majorana property and one would end up with superstring theory with critical dimension $D = 10$ or $D = 11$ for the embedding space. Hence the new interpretation of gamma matrices, proposed already years ago, has very profound consequences and convincingly demonstrates that TGD approach is indeed internally consistent.

In this framework coupling constant evolution would correspond evolution as a function of the scale of CD. It might have interpretation also in terms of addition of intermediate zero energy states corresponding to the generalized Feynman diagrams obtained by the insertion of causal diamonds with a new shorter time scale $T = T_{prev}/2$ to the previous Feynman diagram as the size of CD is increased. p-Adic length scale hypothesis follows naturally. A very close correspondence with ordinary Feynman diagrammatics arises and and ordinary vision about coupling constant evolutions arises. The absence of infinities follows from the symplectic invariance which is genuinely new element. p-Adic and real coupling constant evolutions can be seen as completions of coupling constant evolutions for physics based on rationals and their algebraic extensions.

3.4 Number Theoretic Compactification And $M^8 - H$ Duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally $M^8 - H$ duality was introduced as a number theoretic explanation for $H = M^4 \times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4 \subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M^8_c using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of M^8-H duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of M^8-H duality.

The strong form M^8-H duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M^8_c composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M^8_c identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M^8_c must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere K̈hler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3)\times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4\times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering M^8-H duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M^8_c=O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under multiplication. One can however however define the notion of associativity for the subspace of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q=q_0+q^kI_k$. Hyper-quaternions are expressible as $q=q_0+jq^kI_k$ and form a subspace of complexified quaternions $Q_c=Q\oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O\oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H=q_0+jq^kI_k+jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4 \subset M^8$ implies unique decomposition $M^8 = M^4 \times E^4$ needed to define $M^8 - H$ duality uniquely. This applies also to M^8_c. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCW s associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M^8_c) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

   The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *embedding space level* and *space-time level*. One must have embedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \to H \to H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of Kähler-Dirac gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the Kähler-Dirac gamma matrices need not span the entire tangent space. The space spanned by the Kähler-Dirac gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the Kähler-Dirac gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

   If the space spanned by Kähler-Dirac gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For Kähler-Dirac gamma matrices the notion of co-associativity can produce problems since Kähler-Dirac gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by Kähler-Dirac gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the Kähler-Dirac gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the Kähler-Dirac gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

3.4.1 Basic Idea Behind $M^8 - M^4 \times CP_2$ Duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different ways to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M^8_c. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M^8_c) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \rightarrow S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \rightarrow M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M^8_c: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M^8_c$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a way to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [K14]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [K111] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of Kähler-Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent plane makes it possible to define $M^8 - H$ duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

3.4.2 Hyper-Octonionic Pauli "Matrices" And The Definition Of Associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [K101, K10]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

   Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

   There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \rightarrow H \rightarrow H$.... This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and Kähler-Dirac gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

3.4.3 Are Kähler And Spinor Structures Necessary In M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the M^8-H duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naïve replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler

form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1+1+3+\overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1+kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2+y^2+z^2+t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1+ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of embedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

3.4.4 How Could One Solve Associativity/Co-Associativity Conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \rightarrow H \rightarrow H...$ iteration generating new solutions from existing ones.

Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O \oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1 + io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M^8_c but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M^8_c (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a,b,c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a way that one of the tangent vectors corresponds to real unit (in the embedding map embedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition looks very simple - perhaps too simple!- since it involves only first derivatives of the embedding space vectors.

   One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of embedding space coordinates (rather than involving embedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$\begin{aligned}
e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\
A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\
e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\
\Gamma_k &= e^A_k \gamma_A \ .
\end{aligned} \tag{3.4.1}$$

The very naïve idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{3.4.2}$$

One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see http://tinyurl.com/ya7h3n9z). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD}x^By^Cz^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [A50] (see **Fig. 8.1**) expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product e_1e_2 (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x)e_1e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$e_0 = a \times 1 + b^i e_i$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

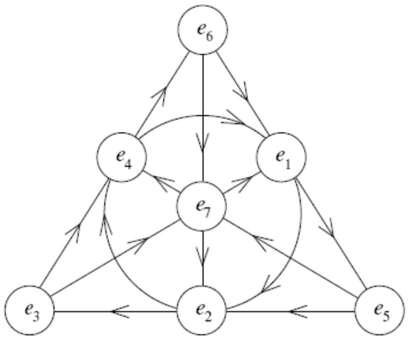

Figure 3.2: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

3.4.5 Quaternionicity At The Level Of Embedding Space Quantum Numbers

From the multiplication table of octonions as illustrated by Fano triangle [A50] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2: s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

3.4.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of $M^8 - H$ duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the Kähler-Dirac gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k\gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the Kähler-Dirac gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection Kähler-Dirac gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2 \times S^2 \subset M^4 \times CP_2$. It seems that associativity is satisfied by all known extremals. Hence Kähler-Dirac gamma matrices are flexible enough to realize associativity in H.

3. Kähler-Dirac gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner $M^8 - H$ duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the Kähler-Dirac gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand $M^8 - H$ correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does $M^8 - H$ correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

$M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since WCW degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K64].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

3.4.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \to H \to H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

3.5 Octo-Twistors And Twistor Space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall $M^8 - H$ duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [K101]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic embedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

3.5.1 Two ways To Twistorialize Embedding Space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1)\times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably, M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-definec operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4\times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

   Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are “real” and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and embedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

3.5.2 Octotwistorialization Of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor

octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

3.5.3 Octonionicity, $SO(1,7)$, G_2, And Non-Associative Malcev Group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1, 7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \rightarrow G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (`http://tinyurl.com/ybd4lcpy`). The basic observation is that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14 + 14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a,b] = ab-ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}\left[\gamma_k, \gamma_l\right] \quad . \tag{3.5.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

   This algebra has decomposition $g = h + t$, $[h,t] \subset t$, $[t,t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1,7)$.

3.5.4 Octonionic Spinors In M^8 And Real Complexified-Quaternionic Spinors In H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.

2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

   The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

   One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc) - (ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

3.5.5 What The Replacement Of $SO(7,1)$ Sigma Matrices With Octonionic Sigma Matrices Could Mean?

The basic implication of octonionization is the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4 \times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4 \times CP_2$ but interpret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ , \ \gamma^i = \gamma^i \otimes \sigma_2 \ , \ i = 1,..,7 \ . \tag{3.5.2}$$

7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1, ..., 6 \ \ , \ \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ \ . \tag{3.5.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \otimes \sigma_1 \ \ , \ \ \gamma_i = e_i \otimes \sigma_2 \ \ . \tag{3.5.4}$$

where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = je_i \times \sigma_3 \ \ , \ \ \Sigma_{ij} = jf_{ij}{}^k e_k \otimes 1 \ \ . \tag{3.5.5}$$

Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A9] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [K15]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the Kähler-Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8?

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://tinyurl.com/y93aerea`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of hyper-complex numbers) such that the second complexified Kähler-Dirac gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

   Does this mean that the difference between quarks and leptons becomes visible only at the embedding space level where ground states of super-conformal representations correspond to embedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

Octo-spinors and their relation to ordinary embedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\Psi_{L,i} = e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$
$$\Psi_{q,i} = e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \quad (3.5.6)$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\}\ , & e_R \text{ and } \nu_R \text{ with spin } 1/2\ ,\\ \{e_2 \pm ie_3\}\ , & e_R \text{ and } \nu_L \text{ with spin -}1/2\ ,\\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2\ ,\\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2\ . \end{array} \quad (3.5.7)$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic Kähler-Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

3.5.6 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K101] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

The case of $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K96] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?

2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.

4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K25].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?

2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as sub-space of complexified octonions.

3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

   (a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

   (b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

(c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

3.6 An attempt to understand preferred extremals of Kähler action

Preferred extremal of Kähler action is one of the basic poorly defined notions of TGD. There are pressing motivations for understanding what "preferred" really means. For instance, the conformal invariance of string models naturally generalizes to 4-D invariance defined by quantum Yangian of quantum affine algebra (Kac-Moody type algebra) characterized by two complex coordinates and therefore explaining naturally the effective 2-dimensionality [K28]. The problem is however how to assign a complex coordinate with the string world sheet having Minkowskian signature of metric. One can hope that the understanding of preferred extremals could allow to identify two preferred complex coordinates whose existence is also suggested by number theoretical vision giving preferred role for the rational points of partonic 2-surfaces in preferred coordinates. The best one could hope is a general solution of field equations in accordance with the hints that TGD is integrable quantum theory.

3.6.1 What "preferred" could mean?

The first question is what preferred extremal could mean.

1. In positive energy ontology preferred extremal would be a space-time surface assignable to given 3-surface and unique in the ideal situation: since one cannot pose conditions to the normal derivatives of embedding space coordinates at 3-surface, there is infinity of extremals. Some additional conditions are required and space-time surface would be analogous to Bohr orbit: hence the attribute "preferred". The problem would be to understand what "preferred" could mean. The non-determinism of Kähler action however destroyed this dream in its original form and led to zero energy ontology (ZEO).

2. In ZEO one considers extremals as space-time surfaces connecting two space-like 3-surfaces at the boundaries. One might hope that these 4-surfaces are unique. The non-determinism of Kähler action suggests that this is not the case. At least there is conformal invariance respecting the light-likeness of the 3-D parton orbits at which the signature of the induced metric changes: the conformal transformations would leave the space-like 3-D ends or at least partonic 2-surfaces invariant. This non-determinism would correspond to quantum criticality.

3. Effective 2-dimensionality follows from strong form of general coordinate invariance (GCI) stating that light-like partonic orbits and space-like 3-surfaces at the ends of space-time surface are equivalent physically: partonic 2-surfaces and their 4-D tangent space data would determine everything. One can however worry about how effective 2-dimensionality relates to the fact that the modes of the induced spinor field are localized at string world sheets and partonic 2-surface. Are the tangent space data equivalent with the data characterizing string world sheets as surfaces carrying vanishing electroweak fields?

   There is however a problem: the hierarchy of Planck constants (dark matter) requires that the conformal equivalence classes of light-like surfaces must be counted as physical degrees of freedom so that either space-like or light-like surfaces do not seem to be quite enough.

   Should one then include also the light-like partonic orbits to the what one calls 3-surface? The resulting connected 3-surfaces would define analogs of Wilson loops. Could the conformal equivalence class of the preferred extremal be unique without any additional conditions? If so, one could get rid of the attribute "preferred". The fractal character of the many-sheeted space-time however suggests that one can have this kind of uniqueness only in given length scale resolution and that "radiative corrections" due to the non-determinism are always present.

These considerations show that the notion of preferred extremal is still far from being precisely defined and it is not even clear whether the attribute “preferred” is needed. If not then the question is what are the extremals of Kähler action.

3.6.2 What is known about extremals?

A lot is is known about properties of extremals and just by trying to integrate all this understanding, one might gain new visions. The problem is that all these arguments are heuristic and rely heavily on physical intuition. The following considerations relate to the space-time regions having Minkowskian signature of the induced metric. The attempt to generalize the construction also to Euclidian regions could be very rewarding. Only a humble attempt to combine various ideas to a more coherent picture is in question.

The core observations and visions are following.

1. Hamilton-Jacobi coordinates for M^4 (discussed in this chapter) define natural preferred coordinates for Minkowskian space-time sheet and might allow to identify string world sheets for X^4 as those for M^4. Hamilton-Jacobi coordinates consist of light-like coordinate m and its dual defining local 2-plane $M^2 \subset M^4$ and complex transversal complex coordinates $(w, \overline{w})$ for a plane E^2_x orthogonal to M^2_x at each point of M^4. Clearly, hyper-complex analyticity and complex analyticity are in question.

2. Space-time sheets allow a slicing by string world sheets (partonic 2-surfaces) labelled by partonic 2-surfaces (string world sheets).

3. The quaternionic planes of octonion space containing preferred hyper-complex plane are labelled by CP_2, which might be called CP_2^{mod} [K96]. The identification $CP_2 = CP_2^{mod}$ motivates the notion of $M^8 - -M^4 \times CP_2$ duality [K25]. It also inspires a concrete solution ansatz assuming the equivalence of two different identifications of the quaternionic tangent space of the space-time sheet and implying that string world sheets can be regarded as strings in the 6-D coset space $G_2/SU(3)$. The group G_2 of octonion automorphisms has already earlier appeared in TGD framework.

4. The duality between partonic 2-surfaces and string world sheets in turn suggests that the $CP_2 = CP_2^{mod}$ conditions reduce to string model for partonic 2-surfaces in $CP_2 = SU(3)/U(2)$. String model in both cases could mean just hypercomplex/complex analyticity for the coordinates of the coset space as functions of hyper-complex/complex coordinate of string world sheet/partonic 2-surface.

The considerations of this section lead to a revival of an old very ambitious and very romantic number theoretic idea.

1. To begin with express octonions in the form $o = q_1 + Iq_2$, where q_i is quaternion and I is an octonionic imaginary unit in the complement of fixed a quaternionic sub-space of octonions. Map preferred coordinates of $H = M^4 \times CP_2$ to octonionic coordinate, form an arbitrary octonion analytic function having expansion with real Taylor or Laurent coefficients to avoid problems due to non-commutativity and non-associativity. Map the outcome to a point of H to get a map $H \to H$. This procedure is nothing but a generalization of Wick rotation to get an 8-D generalization of analytic map.

2. Identify the preferred extremals of Kähler action as surfaces obtained by requiring the vanishing of the imaginary part of an octonion analytic function. Partonic 2-surfaces and string world sheets would correspond to commutative sub-manifolds of the space-time surface and of embedding space and would emerge naturally. The ends of braid strands at partonic 2-surface would naturally correspond to the poles of the octonion analytic functions. This would mean a huge generalization of conformal invariance of string models to octonionic conformal invariance and an exact solution of the field equations of TGD and presumably of quantum TGD itself.

3.6.3 Basic ideas about preferred extremals

The slicing of the space-time sheet by partonic 2-surfaces and string world sheets

The basic vision is that space-time sheets are sliced by partonic 2-surfaces and string world sheets. The challenge is to formulate this more precisely at the level of the preferred extremals of Kähler action.

1. Almost topological QFT property means that the Kähler action reduces to Chern-Simons terms assignable to 3-surfaces. This is guaranteed by the vanishing of the Coulomb term in the action density implied automatically if conserved Kähler current is proportional to the instanton current with proportionality coefficient some scalar function.

2. The field equations reduce to the conservation of isometry currents. An attractive ansatz is that the flow lines of these currents define global coordinates. This means that these currents are Beltrami flows [B8] so that corresponding 1-forms J satisfy the condition $J \wedge dJ = 0$. These conditions are satisfied if

$$J = \Phi \nabla \Psi$$

   hold true for conserved currents. From this one obtains that Ψ defines global coordinate varying along flow lines of J.

3. A possible interpretation is in terms of local polarization and momentum directions defined by the scalar functions involved and natural additional conditions are that the gradients of Ψ and Φ are orthogonal:

$$\nabla \Phi \cdot \nabla \Psi = 0 \ ,$$

   and that the Ψ satisfies massless d'Alembert equation

$$\nabla^2 \Psi = 0$$

   as a consequence of current conservation. If Ψ defines a light-like vector field - in other words

$$\nabla \Psi \cdot \nabla \Psi = 0 \ ,$$

   the light-like dual of Φ -call it Φ_c- defines a light-like like coordinate and Φ and Φ_c defines a light-like plane at each point of space-time sheet.

   If also Φ satisfies d'Alembert equation

$$\nabla^2 \Phi = 0 \ ,$$

   also the current

$$K = \Psi \nabla \Phi$$

   is conserved and its flow lines define a global coordinate in the polarization plane orthogonal to time-lik plane defined by local light-like momentum direction.

   If Φ allows a continuation to an analytic function of the transversal complex coordinate, one obtains a coordinatization of space-time surface by Ψ and its dual (defining hyper-complex coordinate) and $w, \overline{w}$. Complex analyticity and its hyper-complex variant would allow to provide space-time surface with four coordinates very much analogous with Hamilton-Jacobi coordinates of M^4.

   This would mean a decomposition of the tangent space of space-time surface to orthogonal planes defined by light-like momentum and plane orthogonal to it. If the flow lines of J defined Beltrami flow it seems that the distribution of momentum planes is integrable.

4. General arguments suggest that the space-time sheets allow a slicing by string world sheets parametrized by partonic 2-surfaces or vice versa. This would mean a intimate connection with the mathematics of string models. The two complex coordinates assignable to the Yangian of affine algebra would naturally relate to string world sheets and partonic 2-surfaces and the highly non-trivial challenge is to identify them appropriately.

Hamilton-Jacobi coordinates for M^4

The earlier attempts to construct preferred extremals [K14] led to the realization that so called Hamilton-Jacobi coordinates (m, w) for M^4 define its slicing by string world sheets parametrized by partonic 2-surfaces. m would be pair of light-like conjugate coordinates associated with an integrable distribution of planes M^2 and w would define a complex coordinate for the integrable distribution of 2-planes E^2 orthogonal to M^2. There is a great temptation to assume that these coordinates define preferred coordinates for M^4.

1. The slicing is very much analogous to that for space-time sheets and the natural question is how these slicings relate. What is of special interest is that the momentum plane M^2 can be defined by massless momentum. The scaling of this vector does not matter so that these planes are labelled by points z of sphere S^2 telling the direction of the line $M^2 \cap E^3$, when one assigns rest frame and therefore S^2 with the preferred time coordinate defined by the line connecting the tips of CD. This direction vector can be mapped to a twistor consisting of a spinor and its conjugate. The complex scalings of the twistor $(u, \overline{u}) \rightarrow \lambda u, \overline{u}/\lambda)$ define the same plane. Projective twistor like entities defining CP_1 having only one complex component instead of three are in question. This complex number defines with certain prerequisites a local coordinate for space-time sheet and together with the complex coordinate of E^2 could serve as a pair of complex coordinates (z, w) for space-time sheet. This brings strongly in mind the two complex coordinates appearing in the expansion of the generators of quantum Yangian of quantum affine algebra [K28].

2. The coordinate Ψ appearing in Beltrami flow defines the light-like vector field defining M^2 distribution. Its hyper-complex conjugate would define Ψ_c and conjugate light-like direction. An attractive possibility is that Φ allows analytic continuation to a holomorphic function of w. In this manner one would have four coordinates for M^4 also for space-time sheet.

3. The general vision is that at each point of space-time surface one can decompose the tangent space to $M^2(x) \subset M^4 = M^2_x \times E^2_x$ representing momentum plane and polarization plane $E^2 \subset E^2_x \times T(CP_2)$. The moduli space of planes $E^2 \subset E^6$ is 8-dimensional and parametrized by $SO(6)/SO(2) \times SO(4)$ for a given E^2_x. How can one achieve this selection and what conditions it must satisfy? Certainly the choice must be integrable but this is not the only condition.

Space-time surfaces as associative/co-associative surfaces

The idea that number theory determines classical dynamics in terms of associativity condition means that space-time surfaces are in some sense quaternionic surfaces of an octonionic space-time. It took several trials before the recent form of this hypothesis was achieved.

1. Octonionic structure is defined in terms of the octonionic representaton of gamma matrices of the embedding space existing only in dimension $D = 8$ since octonion units are in one-one correspondence with tangent vectors of the tangent space. Octonionic real unit corresponds to a preferred time axes (and rest frame) identified naturally as that connecting the tips of CD. What modified gamma matrices mean depends on variational principle for space-time surface. For volume action one would obtain induced gamma matrices. For Kähler action one obtains something different. In particular, the modified gamma matrices do not define vector basis identical with tangent vector basis of space-time surface.

2. Quaternionicity means that the modified gamma matrices defined as contractions of gamma matrices of H with canonical momentum densities for Kähler action span quaternionic sub-space of the octonionic tangent space [K111, K84]. A further condition is that each quaternionic space defined in this manner contains a preferred hyper-complex subspace of octonions.

3. The sub-space defined by the modified gamma matrices does not co-incide with the tangent space of space-time surface in general so that the interpretation of this condition is far from obvious. The canonical momentum densities need not define four independent vectors at given point. For instance, for massless extremals these densities are proportional to light-like vector so that the situation is degenerate and the space in question reduces to 2-D hyper-complex sub-space since light-like vector defines plane M^2.

The obvious questions are following.

1. Does the analog of tangent space defined by the octonionic modified gammas contain the local tangent space $M^2 \subset M^4$ for preferred extremals? For massless extremals [K14] this condition would be true. The orthogonal decomposition $T(X^4) = M^2 \oplus_\perp E^2$ can be defined at each point if this is true. For massless extremals also the functions Ψ and Φ can be identified.

2. One should answer also the following delicate question. Can M^2 really depend on point x of space-time? CP_2 as a moduli space of quaternionic planes emerges naturally if M^2 is *same* everywhere. It however seems that one should allow an integrable distribution of M^2_x such that M^2_x is same for all points of a given partonic 2-surface.

   How could one speak about fixed CP_2 (the embedding space) at the entire space-time sheet even when M^2_x varies?

   (a) Note first that G_2 (see `http://tinyurl.com/y9rrs7un`) defines the Lie group of octonionic automorphisms and G_2 action is needed to change the preferred hyper-octonionic sub-space. Various $SU(3)$ subgroups of G_2 are related by G_2 automorphism. Clearly, one must assign to each point of a string world sheet in the slicing parameterizing the partonic 2-surfaces an element of G_2. One would have Minkowskian string model with G_2 as a target space. As a matter fact, this string model is defined in the target space $G_2/SU(3)$ having dimension $D = 6$ since $SU(3)$ automorphisms leave given $SU(3)$ invariant.

   (b) This would allow to identify at each point of the string world sheet standard quaternionic basis - say in terms of complexified basis vectors consisting of two hyper-complex units and octonionic unit q_1 with "color isospin" $I_3 = 1/2$ and "color hypercharge" $Y = -1/3$ and its conjugate $\overline{q}_1$ with opposite color isospin and hypercharge.

   (c) The CP_2 point assigned with the quaternionic basis would correspond to the $SU(3)$ rotation needed to rotate the standard basis to this basis and would actually correspond to the first row of $SU(3)$ rotation matrix. Hyper-complex analyticity is the basic property of the solutions of the field equations representing Minkowskian string world sheets. Also now the same assumption is highly natural. In the case of string models in Minkowski space, the reduction of the induced metric to standard form implies Virasoro conditions and similar conditions are expected also now. There is no need to introduce action principle -just the hyper-complex analycitity is enough-since Kähler action already defines it.

3. The WZW model (see `http://tinyurl.com/ydxcvfhv`) inspired approach to the situation would be following. The parameterization corresponds to a map $g : X^2 \to G_2$ for which g defines a flat G_2 connection at string world sheet. WZW type action would give rise to this kind of situation. The transition $G_2 \to G_2/SU(3)$ would require that one gauges $SU(3)$ degrees of freedom by bringing in $SU(3)$ connection. Similar procedure for $CP_2 = SU(3)/U(2)$ would bring in $SU(3)$ valued chiral field and $U(2)$ gauge field. Instead of introducing these connections one can simply introduce $G_2/SU(3)$ and $SU(3)/U(2)$ valued chiral fields. What this observation suggests that this ansatz indeed predicts gluons and electroweak gauge bosons assignable to string like objects so that the mathematical picture would be consistent with physical intuition.

The two interpretations of CP_2

An old observation very relevant for what I have called $M^8 - H$ duality [K25] is that the moduli space of quaternionic sub-spaces of octonionic space (identifiable as M^8) containing preferred hyper-complex plane is CP_2. Or equivalently, the space of two planes whose addition extends hyper-complex plane to some quaternionic subspace can be parametrized by CP_2. This CP_2 can be called it CP_2^{mod} to avoid confusion. In the recent case this would mean that the space $E^2(x) \subset E_x^2 \times T(CP_2)$ is represented by a point of CP_2^{mod}. On the other hand, the embedding of space-time surface to H defines a point of "real" CP_2. This gives two different CP_2s.

1. The highly suggestive idea is that the identification $CP_2^{mod} = CP_2$ (apart from isometry) is crucial for the construction of preferred extremals. Indeed, the projection of the space-time point to CP_2 would fix the local polarization plane completely. This condition for $E^2(x)$ would be purely local and depend on the values of CP_2 coordinates only. Second condition for $E^2(x)$ would involve the gradients of embedding space coordinates including those of CP_2 coordinates.

2. The conditions that the planes M_x^2 form an integrable distribution at space-like level and that M_x^2 is determined by the modified gamma matrices. The integrability of this distribution for M^4 could imply the integrability for X^2. X^4 would differ from M^4 only by a deformation in degrees of freedom transversal to the string world sheets defined by the distribution of M^2s.

   Does this mean that one can begin from vacuum extremal with constant values of CP_2 coordinates and makes them non-constant but allows to depend only on transversal degrees of freedom? This condition is too strong even for simplest massless extremals for which CP_2 coordinates depend on transversal coordinates defined by $\epsilon \cdot m$ and $\epsilon \cdot k$. One could however allow dependence of CP_2 coordinates on light-like M^4 coordinate since the modification of the induced metric is light-like so that light-like coordinate remains light-like coordinate in this modification of the metric.

   Therefore, if one generalizes directly what is known about massless extremals, the most general dependence of CP_2 points on the light-like coordinates assignable to the distribution of M_x^2 would be dependence on either of the light-like coordinates of Hamilton-Jacobi coordinates but not both.

3.6.4 What could be the construction recipe for the preferred extremals assuming $CP_2 = CP_2^{mod}$ identification?

The crucial condition is that the planes $E^2(x)$ determined by the point of $CP_2 = CP_2^{mod}$ identification and by the tangent space of $E_x^2 \times CP_2$ are same. The challenge is to transform this condition to an explicit form. $CP_2 = CP_2^{mod}$ identification should be general coordinate invariant. This requires that also the representation of E^2 as (e^2, e^3) plane is general coordinate invariant suggesting that the use of preferred CP_2 coordinates - presumably complex Eguchi-Hanson coordinates - could make life easy. Preferred coordinates are also suggested by number theoretical vision. A careful consideration of the situation would be required.

The modified gamma matrices define a quaternionic sub-space analogous to tangent space of X^4 but not in general identical with the tangent space: this would be the case only if the action were 4-volume. I will use the notation $T_x^m(X^4)$ about the modified tangent space and call the vectors of $T_x^m(X^4)$ modified tangent vectors. I hope that this would not cause confusion.

$CP_2 = CP_2^{mod}$ condition

Quaternionic property of the counterpart of $T_x^m(X^4)$ allows an explicit formulation using the tangent vectors of $T_x^m(X^4)$.

1. The unit vector pair (e_2, e_3) should correspond to a unique tangent vector of H defined by the coordinate differentials dh^k in some natural coordinates used. Complex Eguchi-Hanson coordinates [K15] are a natural candidate for CP_2 and require complexified octonionic imaginary units. If octonionic units correspond to the tangent vector basis of H uniquely, this is possible.

2. The pair (e_2, e_3) as also its complexification ($q_1 = e_2 + ie_3, \overline{q}_1 = e_2 - ie_3$) is expressible as a linear combination of octonionic units $I_2, ...I_7$ should be mapped to a point of $CP_2^{mod} = CP_2$ in canonical manner. This mapping is what should be expressed explicitly. One should express given (e_2, e_3) in terms of $SU(3)$ rotation applied to a standard vector. After that one should define the corresponding CP_2 point by the bundle projection $SU(3) \rightarrow CP_2$.

3. The tangent vector pair

$$(\partial_w h^k, \partial_{\overline{w}} h^k)$$

defines second representation of the tangent space of $E^2(x)$. This pair should be equivalent with the pair $(q_1, \overline{q}_1)$. Here one must be however very cautious with the choice of coordinates. If the choice of w is unique apart from constant the gradients should be unique. One can use also real coordinates (x, y) instead of ($w = x + iy, \overline{w} = x - iy$) and the pair (e_2, e_3). One can project the tangent vector pair to the standard vielbein basis which must correspond to the octonionic basis

$$(\partial_x h^k, \partial_y h^k) \rightarrow (\partial_x h^k e_k^A e_A, \partial_y h^k e_k^A) e_A) \leftrightarrow (e_2, e_3) \ ,$$

where the e_A denote the octonion units in 1-1 correspondence with vielbein vectors. This expression can be compared to the expression of (e_2, e_3) derived from the knowledge of CP_2 projection.

Formulation of quaternionicity condition in terms of octonionic structure constants

One can consider also a formulation of the quaternionic tangent planes in terms of (e_2, e_3) expressed in terms of octonionic units deducible from the condition that unit vectors obey quaternionic algebra. The expressions for octonionic (see `http://tinyurl.com/5m5lqr`) *resp.* quaternionic (see `http://tinyurl.com/3rr79p9`) structure constants can be found at [A9] *resp.* [A12].

1. The ansatz is

$$\begin{aligned}
\{E_k\} &= \{1, I_1, E_2, E_3\} \ , \\
E_2 &= E_{2k}e^k \equiv \sum_{k=2}^{7} E_{2k}e^k \ , \ E_3 = E_{3k}e^k \equiv \sum_{k=2}^{7} E_{3k}e^k \ , \\
|E_2| &= 1 \ , \ |E_3| = 1 \ .
\end{aligned} \tag{3.6.1}$$

2. The multiplication table for octonionic units expressible in terms of octonionic triangle (see `http://tinyurl.com/5m5lqr`) [A9] gives

$$f^{1kl}E_{2k} = E_{3l} \ , \ f^{1kl}E_{3k} = -E_{2l} \ , \ f^{klr}E_{2k}E_{3l} = \delta_1^r \ . \tag{3.6.2}$$

Here the indices are raised by unit metric so that there is no difference between lower and upper indices. Summation convention is assumed. Also the contribution of the real unit is present in the structure constants of third equation but this contribution must vanish.

3. The conditions are linear and quadratic in the coefficients E_{2k} and E_{3k} and are expected to allow an explicit solution. The first two conditions define homogenous equations which must allow solution. The coefficient matrix acting on (E_2, E_3) is of the form

$$\begin{pmatrix} f_1 & 1 \\ -1 & f_1 \end{pmatrix} \ ,$$

where 1 denotes unit matrix. The vanishing of the determinant of this matrix should be due to the highly symmetric properties of the structure constants. In fact the equations can be written as eigen conditions

$$f_1 \circ (E_2 \pm iE_3) = \mp i(E_2 \pm iE_3) \ ,$$

and one can say that the structure constants are eigenstates of the hermitian operator defined by I_1 analogous to color hyper charge. Both values of color hyper charged are obtained.

Explicit expression for the $CP_2 = CP_2^{mod}$ conditions

The symmetry under $SU(3)$ allows to construct the solutions of the above equations directly.

1. One can introduce complexified basis of octonion units transforming like $(1, 1, 3, \overline{3})$ under SU(3). Note the analogy of triplet with color triplet of quarks. One can write complexified basis as $(1, e_1, (q_1, q_2, q_3), (\overline{q}_1\overline{q}_2, \overline{q}_3))$. The expressions for complexified basis elements are

$$(q_1, q_2, q_3) = \frac{1}{\sqrt{2}}(e_2 + ie_3, e_4 + ie_5, e_6 + ie_7) \ .$$

   These options can be seen to be possible by studying octonionic triangle in which all lines containing 3 units defined associative triple: any pair of octonion units at this kind of line can be used to form pair of complexified unit and its conjugate. In the tangent space of $M^4 \times CP_2$ the basis vectors q_1, and q_2 are mixtures of E^2_x and CP_2 tangent vectors. q_3 involves only CP_2 tangent vectors and there is a temptation to interpret it as the analog of the quark having no color isospin.

2. The quaternionic basis is real and must transform like $(1, 1, q_1, \overline{q}_1)$, where q_1 is any quark in the triplet and $\overline{q}_1$ its conjugate in antitriplet. Having fixed some basis one can perform SU(3) rotations to get a new basis. The action of the rotation is by 3×3 special unitary matrix. The over all phases of its rows do not matter since they induce only a rotation in (e_2, e_3) plane not affecting the plane itself. The action of $SU(3)$ on q_1 is simply the action of its first row on (q_1, q_2, q_3) triplet:

$$\begin{aligned} q_1 &\rightarrow (Uq)_1 = U_{11}q_1 + U_{12}q_2 + U_{13}q_3 \equiv z_1q_1 + z_2q_2 + z_3q_3 \\ &= z_1(e_2 + ie_3) + z_2(e_4 + ie_5) + z_3(e_6 + ie_7) \ . \end{aligned} \quad (3.6.3)$$

   The triplets (z_1, z_2, z_3) defining a complex unit vector and point of S^5. Since overall phase does not matter a point of CP_2 is in question. The new real octonion units are given by the formulas

$$\begin{aligned} e_2 &\rightarrow Re(z_1)e_2 + Re(z_2)e_4 + Re(z_3)e_6 - Im(z_1)e_3 - Im(z_2)e_5 - Im(z_3)e_7 \ , \\ e_3 &\rightarrow Im(z_1)e_2 + Im(z_2)e_4 + Im(z_3)e_6 + Re(z_1)e_3 + Re(z_2)e_5 + Re(z_3)e_7 \ . \end{aligned} \quad (3.6.4)$$

   For instance the CP_2 coordinates corresponding to the coordinate patch (z_1, z_2, z_3) with $z_3 \neq 0$ are obtained as $(\xi_1, \xi_2) = (z_1/z_3, z_2/z_3)$.

Using these expressions the equations expressing the conjecture $CP_2 = CP_2^{mod}$ equivalence can be expressed explicitly as first order differential equations. The conditions state the equivalence

$$(e_2, e_3) \leftrightarrow (\partial_x h^k e^A_k e_A, \partial_y h^k e^A_k e_A) \ , \quad (3.6.5)$$

where e_A denote octonion units. The comparison of two pairs of vectors requires normalization of the tangent vectors on the right hand side to unit vectors so that one takes unit vector in the direction of the tangent vector. After this the vectors can be equated. This allows to expresses the contractions of the partial derivatives with vielbein vectors with the 6 components of e_2 and e_3. Each condition gives 6+6 first order partial differential equations which are non-linear by the presence of the overal normalization factor for the right hand side. The equations are invariant under scalings of (x, y). The very special form of these equations suggests that some symmetry is involved.

It must be emphasized that these equations make sense only in preferred coordinates: ordinary Minkowski coordinates and Hamilton-Jacobi coordinates for M^4 and Eguchi-Hanson complex coordinates in which $SU(2)\times U(1)$ is represented linearly for CP_2. These coordinates are preferred because they carry deep physical meaning.

Does TGD boil down to two string models?

It is good to look what have we obtained. Besides Hamilton-Jacobi conditions, and $CP_2 = CP_2^{mod}$ conditions one has what one might call string model with 6-dimensional $G_2/SU(3)$ as targent space. The orbit of string in $G_2/SU(3)$ allows to deduce the G_2 rotation identifiable as a point of $G_2/SU(3)$ defining what one means with standard quaternionic plane at given point of string world sheet. The hypothesis is that hyper-complex analyticity solves these equations.

The conjectured electric-magnetic duality implies duality between string world sheet and partonic 2-surfaces central for the proposed mathematical applications of TGD [K44, K45, K94, K55]. This duality suggests that the solutions to the $CP_2 = CP_2^{mod}$ conditions could reduce to holomorphy with respect to the coordinate w for partonic 2-surface plus the analogs of Virasoro conditions. The dependence on light-like coordinate would appear as a parametric dependence.

If this were the case, TGD would reduce at least partially to what might be regarded as dual string models in $G_2/SU(3)$ and $SU(3)/U(2)$ and also to string model in M^4 and X^4! In the previous arguments one ends up to string models in moduli spaces of string world sheets and partonic 2-surfaces. TGD seems to yield an inflation of string models! This not actually surprising since the slicing of space-time sheets by string world sheets and partonic 2-surfaces implies automatically various kinds of maps having interpretation in terms of string orbits.

3.7 In What Sense TGD Could Be An Integrable Theory?

During years evidence supporting the idea that TGD could be an integrable theory in some sense has accumulated. The challenge is to show that various ideas about what integrability means form pieces of a bigger coherent picture. Of course, some of the ideas are doomed to be only partially correct or simply wrong. Since it is not possible to know beforehand what ideas are wrong and what are right the situation is very much like in experimental physics and it is easy to claim (and has been and will be claimed) that all this argumentation is useless speculation. This is the price that must be paid for real thinking.

Integrable theories allow to solve nonlinear classical dynamics in terms of scattering data for a linear system. In TGD framework this translates to quantum classical correspondence. The solutions of Kähler-Dirac equation define the scattering data. This data should define a real analytic function whose octonionic extension defines the space-time surface as a surface for which its imaginary part in the representation as bi-quaternion vanishes. There are excellent hopes about this thanks to the reduction of the Kähler-Dirac equation to geometric optics.

In the following I will first discuss briefly what integrability means in (quantum) field theories, list some bits of evidence for integrability in TGD framework, discuss once again the question whether the different pieces of evidence are consistent with other and what one really means with various notions. An an outcome I represent what I regard as a more coherent view about integrability of TGD. The notion of octonion analyticity developed in the previous section is essential for the for what follows.

3.7.1 What Integrable Theories Are?

The following is an attempt to get some bird's eye of view about the landscape of integrable theories.

Examples of integrable theories

Integrable theories are typically non-linear 1+1-dimensional (quantum) field theories. Solitons and various other particle like structures are the characteristic phenomenon in these theories. Scattering matrix is trivial in the sense that the particles go through each other in the scattering and suffer only a phase change. In particular, momenta are conserved. Korteveg- de Vries equation (see http://tinyurl.com/3cyt8hk) [B2] was motivated by the attempt to explain the experimentally discovered shallow water wave preserving its shape and moving with a constant velocity. Sine-Gordon equation (see http://tinyurl.com/yafl243x) [B6] describes geometrically constant curvature surfaces and defines a Lorentz invariant non-linear field theory in 1+1-dimensional space-time, which can be applied to Josephson junctions (in TGD inspired quantum biology it is encountered in the model of nerve pulse [K81]). Non- linear Schrödinger equation (see http://tinyurl.com/y88efbo7) [B5] having applications to optics and water waves represents a further example. All these equations have various variants.

From TGD point of view conformal field theories represent an especially interesting example of integrable theories. (Super-)conformal invariance is the basic underlying symmetry and by its infinite-dimensional character implies infinite number of conserved quantities. The construction of the theory reduces to the construction of the representations of (super-)conformal algebra. One can solve 2-point functions exactly and characterize them in terms of (possibly anomalous) scaling dimensions of conformal fields involved and the coefficients appearing in 3-point functions can be solved in terms of fusion rules leading to an associative algebra for conformal fields. The basic applications are to 2-dimensional critical thermodynamical systems whose scaling invariance generalizes to conformal invariance. String models represent second application in which a collection of super-conformal field theories associated with various genera of 2-surface is needed to describe loop corrections to the scattering amplitudes. Also moduli spaces of conformal equivalence classes become important.

Topological quantum field theories (see http://tinyurl.com/lsvx7g3) are also examples of integrable theories. Because of its independence on the metric Chern-Simons action (see http://tinyurl.com/ydgsqm2c) is in 3-D case the unique action defining a topological quantum field theory. The calculations of knot invariants (for TGD approach see [K44]), topological invariants of 3-manifolds and 4-manifolds, and topological quantum computation (see http://tinyurl.com/dkpo4y) (for a model of DNA as topological quantum computer see [K3]) represent applications of this approach. TGD as almost topological QFT means that the Kähler action for preferred extremals reduces to a surface term by the vanishing of Coulomb term in action and by the weak form of electric-magnetic duality reduces to Chern-Simons action. Both Euclidian and Minkowskian regions give this kind of contribution.

$\mathcal{N} = 4$ SYM is the a four-dimensional and very nearly realistic candidate for an integral quantum field theory. The observation that twistor amplitudes allow also a dual of the 4-D conformal symmetry motivates the extension of this symmetry to its infinite-dimensional Yangian variant [A17]. Also the enormous progress in the construction of scattering amplitudes suggests integrability. In TGD framework Yangian symmetry would emerge naturally by extending the symplectic variant of Kac-Moody algebra from light-cone boundary to the interior of causal diamond and the Kac-Moody algebra from light-like 3-surface representing wormhole throats at which the signature of the induced metric changes to the space-time interior [K101].

About mathematical methods

The mathematical methods used in integrable theories are rather refined and have contributed to the development of the modern mathematical physics. Mention only quantum groups, conformal algebras, and Yangian algebras.

The basic element of integrability is the possibility to transform the non-linear classical problem for which the interaction is characterized by a potential function or its analog to a linear scattering problem depending on time. For instance, for the ordinary Schrödinger function one can solve potential once single solution of the equation is known. This does not work in practice. One can however gather information about the asymptotic states in scattering to deduce the potential. One cannot do without information about bound state energies too.

In TGD framework asymptotic states correspond to partonic 2-surfaces at the two light-like boundaries of CD (more precisely: the largest CD involved and defining the IR resolution for momenta). From the scattering data coding information about scattering for various values of energy of the incoming particle one deduced the potential function or its analog.

1. The basic tool is inverse scattering transform known as Gelfand-Marchenko-Levitan (GML) transform (see `http://tinyurl.com/y9f7ybln`) described in simple terms in [B7].

   (a) In 1+1 dimensional case the S-matrix characterizing scattering is very simple since the only thing that can take place in scattering is reflection or transmission. Therefore the S-matrix elements describe either of these processes and by unitarity the sum of corresponding probabilities equals to 1. The particle can arrive to the potential either from left or right and is characterized by a momentum. The transmission coefficient can have a pole meaning complex (imaginary in the simplest case) wave vector serving as a signal for the formation of a bound state or resonance. The scattering data are represented by the reflection and transmission coefficients as function of time.

   (b) One can deduce an integral equation for a propagator like function $K(t,x)$ describing how delta pulse moving with light velocity is scattered from the potential and is expressible in terms of time integral over scattering data with contributions from both scattering states and bound states. The derivation of GML transform [B7] uses time reversal and time translational invariance and causality defined in terms of light velocity. After some tricks one obtains the integral equation as well as an expression for the time independent potential as $V(x) = K(x,x)$. The argument can be generalized to more complex problems to deduce the GML transform.

2. The so called Lax pair (see `http://tinyurl.com/yc93nw53`) is one manner to describe integrable systems [B3]. Lax pair consists of two operators L and M. One studies what might be identified as "energy" eigenstates satisfying $L(x,t)\Psi = \lambda\Psi$. λ does not depend on time and one can say that the dynamics is associated with x coordinate whereas as t is time coordinate parametrizing different variants of eigenvalue problem with the same spectrum for L. The operator $M(t)$ does not depend on x at all and the independence of λ on time implies the condition

   $$\partial_t L = [L, M] \ .$$

   This equation is analogous to a quantum mechanical evolution equation for an operator induced by time dependent "Hamiltonian" M and gives the non-linear classical evolution equation when the commutator on the right hand side is a multiplicative operator (so that it does not involve differential operators acting on the coordinate x). Non-linear classical dynamics for the time dependent potential emerges as an integrability condition.

   One could say that $M(t)$ introduces the time evolution of $L(t,x)$ as an automorphism which depends on time and therefore does not affect the spectrum. One has $L(t,x) = U(t)L(0,x)U^{-1}(t)$ with $dU(t)/dt = M(t)U(t)$. The time evolution of the analog of the quantum state is given by a similar equation.

3. A more refined view about Lax pair is based on the observation that the above equation can be generalized so that M depends also on x. The generalization of the basic equation for $M(x,t)$ reads as

   $$\partial_t L - \partial_x M - [L, M] = 0 \ .$$

   The condition has interpretation as a vanishing of the curvature of a gauge potential having components $A_x = L, A_t = M$. This generalization allows a beautiful geometric formulation of the integrability conditions and extends the applicability of the inverse scattering transform.

The monodromy of the flat connection becomes important in this approach. Flat connections in moduli spaces are indeed important in topological quantum field theories and in conformal field theories.

4. There is also a connection with the so called Riemann-Hilbert problem (see `http://tinyurl.com/ybay4qjg`) [A13]. The monodromies of the flat connection define monodromy group and Riemann-Hilbert problem concerns the existence of linear differential equations having a given monodromy group. Monodromy group emerges in the analytic continuation of an analytic function and the action of the element of the monodromy group tells what happens for the resulting many-valued analytic function as one turns around a singularity once ("mono-"). The linear equations obviously relate to the linear scattering problem. The flat connection (M, L) in turn defines the monodromy group. What is needed is that the functions involved are analytic functions of (t, x) replaced with a complex or hyper-complex variable. Again Wick rotation is involved. Similar approach generalizes also to higher dimensional moduli spaces with complex structures.

   In TGD framework the effective 2-dimensionality raises the hope that this kind of mathematical apparatus could be used. An interesting possibility is that finite measurement resolution could be realized in terms of a gauge group or Kac-Moody type group represented by trivial gauge potential defining a monodromy group for n-point functions. Monodromy invariance would hold for the full n-point functions constructed in terms of analytic n-point functions and their conjugates. The ends of braid strands are natural candidates for the singularities around which monodromies are defined.

3.7.2 Why TGD Could Be Integrable Theory In Some Sense?

There are many indications that TGD could be an integrable theory in some sense. The challenge is to see which ideas are consistent with each other and to build a coherent picture where everything finds its own place.

1. 2-dimensionality or at least effective 2-dimensionality seems to be a prerequisite for integrability. Effective 2-dimensionality is suggested by the strong form of General Coordinate Invariance implying also holography and generalized conformal invariance predicting infinite number of conservation laws. The dual roles of partonic 2-surfaces and string world sheets supports a four-dimensional generalization of conformal invariance. Twistor considerations [K101, K10] indeed suggest that Yangian invariance and Kac-Moody invariances combine to a 4-D analog of conformal invariance induced by 2-dimensional one by algebraic continuation.

2. Octonionic representation of embedding space Clifford algebra and the identification of the space-time surfaces as quaternionic space-time surfaces would define a number theoretically natural generalization of conformal invariance. The reason for using gamma matrix representation is that vector field representation for octonionic units does not exist. The problem concerns the precise meaning of the octonionic representation of gamma matrices.

   Space-time surfaces could be quaternionic also in the sense that conformal invariance is analytically continued from string curve to 8-D space by octonion real-analyticity. The question is whether the Clifford algebra based notion of tangent space quaternionicity is equivalent with octonionic real-analyticity based notion of quaternionicity.

   The notions of co-associativity and co-quaternionicity make also sense and one must consider seriously the possibility that associativity-co-associativity dichotomy corresponds to Minkowskian-Euclidian dichotomy.

3. Field equations define hydrodynamic Beltrami flows satisfying integrability conditions of form $J \wedge dJ = 0$.

   (a) One can assign local momentum and polarization directions to the preferred extremals and this gives a decomposition of Minkowskian space-time regions to massless quanta analogous to the 1+1-dimensional decomposition to solitons. The linear superposition of modes with 4-momenta with different directions possible for free Maxwell action does not look plausible for the preferred extremals of Kähler action. This rather quantal and solitonic character is in accordance with the quantum classical correspondence giving very concrete connection between quantal and classical particle pictures. For 4-D volume action one does not obtain this kind of decomposition. In 2-D case volume action gives superposition of solutions with different polarization directions so that the situation is nearer to that for free Maxwell action and is not like soliton decomposition.

(b) Beltrami property in strong sense allows to identify 4 preferred coordinates for the space-time surface in terms of corresponding Beltrami flows. This is possible also in Euclidian regions using two complex coordinates instead of hyper-complex coordinate and complex coordinate. The assumption that isometry currents are parallel to the same light-like Beltrami flow implies hydrodynamic character of the field equations in the sense that one can say that each flow line is analogous to particle carrying some quantum numbers. This property is not true for all extremals (say cosmic strings).

(c) The tangent bundle theoretic view about integrability is that one can find a Lie algebra of vector fields in some manifold spanning the tangent space of a lower-dimensional manifolds and is expressed in terms of Frobenius theorem (see `http://tinyurl.com/of6vfz5`) [A4]. The gradients of scalar functions defining Beltrami flows appearing in the ansatz for preferred exremals would define these vector fields and the slicing. Partonic 2-surfaces would correspond to two complex conjugate vector fields (local polarization direction) and string world sheets to light-like vector field and its dual (light-like momentum directions). This slicing generalizes to the Euclidian regions.

4. Infinite number of conservation laws is the signature of integrability. Classical field equations follow from the condition that the vector field defined by Kähler-Dirac gamma matrices has vanishing divergence and can be identified an integrability condition for the Kähler-Dirac equation guaranteeing also the conservation of super currents so that one obtains an infinite number of conserved charges.

5. Quantum criticality is a further signal of integrability. 2-D conformal field theories describe critical systems so that the natural guess is that quantum criticality in TGD framework relates to the generalization of conformal invariance and to integrability. Quantum criticality implies that Kähler coupling strength is analogous to critical temperature. This condition does affects classical field equations only via boundary conditions expressed as weak form of electric magnetic duality at the wormhole throats at which the signature of the metric changes.

   For finite-dimensional systems the vanishing of the determinant of the matrix defined by the second derivatives of potential is similar signature and applies in catastrophe theory. Therefore the existence of vanishing second variations of Kähler action should characterize criticality and define a property of preferred extremals. The vanishing of second variations indeed leads to an infinite number of conserved currents [K14] following the conditions that the deformation of Kähler-Dirac gamma matrix is also divergenceless and that the Kähler-Dirac equation associated with it is satisfied.

3.7.3 Could TGD Be An Integrable Theory?

Consider first the abstraction of integrability in TGD framework. Quantum classical correspondence could be seen as a correspondence between linear quantum dynamics and non-linear classical dynamics. Integrability would realize this correspondence. In integrable models such as Sine-Gordon equation particle interactions are described by potential in 1+1 dimensions. This too primitive for the purposes of TGD. The vertices of generalized Feynman diagrams take care of this. At lines one has free particle dynamics so that the situation could be much simpler than in integrable models if one restricts the considerations to the lines or Minkowskian space-time regions surrounding them.

The non-linear dynamics for the space-time sheets representing incoming lines of generalized Feynman diagram should be obtainable from the linear dynamics for the induced spinor fields defined by Kähler-Dirac operator. There are two options.

1. Strong form of the quantum classical correspondence states that each solution for the linear dynamics of spinor fields corresponds to space-time sheet. This is analogous to solving the potential function in terms of a single solution of Schrödinger equation. Coupling of space-time geometry to quantum numbers via measurement interaction term is a proposal for realizing this option. It is however the quantum numbers of positive/negative energy parts of zero energy state which would be visible in the classical dynamics rather than those of induced spinor field modes.

2. Only overall dynamics characterized by scattering data- the counterpart of S-matrix for the Kähler-Dirac operator- is mapped to the geometry of the space-time sheet. This is much more abstract realization of quantum classical correspondence.

3. Can these two approaches be equivalent? This might be the case since quantum numbers of the state are not those of the modes of induced spinor fields.

What the scattering data could be for the induced spinor field satisfying Kähler-Dirac equation?

1. If the solution of field equation has hydrodynamic character, the solutions of the Kähler-Dirac equation can be localized to light-like Beltrami flow lines of hydrodynamic flow. These correspond to basic solutions and the general solution is a superposition of these. There is no dispersion and the dynamics is that of geometric optics at the basic level. This means geometric optics like character of the spinor dynamics.

   Solutions of the Kähler-Dirac equation are completely analogous to the pulse solutions defining the fundamental solution for the wave equation in the argument leading from wave equation with external time independent potential to Marchenko-Gelfand-Levitan equation allowing to identify potential in terms of scattering data. There is however no potential present now since the interactions are described by the vertices of Feynman diagram where the particle lines meet. Note that particle like regions are Euclidian and that this picture applies only to the Minkowskian exteriors of particles.

2. Partonic 2-surfaces at the ends of the line of generalized Feynman diagram are connected by flow lines. Partonic 2-surfaces at which the signature of the induced metric changes are in a special position. Only the imaginary part of the bi-quaternionic value of the octonion valued map is non-vanishing at these surfaces which can be said to be co-complex 2-surfaces. By geometric optics behavior the scattering data correspond to a diffeomorphism mapping initial partonic 2-surface to the final one in some preferred complex coordinates common to both ends of the line.

3. What could be these preferred coordinates? Complex coordinates for S^2 at light-cone boundary define natural complex coordinates for the partonic 2-surface. With these coordinates the diffeomorphism defining scattering data is diffeomorphism of S^2. Suppose that this map is real analytic so that maps "real axis" of S^2 to itself. This map would be same as the map defining the octonionic real analyticity as algebraic extension of the complex real analytic map. By octonionic analyticity one can make large number of alternative choices for the coordinates of partonic 2-surface.

4. There can be non-uniqueness due to the possibility of $G_2/SU(3)$ valued map characterizing the local octonionic units. The proposal is that the choice of octonionic imaginary units can depend on the point of string like orbit: this would give string model in $G_2/SU(3)$. Conformal invariance for this string model would imply analyticity and helps considerably but would not probably fix the situation completely since the element of the coset space would constant at the partonic 2-surfaces at the ends of CD. One can of course ask whether the $G_2/SU(3)$ element could be constant for each propagator line and would change only at the 2-D vertices?

This would be the inverse scattering problem formulated in the spirit of TGD. There could be also dependence of space-time surface on quantum numbers of quantum states but not on individual solution for the induced spinor field since the scattering data of this solution would be purely geometric.

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Chapter 4

TGD as a Generalized Number Theory III: Infinite Primes

4.1 Introduction

The third part of the multi-chapter discussing the idea about physics as a generalized number theory is devoted to the possible role of infinite primes in TGD.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also for their complexifications and one can speak about infinite primes in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors.

4.1.1 The Notion Of Infinite Prime

The original motivation for the notion of infinite prime came from the first attempts to construct TGD inspired theory of consciousness (around 1995) [K99]. Suppose very naively that the 4-surfaces in a given sector of the "world of classical worlds" (WCW) are labelled by a fixed p-adic prime. The natural expectation is that evolution by quantum jumps means dispersion in the space of these sectors and leads to the increase of the p-adic prime characterizing the Universe. As one moves backwards in subjective time (sequence of quantum jumps) one ends up to the situation in which the prime characterizing the universe was $p = 2$. Should one assume that there was the first quantum jump when everything began? If not, then it would seem that the p-adic prime characterizing the Universe must be infinite. Second problem is that the p-adic length scales are finite and if the size scale of Universe is given by p-adic length scale the Universe has finite sized: this does not make sense in TGD framework. The only way out of the problems is the assumption that the p-adic prime characterizing the entire Universe is literally infinite and that p-adic primes characterizing space-time sheets are finite.

These argument, which are by no means central for the recent view about p-adic primes, motivated the attempt to construct a theory of infinite primes and to extend quantum TGD accordingly. This turns out to be possible. The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. It must be also emphasized that the notion of infinity is relativistic. With respect to the p-adic norm infinite primes have unit norm for all finite and infinite primes so that there is nothing to become scared of!

Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave functions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of embedding space and space-time surface are subject to a number theoretic evolution. In philosophical mood one can of course also ask whether there exists a hierarchy of embedding spaces in which the embedding space at the lower level represents something with infinitesimal size in the sense of real topology and whether this hierarchy is accompanied also by a hierarchy of conscious entities.

This picture suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [A22] providing a rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively. Same generalization could make sense for all classical number fields.

4.1.2 Infinite Primes And Physics In TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this way.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and this cheated me to believe that this supersymmetry is completely absent in TGD Universe.

The progress in the understanding of the properties of the modified Dirac action however led to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [?].

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [K110] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [K25].

Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K110]. the dark matter hierarchy characterized by increasing values of $\hbar$ [K36]. the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory with finite measurement resolution.

Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. This conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

Quantum classical correspondence requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The Kähler-Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries. The notion of finite measurement resolution allows to deduce much more detailed about this correspondence. In particular, the rational defined by the infinite prime classifies the finite sub-manifold geometry defined by the discretization of the partonic 2-surface implied by the finite measurement resolution. Also a direct correlation between integers defining Planck constant and the "fermionic" part of the infinite prime emerges.

4.1.3 Infinite Primes, Cognition, And Intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. One can define the notion of prime also for the algebraic extensions of rationals. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum of infinitesimals (real zeros) is replaced by multiplication of real units meaning that the set of real and also more general units becomes infinitely degenerate.

3. Infinite primes form an infinite hierarchy so that the points of space-time and embedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

4. In zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

5. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower

level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

4.2 Infinite Primes, Integers, And Rationals

The definition of the infinite integers and rationals is a straightforward procedure and structurally similar to a repeated second quantization of a super-symmetric quantum field theory but including also the number theoretic counterparts of bound states.

4.2.1 The First Level Of Hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{array}{l} P = 1 + X \ , \\ X = \prod_p p \ . \end{array} \tag{4.2.1}$$

If P were divisible by finite prime then $P - X = 1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N = P - k$, $k > 1$, are certainly not primes since k can be taken as a factor. The number $P' = P - 2 = -1 + X$ could however be prime. P is certainly not divisible by $P - 2$. It seems that one cannot express P and $P - 2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p\in U} p + q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P - 2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{array}{l} P(\pm, n) = \pm 1 + nX \ , \\ k(p) = 0, 1, \ldots.. \ , \\ n = \prod_p p^{k(p)} \ , \\ X = \prod_p p \ , \end{array} \tag{4.2.2}$$

where $k(p) \neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good candidates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{aligned} &P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ &m = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{4.2.3}$$

Here $x|y$ means "x divides y". To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+, 1, 1, 1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{aligned} &P(\pm, m, n, s|r) = nY^r \pm ms \ , \\ &Y = \frac{X}{s} \ , \\ &m = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{4.2.4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$\begin{aligned} &N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\ &Y = \frac{X}{s} \ , \\ &n_1m_2 - n_2m_1 = 0 \ . \end{aligned}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteeing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n, m, r)(Y) = nY^r + m$ with integer valued coefficients $(n > 0)$ defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{aligned} &P(\pm, m, n, s|r_1, r_2) = nY^{r_1} \pm ms \ , \\ &m = P_{r_2}(Y)Y + m_0 \ , \\ &Y = \frac{X}{s} \ , \\ &m_0 = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{4.2.5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteeing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1: th or r_2: th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteeing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm, m, n, s)$. $P(\pm, 1, 1, 1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm, m, n, s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm, m, n, s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+, m, n, s)$ or $P(-, m, n, s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

4.2.2 Infinite Primes Form A Hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm, m, n, s)$

(or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction recipe. In present case "vacuum primes" at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm,m,n,s)} P(\pm,m,n,s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{4.2.6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm, m, n, s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type "bosons" respectively. The non-negative integer-valued function $K(P) = K(\pm, m, n, s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type "bosons". More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\begin{array}{l} \frac{P_1(\pm,m_1,n_1,s_1 K_1,S_1)}{P_2(\pm,m_2,n_2,s_2,K_2,S_2)} \\ = \frac{n_1 s_2}{n_2 s_1} \prod_{\pm,m,n,s} (\frac{n}{s})^{K_1^+(\pm,n,m,s) - K_2^+(\pm,n,m,s)} \ . \end{array} \tag{4.2.7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

4.2.3 Construction Of Infinite Primes As A Repeated Quantization Of A Super-Symmetric Arithmetic Quantum Field Theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symmetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actually use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the "vacuum" X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ "p-bosons" in mode of type X/s and multiplication of the "vacuum" s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ "p-bosons". in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{4.2.8}$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P\ mod\ 4 = 3$ and $P\ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,...,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the embedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

4.2.4 Construction In The Case Of An Arbitrary Commutative Number Field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [K95] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [K95]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [K95]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

4.2.5 Mapping Of Infinite Primes To Polynomials And Geometric Objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \quad . \tag{4.2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n: th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n: th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

One can speculate with possible connections to TGD physics. The degree n of the polynomial is its basic characterizer. Infinite primes corresponding to polynomials of degree $n > 1$ should correspond to bound states. On the other hand, the hierarchy of Planck constants suggests strongly the interpretation in terms of gravitational bound states. Could one identify $h_{eff}/h = n$ as the degree of the polynomial characterizing infinite prime?

4.2.6 How To Order Infinite Primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the "large" and the "small" part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

4.2.7 What Is The Cardinality Of Infinite Primes At Given Level?

The basic problem is to decide whether Nature allows also integers S, $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric

analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

4.2.8 How To Generalize The Concepts Of Infinite Integer, Rational And Real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \tag{4.2.10}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{4.2.11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{4.2.12}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \tag{4.2.13}$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultra-metricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \tag{4.2.14}$$

Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \tag{4.2.15}$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \tag{4.2.16}$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part.* Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N> \ . \tag{4.2.17}$$

The natural inner product is

$$\langle X,Y\rangle = x_0y_0+\sum_N x_Ny_N \quad . \tag{4.2.18}$$

The inner product is well defined if the number of N: s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X+Y = x_0+y_0+\sum_N (x_N+y_N)p^{-N} \quad , \tag{4.2.19}$$

The product XY is expressible in the form

$$XY = x_0y_0+x_0Y+Xy_0+\sum_{N_1,N_2} x_{N_1}y_{N_2}p^{-N_1-N_2} \quad , \tag{4.2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x=\sum_k x_kp^{-k} \rightarrow x_p=\sum_k x_kp^k \quad ,$$

generalizes to the form

$$x = x_0+\sum_N x_Np^{-N} \rightarrow (x_0)_p+\sum_N (x_N)_pp^N \quad , \tag{4.2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0+\sum_N x_Np_1^{-N} \rightarrow x_0+\sum_N x_Np_2^{-N} \quad . \tag{4.2.22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the embedding space provokes the question whether it might be possible to regard the infinite-dimensional WCW , or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

4.2.9 Comparison With The Approach Of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement "Set is Many allowing to regard itself as One" really means and to the fact that there is no obvious connection with physics.

The proposed approach is based on the introduction of the concept of prime as a basic concept whereas partial ordering is based on the use of ratios: using these one can recursively define partial ordering and get precise quantitative information based on finite reals. The ordering is only partial and there is infinite number of ratios of infinite integers giving rise to same real unit which in turn leads to the idea about number theoretic anatomy of real point.

The "Set is Many allowing to regard itself as One" is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as "One" and its decomposition to a product of primes corresponds to the set as "Many". The concept of prime, the ultimate "One", has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

4.3 Can One Generalize The Notion Of Infinite Prime To TheNon-Commutative And Non-Associative Context?

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [K96] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [K96]. Also the hierarchy of infinite primes should generalize as also the representation of infinite primes as polynomials although associativity is expected to pose technical problems.

4.3.1 Quaternionic And Octonionic Primes And Their Hyper Counterparts

The loss of commutativity and associativity implies that the definitions of quaternionic and octonionic primes are not completely straightforward.

Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space if one uses octonionic representation for the gamma matrices [K96]. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many ways and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio $3:1$. For other Lie-groups this ratio is either 2: 1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic prime has one particualr representative as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$.

Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them if one assumes the entire quaternionic prime as four-momentum: only a system where energy is minimum is possible. The introduction of a preferred hyper-complex plane necessary for several reasons- in particular for the possibility to identify standard model quantum numbers in terms of infinite primes- allows to identify the momentum of particle in the preferred plane as the first two components of the hyper prime in fixed coordinate frame. Note that this leads to a universal spectrum for mass squared.

For time like hyper-primes the momentum is always time like for hyper-primes. In this case it is possible to find a rest frame by applying a hyper-primeness preserving G_2 transformation so that the resulting momentum has no component in the preferred frame. As a matter fact, SU(3) rotation is enough for a suitable choice of SU(3). These transformations form a discrete subgroup of SU(3) since hyper-integer property must be preserved. Massless states correspond to a null norm for the corresponding hyper integer unless one allows also tachyonic hyper primes with minimal representatives $(n_3, n_3 - 1, 0, ...)$, $n_3 = (p-1)/2$. Note that Gaussian primes with $p \ mod 4 \ = 1$

are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$ and would correspond to genuine tachyons. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

The notion of "irreducible" (see Appendix of [K95]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

4.3.2 Hyper-Octonionic Infinite Primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance. It is however not possible to interpret them as as 8-momenta with mass squared equal to prime. The proper identification of standard model quantum numbers will be discussed later.

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes.

Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

Stronger form of associativity and commutativity is obtained if infinite octonionic/quaternionic primes are just ordinary octonionic/quaternonic primes multiplied with ordinary infinite primes. This option is perhaps the more elegant one. For this option the non-commutativity and non-associativity are concentrated on the finite octonionic/quaternionic prime multiplying the commutative infinite prime. This picture allows also the map of infinite octonionic/quaternionic primes to products of finite octonionic/quaternionic primes and of polynomials.

4.4 How To Interpret The Infinite Hierarchy Of Infinite Primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

4.4.1 Infinite Primes And Hierarchy Of Super-Symmetric Arithmetic Quantum Field Theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. It turns also possible to assign to infinite prime space-time surface as a geometric correlate although the original proposal for how to achieve this failed. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-symplectic representations (for instance, ordinary particles correspond to states of this kind).

Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ .$$

2. Form the vacuum states

$$V_\pm = X \pm 1 \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_\pm(m,n,s) = \frac{mX}{s} \pm ns \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed. As found, the conjectured effective 2-dimensionality for hyper-octonionic primes allows the reduction of polynomial representation of hyper-octonionic primes to that for hyper-complex primes. This would be in accordance with the effective 2-dimensionality of the basic objects of quantum TGD.

The physical counterpart of n: th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

How infinite rationals correspond to quantum states and space-time surfaces?

The most promising answer to the question how infinite rationals correspond to space-time surfaces is discussed in detail in the next section. Here it is enough to give only the basic idea.

1. In zero energy ontology hyper-octonionic units (in real sense) defined by ratios of infinite integers have an nterpretation as representations for pairs of positive and negative energy states. Suppose that the quantum number combinations characterizing positive and negative energy quantum states are representable as superpositions of real units defined by ratios of infinite integers at each point of the space-time surface. If this is true, the quantum classical correspondence coded by the measurement interaction term of the Kähler-Dirac action maps the quantum numbers also to space-time geometry and implies a correspondence between infinite rationals and space-time surfaces.

2. The space-time surface associated with the infinite rational is in general not a union of the space-time surfaces associated with the primes composing the integers defining the rational. There the classical description of interactions emerges automatically. The description of classical states in terms of infinite integers would be analogous to the description of many particle states as finite integers in arithmetic quantum field theory. This mapping could in principle make sense both in real and p-adic sectors of WCW .

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime.

What is the interpretation of the higher level infinite primes?

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

4.4.2 Infinite Primes, The Structure Of Many-Sheeted Space-Time, And The Notion Of Finite Measurement Resolution

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in a rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface. It turns out that the notion of finite measurement resolution is the key concept: infinite prime characterizes angle measurement resolution. This gives a direct connection with the p-adicization program relying also on angle measurement resolution as well as a connection with the hierarchy of Planck constants. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory so that the characterization of the finite measurement resolution, which has been the ugly ducling of theoretical physics transforms to a beautiful swan.

The first intuitions

The concrete prediction of the general vision is that the hierarchy of infinite primes should somehow correspond to the hierarchy of space-time sheets or partonic 2-surfaces if one accepts the effective 2-dimensionality. The challenge is to find space-time counterparts for infinite primes at the lowest level of the hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.
2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by flux tubes to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at the lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on.... A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles. The concrete realization of multi-p p-adicity would be in terms of infinite integers coming as power series $\sum x_n N^n$ and having interpretation as p-adic numbers for any prime dividing N.
2. Effective 2-dimensionality would suggest that the individual p-adic topologies could be assigned with the 2-dimensional partonic surfaces. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. There are however reasons to think that even single partonic 2-surface corresponds to a multi-p p-adic topology.

Do infinite primes code for the finite measurement resolution?

The above describe heuristic picture is not yet satisfactory. In order to proceed, it is good to ask what determines the finite prime or set of them associated with a given partonic 2-surface. It is good to recall first the recent view about the p-adicization program relying crucially on the notion of finite measurement resolution.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M and measurement resolution does not depend on on the value of M. Situation is different if one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer. For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$ if modular arithetics is used to define the measurement resolution. This works at the first level of the hierarchy but not at higher levels. The alternative manner to assign a finite measurement resolution to M/N for given p is as $\Delta\phi = 2\pi|N/M|_p = 2\pi/p^n$. In this case the small fermionic part of the infinite prime would fix the measurement resolution. The argument below shows that only this option works also at the higher levels of hierarchy and is therefore more plausible.

3. p-Adicization conditions in their strong form require that the notion of integration based on harmonic analysis in symmetric spaces makes sense even at the level of partonic 2-surfaces. These conditions are satisfied if the partonic 2-surfaces in a given measurement resolution can be regarded as algebraic continuations of discrete surfaces whose points belong to the discrete variant of the $\delta M^4_\pm \times CP_2$. This condition is extremely powerful since it effectively allows to code the geometry of partonic 2-surfaces by the geometry of finite sub-manifold geometries for a given measurement resolution. This condition assigns the integer N to a given partonic surface and all primes appearing as factors of N define possible effective p-adic topologies assignable to the partonic 2-surface.

How infinite primes could then code for the finite measurement resolution? Can one identify the measurement resolution for $M/N = M/(Rp^n)$ as $\Delta\phi = ((M/R)\ mod\ p^n) \times 2\pi/p^n$ or as $\Delta\phi = 2\pi/p^n$? The following argument allows only the latter option.

1. Suppose that p-adic topology makes sense also for infinite primes and that state function reduction selects power of infinite prime P from the product of lower level infinite primes defining the integer N in M/N. Suppose that the rational defined by infinite integer defines measurement resolution also at the higher levels of the hierarchy.

2. The infinite primes at the first level of hierarchy representing Fock states are in one-one correspondence with finite rationals M/N for which integers M and N can be chosen to characterize the infinite bosonic part and finite fermionic part of the infinite prime. This correspondence makes sense also at higher levels of the hierarchy but M and N are infinite integers. Also other option obtained by exchanging "bosonic" and "fermionic" but later it will be found that only the first identification makes sense.

3. The first guess is that the rational M/N characterizing the infinite prime characterizes the measurement resolution for angles and therefore partially classifies also the finite sub-manifold geometry assignable to the partonic 2-surface. One should define what $M/N = ((M/R)\ mod\ P^n) \times P^{-n}$ is for infinite primes. This would require expression of M/R in modular arithmetics modulo P^n. This does not make sense.

4. For the second option the measurement resolution defined as $\Delta\phi = 2\pi|N/M|_P = 2\pi/P^n$ makes sense. The Fourier basis obtained in this manner would be infinite but all states $exp(ik/P^n)$ would correspond in real sense to real unity unless one allows k to be infinite P-adic integer smaller than P^n and thus expressible as $k = \sum_{m<n} k_m P^m$, where k_m are infinite integers smaller than P. In real sense one obtains all roots $exp(iq2\pi)$ of unity with $q < 1$ rational. For instance, for $n = 1$ one can have $0 < k/P < 1$ for a suitably chosen infinite

prime k. Thus one would have essentially continuum theory at higher levels of the hierarchy. The purely fermionic part N of the infinite prime would code for both the number of Fourier components in discretization for each power of prime involved and the ratio characterize the angle resolution.

The proposed relation between infinite prime and finite measurement resolution implies very strong number theoretic selection rules on the reaction vertices.

1. The point is that the vertices of generalized Feynman diagrams correspond to partonic 2-surfaces at which the ends of light-like 3-surfaces describing the orbits of partonic 2-surfaces join together. Suppose that the partonic 2-surfaces appearing a both ends of the propagator lines correspond to same rational as finite sub-manifold geometries. If so, then for a given p-adic effective topology the integers assignable to all lines entering the vertex must contain this p-adic prime as a factor. Particles would correspond to integers and only the particles having common prime factors could appear in the same vertex.

2. In fact, already the work with modelling dark matter [K36] led to ask whether particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions. It also seemed natural to assume that only the space-time sheets containing common primes in this collection can interact. This inspired the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The possibility of multi-p p-adicity raises the question about how to fix the p-adic prime characterizing the mass of the particle. The mass scale of the contribution of a given throat to the mass squared is given by $p^{-n/2}$, where $T = 1/n$ corresponds to the p-adic temperature of throat. Hence the dominating contribution to the mass squared corresponds to the smallest prime power p^n associated with the throats of the particle. This works if the integers characterizing other particles than graviton are divisible by the gravitonic p-adic prime or a product of p-adic primes assignable to graviton. If the smallest power p^n assignable to the graviton is large enough, the mass of graviton is consistent with the empirical bounds on it. The same consideration applies in the case of photons. Recall that the number theoretically very natural condition that in zero energy ontology the number of generalized Feynman graphs contributing to a given process is finite is satisfied if all particles have a non-vanishing but arbitrarily small p-adic thermal mass [K111].

Interpretational problem

The identification of infinite prime as a characterizer of finite measurement resolution looks nice but there is an interpretational problem.

1. The model characterizing the quantum numbers of WCW spinor fields to be discussed in the next section involves a pair of infinite primes P_+ and P_- corresponding to the two vacuum primes $X \pm 1$. Do they correspond to two different measurement resolutions perhaps assignable to CD and CP_2 degrees of freedom?

2. Different measurement resolutions in CD and CP_2 degrees of freedom need not be not a problem as long as one considers only the discrete variants of symmetric spaces involved. What might be a problem is that in the general case the p-adic primes associated with CD and CP_2 degrees of freedom would not be same unless the integers N_+ and N_- are assumed to have have same prime factors (they indeed do if $p^0 = 1$ is formally counted as prime power factors).

3. The idea of assigning different p-adic effective topologies to CD and CP_2 does not look attractive. Both CD and CP_2 and thus also partonic 2-surface could however possess simultaneously both p-adic effective topologies. This kind of option might make sense since the

integers representable as infinite powers series of integer N can be regarded as p-adic integers for all prime factors of N. As a matter fact, this kind of multi-p p-adicity could make sense also for the partonic 2-surfaces characterized by a measurement resolution $\Delta\phi = 2\pi M/N$. One would have what might be interpreted as N_+N_--adicity.

4. It will be found that quantum measurement means also the measurement of the p-adic prime selecting same p-adic prime from N_+ and N_-. If $N_\pm$ is divisible only by $p^0 = 1$, the corresponding angle measurement resolution is trivial. From the point of view of consciousness state function reduction selects also the p-adic prime characterizing the cognitive representation which is very natural since quantum superpositions of different p-adic topologies are not natural physically.

4.4.3 How The Hierarchy Of Planck Constants Could Relate To Infinite Primes And P-Adic Hierarchy?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K110], the dark matter hierarchy characterized by increasing values of $\hbar$ [K34, K32], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations give hopes about developing a more quantitative vision about the relationship between these hierarchies, in particular between the hierarchy of infinite primes, p-adic length scale hierarchy, and the hierarchy if Planck constants.

If infinite primes code for the hierarchy of measurement resolutions, the correlations between the p-adic hierarchy and the hierarchy of Planck constants indeed suggest themselves and allow also to select between two interpretations for the fact that two infinite primes N_+ and N_- are needed to characterize elementary particles (see the next section).

Recall that the hierarchy of Planck constants in the most general situation corresponds to a replacement M^4 and CP_2 factors of the embedding space with singular coverings and factor spaces. The condition that Planck constant is integer valued allows only singular coverings characterized by two integers n_a *resp.* n_b assignable to CD *resp.* CP_2. This condition also guarantees that a given value of Planck constant corresponds to only a finite number of pages of the "Big Book" and therefore looks rather attractive mathematically. This option also forces evolution as a dispersion to the pages of the books characterized by increasing values of Planck constant.

Concerning the correspondence between the hierarchy of Planck constants and p-adic length scale hierarchy there seems to be only single working option. The following assumptions make precise the relationship between finite measurement resolution, infinite primes and hierarchy of Planck constants.

1. Measurement resolution CD *resp.* CP_2 degrees of freedom is assumed to correspond to the rational M_+/N_+ *resp.* M_-/N_-. $N_\pm$ is identified as the integer assigned to the fermionic part of the infinite integer..

2. One must always fix the consideration to a fixed p-adic prime. This process could be regarded as analogous to fixing the quantization axes and p would also characterize the p-adic cognitive space-time sheets involved. The p-adic prime is therefore same for CD and CP_2 degrees of freedom as required by internal consistency.

3. The relationship to the hierarchy of Planck constants is fixed by the identifications $n_a = n_+(p)$ and $n_b = n_-(p)$ so that the number of sheets of the covering equals to the number of bosons in the fermionic mode p of the quantum state defined by infinite prime.

4. A physically attractive hypothesis is that number theoretical bosons *resp.* fermions correspond to WCW orbital *resp.* spin degrees of freedom. The first ones correspond to the symplectic algebra of WCW and the latter one to purely fermionic degrees of freedom.

Consider now the basic consequences of these assumptions from the point of view of physics and cognition.

1. Finite measurement resolution reduces for a given value of p to

$$\Delta\phi = \frac{2\pi}{p^{n_\pm(p)+1}} = \frac{2\pi}{p^{n_{a/b}}} \ ,$$

   where $n_\pm(p) = n_{a/b} - 1$ is the number of bosons in the mode p in the fermionic part of the state. The number theoretical fermions and bosons and also their probably existing physical counterparts are necessary for a non-trivial angle measurement resolution. The value of Planck constant given by

$$\frac{\hbar}{\hbar_0} = n_a n_b = (n_+(p)+1) \times (n_-(p)+1)$$

   tells the total number of bosons added to the fermionic mode p assigned to the infinite prime.

2. The presence of $\hbar > \hbar_0$ partonic 2-surfaces is absolutely essential for a Universe able to measure its own state. This is in accordance with the interpretation of hierarchy of Planck constants in TGD inspired theory of consciousness. One can also say that $\hbar = 0$ sector does not allow cognition at all since $N_\pm = 1$ holds true. For given p $\hbar = n_a n_b = 0$ means that given fermionic prime corresponds to a fermion in the Dirac sea meaning $n_\pm(p) = -1$. Kicking out of fermions from Direac sea makes possible cognition. For purely bosonic vacuum primes one has $\hbar = 0$ meaning trivial measurement resolution so that the physics is purely classical and would correspond to the purely bosonic sector of the quantum TGD.

3. For $\hbar = \hbar_0$ the number of bosons in the fermionic state vanishes and the general expression for the measurement resolution reduces to $\Delta\phi = 2\pi/p$. When one adds $n_\pm(p)$ bosons to the fermionic part of the infinite prime, the measurement resolution increases from $\Delta\phi = 2\pi/p$ to $\Delta\phi = 2\pi/p^{n_\pm(p)+1}$. Adding a sheet to the covering means addition of a number theoretic boson to the fermionic part of infinite prime. The presence of both number theoretic bosons and fermions with the values of p-adic prime $p_1 \neq p$ does not affect the measurement resolution $\Delta\phi = 2\pi/p^n$ for a given prime p.

4. The resolutions in CD and CP_2 degrees of freedom correspond to the same value of the p-adic prime p so that one has dicretizations based on $\Delta\phi = 2\pi/p^{n_a}$ in CD degrees of freedom and $\Delta\phi = 2\pi/p^{n_b}$ in CP_2 degrees of freedom. The finite sub-manifold geometries make sense in this case and since the effective p-adic topology is same, the continuation to continuous p-adic partonic 2-surface is possible.

p-Adic thermodynamics involves the p-adic temperature $T = 1/n$ as basic parameter and the p-adic mass scale of the particle comes as $p^{-(n+1)/2}$. The natural question is whether one could assume the relation $T_\pm = 1/(n_\pm(p)+1)$ between p-adic temperature and infinite prime and thus the relations $T_a = 1/n_a(p)$ and $T_b = 1/n_b(p)$. This identification is not consistent with the recent physical interpretation of the p-adic thermodynamics nor with the view about dark matter hierarchy and must be given up.

1. The minimal non-trivial measurement resolution with $n_i = 1$ and $\hbar = \hbar_0$ corresponds to the p-adic temperature $T_i = 1$. p-Adic mass calculations indeed predict $T = 1$ for fermions for $\hbar = \hbar_0$. In the case of gauge bosons $T \geq 2$ is favored so that gauge bosons would be dark. This would require that gauge bosons propagate along dark pages of the Big Book and become "visible" before entering to the interaction vertex.

2. p-Adic thermodynamics also assumes same p-adic temperature in CD and CP_2 degrees of freedom but the proposed identification allows also different temperatures. In principle the separation of the super-conformal degrees of freedom of CD and CP_2 might allow different p-adic temperatures. This would assign to different p-adic mass scales to the particles and the larger mass scale should give the dominant contribution.

3. For dark particles the p-adic mass scale would be by a factor $1/\sqrt{p}^{n_i(p)-1}$ lower than for ordinary particles. This is in conflict with the assumption that the mass of the particle does not depend on $\hbar$. This prediction would kill completely the recent vision about the dark matter.

4.5 How Infinite Primes Could Correspond To Quantum States And Space-time Surfaces?

The hierarchy of infinite primes is in one-one correspondence with a hierarchy of second quantizations of an arithmetic quantum field theory. The additive quantum number in question is energy like quantity for ordinary primes and given by the logarithm of prime whereas p-adic length scale hypothesis suggests that the conserved quantity is proportional to the inverse of prime or its square root. For infinite primes at the first level of hierarchy these quantum numbers label single particles states having interpretation as ordinary elementary particles. For octonionic and hyper-octonionic primes the quantum number is analogous to a momentum with 8 components. The question is whether these number theoretic quantum numbers could have interpretation as genuine quantum numbers. Quantum classical correspondence raises another question. Is it possible to label space-time surfaces by infinite primes? Could this correspondence be even one-to-one?

I have considered these questions already more than decade ago. The discussion at that time was necessarily highly speculative and just a mathematical exercise. After that time however a lot of progress has taken place in quantum TGD and it is highly interaction to see what comes out from the interaction of the notion of infinite prime with the notions of zero energy ontology and generalized embedding space, and with the recent vision about how Chern-Simons Dirac term in the Kähler-Dirac action allows to code information about quantum numbers to the space-time geometry. The possibility of this coding allows to simplify the discussion dramatically. If one could map infinite hyper-octonionic or hyper-quaternionic primes to quantum numbers of the standard model naturally, then the their map of to the geometry of space-time surfaces would realize the coding of space-time surfaces by infinite primes (and more generally by integers and rationals). Also a detailed realization of number theoretic Brahman=Atman identity would emerge as an outcome.

4.5.1 A Brief Summary About Various Moduli Spaces And Their Symmetries

It is good to sum up the number theoretic symmetries before trying to construct an overall view about the situation. Several kinds of number theoretical symmetry groups are involved corresponding to symmetries in the moduli spaces of hyper-octonionic and hyper-quaternionic structures, symmetries mapping hyper-octonionic primes to hyper-octonionic primes, and translations acting in the space of causal diamonds (CDs) and shifting. The moduli space for CDs labeled by pairs of its tips that its pairs of points of $M^4 \times CP_2$ is also in important role.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit. $SO(7,1)$ acts as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for hyper-quaternionic structures.

2. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

3. Color group $SU(3)$ is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. For given hyper-octonionic prime one can identify a subgroup of SU(3) generating a finite set of hyper-octonionic primes for it at sphere S^7. This suggests wave function at the orbit of given hyper-octonionic prime in turn generalizing to wave functions in the space of infinite primes.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

These symmetries deserve a more detailed discussion.

1. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures. $SO(7)$ respects the choice of the real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as substructures of hyper-octonionic structure. The choice of global hyper-octonionic structures involves also a choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of CD. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture. The choice of both tips of CD reduces $SO(1,3)$ to $SO(3)$.

2. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

3. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit meaning a choice of preferred hyper-complex plane $M^4 \subset M^4$. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement in number theoretical and as it turns also in physical sense. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

4.5.2 Associativity And Commutativity Or Only Their Quantum Variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Hyper-quaternionicity formulated in terms of the Kähler-Dirac gamma matrices defined by Kähler action fixes classical space-time dynamics and a very beautiful algebra formulation of quantum TGD in terms of hyper-octonionic local Clifford algebra of embedding space emerges. There is no need for the use of hyper-octonion real analytic maps although one cannot exclude the possibility that they might be involved with the construction of hyper-quaternionic space-time surfaces.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to complex rational infinite primes. Since one can decompose complex rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative and commutative. In case of associativity this means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

4.5.3 How Space-Time Geometry Could Be Coded By Infinite Primes?

Second key question is whether space-time geometry could be characterized in terms of infinite primes (and integers and rationals in the most general case) and how this is achieved. The question is how the quantum states consisting of fundamental fermions serving as building bricks of elementary particles could be coded by infinite quaternionic integeres to which one can assign ordinary finite quaternionic primes.

The basic idea is roughly that at the first level of the hierarchy the finite primes appearing as building blocks of infinite prime correspond to structures formed by pairs or wormhole contacts assigned with elementary particles.

1. The partonic orbits defined by wormhole throats could be characterized by finite primes specifying the preferred p-adic topology assignable to the p-adic "cognitive representation" of the throat.

2. One could assign hyper-quaternionic integer to the real particle as its four-momentum. In this case the mass shell condition would fix the hyper-quaternionic integer to a high extent. All discrete Lorentz boosts of the particle state taking hyper-quaternionic integers to hyper-quaternionic integers would correspond to the same p-adic integer (prime) defined by the length of the Lorentz boosted hyper-quaternionic integer. The p-adic prime characterizing virtual particle would be one of the primes appearing in the factorization of this integer to a product of powers of prime, most naturally the one whose power is largest.

   Note that p-adic length scale hypothesis suggests that the p-adic primes near powers of two are favored for on mass shell particles and perhaps also for the virtual particles.

3. For fundamental fermions associated with boundaries of string world sheets and appearing as building bricks of particles the masses would vanish on mass shell so that the hyper-quaternionic integer would in this case have vanishing norm.

   The virtual four-momentum assigned to a virtual fermion line as a generalized eigenvalue of Chern-Simons Dirac operator would correspond to hyper-quaternionic integer. In this case p-adic prime would be defined as for physical particles and would depend on the mass of the virtual particle. If the integration over virtual momenta by residue calculus effectively leads to an integral over on mass shell massless virtual momenta with non-physical spinor helicities then also virtual fundamental fermions would correspond to zero norm hyper-quaternionic integers.

4. The correlation between particle's four-momentum and the p-adic prime characterizing corresponding cognitive representation would be in accordance with quantum classical correspondence.

5. The hyperquaternionic primes appearing as largest factors in the factorizion of hyper-quaternionic integers assignable with physical particles could be interpreted as building bricks of an infinite hyperquaternionic prime characterizing the many-particle state and at least the boundaries of string world sheets. The idea that p-adic space-time surfaces defined "cognitive representations" as p-adic chart maps of real space-time surfaces and vice versa (as the TGD based definition of p-adic manifolds assumes) suggests that the p-adic primes in question characterize also space-time regions rather than only the boundaries of string world sheets.

A couple of comments about this speculation are in order.

1. ZEO implies a hierarchy of CDs within CDs and this hierarchy as well as the hierarchy of space-time sheets corresponds naturally to the hierarchy of infinite primes. One can assign standard model quantum numbers to various partonic 2-surfaces with positive and negative energy parts of the quantum state assignable to the light-like boundaries of CD. Also infinite integers and rationals are possible and the inverses of infinite primes would

naturally correspond to elementary particles with negative energy. The condition that zero energy state has vanishing net quantum numbers implies that the ratio of infinite integers assignable to zero energy state equals to real unit in real sense and has has vanishing total quantum numbers.

2. Neither quantum numbers nor infinite primes coding them cannot characterize the partonic 2-surface itself completely since they say nothing about the deformation of the space-time surface but only about labels characterizing the WCW spinor field. Also the topology of partonic 2-surface fails to be coded. Quantum classical correspondence however suggests that this correspondence could be possible in a weaker sense. In the Gaussian approximation for functional integral over the world of classical worlds space-time surface and thus the collection of partonic 2-surfaces is effectively replaced with the one corresponding to the maximum of Kähler function, and in this sense one-one correspondence is possible unless the situation is non-perturbative. In this case the physics implied by the hierarchy of Planck constants could however guarantee uniqueness.

   One of the basic ideas behind the identification of the dark matter as phases with non-standard value of Planck constant is that when perturbative description of the system fails, a phase transition increasing the value of Planck constant takes place and makes perturbative description possible. Geometrically this phase transition means a leakage to another sector of the embedding space realized as a book like structure with pages partially labeled by the values of Planck constant. Anyonic phases and fractionization of quantum numbers is one possible outcome of this phase transition. An interesting question is what the fractionization of the quantum numbers means number theoretically.

4.6 Infinite Primes And Mathematical Consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

4.6.1 Algebraic Brahman=Atman Identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes, which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_{\pm} = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_{\pm} \ mod \ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labeled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.

2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich

number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.

3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense.

Number theoretic anatomy of space-time point

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of embedding space force the conclusion that WCW spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of embedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the WCW spinor fields regarded as wave functions in the set of embedding space points which are equivalent in real sense. Embedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single embedding space point.

To realize this picture would require that WCW spinor fields and perhaps even WCW allow a mapping to the number theoretic anatomies of space-time point. In finite-dimension Euclidian spaces momentum space labelling plane wavs is dual to the space. One could hope that also now the "orbital" quantum numbers of WCW spinor fields could code for WCW in given measurement resolution. The construction of the previous sections realize the mapping of the quantum states defined by WCW spinors fields assignable to given CD to wave function in the space of hyper-octonionic units. These wave functions can be also regarded as linear combinations of these units if the coefficients are complex numbers formed using the commuting imaginary unit of complexified octonions so that the Hilbert space like structure in question would have purely number theoretic meaning. The rationals defined by infinite primes characterize also measurement resolution and classify the finite sub-manifold geometries associated with partonic two-surfaces. At higher levels one has rationals defined by ratios of infinite integers and one can ask whether this interpretation generalizes.

Note that one must distinguish between two kinds of hyper-octonionic units.

1. Already in the case of complex numbers one has rational complex units defined in terms of Pythagorean triangle and their products generate infinite dimensional space. The hyper-octonionic units defined as ratios U of infinite integers and suggested to provide a representation of WCW spinor fields correspond to these. The powers U^m define roots of unity which can be regarded analogous to $exp(i2\pi x)$, where x is not rational but the exponent itself is complex rational.

2. Besides this there are roots of unity which are in general algebraic complex numbers. These roots of unit correspond to phases $exp(i2\pi M/N)$, where M/N is ratio of real infinite integers and i is the commuting hyper-octonionic imaginary unit. These real infinite integers can be assigned to hyper-octonionic integers by replacing everywhere finite hyper-octonionic primes with their norm which is ordinary prime. By the previous considerations only the phases $exp(i2\pi M/P^n)$ make sense p-adically for infinite primes P.

4.6.2 Leaving The World Of Finite Reals And Ending Up To The Ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals at the first level of hierarchy. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. WCW is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of embedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced WCW consisting of the maxima of Kähler function to the anatomy of space-time point. Also WCW spinors and perhaps also the modes of WCW spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of embedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even "simultaneous" time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just "epsilons" if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

4.6.3 Infinite Primes And Mystic World View

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{aligned} P &= nY^{r_1} + mS \ , \quad r = 1, 2, ... \\ m &= m_0 + P_{r_2}(Y) \ , \\ Y &= \frac{X}{S} \ , \\ S &= \prod_i P_i \ . \end{aligned}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightenment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1!$), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the WCW . In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level "beings" (one might call them Angels, Gods, etc...).

4.6.4 Infinite Primes And Evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labeling the WCW sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first WCW spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [K39]. Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two ways.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by.... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are

indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

(a) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

(b) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle: the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

4.7 Does The Notion Of Infinite-P P-Adicity Make Sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing the tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The following list of questions is rather natural with the background provided by the p-adic physics.

1. Can one generalize the notion of p-adic norm and p-adic number field to include infinite primes? Could one define the counterpart of p-adic topology for literally infinite values of p? Does the topology R_P for infinite values of P approximate or is it equivalent with real topology as p-adic topology at the limit of infinite p is assumed to do (at least in the sense that p-adic variants of Diophantine equations at this limit correspond to ordinary Diophantine equations)? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale.

2. Canonical identification maps p-adic numbers of unit norm to real numbers in the range $[0, p]$. Does the canonical identification map the p-adic numbers R_P associated with infinite prime to reals? Could the number fields R_P provide alternative formulations/generalizations of the non-standard analysis based on the hyper-real numbers of Robinson [A22] ?

3. The notion of finite measurement resolution for angle variables given naturally as a hierarchy $2\pi/p^n$ of resolutions for a given p-adic prime defining aa hierarchy of algebraic extension of p-adic numbers is central in the attempts to formulate p-adic variants of quantum TGD and fuse them with real number based quantum TGD [K95] . If p is replaced with an infinite prime, the angular resolution becomes ideal and the roots of unity $exp(2\pi m/p^n)$ are replaced with real units unless also the integer m is replaced with an infinite integer M so that the ratio M/P^n is finite rational number. Could this approach be regarded as alternative for real number based notion of phase angle?

The consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p. Using infinite primes one might obtain the real theory in this approximation.

The question discussed in this section is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can formally introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

4.7.1 Does Infinite-P P-Adicity Reduce To Q-Adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \tag{4.7.1}$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement "infinite integer modulo P" means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime. The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarchy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1+x_1P+..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1+x_1P+...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the "finite field" defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1 + y_1 P + ...)$$

and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the lowest order term in the polymomial expansion. Thus the norm would be infinite for $n < 0$, equal to one for $n = 0$ and vanish for $n > 0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

For the lowest level infinite primes the natural choice of a would be the rational number defined by it so that infinite-P p-adicity would indeed correspond to q-adicity meaning that number field property is lost.

4.7.2 Q-Adic Topology Determined By Infinite Prime As A Local Topology Of WCW?

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the WCW .

This view conforms also with the idea of algebraic holography. The sub-spaces of WCW can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between WCW and number theoretic anatomy of point of the embedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to WCW integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteeing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

4.7.3 The Interpretation Of The Discrete Topology Determined By Infinite Prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naïvely generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naïve sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy of preferred extremals: one can add to any preferred extremal a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also Kähler-Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since $p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

4.8 How Infinite Primes Relate To Other Views About Mathematical Infinity?

Infinite primes is a purely TGD inspired notion. The notion of infinity is number theoretical and infinite primes have well defined divisibility properties. One can partially order them by the real norm. p-Adic norms of infinite primes are well defined and finite. The construction of infinite primes is a hierarchical procedure structurally equivalent to a repeated second quantization of a supersymmetric arithmetic quantum field theory. At the lowest level bosons and fermions are labelled by ordinary primes. At the next level one obtains free Fock states plus states having interpretation as bound many particle states. The many particle states of a given level become the single particle states of the next level and one can repeat the construction ad infinitum. The analogy with quantum theory is intriguing and I have proposed that the quantum states in TGD Universe correspond to octonionic generalizations of infinite primes.

It is interesting to compare infinite primes (and integers) to the Cantorian view about infinite ordinals and cardinals. The basic problems of Cantor's approach which relate to the axiom of choice, continuum hypothesis, and Russell's antinomy: all these problems relate to the definition of ordinals as sets. In TGD framework infinite primes, integers, and rationals are defined purely algebraically so that these problems are avoided. It is not surprising that these approaches are not equivalent. For instance, sum and product for Cantorian ordinals are not commutative unlike for infinite integers defined in terms of infinite primes.

Set theory defines the foundations of modern mathematics. Set theory relies strongly on classical physics, and the obvious question is whether one should reconsider the foundations of mathematics in light of quantum physics. Is set theory really the correct approach to axiomatization?

1. Quantum view about consciousness and cognition leads to a proposal that p-adic physics serves as a correlate for cognition. Together with the notion of infinite primes this suggests that number theory should play a key role in the axiomatics.

2. Algebraic geometry allows algebraization of the set theory and this kind of approach suggests itself strongly in physics inspired approach to the foundations of mathematics. This means powerful limitations on the notion of set.

3. Finite measurement resolution and finite resolution of cognition could have implications also for the foundations of mathematics and relate directly to the fact that all numerical approaches reduce to an approximation using rationals with a cutoff on the number of binary digits.

4. The TGD inspired vision about consciousness implies evolution by quantum jumps meaning that also evolution of mathematics so that no fixed system of axioms can ever catch all the mathematical truths for the simple reason that mathematicians themselves evolve with mathematics.

I will discuss possible impact of these observations on the foundations of physical mathematics assuming that one accepts the TGD inspired view about infinity, about the notion of number, and the restrictions on the notion of set suggested by classical TGD.

4.8.1 Cantorian View About Infinity

The question which I have but repeatedly under the rug during the last fifteen years concerns the relationship of infinite primes to the notion of infinity as Cantor and his followers have understood it. I must be honest: I have been too lazy to even explain to myself what Cantor really said. Therefore the reading of the New Scientist article "The Ultimate logic: to infinity and beyond" (see `http://tinyurl.com/3va48jq`) [A60] was a pleasant surprise since it gave a bird's eye of view about how the ideas about infinity have evolved after Cantor as a response to severe difficulties in the set theoretic formulation for the foundations of Mathematics.

Cantor's paradize

I try to summarize Cantor's view about infinity first. Cantor was the pioneer of set theory, in particular the theory of infinite sets. Cantor started his work around 1870. His goal was to formulate all notions of mathematics in terms of sets, in particular natural numbers. Cardinals and ordinals define two kind of infinite numbers in Cantor's approach.

1. Cantor realized that real numbers are "more numerous" than natural numbers and understood the importance of one-to-one correspondence (bijection) in set theory. One can say that two sets related by bijection have same cardinality. This led to the notion of cardinal number. Cardinals are represented as sets and two cardinals are same if a bijection exists between the corresponding sets. For instance, the infinite cardinals assignable to natural numbers and reals are different since no bijection between them exists.

2. The definition of ordinal relies on successor axiom of natural numbers generalized to allow infinitely large ordinals. Given ordinal can be identified as the union of all ordinals strictly smaller than it. Well ordering is a closely related notion and states that every subset of ordinals has smallest element. One can classify ordinals to three types: 0, elements with precedessor, and elements without precedessor such as ω, which corresponds to the ordinal defined as the union of all natural numbers.

   The number of ordinals much larger than the number of cardinals. This is clear since the notion of ordinal involves additional structure coming from their ordering. A given cardinal corresponds to infinitely many ordinals and one can identify the cardinal as the smallest ordinal of this kind. For instance, ω and $\omega + n$ correspond to same cardinal α_0 (countable infinity) for all finite values of n.

3. Cantor introduced the notion of power set as the set of all subsets of the set and proved that the cardinality of the power set is larger than that of set. Cantor introduced also the continuum hypothesis stating that there are no cardinals between the cardinal $\aleph_0$ *resp.* $\aleph_1$ assignable to natural numbers *resp.* reals. Hilbert represented continuum hypothesis as one of his 23 problems in his talk at the 1900 International Congress of Mathematicians in Paris. Hilbert was also a defender of Cantor and introduced the term Cantor's paradise.

4. Cantor developed the arithmetics of ordinals based on sum, product, and power: each of these operations is expressible in terms of set theoretic concepts. For infinite ordinals multiplication

and sum are not commutative anymore. This looks highly counter intuitive and requires detailed definition of the sum and product. Sum means just writing the ordered sequences representing ordinals in succession. To see the non-commutativity of sum it is enough to notice that the number of elements having precedessor is not the same for $\omega + n$ and $n + \omega$.

To see the non-commutativity of product it is enough to notice that the product is define as cartesian product $S \times T$ of the ordered sets representing the ordinals. This means that every element of T is replaced with S. It is easy to see that $n \times \omega$ and $\omega \times n$ are different.

One can define also the powers (exponentials) in the arithmetics of ordinals: exponent must reduce to the notion of power set X^Y, which can be realized as the set of maps $Y \rightarrow X$ and has formally $\#X^{\#Y}$ elements.

It is pity that the we physicists have so pragmatic attitude to mathematics that we do not have time to realize the beauty of the idea about reduction of all mathematics to set theory. This is even more regrettable since it might well be that the manner to make progress in physics might require replacing the mathematics with a mathematics which does not rely on set theory alone.

Snakes in Cantor's paradise

Cantor's paradise is extremely beautiful place but there are snakes there. Continuum hypothesis looked to Cantor intuitively obvious but the attempts to prove it failed. Betrand Russel showed in 1901 that the logical basis of Cantor's set theory was flawed. This manifested itself via a simple paradox. Assume that it makes sense to speak about the set of all ordinals. This is by definition ordinal itself since ordinal is a set consisting of all ordinals strictly smaller than it. But this would mean that the set of all ordinals is its own member! The famous barber's paradox is a more concrete manner to express Russel's antinomy. One cannot speak of the set of ordinals and must introduce the notion of class. Russell introduced also the notion of types and type theory.

At 1920 Ernst Zermelo and Abraham Fraenkel devised a series of rules for manipulating sets but these rules did not allow to resolve the status of the continuum hypothesis. The stumbling block was the rule known as "axiom of choice" stating that if you have a collection of sets you can form a new set by picking one element from each of them. At first this sounds rather obvious but in the case when there is no obvious rule telling how to do it, situation becomes non-trivial. Then Polish mathematicians Stefan Banach and Alfred Tarski managed to show how the axiom would allow the division of a spherical ball to six subsets which can then be arranged to two balls with the same size as the original ball using only rotations and translations. These six sets are non-measurable in terms of Lebesque measure. The non-intuitive outcome must relate to the definition of the volume of the ball that is integration or measure theory: the axioms of measure theory should bring in constraints preventing construction of the six sets.

Around 1931 Kurt Gödel proved the incompleteness theorem that it is not possible to axiomaticize arithmetics using any axiom system. There always remain unprovable propositions, which are true and cannot be proved to be true. This kind of statement is analogous to "I am a statement which cannot be proved to be true". If this statement could be proved to be true it would not be true.

Constructing logical universes

The attempts to expel the snakes from Cantor's paradise led to the idea that by posing some constraints it might be possible to construct logically consistent set theory obeying Zermelo-Fraenkel axioms such that continuum hypothesis and the axiom of choice would hold true and which would be free of paradoxes such as Banach-Tarski paradox.

Around 1938 Gödel introduced what he called constructible Universe (see `http://tinyurl.com/y43jun`) or L world satisfying these constraints. The structure of L world is hierarchical and one can say that the successor idea manifests itself directly in the construction. The levels are labeled by ordinals and one can always add a new level. The introduction of a new level to the hierarchy means that new axioms are introduced to the system bringing in meta level to the mathematical structure. The axiom system can be extended indefinitely. Gödel's theorem holds true at given level of hierarchy but by adding new levels non-probable truths can be made provable.

1963 Paul Cohen however demonstrated that there is infinite number of this kind of L worlds. In some of them continuum hypothesis holds true, in some of them the number of cardinals between $\aleph_0$ andα_1 can be arbitrary large - even infinite. This initiated a boom of constructions brings in mind the inflation of GUTs in particle physics and the endless variety of brane constructions and the landscape misery of M-theory. From the point of view of physicist the non-uniqueness in foundations of mathematics does not seem to matter much since the everyday mathematics would remain the familiar one. One can of course ask what about quantum theory: should quantum physics replace classical physics in the formulation of fundamental fo mathematics.

For instance, von Neumann (see `http://tinyurl.com/lkndbqo`) proposed one particular L world. In von Neumann unverse one starts from natural numbers and constructs its power set and at each step in the construction one consideres power set assigned to the sete obtained at the previous level. It is clear that one imagine several options. One could consider all subsets, only finite subsets, or only subsets which have cardinality smaller than the set itself. Power sets identified as the set of all finite subsets would give minimal option. Power set identified as the set of all subsets would give the maximal option.

The work of Hugh Woodin represented in 2010 International Congress of Mathematicians in Hyderabad, India represents the last twist in the story. Woodin argues that one must step outside the system that is conventional mathematical world to solve the problem. Woodin has introduced so called Woodin cardinals whose existence implies that all "projective" subsets of reals have a measurable size: it is not an accident that the word "measure" appears here when one recalls what Banach-Tarski paradox states. Woodin was motivated by the problems of set theory. He expresses this by saying "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable".

Woodin proposed his own constructive universe which he calls ultimate L. It has all the desired properties: in particular, continuum hypothesis holds true. Physicists reader need not get frustrated if he fails to intuit why this is the case: for a decade ago Wooding himself did not believe in this. Also this L world is infinite tower to which one can add new levels.

4.8.2 The Notion Of Infinity In TGD Universe

The construction of infinite primes, integers, and rationals brings strongly in mind the L worlds of Gödel and followers and this inspires the idea about concrete comparison of these approaches to see the differences.

Rule of thumb

It is good to start with a rule of thumb allowing to make strong conclusions about the cardinalities of infinite primes. If one considers the set formed by all finite subsets of a countable set you get a countable set because these subsets can be expressed as bit sequences with finite number if non-vanishing binary digits telling whether given element of set belongs to the subset or not: this bit sequence corresponds to a unique integer. If *all* subsets (also infinite) are allowed the set is not countably finite. If continuum hypothesis holds true it has at least as many elements as real line.

2-adic integers are good example. Consider first all 2-adic numbers with a *finite* number of non-vanishing bits (finite as real numbers). You get a countably infinite set since you can map these bit sequences to natural numbers in an obvious manner.

Consider next all possible bit sequences: most of them have infinite number bits. These numbers form naturally 2-adic continuum with 2-adic topology and differentiability. 2-adics can be mapped to real continuum in simple manner: canonical identification allows to do this continuously. The cardinality of these bit sequences is same as for reals as the rule of thumb would predict.

The hierarchy of infinite integers is based on number theoretical view about infinity and it would seem that these infinities are between the countable infinity and infinity defining the number of points of real axis. This reflects the fact that number theoretic infinity is much more refined notion than the infinities associated with cardinals and even ordinals. For instance, one can divide these infinities whereas Cantorian arithmetics contains only sum, product and power.

How Cantor's ordinals relate to the construction of infinite primes?

The fascinating question is whether the comparison of the construction of infinite primes, integers and rationals could relate to the work of Cantor and Gödel and his followers could provide new insights about infinite primes themselves.

1. What is intriguing that L-worlds are defined as infinite hierarchies just as the hierarchy of infinite primes and associated hierarchies. The great idea is that these constructions are essentially set theoretic in accordance with the vision that mathematics should reduce to set theory. As already noticed, naïve set theory however leads to paradoxes which motivates the work of Gödel and followers. The basic physical philosophy is the identification of physical state as a set: this is essentially a notion belonging to classical physics.

2. TGD approach is algebraic rather than set theoretic. The construction is based on explicit formulas assuming the existence of weird quantities defined as product of all primes at previous level. These quantities can be taken as purely algebraic notions without any attempt to find a set theoretic definition.

   The possibility to interpret the construction as a repeated second quantization of a supersymmetric arithmetic quantum field theory with bosons and fermions labeled by ordinary primes at the loweset level of hierarchy replaces the set theoretic picture. These weird products of all primes represent Dirac sea at a given level of hierarchy and the many particles states of previous level become elementary particles at the new level of hierarchy. This construction is proposed to have a direct physical realization in terms of many-sheeted space-time and generalized to the level of octonionic primes is suggested to allow number theoretic interpretation of standard model quantum numbers.

   Perhaps it is not mere arrogance of quantum physics to argue that the classical set theoretic view about physical state is replaced with quantum view about it. Algebra replaces set theory and real and p-adic topologies are essential: for instance, infinite primes are infinite only in real topology.

One can raise many interesting questions. Although the underlying philosophies are very different, one can ask whether it might be possible to reduce TGD inspired construction to set theory playing key role in the construction of ordinals?

1. Can one assign to a given infinite integer a set in a natural manner? At the lowest level of hierarchy infinite prime can be mapped to a rational. Could one assign to this rational a set in cartesian product $N \times N$? Does this argument generalize to higher levels? Could the construction discussed in [K59] allow to realize the set theoretic representation?

2. The notion of divisibility and explicit formulas for infinite integers obviously imply that the number of infinite numbers is much larger than cardinals of Cantor. This is true also for the ordinals of Cantor. How infinite integers relate to the ordinals of Cantor for which successor axiom is true? Also now it makes sense to form successors and in general they correspond to products of infinite primes which can be mapped to polynomials of several variables. For infinite integers however also the precedessor always exists. For instance $X \pm 1$ are infinite primes, where X represents the product of primes at previous level. Only zero fails to have precedessor for infinite natural numbers.

3. In TGD framework one loses the very essential notion of well-orderedness stating that every ordinal corresponds to a set with smallest element: that is element without precedessor. For instance, the infinite numbers known as limits and by definition are infinite and have no precedessor, the simplest example about limit is ω, which corresponds to the union of all natural numbers. The study of precedessors allowed to conclude that the sum and product are non-commutative for ordinals. Since the notion of well-ordered set does not make sense for infinite integers, one cannot identify infinite integers as ordinals.

   One must however remember that just the well-orderedness hypothesis together with successor axiom allows to express ordinal as a union of strictly smaller ordinals. This in turn leads to the conclusion that ordinals cannot form a set and to Russel's antinomy and are responsible for the many problems of set theory forcing Wooding to sigh "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable". Maybe the well-orderedness axiom is simply too strong for infinite ordinals.

4. Sum, product, and power are the basic operations in the arithmetics of ordinals. All they reduce to set theoretic constructions. One can however define these operations purely algebraically. The algebraic definition of sum and product makes sense since one can map the infinite integers to polynomials of several variables. The possibly existing set theoretic definition of infinite integers using infinite sets cannot be consistent with the commutativity of sum and product defined algebraically. Either algebra or set theory but not both!

5. Also the notion of power makes sense for ordinals and relies on the notion of power set. Could the algebraic definition of exponential make sense? If the exponent N of M^N is finite integer, then the exponent makes sense for infinite M. If N is infinite integer it does not. Hence it seems that the analogs of numbers like ω^ω do not exist in TGD inspired L universe.

6. The failure of set theoretic reductionism brings in mind the motivic approach to integration as purely algebraic approach applied to the symbol defining the integral instead of a number approach based on set theoretic notions. The motivation of the motivic approach in p-adic context is that p-adic numbers are not well-ordered so that one loses the notion of boundary and orientation as topological concepts although they can make sense algebraically.

For the hierarchy infinite integers the notion of infinity relies on real norm, which is essentially length rather than on the cardinality of a set. This infinity is essentially non-Cantorian and it is perhaps useless to try to relate it to that of ordinal or cardinal. There is just an infinite hierarchy of infinities which replaces the hierarchy of ordinals and for which the real norm of ratio makes possible partial ordering. Clearly the notion of infinity is extremely slippery and one must carefully specify what one means with infinite.

Cardinals in TGD Universe

What about cardinals in TGD framework? There seems to be no reason for giving them up and the first guess is that TGD replaces cardinals and ordinals of Cantor with cardinals and the hierarchy of infinite primes, integers, and rationals.

1. The first question is what is the cardinal assignable to infinite primes at the first level of hierarchy. For the set of finite primes the cardinal is $\aleph_0$. For the first level of infinite primes the situation is not so simple. The simple infinite primes correspond to free Fock states constructed from fermions and bosons labelled by primes. They are in one-one correspondence with rationals. There is however infinite number of many particle bound states representable as products of irreducible polynomials of one variable with integer coefficients and having finite number of roots which are algebraic numbers. The set of algebraic numbers is countable. This suggests that the cardinality of set of infinite primes at the first level of hierarchy corresponds to $\aleph_0$. This if course assuming that infinite integers and rationals for a set although they themselves cannot be described as sets.

   If one allows Fock states containing infinite number of particles and having thus infinite energy one obtains formally polynomials of infinite degree identifiable as Taylor expansions. In this case the roots can be transcendental numbers and one expects that a cardinal larger than $\aleph_0$, say $\aleph_1$ emerges. In von Neumann's Universe one indeed allows all subsets and $\aleph_1$ appears already at the first level. The higher cardinals appear at higher levels.

   One cannot exclude the Fock states containing infinite number of quanta if one accepts the idea that infinite prime representing quantum state characterizing entire Universe make sense. Does this mean that $\aleph_1$ has meaning only for entire universe and for states carrying infinite energy (in ZEO the positive energy part of zero energy state would carry the infinite energy)?

2. What happens at the next levels of the hierarchy? One possibility is that infinite primes at each level define a countable set. The point is that in polynomials representation one

considers only finite degree polynomials depending on finite number of variables, having rational coefficients. Therefore everything at the level of definitions is countable and finite and the product X of primes of previous level is just an algebraic symbol identifiable as a variable of polynomial.

3. In an alternative construction of infinite integers suggested in [**?**]ne considers the first level of the hierarchy the set of finite subsets of algebraic numbers and the set of finite subsets of this set at the next level and so on. All these sets are countable which suggests that the number of infinite primes at each level of the hierarchy is countable and that only the completion of algebraic number to reals or p-adic can give rise to $\aleph_1$. This would conform with the fact that quantum physics is basically based on counting of quanta and that finite measurement resolution is an essential restriction on what we can know.

What about the axiom of choice?

Axiom of choice has several variants. One variant is axiom of countable choice. Second variant is generalized continuum hypothesis states that the cardinality of an infinite set is between that of infinite set S and its power set: in other words there is no cardinal satisfying $\aleph_\alpha < \lambda < 2^{\aleph_\alpha}$ or equivalently: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. For a finite collection of sets it can be proved but already when on has a countable collection of nonempty set and in the case that one cannot uniquely specify some preferred element of each set, axiom of choice must be postulated. For instance, each subset of natural numbers has smallest element so that there is no need to postulate axiom of choice separately. Also closed intervals of real axis have smallest element.

What happens to the axiom of choice in TGD Universe. TGD is a physical theory and this means that the laws of classical physics strong considerations on the allowed sets. Classical physics is in TGD framework the dictated by the Kähler action and by a principle selecting its preferred extremals. Although several almost formulations for this principle exist, it is far from being well-understood and it is not clear whether one can give explicit formula for preferred extremals. One formulation is as quaternionic sub-manifolds of 8-D embedding space allowing octonionic structure in its tangent space and defined by octonionic representation of the gamma matrices defining the Clifford algebra.

1. The world of classical worlds can be regarded as the space of preferred extremals of Kähler action identifiable as certain 4-surfaces in $M^4 \times CP_2$. The mere extremal property implies also smoothness of the partonic 2-surfaces so that very powerful constraints are involved: therefore situation is very far from the extreme generality of set theory where one does assumes neither continuity nor smoothness. Zero energy ontology means that this space effectively reduces to a collection of spaces assignable to causal diamonds. Strong form of holography reduces this space effectively to the space consisting of collectinons of partonic 2-surfaces at the light-like boundaries of CD plus 4-D tangent space data at them which very probably cannot be chosen freely.

2. In this kind of situation it might well happen that all collections of sets, say are finite or in the case that they are countable they allow a unique choice of preferred point. Axiom of choice would not be needed. The specification of a preferred point of every 4-surface in the collection does not look a problem for a pragmatic physicist, since one can restrict the consideration to the boundaries of causal diamonds and consider for instance minimum of light-like radial coordinate. In fact, finite measurement resolution leads to the effective replacement of partonic 2-surfaces with the collection of ends of braid strands and the ends of braid strands define the preferred points. One might say, that physics with finite measurement resolution performs the choice automatically. A stronger form of this choice is that the points in question are rational points for a natural choice of the embedding space coordinates.

Generalization of real numbers inspired by infinite integers

Surreal numbers define a generalization of reals obtained by introducing a hierarchy of infinite reals and infinitesimals as their inverses. Infinite integers and rationals in TGD sense could give rise to a similar generalization so that one would have an infinite hierarchy of 8-D embedding space such that at given level previous level would represent infinitesimals.

TGD suggests another generalization of reals. One can construct from infinite integers rationals with unit norm. A possible interpretation would be as zero energy states with denominator and numerator representation positive and negative energy parts of the zero energy state and vanishing of total quantum numbers represented by real unit property. These numbers would have arbitrarily complex number theoretical anatomy however.

This structure has enormous representative power and one could dream that the world of classical worlds and spinor fields in this space could allow representation in terms of these real units. Brahman Atman Identity would be realized: the structure of single space-time point invisible to ordinary physics would represent the world of classical worlds! Single space-time point would be the Platonia!

Could one say that the space of all infinite rationals which are real units is countable? If previous arguments are correct this would seem to be true. If this is true, then TGD inspired notion of infinity would be extremely conservative as compared to the view proposed by Cantor and his followers using the Cantorian criteria. Just $\aleph_n$, $n = 0, 1$ and hierarchy of infinite integers which are countable sets. One can of course, ask how many surfaces WCW contains, what $\aleph$ is in question. This depends on the properties of preferred extremals. If partonic 2-surfaces can be chosen freely at the boundaries of CDs the restrictions come only from smoothness of the embedding of the partonic 2-surfaces and tangent space data. The space of all functions from reals to reals has cardinality $2^{\aleph_1}$ which suggests that the cardinality is not larger than this, perhaps smaller since continuity and smoothness poses strong conditions. The natural guess is that the tangent space of WCW can be modelled as and infinite-dimensional separable Hilbert space which has cardinality $\aleph_1$.

TGD leads also a second generalization of the number concept motivated by number theoretical universality inspiring the attempt to glue different number fields (reals and various p-adics) together among common numbers -rationals in particular- to form a larger structure [K95].

To sum up, the distinctions between Cantorian and TGD inspired approaches are clear. Cantorian approach relies on set theory and TGD on number theory. What is common is the hierarchy of infinities.

4.8.3 What Could Be The Foundations Of Physical Mathematics?

Theoretical physicists do not spend normally their time for questioning the foundations of mathematics. They calculate. There are exceptions: Von Neumann was both a theoretical physicist developing mathematical foundations of quantum theory and mathematician building the mathematics of quantum theory and also proposing his own L world for foundations of mathematics.

A physicist posing the question "What should be done for the foundations of mathematics?" sounds blasphemous and the physicist should add the attribute "physical" to "mathematics" to avoid irritation. In any case, the fact is that the problems plaguing set theory and therefore the foundations of mathematics had been discovered roughly century ago and no commonly accepted solution to these problems have been found. The foundations of mathematics rely on classical physics and quantum view about existence suggests that the foundations of mathematics might need a revision.

Again the work of von Neumann comes readily into mind. The goal of von Neuman was to build a non-commutative measure theory: the outcome was the three algebras bearing his name and defining the mathematical backbone of three kinds of quantum theories. Factors of type I are natural for wave mechanism with finite number of degrees of freedom. In QFT hyperfinite factors of type III appear. In TGD framework hyperfinite factors of type II (and possibly of type III) are natural.

Connes who has studied von Neumann algebras highly relevant to quantum physics proposed the notion of non-commutative geometry in terms of a spectral triplet defined by C^* algebra A, Hilbert space H, and Dirac operator D with some additional properties. As a special case one re-discovers Riemannian manifolds using commutative function algebra, the Hilbert space of continuous functions, and certain kind of Dirac operator.

Physicists are usually mathematical opportunists and do not want to use time to ponder the foundations of mathematics My belief is that physicists should get rid of this attitude and make fool of themselves by posing the childish questions of physicist in the hope that some real mathematician might get interested. In order to not irritate mathematicians too much I will talk about physical mathematics instead of mathematics in the sequel.

The proposal that infinite primes, integers, and rationals should replace Cantor's ordinals and surreal numbers [K59] has been already made. This would allow to get rid of Russell's antinomy, leave the notion of cardinal intact. Also axiom of choice looks too strong from the point of view of physics.

Does it make sense to speak about physical set theory?

For the physicist set theory looks quite too general. In the recent day physical theories sets are typically manifolds, sub-manifolds, or orbifolds. Feynman diagrams represent example of 1-D singular manifolds and in TGD generalized Feynman diagrams of TGD fail to be 3-manifolds only at the vertices represented as 2-D partonic surfaces. In string theories and in twistor approach to gauge theories algebraic geometry is important. Branes are typically algebraic surfaces. The spaces are endowed with various structures: besides metric induced topology one differential structure, differential forms, metric, spinor structure, complex and Kähler structure, etc...

1. In algebraic geometry sets are replaced with varieties and basic set theoretic operations such as intersection and union are algebraized. Physicists should not fail to realize how profound this algebraization of the set theory is. The price that must be paid is that varieties are manifolds only locally. What limitations does this mean for set theory? Is it enough to formulate set theory algebraically? In TGD framework this could be possible in the intersection of real and p-adic worlds (WCW s) since set theoretic operations would have algebraic representation. For instance, $A \subset B$ would be formulated by adding additional functions for which the intersection of zero locus with B defines A.

   The algebraic notion of set as a variety is extremely restrictive: maybe the problems of set theory are partially due to the neglect of the fact that allowed sets must have a physical realization. Every physicists of course has her own pet theory, which he regards as the real physics, and one natural condition on any acceptable physics is that it can emulate sufficiently general spaces - to act as a kind of mathematical Turing machine. At least real and complex manifolds with arbitrary dimension should have some kind of physical representation. One can imagine this kind of representation in terms of unions of partonic 2-surfaces since union can be regarded also as a Cartesian product as long as the surfaces do not intersect.

2. The introduction of topology is the first step in bringing structure to the set theoretic primordial chaos. Metric topology is standard in physics at space-time level. More refined topologies can be certainly found in highly technical mathematical physics articles. In algebraic geometry Zariski topology is important but has its problems realized by Groethendienck in his attempts to build a universal cohomology theory working in all number fields. The closed sets of Zariski topology are varieties. Their complements would be open sets open also in norm based topology. Zariski topology is obviously much rougher than the metric topology. Zariski topology makes sense also for p-adic number fields. This kind of topology might make sense in TGD framework if one restricts the consideration to the intersection of real and p-adic worlds identified at the level of WCW as the space of algebraic surfaces defined using polynomials with rational coefficients and having finite degree.

3. In TGD framework preferred extremals of Kähler action define space-time surfaces and strong form of holography makes the situation effectively 2-dimensional. The conjecture is that preferred extremals correspond to quaternionic surfaces of octonionic 8-space. Octonionic structure is associated with the octonionic representation of the embedding space gamma matrices (not actually matrices any more!) defining the Clifford algebra. Associativity would be the basic dynamical principle. Does this mean that number theory- in particular classical number fields- should appear in the formulation of the foundations of physical mathematics? This idea is attractive even when one does not assume that TGD Universe is the Universe.

What is beautiful that algebraic geometry brings in also number theory. One might hope that the foundations of physical mathematics could be based on the fusion of set theory, geometry, algebra, and number theory.

Quantum Boolean algebra instead of Boolean algebra?

Mathematical logic relies on the notion of Boolean algebra, which has a well-known representation as the algebra of sets which in turn has in algebraic geometry a representation in terms of algebraic varieties. This is not however attractive at space-time level since the dimension of the algebraic variety is different for the intersection *resp.* union representing AND *resp.* OR so that only only finite number of ANDs can appear in the Boolean function. TGD inspired interpretation of the fermionic sector of the theory in terms of Boolean algebra inspires more concrete ideas about the realization of Boolean algebra at both quantum level and classical space-time level and also suggests a geometric realization of the basic logical functions respecting the dimension of the representative objects.

1. In TGD framework WCW spinors correspond to fermionic Fock states and an attractive interpretation for the basis of fermionic Fock states is as Boolean algebra. In zero energy ontology one consider pairs of positive and negative energy states and zero energy states could be seen as physical correlates for statements $A \rightarrow B$ or $A \leftrightarrow B$ with individual state pairs in the quantum superposition representing various instances of the rule $A \rightarrow B$ or $A \leftrightarrow B$. The breaking of time reversal invariance means that either the positive or negative energy part of the state (but not both) can correspond to a state with precisely define number of particles with precisely defining quantum numbers such as four-momentum. At the second end one has scattered state which is a superposition of this kind of many-particle states. This would suggest that $A \rightarrow B$ is the correct interpretation.

2. In quantum group theory (see `http://tinyurl.com/3tors5`) [A11] the notion of co-algebra (see `http://tinyurl.com/27dkk4y`) [A3] is very natural and the binary algebraic operations of co-algebra are in a well-defined sense time reversals of those of algebra. Hence there is a great temptation to generalize Boolean algebra to include its co-algebra (see `http://tinyurl.com/y8s585p8`) [A64] so that one might speak about quantum Boolean algebra. The vertices of generalized Feynman diagrams represent two topological binary operations for partonic two surfaces and there is a strong temptation to interpret them as representations for the operations of Boolean algebra and its co-algebra.

    (a) The first vertex corresponds to the analog of a stringy trouser diagram in which partonic 2-surface decays to two and the reversal of this representing fusion of partonic 2-surfaces. In TGD framework this diagram does not represent classically particle decay or fusion but the propagation of particle along two paths after the decay or the reversal of this process. The Boolean analog would be logical OR ($A \vee B$) or set theoretical union $A \cup B$ *resp.* its co-operation. The partonic two surfaces would represent the arguments (*resp.* co-arguments) A and B.

    (b) Second one corresponds to the analog of 3-vertex for Feynman diagram: the three 3-D "lines" of generalized Feynman diagram meet at the partonic 2-surface. This vertex (co-vertex) is the analog of Boolean AND ($A \wedge B$) or intersection $A \cap B$ of two sets *resp.* its co-operation.

    (c) I have already earlier ended up with the proposal that only three-vertices appear as fundamental vertices in quantum TGD [K23]. The interpretation of generalized Feynman diagrams as a representation of quantum Boolean algebra would give a deeper meaning for this proposal.

    These vertices could therefore have interpretation as a space-time representation for operations of Boolean algebra and its co-algebra so that the space-time surfaces could serve as classical correlates for the generalized Boolean functions defined by generalized Feynman diagrams and expressible in terms of basic operations of the quantum Boolean algebra. For this representation the dimension of the variety representing the value of Boolean function at classical level is the same as as the dimension of arguments: that is two. Hence this representation is not equivalent with the representation provided by algebraic geometry for which the dimension of the geometric variety representing $A \wedge B$ and $A \vee B$ in general differs from that for A and B. If one however restricts the algebra to that assignable to braid strands, statements would correspond to points at partonic level, so that one would have discrete sets and the set theoretic representation of quantum Boolean algebra could make sense. Discrete sets are indeed the only possibility since otherwise the dimension of intersection and union are different if algebraic varieties are in question.

3. The breaking of time reversal invariance is accompanied by a generation of entropy and loss of information. The interpretation at the level of quantum Boolean algebra would be following. The Boolean functionandOR assign to two statements a single statement: this means a gain of information and at the level of physics this is indeed the case since entropy is reduced in the process reducing the number of particles. The occurrence of co-operations ofandOR corresponds to particle decays and uncertainty about the path along which particle travels (dispersion of wave packet) and therefore loss of information.

   (a) The "most logical" interpretation for the situation is in conflict with the identification of the arrow of logic implication with the arrow of time: the direction of Boolean implication arrow and the arrow of geometric time would be opposite so that final state could be said to imply the initial state. The arrow of time would weaken logical equivalence to implication arrow.

   (b) If one naïvely identifies the arrows of logical implication and geometric time so that initial state can be said to imply the final state, second law implies that logic becomes fuzzy. Second law would weak logical equivalence to statistical implication arrow.

   (c) The natural question is whether just the presence of both algebra and co-algebra operations causing a loss of information in generalized Feynman diagrams could lead to what might be called fuzzy Boolean functions expressing the presence of entropic element appears at the level of Boolean cognition.

4. This picture requires a duality between Boolean algebra and its co-algebra and this duality would naturally correspond to time reversal. Skeptic can argue that there is no guarantee about the existence of the extended algebra analogous to Drinfeld double [A59] (see `http://tinyurl.com/y7tpshkp`) that would unify Boolean algebra and its dual. Only the physical intuition suggests its existence.

These observations suggest that generalized Feynman diagrams and their space-time counterparts could have a precise interpretation in quantum Boolean algebra and that one should perhaps consider the extension of the mathematical logic to quantum logic. Alternatively, one could argue that quantum Boolean algebra is more like a model for what mathematical cognition could be in the real world.

The restrictions of mathematical cognition as a guideline?

With the birth of quantum theory physicists ceased to be outsiders since it was impossible to consider quantum measurement as something not affecting the measured system in any way. With the advent of consciousness theory physicists have been forced to give up the idea about unidirectional action with with reality and have become a part of quantum Universe - self. This also requires dramatic modification of the basic ontology forcing to give up the physicalistic dogmas. Consciousness involves free will manifested in ability to select and create something completely new in each quantum jump. Physical Universe is not given but is re-created again and again and evolves.

In standard mathematics mathematician is still a complete outsider, and the possible limitations of mathematical cognition are not considered seriously in the attempts to formulate the foundations of mathematics. Mathematicians still choose effortlessly one element from each set of infinite collection of sets. We know that in numerics one is always bound to introduce cutoff on the number of bits and use finite subset of rational numbers but also this has not been taken into account in the formulation of foundations as far as I know. If one takes consciousness theory seriously one is led to wonder what are the physical restrictions on mathematical cognition and therefore on physical mathematics. What looks obvious that the idea about mathematics based on fixed axiomatics must be given up. The evolution of the physical universe and of consciousness means also the evolution of (at least physical) mathematics. The paradox of self reference plaguing conventional view about consciousness and leading to infinite regress disappears when this regress is replaced with evolution.

Suppose that life resides and cognitive representations are realized in the intersection of real and p-adic worlds reducing to intersections of real and p-adic variants of partonic 2-surfaces at space-time level. At the level of WCW the intersection of real and p-adic worlds could correspond to the space of partonic 2-surfaces defined by rational functions constructed using polynomials of finite degree with rational coefficients.

What kind of restrictions of this picture poses set theory, topology, and logic? The reader can of course imagine restrictions on some other fields of mathematics involved. The question in the case of the set theory and topology has been already touched. In the case of logic the key question seems to concern the operational meaning of ∀ and ∃, when the finite resolution of measurements and cognitive representation are taken into account. What these universal quantors really mean: what is their domain of definition?

Consider first the domain of definition at space-time level.

1. Should all theorems be formulated using ∀ and ∃ restricted to the dense subset rationals of 8-D embedding space. Since continuous function is fixed from its values in a dense subset, this assumption is not so strong unless there are other restrictions.

2. At space-time surface and partonic 2-surfaces the situation is different. The assumption that only the common rational points of real and p-adic surfaces define cognitive representations poses a strong limitation since typically the number of rational points of 2-surface is expected to be finite. Algebraic extensions of p-adic numbers extend the number of common points and one can imagine an evolutionary hierarchy of mathematics realized in terms of geometry of partonic 2-surfaces reflecting itself as the geometry of space-time surfaces by strong form of holography.

3. The orbits of the rational points selected at the ends of partonic 2-surfaces are braids along light-like 3-surfaces. At space-time level one has world sheets or strings which form in general case 2-braids. This picture leads to a what I have used to call almost topological QFT.

What about the domain of definition of existence quantors at the level of WCW ? The natural conjecture is that the surfaces in the intersection of real and p-adic worlds form a dense set of full WCW so that everything holding true in the intersection would hold true generally and one coul dhope that systems which are living in the proposed sense are able to discover interesting mathematics.

Suppose that the partonic 2-surfaces decompose into patches such that in each patch the surface is a zero locus of polynomials with rational coefficients. Since polynomials can be seen as Taylor series with cutoff one can hope that they form a dense subset. Since rationals are dense subset of reals, one can hope that also the restriction to rational coefficients preserves the dense subset property and living subsystems are able to represent all that is needed and completion takes care of the rest as it does for rationals. The notion of completion leading from rationals to various algebraic numbers fields and also to reals and complex numbers would become the fundamental principle leading from number theory to metric topology.

Physicist reader has certainly noticed that "rational point" does not represent a general coordinate invariant notion.

1. The coordinates of point are rational in preferred coordinates and the symmetries of the 8-D embedding space suggest families of preferred coordinates. The moduli space for CDs would be characterized by the choice of these preferred coordinates dictating also the choice of quantization axes so that quantum measurement theory would be realized as a decomposition of WCW to a union corresponding to different choices. State function reduction would involve also a localization determining quantization axes.

2. There are many possible choices of quantization axes/preferred coordinates and this means a restriction of general coordinate invariance from group of all coordinate transformations to a discrete subgroup of isometries which is not unique. Cognition would break the general coordinate invariance. The world in which the mathematician thinks using spherical coordinates differs in some subtle manner from the world in which she thinks using Cartesian coordinates.

Mathematician does not remain outside Platonia anymore just as quantum physicists is not outside the physical Universe!

Axiom of choice relates to selection, which can be regarded as a cognitive act. The question whether axiom of choice is needed at all has been already discussed but a couple of clarifying comments are in order.

1. At quantum level selection would be naturally assigned with state function reduction, also the state function reduction selecting quantization axes. The cascade of state function reductions - starting from the scale of CD and proceeding fractally downwards sub-CD by sub-CD and stopping when only negentropic entanglement stabilized by NMP remains - could be how Nature performs the choice. State function reduction would involve also the choice of quantization axes dictating possible subsequent choices. Note that non-deterministic element would be involved with the quantum choice.

2. If life and cognitive representations are at the intersection of real and p-adic worlds, it would seem that rational points are chosen at space-time level and algebraic 2-surfaces at WCW level. As explained, it is easy to imagine the collection of sets from which one selects points is always finite or that there is a natural explicit criterion allowing to select preferred point from each set. Finite measurement resolution implying braids and string world sheets could provide this criterion. If so, the axiom of choice would be un-necessary in physical mathematics. Finite measurement resolution suggests that for partonic 2-surfaces the ends of braid strands define preferred points.

Platonia is a strange place about which many mathematicians claim to visit regularly. I already proposed that the generalization of space-time point by bringing in the infinite number theoretical anatomy of real (and octonionic) units might allow to realize number theoretical Brahman=Atman identity by representing WCW in terms of the number theoretic anatomy of space-time points. This kind of representation would certainly be the most audacious idea that physical mathematician could dare to think of.

Is quantal Boolean reverse engineering possible?

The quantal version of Boolean algebra means that the basic logical functions have quantum inverses. The inverse of $C = A \wedge B$ represents the quantum superposition of all pairs A and B for which $A \wedge B = C$ hols true. Same is true for $\vee$. How could these additional quantum logical functions with no classical counterparts extend the capacities of logician?

What comes in mind is logical reverse engineering. Consider the standard problem solving situation repeatedly encountered by my hero Hercule Poirot. Someone has been murdered. Who could have done it? Who did it? Actually scientists who want to explain instead of just applying the method to get additional items to the CVC, meet this kind of problem repeatedly. One has something which looks like an experimental anomaly and one has to explain it. Is this anomaly genuine or is it due to a systematic error in the information processing? Could the interpretation of data be somehow wrong? Is the model behind experiments based on existing theory really correct or has something very delicate been neglected? If a genuine anomaly is in question (someone has been really murdered- this is always obvious in the tales about the deeds of Hercule Poirot since the mere presence of Hercule guarantees the murder unless it has been already done), one encounters what might be called Poirot problem in honor of my hero. As a matter fact, from the point of view of Boolean algebra, one has the same reverse Boolean engineering problem irrespective of whether it was a genuine anomaly or not.

This brings in my mind the enormously simplified problem. The logical statement C is found to be true. Which pairs A, B could have implied C as $C = A \wedge B$ (or $A \vee B$). Of course, much more complex situations can be considered where C corresponds to some logical function $C = f(A_1, A_2, ..., A_n)$. Quantum Poirot could use quantum computer able to realize the co-gates for gatesandOR (essentially time reversals) and write a quantum computer program solving the problem by constructing the Boolean co-function of Boolean function f.

What would happen in TGD Universe obeying zero energy ontology (ZEO) is following.

1. The statement C is represented as as positive energy part of zero energy state (analogous to initial state of physical event) and $A_1,..A_n$ is represented as one state in the quantum superposition of final states representing various value combinations for $A_1,...,A_n$. Zero energy states (rather than only their evolution) represents the arrow of time. The M-matrix characterizing time-like entanglement between positive and negative energy states generalizes generalizes S-matrix. S-matrix is such that initial states have well defined particle numbers and other quantum numbers whereas final states do not. They are analogous to the outcomes of quantum measurement in particle physics.

2. Negentropy Maximization Principle [K56] maximizing the information contents of conscious experience (sic!) forces state function reduction to one particular $A_1,...,A_n$ and one particular value combination consistent with C is found in each state function reduction. At the ensemble level one obtains probabilities for various outcomes and the most probable combination might represent the most plausible candidate for the murderer in quantum Poirot problem. Also in particle physics one can only speak about plausibility of the explanation and this leads to the endless n sigma talk. Note that it is absolutely essential that state function reduction occurs. Ironically, quantum problem solving causes dissipation at the level of ensemble but the ensemble probabilities carry actually information! Second law of thermodynamics tells us that Nature is a pathological problem solver- just like my hero!

3. In TGD framework basic logical binary operations have a representation at the level of Boolean algebra realized in terms of fermionic oscillator operators. They have also space-time correlates realized topologically. $\wedge$ has a representation as the analog of three-vertex of Feynman graph for partonic 2-surfaces: partonic 2-surfaces are glued along the ends to form outgoing partonic 2-surface. $\vee$ has a representation as the analog of stringy trouser vertex in which partonic surfaces fuse together. Here TGD differs from string models in a profound manner.

To conclude, I am a Boolean dilettante and know practically nothing about what quantum computer theorists have done- in particular I do not know whether they have considered quantum inverse gages. My feeling is that only the gates with bits replaced with qubits are considered: very natural when one thinks in terms of Boolean logic. If this is really the case, quantal co-AND and co-OR having no classical counterparts would bring a totally new aspect to quantum computation in solving problems in which one cannot do without (quantum) Poirot and his little gray (quantum) brain cells.

How to understand transcendental numbers in terms of infinite integers?

Santeri Satama made in my blog (see `http://tinyurl.com/yd9nh9fy`) a very interesting question about transcendental numbers. The reformulation of Santeri's question could be "How can one know that given number defined as a limit of rational number is genuinely algebraic or transcendental?". I answered to the question and since it inspired a long sequence of speculations during my morning walk on sands of Tullinniemi I decided to expand my hasty answer to a blog posting.

The basic outcome was the proposal that by bringing TGD based view about infinity based on infinite primes, integers, and rationals one could regard transcendental numbers as algebraic numbers by allowing genuinely infinite numbers in their definition.

1. In the definition of any transcendental as a limit of algebraic number (root of a polynomial and rational in special case) in which integer n approaches infinity one can replace n with any infinite integer. The transcendental would be an algebraic number in this generalized sense. Among other things this might allow polynomials with degree given by infinite integer if they have finite number of terms. Also mathematics would be generalized number theory, not only physics!

2. Each infinite integer would give a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to real norm they would be identical.

3. This would extend further the generalization of number concept obtained by allowing all infinite rationals which reduce to units in real sense and would further enrich the infinitely rich number theoretic anatomy of real point and also of space-time point. Space-time point would be the Platonia. One could call this number theoretic Brahman=Atman identity or algebraic holography.

1. How can one know that the real number is transcendental?

The difficulty of telling whether given real number defined as a limit of algebraic number boils down to the fact that there is no numerical method for telling whether this kind of number is rational, algebraic, or transcendental. This limitation of numerics would be also a restriction of cognition if p-adic view about it is correct. One can ask several questions. What about infinite-P p-adic numbers: if they make sense could it be possible to cognize also transcendentally? What can we conclude from the very fact that we cognize transcendentals? Transcendentality can be proven for some transcendentals such as π. How this is possible? What distinguishes "knowably transcendentals" like π and e from those, which are able to hide their real number theoretic identity?

1. Certainly for "knowably transcendentals" there must exist some process revealing their transcendental character. How π and e are proven to be transcendental? What in our mathematical cognition makes this possible? First of all one starts from the definitions of these numbers. e can be defined as the limit of the rational number $(1 + 1/n)^n$ and 2π could be defined as the limit for the length of the circumference of a regular n-side polygon and is a limit of an algebraic number since Pythagoras law is involved in calculating the length of the side. The process of proving "knowable transcendentality" would be a demonstration that these numbers cannot be solutions of any polynomial equation.

2. Squaring of circle is not possible because π is transcendental. When I search Wikipedia for squaring of circle (see `http://tinyurl.com/yaf6nf99`) I find a link to Weierstrass theorem (see `http://tinyurl.com/5y2gfr`) allowing to prove that π and e are transcendentals. In the formulation of Baker this theorem states the following: If α_1, ..., α_n are distinct algebraic numbers then the numbers e^{α_1}, ..., e^{α_n} are linearly independent over algebraic numbers and therefore transcendentals. One says that the extension $Q(e^{\alpha_1}, ..., e^{\alpha_n})$ of rationals has transcendence degree n over Q. This is something extremely deep and unfortunately I do not know what is the gist of the proof. In any case the proof defines a procedure of demonstrating "knowable transcendentality" for these numbers. The number of these transcendentals is huge but countable and therefore vanishingly small as compared to the uncountable cardinality of all transcendentals.

3. This theorem allows to prove that π and e are transcendentals. Suppose on the contrary that π is algebraic number. Then also $i\pi$ would be algebraic and the previous theorem would imply that $e^{i\pi} = -1$ is transcendental. This is of course a contradiction. Theorem also implies that e is transcendental. But how do we know that $e^{i\pi} = -1$ holds true? Euler deduced this from the connection between exponential and trigonometric functions understood in terms of complex analysis and related number theory. Clearly, rational functions and exponential function and its inverse -logarithm- continued to complex plane are crucial for defining e and π and proving also $e^{i\pi} = -1$. Exponent function and logarithm appear everywhere in mathematics: in group theory for instance. All these considerations suggest that "knowably transcendental" is a very special mathematical property and deserves a careful analysis.

2. Exponentiation and formation of set of subsets as transcendence

What is so special in exponentiation? Why it sends algebraic numbers to "knowably transcendentals". One could try to understand this in terms of exponentiation which for natural numbers has also an interpretation in terms of power set just as product has interpretation in terms of Cartesian product.

1. In Cantor's approach to the notion of infinite ordinals exponentiation is involved besides sum and product. All three binary operations - sum, product, exponent are expressed set theoretically. Product and sum are "algebraic" operations. Exponentiation is "non-algebraic"

binary operation defined in terms of power set (set of subsets). For m and n defining the cardinalities of sets X and Y, m^n defines the cardinality of the set Y^X defining the number of functions assigning to each point of Y a point of X. When X is two-element set (bits 0 and 1) the power set is just the set of all subsets of Y which bit 1 assigned to the subset and 0 with its complement. If X has more than two elements one can speak of decompositions of Y to subsets colored with different colors- one color for each point of X.

2. The formation of the power set (or of its analog for the number of colors larger than 2) means going to the next level of abstraction: considering instead of set the set of subsets or studying the set of functions from the set. In the case of Boolean algebras this means formation of statements about statements. This could be regarded as the set theoretic view about transcendence.

3. What is interesting that 2-adic integers would label the elements of the power set of integers (all possible subsets would be allowed, for finite subsets one would obtain just natural numbers) and p-adic numbers the elements in the set formed by coloring integers with p colors. One could thus say that p-adic numbers correspond naturally to the notion of cognition based on power sets and their finite field generalizations.

4. But can one naïvely transcend the set theoretic exponent function for natural numbers to that defined in complex plane? Could the "knowably transcendental" property of numbers like e and π reduce to the transcendence in this set theoretic sense? It is difficult to tell since this notion of power applies only to integers m, n rather than to a pair of transcendentals e, π. Concretization of $e^{i\pi}$ in terms of sets seems impossible: it is very difficult to imagine what sets with cardinality e and π could be.

3. Infinite primes and transcendence

TGD suggests also a different identification of transcendence not expressible as formation of a power set or its generalizations.

1. The notion of infinite primes replaces the set theoretic notion of infinity with purely number theoretic one.

   (a) The mathematical motivation could be the need to avoid problems like Russell's antinomy. In Cantorian world a given ordinal is identified as the ordered set of all ordinals smaller than it and the set of all ordinals would define an ordinal larger than every ordinal and at the same time member of all ordinals.

   (b) The physical motivation for infinite primes is that their construction corresponds to a repeated second quantization of an arithmetic supersymmetric quantum field theory such that the many particle states of the previous level become elementary particles of the new level. At the lowest level finite primes label fermionic and bosonic states. Besides free many-particle states also bound states are obtained and correspond at the first level of the hierarchy to genuinely algebraic roots of irreducible polynomials.

   (c) The allowance of infinite rationals which as real numbers reduce to real units implies that the points of real axes have infinitely rich number theoretical anatomy. Space-time point would become the Platonia. One could speak of number theoretic Brahman=Atman identity or algebraic holography. The great vision is that the World of Classical Worlds has a mathematical representation in terms of the number theoretical anatomy of space-time point.

2. Transcendence in purely number theoretic sense could mean a transition to a higher level in the hierarchy of infinite primes. The scale of new infinity defined as the product of all prime at the previous level of hierarchy would be infinitely larger than the previous one. Quantization would correspond to abstraction and transcendence.

This idea inspires some questions.

1. Could infinite integers allow the reduction of transcendentals to algebraic numbers when understood in general enough sense. Could real algebraic numbers be reduced to infinite rationals with finite real values (for complex algebraic numbers this is certainly not the case)? If so, then all real numbers would be rationals identified as ratios of possibly infinite integers and having finite value as real numbers? This turns out to be too strong a statement. The statement that all real numbers can be represented as finite or infinite algebraic numbers looks however sensible and would reduce mathematics to generalized number theory by reducing limiting procedure involved with the transition from rationals to reals to algebraic transcendence. This applies also to p-adic numbers.

2. p-Adic cognition for finite values of prime p does not explain why we have the notions of π and e and more generally, that of transcendental number. Could the replacement of finite-p p-adic number fields with infinite-P p-adic number fields allow us to understand our own mathematical cognition? Could the infinite-P p-adic number fields or at least integers and corresponding space-time sheets make possible mathematical cognition able to deduce analytic formulas in which transcendentals and transcendental functions appear making it possible to leave the extremely restricted realm of numerics and enter the realm of mathematics? Lie group theory would represent a basic example of this transcendental aspect of cognition. Maybe this framework might allow to understand why we can have the notion of transcendental number!

4. Identification of real transcendentals as infinite algebraic numbers with finite value as real numbers

The following observations suggests that it could be possible to reduce transcendentals to generalized algebraic numbers in the framework provided by infinite primes. This would mean that not only physics but also mathematics (or at least "physical mathematics") could be seen as generalized number theory.

1. In the definition of any transcendental as an $n \to \infty$ limit of algebraic number (root of a polynomial and rational in special case), one can replace n with any infinite integer if n appears as an argument of a function having well defined value at this limit. If n appears as the number of summands or factors of product, the replacement does not make sense. For instance, an algebraic number could be defined as a limit of Taylor series by solving the polynomial equation defining it. The replacement of the upper limit of the series with infinite integer does not however make sense. Only transcendentals (and possibly also some algebraic numbers) allowing a representation as $n \to \infty$ limit with n appearing as argument of expression involving a finite number of terms can have representation as infinite algebraic number. The rule would be simple.

   Transcendentals or algebraic numbers allowing an identification as infinite algebraic number must correspond to a term of a sequence with a fixed number of terms rather than sum of series or infinite product.

2. Each infinite integer gives a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to the real norm they would be identical.

3. The heuristic guess is that any genuine algebraic number has an expression as Taylor series obtained by writing the solution of the polynomial equation as Tarylor expansion. If so, algebraic numbers must be introduced in the standard manner and do not allow a representation as infinite rationals. Only transcendentals would allow a representation as infinite rationals or infinite algebraic numbers. The infinite variety of representation in terms of infinite integers would enormously expand the number theoretical anatomy of the real point. Do all transcendentals allow an expression containing a finite number of terms and N appearing as argument? Or is this the defining property of only "knowably transcendentals" ?

One can consider some examples to illustrate the situation.

1. The transcendental π could be defined as $\pi_N = -iN(e^{i\pi/N} - 1)$, where $e^{i\pi/N}$ is N: th root of unity for infinite integer N and as a real number real unit. In real sense the limit however gives π. There are of course very many definitions of π as limits of algebraic numbers and each gives rise to infinite variety of number theoretic anatomies of π.

2. One can also consider the roots $exp(i2\pi n/N)$ of the algebraic equation $x^N = 1$ for infinite integer N. One might define the roots as limits of Taylor series for the exponent function but it does not make sense to define the limit when the cutoff for the Taylor series approaches some infinite integer. These roots would have similar multiplicative structure as finite roots of unity with p^n: th roots with p running over primes defining the generating roots. The presence of N^{th} roots of unity f or infinite N would further enrich the infinitely rich number theoretic anatomy of real point and therefore of space-time points.

3. There would be infinite variety of Neper numbers identified as $e_N = (1 + 1/N)^N$, N any infinite integer. Their number theoretic anatomies would be different but as real numbers they would be identical.

To conclude, the talk about infinite primes might sound weird in the ears of a layman but mathematicians do not lose their peace of mind when they here the word "infinity". The notion of infinity is relative. For instance, infinite integers are completely finite in p-adic sense. One can also imagine completely "real-worldish" realizations for infinite integers (say as states of repeatedly second quantized arithmetic quantum field theory and this realization might provide completely new insights about how to understand bound states in ordinary QFT).

4.9 Local Zeta Functions, Galois Groups, And Infinite Primes

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.

2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.

3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

4.9.1 Zeta Function And Infinite Primes

Fermionic Zeta function is expressible as a product of fermionic partition functions $Z_{F,p} = 1 + p^{-z}$ and could be seen as an image of X under algebraic homomorphism mapping prime p to $Z_{F,p}$ defining an analog of prime in the commutative function algebra of complex numbers. For hyper-octonionic infinite primes the contribution of each p to the norm of X is same finite power of p since only single representative from each Lorentz equivalence class is included, and there is one-one correspondence with ordinary primes so that an appropriate power of ordinary ζ_F might be regarded as a representation of X also in the case of hyper-octonionic primes.

Infinite primes suggest a generalization of the notion of ζ function. Real-rational infinite prime $X \pm 1$ would correspond to $\zeta_F \pm 1$. General infinite prime is mapped to a generalized zeta function by dividing ζ_F with the product of partition functions $Z_{F,p}$ corresponding to fermions kicked out from sea. The same product multiplies "1". The powers p^n present in either factor correspond to the presence of n bosons in mode p and to a factor $Z^n_{p,B}$ in corresponding summand of the generalized Zeta.

For zeros of ζ_F which are same as those of Riemann ζ the image of infinite part of infinite prime vanishes and only the finite part is represented faithfully. Whether this might have some physical meaning is an interesting question.

4.9.2 Local Zeta Functions And Weil Conjectures

Riemann Zeta is not the only zeta [A1, A18]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A15] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A15].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

4.9.3 Galois Groups, Jones Inclusions, And Infinite Primes

Langlands program [K45, A33] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [A34]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW -spinor fields. One can also speak about WCW spinor s invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \rightarrow K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths assignable to Gaussian Mersennes $M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$, which suggests that the emergence of living matter means an improved cognitive resolution.

Galois groups and infinite primes

In particular, the notion of infinite prime suggests a way to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [A8]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L3] allows the embedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of embedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of embedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L3].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L5] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the embedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the embedding of space-time surface to embedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

4.9.4 Prime Hilbert Spaces And Infinite Primes

There is a result of quantum information science providing an additional reason why for p-adic physics. Suppose that one has N-dimensional Hilbert space which allows $N+1$ unbiased basis. This means that the moduli squared for the inner product of any two states belonging to different basis equals to $1/N$. If one knows all transition amplitudes from a given state to all states of all $N+1$ mutually unbiased basis, one can fully reconstruct the state. For $N=p^n$ dimensional $N+1$ unbiased basis can be found and the article of Durt [A69] gives an explicit construction of these basis by applying the properties of finite fields. Thus state spaces with p^n elements - which indeed emerge naturally in p-adic framework - would be optimal for quantum tomography. For instance, the discretization of one-dimensional line with length of p^n units would give rise to p^n-dimensional Hilbert space of wave functions.

The observation motivates the introduction of prime Hilbert space as as a Hilbert space possessing dimension which is prime and it would seem that this kind of number theoretical structure for the category of Hilbert spaces is natural from the point of view of quantum information theory. One might ask whether the tensor product of mutually unbiased bases in the general case could be constructed as a tensor product for the bases for prime power factors. This can be done but since the bases cannot have common elements the number of unbiased basis obtained in this manner is equal to $M+1$, where M is the smallest prime power factor of N. It is not known whether additional unbiased bases exists.

Hierarchy of prime Hilbert spaces characterized by infinite primes

The notion of prime Hilbert space provides also a new interpretation for infinite primes, which are in 1-1 correspondence with the states of a supersymmetric arithmetic QFT. The earlier interpretation was that the hierarchy of infinite primes corresponds to a hierarchy of quantum states. Infinite primes could also label a hierarchy of infinite-D prime Hilbert spaces with product and sum for infinite primes representing unfaithfully tensor product and direct sum.

1. At the lowest level of hierarchy one could interpret infinite primes as homomorphisms of Hilbert spaces to generalized integers (tensor product and direct sum mapped to product and sum) obtained as direct sum of infinite-D Hilbert space and finite-D Hilbert space. (In)finite-D Hilbert space is (in)finite tensor product of prime power factors. The map of N-dimensional Hilbert space to the set of N-orthogonal states resulting in state function reduction maps it to N-element set and integer N. Hence one can interpret the homomorphism as giving rise to a kind of shadow on the wall of Plato's cave projecting (shadow quite literally!) the Hilbert space to generalized integer representing the shadow. In category theoretical setting one could perhaps see generalize integers as shadows of the hierarchy of Hilbert spaces.

2. The interpretation as a decomposition of the universe to a subsystem plus environment does not seem to work since in this case one would have tensor product. Perhaps the decomposition could be to degrees of freedom to those which are above and below measurement resolution. One could of course consider decomposition to a tensor product of bosonic and fermionic state spaces.

3. The construction of the Hilbert spaces would reduce to that of infinite primes. The analog of the fermionic sea would be infinite-D Hilbert space which is tensor product of all prime Hilbert spaces H_p with given prime factor appearing only once in the tensor product. One can "add n bosons" to this state by replacing of any tensor factor H_p with its n+1: th tensor power. One can "add fermions" to this state by deleting some prime factors H_p from the tensor product and adding their tensor product as a finite-direct summand. One can also "add n bosons" to this factor.

4. At the next level of hierarchy one would form infinite tensor product of all infinite-dimensional prime Hilbert spaces obtained in this manner and repeat the construction. This can be continued ad infinitum and the construction corresponds to abstraction hierarchy or a hierarchy of statements about statements or a hierarchy of n: th order logics. Or a hierarchy of space-time sheets of many-sheeted space-time. Or a hierarchy of particles in which certain many-particle states at the previous level of hierarchy become particles at the new level (bosons and fermions). There are many interpretations.

5. Note that at the lowest level this construction can be applies also to Riemann Zeta function. ζ would represent fermionic vacuum and the addition of fermions would correspond to a removal of a product of corresponding factors ζ_p from ζ and addition of them to the resulting truncated ζ function. The addition of bosons would correspond to multiplication by a power of appropriate ζ_p. The analog of ζ function at the next level of hierarchy would be product of all these modified ζ functions and might well fail to exist since the product might typically converge to either zero or infinity.

Hilbert spaces assignable to infinite integers and rationals make also sense

1. Also infinite integers make sense since one can form tensor products and direct sums of infinite primes and of corresponding Hilbert spaces. Also infinite rationals exist and this raises the question what kind of state spaces inverses of infinite integers mean.

2. Zero energy ontology suggests that infinite integers correspond to positive energy states and their inverses to negative energy states. Zero energy states would be always infinite rationals with real norm which equals to real unit.

3. The existence of these units would give for a given real number an infinite rich number theoretic anatomy so that single space-time point might be able to represent quantum states of the entire universe in its anatomy (number theoretical Brahman=Atman). Also the world of classical worlds (light-like 3-surfaces of the embedding space) might be imbeddable to this anatomy so that basically one would have just space-time surfaces in 8-D space and WCW would have representation in terms of space-time based on generalized notion of number. Note that infinitesimals around a given number would be replaced with infinite number of number-theoretically non-equivalent real units multiplying it.

Should one generalize the notion of von Neumann algebra?

Especially interesting are the implications of the notion of prime Hilbert space concerning the notion of von Neumann algebra -in particular the notion of hyper-finite factors of type II_1 playing a key role in TGD framework. Does the prime decomposition bring in additional structure? Hyper-finite factors of type II_1 are canonically represented as infinite tensor power of 2×2 matrix algebra having a representation as infinite-dimensional fermionic Fock oscillator algebra and allowing a natural interpretation in terms of spinors for the world of classical worlds having a representation as infinite-dimensional fermionic Fock space.

Infinite primes would correspond to something different: a tensor product of all $p\times p$ matrix algebras from which some factors are deleted and added back as direct summands. Besides this some factors are replaced with their tensor powers. Should one refine the notion of von Neumann algebra so that one can distinguish between these algebras as physically non-equivalent? Is the full algebra tensor product of this kind of generalized hyper-finite factor and hyper-finite factor of type II_1 corresponding to the vibrational degrees of freedom of 3-surface and fermionic degrees of freedom? Could p-adic length scale hypothesis - stating that the physically favored primes are near powers of 2 - relate somehow to the naturality of the inclusions of generalized von Neumann algebras to HFF of type II_1?

4.10 Miscellaneous

This section is devoted to what might be called miscellaneous since it does not relate directly to quantum TGD.

4.10.1 The Generalization Of The Notion Of Ordinary Number Field

The notion of infinite rationals leads also to the generalization of the notion of a finite number. The obvious generalization would be based on the allowance of infinitesimals. Much more interesting approach is however based on the observation that one obtains infinite number of real units by taking two infinite primes with a finite rational valued ratio q and by dividing this ratio by ordinary rational number q. As a real number the resulting number differs in no manner from ordinary unit of real numbers but in p-adic sense the points are not equivalent. This construction generalizes also to quaternionic and octonionic case.

Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving in well-define sense as Mother of All Algebras. The units of the algebra multiplying ordinary rational numbers (and also other elements) of various number fields are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure. Infinite rationals would allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time.

The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions and hyper-quaternions group becomes non-commutative and in case of unit hyper-octonions the group is replaced by a kind non-associative generalization of group.

1. For infinite primes for which only finite number of bosonic and fermionic modes are excited it is possible to tell how the products AB and BA of two infinite primes explicitly since finite hyper-octonionic primes can be assumed to multiply the infinite integer X from say left.

2. Situation changes if an infinite number of bosonic excitations are present since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units with infinitely many bosonic excitations is a free group. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary hyper-octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are real-rationals or possibly complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement or negentropic entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \to A_1 = A/I_1 \to A_1/I_2...$ where the ideal I_k is defined by $k:th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface could have the same value for all points of A which differ by an element of the ideal. At WCW level, the WCW spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i \Pi_i$.

The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures. Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical cognition. Leibniz might have been right about his monads! The idealization is in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One might perhaps say that each point represents an equation.

One could also try interpret generalized Feynman diagrams as sequences of mathematical operations. For instance, the scattering $AB \rightarrow CD$ by exchange of particle C could be seen as an arithmetic operation $AB \rightarrow (AE^{-1})(EB) = CD$. If this is really the case, then at least tree diagrams might allow interpretation in terms of arithmetic operations for the complexified octonionic units. In case of loop diagrams it seems that one must allow sums over units.

When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n+1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [A58] poses a question which at first sounds absurd. What might be the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$
\begin{aligned}
Y_I = \frac{\prod_i Y(q^I_{1i})}{\prod_i Y(q^I_{2i})} \ , \quad Y_F = \frac{\prod_i Y(q^F_{1i})}{\prod_i Y(q^F_{2i})} \ , \\
q^I_{ki} = \frac{n^I_{k_i}}{m^I_{k_i}} \ , \qquad q^F_{ki} = \frac{n^F_{k_i}}{m^F_{k_i}} \ ,
\end{aligned}
\tag{4.10.1}
$$

Here $m^{\cdot}_{\cdot}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{\cdot}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{\cdot}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$
\frac{\prod_i q^I_{1i}}{\prod_i q^I_{2i}} = \frac{\prod_i q^F_{1i}}{\prod_i q^F_{2i}} \ . \tag{4.10.2}
$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labeled by the prime p is $E_p = log(p)$:

$$
\begin{aligned}
E^I &= \sum_i log(n^I_{1i}) - \sum_i log(n^I_{2i}) - \sum_i log(m^I_{1i}) + \sum_i log(m^I_{2i}) = \\
&= \sum_i log(n^F_{1i}) - \sum_i log(n^F_{2i}) - \sum_i log(m^F_{1i}) + \sum_i log(m^F_{2i}) = E^F \ .
\end{aligned}
\tag{4.10.3}
$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

TGD inspired analog for d-algebras

Maxim Kontsevich has done deep work with quantizations interpreted as a deformation of algebraic structures and there are deep connections with this work and braid group [A54]. In particular, the Grothendienck-Teichmueller algebra believed to act as automorphisms for the deformation structures acts as automorphisms of the braid group at the limit of infinite number of strands. I must admit that my miserable skills in algebra do not allow to go to the horrendous technicalities but occasionally I have the feeling that I have understood some general ideas related to this work. In his article "Operads and Motives in Deformation Quantization" Kontsevich introduces the notions of operad and d-algebras over operad. Without going to technicalities one can very roughly say that d-algebra is essentially d-dimensional algebraic structure, and that the basic conjecture of Deligne generalized and proved by Kontsevich states in its generalized form that $d+1$-algebras have a natural action in all d-algebras.

In the proposed extension of various rationals a notion resembling that of universal d-algebra to some degree but not equivalent with it emerges naturally. The basic idea is simple.

1. Points correspond to the elements of the assumed to be universal algebra A which in this sense deserves the attribute $d = 0$ algebra. By its universality A should be able to represent any algebra and in this sense it cannot correspond $d = 0$-algebra of Kontsevich defined as a complex, that is a direct sum of vector spaces V_n and possessing d operation $V_n \rightarrow V_{n+1}$, satisfying $d^2 = 0$. Each point of a manifold represents one particular element of 0-algebra and one could loosely say that multiplication of points represents algebraic multiplication. This algebra has various subalgebras, in particular those corresponding to reals, complex numbers and quaternions. One can say that sub-algebra is non-associative, non-commutative, etc.. if its real evaluation has this property.

2. Lines correspond to evolutions for the elements of A which are continuous with respect to real (trivially) and all p-adic number fields. The latter condition is nontrivial and allows to interpret evolution as an evolution conserving number theoretical analog of total energy. Universal 1-group would consist of curves along which one has the analog of group valued field (group being the group of generalized units) having values in the universal 0-group G. The action of the 1-group in 0-group would simply map the element of 0-group at the first end of the curve its value at the second end. Curves define a monoid in an obvious manner. The interpretation as a map to A allows pointwise multiplication of these mappings which generalizes to all values of d.

   One could also consider the generalization of local gauge field so that there would be gauge potential defined in the algebra of units having values on A. This potential would define holonomy group acting on 0-algebra and mapping the element at the first end of the curve to its gauge transformed variant at the second end. In this case also closed curves would define non-trivial elements of the holonomy group. In fact, practically everything is possible since probably any algebra can be represented in the algebra generated by units.

3. Two-dimensional structures correspond to dynamical evolutions of one-dimensional structures. The simplest situation corresponds to 2-cubes with the lines corresponding to the initial and final values of the second coordinate representing initial and final states. One can also consider the possibility that the two-surface is topologically non-trivial containing handles and perhaps even holes. One could interpret this cognitive evolution as a 1-dimensional flow so that the initial points travel to final points. Obviously there is symmetry breaking involved since the second coordinate is in the role of time and this defines kind of time orientation for the surface.

4. The generalization to 4- and higher dimensional cases is obvious. One just uses d-manifolds with edges and uses their time evolution to define $d + 1$-manifolds with edges. Universal 3-algebra is especially interesting from the point of view of braid groups and in this case the maps between initial and final elements of 2-algebra could be interpreted as braid operations if the paths of the elements along 3-surface are entangled. For instance field lines of Kähler gauge potential or of magnetic field could define this kind of braiding.

5. The d-evolutions define a monoid since one can glue two d-evolutions together if the outcome of the first evolution equals to the initial state of the second evolution. $d+1$-algebra also acts naturally in d-algebra in the sense that the time evolution $f(A \rightarrow B)$ assigns to the d-algebra valued initial state A a d-algebra valued final state and one can define the multiplication as $f(A \rightarrow B)C = B$ for $A = C$, otherwise the action gives zero. If time evolutions correspond to standard cubes one gets more interesting structure in this manner since the cubes differing by time translation can be identified and the product is always non-vanishing.

6. It should be possible to define generalizations of homotopy groups to what might be called "cognitive" homotopy groups. Effectively the target manifold would be replaced by the tensor product of an ordinary manifold and some algebraic structure represented in A. All kinds of "cognitive" homotopy groups would result when the image is cognitively non-contractible. Also homology groups could be defined by generalizing singular complex consisting of cubes with cubes having the hierarchical decomposition into time evolutions of time evolutions of... in some sub-algebraic structure of A. If one restricts time evolutions to sub-algebraic structures one obtains all kinds of homologies. For instance, associativity reduces 3-evolutions

to paths in rational $SU(3)$ and since $SU(3)$ just like any Lie group has non-trivial 3-homology, one obtains nontrivial "cognitive" homology for 3-surfaces with non-trivial 3-homology.

The following heuristic arguments are inspired by the proposed vision about algebraic cognition and the conjecture that Grothendienck-Teichmueller group acts as automorphisms of Feynman diagrammatics relating equivalent quantum field theories to each other.

1. The operations of $d+1$-algebra realized as time evolution of d-algebra elements suggests an interpretation as cognitive counterparts for sequences of algebraic manipulations in d-algebra which themselves become elements of $d+1$ algebra. At the level of paths of points the sequences of algebraic operations correspond to transitions in which the number of infinite primes defining an infinite rational can change in discrete steps but is subject to the topological energy conservation guaranteeing the p-adic continuity of the process for all primes. Different paths connecting a and b represent different but equivalent manipulations sequences.

   For instance, at $d=2$ level one has a pile of these processes and this in principle makes it possible an abstraction to algebraic rules involved with the process by a pile of examples. Higher values of d in turn make possible further abstractions bringing in additional parameters to the system. All kinds of algebraic processes can be represented in this manner. For instance, multiplication table can be represented as paths assigning to an the initial state product of elements a and b represented as infinite rationals and to the final state their product ab represented as single infinite rational. Representation is of course always approximate unless the algebra is finite. All kinds abstract rules such as various commutative diagrams, division of algebra by ideal by choosing one representative from each equivalence class of A/I as end point of the path, etc... can be represented in this manner.

2. There is also second manner to represent algebraic rules. Entanglement is a purely algebraic notion and it is possible to entangle the many-particle states formed as products of infinite rationals representing inputs of an algebraic operation A with the outcomes of A represented in the same manner such that the entanglement is consistent with the rule.

3. There is nice analogy between Feynman diagrams and sequences of algebraic manipulations. Multiplication ab corresponds to a map $A \otimes A \rightarrow A$ is analogous to a fusion of elementary particles since the product indeed conserves the number theoretical energy. Co-algebra operations are time reversals of algebra operations in this evolution. Co-multiplication Δ assigns to $a \in A$ an element in $A \otimes A$ via algebra homomorphism and corresponds to a decay of initial state particle to two final state particles. It defines co-multiplication assign to $a \otimes b \in A \otimes A$ an element of $A \otimes A \rightarrow A \otimes A \otimes A$ and corresponds to a scattering of elementary particles with the emission of a third particle. Hence a sequence of algebraic manipulations is like a Feynman diagram involving both multiplications and co-multiplications and thus containing also loops. When particle creation and annihilation are absent, particle number is conserved and the process represents algebra endomorphism $A \rightarrow A$. Otherwise a more general operation is in question. This analogy inspires the question whether particle reactions could serve as a blood and flesh representation for $d=4$ algebras.

4. The dimension $d=4$ is maximal dimension of single space-time evolution representing an algebraic operation (unless one allows the possibility that space-time and embedding space dimensions are come as multiples of four and 8). Higher dimensions can be effectively achieved only if several space-time sheets are used defining $4n$-dimensional WCW . This could reflect some deep fact about algebras in general and also relate to the fact that 3- and 4-dimensional manifolds are the most interesting ones topologically.

4.10.2 One Element Field, Quantum Measurement Theory And ItsQ-Variant, And The Galois Fields Associated With Infinite Primes

John Baez talked in This Weeks Finds (Week 259) [B10] about one-element field - a notion inspired by the $q = exp(i2\pi/n) \rightarrow 1$ limit for quantum groups. This limit suggests that the notion of one-element field for which 0=1 - a kind of mathematical phantom for which multiplication and sum should be identical operations - could make sense. Physicist might not be attracted by this kind of identification.

In the following I want to articulate some comments from the point of view of quantum measurement theory and its generalization to q-measurement theory which I proposed for some years ago and which is represented above.

I also consider and alternative interpretation in terms of Galois fields assignable to infinite primes which form an infinite hierarchy. These Galois fields have infinite number of elements but the map to the real world effectively reduces the number of elements to 2: 0 and 1 remain different.

$q \to 1$ limit as transition from quantum physics to effectively classical physics?

The $q \to 1$ limit of quantum groups at q-integers become ordinary integers and n-D vector spaces reduce to n-element sets. For quantum logic the reduction would mean that 2^N-D spinor space becomes 2^N-element set. N qubits are replaced with N bits. This brings in mind what happens in the transition from wave mechanism to classical mechanics. This might relate in interesting manner to quantum measurement theory.

Strictly speaking, $q \to 1$ limit corresponds to the limit $q = exp(i2\pi/n)$, $n \to \infty$ since only roots of unity are considered. This also correspond to Jones inclusions at the limit when the discrete group Z_n or or its extension-both subgroups of $SO(3)$- to contain reflection has infinite elements. Therefore this limit where field with one element appears might have concrete physical meaning. Does the system at this limit behave very classically?

In TGD framework this limit can correspond to either infinite or vanishing Planck constant depending on whether one consider orbifolds or coverings. For the vanishing Planck constant one should have classicality: at least naïvely! In perturbative gauge theory higher order corrections come as powers of $g^2/4\pi\hbar$ so that also these corrections vanish and one has same predictions as given by classical field theory.

Q-measurement theory and $q \to 1$ limit

Q-measurement theory differs from quantum measurement theory in that the coordinates of the state space, say spinor space, are non-commuting. Consider in the sequel q-spinors for simplicity.

Since the components of quantum spinor do not commute, one cannot perform state function reduction. One can however measure the modulus squared of both spinor components which indeed commute as operators and have interpretation as probabilities for spin up or down. They have a universal spectrum of eigen values. The interpretation would be in terms of fuzzy probabilities and finite measurement resolution but may be in different sense as in case of HFF: s. Probability would become the observable instead of spin for q not equal to 1.

At $q \to 1$ limit quantum measurement becomes possible in the standard sense of the word and one obtains spin down or up. This in turn means that the projective ray representing quantum states is replaced with one of n possible projective rays defining the points of n-element set. For HFF: s of type II_1 it would be N-rays which would become points, N the included algebra. One might also say that state function reduction is forced by this mapping to *single* object at $q \to 1$ limit.

On might say that the set of orthogonal coordinate axis replaces the state space in quantum measurement. We do this replacement of space with coordinate axis all the time when at blackboard. Quantum consciousness theorist inside me adds that this means a creation of symbolic representations and that the function of quantum classical correspondences is to build symbolic representations for quantum reality at space-time level.

$q \to 1$ limit should have space-time correlates by quantum classical correspondence. A TGD inspired geometro-topological interpretation for the projection postulate might be that quantum measurement at $q \to 1$ limit corresponds to a leakage of 3-surface to a dark sector of embedding space with $q \to 1$ (Planck constant near to 0 or ∞ depending on whether one has $n \to \infty$ covering or division of M^4 or CP_2 by a subgroup of $SU(2)$ becoming infinite cyclic - very roughly!) and Hilbert space is indeed effectively replaced with n rays. For $q \neq 1$ one would have only probabilities for different outcomes since things would be fuzzy.

In this picture classical physics and classical logic would be the physical counterpart for the shadow world of mathematics and would result only as an asymptotic notion.

Could 1-element fields actually correspond to Galois fields associated with infinite primes?

Finite field G_p corresponds to integers modulo p and product and sum are taken only modulo p. An alternative representation is in terms of phases $exp(ik2\pi/p)$, $k = 0, ..., p-1$ with sum and product performed in the exponent. The question is whether could one define these fields also for infinite primes by identifying the elements of this field as phases $exp(ik2\pi/\Pi)$ with k taken to be finite integer and Π an infinite prime (recall that they form infinite hierarchy). Formally this makes sense. 1-element field would be replaced with infinite hierarchy of Galois fields with infinite number of elements!

The probabilities defined by components of quantum spinor make sense only as real numbers and one can indeed map them to real numbers by interpreting q as an ordinary complex number. This would give same results as $q \rightarrow 1$ limit and one would have effectively 1-element field but actually a Galois field with infinite number of elements.

If one allows k to be also infinite integer but not larger than than Π in the real sense, the phases $exp(ik2\pi/\Pi)$ would be well defined as real numbers and could differ from 1. All real numbers in the range $[-1,1]$ would be obtained as values of $cos(k2\pi/\Pi)$ so that this limit would effectively give real numbers.

This relates also interestingly to the question whether the notion of p-adic field makes sense for infinite primes. The p-adic norm of any infinite-p p-adic number would be power of π either infinite, zero, or 1. Excluding infinite normed numbers one would have effectively only p-adic integers in the range $1, ...\Pi - 1$ and thus only the Galois field G_Π representable also as quantum phases.

I conclude with a nice string of text from John'z page:

What's a mathematical phantom? According to Wraith, it's an object that doesn't exist within a given mathematical framework, but nonetheless "obtrudes its effects so convincingly that one is forced to concede a broader notion of existence".

and unashamedly propose that perhaps Galois fields associated with infinite primes might provide this broader notion of existence! In equally unashamed tone I ask whether there exists also hierarchy of conscious entities at $q = 1$ levels in real sense and whether we might identify ourselves as this kind of entities? Note that if cognition corresponds to p-adic space-time sheets, our cognitive bodies have literally infinite geometric size in real sense.

One-element field realized in terms of real units with number theoretic anatomy

One-element field looks rather self-contradictory notion since 1 and 0 should be represented by same element. The real units expressible as ratios of infinite rationals could however provide a well-defined realization of this notion.

1. The condition that same element represents the neutral element of both sum and product gives strong constraint on one-element field. Consider an algebra formed by reals with sum and product defined in the following manner. Sum, call it $\oplus$, corresponds to the ordinary product $x \times y$ for reals whereas product, call it $\otimes$, is identified as the non-commutative product $x \otimes y = x^y$. $x = 1$ represents both the neutral element (0) of $\oplus$ and the unit of $\otimes$. The sub-algebras generated by 1 and multiple powers $P_n(x) = P_{n-1}(x) \otimes x = x \otimes ... \otimes x$ form commutative sub-algebras of this algebra. When one restricts the consideration to $x = 1$ one obtains one-element field as sub-field which is however trivial since $\oplus$ and $\otimes$ are identical operations in this subset.

2. One can get over this difficulty by keeping the operations $\oplus$ and $\otimes$, by assuming one-element property only with respect to the real and various p-adic norms, and by replacing ordinary real unit 1 with the algebra of real units formed from infinite primes by requiring that the real and various p-adic norms of the resulting numbers are equal to one. As far as real and various p-adic norms are considered, one has commutative one-element field. When number theoretic anatomy is taken into account, the algebra contains infinite number of elements and is non-commutative with respect to the product since the number theoretic anatomies of x^y and y^x are different.

4.10.3 A Little Crazy Speculation About Knots And Infinite Primes

D-dimensional knots correspond to the isotopy equivalence classes of the embeddings of spheres S^d to S^{d+2}. One can consider also the isotopy equivalence classes of more general manifolds $M^d \subset M^{d+2}$. Knots [A6] are very algebraic objects. The product (or sum, I prefer to talk about product) of knots is defined in terms of connected sum. Connected sum quite generally defines a commutative and associative product, and one can decompose any knot into prime knots.

Knots can be mapped to Jones polynomials $J(K)$ (for instance - there are many other polynomials and there are very general mathematical results about them [A6]) and the product of knots is mapped to a product of corresponding polynomials. The polynomials assignable to prime knots should be prime in a well-defined sense, and one can indeed define the notion of primeness for polynomials $J(K)$: prime polynomial does not factor to a product of polynomials of lower degree in the extension of rationals considered.

This raises the idea that one could define the notion of zeta function for knots. It would be simply the product of factors $1/(1-J(K)^{-s})$ where K runs over prime knots. The new (to me) but very natural element in the definition would be that ordinary prime is replaced with a polynomial prime. This observation led to the idea that the hierarchy of infinite primes could correspond to the hierarchy of knots in various dimensions and this in turn stimulated quite fascinating speculations.

Do knots correspond to the hierarchy of infinite primes?

A very natural question is whether one could define the counterpart of zeta function for infinite primes. The idea of replacing primes with prime polynomials would resolve the problem since infinite primes can be mapped to polynomials. For some reason this idea however had not occurred to me earlier.

The correspondence of both knots and infinite primes with polynomials inspires the question whether $d = 1$-dimensional prime knots might be in correspondence (not necessarily 1-1) with infinite primes. Rational or Gaussian rational infinite primes would be naturally selected these are also selected by physical considerations as representatives of physical states although quaternionic and octonionic variants of infinite primes can be considered.

If so, knots could correspond to the subset of states of a super-symmetric arithmetic quantum field theory with bosonic single particle states and fermionic states labeled by quaternionic primes.

1. The free Fock states of this QFT are mapped to first order polynomials and irreducible polynomials of higher degree have interpretation as bound states so that the non-decomposability to a product in a given extension of rationals would correspond physically to the non-decomposability into many-particle state. What is fascinating that apparently free arithmetic QFT allows huge number of bound states.

2. Infinite primes form an infinite hierarchy, which corresponds to an infinite hierarchy of second quantizations for infinite primes meaning that n-particle states of the previous level define single particle states of the next level. At space-time level this hierarchy corresponds to a hierarchy defined by space-time sheets of the topological condensate: space-time sheet containing a galaxy can behave like an elementary particle at the next level of hierarchy.

3. Could this hierarchy have some counterpart for knots?In one realization as polynomials, the polynomials corresponding to infinite prime hierarchy have increasing number of variables. Hence the first thing that comes into my uneducated mind is as the hierarchy defined by the increasing dimension d of knot. All knots of dimension d would in some sense serve as building bricks for prime knots of dimension $d+1$ or possibly $d+2$ (the latter option turns out to be the more plausible one). A canonical construction recipe for knots of higher dimensions should exist.

4. One could also wonder whether the replacement of spherical topologies for d-dimensional knot and $d+2$-dimensional embedding space with more general topologies could correspond to algebraic extensions at various levels of the hierarchy bringing into the game more general infinite primes. The units of these extensions would correspond to knots which involve in an essential manner the global topology (say knotted non-contractible circles in 3-torus). Since the knots defining the product would in general have topology different from spherical

topology the product of knots should be replaced with its category theoretical generalization making higher-dimensional knots a groupoid in which spherical knots would act diagonally leaving the topology of knot invariant. The assignment of d-knots with the notion of n-category, n-groupoid, etc.. by putting d=n is a highly suggestive idea. This is indeed natural since are an outcome of a repeated abstraction process: statements about statements about.....

5. The lowest ($d = 1, D = 3$) level would be the fundamental one and the rest would be (somewhat boring!) repeated second quantization. This is why the dimension $D = 3$ (number theoretic braids at light-like 3-surfaces!) would be fundamental for physics.

Further speculations

Some further speculations about the proposed structure of all structures are natural.

1. The possibility that algebraic extensions of infinite primes could allow to describe the refinements related to the varying topologies of knot and embedding space would mean a deep connection between number theory, manifold topology, sub-manifold topology, and n-category theory.

2. Category theory appears already now in fundamental role in the construction of the generalization of M-matrix unifying the notions of density matrix and S-matrix. Generalization of category to n-category theory and various n-structures would have very direct correspondence with the physics of TGD Universe if one assumes that repeated second quantization makes sense and corresponds to the hierarchical structure of many-sheeted space-time where even galaxy corresponds to elementary fermion or boson at some level of hierarchy.

   This however requires that the unions of light-like 3-surfaces and of their sub-manifolds at different levels of topological condensate are able to represent higher-dimensional manifolds physically albeit not in the standard geometric sense since embedding space dimension is just 8. This might be possible.

3. As far as physics is considered, the disjoint union of sub-manifolds of dimensions d_1 and d_2 behaves like a $d_1 + d_2$-dimensional Cartesian product of the corresponding manifolds. This is of course used in standard manner in wave mechanics (the WCW of n-particle system is identified as E^{3n}/S_n with division coming from statistics).

4. If the surfaces have intersection points, one has a union of Cartesian product with punctures (intersection points) and of lower-dimensional manifold corresponding to the intersection points.

5. Note also that by posing symmetries on classical fields one can effectively obtain from a given n-manifold manifolds (and orbifolds) with quotient topologies.

The megalomanic conjecture is that this kind of physical representation of d-knots and their embedding spaces is possible using many-sheeted space-time. Perhaps even the entire magnificent mathematics of n-manifolds and their sub-manifolds might have a physical representation in terms of sub-manifolds of 8-D $M^4 \times CP_2$ with dimension not higher than space-time dimension $d = 4$.

The idea survives the most obvious killer test

All this looks nice and the question is how to give a death blow to all this reckless speculation. Torus knots are an excellent candidate for performing this unpleasant task but the hypothesis survives!

1. Torus knots [A14] are labeled by a pair integers (m, n), which are relatively prime. These are prime knots. Torus knots for which one has $m/n = r/s$ are isotopic so that any torus knot is isotopic with a knot for which m and n have no common prime power factors.

2. The simplest infinite primes correspond to free Fock states of the supersymmetric arithmetic QFT and are labeled by pairs (m, n) of integers such that m and n do not have any common prime factors. Thus torus knots would correspond to free Fock states! Note that the prime power $p^{k(p)}$ appearing in m corresponds to $k(p)$-boson state with boson "momentum" p and the corresponding power in n corresponds to fermion state plus $k(p) - 1$ bosons.

3. A further property of torus knots is that (m, n) and (n, m) are isotopic: this would correspond at the level of infinite primes to the symmetry $mX + n \rightarrow nX + m$, X product of all finite primes. Thus infinite primes are in $2 \rightarrow 1$ correspondence with torus knots and the hypothesis survives also this murder attempt. Probably the assignment of orientation to the knot makes the correspondence 1-1 correspondence.

How to realize the representation of the braid hierarchy in many-sheeted space-time?

One can consider a concrete construction of higher-dimensional knots and braids in terms of the many-sheeted space-time concept.

1. The basic observation is that ordinary knots can be constructed as closed braids so that everything reduces to the construction of braids. In particular, any torus knot labeled by (m, n) can be made from a braid with m strands: the braid word in question is $(\sigma_1 ....\sigma_{m-1})^n$ or by $(m, n) = (n, m)$ equivalence from n strands. The construction of infinite primes suggests that also the notion of d-braid makes sense as a collection of d-braids in $d + 2$-space, which move and and define $d + 1$-braid in $d + 3$ space (the additional dimension being defined by time coordinate).

2. The notion of topological condensate should allow a concrete construction of the pairs of d- and $d + 2$-dimensional manifolds. The 2-D character of the fundamental objects (partons) might indeed make this possible. Also the notion of length scale cutoff fundamental for the notion of topological condensate is a crucial element of the proposed construction.

3. Infinite primes have also interpretation as physical states and the representation in terms of knots would mean a realization of quantum classical correspondence.

The concrete construction would proceed as follows.

1. Consider first the lowest non-trivial level in the hierarchy. One has a collection of 3-D light-like 3-surfaces X^3_i representing ordinary braids. The challenge is to assign to them a 5-D embedding space in a natural manner. Where do the additional two dimensions come from? The obvious answer is that the new dimensions correspond to the partonic 2-surface X^2 assignable to the $3 - D$ light-like surface X^3 at which these surfaces have suffered topological condensation. The geometric picture is that X^3_i grow like plants from ground defined by X^2 at 7-dimensional $\delta M^4_+ \times CP_2$.

2. The degrees of freedom of X^2 should be combined with the degrees of freedom of X^3_i to form a 5-dimensional space X^5. The natural idea is that one first forms the Cartesian products $X^5_i = X^3_i \times X^2$ and then the desired 5-manifold X^5 as their union by posing suitable additional conditions. Braiding means a translational motion of X^3_i inside X^2 defining braid as the orbit in X^5. It can happen that X^3_i and X^3_j intersect in this process. At these points of the union one must obviously pose some additional conditions. Same applies to intersection of more than two X^3_i.

   Finite (p-adic) length scale resolution suggests that all points of the union at which an intersection between two or more light-like 3-surfaces occurs must be regarded as identical. In general the intersections would occur in a 2-d region of X^2 so that the gluing would take place along 5-D regions of X^5_i and there are therefore good hopes that the resulting 5-D space is indeed a manifold. The embedding of the surfaces X^3_i to X^5 would define the braiding.

3. At the next level one would consider the 5-d structures obtained in this manner and allow them to topologically condense at larger 2-D partonic surfaces in the similar manner. The outcome would be a hierarchy consisting of $2n + 1$-knots in $2n + 3$ spaces. A similar construction applied to partonic surfaces gives a hierarchy of $2n$-knots in $2n + 2$-spaces.

4. The notion of length scale cutoff is an essential element of the many-sheeted space-time concept. In the recent context it suggests that d-knots represented as space-time sheets topologically condensed at the larger space-time sheet representing $d+2$-dimensional embedding space could be also regarded effectively point-like objects (0-knots) and that their d-knottiness and internal topology could be characterized in terms of additional quantum numbers. If so then d-knots could be also regarded as ordinary colored braids and the construction at higher levels would indeed be very much analogous to that for infinite primes.

Part II

TGD, P-ADIC NUMBERS, AND ADELES

Chapter 5

p-Adic Numbers and Generalization of Number Concept

5.1 Introduction

In this chapter basic facts about p-adic numbers and the question about their relation to real numbers are discussed. Also the basic technicalities related to the notion of p-adic physics are discussed.

5.1.1 Problems

It is far from obvious what the p-adic counterpart of real physics could mean and how one could fuse together real and p-adic physics. Therefore it is good to list the basic problems and proposals for their solution.

The first problem concerns the correspondence between real and p-adic numbers.

1. The success of p-adic mass calculations involves the notions of p-adic probability, thermodynamics, and the mapping of p-adic probababilities to the real ones by a continuous correspondence $x = \sum x_n p^n \rightarrow Id(x) = \sum x_n p^{-n}$ that I have christened canonical identification. The problem is that I n does not respect symmetries defined by isometries and also general coordinate invariance is possible only if one can identify preferred embedding space coordinates. The reason is that I does not commute with the basic arithmetic operations. I allows several variants and it is possible to have correspondence which respects symmetries in arbitrary accuracy in preferred coordinates. Thus I can play a role at space-time level only if one defines symmetries modulo measurement resolution. I would make sense only in the interval defining the measurement resolution for a given coordinate variable and the p-adic effective topology would make sense just because the finite measurement resolution does not allow to well-order the points.

2. The identification of real and p-adic numbers via rationals common to all number fields - or more generally along algebraic extension of rationals- respects symmetries and algebra but is not continuous. At the embedding space level preferred coordinates are required also now. The maximal symmetries of the embedding space allow identification of this kind of coordinates. They are not unique. For instance, M^4 linear coordinates look very natural but for CP_2 trigonometric functions of angle like coordinates look more suitablel and Fourier analysis suggests strongly the introduction of algebraic extensions involving roots of unity. Partly the non-uniqueness has an interpretation as an embedding space correlate for the selection of the quantization axes. The symmetric space property of WCW gives hopes that general coordinate invariance in quantal sense can be realized. The existence of p-adic harmonic analysis suggests a discretization of the p-adic variant of embedding space and WCW based on roots of unity.

3. One can consider a compromise between the two correspondences. Discretization via common algebraic points can be completed to a p-adic continuum by assigning to each real discretization interval (say angle increment $2\pi/N$) p-adic numbers with norm smaller than one.

Second problem relates to integration and Fourier analysis. Both these procedures are fundamental for physics -be it classical or quantum. The p-adic variant of definite integral does not exist in the sense required by the action principles of physics although classical partial differential equations assigned to a particular variational principle make perfect sense. Fourier analysis is also possible only if one allows algebraic extension of p-adic numbers allowing a sufficient number of roots of unity correlating with the measurement resolution of angle. The finite number of them has interpretation in terms of finite angle resolution. Fourier analysis provides also an algebraic realization of definite integral when one integrates over the entire manifold as one indeed does in the case of WCW. If the space in question allows maximal symmetries as WCW and embedding space do, there are excellent hopes of having p-adic variants of both integration and harmonic analysis and the above described procedure allows a precise completion of the discretized variant of real manifold to its continuous p-adic variant.

The third problem relates to the definitions of the p-adic variants of Riemannian, symplectic, and Kähler geometries. It is possible to generalize formally the notion of Riemann metric although non-local quantities like areas and total curvatures do not make sense if defined in terms of integrals. If all relevant quantities assignable to the geometry (family of Hamiltonians defining isometries, Killing vector fields, components of metric and Kähler form, Kähler function, etc...) are expressible in terms of rational functions involving only rational numbers as coefficients of polynomials, they allow an algebraic continuation to the p-adic context and the p-adic variant of the geometry makes sense.

The fourth problem relates to the question what one means with p-adic quantum mechanics. In TGD framework p-adic quantum theory utilizes p-adic Hilbert space. The motivation is that the notions of p-adic probability and unitarity are well defined. From the beginnning it was clear that the straightforward generalization of Schrödinger equation is not very interesting physically and gradually the conviction has developed that the most realistic approach must rely on the attempt to find the p-adic variant of the TGD inspired quantum physics in all its complexity. The recent approach starts from a rather concrete view about generalized Feynman diagrams defining the points of WCW and leads to a rather detailed view about what the p-adic variants of QM could be and how they could be fused with real QM to a larger structure. Even more, just the requirement that this p-adicization exists, gives very powerful constraints on the real variant of the quantum TGD.

The fifth problem relates to the notion of information in p-adic context. p-Adic thermodynamics leads naturally to the question what p-adic entropy might mean and this in turn leads to the realization that for rational or even algebraic probabilities p-adic variant of Shannon entropy can be negative and has minimum for a unique prime. One can say that the entanglement in the intersection of real and p-adic worlds is negentropic. This leads to rather fascinating vision about how negentropic entanglement makes it possible for living systems to overcome the second law of thermodynamics. The formulation of quantum theory in the intersection of real and living worlds becomes the basic challenge.

The proposed solutions to the technical problems could be rephrased in terms of the notion of algebraic universality. Various p-adic physics are obtained as algebraic continuation of real physics through the common algebraic points of real and p-adic worlds and by performing completion in the sense that the interval corresponding to finite measurement resolution are replaced with their p-adic counterpart via canonical identification. This allows to have exact symmetries as their discrete variants and also a continuous correspondence if desired. Particular p-adicization is characterized by a choice fo preferred embedding space coordinates, which has interpretation in terms of a particular cognitive representation. Hence one is forced to refine the view about general coordinate invariance. Different coordinate choices correspond to different cognitive representations having delicate effects on physics if it is assumed to include also the effects of cognition.

5.1.2 Program

These ideas lead to a reasonably well defined p-adicization program. Try to define precisely the concepts of the p-adic space-time and configuration space (WCW), formulate the finite-p p-adic versions of quantum TGD. Try to fuse together real and various p-adic quantum TGDs are to form a full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, and the notions of probability and unitarity to the p-adic context. Also new physical thinking and philosophy is needed. The notions of zero energy ontology, hierarchy of Planck constants and the generalization of the notion of embedding space required by it are essential but not discussed in detail in this chapter.

5.1.3 Topics Of The Chapter

The topics of the chapter are the following:

1. p-Adic numbers, their extensions (also those involving transcendentals) are described. The existence of a square root of an ordinary p-adic number is necessary in many applications of the p-adic numbers (p-adic group theory, p-adic unitarity, Riemannian geometry) and its existence implies a unique algebraic extension, which is 4-dimensional for $p > 2$ and 8-dimensional for $p = 2$. Contrary to the first expectations, all possible algebraic extensions are possible and one cannot interpret the algebraic dimension of the algebraic extension as a physical dimension.

2. The concepts of the p-adic differentiability and analyticity are discussed. The notion of p-adic fractal is introduced the properties of the fractals defined by p-adically differentiable functions are discussed.

3. Various approaches to the problem of defining p-adic valued definite integral are discussed. The only reasonable generalizations rely on algebraic continuation and correspondence via common rationals. p-Adic field equations do not necessitate p-adic definite integral but algebraic continuation allows to assign to a given real space-time sheets a p-adic space-time sheets if the definition of space-time sheet involves algebraic relations between embedding space coordinates. There are also hopes that one can algebraically continue the value of Kähler action to p-adic context if finite-dimensional extensions are allowed.

4. Symmetries are discussed from p-adic point of view starting from the identification via common rationals. Also possible p-adic generalizations of Fourier analysis are considered. Besides a number theoretical approach, group theoretical approach providing a direct generalization of the ordinary Fourier analysis based on the utilization of exponent functions existing in algebraic extensions containing some root of e and its powers up to e^{p-1} is discussed. Also the generalization of Fourier analysis based on the Pythagorean phases is considered.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

5.2 Summary Of The Basic Physical Ideas

In the following various ways to end up with p-adic physics and with the idea about p-adic topology as an effective topology of space-time surface are described.

5.2.1 P-Adic Mass Calculations Briefly

p-Adic mass calculations based on p-adic thermodynamics with energy replaced with the generator $L_0 = zd/dz$ of infinitesimal scaling are described in the first part of [K62].

1. p-Adic thermodynamics is justified by the randomness of the motion of partonic 2-surfaces restricted only by the light-likeness of the orbit.

2. It is essential that the conformal symmetries associated with the light-like coordinates of parton and light-cone boundary are not gauge symmetries but dynamical symmetries. The point is that there are two kinds of conformal symmetries: the super-symplectic conformal symmetries assignable to the light-like boundaries of $CD \times CP_2$ and super Kac-Moody symmetries assignable to light-like 3-surfaces defining fundamental dynamical objects. In so called coset construction the differences of super-conformal generators of these algebras annihilate the physical states. This leads to a generalization of equivalence principle since one can assign four-momentum to the generators of both algebras identifiable as inertial *resp.* gravitational four-momentum. A second important consequence is that the generators of either algebra do not act like gauge transformations so that it makes sense to construct p-adic thermodynamics for them.

3. In p-adic thermodynamics scaling generator L_0 having conformal weights as its eigen values replaces energy and Boltzmann weight $exp(H/T)$ is replaced by p^{L_0/T_p}. The quantization $T_p = 1/n$ of conformal temperature and thus quantization of mass squared scale is implied by number theoretical existence of Boltzmann weights. p-Adic length scale hypothesis states that primes $p \simeq 2^k$, k integer. A stronger hypothesis is that k is prime (in particular Mersenne prime or Gaussian Mersenne) makes the model very predictive and fine tuning is not possible.

The basic mystery number of elementary particle physics defined by the ratio of Planck mass and proton mass follows thus from number theory once CP_2 radius is fixed to about 10^4 Planck lengths. Mass scale becomes additional discrete variable of particle physics so that there is not more need to force top quark and neutrinos with mass scales differing by 12 orders of magnitude to the same multiplet of gauge group. Electron, muon, and τ correspond to Mersenne prime $k = 127$ (the largest non-super-astrophysical Mersenne), and Mersenne primes $k = 113, 107$. Intermediate gauge bosons and photon correspond to Mersenne M_{89}, and graviton to M_{127}.

Mersenne primes are very special also number theoretically because bit as the unit of information unit corresponds to $log(2)$ and can be said to exists for M_n-adic topology. The reason is that $log(1+p)$ existing always p-adically corresponds for $M_n = 2^n - 1$ to $log(2^n) \equiv nlog(2)$ so that one has $log(2 \equiv log(1 + M_n)/n$. Since the powers of 2 modulo p give all integers $n \in \{1, p-1\}$ by Fermat's theorem, one can say that the logarithms of all integers modulo M_n exist in this sense and therefore the logarithms of all p-adic integers not divisible by p exist. For other primes one must introduce a transcendental extension containing $log(a)$ where are is so called primitive root. One could criticize the identification since $log(1 + M_n)$ corresponding in the real sense to n bits corresponds in p-adic sense to a very small information content since the p-adic norm of the p-adic bit is $1/M_n$.

The value of k for quark can depend on hadronic environment [K64] and this would produce precise mass formulas for low energy hadrons. This kind of dependence conforms also with the indications that neutrino mass scale depends on environment [C2]. Amazingly, the biologically most relevant length scale range between 10 nm and 4 μm contains four Gaussian Mersennes $(1 + i)^n - 1$, $n = 151, 157, 163, 167$ and scaled copies of standard model physics in cell length scale could be an essential aspect of macroscopic quantum coherence prevailing in cell length scale.

p-Adic mass thermodynamics is not quite enough: also Higgs boson is needed and wormhole contact carrying fermion and anti-fermion quantum numbers at the light-like wormhole throats is excellent candidate for Higgs [K51]. The coupling of Higgs to fermions can be small and induce only a small shift of fermion mass: this could explain why Higgs has not been observed. Also the Higgs contribution to mass squared can be understood thermodynamically if identified as absolute value for the thermal expectation value of the eigenvalues of the Kähler-Dirac operator having interpretation as complex square root of conformal weight.

The original belief was that only Higgs corresponds to wormhole contact. The assumption that fermion fields are free in the conformal field theory applying at parton level forces to identify all gauge bosons as wormhole contacts connecting positive and negative energy space-time sheets [K51]. Fermions correspond to topologically condensed CP_2 type extremals with single light-like wormhole throat. Gravitons are identified as string like structures involving pair of fermions or gauge bosons connected by a flux tube. Partonic 2-surfaces are characterized by genus which explains family replication phenomenon and an explanation for why their number is three emerges [K23]. Gauge bosons are labeled by pairs (g_1, g_2) of handle numbers and can be arranged to octet and singlet representations of the resulting dynamical SU(3) symmetry. Ordinary gauge bosons are SU(3) singlets and the heaviness of octet bosons explains why higher boson families are effectively absent. The different character of bosons could also explain why the p-adic temperature for bosons is $T_p = 1/n < 1$ so that Higgs contribution to the mass dominates.

5.2.2 P-Adic Length Scale Hypothesis, Zero Energy Ontology, And Hierarchy Of Planck Constants

Zero energy ontology and the hierarchy of Planck constants realized in terms of the generalization of the embedding space lead to a deeper understanding of the origin of the p-adic length scale hypothesis.

Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the light-like boundaries of CD. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. "Any physical state is creatable from vacuum" becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe?, Is theory building completely useless if only single solution of field equations is realized?). At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events.

Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [K24] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the "world of classical worlds" represents a von Neumann algebra [A48] known as hyperfinite factor of type II_1 (HFF) [K24, K110, K36]. HFF [A28, A41] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [A2]. The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [A71], anyons [D2] , quantum groups and conformal field theories [A72], and knots and topological quantum field theories [A65, A34].

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

One can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of CD corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is.1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [K24]. The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics. Connes tensor product [A28] provides a mathematical description of the finite measurement resolution but does not fix the M-matrix as was the original hope. The remaining challenge is the calculation of M-matrix and the progress induced by zero energy ontology during last years has led to rather concrete proposal for the construction of M-matrix.

How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

In zero energy ontology zero energy states have as embedding space correlates causal diamonds for which the distance between the tips of the intersecting future and past directed light-cones comes as integer multiples of a fundamental time scale: $T_n = n \times T_0$. p-Adic length scale hypothesis allows to consider a stronger hypothesis $T_n = 2^n T_0$ and its generalization a slightly more general hypothesis $T_n = p^n T_0$, p prime. It however seems that these scales are dynamically favored but that also other scales are possible.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ (or $T_p = pT_0$) induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around.1 eV would correspond to $L(169) \simeq 5\ \mu$m (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of WCW .

Mersenne primes and Gaussian Mersennes

The generalization of the embedding space required by the postulated hierarchy of Planck constants [K36] means a book like structure for which the pages are products of singular coverings or factor spaces of CD (causal diamond defined as intersection of future and past directed light-cones) and of CP_2 [K36]. This predicts that Planck constants are rationals and that a given value of Planck constant corresponds to an infinite number of different pages of the Big Book, which might be seen as a drawback. If only singular covering spaces are allowed the values of Planck constant are products of integers and given value of Planck constant corresponds to a finite number of pages given by the number of decompositions of the integer to two different integers. The definition of the book like structure assigns to a given CD preferred quantization axes and so that quantum measurement has direct correlate at the level of moduli space of CDs.

TGD inspired quantum biology and number theoretical considerations suggest preferred values for $r = \hbar/\hbar_0$. For the most general option the values of $\hbar$ are products and ratios of two integers n_a and n_b. Ruler and compass integers defined by the products of distinct Fermat primes and power of two are number theoretically favored values for these integers because the phases $exp(i2\pi/n_i)$, $i \in \{a, b\}$, in this case are number theoretically very simple and should have emerged first in the number theoretical evolution via algebraic extensions of p-adics and of rationals. p-Adic length scale hypothesis favors powers of two as values of r.

One can however ask whether a more precise characterization of preferred Mersennes could exist and whether there could exists a stronger correlation between hierarchies of p-adic length scales and Planck constants. Mersenne primes $M_k = 2^k - 1$, $k \in \{89, 107, 127\}$, and Gaussian Mersennes $M_{G,k} = (1+i)k-1$, $k \in \{113, 151, 157, 163, 167, 239, 241..\}$ are expected to be physically highly interesting and up to $k = 127$ indeed correspond to elementary particles. The number theoretical miracle is that all the four p-adic length scales with $k \in \{151, 157, 163, 167\}$ are in the biologically highly interesting range 10 nm-2.5 μm). The question has been whether these define scaled up copies of electro-weak and QCD type physics with ordinary value of $\hbar$. The proposal that this is the case and that these physics are in a well-defined sense induced by the dark scaled up variants of corresponding lower level physics leads to a prediction for the preferred values of $r = 2^{k_d}$, $k_d = k_i - k_j$.

What induction means is that dark variant of exotic nuclear physics induces exotic physics with ordinary value of Planck constant in the new scale in a resonant manner: dark gauge bosons transform to their ordinary variants with the same Compton length. This transformation is natural since in length scales below the Compton length the gauge bosons behave as massless and free particles. As a consequence, lighter variants of weak bosons emerge and QCD confinement scale becomes longer.

This proposal will be referred to as Mersenne hypothesis. It leads to strong predictions about EEG [K33] since it predicts a spectrum of preferred Josephson frequencies for a given value of membrane potential and also assigns to a given value of $\hbar$ a fixed size scale having interpretation as the size scale of the body part or magnetic body. Also a vision about evolution of life emerges. Mersenne hypothesis is especially interesting as far as new physics in condensed matter length scales is considered: this includes exotic scaled up variants of the ordinary nuclear physics and their dark variants. Even dark nucleons are possible and this gives justification for the model of dark nucleons predicting the counterparts of DNA, RNA, tRNa, and amino-acids as well as realization of vertebrate genetic code [K105].

These exotic nuclear physics with ordinary value of Planck constant could correspond to ground states that are almost vacuum extremals corresponding to homologically trivial geodesic sphere of CP_2 near criticality to a phase transition changing Planck constant. Ordinary nuclear physics would correspond to homologically non-trivial geodesic sphere and far from vacuum extremal property. For vacuum extremals of this kind classical Z^0 field proportional to electromagnetic field is present and this modifies dramatically the view about cell membrane as Josephson junction. The model for cell membrane as almost vacuum extremal indeed led to a quantitative breakthrough in TGD inspired model of EEG and is therefore something to be taken seriously. The safest option concerning empirical facts is that the copies of electro-weak and color physics with ordinary value of Planck constant are possible only for almost vacuum extremals - that is at criticality against phase transition changing Planck constant.

5.2.3 P-Adic Physics And The Notion Of Finite Measurement Resolution

Canonical identification mapping p-adic numbers to reals in a continuous manner plays a key role in some applications of TGD and together with the discretization necessary to define the p-adic variants of integration and harmonic analysis suggests that p-adic topology identified as an effective topology could provide an elegant manner to characterize finite measurement resolution.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with a pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

This interpretation would suggest that there is an infinite hierarchy of measurement resolutions characterized by the value of the p-adic prime. This would mean quite interesting refinement of the notion of finite measurement resolution. At the level of quantum theory it could be interpreted as a maximization of p-adic entanglement negentropy as a function of the p-adic prime. Perhaps one might say that there is a unique p-adic effective topology allowing to maximize the information content of the theory relying on finite measurement resolution.

5.2.4 P-Adic Numbers And The Analogy Of TGD With Spin-Glass

The vacuum degeneracy of the Kähler action leads to a precise spin glass analogy at the level of the WCW geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity could be realized at the level of WCW . Also the concept of p-adic space-time surface emerges rather naturally.

Spin glass briefly

The basic characteristic of the spin glass phase [B12] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines "energy landscape" and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [A67].

Vacuum degeneracy of Kähler action

The Kähler action defining WCW geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrange manifold of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new

Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ , \ \ i = 1, 2 \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by the classical gravitational interaction there are good reasons to expect large ground state degeneracy, when the system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass but is 4-dimensional.

Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Lagrangian sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a given preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surfaces which is vacuum extremal and deforming them.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function become possible. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe [A35]. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multi-furcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate preferred extremals. This non-determinism is expected to make the preferred extremals of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{5.2.1}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the preferred extremals are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible "engineering" at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible "engineering" at the level of the real world.

5.2.5 Life As Islands Of Rational/Algebraic Numbers In The Seas Of Real And P-Adic Continua?

NMP and negentropic entanglement demanding entanglement probabilities which are equal to inverse of integer, is the starting point. Rational and even algebraic entanglement coefficients make sense in the intersection of real and p-adic words, which suggests that in some sense life and conscious intelligence reside in the intersection of the real and p-adic worlds.

What could be this intersection of realities and p-adicities?

1. The facts that fermionic oscillator operators are correlates for Boolean cognition and that induced spinor fields are restricted to string world sheets and partonic 2-surfaces suggests that the intersection consists of these 2-surfaces.

2. Strong form of holography allows a rather elegant adelization of TGD by a construction of space-time surfaces by algebraic continuations of these 2-surfaces defined by parameters in algebraic extension of rationals inducing that for various p-adic number fields to real or p-adic number fields. Scattering amplitudes could be defined also by a similar algebraic contination. By conformal invariance the conformal moduli characterizing the 2-surfaces would defined the parameters.

This suggests a rather concrete view about the fundamental quantum correlates of life and intelligence.

1. For the minimal option life would be effectively 2-dimensional phenomenon and essentially a boundary phenomenon as also number theoretical criticality suggests. There are good reasons to expect that only the data from the intersection of real and p-adic string world sheets partonic two-surfaces appears in U-matrix so that the data localizable to strings connecting partonic 2-surfaces would dictate the scattering amplitudes.

A good guess is that algebraic entanglement is essential for quantum computation, which therefore might correspond to a conscious process. Hence cognition could be seen as a quantum computation like process, a more appropriate term being quantum problem solving [K3]. Living-dead dichotomy could correspond to rational-irrational or to algebraic-transcendental dichotomy: this at least when life is interpreted as intelligent life. Life would in a well defined sense correspond to islands of rationality/algebraicity in the seas of real and p-adic continua. Life as a critical phenomenon in the number theoretical sense would be one aspect of quantum criticality of TGD Universe besides the criticality of the space-time dynamics and the criticality with respect to phase transitions changing the value of Planck constant and other more familiar criticalities. How closely these criticalities relate remains an open question [K83].

The view about the crucial role of rational and algebraic numbers as far as intelligent life is considered, could have been guessed on very general grounds from the analogy with the orbits of a dynamical system. Rational numbers allow a predictable periodic decimal/pinary expansion and are analogous to one-dimensional periodic orbits. Algebraic numbers are related to rationals by a finite number of algebraic operations and are intermediate between periodic and chaotic orbits allowing an interpretation as an element in an algebraic extension of any p-adic number field. The projections of the orbit to various coordinate directions of the algebraic extension represent now periodic orbits. The decimal/pinary expansions of transcendentals are un-predictable being analogous to chaotic orbits. The special role of rational and algebraic numbers was realized already by Pythagoras, and the fact that the ratios for the frequencies of the musical scale are rationals supports the special nature of rational and algebraic numbers. The special nature of the Golden Mean, which involves $\sqrt{5}$, conforms the view that algebraic numbers rather than only rationals are essential for life.

Later progress in understanding of quantum TGD allows to refine and simplify this view dramatically. The idea about p-adic-to-real transition for space-time sheets as a correlate for the transformation of intention to action has turned out to be un-necessary and also hard to realize mathematically. In adelic vision real and p-adic numbers are aspects of existence in all length scales and mean that cognition is present at all levels rather than emerging. Intentions have interpretation in terms of state function reductions in ZEO and there is no need to identify p-adic space-time sheets as their correlates.

5.2.6 P-Adic Physics As Physics Of Cognition

The vision about p-adic physics as physics of cognition has gradually established itself as one of the key idea of TGD inspired theory of consciousness. There are several motivations for this idea.

The strongest motivation is the vision about living matter as something residing in the intersection of real and p-adic worlds. One of the earliest motivations was p-adic non-determinism identified tentatively as a space-time correlate for the non-determinism of imagination. p-Adic non-determinism follows from the fact that functions with vanishing derivatives are piecewise constant functions in the p-adic context. More precisely, p-adic pseudo constants depend on the pinary cutoff of their arguments and replace integration constants in p-adic differential equations. In the case of field equations this means roughly that the initial data are replaced with initial data given for a discrete set of time values chosen in such a way that unique solution of field equations results. Solution can be fixed also in a discrete subset of rational points of the embedding space. Presumably the uniqueness requirement implies some unique pinary cutoff. Thus the space-time surfaces representing solutions of p-adic field equations are analogous to space-time surfaces consisting of pieces of solutions of the real field equations. p-Adic reality is much like the dream reality consisting of rational fragments glued together in illogical manner or pieces of child's drawing of body containing body parts in more or less chaotic order.

The obvious interpretation for the solutions of the p-adic field equations is as a geometric correlate of imagination. Plans, intentions, expectations, dreams, and cognition in general are expected to have p-adic cognitive space-time sheets as their geometric correlates. A deep principle seems to be involved: incompleteness is characteristic feature of p-adic physics but the flexibility made possible by this incompleteness is absolutely essential for imagination and cognitive consciousness in general.

If one accepts the idea that real and p-adic space-time regions are correlates for matter and cognitive mind, one encounters the question how matter and mind interact. The original candidate for this interaction was as a phase transition leading to a transformation of the real space-time regions to p-adic ones and vice versa. These transformations would take place in quantum jumps. p-Adic-to-real phase transition would have interpretation as a transformation of thought into a sensory experience (dream or hallucination) or to an action. The reverse phase transition might relate to the transformation of the sensory experience to cognition. Sensory experiences could be also transformed to cognition by initial values realized as common rational points of a real space-time sheet representing sensory input and a p-adic space-time sheet representing the cognitive output. In this case the cognitive mental image is unique only in case that p-adic pseudo constants are ordinary constants.

It turned out that this interpretation leads to grave mathematical difficulties: one should construct U-matrix and M-matrix for transitions between different number fields, and this makes sense only if all the parameters involved are rational or algebraic. A more realistic view is that the interaction between real and p-adic number fields is that p-adic space-time surfaces define cognitive representations of real space-time surfaces (preferred extremals). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics. The notion of p-adic manifold [K112] tries to catch this idea mathematically.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. Mathematically this formulation is much more elegant that based on p-adic manifold since discretization seems to be un-necessary at space-time level and applies only to the parameters characterizing string world sheet.

String world sheets and partonic 2-surfaces would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime.

The idea about p-adic pseudo constants as correlates of imagination is however too nice to be thrown away without trying to find an alternative interpretation consistent with strong form of holography. Could the following argument allow to save p-adic view about imagination in a mathematically respectable manner?

1. The construction of preferred extremals from data at 2-surfaces is like boundary value problem. Integration constants are replaced with pseudo-constants depending on finite number pinary digits of variables depending on coordinates normal to string world sheets and partonic 2-surfaces.

2. Preferred extremal property in real context implies strong correlations between string world sheets and partonic 2-surfaces by boundary conditions a them. One cannot choose these 2- surfaces completely independently. Pseudo-constant could allow a large number of p-adic configurations involving string world sheets and partonic 2-surfaces not allowed in real context and realizing imagination.

3. Could imagination be realized as a larger size of the p-adic sectors of WCW? Could the realizable intentional actions belong to the intersection of real and p-adic WCWs? Could the modes of WCW spinor fields for which 2-surfaces are extendable to space-time surfaces

only in some p-adic sectors make sense? The real space-time surface for them be somehow degenerate, for instance, consisting of string world sheets only.

Could imagination be search for those collections of string world sheets and partonic 2-surfaces, which allow extension to (realization as) real preferred extremals? p-Adic physics would be there as an independent aspect of existence and this is just the original idea. Imagination could be realized in state function reduction, which always selects only those 2-surfaces which allow continuation to real space-time surfaces. The distinction between only imaginable and also realizable would be the extendability by using strong form of holography.

I have the feeling that this view allows respectable mathematical realization of imagination in terms of adelic quantum physics. It is remarkable that strong form of holography derivable from - you can guess, strong form of General Coordinate Invariance (the Big E again!), plays an absolutely central role in it.

5.2.7 P-Adic Numbers

Basic properties of p-adic numbers

p-Adic numbers (p is prime: 2,3,5,...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A25] . p-Adic numbers are representable as power expansion of the prime number p of form:

$$x = \sum_{k\geq k_0} x(k)p^k ,\ x(k) = 0, ..., p-1 \ . \tag{5.2.2}$$

The norm of a p-adic number is given by

$$|x| = p^{-k_0(x)} \ . \tag{5.2.3}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x = p^{k_0}\varepsilon(x) \ , \tag{5.2.4}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x,z) \leq max\{d(x,y), d(y,z)\} \ . \tag{5.2.5}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x,y) \leq D \ . \tag{5.2.6}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.

2. Distances of points x and y inside single class are smaller than distances between different classes.

3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B20] . The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Extensions of p-adic numbers

Algebraic democracy suggests that all possible real algebraic extensions of the p-adic numbers are possible. This conclusion is also suggested by various physical requirements, say the fact that the eigenvalues of a Hamiltonian representable as a rational or p-adic $N \times N$-matrix, being roots of N:th order polynomial equation, in general belong to an algebraic extension of rationals or p-adics. The dimension of the algebraic extension cannot be interpreted as physical dimension. Algebraic extensions are characteristic for cognitive physics and provide a new manner to code information. A possible interpretation for the algebraic dimension is as a dimension for a cognitive representation of space and might explain how it is possible to mathematically imagine spaces with all possible dimensions although physical space-time dimension is four. The idea of algebraic hologram and other ideas related to the physical interpretation of the algebraic extensions of p-adic numbers are discussed in [K95] .

It seems however that algebraic democracy must be extended to include also transcendentals in the sense that finite-dimensional extensions involving also transcendental numbers are possible: for instance, Neper number e defines a p-dimensional extension. It has become clear that these extensions fundamental for understanding how p-adic physics as physics of cognition is able to mimic real physics. The evolution of mathematical cognition can be seen as a process in which p-adic space-time sheets involving increasing value of p-adic prime p and increasing dimension of algebraic extension appear in quantum jumps.

1. Recipe for constructing algebraic extensions

Real numbers allow only complex numbers as an algebraic extension. For p-adic numbers algebraic extensions of arbitrary dimension are possible
[A25] . The simplest manner to construct (n+1)-dimensional extensions is to consider irreducible polynomials $P_n(t)$ in R_p assumed to have rational coefficients: irreducibility means that the polynomial does not possess roots in R_p so that one cannot decompose it into a product of lower order R_p valued polynomials. This condition is equivalent with the condition with irreducibility in the finite field $G(p,1)$, that is modulo p in R_p.

Denoting one of the roots of $P_n(t)$ by θ and defining $\theta^0 = 1$ the general form of the extension is given by

$$Z = \sum_{k=0,..,n-1} x_k\theta^k \ . \tag{5.2.7}$$

Since θ is root of the polynomial in R_p it follows that θ^n is expressible as a sum of lower powers of θ so that these numbers indeed form an n-dimensional linear space with respect to the p-adic topology.

Especially simple odd-dimensional extensions are cyclic extensions obtained by considering the roots of the polynomial

$$\begin{aligned} P_n(t) &= t^n + \epsilon d \ , \\ \epsilon &= \pm 1 \ . \end{aligned} \tag{5.2.8}$$

For $n = 2m+1$ and $(n = 2m, \epsilon = +1)$ the irreducibility of $P_n(t)$ is guaranteed if d does not possess n:th root in R_p. For $(n = 2m, \epsilon = -1)$ one must assume that $d^{1/2}$ does not exist p-adically. In this case θ is one of the roots of the equation

$$t^n = \pm d \ , \tag{5.2.9}$$

where d is a p-adic integer with a finite number of pinary digits. It is possible although not necessary to identify the roots as complex numbers. There exists n complex roots of d and θ can be chosen to be one of the real or complex roots satisfying the condition $\theta^n = \pm d$. The roots can be written in the general form

$$\begin{aligned} \theta &= d^{1/n} exp(i\phi(m)), \ m = 0, 1, ..., n-1 \ , \\ \phi(m) &= \frac{m2\pi}{n} \text{ or } \frac{m\pi}{n} \ . \end{aligned} \tag{5.2.10}$$

Here $d^{1/n}$ denotes the real root of the equation $\theta^n = d$. Each of the phase factors $\phi(m)$ gives rise to an algebraically equivalent extension: only the representation is different. Physically these extensions need not be equivalent since the identification of the algebraically extended p-adic numbers with the complex numbers plays a fundamental role in the applications. The cases $\theta^n = \pm d$ are physically and mathematically quite different.

2. p-Adic valued norm for numbers in algebraic extension

The p-adic valued norm of an algebraically extended p-adic number x can be defined as some power of the ordinary p-adic norm of the determinant of the linear map $x :^e R_p^n \rightarrow^e R_p^n$ defined by the multiplication with x: $y \rightarrow xy$

$$N(x) = det(x)^{\alpha} \ , \ \alpha > 0 \ . \tag{5.2.11}$$

Real valued norm can be defined as the p-adic norm of $N(x)$. The requirement that the norm is homogenous function of degree one in the components of the algebraically extended 2-adic number (like also the standard norm of R^n) implies the condition $\alpha = 1/n$, where n is the dimension of the algebraic extension.

The canonical correspondence between the points of R_+ and R_p generalizes in obvious manner: the point $\sum_k x_k \theta^k$ of algebraic extension is identified as the point $(x_R^0, x_R^1, ..., x_R^k, ..,)$ of R^n using the pinary expansions of the components of p-adic number. The p-adic linear structure of the algebraic extension induces a linear structure in R_+^n and p-adic multiplication induces a multiplication for the vectors of R_+^n.

3. Algebraic extension allowing square root of ordinary p-adic numbers

The existence of a square root of an ordinary p-adic number is a common theme in various applications of the p-adic numbers and for long time I erratically believed that only this extension is involved with p-adic physics. Despite this square root allowing extension is of central importance and deserves a more detailed discussion.

1. The p-adic generalization of the representation theory of the ordinary groups and Super Kac Moody and Super Virasoro algebras exists provided an extension of the p-adic numbers allowing square roots of the "real" p-adic numbers is used. The reason is that the matrix elements of the raising and lowering operators in Lie-algebras as well as of oscillator operators typically involve square roots. The existence of square root might play a key role in various p-adic considerations.

2. The existence of a square root of a real p-adic number is also a necessary ingredient in the definition of the p-adic unitarity and probability concepts since the solution of the requirement that $p_{mn} = S_{mn}\bar{S}_{mn}$ is ordinary p-adic number leads to expressions involving square roots.

3. p-Adic length scales hypothesis states that the p-adic length scale is proportional to the square root of p-adic prime.

4. Simple metric geometry of polygons involves square roots basically via the theorem of Pythagoras. p-Adic Riemannian geometry necessitates the existence of square root since the definition of the infinitesimal length ds involves square root. Note however that p-adic Riemannian geometry can be formulated as a mere differential geometry without any reference to global concepts like lengths, areas, or volumes.

The original belief that square root allowing extensions of p-adic numbers are exceptional seems to be wrong in light of TGD as a generalized number theory vision. All algebraic extensions of p-adic numbers a possible and the interpretation of algebraic dimension of the extension as a physical dimension is not the correct thing to do. Rather, the possibility of arbitrarily high algebraic dimension reflects the ability of mathematical cognition to imagine higher-dimensional spaces. Square root allowing extension of the p-adic numbers is the simplest one imaginable, and it is fascinating that it indeed is the dimension of space-time surface for $p > 2$ and dimension of embedding space for $p = 2$. Thus the square root allowing extensions deserve to be discussed.

The results can be summarized as follows.

1. In $p > 2$ case the general form of extension is

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ , \tag{5.2.12}$$

where the condition $\theta^2 = x$ for some p-adic number x not allowing square root as a p-adic number. For $p \ mod \ 4 = 3$ θ can be taken to be imaginary unit. This extension is natural for p-adication of space-time surface so that space-time can be regarded as a number field locally. Embedding space can be regarded as a cartesian product of two 4-dimensional extensions locally.

2. In $p = 2$ case 8-dimensional extension is needed to define square roots. The extension is defined by adding $\theta_1 = \sqrt{-1} \equiv i$, $\theta_2 = \sqrt{2}$, $\theta_3 = \sqrt{3}$ and the products of these so that the extension can be written in the form

$$Z = x_0 + \sum_k x_k \theta_k + \sum_{k<l} x_{kl}\theta_{kl} + x_{123}\theta_1\theta_2\theta_3 \ . \tag{5.2.13}$$

Clearly, $p = 2$ case is exceptional as far as the construction of the conformal field theory limit is considered since the structure of the representations of Virasoro algebra and groups in general changes drastically in $p = 2$ case. The result suggest that in $p = 2$ limit space-time surface and H are in same relation as real numbers and complex numbers: space-time surfaces defined as the absolute minima of 2-adiced Kähler action are perhaps identifiable as surfaces for which the imaginary part of 2-adically analytic function in H vanishes.

The physically interesting feature of p-adic group representations is that if one doesn't use $\sqrt{p}$ in the extension the number of allowed spins for representations of $SU(2)$ is finite: only spins $j < p$ are allowed. In $p = 3$ case just the spins $j \leq 2$ are possible. If 4-dimensional extension is used for $p = 2$ rather than 8-dimensional then one gets the same restriction for allowed spins.

4. Is e an exceptional transcendental?

One can consider also the possibility of transcental extensions of p-adic numbers and an open problem is whether the infinite-dimensional extensions involving powers of π and logarithms of primes make sense and whether they should be allowed. For instance, it is not clear whether the allowance of powers of π is consistent with the extensions based on roots of unity. This question is not academic since Feynman amplitudes in real context involve powers of π and algebraic universality forces the consider that also they p-adic variants might involve powers of π.

Neper number obviously defines the simplest transcendental extension since only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$ so that the extension is finite-dimensional. In the case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

1. Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

2. The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

3. The differential equation allows to develop $f(x)$ in power series, say in origin

$$f(x) = \sum f_n \frac{x^n}{n!}$$

   such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

   The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

p-Adic Numbers and finite fields

Finite fields (Galois fields) consists of finite number of elements and allow sum, multiplication and division. A convenient representation for the elements of a finite field is as the roots of the polynomial equation $t^{p^m} - t = 0 \ mod \ p$, where p is prime, m an arbitrary integer and t is element of a field of characteristic p ($pt = 0$ for each t). The number of elements in a finite field is p^m, that is power of prime number and the multiplicative group of a finite field is group of order $p^m - 1$. $G(p, 1)$ is just cyclic group Z_p with respect to addition and $G(p, m)$ is in rough sense m:th Cartesian power of $G(p, 1)$.

The elements of the finite field $G(p, 1)$ can be identified as the p-adic numbers $0, ..., p-1$ with p-adic arithmetics replaced with modulo p arithmetics. The finite fields $G(p, m)$ can be obtained from m-dimensional algebraic extensions of the p-adic numbers by replacing the p-adic arithmetics with the modulo p arithmetics. In TGD context only the finite fields $G(p > 2, 2)$, $p\ mod\ 4 = 3$ and $G(p = 2, 4)$ appear naturally. For $p > 2$, $p\ mod\ 4 = 3$ one has: $x+iy+\sqrt{p}(u+iv) \rightarrow x_0+iy_0 \in G(p, 2)$.

An interesting observation is that the unitary representations of the p-adic scalings $x \rightarrow p^k x$ $k \in Z$ lead naturally to finite field structures. These representations reduce to representations of a finite cyclic group Z_m if $x \rightarrow p^m x$ acts trivially on representation functions for some value of m, $m = 1, 2, ...$ Representation functions, or equivalently the scaling momenta $k = 0, 1, ..., m - 1$ labelling them, have a structure of cyclic group. If $m \neq p$ is prime the scaling momenta form finite field $G(m, 1) = Z_m$ with respect to the summation and multiplication modulo m. Also the p-adic counterparts of the ordinary plane waves carrying p-adic momenta $k = 0, 1..., p - 1$ can be given the structure of Finite Field $G(p, 1)$: one can also define complexified plane waves as square roots of the real p-adic plane waves to obtain Finite Field $G(p, 2)$.

5.3 What Is The Correspondence Between P-Adic And Real Numbers?

There must be some kind of correspondence between reals and p-adic numbers. This correspondence can depend on context. In p-adic mass calculations one must map p-adic mass squared values to real numbers in a continuous manner and canonical identification is a natural guess. Presumably also p-adic probabilities should be mapped to their real counterparts. One can wonder whether p-adic valued S-matrix has any physical meaning or whether one should assume that the elements of S-matrix are algebraic numbers allowing interpretation as real or p-adic numbers: this would pose extremely strong constraints on S-matrix. If one wants to introduce p-adic physics at space-time level one must be able to relate p-adic and real space-time regions to each other and the identification along common rational points of real and various p-adic variants of the embedding space suggests itself here.

5.3.1 Generalization Of The Number Concept

The recent view about the unification of real and p-adic physics is based on the generalization of number concept obtained by fusing together real and p-adic number fields along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book).

Rational numbers as numbers common to all number fields

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers appearing in the extension of p-adic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional algebraic and perhaps even transcendental extensions of p-adic numbers adds additional pages to this "Big Book".

This leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common points. The outcome of experimentation is that this generalization makes sense under very high symmetries and that it is safest to lean strongly on the physical picture provided by quantum TGD.

1. The most natural guess is that the coordinates of common points are rational or in some algebraic extension of rational numbers. General coordinate invariance and preservation of symmetries require preferred coordinates existing when the manifold has maximal number of isometries. This approach is especially natural in the case of linear spaces- in particular Minkowski space M^4. The natural coordinates are in this case linear Minkowski coordinates. The choice of coordinates is not completely unique and has interpretation as a geometric correlate for the choice of quantization axes for a given CD.

2. As will be found, the need to have a well-defined integration based on Fourier analysis (or its generalization to harmonic analysis in symmetric spaces) poses very strong constraints and allows p-adicization only if the space has maximal symmetries. Fourier analysis requires the

introduction of an algebraic extension of p-adic numbers containing sufficiently many roots of unity.

(a) This approach is especially natural in the case of compact symmetric spaces such as CP_2.

(b) Also symmetric spaces such the 3-D proper time $a = constant$ hyperboloid of M^4- call it $H(a)$ -allowing Lorentz group as isometries allows a p-adic variant utilizing the hyperbolic counterparts for the roots of unity. $M^4 \times H(a = 2^n a_0)$ appears as a part of the moduli space of CDs.

(c) For light-cone boundaries associated with CDs $SO(3)$ invariant radial coordinate r_M defining the radius of sphere S^2 defines the hyperbolic coordinate and angle coordinates of S^2 would correspond to phase angles and $M^4_\pm$ projections for the common points of real and p-adic variants of partonic 2-surfaces would be this kind of points. Same applies to CP_2 projections. In the "intersection of real and p-adic worlds" real and p-adic partonic 2-surfaces would obey same algebraic equations and would be obtained by an algebraic continuation from the corresponding equations making sense in the discrete variant of $M^4_\pm \times CP_2$. This connection with discrete sub-manifolds geometries means very powerful constraints on the partonic 2-surfaces in the intersection.

3. The common algebraic points of real and p-adic variant of the manifold form a discrete space but one could identify the p-adic counterpart of the real discretization intervals $(0, 2\pi/N)$ for angle like variables as p-adic numbers of norm smaller than 1 using canonical identification or some variant of it. Same applies to the the hyperbolic counterpart of this interval. The non-uniqueness of this map could be interpreted in terms of a finite measurement resolution. In particular, the condition that WCW allows Kähler geometry requires a decomposition to a union of symmetric spaces so that there are good hopes that p-adic counterpart is analogous to that assigned to CP_2.

The idea about astrophysical size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves astrophysical length scales.

Generalizing complex analysis by replacing complex numbers by generalized numbers

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions for which polynomials have rational coefficients are obviously functions satisfying this constraint. Algebraic functions for which polynomials have rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed.

For instance, one can ask whether residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the back of the book like structure (in very metaphorical sense) having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "Big Book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense. Contrary to the first expectations the algebraically continued residue calculus does not seem to have obvious applications in quantum TGD.

5.3.2 Canonical Identification

Canonical There exists a natural continuous map $Id : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$

this correspondence reads

$$\begin{aligned} y &= \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \ , \\ y_k &\in \{0,1,..,p-1\} \ . \end{aligned} \tag{5.3.1}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique ($1 = 0.999...$) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{5.3.2}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^k \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^k + (x_N - 1)p^N + (p-1)p^{N+1} \sum_{k=0,..} p^k \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{5.3.3}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite number of pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite number of pinary digits. The finite number of pinary digits expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

Canonical identification is a continuous map of non-negative reals to p-adics

The topology induced by the inverse of the canonical identification map in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. This allows two alternative interpretations. Either p-adic image of a physical systems provides a good representation of the system above some pinary cutoff or the physical system can be genuinely p-adic below certain length scale L_p and become in good approximation real, when a length scale resolution L_p is used in its description. The first interpretation is correct if canonical identification is interpreted as a cognitive map. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right, see **Fig. A-6.1**). This feature is one clear signature of the p-adic topology.

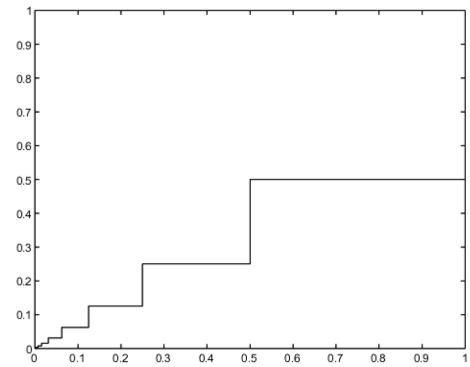

Figure 5.1: The real norm induced by canonical identification from 2-adic norm.

If one considers seriously the application of canonical identification to basic quantum TGD one cannot avoid the question about the p-adic counterparts of the negative real numbers. There is no satisfactory manner to circumvent the fact that canonical images of p-adic numbers are naturally non-negative. This is not a problem if canonical identification applies only to the coordinate interval $(0, 2\pi/N)$ or its hyperbolic variant defining the finite measurement resolution. That p-adicization program works only for highly symmetric spaces is not a problem from the point of view of TGD.

The interpretation of canonical identification in terms of finite measurement resolution

The question what the canonical identification really means could be a key to the understanding of the special aspects of this map. The notion of finite measurement resolution is a good candidate for the needed principle.

1. Finite measurement resolution can be characterized as an interval of minimum length. Below this length scale one cannot distinguish points from each other. A natural definition for this inability could be as an inability to well-order the points. The real topology is too strong in the modelling in kind of situation since it brings in large amount of processing of pseudo information whereas p-adic topology which lacks the notion of well-ordering could be more appropriate as effective topology and together with pinary cutoff could allow to get rid of the irrelevant information.

2. This suggest that canonical identification applies only inside the intervals defining finite measurement resolution in a given discretization of the space considered by say small cubes. The canonical identification is unique only modulo diffeomorphism applied on both real and p-adic side but this is not a problem since this would only reflect the absence of the well-ordering lost by finite measurement resolution. Also the fact that the map makes sense only at positive real axis would be natural if one accepts this identification.

The notion of p-adic linearity

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

Does canonical identification define a generalized norm?

Canonical correspondence is quite essential in TGD applications. Canonical identification makes it possible to define a p-adic valued definite integral. Canonical identification is in a key role in the successful predictions of the elementary particle masses. Canonical identification makes also possible to understand the connection between p-adic and real probabilities. These and many other successful applications suggests that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{5.3.4}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{5.3.5}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{5.3.6}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some nonlinear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

5.3.3 The Interpretation Of Canonical Identification

During the development of p-adic TGD two seemingly mutually inconsistent competing identifications of reals and p-adics have caused a lot of painful tension. Canonical identification provides one possible identification map respecting continuity whereas the identification of rationals as points common to p-adics and reals respects algebra of rationals. The resolution of the tension came from the realization that canonical identification naturally maps the predictions of p-adic probability theory and thermodynamics to real numbers. Canonical identification also maps p-adic cognitive representations to symbolic ones in the real world world or vice versa. The identification by common rationals is in turn the correspondence implied by the generalized notion of number and natural in the construction of quantum TGD proper.

Canonical identification maps the predictions of the p-adic probability calculus and statistical physics to real numbers

p-Adic mass calculations based on p-adic thermodynamics were the first and rather successful application of the p-adic physics (see the four chapters in [K62]. The essential element of the approach was the replacement of the Boltzmann weight $e^{-E/T}$ with its p-adic generalization p^{L_0/T_p}, where L_0 is the Virasoro generator corresponding to scaling and representing essentially mass squared operator instead of energy. T_p is inverse integer valued p-adic temperature. The predicted mass squared averages were mapped to real numbers by canonical identification.

One could also construct a real variant of this approach by considering instead of the ordinary Boltzmann weights the weights p^{-L_0/T_p}. The quantization of temperature to $T_p = log(p)/n$ would be a completely ad hoc assumption. In the case of real thermodynamics all particles are predicted to be light whereas in case of p-adic thermodynamics particle is light only if the ratio for the degeneracy of the lowest massive state to the degeneracy of the ground state is integer. Immense number of particles disappear from the spectrum of light particles by this criterion. For light particles the predictions are same as of p-adic thermodynamics in the lowest non-trivial order but in the next order deviations are possible.

Also p-adic probabilities and the p-adic entropy can be mapped to real numbers by canonical identification. The general idea is that a faithful enough cognitive representation of the real physics can by the number theoretical constraints involved make predictions, which would be extremely difficult to deduce from real physics.

The variant of canonical identification commuting with division of integers

The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity.

A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p\rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries. It seems that this option must be used to relate p-adic transition amplitudes to real ones and vice versa [K57]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

Generalized numbers would be regarded in this picture as a generalized manifold obtained by gluing different number fields together along rationals. Instead of a direct identification of real and p-adic rationals, the p-adic rationals in R_p are mapped to real rationals (or vice versa) using a variant of the canonical identification $I_{R\rightarrow R_p}$ in which the expansion of rational number $q = r/s = \sum r_n p^n / \sum s_n p^n$ is replaced with the rational number $q_1 = r_1/s_1 = \sum r_n p^{-n}/\sum s_n p^{-n}$ interpreted as a p-adic number:

$$q = \frac{r}{s} = \frac{\sum_n r_n p^n}{\sum_m s_n p^n} \rightarrow q_1 = \frac{\sum_n r_n p^{-n}}{\sum_m s_n p^{-n}} \quad . \qquad (5.3.7)$$

R_{p1} and R_{p_2} are glued together along common rationals by an the composite map $I_{R\rightarrow R_{p_2}} I_{R_{p_1}\rightarrow R}$.

This variant of canonical identification seems to be excellent candidate for mapping the predictions of p-adic mass calculations to real numbers and also for relating p-adic and real scattering amplitudes to each other [K57].

p-Adic fractality, canonical identification, and symmetries

The original motivation for the canonical identification and its variants- in particular the variant mapping real rationals with the defining integers below a pinary cutoff to p-adic rationals- was that it defines a continuous map from p-adics to reals and produces beautiful p-adic fractals as a map from reals to p-adics by canonical identification followed by a p-adically smooth map in turn followed by the inverse of the canonical identification.

The first drawback was that the map does not commute with symmetries. Second drawback was that the standard canonical identification from reals to p-adics with finite pinary cutoff is two-valued for finite integers. The canonical real images of these transcendentals are also transcendentals. These are however countable whereas p-adic algebraics and transcendentals having by definition a non-periodic pinary expansion are uncountable. Therefore the map from reals to p-adics is single valued for almost all p-adic numbers.

On the other hand, p-adic rationals form a dense set of p-adic numbers and define “almost all” for the purposes of numerics! Which argument is heavier? The direct identification of reals and p-adics via common rationals commutes with symmetries in an approximation defined by the pinary cutoff an is used in the canonical identification with pinary cutoff mapping rationals to rationals.

Symmetries are of extreme importance in physics. Is it possible to imagine the action of say Poincare transformations commuting with the canonical identification in the sets of p-adic and real transcendentals? This might be the case.

1. Wick rotation (see `http://tinyurl.com/qh8jvoj`) is routinely used in quantum field theory to define Minkowskian momentum integrals. One Wick rotates Minkowski space to Euclidian space, performs the integrals, and returns to Minkowskian regime by using the inverse of Wick rotation. The generalization to the p-adic context is highly suggestive. One could map the real Minkowski space to its p-adic counterpart, perform Poincare transformation there, and return back to the real Minkowski space using the inverse of the rational canonical identification.

2. For p-adic transcendentals one would a formal automorph of Poincare group as IPI^{-1} and these Poincare group would be the fractal counterpart of the ordinary Poincare group. Mathematician would regard I as the analog of intertwining operator, which is linear map between Hilbert spaces. This variant of Poincare symmetry would be exact in the transcendental realm since canonical identification is continuous. For rationals this symmetry would fail.

3. For rationals which are constructed as ratios of small enough integers, the rational Poincare symmetry with group elements involving rationals constructed from small enough integers would be an exact symmetry. For both options the use of preferred coordinates, most naturally linear Minkowski coordinates would be essential since canonical identification does not commute with general coordinate transformations.

4. Which of these Poincare symmetries corresponds to the physical Poincare symmetry? The above argument does not make it easy to answer the question. One can however circumvent it. Maybe one could distinguish between rational and transcendental regime in the sense that Poincare group and other symmetries would be realized in different manner in these regimes?

Note that the analog of Wick rotation could be used also to define p-adic integrals by mapping the p-adic integration region to real one by some variant of canonical identification continuously, performing the integral in the real context, and mapping the outcome of the integral to p-adic number by canonical identification. Again preferred coordinates are essential and in TGD framework such coordinates are provided by symmetries. This would allow a numerical treatment of the p-adic integral but the map of the resulting rational to p-adic number would be two valued. The difference between the images would be determined by the numerical accuracy when p-adic expansions are used. This method would be a numerical analog of the analytic definition of p-adic integrals by analytic continuation from the intersection of real and p-adic worlds defined by rational values of parameters appearing in the expressions of integrals.

5.4 P-Adic Differential And Integral Calculus

p-Adic differential calculus differs from its real counterpart in that piecewise constant functions depending on a finite number of pinary digits have vanishing derivative. This property implies p-adic nondeterminism, which has natural interpretation as making possible imagination if one identifies p-adic regions of space-time as cognitive regions of space-time.

One of the stumbling blocks in the attempts to construct p-adic physics have been the difficulties involved with the definition of the p-adic version of a definite integral. There are several alternative options as how to define p-adic definite integral and it is quite possible that there is simply not a single correct version since p-adic physics itself is a cognitive model.

1. The first definition of the p-adic integration is based on three ideas. The ordering for the limits of integration is defined using canonical correspondence. $x < y$ holds true if $x_R < y_R$ holds true. The integral functions can be defined for Taylor series expansion by defining indefinite integral as the inverse of the differentiation. If p-adic pseudo constants are present in the integrand one must divide the integration range into pieces such that p-adic integration constant changes its value in the points where new piece begins.

2. Second definition is based on p-adic Fourier analysis based on the use of p-adic plane waves constructed in terms of Pythagorean phases. This definition is especially attractive in the definition of p-adic QFT limit and is discussed in detail later in the section "p-Adic Fourier analysis". In this case the integral is defined in the set of rationals and the ordering of the limits of integral is therefore not a problem.

3. For p-adic functions which are direct canonical images of real functions, p-adic integral can be defined also as a limit of Riemann sum and this in principle makes the numerical evaluation of p-adic integrals possible. As found in the chapter "Mathematical Ideas", Riemann sum representation leads to an educated guess for an *exact formula for the definite integral* holding true for functions which are p-adic counterparts of real-continuous functions and for p-adically analytic functions. The formula provides a calculational recipe of p-adic integrals, which converges extremely rapidly in powers of p. Ultrametricity guarantees the absence of divergences in arbitrary dimensions provided that integrand is a bounded function. It however seems that this definition of integral cannot hold true for the p-adically differentiable function whose real images are not continuous.

5.4.1 P-Adic Differential Calculus

The rules of the p-adic differential calculus are formally identical to those of the ordinary differential calculus and generalize in a trivial manner for the algebraic extensions.

The class of the functions having vanishing p-adic derivatives is larger than in the real case: any function depending on a finite number of positive pinary digits of p-adic number and of arbitrary number of negative pinary digits has a vanishing p-adic derivative. This becomes obvious, when one notices that the p-adic derivative must be calculated by comparing the values of the function at nearby points having the same p-adic norm (here is the crucial difference with respect to real case!). Hence, when the increment of the p-adic coordinate becomes sufficiently small, p-adic constant doesn't detect the variation of x since it depends on finite number of positive p-adic pinary digits only. p-Adic constants correspond to real functions, which are constant below some length scale $\Delta x = 2^{-n}$. As a consequence p-adic differential equations are non-deterministic: integration constants are arbitrary functions depending on a finite number of the positive p-adic pinary digits. This feature is central as far applications are considered and leads to the interpretation of p-adic physics as physics of cognition which involves imagination in essential manner. The classical non-determinism of the Kähler action, which is the key feature of quantum TGD, corresponds in a natural manner to the non-determinism of volition in macroscopic length scales.

p-analytic maps $g : R_p \rightarrow R_p$ satisfy the usual criterion of differentiability and are representable as power series

$$g(x) = \sum_k g_k x^k \ . \qquad (5.4.1)$$

Also negative powers are in principle allowed.

5.4.2 P-Adic Fractals

p-Adically analytic functions induce maps $R_+ \rightarrow R_+$ via the canonical identification map. The simplest manner to get some grasp on their properties is to plot graphs of some simple functions (see **Fig. 5.2** for the graph of p-adic x^2 and **Fig. 5.3** for the graph of p-adic $1/x$). These functions have quite characteristic features resulting from the special properties of the p-adic topology. These features should be universal characteristics of cognitive representations and should allow to deduce the value of the p-adic prime p associated with a given cognitive system.

1. p-Analytic functions are continuous and differentiable from right: this peculiar asymmetry is a completely general signature of the p-adicity. As far as time dependence is considered, the interpretation of this property as a mathematical counterpart of the irreversibility looks attractive. This suggests that the transition from the reversible microscopic dynamics to irreversible macroscopic dynamics could correspond to the transition from the ordinary topology to an effective p-adic topology.

2. There are large discontinuities associated with the points $x = p^n$. This implies characteristic threshold phenomena. Consider a system whose output $f(n)$ is a function of input, which is integer n. For $n < p$ nothing peculiar happens but for $n = p$ the real counterpart of the output becomes very small for large values of p. In the bio-systems threshold phenomena are typical and p-adicity might be the key to their understanding. The discontinuities associated with the powers of $p = 2$ are indeed encountered in many physical situations. Auditory experience has the property that a given frequency ω_0 and its multiples $2^k\omega_0$, octaves, are experienced as the same frequency, this suggests that the auditory response function for a given frequency ω_0 is a 2-adicallly analytic function. Titius-Bode law states that the mutual distances of planets come in powers of 2, when suitable unit of distance is used. In turbulent systems period doubling spectrum has peaks at frequencies $\omega = 2^k\omega_0$.

3. A second signature of the p-adicity is "p-plicity" appearing in the graph of simple p-analytic functions. As an example, consider the graph of the p-adic x^2 demonstrating clearly the decomposition into p steps at each interval $[p^k, p^{k+1})$.

4. The graphs of the p-analytic functions are in general ordered fractals as the examples demonstrate. For example, power functions x^n are self-similar (the values of the function at some any interval (p^k, p^{k+1}) determines the function completely) and in general p-adic x^n with non-negative (negative) n is smaller (larger) than real x^n expect at points $x = p^n$ as the graphs of p-adic x^2 and $1/x$ show (see **Fig. 5.2** and **5.3**) These properties are easily understood from the properties of the p-adic multiplication. Therefore the first guess for the behavior of a p-adically analytic function is obtained by replacing x and the coefficients g_k with their p-adic norms: at points $x = p^n$ this approximation is exact if the coefficients of the power series are powers of p. This step function approximation is rather reasonable for simple functions such as x^n as the figures demonstrate. Since p-adically analytic function can be approximated with $f(x) \sim f(x_0) + b(x - x_0)^n$ or as $a(x - x_0)^n$ (allowing non-analyticity at x_0) around any point the fractal associated with p-adically analytic function has universal geometrical form in sufficiently small length scales.

p-Adic analyticity is well defined for the algebraic extensions of R_p, too. The figures **??** and **??** visualize the behavior of the real and imaginary parts of the 2-adic z^2 function as a function of the real x and y coordinates in the parallelpiped I^2, $I = [1 + 2^{-7}, 2 - 2^{-7}]$. An interesting possibility is that the order parameters describing various phases of some physical systems are p-adically differentiable functions. The p-analyticity would therefore provide a means for coding the information about ordered fractal structures.

The order parameter could be one coordinate component of a p-adically analytic map $R^n \rightarrow R^n$, $n = 3, 4$. This is analogous to the possibility to regard the solution of the Laplace equation in two dimensions as a real or imaginary part of an analytic function. A given region V of the order parameter space corresponds to a given phase and the volume of the ordinary space occupied by this phase corresponds to the inverse image $g^{-1}(V)$ of V. Very beautiful images are obtained if the order parameter is the real or imaginary part of a p-analytic function $f(z)$. A good example is p-adic z^2 function in the parallelpiped $[a, b] \times [a, b]$, $a = 1 + 2^{-9}$, $b = 2 - 2^9$ of C-plane. The value range of the order parameter can be divided into, say, 16 intervals of the same length so that each interval corresponds to a unique color. The resulting fractals possess features, which probably generalize to higher-dimensional extensions.

1. The inverse image is an ordered fractal and possesses lattice/cell like structure, with the sizes of cells appearing in powers of p. Cells are however not identical in analogy with the differentiation of the biological cells.

2. p-Analyticity implies the existence of a local vector valued order parameter given by the p-analytic derivative of $g(z)$: the geometric structure of the phase portrait indeed exhibits the local orientation clearly.

A second representation of the fractals is obtained by dividing the value range of z into a finite number of intervals and associating different color to each interval. In a given resolution this representation makes obvious the presence of 0, 1- and 2-dimensional structures not obvious from the graph representation used in the figures of this book.

These observations suggests that p-analyticity might provide a means to code the information about ordered fractal structures in the spatial behavior of order parameters (such as enzyme concentrations in bio-systems). An elegant manner to achieve this is to use purely real algebraic extension for 3-space coordinates and for the order parameter: the image of the order parameter $\Phi = \phi_1 + \phi_2\theta + \phi_3\theta^2$ under the canonical identification is real and positive number automatically and might be regarded as concentration type quantity.

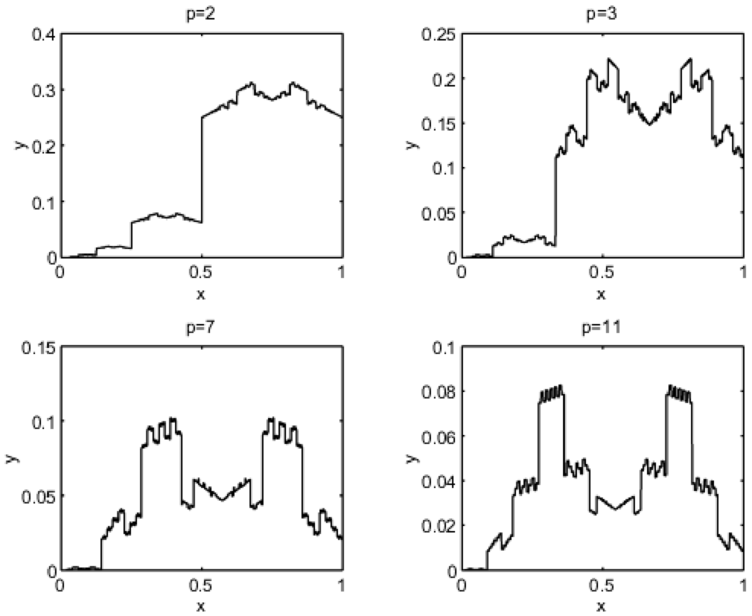

Figure 5.2: p-Adic x^2 function for some values of p

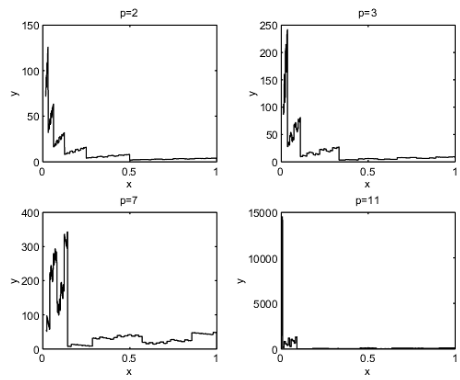

Figure 5.3: p-Adic $1/x$ function for some values of p

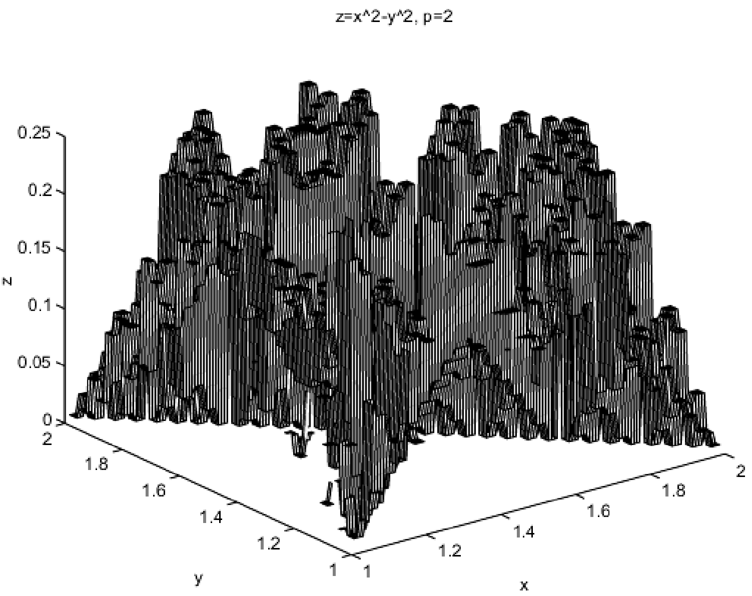

Figure 5.4: The graph of the real part of 2-adically analytic $z^2 =$ function.

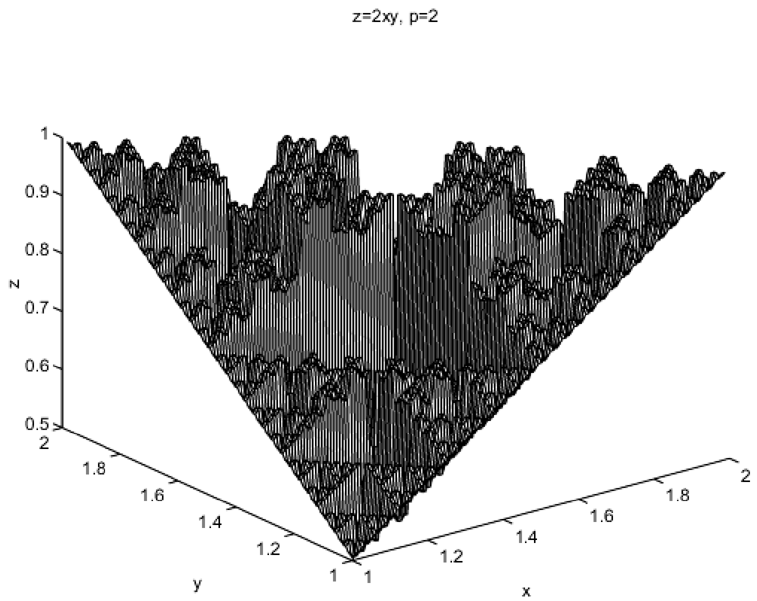

Figure 5.5: The graph of 2-adically analytic $Im(z^2) = 2xy$ function.

5.4.3 P-Adic Integral Calculus

The basic problems of the integration with p-adic values of integral are caused by the facts that p-adic numbers are not well-ordered and by the properties of p-adic norm. The general idea that p-adic physics can mimic real physics only at the algebraic level, leads to the idea that p-adic integration could be algebraized whereas numerical approaches analogous to Riemann sum are not possible. In the following three examples are discussed.

1. Definite integral can be defined using integral function and by defining integration limits via canonical identification: the drawback is the loss of general coordinate invariance. A more elegant general coordinate invariant approach is based on the identification of rationals as common to both reals and p-adics. This works for rational valued integration limits.

2. residue calculus allows to realize integrals of analytic functions over closed curves of complex plane. The generalization of the residue calculus makes possible to realize conformal invariance at elementary particle horizons which are metrically 2-dimensional and allow conformal invariance and has also p-adic counterpart.

3. The perturbative series using Gaussian integration is the only to perform in practice infinite-dimensional functional integrals and being purely algebraic procedure, allows a straightforward p-adic generalization. This is the only option for p-adicizing configuration space integral.

Definition of the definite integral using integral function concept and canonical identification or identification by common rationals

The concept of the p-adic definite integral can be defined for functions $R_p \rightarrow C$ [A26] using translationally invariant Haar measure for R_p. In present context one is however interested in defining a p-adic valued definite integral for functions $f: R_p \rightarrow R_p$: target and source spaces could of course be also some some algebraic extensions of the p-adic numbers.

What makes the definition nontrivial is that the ordinary definition as the limit of a Riemann sum doesn't seem to work: it seems that Riemann sum approaches to zero in the p-adic topology since, by ultra-metricity, the p-adic norm of a sum is never larger than the maximum p-adic norm for the summands. The second difficulty is related to the absence of a well-ordering for the p-adic numbers. The problems might be avoided by defining the integration essentially as the inverse of the differentiation and using the canonical correspondence to define ordering for the p-adic numbers. More generally, the concepts of the form, cohomology and homology are crucially based on the concept of the boundary. The concept of boundary reduces to the concept of an ordered interval and canonical identification makes it indeed possible to define this concept.

The definition of the p-adic integral functions defining integration as inverse of the differentiation is straightforward and one obtains just the generalization of the standard calculus. For instance, one has $\int z^n = \frac{z^{n+1}}{(n+1)} + C$ and integral of the Taylor series is obtained by generalizing this. One must however notice that the concept of integration constant generalizes: any function $R_p \rightarrow R_p$ depending on a finite number of pinary digits only, has a vanishing derivative.

Consider next the definite integral. The absence of the well ordering implies that the concept of the integration range (a, b) is not well defined as a purely p-adic concept. As already mentioned there are two solutions of the problem.

1. The identification of rational numbers as common to both reals and p-adics allows to order the integration limits when the end points of the integral are rational numbers. This is perhaps the most elegant solution of the problem since it is consistent with the restricted general coordinate invariance allowing rational function based coordinate changes. This approach works for rational functions with rational coefficients and more general functions if algebraic extension or extension containing transcendentals like e and logarithms of primes are allowed. The extension containing e, π, and $log(p)$ is finite-dimensional if e/π and $\pi/log(p)$ are rational numbers for all primes p. Essentially algebraic continuation of real integral to p-adic context is in question.

2. An alternative resolution of the problem is based on the canonical identification. Consider p-adic numbers a and b. It is natural to define a to be smaller than b if the canonical images of a and b satisfy $a_R < b_R$. One must notice that $a_R = b_R$ does not imply $a = b$, since the inverse of the canonical identification map is two-valued for the real numbers having a finite number of pinary digits. For two p-adic numbers a, b with $a < b$, one can define the integration range (a, b) as the set of the p-adic numbers x satisfying $a \leq x \leq b$ or equivalently $a_R \leq x_R \leq b_R$. For a given value of x_R with a finite number of pinary digits, one has two values of x and x can be made unique by requiring it to have a finite number of pinary digits.

One can define definite integral $\int_a^b f(x)dx$ formally as

$$\int_a^b f(x)dx = F(b) - F(a) \ , \tag{5.4.2}$$

where $F(x)$ is integral function obtained by allowing only ordinary integration constants and $b_R > a_R$ holds true. One encounters however a problem, when $a_R = b_R$ and a and b are different. Problem is avoided if the integration limits are assumed to correspond to p-adic numbers with a finite number of pinary digits.

One could perhaps relate the possibility of the p-adic integration constants depending on finite number of pinary digits to the possibility to decompose integration range $[a_R, b_R]$ as $a = x_0 < x_1 < ....x_n = b$ and to select in each subrange $[x_k, x_{k+1}]$ the inverse images of $x_k \leq x \leq x_{k+1}$, with x having finite number of pinary digits in two different ways. These different choices correspond to different integration paths and the value of the integral for different paths could correspond to the different choices of the p-adic integration constant in integral function. The difference between a given integration path and "standard" path is simply the sum of differences $F(x_k) - F(y_k)$, $(x_k)_R = (y_k)_R$.

This definition has several nice features.

1. The definition generalizes in an obvious manner to the higher dimensional case. The standard connection between integral function and definite integral holds true and in the higher-dimensional case the integral of a total divergence reduces to integral over the boundaries of the integration volume. This property guarantees that p-adic action principle leads to same field equations as its real counterpart. It this in fact this property, which drops other alternatives from the consideration.

2. The basic results of the real integral calculus generalize as such to the p-adic case. For instance, integral is a linear operation and additive as a set function.

The ugly feature is the loss of the general coordinate invariance due to the fact that canonical identification does not commute with coordinate changes (except scalings by powers of p) and it seems that one cannot use canonical identification at the fundamental level to define definite integrals.

Definite integrals in p-adic complex plane using residue calculus

residue calculus allows to calculate the integrals $\oint_C f(z)dz$ around complex curves as sums over poles of the function inside the curve:

$$\oint f(z)dz = i2\pi \sum_k Res(f(z_k)) \ , \tag{5.4.3}$$

where $Res(f(z_k))$ at pole $z = z_k$ is defined as $Res(f(z_k)) = \lim_{z \to z_k}(z - z_k)f(z)$. This definition applies in case of 2-dimensional $\sqrt{-1}$-containing algebraic extension of p-adic numbers ($p \ mod \ 4 = 3$) but its seems that this is not relevant for quantum TGD.

Quaternion conformal invariance corresponds to the conformal invariance associated with topologically 3-dimensional elementary particle horizons surrounding wormhole contacts which have Euclidian signature of induced metric. The induced metric is degenerate at the elementary particle horizon so that these surfaces are metrically two-dimensional. This implies a generalization of conformal invariance analogous to that at light cone cone boundary. In particular, a subfield of quaternions isomorphic with complex numbers is selected. One expects that residue calculus generalizes.

Elementary particle horizons are defined by a purely algebraic condition stating that the determinant of the induced metric vanishes, and thus the notion makes sense for p-adic space-time sheets too. Also residue calculus should make sense for all algebraic extensions of p-adic numbers and the algebra of quaternion conformal invariance would generalize to the p-adic context too. Note however that the notion of p-adic quaternions does not make sense: the reason is that p-adic Euclidian length squared for a non-vanishing p-adic quaternion can vanish so that the inverse of quaternion is not well defined always. In the set of rational numbers this failure does not however occur and this might be enough for p-adicization to work.

Definite integrals using Gaussian perturbation theory

In quantum field theories functional integrals are defined by Gaussian perturbation theory. For real infinite-dimensional Gaussians the procedure has a rigorous mathematical basis deriving from measure theory. For the imaginary infinite-dimensional Gaussians defining the Feynman path integrals of quantum field theory the rigorous mathematical justification is lacking.

In TGD framework the integral over WCW of three surface can be reduced to a real Gaussian perturbation theory around the maxima of Kähler function. The integration is over quantum fluctuating degrees of freedom defining infinite-dimensional symmetric space for given values of zero modes. According to the more detailed arguments about how to construct p-adic counterpart of real WCW physics described in the chapter "Construction of Quantum Theory", the following conjectures are trued.

1. The symmetric space property implies that there is only one maximum of Kähler function for given values of zero modes.

2. The generalization of Duistermaat-Heecke theorem holding true in finite-dimensional case suggests that by symmetric space property the integral of the exponent of Kähler gives just the exponent of Kähler function at the maximum and Gaussian determinant and metric determinant cancel each other.

3. The fact that free Gaussian field theory corresponds to a flat symmetric space inspires the hypothesis that S-matrix elements involving WCW spinor fields in the representations of the isometry group reduce to those given by free field theory with propagator defined by the inverse of WCW covariant Kähler metric evaluated in the tangent space basis defined by the isometry currents at the maximum of Kähler function. This implies that there is no perturbation series which would spoil any hopes about proving the rationality. The reduction to a free field theory does not make quantum TGD non-interacting since interactions are described as topologically (as decays and fusions of 3-surfaces) rather than algbraically as non-linearities of local action.

4. If the exponent function is a rational function with rational coefficients in the sense that for the points of WCW having finite number of rational valued coordinates (also zero modes), then the exponent $e^{K_{max}}$ is a rational number for rational values of zero modes. From the rationality of the exponent of the Kähler function follows the rational valuedness of the matrix elements of the metric. The undeniably very optimistic conclusion is that for rational values of the zero modes the S-matrix elements would be rational valued or have values if finite extension of rationals, so that they could be continued to the p-adic sectors of WCW . The S-matrix would have the same form in all number fields.

5. One could also interpret the outcome as an algebraic continuation of the rational quantum physics to real and p-adic physics. WCW -integrals can be thought of as being performed in the rational WCW . Of course, one can define also ordinary integrals over R^n numerically using Riemann sums by considering the division of the integration region to very small n-cubes for which the sides have rational-number valued lengths and such that the value of the function is taken at rational valued point inside each cube.

The finite-dimensional real one-dimensional Gaussian $exp(-ax^2/2)$ provides a natural testing ground for this rather speculative picture. The integral of the Gaussian is $(2\pi)^{1/2}/\sqrt{a}$: in n-dimensional case where a is replaced by a quadratic form defined by a matrix A one obtains $(2\pi)^{n/2}/\sqrt{det(A)}$ in n-dimensional case. The integral of a function $exp(-ax^2+kx^n)x^k$ reduces to a perturbation series as sum of graphs containing single vertex containing k lines and arbitrary number of vertices containing n lines and endowed with a factor k, and assigning with the lines the propagator factor $1/a$. For n-dimensional case the propagator factor would be inverse of the matrix A.

The result makes sense in the p-adic context if a and k are rational numbers. In the n-dimensional case matrix A and the coefficients defining the polynomial defining the interaction term must be rational numbers. The only problematic factor is the power of 2π, which seems to require algebraic extension containing π. Of course, one could define the normalization of the functional integral by dividing it by $(2\pi)^{n/2}$ to get rid of this fact. In the definition of S-matrix elements this normalization factor always disappears so that this problem has no physical significance.

In the case of free scalar quantum field theory n-point functions the perturbation theory are simply products of 2-point functions defined by the inverse of the infinite-dimensional Gaussian matrix. For plane wave basis for scalar field labelled by 4-momentum k the inverse of the Gaussian matrix reduces to the propagator ($i/(k^2+i\epsilon)$ for scalar field), which is rational function of the square of 4-momentum vector. In case of interacting quantum field the infinite summation over graphs spoils the hopes of obtaining end result which could be proven to be rational valued for rational values of incoming and outgoing four-momenta. The loop integrals are source of divergence problems and also number-theoretically problematic.

5.5 P-Adic Symmetries And Fourier Analysis

5.5.1 P-Adic Symmetries And Generalization Of The Notion Of Group

The most basic questions physicist can ask about the p-adic numbers are related to symmetries. It seems obvious that the concept of a Lie-group generalizes: nothing prevents from replacing the real or complex representation spaces associated with the definitions of the classical Lie-groups with the linear space associated with some algebraic extension of the p-adic numbers: the defining algebraic conditions, such as unitarity or orthogonality properties, make sense for the algebraically extended p-adic numbers, too.

For orthogonal groups one must replace the ordinary real inner product with the inner product $\sum_k X_k^2$ with a Cartesian power of a purely real extension of p-adic numbers. In the unitary case one must consider the complexification of a Cartesian power of a purely real extension with the inner product $\sum \bar{Z}_k Z_k$. Here $p \ mod \ 4 = 3$ is required. It should be emphasized however that the p-adic inner product differs from the ordinary one so that the action of, say, p-adic counterpart of a rotation group in R_p^3 induces in R^3 an action, which need not have much to do with ordinary rotations so that the generalization is physically highly nontrivial. Extensions of p-adic numbers also mean extreme richness of structure.

The exponentiation $t \rightarrow exp(tJ)$ of the Lie-algebra element J is a central element of Lie group theory and allows to coordinatize that elements of Lie group by mapping tangent space points the points representing group elements. Without algebraic extensions involving e or its roots one can exponentiate only the group parameters t satisfying $|t|_p < 1$. Thus the values of the exponentiation parameter which are too small/large in real/p-adic sense are not possible and one can say that the standard p-adic Lie algebra is a ball with radius $|t|_p = 1/p$.

The study of ordinary one-dimensional translations gives an idea about what it is involved. For finite values of the p-adic integer t the exponentiated group element corresponds in the case of translation group to a power of e so that the points reached by exponentiation cannot correspond to rational points. Since logarithm function exist as an inverse of p-adic exponent and since rationals correspond to infinite but periodic pinary expansions, rational points having the same p-adic norm can be reached by p-adic exponentials using t which is infinite as ordinary integer. This result is expected to generalize to the case of groups represented using rational-valued matrices.

One can define a hierarchy of p-adic Lie-groups by allowing extensions allowing e and even its roots such that the algebras have p-adic radii p^k. Hence the fact that the powers $e, ..., e^{p-1}$ define a finite-dimensional extensions of p-adic numbers seems to have a deep group theoretical meaning. One can define a hierarchy of increasingly refined extensions by taking the generator of extension to be $e^{1/n}$. For instance, in the case of translation group this makes possible p-adic variant of Fourier analysis by using discrete plane wave basis.

One can generalize also the notion of group by using the generalized notion of number. This means that one starts from the restriction of the group in question to a group acting in say rational and complex rational linear space and requires that real and p-adic groups have rational group transformations as common. By performing various completions one obtains a generalized group having the characteristic book like structure. In this kind of situation the relationship between various groups is clear and also the role of extensions of p-adic numbers can be understood. The notion of Lie-algebra generalizes also to form a book like structure. Coefficients of the pages of the Lie-algebra belong to various number fields and rational valued coefficients correspond to a part partially (because of the restriction $|t|_p < p^k$) common to all Lie-algebras.

$SO(2)$ as example

A simple example is provided by the generalization of the rotation group $SO(2)$. The rows of a rotation matrix are in general n orthonormalized vectors with the property that the components of these vectors have p-adic norm not larger than one. In case of $SO(2)$ this means the matrix elements $a_{11} = a_{22} = a, a_{12} = -a_{21} = b$ satisfy the conditions

$$\begin{aligned} a^2 + b^2 &= 1 \ , \\ |a|_p &\leq 1 \ , \\ |b|_p &\leq 1 \ . \end{aligned} \tag{5.5.1}$$

One can formally solve a as $a = \sqrt{1 - b^2}$ but the solution doesn't exists always. There are various possibilities to define the orthogonal group.

1. One possibility is to allow only those values of a for which square root exists as p-adic number. In case of orthogonal group this requires that both $b = sin(\Phi)$ and $a = cos(\Phi)$ exist as p-adic numbers. If one requires further that a and b make sense also as ordinary rational numbers, they define a Pythagorean triangle (orthogonal triangle with integer sides) and the group becomes discrete and cannot be regarded as a Lie-group. Pythagorean triangles emerge for rational counterpart of any Lie-group.

2. Other possibility is to allow an extension of the p-adic numbers allowing a square root of any ordinary p-adic number. The minimal extensions has dimension 4 (8) for $p > 2$ ($p = 2$). Therefore space-time dimension and embedding space dimension emerge naturally as minimal dimensions for spaces, where p-adic $SO(2)$ acts "stably". The requirement that a and b are real is necessary unless one wants the complexification of $so(2)$ and gives constraints on the values of the group parameters and again Lie-group property is expected to be lost.

3. The Lie-group property is guaranteed if the allowed group elements are expressible as exponents of a Lie-algebra generator Q. $g(t) = exp(iQt)$. This exponents exists only provided the p-adic norm of t is smaller than one. If one uses square root allowing extension, one can require that t satisfies $|t| \leq p^{-n/2}$, $n > 0$ and one obtains a decreasing hierarchy of groups $G_1, G_2, ...$ For the physically interesting values of p (typically of order $p = 2^{127} - 1$) the real counterparts of the transformations of these groups are extremely near to the unit element of the group. These conclusions hold true for any group. An especially interesting example physically is the group of "small" Lorentz transformations with $t = O(\sqrt{p})$. If the rest energy of the particle is of order $O(\sqrt{p})$: $E_0 = m = m_0\sqrt{p}$ (as it turns out) then the Lorentz boost with velocity $\beta = \beta_0\sqrt{p}$ gives particle with energy $E = m/\sqrt{1 - \beta_0^2 p} = m(1 + \frac{\beta_0^2 p}{2} + ..)$ so that $O(p^{1/2})$ term in energy is Lorentz invariant. This suggests that non-relativistic regime corresponds to small Lorentz transformations whereas in genuinely relativistic regime one must include also the discrete group of "large" Lorentz transformations with rational transformations matrices.

4. One can extend the group to contain products G_1G_2, such that G_1 is a rational matrix belonging to the restriction of the Lie-group to rational matrices not obtainable from a unit matrix p-adically by exponentiation, and G_2 is a group element obtainable from unit element by exponentiation. For instance, rational CP_2 is obtained from the group of rational 3×3 unitary matrices as by dividing it by the $U(2)$ subgroup of rational unitary matrices.

Even the construction of the representations of the translation group raises nontrivial issues since the construction of p-adic Fourier analysis is by no means a nontrivial task. One can however define the concept of p-adic plane wave group theoretically and p-adic plane waves are orthogonal with respect to the inner product defined by the proposed p-adic integral.

The representations of 3-dimensional rotation group $SO(3)$ can be constructed as homogenous functions of Cartesian coordinates of E^3 and in this case the phase factors $exp(im\phi)$ typically appearing in the expressions of spherical harmonics do not pose any problems. The construction of p-adic spherical harmonics is possible if one assumes that allowed spherical angles (θ, ϕ) correspond to Pythagorean triangles.

A similar situation is encountered also in the case of CP_2 spherical harmonics in in fact, quite generally. This number theoretic quantization of angles could be perhaps interpreted as a kind of cognitive quantum effect consistent with the fact that only rationals can be visualized concretely and relate directly to the sensory experience. More generally, the possibility to realize only rationals numerically might reflect the facts that only rationals are common to reals and p-adics and that cognition is basically p-adic.

Fractal structure of the p-adic Poincare group

p-Adic Poincare group, just as any other p-adic Lie group, contains entire fractal hierarchy of subgroups with the same Lie-algebra. For instance, translations $m^k \rightarrow m^k + p^N a^k$, where a^k has p-adic norm not larger than one form subgroup for all values of N. The larger the value of N is, the smaller this subgroup is. Quite generally this implies orbits within orbits and representations within representations like structure so that p-adic symmetry concept contains hologram like aspect. This property of the p-adic symmetries conforms nicely with the interpretation of p-adic symmetries as cognitive representations of real symmetries since the symmetries can be realized in a p-adically finite spatiotemporal volume of the cognitive space-time sheet. Even more, this volume can be p-adically arbitrarily small. If one identifies both p-adics and reals as a completion of rationals, the corresponding real volumes are however strictly speaking infinite in absence of a pinary cutoff.

The hierarchy of subgroups implies that M^4_+ decomposes in a natural manner to 4-cubes with side $L_0 = N_p(L)L_p$, where $N_p(L) = p^{-N}$ denotes the p-adic norm of L such that these 4-cubes are invariant under the group of sufficiently small Poincare transformations. In real context these cubes define a hierarchy of exteriors of cubes with decreasing sizes. One can have full p-adic Poincare invariance in p-adically arbitrarily small volume. Only those Poincare transformations, which leave the minimal p-adic cube invariant are symmetries. Also this picture suggest that the p-adic space-time sheets providing cognitive representations about finite space-time regions by canonical identification can have very large size.

The construction of the p-adic Fourier analysis is a nontrivial problem. The usual exponent functions $f_P(x) = exp(iPx)$, providing a representation of the p-adic translations do not make sense as a Fourier basis: f_P is not a periodic function; f_P does not converge if the norm of Px is not smaller than one and the natural orthogonalization of the different momentum eigenstates does not seem to be possible using the proposed definition of the definite integral.

This state of affairs suggests that p-adic Fourier analysis involves number theory. It turns out that one can construct what might be called number theoretical plane waves and that p-adic momentum space has a natural fractal structure in this case. The basic idea is to reduce p-adic Fourier analysis to a Fourier analysis in a finite field $G(p, 1)$ plus fractality in the sense that all p^m-scaled versions of the $G(p, 1)$ plane waves are used. This means that p-adic plane waves in a given interval $[n, n + 1)p^m$ are piecewise constant plane waves in a finite field $G(p, 1)$. Number theoretical p-adic plane waves are pseudo constants so that the construction does not work for p-adically differentiable functions. The pseudo-constancy however turns out to be a highly desirable feature in the construction of the p-adic QFT limit of TGD based on the mapping of the real H-quantum fields to p-adic quantum fields using the canonical identification.

The unsatisfactory feature of this approach is that number theoretic p-adic plane waves do not behave in the desired manner under translations. It would be nice to have a p-adic generalization of the plane wave concept allowing a generalization of the standard Fourier analysis and a direct connection with the theory of the representations of the translation group. A natural idea is to define exponential function as a solution of a p-adic differential equation representing the action of a translation generator and to introduce multiplicative pseudo constant making possible to define exponential function for all values of its argument. One can develop an argument suggesting that the plane waves obtained in this manner are indeed orthogonal.

Infinitesimal form of translational symmetry might be argued to be too strong requirement since p-adically infinitesimal translations typically correspond to real translations which are arbitrarily large: this is not consistent with the idea that cognitive representations with a finite spatial resolution are in question. This motivates a third approach to the p-adic Fourier analysis. The basic requirement is that discrete subgroup of translations commutes with the map of the real plane waves to their p-adic counterparts. This means that the products of the real phase factors are mapped to the products of the corresponding p-adic phase factors. This is possible if the phase factor is a rational complex number so that the phase angle corresponds to a Pythagorean triangle. The p-adic images of the real plane waves are defined for the momenta $k = nk_G$, $k_G = \phi_G/\Delta x$, where $\phi_G \in [0, 2\pi]$ is a Pythagorean phase angle and where the points $x_n = n\Delta x$ define a discretization of x-space, Δx being a rational number. These plane waves form a complete and orthogonalized set.

5.5.2 P-Adic Fourier Analysis: Number Theoretical Approach

Contrary to the original expectations, number theoretical Fourier analysis is probably not basic mathematical tools of p-adic QFT since it fails to provide irreducible representation for the translational symmetries. Despite this it deserves documentation.

Fourier analysis in a finite field $G(p,1)$

The p-adic numbers of unit norm modulo p reduce to a finite field $G(p,1)$ consisting of the integers $0, 1, ..., p-1$ with arithmetic operations defined by those of the ordinary integers taken modulo p. Since the elements $1, ..., p-1$ form a multiplicative group there must exists an element a of $G(p,1)$ (actually several) such that $a^{p-1} = 1$ holds true in $G(p,1)$. This kind of element is called primitive root. If n is a factor of $p-1$: $(p-1) = nm$, then also $a^m = 1$ holds true. This reflects the fact that Z_{p-1} decomposes into a product $Z_{m_1}^{n_1} Z_{m_2}^{n_2} ... Z_{m_s}^{n_s}$ of commuting factors Z_{m_i}, such that $m_i^{n_i}$ divides $p-1$.

A Fourier basis in $G(p,1)$ can be defined using p functions $f_k(n)$, $k = 0, .., p-1$. For $k = 0, 1, ..., p-2$ these functions are defined as

$$f_k(n) \quad = \quad a^{nk} \ , \ n = 0 \ , ..., p-1 \ , \tag{5.5.2}$$

and satisfy the periodicity property

$$f_k(0) = f_k(p-1) \ .$$

The problem is to identify the lacking p: th function. Since $f_k(n)$ transforms irreducibly under translations $n \rightarrow n+m$ it is natural to require that also the p: th function transforms in a similar manner and satisfies the periodicity property. This is achieved by defining

$$f_{p-1}(n) \quad = \quad (-1)^n \ . \tag{5.5.3}$$

The counterpart of the complex conjugation for f_k for $k \neq p-1$ is defined as $f_k \rightarrow f_{p-1-k}$. f_{p-1} is invariant under the conjugation. The inner product is defined as

$$\langle f_k, f_l \rangle = \sum_{n=0}^{p-2} f_{p-1-k}(n) f_l(n) = \delta(k,l)(p-1) \ . \tag{5.5.4}$$

The dual basis $\hat{f}_k$ clearly differs only by the normalization factor $1/(p-1)$ from the basis f_{p-k}. The counterpart of Fourier expansion for any real function in $G(p,1)$ can be obviously constructed using this function basis and Fourier components are obtained as the inner products of the dual Fourier basis with the function in question.

A natural interpretation for the integer k is as a p-adic momentum since in the translations $n \rightarrow n+m$ the plane wave with $k \neq p-1$ changes by a phase factor a^{km}. For $k = p-1$ it transforms by $(-1)^m$ so that also now an eigen state of finite field translations is in question.

p-Adic Fourier analysis based on p-adic plane waves

The basic idea is to reduce p-adic Fourier analysis to the Fourier analysis in $G(p,1)$ by using fractality.

1. Let the function $f(x)$ be such that the maximum p-adic norm of $f(x)$ is p^{-m}. One can uniquely decompose $f(x)$ to a sum of functions $f_n(x)$ such that $|f_n(x)|_p = p^n$ or vanishes in the entire range of definition for f:

$$\begin{aligned} f(x) &= \sum_{n\geq m} f_n(x) \ , \\ f_n(x) &= g_n(x)p^n \ , \\ |g_n(x)| &= 1 \ \text{for} \ g(x) \neq 0 \ . \end{aligned} \qquad (5.5.3)$$

   The higher the value of n, the smaller the contribution of f_n. The expansion converges extremely rapidly for the physically interesting large values of p.

2. Assume that $f(x)$ is such that for each value of n one can find some resolution $p^{m(n)}$ below which $g_n(x)$ is constant in the sense that for all intervals $[r, r+1)p^{m(n)}$ (defined in terms of the canonical identification) the function $f_n(x)$ is constant. For p-adically differentiable functions this cannot be the case since they would be pseudo constants if this were true. In the physical situation CP_2 size provides a natural p-adic cutoff so that only a finite number of f_n: s are needed and the resolution in question corresponds to CP_2 length scale. Hence ordinary plane waves (possibly with a natural UV cutoff) should have an expansion in terms of the p-adic plane waves.

3. The assumption implies that in each interval $(r, r+1)p^{m(n)-1}$, g_n can be regarded as a function in $G(p,1)$ identified as the set $x = (r+sp)p^{m(n)-1}$, $s = 0, 1, ..., p-1$. Hence one can Fourier expand $f_n(x)$ using $G(p,1)$ plane waves f^{ks}. In this manner one obtains a rapidly converging expansion using p-adic plane waves.

Periodicity properties of the number theoretic p-adic plane waves

The periodicity properties of the p-adic plane waves make it possible to associate a definite wavelength with a given p-adic plane wave. For the p-adic momenta k not dividing $p-1$, the wavelength corresponds to the entire range $(n, n+1)p^m$ and its real counterpart is

$$\lambda = p^{-m-1/2} l \ ,$$

where $l \sim 10^4\sqrt{\hbar G}$ is the fundamental p-adic length scale. If k divides $p-1 = \prod_i m_i^{n_i}$, the period is m_i and the real wavelength is

$$\lambda(m_i) = m_i p^{-m-1-1/2} l \ .$$

One might wonder whether this selection of preferred wavelengths has some physical consequences. The first thing to notice is that p-adic plane waves do *not* replace ordinary plane waves in the construction of the p-adic QFT limit of TGD. Rather, ordinary plane waves are expanded using the p-adic plane waves so that the selection of the preferred wavelengths, if it occurs at all, must be a dynamical process. The average value of the prime divisors, and hence the number of different wavelengths for a given value of p, counted with the degeneracy of the divisor is given by [A57]

$$\Omega(n) = ln(ln(n)) + 1.0346 \ ,$$

and is surprisingly small, or order 6 for numbers of order M_{127}! If one can apply probabilistic arguments or [A57] to the numbers of form $p-1$, too then one must conclude that very few wavelengths are possible for general prime p! This in turn means that to each p there are associated only very few characteristic length scales, which are predictable. Furthermore, all the p^k-multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes M_n can be considered as an illustrative example of the phenomenon. From [A39] one finds that $M_{127}-1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively. The number of distinct length scales is $3 \cdot 2^{11} - 1 \sim 2^{12}$. $M_{107}-1$ and $M_{89}-1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7 - 1 = 127 = M_7$ and $2^{11} - 1$. Note that for hadrons (M_{107}) the number of possible wavelengths is especially small: does this have something to do with the collective behavior of color confined quarks and gluons? An interesting possibility is that this length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see the third part of the book). One cannot exclude the possibility that long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

5.5.3 P-Adic Fourier Analysis: Group Theoretical Approach

The problem with the straightforward generalization of the Fourier analysis is that the standard Taylor expansion of the plane wave $exp(ikx)$ converges only provided x has p-adic norm smaller than one and that the p-adic exponential function does not have the periodicity properties of the ordinary exponential function guaranteeing orthogonality of the functions of the Fourier basis. Besides this one must assume $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as ordinary p-adic number.

The approach based on algebraic extensions allowing trigonometry

In an attempt to construct Fourier analysis the safest approach is to start from the ordinary Fourier analysis at circle or that for a particle in a one-dimensional box. The function basis uses as the basic building blocks the functions $e^{in\phi}$ in the case of circle and functions $e^{in\pi x/L}$ in the case of a particle in a box of side L.

The view about rationals as common to both reals and p-adics, and the possibility of finite-dimensional extensions of p-adics generated by the roots $e^{i2\pi/p^k}$ suggest how to realize this idea.

1. Consider first the case of the circle. Fix some value of N and select a set of points $\phi_n = in2\pi/p^k$ at which the phases are defined meaning p^{k+1}-dimensional algebraic extension. That powers of p appear is consistent with p-adic fractality. If so spin 1/2 *resp.* spin 1 particles would be inherently 2-adic *resp.* 3-adic. The plane wave basis corresponds $exp(ik\phi_n)$, $k = 0, ..., N-1$. In the case of particle in the one-dimensional box such that L corresponds to a rational number, the box is decomposed into N intervals of length L/N.

2. One can assign to the phases a well defined angular momentum as integer $n = 0, ..., N-1$ whereas the momentum spectrum for a particle in a box are given by $n\pi/L$. It is possible to continue the phase factor to the neighborhood of each point by requiring that the differential equation

$$\frac{d}{dx} exp(ikx) = ikexp(ikx)$$

defining the exponential function is satisfied.

3. The inner product of the plane waves f_{k_1}) and f_{k_2} can be defined as the sum

$$\langle k_1\rangle \equiv \sum_n \overline{f}_{k_1}(x_n)f_{k_2}(x_n) \ , \tag{5.5.4}$$

and orthogonality and completeness differ by no means from those of ordinary Fourier analysis.

p-Adic Fourier analysis, Pythagorean phases, and Gaussian primes

An alternative approach is based on Pythagorean phases and discretization in x-space, which might be a natural thing to do if p-adic field theory is taken as a cognitive model rather than "real" physics. This is also natural because rational Minkowski space is in the algebraic approach the fundamental object and reals and p-adics emerge as its completions.

Rational phase factors are common to the complexified p-adics ($p\ mod\ 4=3$) and reals and this suggests that one should define p-adic plane waves so that their values are in the set of the Pythagorean phases. Pythagorean phases are in one-one correspondence with the phases of the squares of Gaussian integers N_G and thus generated as products of squares of Gaussian primes π_G, which are complex integers with modulus squared equal to prime $p\ mod\ 4=1$. Thus the set of phases $\phi(\pi_G)$ for the phases for π_G^2 form an algebraically infinite-dimensional linear space in the sense that the phases representable as superpositions

$$2\phi_G=\sum_{\pi_G} n_{\pi_G}2\phi(\pi_G)$$

of these phases with integer coefficients belong to the set.

Consider now the definition of the plane wave basis based on Pythagorean phases and the identification of the p-adics and reals via common rationals.

1. Let $x_0=q=m/n$ denote a value of x-coordinate and let k denote some value of momentum. If $exp(ikx_0)$ is a Pythagorean phase then also the multiples nk correspond to Pythagorean phases. k itself cannot be a rational number so that k is not defined as an ordinary p-adic number: this could be seen as a defect of the approach since one cannot speak of a well-defined momentum. Neither can k be a rational multiple of π so that Pythagorean phases have nothing to do with the phases defined by algebraic extensions containing the phase $exp(i\pi/n)$ already discussed.

   For a given value of $x_0=q$ the momenta k for which $exp(ikq)$ is a Pythagorean phase are in one-one correspondence with Pythagorean phases. Moreover, Pythagorean phases result in the lattice defined by the multiples of the x_0. Thus a natural definition of the p-adic plane waves emerges predicting a maximal momentum spectrum with one-one correspondence with Pythagorean phases, and selecting a preferred lattice of points at the real axis. This definition is also in accordance with the idea that p-adic plane waves are related with a cognitive representation for real physics.

2. Pythagorean phases are in one-one correspondence with the phase factors associated with the squares of the Gaussian integers and generating phases correspond to the phases $\phi(\pi_G)$ associated with the squares of Gaussian primes π_G. The moduli squared for the Gaussian primes correspond to squares of rational primes $p\ mod\ 4=1$. Thus set of allowed momenta k_G for given spatial resolution m/n is the set

$$\{k_G(q)\}=\{\tfrac{2\phi_G/}{q}+\tfrac{2\pi n}{q}|n\in Z\} \ ,$$
$$\{\phi_G\}=\{\textstyle\sum_{\pi_G} n_{\pi_G}\phi(\pi_G)\} \ .$$

   When the spatial resolution $x_0=q$ is replaced with $q_1=r/s$, the spectrum is scaled by a rational factor q/q_1. The set of momenta is a dense subset of the real axis. There is no observable difference between the real momenta differing by a multiple of $2\pi/q$ and one must drop them from consideration. This conclusion is forced also by the fact that p-adically the momenta $k = nk_0$ do not exist, it is only the phase factors which exist.

3. It is easy to see that the p-adic plane waves with different momenta are orthogonal to each other as complex rational numbers:

$$\sum_n exp\left[in(k_G(1) - k_G(2))\right] = 0 \ .$$

4. Also completeness relations are satisfied in the sense that the condition

$$\sum_{k_G} exp\left[i(n_1 - n_2)k_G\right] = 0$$

is satisfied for $n_1 \neq n_2$. This is due to the fact that all integer multiples of k_G define Pythagorean phases. This means that the Fourier series of a function with respect to Pythagorean phases makes sense and one can expand p-adic-valued functions of space-time coordinates as Fourier series using Pythagorean phases. In particle expansion of the embedding space coordinates as functions of p-adic space-time coordinates might be carried out in this manner.

5. One can criticize this approach for the fact that there is no unique continuation of the phase factors from the set of the rationals $x_n = nx_0$ to p-adic numbers neighborhoods of these points. Although eigen states of finite translations are in question one cannot regard the states as eigen states of infinitesimal translations since the momenta are not well defined as p-adic numbers. One could of course arbitrarily assign momentum eigenstate $e^{in\pi(x-x_k)}$ the point x_k to the eigenstate characterized by the dimensionless momentum n but the momentum spectrum associated with different Pythagorean phases would be same.

5.5.4 How To Define Integration And P-Adic Fourier Analysis, Integral Calculus, And P-Adic Counterparts Of Geometric Objects?

p-Adic differential calculus exists and obeys essentially the same rules as ordinary differential calculus. The only difference from real context is the existence of p-adic pseudo-constants: any function which depends on finite number of pinary digits has vanishing p-adic derivative. This implies non-determinism of p-adic differential equations. One can defined p-adic integral functions using the fact that indefinite integral is the inverse of differentiation. The basis problem with the definite integrals is that p-adic numbers are not well-ordered so that the crucial ordering of the points of real axis in definite integral is not unique. Also p-adic Fourier analysis is problematic since direct counterparts of ep(ix) and trigonometric functions are not periodic. Also exp(-x) fails to converse exponentially since it has p-adic norm equal to 1. Note also that these functions exists only when the p-adic norm of x is smaller than 1.

The following considerations support the view that the p-adic variant of a geometric objects, integration and p-adic Fourier analysis exists but only when one considers highly symmetric geometric objects such as symmetric spaces. This is welcome news from the point of view of physics. At the level of space-time surfaces this is problematic. The field equations associated with Kähler action and Kähler-Dirac equation make sense. Kähler action defined as integral over p-adic space-time surface fails to exist. If however the Kähler function identified as Kähler for a preferred extremal of Kähler action is rational or algebraic function of preferred complex coordinates of WCW with rational coefficients, its p-adic continuation is expected to exist.

Circle with rotational symmetries and its hyperbolic counterparts

Consider first circle with emphasis on symmetries and Fourier analysis.

1. In this case angle coordinate ϕ is the natural coordinate. It however does not make sense as such p-adically and one must consider either trigonometric functions or the phase $exp(i\phi)$

instead. If one wants to do Fourier analysis on circle one must introduce roots $U_{n,N} = exp(in2\pi/N)$ of unity. This means discretization of the circle. Introducing all roots $U_{n,p} = exp(i2\pi n/p)$, such that p divides N, one can represent all $U_{k,n}$ up to $n = N$. Integration is naturally replaced with sum by using discrete Fourier analysis on circle. Note that the roots of unity can be expressed as products of powers of roots of unity $exp(in2\pi/p^k)$, where p^k divides N.

2. There is a number theoretical delicacy involved. By Fermat's theorem $a^{p-1}\ mod\ p = 1$ for $a = 1, ...p - 1$ for a given p-adic prime so that for any integer M divisible by a factor of $p - 1$ the M: th roots of unity exist as ordinary p-adic numbers. The problem disappears if these values of M are excluded from the discretization for a given value of the p-adic prime. The manner to achieve this is to assume that N contains no divisors of $p-1$ and is consistent with the notion of finite measurement resolution. For instance, $N = p^n$ is an especially natural choice guaranteeing this.

3. The p-adic integral defined as a Fourier sum does not reduce to a mere discretization of the real integral. In the real case the Fourier coefficients must approach to zero as the wave vector $k = n2\pi/N$ increases. In the p-adic case the condition consistent with the notion of finite measurement resolution for angles is that the p-adic valued Fourier coefficients approach to zero as n increases. This guarantees the p-adic convergence of the discrete approximation of the integral for large values of N as n increases. The map of p-adic Fourier coefficients to real ones by canonical identification could be used to relate p-adic and real variants of the function to each other.

This finding would suggests that p-adic geometries -in particular the p-adic counterpart of CP_2, are discrete. Variables which have the character of a radial coordinate are in natural manner p-adically continuous whereas phase angles are naturally discrete and described in terms of algebraic extensions. The conclusion is disappointing since one can quite well argue that the discrete structures can be regarded as real. Is there any manner to escape this conclusion?

1. Exponential function $exp(ix)$ exists p-adically for $|x|_p \leq 1/p$ but is not periodic. It provides representation of p-adic variant of circle as group $U(1)$. One obtains actually a hierarchy of groups $U(1)_{p,n}$ corresponding to $|x|_p \leq 1/p^n$. One could consider a generalization of phases as products $Exp_p(N, n2\pi/N + x) = exp(in2\pi n/N)exp(ix)$ of roots of unity and exponent functions with an imaginary exponent. This would assign to each root of unity p-adic continuum interpreted as the analog of the interval between two subsequent roots of unity at circle. The hierarchies of measurement resolutions coming as $2\pi/p^n$ would be naturally accompanied by increasingly smaller p-adic groups $U(1)_{p,n}$.

2. p-Adic integration would involve summation plus possibly also an integration over each p-adic variant of discretization interval. The summation over the roots of unity implies that the integral of $\int exp(inx)dx$ would appear for $n = 0$. Whatever the value of this integral is, it is compensated by a normalization factor guaranteeing orthonormality.

3. If one interprets the p-adic coordinate as p-adic integer without the identification of points differing by a multiple of n as different points the question whether one should require p-adic continuity arises. Continuity is obtained if $U_n(x + mp^m) = U_n(x)$ for large values of m. This is obtained if one has $n = p^k$. In the spherical geometry this condition is not needed and would mean quantization of angular momentum as $L = p^k$, which does not look natural. If representations of translation group are considered the condition is natural and conforms with the spirit of the p-adic length scale hypothesis.

The hyperbolic counterpart of circle corresponds to the orbit of point under Lorentz group in two 2-D Minkowski space. Plane waves are replaced with exponentially decaying functions of the coordinate η replacing phase angle. Ordinary exponent function $exp(x)$ has unit p-adic norm when it exists so that it is not a suitable choice. The powers p^n existing for p-adic integers however approach to zero for large values of $x = n$. This forces discretization of η or rather the hyperbolic phase as powers of p^x, $x = n$. Also now one could introduce products of $Exp_p(nlog(p) + z) = p^n exp(x)$ to achieve a p-adic continuum. Also now the integral over the discretization interval is compensated by orthonormalization and can be forgotten. The integral of exponential function would reduce to a sum $\int Exp_p dx = \sum_k p^k = 1/(1-p)$. One can also introduce finite-dimensional but non-algebraic extensions of p-adic numbers allowing e and its roots $e^{1/n}$ since e^p exists p-adically.

Plane with translational and rotational symmetries

Consider first the situation by taking translational symmetries as a starting point. In this case Cartesian coordinates are natural and Fourier analysis based on plane waves is what one wants to define. As in the previous case, this can be done using roots of unity and one can also introduce p-adic continuum by using the p-adic variant of the exponent function. This would effectively reduce the plane to a box. As already noticed, in this case the quantization of wave vectors as multiples of $1/p^k$ is required by continuity.

One can take also rotational symmetries as a starting point. In this case cylindrical coordinates (ρ, ϕ) are natural.

1. Radial coordinate can have arbitrary values. If one wants to keep the connection $\rho = \sqrt{x^2+y^2}$ with the Cartesian picture square root allowing extension is natural. Also the values of radial coordinate proportional to odd power of p are problematic since one should introduce $\sqrt{p}$: is this extension internally consistent? Does this mean that the points $\rho \propto p^{2n+1}$ are excluded so that the plane decomposes to annuli?

2. As already found, angular momentum eigen states can be described in terms of roots of unity and one could obtain continuum by allowing also phases defined by p-adic exponent functions.

3. In radial direction one should define the p-adic variants for the integrals of Bessel functions and they indeed might make sense by algebraic continuation if one consistently defines all functions as Fourier expansions. Delta-function renormalization causes technical problems for a continuum of radial wave vectors. One could avoid the problem by using exponentially decaying variants of Bessel function in the regions far from origin, and here the already proposed description of the hyperbolic counterparts of plane waves is suggestive.

4. One could try to understand the situation also using Cartesian coordinates. In the case of sphere this is achieved by introducing two coordinate patches with Cartesian coordinates. Pythagorean phases are rational phases (orthogonal triangles for which all sides are integer valued) and form a dense set on circle. Complex rationals (orthogonal triangles with integer valued short sides) define a more general dense subset of circle. In both cases it is difficult to imagine a discretized version of integration over angles since discretization with constant angle increment is not possible.

The case of sphere and more general symmetric space

In the case of sphere spherical coordinates are favored by symmetry considerations. For spherical coordinates $sin(\theta)$ is analogous to the radial coordinate of plane. Legedre polynomials expressible as polynomials of $sin(\theta)$ and $cos(\theta)$ are expressible in terms of phases and the integration measure $sin^2(\theta)d\theta d\phi$ reduces the integral of S^2 to summation. As before one can introduce also p-adic continuum. Algebraic cutoffs in both angular momentum l and m appear naturally. Similar cutoffs appear in the representations of quantum groups and there are good reasons to expect that these phenomena are correlated.

Exponent of Kähler function appears in the integration over WCW . From the expression of Kähler gauge potential given by $A_\alpha = J_\alpha{}^\theta \partial_\theta K$ one obtains using $A_\alpha = cos(\theta)\delta_{\alpha,\phi}$ and $J_{\theta\phi} = sin(\theta)$ the expression $exp(K) = sin(\theta)$. Hence the exponent of Kähler function is expressible in terms of spherical harmonics.

The completion of the discretized sphere to a p-adic continuum- and in fact any symmetric space- could be performed purely group theoretically.

1. Exponential map maps the elements of the Lie-algebra to elements of Lie-group. This recipe generalizes to arbitrary symmetric space G/H by using the Cartan decomposition $g = t + h$,

$[h,h] \subset h$, $[h,t] \subset t$, $[t,t] \subset h$. The exponentiation of t maps t to G/H in this case. The exponential map has a p-adic generalization obtained by considering Lie algebra with coefficients with p-adic norm smaller than one so that the p-adic exponent function exists. As a matter fact, one obtains a hierarchy of Lie-algebras corresponding to the upper bounds of the p-adic norm coming as p^{-k} and this hierarchy naturally corresponds to the hierarchy of angle resolutions coming as $2\pi/p^k$. By introducing finite-dimensional transcendental extensions containing roots of e one obtains also a hierarchy of p-adic Lie-algebras associated with transcendental extensions.

2. In particular, one can exponentiate the complement of the $SO(2)$ sub-algebra of $SO(3)$ Lie-algebra in p-adic sense to obtain a p-adic completion of the discrete sphere. Each point of the discretized sphere would correspond to a p-adic continuous variant of sphere as a symmetric space. Similar construction applies in the case of CP_2. Quite generally, a kind of fractal or holographic symmetric space is obtained from a discrete variant of the symmetric space by replacing its points with the p-adic symmetric space.

3. In the N-fold discretization of the coordinates of M-dimensional space t one $(N-1)^M$ discretization volumes which is the number of points with non-vanishing t-coordinates. It would be nice if one could map the p-adic discretization volumes with non-vanishing t-coordinates to their positive valued real counterparts by applying canonical identification. By group invariance it is enough to show that this works for a discretization volume assignable to the origin. Since the p-adic numbers with norm smaller than one are mapped to the real unit interval, the p-adic Lie algebra is mapped to the unit cell of the discretization lattice of the real variant of t. Hence by a proper normalization this mapping is possible.

The above considerations suggest that the hierarchies of measurement resolutions coming as $\Delta\phi = 2\pi/p^n$ are in a preferred role. One must be however cautious in order to avoid too strong assumptions. The following arguments however support this identification.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier basis irrespective of the value of M unless one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. Otherwise the measurement resolution is just $\Delta\phi = 2\pi/p^n$. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer and assume that state function reduction selects one particular prime (no superposition of quantum states with different p-adic topologies). For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$. One could assign to a given measurement resolution all the p-adic primes appearing as factors in N so that the notion of multi-p p-adicity would make sense. One can also consider the identification of the measurement resolution as $\Delta\phi = |N/M|_p = 2\pi/p^k$. This interpretation is supported by the approach based on infinite primes [K94].

What about integrals over partonic 2-surfaces and space-time surface?

One can of course ask whether also the integrals over partonic 2-surfaces and space-time surface could be p-adicized by using the proposed method of discretization. Consider first the p-adic counterparts of the integrals over the partonic 2-surface X^2.

1. WCW Hamiltonians and Kähler form are expressible using flux Hamiltonians defined in terms of X^2 integrals of JH_A, where H_A is $\delta CD \times CP_2$ Hamiltonian, which is a rational function of the preferred coordinates defined by the exponentials of the coordinates of the sub-space

t in the appropriate Cartan algebra decomposition. The flux factor $J = \epsilon^{\alpha\beta}J_{\alpha\beta}\sqrt{g_2}$ is scalar and does not actually depend on the induced metric.

2. The notion of finite measurement resolution would suggest that the discretization of X^2 is somehow induced by the discretization of $\delta CD \times CP_2$. The coordinates of X^2 could be taken to be the coordinates of the projection of X^2 to the sphere S^2 associated with $\delta M^4_\pm$ or to the homologically non-trivial geodesic sphere of CP_2 so that the discretization of the integral would reduce to that for S^2 and to a sum over points of S^2.

3. To obtain an algebraic number as an outcome of the summation, one must pose additional conditions guaranteeing that both H_A and J are algebraic numbers at the points of discretization (recall that roots of unity are involved). Assume for definiteness that S^2 is $r_M = constant$ sphere. If the remaining preferred coordinates are functions of the preferred S^2 coordinates mapping phases to phases at discretization points, one obtains the desired outcome. These conditions are rather strong and mean that the various angles defining CP_2 coordinates -at least the two cyclic angle coordinates- are integer multiples of those assignable to S^2 at the points of discretization. This would be achieved if the preferred complex coordinates of CP_2 are powers of the preferred complex coordinate of S^2 at these points. One could say that X^2 is algebraically continued from a rational surface in the discretized variant of $\delta CD \times CP_2$. Furthermore, if the measurement resolutions come as $2\pi/p^n$ as p-adic continuity actually requires and if they correspond to the p-adic group $G_{p,n}$ for which group parameters satisfy $|t|_p \leq p^{-n}$, one can precisely characterize how a p-adic prime characterizes the real partonic 2-surface. This would be a fulfilment of one of the oldest dreams related to the p-adic vision.

A even more ambitious dream would be that even the integral of the Kähler action for preferred extremals could be defined using a similar procedure. The conjectured slicing of Minkowskian space-time sheets by string world sheets and partonic 2-surfaces encourages these hopes.

1. One could introduce local coordinates of H at both ends of CD by introducing a continuous slicing of $M^4 \times CP_2$ by the translates of $\delta M^4_\pm \times CP_2$ in the direction of the time-like vector connecting the tips of CD. As space-time coordinates one could select four of the eight coordinates defining this slicing. For instance, for the regions of the space-time sheet representable as maps $M^4 \rightarrow CP_2$ one could use the preferred M^4 time coordinate, the radial coordinate of δM^4_+, and the angle coordinates of $r_M = constant$ sphere.

2. Kähler action density should have algebraic values and this would require the strengthening of the proposed conditions for X^2 to apply to the entire slicing meaning that the discretized space-time surface is a rational surface in the discretized $CD \times CP_2$. If this condition applies to the entire space-time surface it would effectively mean the discretization of the classical physics to the level of finite geometries. This seems quite strong implication but is consistent with the preferred extremal property implying the generalized Bohr rules. The reduction of Kähler action to 3-dimensional boundary terms is implied by rather general arguments. In this case only the effective algebraization of the 3-surfaces at the ends of CD and of wormhole throats is needed [K43]. By effective 2-dimensionality these surfaces cannot be chosen freely.

3. If Kähler function and WCW Hamiltonians are rational functions, this kind of additional conditions are not necessary. It could be that the integrals of defining Kähler action flux Hamiltonians make sense only in the intersection of real and p-adic worlds assumed to be relevant for the physics of living systems.

Tentative conclusions

These findings suggest following conclusions.

1. Exponent functions play a key role in the proposed p-adicization. This is not an accident since exponent functions play a fundamental role in group theory and p-adic variants of real geometries exist only under symmetries- possibly maximal possible symmetries- since otherwise the notion of Fourier analysis making possible integration does not exist. The inner product defined in terms of integration reduce for functions representable in Fourier

basis to sums and can be carried out by using orthogonality conditions. Convolution involving integration reduces to a product for Fourier components. In the case of embedding space and WCW these conditions are satisfied but for space-time surfaces this is not possible.

2. There are several ways to choose the Cartan algebra already in the case of sphere. In the case of plane one can consider either translations or rotations and this leads to different p-adic variants of plane. Also the realization of the hierarchy of Planck constants leads to the conclusion that the extended embedding space and therefore also WCW contains sectors corresponding to different choices of quantization axes meaning that quantum measurement has a direct geometric correlate.

3. The above described 2-D examples represent symplectic geometries for which one has natural decomposition of coordinates to canonical pairs of cyclic coordinate (phase angle) and corresponding canonical conjugate coordinate. p-Adicization depends on whether the conjugate corresponds to an angle or non-compact coordinate. In both cases it is however possible to define integration. For instance, in the case of CP_2 one would have two canonically conjugate pairs and one can define the p-adic counterparts of CP_2 partial waves by generalizing the procedure applied to spherical harmonics. Products of functions expressible using partial waves can be decomposed by tensor product decomposition to spherical harmonics and can be integrated. In particular inner products can be defined as integrals. The Hamiltonians generating isometries are rational functions of phases: this inspires the hope that also WCW Hamiltonians also rational functions of preferred WCW coordinates and thus allow p-adic variants.

4. Discretization by introducing algebraic extensions is unavoidable in the p-adicization of geometrical objects but one can have p-adic continuum as the analog of the discretization interval and in the function basis expressible in terms of phase factors and p-adic counterparts of exponent functions. This would give a precise meaning for the p-adic counterparts of the embedding space and WCW if the latter is a symmetric space allowing coordinatization in terms of phase angles and conjugate coordinates.

5. The intersection of p-adic and real worlds would be unique and correspond to the points defining the discretization.

5.6 Generalization Of Riemann Geometry

Geometrization of physics program requires Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be also understood. In this section the basic problems and ideas related to these challenges are discussed.

5.6.1 P-Adic Riemannian Geometry Depends On CognitiveRepresentation

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal purely algebraic generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. Canonical identification would make it possible to define p-adic variant of Riemann integral formally allowing to calculate arc lengths and similar quantities but looks like a trick. The realization that the p-adic variant of harmonic analysis makes it possible to define definite integrals in the case of symmetric space became possible only after a detailed vision about what quantum TGD is [K111] had emerged.

Symmetry considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ to a high degree but not uniquely. This non-uniqueness might relate to the distinction between different cognitive representations. For instance, in the case of Euclidian plane one can introduce linear or cylindrical coordinates and the manifest symmetries dictating the preferred coordinates correspond to translational and rotational symmetries in these two cases and give rise to different p-adic variants of the plane. Both linear and cylindrical coordinates are fixed only modulo the action of group consisting of translations and rotations and the degeneracy of choices can be interpreted in terms of a choice of quantization axes of angular momentum and momenta.

The most natural looking manner to define the p-adic counterpart of M^4 is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of M^4 linear Minkowski coordinates are an obvious choice but also the counterparts of Robertson-Walker coordinates for M^4_+ defined as $[t,(z,x,y)] = a\times[cosh(\eta), sinh(\eta)(cos(\theta), sin(\theta)cos(\phi), sin(\theta)sin($ expressible in terms of phases and their hyperbolic counterparts and transforming nicely under the Cartan algebra of Lorentz group are possible. p-Adic variant is obtained by introducing finite measurement resolution for angle and replacing angle range by finite number of roots of unity. Same applies to hyperbolic angles.

Rational CP_2 could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary 3×3-matrices. CP_2 could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational 3×3-matrix representable in terms of Pythagorean phases [A10] and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

This option is not very natural as far as integration is considered. CP_2 however allows the analog of spherical coordinates for S^2 expressible in terms of angle variables alone and this suggests the introduction of the variant of CP_2 for which the coordinate values correspond to roots of unity. Completion would be performed in the same manner as for rational CP_2. This non-uniqueness need not be a drawback but could reflect the fact that the p-adic cognitive representation of real geometry are geometrically non-equivalent. This means a refinement of the principle of General Coordinate Invariance taking into account the fact that the cognitive representation of the real world affects the world with cognition included in a delicate manner.

5.6.2 P-Adic Embedding Space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the embedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the embedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

The identification of the fundamental p-adic length scale

The fundamental p-adic length scale corresponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the "radius" R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the "cosmological constant" Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda \ . \tag{5.6.1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} \ . \tag{5.6.2}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π: s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod \ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

   (b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R \ ,$$

   rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counterpart of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal plane waves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naïvely expected p, in the expression of the p-adic length scale can be understood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{5.6.2}$$

where m_{kl} is the standard M^4 metric $(1,-1,-1,-1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k: th coordinate axis the expression

$$s = R\sqrt{p}m^k \ . \tag{5.6.3}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{5.6.4}$$

The p-adic distance along the k: th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p}l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R\frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{5.6.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1,..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

The two variants of CP_2

As noticed, CP_2 allows two variants based on rational discretization and on the discretiation based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rational variant of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/columns are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_k X_k) \ ; \ |t_k|_p < 1\} = \{x = 1 + iy \ ; \ |y|_p < 1\} \ . \tag{5.6.5}$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2))\} \ . \tag{5.6.6}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1 + O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the "physical" $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod \ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic sub-manifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{5.6.6}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1+r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \quad . \tag{5.6.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1+r^2)^2} \quad . \tag{5.6.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

5.6.3 Topological Condensate As A Generalized Manifold

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlates for cognition and intentionality. The vision about quantum TGD as a generalized number theory provides possible solutions to the basic problems associated with the precise definition of topological condensate.

Generalization of number concept and fusion of real and p-adic physics

The unification of real physics of material work and p-adic physics of cognition leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals (see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg` or **Fig. ??** in the appendix of this book). The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

This notion of manifold implies a generalization of the notion of embedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in a discrete set consisting of rational points. This view in which cognition would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

It took some time to end up with this vision. The first picture was based on the notion of real and p-adic space-time sheets glued together by using canonical identification or some of its variants but led to insurmountable difficulties since p-adic topology is so different from real topology. One can of course ask whether one can speak about p-adic counterparts of notions like boundary of 3-surface or genus of 2-surface crucial for TGD based model of family replication phenomenon. It seems that these notions generalize as purely algebraically defined concepts which supports the view that p-adicization of real physics must be a purely algebraic procedure.

How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \rightarrow \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n + y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about infinite size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves cosmic length scales.

What determines the p-adic primes assignable to a given real space-time sheet?

The p-adic realization of the Slaving Principle suggests that various levels of the topological condensate correspond to real matter like regions and p-adic mind like regions labelled by p-adic primes p. The larger the length scale, the larger the value of p and the course the induced real topology. If the most interesting values of p indeed correspond Mersenne primes, the number of most interesting levels is finite: at most 12 levels below electron length scale: actually also primes near prime powers of two seem to be physically important.

The intuitive expectation is that the p-adic prime associated with a given real space-time sheet characterizes its effective p-adic topology. As a matter fact, several p-adic effective topologies can be considered and the attractive hypothesis is that elementary particles are characterized by integers defined by the product of these p-adic primes and the integers for particles which can have direct interactions possess common prime factors.

The intuitive view is that those primes are favored for with the p-adic space-time sheet obtained by an algebraic continuation has as many rational or algebraic space-time points as possible in common with the real space-time sheet. The rationale is that if the real space-time sheet is generated in a quantum jump in which p-adic space-time sheet is transformed to a real one, it must have a large number of points in common with the real space-time sheet if the probability amplitude for this process involves a sum over the values of an n-point function of a conformal field theory over all common n-tuples and vanishes when the number of common points is smaller than n.

5.7 Appendix: P-Adic Square Root Function And Square Root Allowing Extension Of P-Adic Numbers

The following arguments demonstrate that the extension allowing square roots of ordinary p-adic numbers is 4-dimensional for $p < 2$ and 8-dimensional for $p = 2$.

5.7.1 $P > 2$ Resp. $P = 2$ Corresponds To $D = 4$ Resp. $D = 8$ Dimensional Extension

What is important is that only the square root of ordinary p-adic numbers is needed: the square root need not exist outside the real axis. It is indeed impossible to find a finite-dimensional extension allowing square root for all ordinary p-adic numbers numbers. For $p > 2$ the minimal dimension for algebraic extension allowing square roots near real axis is $D = 4$. For $p = 2$ the dimension of the extension is $D = 8$.

For $p > 2$ the form of the extension can be derived by the following arguments.

1. For $p > 2$ a p-adic number y in the range $(0, p-1)$ allows square root only provided there exists a p-adic number $x \in \{0, p-1\}$ satisfying the condition $y = x^2 \ mod\ p$. Let x_0 be the smallest integer, which does not possess a p-adic square root and add the square root θ of x_0 to the number field. The numbers in the extension are of the form $x + \theta y$. The extension allows square root for every $x \in \{0, p-1\}$ as is easy to see. p-adic numbers $mod\ p$ form a finite field $G(p, 1)$ [A25] so that any p-adic number y, which does not possess square root can be written in the form $y = x_0 u$, where u possesses square root. Since θ is by definition the square root of x_0 then also y possesses square root. The extension does not depend on the choice of x_0.

   The square root of -1 does not exist for $p\ mod\ 4 = 3$ [A23] and $p = 2$ but the addition of θ guarantees its existence automatically. The existence of $\sqrt{-1}$ follows from the existence of $\sqrt{p-1}$ implied by the extension by θ. $\sqrt{(-1+p)-p}$ can be developed in power in powers of p and series converges since the p-adic norm of coefficients in Taylor series is not larger than 1. If $p - 1$ does not possess a square root, one can take θ to be equal to $\sqrt{-1}$.

2. The next step is to add the square root of p so that the extension becomes 4-dimensional and an arbitrary number in the extension can be written as

$$Z = (x + \theta y) + \sqrt{p}(u + \theta v) \ . \qquad (5.7.1)$$

In $p = 2$ case 8-dimensional extension is needed to define square roots. The addition of $\sqrt{2}$ implies that one can restrict the consideration to the square roots of odd 2-adic numbers. One must be careful in defining square roots by the Taylor expansion of square root $\sqrt{x_0 + x_1}$ since n: the Taylor coefficient is proportional to 2^{-n} and possesses 2-adic norm 2^n. If x_0 possesses norm 1 then x_1 must possess norm smaller than $1/8$ for the series to converge. By adding square roots $\theta_1 = \sqrt{-1}, \theta_2 = \sqrt{2}$ and $\theta_3 = \sqrt{3}$ and their products one obtains 8-dimensional extension.

The emergence of the dimensions $D = 4$ and $D = 8$ for the algebraic extensions allowing the square root of an ordinary p-adic number stimulates an obvious question: could one regard space-time as this kind of an algebraic extension for $p > 2$ and the embedding space $H = M^4_+ \times CP_2$ as a similar 8-dimensional extension of the 2-adic numbers? Contrary to the first expectations, it seems that algebraic dimension cannot be regarded as a physical dimension, and that quaternions and octonions provide the correct framework for understanding space-time and embedding space dimensions. One could perhaps say that algebraic dimensions are additional dimensions of the world of cognitive physics rather than those of the real physics and there presence could perhaps explain why we can imagine all possible dimensions mathematically.

By construction, any ordinary p-adic number in the extension allows square root. The square root for an arbitrary number sufficiently near to p-adic axis can be defined through Taylor series expansion of the square root function $\sqrt{Z}$ at a point of p-adic axis. The subsequent considerations show that the p-adic square root function does not allow analytic continuation to R^4 and the points of the extension allowing a square root consist of disjoint converge cubes forming a structure resembling future light cone in certain respects.

5.7.2 P-Adic Square Root Function For $P > 2$

The study of the properties of the series representation of a square root function shows that the definition of the square root function is possible in certain region around the real p-adic axis. What is nice that this region can be regarded as the p-adic analog (not the only one) of the future light cone defined by the condition

$$N_p(Im(Z)) \quad < \quad N_p(t = Re(Z)) = p^k \ , \tag{5.7.2}$$

where the real p-adic coordinate $t = Re(Z)$ is identified as a time coordinate and the imaginary part of the p-adic coordinate is identified as a spatial coordinate. The p-adic norm for the four-dimensional extension is analogous to ordinary Euclidian distance. p-Adic light cone consists of cylinders parallel to time axis having radius $N_p(t) = p^k$ and length $p^{k-1}(p-1)$. As a real space (recall the canonical correspondence) the cross section of the cylinder corresponds to a parallelpiped rather than ball.

The result can be understood heuristically as follows.

1. For the four-dimensional extension allowing square root ($p > 2$) one can construct square root at each point $x(k,s) = sp^k$ represented by ordinary p-adic number, $s = 1, ..., p-1$, $k \in Z$. The task is to show that by using Taylor expansion one can define square root also in some neighbourhood of each of these points and find the form of this neighbourhood.

2. Using the general series expansion of the square root function one finds that the convergence region is p-adic ball defined by the condition

   $$N_p(Z - sp^k) \quad \leq \quad R(k) \ , \tag{5.7.3}$$

   and having radius $R(k) = p^d, d \in Z$ around the expansion point.

3. A purely p-adic feature is that the convergence spheres associated with two points are either disjoint or identical! In particular, the convergence sphere $B(y)$ associated with any point inside convergence sphere $B(x)$ is identical with $B(x)$: $B(y) = B(x)$. The result follows directly from the ultra-metricity of the p-adic norm. The result means that stepwise analytic continuation is not possible and one can construct square root function only in the union of p-adic convergence spheres associated with the points $x(k,s) = sp^k$ which correspond to ordinary p-adic numbers.

4. By the scaling properties of the square root function the convergence radius $R(x(k,s)) \equiv R(k)$ is related to $R(x(0,s)) \equiv R(0)$ by the scaling factor p^{-k}:

   $$R(k) \quad = \quad p^{-k}R(0) \ , \tag{5.7.4}$$

   so that the convergence sphere expands as a function of the p-adic time coordinate. The study of the convergence reduces to the study of the series at points $x = s = 1, ..., k-1$ with a unit p-adic norm.

5. Two neighboring points $x = s$ and $x = s+1$ cannot belong to the same convergence sphere: this would lead to a contradiction with the basic results of about square root function at integer points. Therefore the convergence radius satisfies the condition

   $$R(0) \quad < \quad 1 \ . \tag{5.7.5}$$

   The requirement that the convergence is achieved at all points of the real axis implies

$$\begin{aligned} R(0) &= \frac{1}{p} \ , \\ R(p^k s) &= \frac{1}{p^{k+1}} \ . \end{aligned} \tag{5.7.5}$$

If the convergence radius is indeed this, then the region, where the square root is defined, corresponds to a connected light cone like region defined by the condition $N_p(Im(Z)) = N_p(Re(Z))$ and $p > 2$-adic space time is the p-adic analog of the M^4 light-cone. If the convergence radius is smaller, the convergence region reduces to a union of disjoint p-adic spheres with increasing radii.

How the p-adic light cone differs from the ordinary light cone can be seen by studying the explicit form of the p-adic norm for $p > 2$ square root allowing extension $Z = x + iy + \sqrt{p}(u + iv)$

$$\begin{aligned} N_p(Z) &= (N_p(det(Z)))^{\frac{1}{4}} \ , \\ &= (N_p((x^2+y^2)^2 + 2p^2((xv-yu)^2 + (xu-yv)^2) + p^4(u^2+v^2)^2))^{\frac{1}{4}} \ , \end{aligned} \tag{5.7.4}$$

where $det(Z)$ is the determinant of the linear map defined by a multiplication with Z. The definition of the convergence sphere for $x = s$ reduces to

$$N_p(det(Z_3)) = N_p(y^4 + 2p^2y^2(u^2+v^2) + p^4(u^2+v^2)^2)) < 1 \ . \tag{5.7.5}$$

For physically interesting case $p \ mod \ 4 = 3$ the points (y, u, v) satisfying the conditions

$$\begin{aligned} N_p(y) &\leq \frac{1}{p} \ , \\ N_p(u) &\leq 1 \ , \\ N_p(v) &\leq 1 \ , \end{aligned} \tag{5.7.4}$$

belong to the sphere of convergence: it is essential that for all u and v satisfying the conditions one has also $N_p(u^2 + v^2) \leq 1$. By the canonical correspondence between p-adic and real numbers, the real counterpart of the sphere $r = t$ is now the parallelpiped $0 \leq y < 1, 0 \leq u < p, 0 \leq v < p$, which expands with an average velocity of light in discrete steps at times $t = p^k$.

5.7.3 Convergence Radius For Square Root Function

In the following it will be shown that the convergence radius of $\sqrt{t+Z}$ is indeed non-vanishing for $p > 2$. The expression for the Taylor series of $\sqrt{t+Z}$ reads as

$$\begin{aligned} \sqrt{t+Z} &= = \sqrt{x}\sum_n a_n \ , \\ a_n &= (-1)^n \frac{(2n-3)!!}{2^n n!} x^n \ , \\ x &= \frac{Z}{t} \ . \end{aligned} \tag{5.7.3}$$

The necessary criterion for the convergence is that the terms of the power series approach to zero at the limit $n \to \infty$. The p-adic norm of the n: th term is for $p > 2$ given by

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{n!})N_p(x^n) < N_p(x^n)N_p(\frac{1}{n!}) \ . \tag{5.7.4}$$

The dangerous term is clearly the $n!$ in the denominator. In the following it will be shown that the condition

$$U \equiv \frac{N_p(x^n)}{N_p(n!)} < 1 \ for \ \ N_p(x) < 1 \ , \tag{5.7.5}$$

holds true. The strategy is as follows:
a) The norm of x^n can be calculated trivially: $N_p(x^n) = p^{-Kn}, K \geq 1$.
b) $N_p(n!)$ is calculated and an upper bound for U is derived at the limit of large n.

p-Adic norm of $n!$ for $p > 2$

Lemma 1: Let $n = \sum_{i=0}^{k} n(i)p^i$, $0 \leq n(i) < p$ be the p-adic expansion of n. Then $N_p(n!)$ can be expressed in the form

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} N(i)^{n(i)} \ , \\ N(1) &= \frac{1}{p} \ , \\ N(i+1) &= N(i)^{p-1}p^{-i} \ . \end{aligned} \tag{5.7.4}$$

An explicit expression for $N(i)$ reads as

$$N(i) = p^{-\sum_{m=0}^{i} m(p-1)^{i-m}} \ . \tag{5.7.5}$$

Proof: $n!$ can be written as a product

$$\begin{aligned} N_p(n!) &= \prod_{i=1}^{k} X(i, n(i)) \ , \\ X(k, n(k)) &= N_p((n(k)p^k)!) \ , \\ X(k-1, n(k-1)) &= N_p(\prod_{i=1}^{n(k-1)p^{k-1}} (n(k)p^k + i)) = N_p((n(k-1)p^{k-1})!) \ , \\ X(k-2, n(k-2)) &= N_p(\prod_{i=1}^{n(k-2)p^{k-2}} (n(k)p^k + n(k-1)p^{k-1} + i) \ \ ,) \\ &= N_p((n(k-2)p^{k-2})!) \ , \\ X(k-i, n(k-i)) &= N_p((n(k-i)p^{k-i})!) \ . \end{aligned} \tag{5.7.1}$$

The factors $X(k, n(k))$ reduce in turn to the form

$$\begin{aligned} X(k, n(k)) &= \prod_{i=1}^{n(k)} Y(i, k) \ , \\ Y(i, k) &= \prod_{m=1}^{p^k} N_p(ip^k + m) \ . \end{aligned} \tag{5.7.1}$$

The factors $Y(i, k)$ in turn are indentical and one has

$$\begin{aligned} X(k, n(k)) &= X(k)^{n(k)} \ , \\ X(k) &= N_p(p^k!) \ . \end{aligned} \tag{5.7.1}$$

The recursion formula for the factors $X(k)$ can be derived by writing explicitely the expression of $N_p(p^k!)$ for a few lowest values of k:
1) $X(1) = N_p(p!) = p^{-1}$.
2) $X(2) = N_p(p^2!) = X(1)^{p-1}p^{-2}$ ($p^2!$ decomposes to $p-1$ products having same norm as $p!$ plus the last term equal to p^2.
i) $X(i) = X(i-1)^{p-1}p^{-i}$

Using the recursion formula repeatedly the explicit form of $X(i)$ can be derived easily. Combining the results one obtains for $N_p(n!)$ the expression

$$\begin{aligned} N_p(n!) &= p^{-\sum_{i=0}^{k} n(i)A(i)} \ , \\ A(i) &= \sum_{m=1}^{i} m(p-1)^{i-m} \ . \end{aligned} \tag{5.7.1}$$

The sum $A(i)$ appearing in the exponent as the coefficient of $n(i)$ can be calculated by using geometric series

$$\begin{aligned} A(i) &= (\frac{p-1}{p-2})^2(p-1)^{i-1}(1+\frac{i}{(p-1)^{i+1}}-\frac{(i+1)}{(p-1)^i}) \ , \\ &\leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \end{aligned} \tag{5.7.1}$$

Upper bound for $N_p(\frac{x^n}{n!})$ for $p>2$

By using the expressions $n=\sum_i n(i)p^i$, $N_p(x^n) = p^{-Kn}$ and the expression of $N_pn!$ as well as the upper bound

$$A(i) \leq (\frac{p-1}{p-2})^2(p-1)^{i-1} \ . \tag{5.7.2}$$

For $A(i)$ one obtains the upper bound

$$N_p(\frac{x^n}{n!}) \leq p^{-\sum_{i=0}^{k} n(i)p^i(K-(\frac{(p-1)}{(p-2)})^2(\frac{(p-1)}{p})^{i-1})} \ . \tag{5.7.2}$$

It is clear that for $N_p(x)<1$ that is $K\geq 1$ the upper bound goes to zero. For $p>3$ exponents are negative for all values of i: for $p=3$ some lowest exponents have wrong sign but this does not spoil the convergence. The convergence of the series is also obvious since the real valued series $\frac{1}{1-\sqrt{N_p(x)}}$ serves as a majorant.

5.7.4 $P=2$ Case

In $p=2$ case the norm of a general term in the series of the square root function can be calculated easily using the previous result for the norm of $n!$:

$$N_p(a_n) = N_p(\frac{(2n-3)!!}{2^n n!})N_p(x^n) = 2^{-(K-1)n+\sum_{i=1}^{k} n(i)\frac{i(i+1)}{2^{i+1}}} \ . \tag{5.7.3}$$

At the limit $n\rightarrow\infty$ the sum term appearing in the exponent approaches zero and convergence condition gives $K>1$, so that one has

$$N_p(Z) \equiv (N_p(det(Z)))^{\frac{1}{8}} \leq \frac{1}{4} \ . \tag{5.7.4}$$

The result does not imply disconnected set of convergence for square root function since the square root for half odd integers exists:

$$\sqrt{s+\frac{1}{2}} = \frac{\sqrt{2s+1}}{\sqrt{2}} \ , \tag{5.7.5}$$

so that one can develop square as a series in all half odd integer points of the p-adic axis (points which are ordinary p-adic numbers). As a consequence, the structure for the set of convergence is just the 8-dimensional counterpart of the p-adic light cone. Space-time has natural binary structure in the sense that each $N_p(t) = 2^k$ cylinder consists of two identical p-adic 8-balls (parallelpipeds as real spaces).

Chapter 6

p-Adic Physics: Physical Ideas

6.1 Introduction

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic and real space-time sheets of increasing size and increasing value of prime p. These surfaces are glued together using topological sum or join along boundaries bonds. Contrary to the original expectations, p-adic space-time regions represent "mind-stuff" rather than "matter" which is also present and represented by real and infinite-p p-adic regions. Thus p-adic provide "cognitive representations" for matter like regions and this is why their physics provides a way to understand real physics. If p-adic-to-real phase transitions are possible, one can understand why it is possible to assign p-adic prime even to real regions. In fact, the hypothesis that p-adic regions provide a cognitive model for real physics, poses very strong constraints on real physics.

There is a "holy trinity" of non-determinisms in TGD in the sense that there is the non-determinism associated with the quantum jumps, the classical non-determinism of the Kähler action and p-adic non-determinism. The non-determinism of quantum jumps can involve also a selection between various multi-furcations for various absolute minima of the Kähler action in which case it represents a genuine volitional act. p-Adic non-determinism in turn corresponds to the non-determinism of pure imagination with no material consequences. Also real space-time sheets with finite time duration are also possible and they might represent what might be called "sensory space-time sheets" as opposed to cognitive space-time sheets. Cognitive space-time sheets can be transformed to real ones in quantum jumps inducing change of control parameters of the polynomial defining space-time surface: if the change is such that the p-adic root is replaced with a real root, one can say that thought is transformed into action. The reverse of this process is the transformation of sensory input into cognition.

"Holy trinity" implies that it should be possible to determine the p-adic prime characterizing a given space-time region (or space-time sheet) by observing a large number of quantum time developments of this system. The characteristic p-adic fractality, that is the presence of time scales $T(p,k) = p^k T_p$, should become manifest in the statistical properties of the cognitive time developments which in should turn reflect the properties of the real physics since cognitive representations are in question. For instance, quantum jumps with especially large amplitude would tend to occur at time scales $T(p,k) = p^k T_p$. $T(p,k)$ could also provide series of characteristic correlation times. Needless to say, this prediction means definite departure from the non-determinism of ordinary quantum mechanics and only at the limit of infinite p the predictions should be identical. An interesting possibility is that $1/f$ noise [D1] is a direct manifestation of the classical non-determinism: if this is the case, it should be possible to associate a definite value of p to $1/f$ noise. Also transformations of the p-adic cognitive space-time sheets to real space-time sheets of a finite time duration and vice versa might be involved with the $1/f$ noise so that $1/f$ noise would be a direct signature of cognitive consciousness.

The "physical" building blocks of p-adic TGD, as opposed to the philosophical mathematical ones briefly summarized above, and in more detail in previous chapters, are spin glass analogy leading to the general picture about how finite-p p-adicity emerges from quantum TGD, the identification of elementary particles as CP_2 type extremals, and elementary particle black hole analogy. These building blocks have been present as stable pieces of theory from beginning whereas philosophical ideas and interpretations have undergone rather wild fluctuations during an almost last decade of p-adic TGD.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

6.2 P-Adic Numbers And Spin Glass Analogy

Spin glass phase decomposes into regions in which the direction of the magnetization varies randomly with respect to spatial coordinates but remains constant in time. What makes spin glass special is that the boundary regions between regions of different magnetization do not give rise to large surface energies. Spin glass structure emerges in two ways in TGD framework.

1. Spin glass behavior at the level of real physics is encountered in TGD framework because of the classical non-determinism of the Kähler action. The classical non-determinism of CP_2 type extremals represents the manifestation of the spin glass analogy at the level of elementary particle physics. In macroscopic length scales real physics spin glass analogy makes possible "real world engineering".

2. Spin glass behavior at the level of cognition is encountered because of the p-adic non-determinism and makes possible what might be called imagination or "cognitive engineering". The point is that any piecewise constant function has a vanishing p-adic derivative. Therefore any function of the spatial coordinates depending on a finite number of the pinary digits is a pseudo constant. The discontinuities of this kind in the field variables do not lead to infinite surface energies in the p-adic context as they would in the real context.

Spin glass energy landscape is characterized by an ultra-metric distance function. The reduced WCW CH_{red} consisting of the maxima of the Kähler function with respect to quantum fluctuating degrees of freedom and zero modes defines the TGD counterpart of the spin glass energy landscape. This notion makes sense only in real context since p-adic space-time regions do not contribute to the Kähler function and all p-adic configurations are equally probable. The original vision was that if the ultra-metric distance function in CH_{red} is induced from a p-adic norm, a connection between p-adic physics and real physics also at the level of space-time might emerge somehow. It seems however that the ultra-metricity of CH_{red} need not directly relate to the p-adicity at the space-time level which can be understood if p-adic space-time regions give rise to cognitive representations of the real regions.

Of course, it *might* be that the p-adic prime characterizing cognitive representation of a real region characterizes also the reduced WCW associated with the region in question (one must of course assume that the reduced WCW approximately decomposes into a Cartesian product of the reduced WCW s associated with real regions).

6.2.1 General View About How P-Adicity Emerges

In TGD classical theory is exact part of the quantum theory and in a well defined sense appears already at the level of the configuration space geometry: the definition of WCW Kähler metric [K43] associates a unique space-time surface to a given 3-surface. The vacuum functional of the theory (exponent of the Kähler function) is analogous to the exponent $exp(H/T_c)$ appearing in the definition of the partition function of a critical system so that the Universe described by TGD is quantum critical system. Critical system is characterized by the presence of two phases, which can be present in arbitrary large volumes. The TGD counterpart of this seems to be the presence of two kinds of 3-surfaces for which either Kähler electric or Kähler magnetic field energy dominates. These 3-surfaces have outer boundaries for purely topological reasons and these boundaries can be of a macroscopic size. Therefore it seems that 3-space should be regarded as what could be called topological condensate with a hierarchical, fractal like structure: there are 3-surfaces (with boundaries) condensed on 3-surfaces condensed on...... .

This leads to a radically new manner to see the world around us. The outer surfaces of the macroscopic bodies correspond to the boundaries of 3-surfaces in the condensate so that one can see the 3-topology in all its complexity just by opening one's eyes! A rather compelling evidence for the basic ideas of TGD if one is willing to give up the nebulous concept of "material object in topologically trivial 3-space" and to allow nontrivial 3-topology in macroscopic length scales. A second rather radical departure from the conventional picture of the 3-space is that 3-space is not connected in TGD Universe but contains arbitrary many disjoint components. In fact the actual Universe should consist of infinitely many 3-surfaces condensed on each other.

In two-dimensional critical systems conformal transformations act as symmetries and conformal invariance implies the Universality of critical systems. This suggests that one should try to find the generalization of the conformal invariance to higher dimensional, in particular, 4-dimensional case. If finally turned out that quaternion-conformal invariance realizes quantum criticality four 4-surfaces imbedded to 8-dimensional space. As a by product an explanation for space-time and embedding space dimensions results.

In this approach the p-adic regions of the space-time surface result dynamically. Space-time surface is defined by the vanishing condition of a polynomial of two quaternion-valued variables q and p. This condition gives p as a function of q. It can however occur that some components of p become complex numbers. They must be however real so that the solution fails to exist in the real sense. It might be however possible to perform the completion of the rational space-time surface to a p-adic space-time surface and for some values of the p-adic prime the series defining the power series representing $p = f(q)$ might converge to a number in some algebraic extension of the ordinary p-adic numbers. Even more general rational-adic topologies in which norm is power of a rational number are possible. p-Adic numbers would thus be very closely related with quaternion-conformal invariance and criticality.

p-Adic topologies form an infinite hierarchy and p-adic physics leads to a vision about many-sheeted space-time as a hierarchical structure consisting of p-adic 4-surfaces of increasing size and increasing value of prime p. These surfaces are glued together using topological sum operation. Contrary to the original expectations, this hierarchy is the hierarchy for the regions of space-time representing "mind-stuff" rather than "matter" which is also present and represented by real and infinite-p p-adic regions. p-Adic provide "cognitive representations" for matter-like regions and this is why their physics provides a way to understand real physics.

6.2.2 P-Adic Numbers And The Analogy Of TGD With Spin-Glass

The vacuum degeneracy of the Kähler action leads to precise spin glass analogy at the level of the WCW geometry and the generalization of the energy landscape concept to TGD context leads to the hypothesis about how p-adicity is realized at the level of the WCW . Also the concept of p-adic space-time surface emerges rather naturally.

Spin glass briefly

The basic characteristic of the spin glass phase [B12] is that the direction of the magnetization varies spatially, being constant inside a given spatial region, but does not depend on time. In the real context this usually leads to large surface energies on the surfaces at which the magnetization direction changes. Regions with different direction of magnetization clearly correspond non-vacuum regions separated by almost vacuum regions. Amusingly, if 3-space is effectively p-adic and if magnetization direction is p-adic pseudo constant, no surface energies are generated so that p-adics might be useful even in the context of the ordinary spin glasses.

Spin glass phase allows a great number of different ground states minimizing the free energy. For the ordinary spin glass, the partition function is the average over a probability distribution of the coupling constants for the partition function with Hamiltonian depending on the coupling constants. Free energy as a function of the coupling constants defines "energy landscape" and the set of free energy minima can be endowed with an ultra-metric distance function using a standard construction [A67].

Vacuum degeneracy of Kähler action

The Kähler action defining WCW geometry allows enormous vacuum degeneracy: any four-surface for which the induced Kähler form vanishes, is an extremal of the Kähler action. Induced Kähler form vanishes if the CP_2 projection of the space-time surface is Lagrange manifold of CP_2: these manifolds are at most two-dimensional and any canonical transformation of CP_2 creates a new Lagrange manifold. An explicit representation for Lagrange manifolds is obtained using some canonical coordinates P_i, Q_i for CP_2: by assuming

$$P_i = \partial_i f(Q_1, Q_2) \ ,$$

where f arbitrary function of its arguments. One obtains a 2-dimensional sub-manifold of CP_2 for which the induced Kähler form proportional to $dP_i \wedge dQ^i$ vanishes. The roles of P_i and Q_i can obviously be interchanged. A familiar example of Lagrange manifolds are $p_i = constant$ surfaces of the ordinary (p_i, q_i) phase space.

Since vacuum degeneracy is removed only by classical gravitational interaction there are good reasons to expect large ground state degeneracy, when system corresponds to a small deformation of a vacuum extremal. This degeneracy is very much analogous to the ground state degeneracy of spin glass.

Vacuum degeneracy of the Kähler action and physical spin glass analogy

Quite generally, the dynamical reason for the physical spin glass degeneracy is the fact that Kähler action has a huge vacuum degeneracy. Any 4-surface with CP_2 projection, which is a Legendre sub-manifold (generically two-dimensional), is vacuum extremal. This implies that space-time decomposes into non-vacuum regions characterized by non-vanishing Kähler magnetic and electric fields such that the (presumably thin) regions between the non-vacuum regions are vacuum extremals. Therefore no surface energies are generated. Also the fact that various charges and momentum and energy can flow to larger space-time sheets via wormholes is an important factor making possible strong field gradients without introducing large surfaces energies. From a preferred extremal of Kähler action one obtains a new one by adding arbitrary space-time surface which is vacuum extremal. Uniqueness of the absolute minima in the sense that real regions of space-time $X^4(X^3)$ are unique could be achieved by requiring that vacuum regions are p-adic and represent thus cognitive regions whereas real regions carry non-vanishing induced Kähler field.

The symplectic invariance of the Kähler action for vacuum extremals allows a further understanding of the vacuum degeneracy. The presence of the classical gravitational interaction spoils the canonical group $Can(CP_2)$ as gauge symmetries of the action and transforms it to the isometry group of CH. As a consequence, the $U(1)$ gauge degeneracy is transformed to a spin glass type degeneracy and several, perhaps even infinite number of maxima of Kähler function for given values of the zero modes, become possible. Thus locally, the space maxima of Kähler function should look like a union of copies of the space of zero modes. Given sheet has naturally as its boundary the 3-surfaces for which two maxima of the Kähler function coalesce or are created from single maximum by a cusp catastrophe. In catastrophe regions there are several sheets and the value of the maximum Kähler function determines which give a measure for the importance of various sheets. The quantum jumps selecting one of these sheets can be regarded as phase transitions.

In TGD framework classical non-determinism forces to generalize the notion of the 3-surface by replacing it with a sequence of space like 3-surfaces having time like separations such that the sequence characterizes uniquely one branch of multi-furcation. This characterization works when non-determinism has discrete nature. For CP_2 type extremals which are bosonic vacua, basic objects are essentially four-dimensional since M^4_+ projection of CP_2 type extremal is random light like curve. This effective four-dimensionality of the basic objects makes it possible to topologize Feynman diagrammatics of quantum field theories by replacing the lines of Feynman diagrams with CP_2 type extremals.

In TGD framework spin glass analogy holds true also in the time direction, which reflects the fact that the vacuum extremals are non-deterministic. For instance, by gluing vacuum extremals with a finite space-time extension (also in time direction!) to a non-vacuum extremal and deforming slightly, one obtains good candidates for the degenerate absolute minima. This non-determinism is expected to make the absolute minima of the Kähler action highly degenerate. The construction of S-matrix at the high energy limit suggests that since a localization selecting one degenerate maximum occurs, one must accept as a fact that each choice of the parameters corresponds to a particular S-matrix and one must average over these choices to get scattering rates. This averaging for scattering rates corresponds to the averaging over the thermodynamical partition functions for spin glass. A more general is that one allows final state wave functions to depend on the zero modes which affect S-matrix elements: in the limit that wave functions are completely localized, one ends up with the simpler scenario.

The real effective action is expected to be Einstein-Yang-Mills action for the induced gauge fields. This action does not possess any vacuum degeneracy. The space-time surfaces are certainly absolute minima of the Kähler action and EYM-action could take a dynamical role only in the sense that extremality with respect to classical part of EYM action selects one of the degenerate absolute minima of the Kähler action. On the other hand, the construction of S-matrix suggests that the choice of particular parameter values characterizing zero modes affects only the coupling constants and propagators of the effective Einstein-Yang-Mills theory, and that one must perform averaging over the predictions of these theories. Thus EYM action could at most fix a gauge.

p-Adic non-determinism and spin glass analogy

One must carefully distinguish between cognitive and physical spin-glass analogy. Cognitive spin-glass analogy is due to the p-adic non-determinism. p-Adic pseudo constants induce a non-determinism which essentially means that p-adic extrema depend on the p-adic pseudo constants which depend on a finite number of positive pinary digits of their arguments only. Thus p-adic extremals are glued from pieces for which the values of the integration constants are genuine constants. Obviously, an optimal cognitive representation is achieved if pseudo constants reduce to ordinary constants.

More precisely, any function

$$\begin{aligned} f(x) &= f(x_N) \ , \\ x_N &= \sum_{k\leq N} x_k p^k \ , \end{aligned} \tag{6.2.0}$$

which does not depend on the pinary digits x_n, $n > N$ has a vanishing p-adic derivative and is thus a pseudo constant. These functions are piecewise constant below some length scale, which in principle can be arbitrary small but finite. The result means that the constants appearing in the solutions the p-adic field equations are constants functions only below some length scale. For instance, for linear differential equations integration constants are arbitrary pseudo constants. In particular, the p-adic counterparts of the absolute minima (defined by the correspondence with infinite primes) are highly degenerate because of the presence of the pseudo constants. This in turn means a characteristic randomness of the spin glass also in the time direction since the surfaces at which the pseudo constants change their values do not give rise to infinite surface energy densities as they would do in the real context.

The basic character of cognition would be spin glass like nature making possible "engineering" at the level of thoughts (planning) whereas classical non-determinism of the Kähler action would make possible "engineering" at the level of the real world.

Localization in zero modes

The Kähler function defining WCW metric possesses infinite number of zero modes which represent non-quantum-fluctuating degrees of freedom. The requirement that physics is local at the level of zero modes implies that each quantum jump involves a localization in zero modes. This localization could be complete or in a region whose size is determined by the p-adic length scale hypothesis.

Localization would mean an enormous calculational simplification: functional integral reduces into ordinary functional integral over the quantum-fluctuating degrees of freedom and there is no need to integrate over the zero modes. The complete or partial localization in zero modes would explain why the world of conscious experience looks classical. Perhaps the complete localization is however too much to wish for: it could however be that one must use wave functionals in the zero modes only in the case that one is interested in a comparison of the transition rates associated with different values of zero modes rather than in transition rates with the condition that a localization has occurred to definite values of zero modes.

The functional integral over the fiber degrees of freedom can be approximated by a Gaussian integrals around maxima. Classical non-determinism would suggest the possibility of several maxima in fiber degrees of freedom but the symmetric space property of the fiber suggests that there is only single maximum of Kähler function. The existence of single maximum gives good hopes that the configuration space integration reduces effectively to Gaussian integration of free field theory.

6.2.3 The Notion Of The Reduced WCW

Quantum jumps occur with highest probability to those values of zero modes which correspond to the maxima of the Kähler function and a simplified description of the situation is obtained by considering the reduced WCW CH_{red} consisting of the maxima of Kähler function with respect to both zero modesandquantum fluctuating degrees of freedom.

The hypothesis that the space CH_{red} is an enumerable set is a natural first guess. In macroscopic length scales, one might indeed hope that the generation of Kähler electric fields reducing the vacuum degeneracy could imply a discrete degeneracy for the maxima of the Kähler action.

In elementary particle length scales this hypothesis fails and it is good to analyze the situation in more detail since it gives some about how complex the situation can be. For the so called CP_2 type extremals the classical non-determinism gives rise to a functional continuum of degenerate maxima of the Kähler function. The degenerate maxima correspond to random zitterbewegung orbits for which the "time parameter" u is an arbitrary function of CP_2 coordinates. In this case however zero modes characterizing light like random curve representing the zitterbewegung orbit behave exactly like conformal gauge degrees of freedom. The choice of the "time parameter" u however affects S-matrix elements: dependence is very weak and only through the volumes of the propagator lines determined by the selection of u (Kähler action for CP_2 type extremal is proportional to its volume) occurring in quantum jump. Effectively the functional continuum is replaced with the real continuum of the volume of the propagator line varying from zero to the volume of CP_2.

A localization for the positions of the vertices of the Feynman diagrams defined by CP_2 type extremals cannot however be assumed. Neither can one assume that only single Feynman diagram is selected if one wants that a generalization of ordinary Feynman diagrammatics results. There are several alternative identifications.

1. The degrees represented by Feynman diagrams with varying positions of vertices represent fiber degrees of freedom so that there would be slight dependence of the Kähler function on the positions of the vertices. Certainly the Feynman diagrams with different topologies have different value of Kähler action and must correspond to fiber degrees of freedom. The reason is that vertex regions of the Feynman diagrams must involve deformations of CP_2 extremals since otherwise Feynman diagrams are singular as 4-manifolds. Note that the idea about localization in fiber degrees of freedom is not favored by this example.

2. The positions for the vertices of the Feynman diagram are excellent candidates for zero modes and localization is not possible now. The fact that these degrees of freedom correspond to center of mass degrees of freedom related to the isometries of the theory might distinguish between them and other zero modes. One can consider also a refinement for localization in the zero modes hypothesis: localization occurs only in length scale resolution defined by the p-adic length scale. In fact, the assumption that CP_2 type extremals have suffered topological condensation on space-time sheets with size of order p-adic length scale characterizing the elementary particle implies this.

Whether the notion of CH_{red} makes sense for the p-adic space-time regions is not at all obvious. For the proposed construction of the WCW metric p-adic regions do not contribute to the Kähler function which is real-valued. Only in case that the p-adic contribution is rational number, it could be interpreted as a real valued contribution to the Kähler function. In case of CP_2 type extremals this is not the case although the exponent of the Kähler function for a full

CP_2 type extremal is a rational number if the proposed model for the p-adic evolution of Kähler coupling strength is correct. If it does not make sense to distinguish between the maxima of the Kähler function in the p-adic context, one cannot define CH_{red} on basis of this criterion. From the point of view of cognition this means maximal freedom of imagination.

An interesting question is whether one must count the cognitive degeneracy as a degeneracy of physical states. If localization occurs in each quantum jump with respect to both real and p-adic zero mode degeneracy, and if all cognitive options are equally probable, then the only conclusion seems to be that space-time surfaces for which the cognitive degeneracy is highest, represent the most probable final states. This would mean that the systems with the highest cognitive resources would be winners in the struggle for survival.

Explicit definition of the ultra-metric distance function for energy landscape

The points of CH_{red} are completely analogous to the minima of the free energy and the precise analogy with spin glass suggests that CH_{red} must possess naturally an ultra-metric topology. One can quite generally construct an explicit ultra-metric distance function for the set of energy minima in a given energy landscape describing energy as a function of the coordinates of some WCW using existing recipes [B27]. The concept is useful when the energy landscape has fractal like structure. An attractive metaphor is to regard energy as a height function for a landscape with mountains.

The distance function between two energy minima should describe the difficulty of getting from a given minimum to another one. A concrete measure for this difficulty is obtained by considering all possible paths from x to y. The height for the highest point on this path, absolute maximum $h_{max}(\gamma)$ of the height function on this path gives the measure for the difficulty for reaching y along the path γ. There exists some easiest path from x to y. The difficulty to reach y from x can be defined as the height of the highest point associated with the easiest path and hence the minimum of $h_{max}(\gamma)$ in the set of all possible paths from x to y:

$$d(x,y) = Min(h_{max}(\gamma(x,y)) \ .$$

It is easy check that this distance function is ultra-metric:

$$d(x,z) \leq Max\{d(x,y),d(y,z)\} \ .$$

All what is needed is to notice that for any path $x \rightarrow z$ going through y highest point of the path is either the highest point associated with the path from $x \rightarrow y$ or $y \rightarrow z$: from this the inequality follows trivially since one can in principle find also easier paths.

Identification of the height function in the case of the reduced WCW ?

Obviously the negative for the maximum of Kähler function as function of zero modes is the counterpart of free energy. This function could well be many valued but this is an unessential complication. It is not clear whether K is negative definite (there are strong reasons to believe that this is the case). One can however consider any positive definite function of K as a height function defining an ultra-metric norm in the manner suggested. The requirement that p-adic norm results should fix the definition uniquely.

The exponential $exp(-K_{max})$ of the maximum of Kähler function as function of the zero modes, which is the inverse for the vacuum functional of the theory, is the first guess for the height function defining the ultra-metric norm (the wandering from 3-surface X^3 to Y^3 corresponds to quantum tunnelling physically.). The justification for this identification is that the integration over the fiber degrees of freedom gives Gaussian determinant cancelling the metric determinant and leaves on the exponent of Kähler function to the functional integral over zero modes. The intuitive expectation is that ultra-metric norm is p-adic for some p and that the space of zero modes decomposes into regions D_p In order to get a power of p as required by p-adicity, one can expand h as powers of p and identify p-adic norm as p^n for the highest pinary digit n with non-vanishing coefficient.

The height function can have a normalization factor and this factor could be chosen so that the ultra-metric norm is a power of p for CP_2 type extremals, which are certainly very important building blocks of preferred extremals The argument relating the gravitational coupling constant to the Kähler coupling strength and fixing the dependence of the Kähler coupling strength on the prime p, suggests that one must define the height function as

$$h_p = \frac{exp(-K(p))}{exp(-K(p=1))} \quad ,$$

where the Kähler function at $p = 1$ is formally obtained by regarding the value of the Kähler coupling strength as a function in the set of all natural numbers.

Does the proposed height function h_p define p-adic topology?

The great question is whether one can obtain p-adic ultra-metricity in this manner. There is some evidence for this.

1. Criticality and spin glass analogy suggests that $exp(K)$ as a function of zero modes is fractal. If it is p-adic fractal then p-adic topology is expected to be a natural consequence: in this case the map of CH_{red} to its p-adic counterpart could make it possible to replaced CH_{red} with a smooth function.

2. CP_2 type extremals, the counterparts of black holes and a model of elementary particle in TGD, have finite negative Kähler action. One can glue CP_2 type extremals to any space-time surface to lower the Kähler action. 3-surfaces Z^3 on path from X^3 to Y^3 containing CP_2 type extremals on $X^4(Z^3)$ are excellent candidates for "mountains" in the landscape metaphor. The height of Z^3 is roughly described by the number of CP_2 type extremals glued on $X^4(Z^3)$.

3. The argument leading to a correct prediction of gravitational constant in terms of assuming that Kähler coupling strength α_K depends on zero modes only through the p-adic prime assumed to characterize a given region D_p of WCW for which the set of maxima of Kähler function as function of zero modes should obey has p-adic topology. The crucial input is the relationship

$$exp(K_p(CP_2))\frac{R^2}{G} = \frac{1}{p} \quad ,$$

which is equivalent with $G = exp(K_p(CP_2)L_p^2$, where $L_p \simeq \sqrt{p} \times R$ is the p-adic length scale and $R \simeq 10^4\sqrt{G}$ is CP_2 size and the fundamental p-adic length scale. This formula is a dimensional estimate for gravitational coupling strength in terms of the p-adic length scale squared and the exponential of Kähler function for CP_2 type extremal describing graviton. The exponent gives the probability for the appearance of one virtual graviton in a given quantum state. The probability is very small since the exponent is negative for CP_2type extremal and gravitation is consequently a very weak interaction.

4. If one makes the identification

$$\frac{R^2}{G}(\sim 10^8) = exp(-K_{p=1}),$$

then the function

$$h_p = \frac{exp(-K_p)}{exp(-K_{p=1})}/$$

is the n: th power of p for a vacuum extremal to which n CP_2 type extremals are glued. This is just the p-adic norm p^n! If h_p were p^n-valued in the general case it would be a p-adic pseudo constant and rather tame as a fractal. Very probably, this is not true in the general case and the p-adic norm of the p-adic counterpart of h_p in the canonical identification

$$\begin{array}{l} N_p \equiv |Id(h_p)|_p \quad , \\ Id(\sum x_n p^n) = \sum_n x_n p^{-n} \quad . \end{array}$$

depending on the most significant pinary digit of h_p only, is a good candidate for a p-adically ultra-metric height function having also a correct normalization. In any case, it seems that the number of virtual CP_2 type extremals (gravitons!) glued to an preferred extremal $X^4(X^3)$ could define the height function. p-Adicity would emerge naturally and would have a direct physical meaning. Of course, this identification works for $n \geq 0$ only: the physical interpretation of the p-adic norm in $n < 0$ case is open.

A possible interpretation in terms of virtual graviton emission suggests the interpretation of the factor $\frac{R^2}{G} = exp(-K_{p=1})$ as a Gaussian determinant $\sqrt{det_G}$ associated with the integration over the zero modes around the maximum. The definition of Gaussian determinant in the real context is problematic and p-adicization plus adelic decomposition of the functional integral might provide a precise definition of $\sqrt{det_G}$. The divergence of the Gaussian determinant in the real context would lead to the vanishing of the gravitational constant. This picture is in accordance with the assumption that gravitational constant does not appear in quantum TGD as a fundamental constant and that the curvature scalar term in the low energy effective action essentially results from radiative corrections and hence derives from the logarithm of det_G.

6.3 P-Adic Numbers And Quantum Criticality

TGD Universe is quantum critical in the sense that the value of Kähler coupling constant is completely analogous to critical temperature. Therefore the obvious question is how p-adicity might relate to quantum criticality.

6.3.1 Connection With Quantum Criticality

p-Adicization of the reduced WCW relates in an interesting manner to quantum criticality. At quantum criticality the number of the absolute minima of Kähler action for a surface Y^3 belonging to light cone boundary measures the cognitive resources of this surface and of its diffeomorphs. N_d is assumed to behave as $N_d \sim exp(-K_{cr})$, where Kähler function is evaluated for the critical value α_{cr} of the Kähler coupling strength. α_{cr} is like Hagedorn temperature appearing in the thermodynamics of strings. Above α_{cr} the theory might not be mathematically well defined since (at least real) the sum over the WCW integrals associated with the maxima of Kähler function would diverge exponentially at the limit when the value of Kähler function increases. In string thermodynamics this corresponds to the growth of number $g(E)$ of the states of given energy more rapidly than the inverse of the Boltzmann factor $exp(-E/T_H)$. Below α_{cr} the theory is certainly well defined but in TGD framework the cognitive resources of the Universe would not be maximal since vacuum functional would differ significantly from zero for very few space-time surfaces only. At quantum criticality the situation is optimal but it is not clear whether the real theory makes sense at quantum criticality: at least in string thermodynamics the partition function diverges also at Hagedorn temperature.

The cognitive resources of p-adic space-time sheet are measured by the entropy type quantity $log(N_d)/log(2)$ having lower bound $log(p)/log(2)$ bits for the 3-surfaces allowed by the vacuum functional. For instance, the maximal cognitive resources of electronic space-time sheet ($M_{127} = 2^{127} - 1$) would be 127 bits. In TGD one must allow even infinite primes and for these cognitive resources can be literally infinite.

6.3.2 Geometric Description Of The Critical Phenomena?

The idea that critical systems might have a geometric description is not new. There is a lot of evidence that simple, purely geometric lattice models based on the bond concept reproduce same critical exponents as the thermal models [B25]. The probability for a bond to exist corresponds to temperature in these models. For example, in a bond percolation model it is possible to relate the critical exponents to various fractal dimensions. This provides a nice manner to reduce the problem of predicting critical temperature to that of predicting the critical probability for the bond. This problem is local and once the temperature dependence of the bond probability and critical bond probability are known one can calculate the critical temperature.

What is new that in TGD approach the concept of bond ceases to be a phenomenological concept related to the simple modelling of the critical systems. TGD predicts that the boundaries of 3-surfaces can have arbitrarily large sizes. Furthermore, the formation of the join along boundaries bonds connecting the boundaries of two disjoint 3-surfaces seems to provide the basic mechanism for the formation of macroscopic quantum systems with long range correlations. This means that phase transitions should basically correspond to changes in the connectedness of the boundary of the 3-space. The description of the super fluidity, super conductivity and Quantum Hall effect based on the join along boundaries bond concept is suggested in [K48, K4] and also other phase transitions might be describable in the same manner. In hadronic length scale flux tubes correspond to color flux tubes connecting valence quarks. In nuclear length scale the short range part of the nuclear force corresponds to the formation of join along boundaries bonds between nucleons.

p-Adic approach suggests a concrete description for the phase transition changing the connectedness of the 3-surface. Disjoint 3-surfaces are labelled by p-adic numbers, whose p-adic expansion does not contain powers p^n with $n > N$, where N is some finite integer: the larger the value of N the larger the degree of disjointness. This means that phase transitions (say evaporation or condensation) changing the connectedness of the 3-surface should correspond to transitions changing the value of N. In evaporation process N increases and in condensation process N decreases. Also catastrophic processes like the breaking of a solid object to pieces might correspond to increase in N. Typical self organization processes such as biological growth and healing might correspond to a gradual decrease of N.

Fractal like configurations with a discrete scale invariance are known to play important role in the description of the critical phenomena: they are the most probable configurations at the critical point. The idea that fractal corresponds to a fixed point of a discrete scaling transformation, is in accordance with the definition of the fractals as fixed points for a set of affine transformations acting on subsets of some metric space [A52]. A natural candidate for the discrete scaling transformation is the transformation of the 4-surface induced by the multiplication of the p-adic argument Z of H-coordinate $h(Z)$ by a power of p: $Z \rightarrow p^n Z$. A tempting idea is that most probable 3-spaces indeed are invariant under these scalings. This even suggests that something, which might be called "Mandelbrot cosmology", might provide a description of the Universe in all length scales as a 4-dimensional analog of Mandelbrot set. The breaking of the discrete scaling invariance is bound to occur, when one considers finite subsystem instead of the whole Universe. p-Adic cutoff might provide an elegant description for the breaking of the exact scaling invariance: 3-surface in question depends on finite number of the pinary digits of Z only.

6.3.3 Initial Value Sensitivity And P-Adic Differentiability

Initial value sensitivity is one of the basic properties of the critical systems and implies unpredictability in practice. p-Adic differentiability seems to be related to this property in a very general manner. Consider a configuration of an initial value sensitive system, which can possess very high dimension. For definiteness, assume that the dynamics is described by some differential equations, which can be reduced to equations of first order for WCW coordinates X (we do not bother to write indices):

$$\frac{dX}{dt} = J(X) \ . \quad (6.3.1)$$

Space-time coordinate is a p-adic number one can assume that time coordinate is a p-adic number, too.

The purely p-adic feature of this differential equation follows from the fact that any function depending on a finite number of pinary digits of a p-adic number possesses a vanishing p-adic derivative! This implies that the integration constants are not just ordinary constants but functions of the p-adic number t depending on finite number of pinary digits of t! Obviously this implies classical non-determinism in long time scales! One can construct solutions of the differential equation in the form $X(t) = X_0(t) + X_1(t)$, where $X_0(t)$ depends on a finite number of pinary digits of the p-adic time t and equations reduce to

$$\frac{dX_1}{dt} = J(X_0 + X_1) \ . \quad (6.3.2)$$

Of course, one must be careful in defining what "finite number of pinary digits" means, when p-adic cutoff is actually present. The simplest integration constants depend on the p-adic norm of t (or on the lowest pinary digit of t) only.

The result is in accordance with the so called Slaving Principle [B21]. One can think that the dynamics in long time scales (low pinary digits of p-adic number t) is given by the integration constants having arbitrary dependence on these pinary digits and the dynamics in short length scales is determined by the differential equations in the "background" given by these time dependent integration constants.

Initial value sensitivity implies effectively non-deterministic behavior and p-adic numbers perhaps provide a possibility to describe it properly. The properties of the Kähler function suggests that the classical non-determinism might be in fact actual. The point is that the classical space time surface associated with a given 3-surface need not be unique. This surface is determined as a preferred extremal of the so called Kähler action and Kähler action possesses enormous vacuum degeneracy [K14]: the most general vacuum extremal has 2- dimensional CP_2 projection, which is so called Lagrange manifold possessing a vanishing induced Kähler form. Symplectic transformations and $Diff(M^4)$ act as exact dynamical symmetries of the vacuum extremals and $Diff(M^4)$ contains p-adically analytic transformations of M^4 as subgroup. It might well happen that those absolute minima, which are obtainable as small deformations of the vacuum extremals inherit the characteristic degeneracy of the vacuum extremals.

The classical macroscopic non-determinism might be essential to the possibility of the quantum measurements. In TGD the state function reduction is described as "jump between histories" that is two deterministic time developments [K53]. In quantum measurement microscopic and macroscopic system are strongly correlated and microscopic transition induces a phase transition like phenomenon in a macroscopic critical system. The general belief is that quantum effects become unimportant in macroscopic systems. The situation need not be this if macroscopic system is critical, or even non-deterministic.

In the TGD inspired theory of "thinking systems", conscious thoughts correspond to quantum jumps selecting one of the possible time developments in the quantum superposition of several quantum average effective space-time times allowed by the non-determinism. p-Adic pseudo constants could provide a mathematical description for this non-determinism. These "cognitive" quantum jumps are certainly involved with a realistic description of a quantum measurement modelling also the presence of the observer quantum mechanically.

In turns out that quantum non-determinism, classical non-determinism of Kähler action and p-adic non-determinism are very closely related in quantum TGD: one could even speak of a holy trinity of non-determinisms. Quantum non-determinism corresponds closely to the classical non-determinism of Kähler action: quantum jumps select between various branches of the branches of multi-furcations of classical space-time surface. The p-adic counterparts of these branches are in turn obtained by varying pseudo constants in the solution of the p-adic Euler-Lagrange equations for the Kähler action: this requirement in fact makes it possible to assign unique p-adic prime to a given, sufficiently small space-time region.

6.3.4 There Are Very Many P-Adic Critical Orbits

An interesting connection between the p-adicity and initial value sensitive systems is related to the possibility to replace also the WCW (possibly infinite dimensional) with an algebraic extension of the p-adic numbers. The underlying motivation is the need to get a proper mathematical description of the finite accuracy for the observables and p-adic cutoff provides this description.

This in turn suggests Universality in some aspects of the dynamical behavior. The dynamical equations $dX/dt = J(X)$ define a flow that is a diffeomorphism $X \rightarrow F(X,t)$ of WCW . This flow contains as integration constants arbitrary functions of the p-adic time coordinate t depending on a finite number of pinary digits of t so that classical non-determinism is present. By p-adic conformal invariance this diffeomorphism ought to be p-adically analytic map that is representable as a power series of the algebraically extended p-adic numbers x and t.

The p-adic analyticity of the dynamic diffeomorphism gives strong constraints on the properties of the dynamic map. A particularly interesting map is in this respect Poincare map. One can ask several interesting questions. How does the Universal behavior of one- dimensional and 2-dimensional analytic iterated maps generalize to the p-adic case? What do attractors look like?

What are the counterparts of Julia set and Mandelbrot set? What about routes to chaos? Could p-adic hypothesis provide deeper explanation for the fact that period doubling seems to be a rather general mechanism for the transition to turbulence. It might be possible to answer these questions since p-adic analyticity is very strong constraint on the behavior of the maps.

Already the study of the simplest p-adic complex maps reveal some surprises. The simplest map to study is the map $Z \rightarrow Z^n$ for any extension of p-adic numbers (dimension is arbitrary!). The repeller consists of the points p-adic norm equal to one. Due to the roughness of the p-adic topology, the real counterpart of the repeller is of same dimension as WCW itself so that the critical orbits form a set with a non-vanishing measure! For example, in the 2-dimensional case and for the 2-adic extension, the set of the critical orbits corresponds in the real plane to a square $(1/2, 1] \times (1/2, 1]$.

How do the small deformations of $Z \rightarrow Z^n$ of form $Z \rightarrow Z^n + \epsilon Z^m$ affect the set of the critical orbits? If the norm of the parameter ϵ is sufficiently small, the previous repeller belongs to the repeller also now. Also new points can appear in repeller. These considerations suggest that the repellers/attractors of the p-adically analytic maps have rather simple structure as compared to their real and complex counterparts. An interesting possibility is that in general case these sets are fractal like objects resembling the fractals associated with p-adic order parameters.

The fact that set of critical orbits is n-dimensional rather than $(n-1)$ or lower-dimensional in the p-adic case suggests an interesting physical interpretation in accordance with the general idea that p-adic topology corresponds to criticality. In ordinary situation these orbits are not very interesting because a small deformation spoils their criticality. In p-adic case the situation is different since the critical orbits are meta-stable and their are very many of them. In TGD one can even identify good candidates for the set of of these meta-stable critical orbits as small deformations of the vacuum extremals of the Kähler action. Needless to emphasize, this vacuum degeneracy is a phenomenon not encountered in the standard field theories.

6.4 P-Adic Slaving Principle And Elementary Particle Mass Scales

The understanding of the elementary particle mass scales is a fundamental problem in the unified field theories. The attempts to understand the generation of the mass scales dynamically have not been successful. The basic problem is the fine tuning difficulty: the predicted mass scale hierarchy is not stable under the small changes of the model parameters. A possible explanation for the failure is that the fundamental mass scales are really fundamental and therefore cannot depend on the details of the dynamical model.

Criticality is known to imply Universality and criticality indeed is the fundamental property of Kähler action. Therefore the derivation of the elementary particle length scale(s) should be based on a proper formulation of the criticality concept. p-Adic numbers indeed provide a promising tool in this respect and the following arguments show that it is possible not only to understand some general elementary particle length scale but leptonic, hadronic and intermediate gauge boson length scales plus a small number of shorter length scales in terms of primes near prime powers of two. The most important length scales correspond to Mersenne primes: there are only sixteen Mersenne primes below electron length scale and the remaining Mersenne primes correspond to super astronomical length scales.

What is nice that the p-adic hypothesis makes possible to express these length scales as square roots of Mersenne primes and possibly Fermat primes, that is prime numbers of type $p = 2^m \pm 1$. What is amusing is that Mersenne primes are closely related to the so called Perfect Numbers $n = 2^{m-1}(2^m - 1)$ representable not only as a product of their prime factors but also as a sum of their proper divisors. The ancient number mystics believed that this property makes these numbers very exceptional in the World Order!

6.4.1 P-Adic Length Scale Hypothesis

p-Adic length scale hypothesis has served as a basic hypothesis of p-adic TGD for several years. This hypothesis states that the scales $L_p = \sqrt{p}l$, $l = 1.376 \cdot 10^4 \sqrt{G}$ are fundamental length scale at p-adic condensate level p. The original interpretation of the hypothesis was following:

1. Above the length scale L_p p-adicity sets on and effective course grained space-time topology is p-adic rather than ordinary real topology.

2. The length scale L_p serves as a p-adic length scale cutoff for the field theory description of particles. This means that space-time begins to look like Minkowski space so that quantum field theory $M^4 \rightarrow CP_2$ becomes a realistic approximation. Below this length scale string like objects and other particle like 3-surfaces dominate.

3. It is un-natural to assume that just single p-adic field would be chosen from the infinite number of possibilities. Rather, there is an infinite number of cutoff length scales. To each prime p there corresponds a cutoff length scale L_p above which p-adic quantum field theory $M^4 \rightarrow CP_2$ makes sense and one has a hierarchy of p-adic quantum field theories. These different p-adic field theories correspond to different hierarchically levels possibly present in the topological condensate. Hierarchical ordering $< p_1 < p_2 < ...$ means that only the surface $p_1 < p_2$ can condense on the surface p_2. The condensed surface can in practice be regarded as a point like particle at level p_2 described by the p-adic conformal field theory below length scale L_{p_2}.

The work with p-adic QFT has however demonstrated that the hypothesis a) and b) are probably wrong and the following interpretation is closer to the truth.

1. The length scale $L_p = \sqrt{p}l$ defines an *infrared* cutoff rather than ultraviolet cutoff for a p-adic quantum field theory formulated in terms of quarks and leptons and gauge bosons. For instance, for hadrons this length scale is of order hadron size and L_p defines UV cutoff for possibly existing field theory describing hadrons as basic objects. Above L_p real topology effectively replaces the p-adic one (real continuity implies p-adic continuity) and if length scale resolution L_p is used real physics is excellent approximation.

2. p-Adic QFT is free of UV divergences with any UV cutoff and there is no need to assume that p-adicity fails below some length scale. Rather, p-adicity is completely general property of the effective quantum average space-time defined by the Quantum TGD, which is based on the real number field. The concept of the effective space-time, or topological condensate, is in turn necessary for the formulation of field theory limit of TGD. The analogy of Quantum TGD with spin glass phase gives strong support for the p-adic topological condensate consisting of p-adic regions with different p glued together along their boundaries.

p-Adic topologies form a hierarchy of increasingly coarser topologies. The p-adic norm $N(x_p)$ defines a function of a real argument via the canonical identification of the non-negative real numbers and p-adic numbers. The p-adic norm is same as ordinary real norm for $x = p^k$ and is constant at each interval $[p^k, p^{k+1})$. This means that

1. p-adic topologies are coarser than real topologies so that the functions, which are continuous in the p-adic topology need not be continuous in the real topology.

2. p-adic topologies are ordered: the larger the value of p, the coarser the topology in the long length scales. In short length scales the situation is just the opposite.

6.4.2 Slaving Principle And P-Adic Length Scale Hypothesis

Slaving Principle states that there exists a hierarchy of dynamics with increasing characteristic length (time) scales and the dynamical variables of a given length scale obey dynamics, where the dynamical variables of the longer length (time) scale serve as "masters" that is effectively as external parameters or integration constants. The dynamics of the "slave" corresponds to a rapid adaptation to the conditions posed by the "master".

p-Adic length scale hierarchy suggests a quantitive realization of this philosophy.

1. By the previous considerations there is an infinite hierarchy of length scales L_p such that the space-time surfaces below the length scale L_p look like Minkowski space and p-adic quantum field theory $M^4 \rightarrow CP_2$ makes sense below the length scale L_p. These length scales are

associated with the different condensation levels present in the topological condensate and define the typical size of the p-adic surface in absence of the collective quantum effects, which should correspond to the formation of the flux tubes between objects with size of order L_p. The reason why the typical size is just this is that the embedding of the p-adic coordinate space into space H has strongest discontinuities in the real topology, when coordinate values correspond to powers of p so that a typical embedding decomposes into separate pieces with size of order L_p. Of course, this kind of discontinuity is possible for all powers of p but is not observable in shorter length scales for the physically most interesting values of p due to the extreme smallness of the corresponding length scales.

2. The lowest level of the hierarchy corresponds to 2-adic dynamics and this field theory makes sense below the cutoff length scale $L_2 = \sqrt{2}l$ defining the typical size for a 2-adic surface. Solutions of the 2-adic field equations are non-deterministic due to the possibility of the integration constants depending on finite number of binary digits. The dependence on a finite number of positive bits of the real coordinates only means that they are genuine constants below some length scale $L_2(lower) < L_2$, which in principle depends on the state of the system.

3. 2-adic pseudo-constants are analogous to external parameters and should be determined by the dynamics associated with the longer length and time scales. The properties of the p-adic numbers suggest that these constants in turn are p-adically differentiable functions of their argument with some value of $p_1 > 2$ determined by the p_1-adic dynamics describing the interaction between $p = 2$ surface condensed on $p = p_1$ level and $p = p_1$ background surface. The p_1-adic integration constants associated with these functions are actual constants above the length scale $L_{p_1}(lower) \geq L_2(lower)$ but also these in principle depend on a finite number of pinary digits and their values are determined by the interaction of p_1 level with the next level in the condensation hierarchy.

4. At the next level p_1 one encounters p_1-adic dynamics and new p-adic integration constants. The net effect is that one obtains a hierarchy of p-adic numbers $2 < p_1 < p_2 < ...$ in correspondence with the length and time scales $L_2 < L_{p_1} < L_{p_2} < ...$: the higher the boss the larger the p. In TGD it is very tempting to interpret the various levels of the slaving hierarchy as the levels of the topological condensate so that the surfaces at level p are condensed on the surfaces of level $p_1 > p$ (see **Fig. 6.1**). Not all values of p need be present in the hierarchy and it might well happen that certain values of p are in an exceptional position physically.

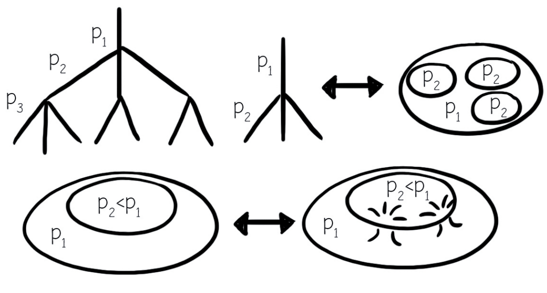

Figure 6.1: Two-dimensional visualization of topological condensate concept

6.4.3 Primes Near Powers Of Two And Slaving Hierarchy: Mersenne Primes

All values of p are in principle present in the Slaving Hierarchy but the assumption that all values of p are equally important physically is not realistic. The point is that the number $N(n)$ of primes smaller than n behaves as $N(n) \sim n/ln(n)$ and there are just too many prime numbers. For example, for $n = 10^{38}$ there are about one prime number per 87 natural numbers!

A natural looking assumption is that a new physically important length scale emerges, when a fixed number of powers of 2 combine to form a new length scale. The reason is that a given interval $[2^k, 2^{k+1})$ forms an independent fractal unit (for the simplest fractals these intervals are related by a similarity, see figures in [K63] and it is therefore unnatural to cut this unit into pieces as would happen if p were far from a power of two. This breaking would indeed happen since p-adically differentiable functions have sharp gradients at points p^k. This non-breaking or "synergy" is reached provided the allowed primes are as close as possible to powers of 2: $p \simeq 2^m$. It should be noticed that this condition also guarantees that the frequency peaks associated with various powers of p in good approximation correspond to period doubling frequencies characteristic to fractal and chaotic systems.

The best approximation achievable corresponds to Fermat and Mersenne primes

$$p = 2^m \pm 1 \ . \tag{6.4.1}$$

It can be shown that for Fermat primes (+) the condition $m = 2^k$ must be satisfied and for Mersenne primes (-) m must be itself prime.

How abundant are the prime numbers of type $p = 2^m \pm 1$? The great surprise was that there are very few numbers of this kind!

1. The primes of type $2^m + 1$, Fermat primes, are very rare: only 5 numbers in the range $1 < n < 2^{2^{21}} \simeq 10^{10^6}$(!) [A23] and there are good arguments suggesting that the number of the Fermat primes is finite! The known Fermat primes correspond to $m = 2^k$, with $k = 0, 1, 2, 3, 4$. The corresponding primes are $p = 3, 5, 17, 257, 65537$. Note that the lowest Fermat prime 3 is also a Mersenne prime. It will be later found that p-adic conformal invariance is in TGD possible for primes p satisfying the condition $p \ mod \ 4 = 3$ and this condition is not satisfied by Fermat primes $F > 3$.

2. The primes of form $2^m - 1$, Mersenne primes, are also there as follows from the requirement that m is prime. The list of allowed exponents of m consists of the following numbers:

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, ...$$

.

One can make two observations about these numbers:

1. $m = 127$ corresponds to the number 10^{38} fundamental to Physics. The square root of this number gives the ratio of the proton length scale to Planck length scale. This suggests the possibility that fundamental physical length scales are given by square roots of Mersenne and possibly Fermat primes using some length scale of order Planck scale as a unit.

2. $m = 61$ corresponds to the number of order 10^{19}: this in turn allows the possibility that fundamental physical length scales are linearly related to Fermat and Mersenne primes. This alternative however turns out to be not the correct one.

These observations lead to following scenario for the fundamental length scales:

1. The p-adic length scale L_p, below which p-adic quantum field theory approximation makes sense, is proportional to the square root of p and these length scales are p-adically the most interesting length scales:

$$L_p = \sqrt{p}l \ ,$$
$$l \sim k \cdot 10^4\sqrt{G} \ ,$$
$$k \simeq 1.376 \ . \qquad (6.4.0)$$

Only quite recently the physical interpretation of the length scale l was found. Contrary to the original expectations, CP_2 is not of order Planck length but of order l. At this length scale Euclidian regions of space-time, in particular CP_2 type extremals representing elementary particles, become important. Above this length scale a field theory in Minkowski space is expected to be a good approximation to quantum physics.

2. Physically the most interesting length scales correspond to the p-adic cutoff length scales L_p associated with the Mersenne primes M_n.

3. The fact that l is of the same order of magnitude as the length scale at which the coupling constants of the standard model become approximately equal, is not probably an accident. Below l it is not anymore sensible to speak about the topological condensation of CP_2 type extremals since CP_2 type extremals themselves have size of order l. Hence the symmetry breaking effects caused by the topological condensation cannot be present in the string model type description applying below l.

The predictions are as follows:

1. $m = 127$ corresponds to electron Compton length.

2. $m = 107$ corresponds to proton Compton length L_P.

3. $m = 89$ corresponds to length scale of order 1/256 times proton Compton length and is identifiable approximately as $L_W/2\sqrt{2}$, where L_W is intermediate boson length scale of about $L_P/100$.

4. $m = 61$ corresponds to length scale of the order of $10^{-6}L_P$ is not reachable by the present day accelerators.

5. $m = 521$ corresponds to a completely super-astronomical length scale of order 10^{27} light years!

It seems that the proposed scenario might have caught something essential in the problem of the elementary particle mass scales: it predicts correctly 3 fundamental length scales associated with leptons, hadrons and intermediate gauge bosons from number theory; there is extremely large gap in the length scale hierarchy after electron Compton length and new shorter length scales exist but unfortunately they are outside the reach of the present day experiments. The calculations of the third part of the book show that not only the mass scales can be understood but also particle masses can be predicted with errors below one per cent using the length scale hypothesis combined with the p-adic Super Virasoro invariance and p-adic thermodynamics.

6.4.4 Length Scales Defined By Prime Powers Of Two And Finite Fields

Above M_{127} there is an extremely large gap for Mersenne primes and this suggests that there must be also other physically important primes. Certainly all primes near powers of two define physically interesting length scales by 2-adic fractality but there are two many of them. The first thing, which comes into mind is to consider the set of primes near prime powers of two containing as special case Mersenne primes. The following argument is one of the many arguments in favor of these length scales developed during last years.

TGD Universe is critical at quantum level and criticality is related closely to the scaling invariance. This suggests that unitary irreducible representations of p-adic scalings $x \rightarrow p^m x$, $m \in Z$ should play central role in quantum theory. Unitarity requires that scalings are represented by a multiplication with phase factor and the reduction to a representation of a finite cyclic group

Z_m requires that scalings $x \rightarrow p^m x$, m some integer, act trivially. In ordinary complex case the representations in question correspond to the phase factors $\Psi_k(x) = |x|^{(\frac{ik2\pi}{ln(p)})} = exp(iln(|x|)\frac{k2\pi}{ln(p)})$, $k \in Z$ and the reduction to a representation of Z_m is also possible but there is no good reason for restricting the consideration to discrete scalings.

1. The Schrödinger amplitudes in question are p-adic counterparts of the ordinary complex functions $\Psi_k(x) = exp(iln(|x|)k\frac{ik2\pi}{ln(p)})$, $k \in Z$. They have a unit p-adic norm, they are analogous to plane waves, they depend on p-adic norm only and satisfy the scaling invariance condition

$$
\begin{aligned}
\Psi_k(p^m x|p \rightarrow p_1) &= \Psi_k(x|p \rightarrow p_1) \ , \\
\Psi_k(x|p \rightarrow p_1) &= \Psi_k(|x|_p|p \rightarrow p_1) \ , \\
|\Psi_k(x|p \rightarrow p_1)|_p &= 1 \ ,
\end{aligned} \tag{6.4.-1}
$$

   which guarantees that these functions are effectively functions on the set of the p-adic numbers with cutoff performed in m: th power.

2. The solution to the conditions is suggested by the analogy with the real case:

$$
\begin{aligned}
\Psi_k(x|p \rightarrow p_1) &= exp(i\frac{kn(x)2\pi}{m}) \ , \\
n(x) &= ln_p(N(x)) \in N \ ,
\end{aligned} \tag{6.4.-1}
$$

   where $n(x)$ is integer (the exponent of the lowest power of the p-adic number) and $k = 0, 1, ..., m-1$ is integer. The existence of the functions is however not obvious. It will be shortly found that the functions in question exist in $p > 2$-adic for all m relatively prime with respect to p but exist for all odd m and $m = 2$ in the 2-adic case.

3. If m is prime (!) the functions $K = \Psi_k$ form a finite field $G(m, 1) = Z_m$ with respect to the p-adic sum defined as the p-adic product of the Schrödinger amplitudes

$$
K + L = \Psi_{k+l} = \Psi_k \Psi_l \ , \tag{6.4.0}
$$

   and multiplication defined as

$$
KL = \Psi_{kl} \ . \tag{6.4.1}
$$

   Hence, if the proposed Schödinger amplitudes possessing definite scaling invariance properties are physically important, then the length scales defined by the prime powers of two must be physically special since Schrödinger amplitudes or equivalently, the p-adic scaling momenta k labeling them, have a natural finite field structure. By the Slaving Hierarchy Hypothesis, also the p-adic length scales near prime powers of two (and perhaps of prime $p > 2$, too) are therefore physically interesting. p-Adic scalings correspond to p-adic translations if p-adic coordinates correspond to exponentials of the ordinary linear coordinates so that translations are represented by scalings.

The generalized plane waves exist p-adically if nontrivial $N = p$: th root of the quantity $exp(i2\pi) = 1$ exists.

1. $N = 2$: th roots of 1 exist trivially for all values of p.

2. In 2-adic case the roots exist always for odd values of N and especially so for prime values of N: the trick is to write $1^{1/N} = -(-1)^{1/N} = -(1-2)^{1/N}$ and use the Taylor series

$$
\begin{aligned}
(1+x)^{1/N} &= \sum_n \frac{A_n}{n!}x^n \ , \\
A_n &= \prod_{k=0}^{n-1}(\frac{1}{N}-k)(-1)^n \ , \\
x &= -2 \ .
\end{aligned}
\tag{6.4.0}
$$

to show the existence of one root different from the trivial root. In 2-adic case the powers of $x = 2$ converge to zero rapidly and compensate the powers of 2 coming from $n!$ in the denominator. The coefficients A_n possess 2-adic norm not larger than 1.

3. For $p > 2$ nontrivial $N = p$: th roots do not allow representation as plane waves for the simple reason that only the trivial p: th root of 1 exists p-adically. Roots of unity must have p-adic norm equal to one and by writing the condition modulo p one obtains a condition $a^N \ mod \ p = 1$ in $G(p,1)$. The roots of unity in $G(p,1)$ satisfy always $a^{p-1} = 1$ and the possible orders N are factors of $p-1$. In particular, prime roots with $p_1 > p-1$ are not possible. The number of prime factors is typically quite small. For instance, for primes of order $p = 2^{127}$ the number of prime roots is of order 6.

The conclusion is that for $p > 2$ only those finite fields $G(p_1,1)$ for which p_1 is factor of $p-1$ are realizable as representation of phase factors whereas for $p = 2$ all fields $G(p_1,1)$ allow this kind of representation. Therefore $p = 2$-adic numbers are clearly exceptional. In the p-adic case the functions $\Psi_p(x,|p \rightarrow p_1)$ give irreducible representations for the group of p-adic scalings $x \rightarrow p^m x$, $m \in Z$ and the integers k can be regarded as scaling momenta. This suggests that these functions should play the role of the ordinary momentum eigenstates in the quantum theory of fractal structures. The result motivates the hypothesis that prime powers of two and also of p define physically especially interesting p-adic length scales: this hypothesis will be of utmost importance in future applications of TGD.

The ordinary (number theoretic) p-adic plane waves associated with the translations can be constructed as functions $f_k(x) = a^{kx}$, $k = 0,...,n$, $a^n = 1$. For $p > 2$ these plane waves are periodic with period n, which is factor of $p-1$ so that wavelengths correspond to factors of $p-1$ and generate a finite number of physically favored length scales. The p-adic plane waves with the momenta $k = 0,...,p-2$ form finite field $G(p,1)$, when p-adic arithmetics is replaced with the modulo p arithmetics, that is to accuracy $O(p)$ (note that the definition of the arithmetic operations is *not* the same as in the previous case). The square roots of the p-adic plane waves are also well defined

The important property of the p-adic plane waves is that they are pseudo constants: this property played profound role in the earlier formulations of the p-adic QFT limit. It took a considerable time to discover that the counterparts of the ordinary real plane waves providing representations for translation group exists and satisfy the appropriate orthogonality relations. Therefore number theoretic plane waves do not play so essential role in p-adic QFT as was originally believed.

6.5 CP_2 Type Extremals

CP_2 type extremals are perhaps the most important vacuum extremals of the Kähler action. The reason is that they are vacuum extremals with a negative and finite Kähler action and hence favored both by the absolute minimization of the Kähler action and criticality (randomness of light-like projection to M^4 implies criticality). It seems that also other identification of preferred extremals allow CP_2 type vacuum extremals and actually all known extremals. On the other hand, maximization of Kähler function does not favor CP_2 type extremals because the virtual CP_2 type extremals are exponentially suppressed. CP_2 type extremals seem to play the same role as black holes possess in General Relativity. p-Adic thermodynamics, leading to excellent predictions for the masses of the elementary particles, predicts that elementary particles should possess p-adic entropy and Hawking-Bekenstein law for the entropy generalizes.

In GRT based cosmology black holes populate the most probable Universe, which is of course a problem: in TGD black holes are replaced by elementary particles. The second law of thermodynamics requires that the very early Universe should have a low entropy and hence that black holes should populate the recent day Universe: in TGD the very early cosmology is dominated by cosmic strings, which is a low entropy state. The absolute minimization of the Kähler action would imply that most cosmic strings would decay to elementary particles and produce p-adic entropy. It is not clear whether also criticality implies this. To get a grasp of the orders of magnitude, it is good to notice that electron, which corresponds to $p = M_{127} = 2^{127} - 1$, has entropy equal to 127 bits.

The basic observation is that the M^4_+ projection of the CP_2 type extremal corresponds to a light like random curve and the quantization of this motion leads to Virasoro algebra and Kac Moody algebra characterizing quantized transversal motion superposed with the cm motion. CP_2 type extremals allow covariantly constant right handed neutrino spinors as solutions of the Dirac equation for the induced spinors in the interior and this leads to $N = 1$ super symmetry and a generalization of the Virasoro invariance to Super Virasoro invariance.

The previous p-adic mass calculations were based on this picture but it turned out that the Super Virasoro invariance and related Kac Moody symmetries generalize to the level of WCW geometry and in an extended form provide the basic symmetries of the quantum TGD. Although the quantization of the zitterbewegung motion of the CP_2 type extremals is a phenomenological procedure only, and is not needed in the fundamental theory, it deserves to be described because of its key role in the development of quantum TGD. There were however some strange features involved: for instance, $N = 1$ super-symmetry generated by right-handed neutrino was exact only for minimal surfaces.

The realization that super-symmetry requires Kähler-Dirac action led to the final breakthrough. CP_2 type extremals allow quaternion-conformal symmetries and the super-generators associated with quark and lepton numbers are non-vanishing despite the fact that vacuum extremals are in question. Even Super-Kac-Moody generators are non-vanishing. Even more, CP_2 type extremals cease to be vacua for Dirac action. Especially beautiful feature of CP_2 type extremals is that they can describe also massive states and zitterbewegung is the geometric correlate of massivation.

6.5.1 Zitterbewegung Motion Classically

The M^4_+ projection of a CP_2 type extremal is a random light like curve. Also Dirac equation, which gives also classically rise to a motion with light velocity and this motivates the term "zitterbewebung". Zitterbewegung occurs at the light of velocity and any given 3-velocity gives rise to the solution of light likeness condition if one fixes the time component of velocity to be

$$\frac{dm^0}{d\tau} = \sqrt{m_{ij}\frac{dm^i}{d\tau}\frac{dm^j}{d\tau}} \ . \tag{6.5.0}$$

The vanishing of CP_2 part of the second fundamental form requires that velocity and acceleration are orthogonal:

$$m_{kl}\frac{dm^k}{d\tau}\frac{d^2m^l}{d\tau^2} = 0 \ . \tag{6.5.1}$$

This condition is identically satisfied.

A very general solution to the conditions is provided by the equations

$$\frac{d^2m^k}{d\tau^2} = F^{kl}\frac{dm^l}{d\tau} \ , \tag{6.5.2}$$

describing the motion the of massless charged particle in external Maxwell field.

6.5.2 Basic Properties Of CP_2 Type Extremals

CP_2 type extremal has the following explicit representation

$$m^k = f^k(u(s^k)) \ , \quad m_{kl}\frac{df^k}{du}\frac{df^l}{du} = 0 \ . \tag{6.5.3}$$

The function $u(s^k)$ is an arbitrary function of CP_2 coordinates and serves effectively as a time parameter in CP_2 defining a slicing of CP_2 to time=constant sections. The functions f^k are arbitrary apart from the restriction coming from the light likeness. When one expands the functions f^k to Fourier series with respect to the parameter u, light likeness conditions reduce to classical Virasoro conditions $L_n = 0$.

It is possible to write the expression for m^k in a physically more transparent form by separating the center of mass motion and by introducing p-adic length scale L_p as a normalization factor.

$$\frac{m^k}{L_p} = m_0^k + p_0^k u + \sum_n \frac{1}{\sqrt{n}} a_n^k exp(i2\pi nu) + c.c. \ . \tag{6.5.4}$$

The first term corresponds to the center of mass term responsible for rectilinear motion along geodesic line and second term corresponds to the zitterbewegung motion. p^k serves as an effective classical momentum which can be normalized as $p_k p^k = \epsilon$, $\epsilon = \pm 1$ or $\epsilon = 0$. What has significance is whether p^k is time like, light like, or space like. Conformal invariance corresponds to the freedom to replace u with a new "time parameter" $f(u)$.

The physically most natural representation of u is as a function $f(U)$ of the fractional volume U for a 4-dimensional sub-manifold of CP_2 spanned by the 3-surfaces $X^3(U=0)$ and $X^3(U)$:

$$u = f(U) \ , \quad U = \frac{V(s^k)}{V(CP_2)} = \frac{S_K(u)}{S_K(CP_2)} \ . \tag{6.5.5}$$

The range of the values for U is bounded from above: $U \leq V_{max}/V(CP_2)$ and the value $U = 1$ is possible only if CP_2 type extremal begins and ends as a point. U represents also Kähler action using the value of the Kähler action for CP_2 as a unit.

The requirement that CP_2 type extremal extends over an infinite time and spatial scale implies the requirement

$$f(U_{max}) = \infty \ . \tag{6.5.6}$$

For $f(U_{max}) < \infty$ CP_2 type extremal can exist only in a finite temporal and spatial interval for finite values of "momentum" components p^k. This suggest a precise geometric distinction between real and virtual particles: virtual particles correspond to the functions $f(U_{max}) < \infty$ in contrast to the incoming and outgoing particles for which one has $f(U_{max}) = \infty$. This hypothesis, although it looks like an ad hoc assumption, is at least worth of studying.

The mere requirement that virtual CP_2 type extremal extends over a temporal or spatial distance of order $L > L_p$ implies that for $L < L_p$ the value of U is smaller than one. Kähler action, which is given by

$$S_K(X^4) = U \times S_K(CP_2) \ , \tag{6.5.7}$$

remains small for distances much smaller than L. For $f(U_{max}) = \infty$ this is even more true. This has an important implication: below a certain length scale the exponential of the Kähler action associated with the internal line of a Feynman diagram does not give rise to a suppression factor whereas above some characteristic length L and time scale there is an exponential suppression of the propagator by the factor $exp(-S_K(CP_2))$ practically hindering the propagation over distances larger than this length scale.

The presence of the exponential obviously introduces an effective infrared cutoff: this cutoff is prediction of the fundamental theory rather than ad hoc input as in quantum field theories.

Of course, infrared cutoff results also from the condition $f(U_{max}) < \infty$. Physically the infrared cutoff results from the topological condensation of the CP_2 type extremals to larger space-time sheets. These could correspond to massless extremals (MEs). p-Adic length scale L_p is an excellent candidate for the cutoff length scale in the directions transversal to ME.

The suppression factor coming from the exponent of the Kähler action implies a distance dependent renormalization of the propagators. In the long length scale limit the suppression factor approaches to a constant value

$$exp\left[-\frac{V_{max}}{V(CP_2)}S_K(CP_2)\right] \quad ,$$

and can be absorbed to the coupling constant so that the dependence on the maximal length of the internal lines can be interpreted as an effective coupling constant evolution. For instance, the smallness of the gravitational constant could be understood as follows. Since gravitons propagate over macroscopic distances, the virtual CP_2 type extremals develops a full Kähler action and there is huge suppression factor reducing the value of the gravitational coupling to its observed value: at short length scales the values of the gravitational coupling approaches to $G_{short} = L_p^2$ which means strong gravitation for momentum transfers $Q^2 > 1/L_p^2$. The values of V_{max} and thus those of the suppression factor can vary: only at the limit when CP_2 type extremal has point like contact with the lines it joins together, one has $V_{max} = V(CP_2)$. If the boundary component characterizing elementary particle family belongs to CP_2 type extremal (it could be associated with a larger space-time sheet), CP_2 type extremal contains a hole: also this reduces the maximal volume of the CP_2 type extremal.

6.5.3 Quantized Zitterbewegung And Super Virasoro Algebra

Calculating various Fourier components of right left hand side of the light likeness condition $m_{kl}p^kp^l = 0$ for $p^k = dm^k/du$ explicitly using the general expansion for m^k separating center of mass motion from zitterbewegung, one obtains classical Virasoro conditions

$$\begin{aligned} p_0^2 &= L_0 \ , \\ L_n|phys\rangle &= 0 \ , \ . \end{aligned} \qquad (6.5.7)$$

where L_n are defined by by their classical expressions as bi-linears of the Fourier coefficients. Therefore interior degrees of freedom give Virasoro algebra and zitterbewegung is more or less equivalent with the classical string dynamics.

It is not however not obvious whether a quantization of this dynamics is needed. If quantization is needed (perhaps to formulate the unitarity conditions in zero modes properly), it corresponds to the construction of the bosonic wave functionals in zero modes defined by the zitterbewegung degrees of freedom. Quantization could be carried out in the same manner as in string models.

The simplest assumption motivated by the Euclidian metric of CP_2 type extremal is that the commutator of p^k and m^k is proportional to a delta function as in ordinary quantization. One can Fourier expand m^k and p_k in the form

$$\begin{aligned} m^k &= m_0^k + p_0^k s + \frac{1}{K}\sum \frac{1}{n}a_n^{k,\dagger}exp(inKs) + \sum \frac{1}{n}a_n^k exp(-inKs) \ , \\ p^k &= p_0^k + i\sum a_n^{k,\dagger}exp(inKs) - i\sum a_n^k exp(-inKs) \ . \end{aligned} \qquad (6.5.7)$$

Here cm motion has been extracted and the formula is identical with the formula expressing the motion for a fixed point of string. The parameter K is Kac Moody central charge. Note that the exponents $exp(iKns)$ exist provided that Ks is p-adically of order $O(p)$ or, if algebraic extension by introducing $\sqrt{p}$ is allowed, of order $O(\sqrt{p})$.

The commutator of p_i and m^j is of the standard form if the oscillator operators obey Kac-Moody algebra

$$\begin{aligned} \left[p_{i,0}, m_0^j\right] &= m_i^{\ j} \ , \\ Comm(a_{i,m}^\dagger, a_n^j) &= Km\delta(m,n)m_i^{\ j} \ . \end{aligned} \tag{6.5.7}$$

Here K appears Kac-Moody central charge, which must be integer in the real context at least.

Expressing the light likeness condition as quantum condition, one obtains an infinite series of conditions, which give the quantum counterparts of the Virasoro conditions

$$\begin{aligned} p_0^2 &= kL_0 \ , \\ L_n|phys\rangle &= 0 \ , \ n<0 \ . \end{aligned} \tag{6.5.7}$$

k is some proportionality constant. One can solve these conditions by going to the transverse gauge in which physical states are created by oscillator operators orthogonal to an arbitrarily chosen light like vector. What quantization means physically is that zitterbewegung amplitudes are constrained by a Gaussian vacuum functional. A good guess motivated by the p-adic considerations is that the width of the ground state Gaussian is given by a p-adic length scale L_p: this is achieved if m^k is replaced with m^k/L_p in the general expression for $m^k(u)$. The experience with string models would suggests that vacuum functionals might be crucial for the understanding of graviton emission.

6.5.4 Zitterbewegung At The Level Of The Kähler-Dirac Action

At the level of the Kähler-Dirac action zitterbewegung motion implies that the conserved momentum associated with CP_2 type extremal, besides being conserved and non-vanishing, is also time like. This means that zitterbewegung creates massive particles besides massless particles as well as off-mass-shell versions of both and Super Virasoro conditions imply the quantization of the mass squared spectrum.

This means that in quantum TGD Feynman diagrammatics is topologized in the sense that the lines of Feynman diagram correspond to CP_2 type extremals which in general performing zitterbewegung. The non-determinism of the CP_2 type extremals means that one obtains a sum over over all possible diagrams with vertices at arbitrary space-time locations just as in quantum field theory approach. What is so nice that the time-development operator associated with an individual line of the diagram is the exponent of the Hamiltonian operator identified as the Poincare energy associated with the modified Dirac action. This operator is that associated with a free theory and contains no nonlinear terms. Interactions result from criticality property of the extremals of Kähler action. In particular, one gets rid of the divergences of the interacting quantum field theories by the topologization of the Feynman diagrammatics.

6.6 Black-Hole-Elementary Particle Analogy

String models have provided considerable insights into black hole thermodynamics by reducing it to ordinary thermodynamics for stringy black holes [B22] although one still does not understand, which is the mechanism of the thermalization. In TGD context elementary particles are regarded as thermodynamical systems in p-adic sense. This is something new since the standard theories of particle physics describe elementary particles as pure quantum states. The resulting thermal description of the the particle massivation is extremely successful. The fact that one can associate a well defined entropy to an elementary particle, suggests an analogy between black holes and elementary particles and this analogy indeed exists in a quite precise form as will be found. It also leads to a partial explanation for the p-adic length scale hypothesis serving as the corner stone of the p-adic mass calculations. The identification of the CP_2 type extremal as a cognitive representation of elementary particle suggests that p-adic entropy characterizes information associated with a cognitive representation provided by CP_2 type extremal.

6.6.1 Generalization Of The Hawking-Bekenstein Law Briefly

In TGD elementary particles are modelled as so called CP_2 type extremals, which are surfaces with a size of order Planck length having metric with Euclidian signature. These vacuum surfaces are isometric with CP_2 itself and have a one-dimensional, random light like curve as the M^4_+ projection. A natural candidate for the TGD counterpart of the black hole horizon is the surface at which the Euclidian signature of the metric associated with the CP_2 type extremal is changed to the Minkowskian signature of the background space-time. The radius r of this surface is the crucial length scale for the topological condensation and the simplest guess is that it is of the order of the size of the CP_2 radius and hence of the fundamental p-adic length scale. The hope is that the generalization of the black hole thermodynamics, with r replacing the radius of the black hole horizon, could give this information.

p-Adic mass calculations indeed give the p-adic counterpart of the Hawking-Bekenstein formula $S \propto GM^2$ as an identity at p-adic level:

$$S_p = -\frac{1}{T_p}(M_p^2/m_0^2) \ ,$$

where $1/T_p = n$ is the the integer valued inverse of the p-adic temperature and the mass scale $m_0^2/3$ corresponds to unit p-adic number in the unit used. The peculiar looking sign of S_p does not have in the p-adic context the same significance as in real context since the real counterpart of S_p is positive. Although p-adic entropy and mass squared are linearly related, the real counterparts are not in such a simple relation. In case of massive particles the real counterpart of the entropy is in excellent approximation equal to $S = log(p)$ whereas the mass is of order $1/p$ (p is of order 10^{38} for electron!). For massless (or nearly massless) particles one has $S \leq log(p)/p$. The large difference between fermionic and photonic entropies does not favor pair annihilation and this suggests that matter antimatter asymmetry is generated thermodynamically. For instance, via the topological condensation of fermions and anti-fermions on different space-time sheets during the early cosmology.

The generalization of the Hawking-Bekenstein formula in the form of the area law $S = A/4G$ reads as

$$S = \frac{xA}{4l^2} \ ,$$

where the fundamental p-adic length scale $l \simeq 1.376 \cdot 10^4\sqrt{G}$ replaces Planck length $\sqrt{G}$ and x is a numerical constant near unity. The radius of the elementary particle horizon is in an excellent approximation given by $r(p) = \sqrt{\frac{log(p)}{\pi x}}l$. Particles are thus surrounded by an Euclidian region of the space-time with radius r. Thus the fundamental p-adic length scale l of order CP_2 size has a direct geometric meaning. For instance, in the energy scales below $1/l$ the induced metric of the space-time becomes Euclidian and it might be possible to describe particle physics using Euclidian field theory: essentially QFT in a small deformation of CP_2 would be in question. It is encouraging, that l is also the length scale at which the standard model couplings become identical and super symmetry is expected to become manifest.

The p-adic length scale hypothesis stating that the primes p near prime powers of two are the physically most interesting p-adic primes, is the cornerstone of p-adic mass calculations but there is no really convincing argument for why should it be so. The proportionality of r to $\sqrt{log(p)}$ suggests an explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime, one has $r \propto L(k)$ and if the numerical constant x is chosen to be $x = \frac{log(2)}{\pi}$, the radius of elementary particle horizon is in excellent approximation $r(p \simeq 2^k) = L(k)$. Note also that the area of the elementary particle horizon becomes quantized in multiples of prime. This suggests that the precise value of $p \simeq 2^k$ is such that this condition is satisfied optimally and that physics is k-adic below r and $p \simeq 2^k$-adic above r.

$M^4_+ \times CP_2$ allows the embedding of Schwartshild metric in the region below Schwartchild radius but the embedding fails for too small values of the radial variable [K104]. An interesting possibility is that black hole entropy is just the sum of the elementary particle entropies topologically condensed below the horizon. This would give $S_{TGD} \propto \sum m_i^2 < S_{GRT} \propto (\sum m_i)^2$. An interesting problem is related to the detailed definition of p-adic entropy: are the entropies of particles with same value of p additive as p-adic numbers or does the additivity hold true for the real counterparts of the p-adic entropies. A related question is whether it might be that also in case of black holes additivity holds true, not for the mass as it is usually assumed, but for the p-adic mass squared for a given p (in TGD inspired model of hadron this is true for quark masses). This could be understood as a result of strong gravitational interactions. The additivity with respect to mass squared would give an upper bound of order $10^{-4}/\sqrt{G}$ for the contribution of a given p-adic prime to the total mass. For instance, the total contribution of electrons to the mass would be always below this mass irrespective of the number of electrons!

6.6.2 In What Sense CP_2 Type Extremals Behave Like Black Holes?

CP_2 type extremals are in some respects classically black hole like objects since their metric is Euclidian. When this kind of surface is glued to Minkowskian background there must exist a two-dimensional surface, where the signature of the induced metric changes from the Minkowskian $(1,-1,-1,-1)$ to the Euclidian $(-1,-1,-1,-1)$. On this surface, which could be called elementary particle horizon, the metric is degenerate and has the signature $(0,-1,-1,-1)$. Physically elementary particle horizon can be visualized as the throat of the wormhole feeding the elementary particle gauge fluxes to the background space-time. Of course, one cannot exclude the presence of several wormholes for a given space-time sheet.

This surface indeed behaves in certain respects like horizon. Time like geodesic lines cannot go through this surface. The reason is that the square of the four velocity associated with the geodesic is conserved:

$$v_\mu v^\mu = 1 \ , \ 0 \ \text{or} \ -1 \ ,$$

depending on whether the geodesic is time like, light like or space like. Clearly, a time like geodesic cannot enter from the external world to the interior of the CP_2 type extremal. If a space like geodesic starts from the interior of the CP_2 type extremal it can in principle continue as a space like geodesic into the exterior. These analogies should not be taken too seriously: it does not make sense to identify particles orbits as geodesics in these length scales shorter than the actual sizes of particle.

These analogies suggest that Hawking-Bekenstein formula $S = A/4G$ relating black hole entropy to the area of the black hole horizon, might have a generalization to the elementary particle context with the radius of the elementary particle horizon replacing the black hole horizon. The unit of the area need not be determined by Planck length $\sqrt{G}$, it could be replaced by the fundamental p-adic length scale $l \sim 10^4\sqrt{G}$: this length scale indeed replaces Planck length as a fundamental length scale in TGD.

6.6.3 Elementary Particles As P-Adically Thermal Objects?

In the p-adic mass calculations elementary particles were assumed to be thermal objects in the p-adic sense. What is new that energy is replaced with mass squared and the thermalization is believed to result from the interactions of a topologically condensed CP_2 type extremal with the background space-time surface of a much larger size. The thermalization mixes massless states with Planck mass states and gives rise to particle massivation. Super Virasoro invariance − abstracted from the Virasoro invariance of the CP_2 type extremals − together with the general symmetry considerations based on the symmetries of $M^4_+ \times CP_2$, leads to the realization of the mass squared operator essentially as the Virasoro generator L_0 in certain representations of the Super Virasoro algebra constructed using the representations of various Kac Moody algebras associated with Lorentz group, electro-weak group and color group.

$-L_0$ takes thus the role of a Hamiltonian in the partition function:

$$exp(-H/T) \rightarrow p^{L_0/T_p} \ ,$$

where T_p is the p-adic temperature, which by number theoretic reasons is quantized to $1/T_p = n$, n a positive integer. Mass squared is essentially the thermal expectation of L_0. The real mass squared is the real counterpart of the p-adic mass squared in the canonical identification $x = \sum x_n p^n \rightarrow \sum x_n p^{-n} \equiv x_R$ mapping p-adics to reals. Assuming that elementary particles correspond to p-adic primes near prime powers of two, one obtains excellent predictions, not only for the mass scales of elementary particles but also for the particle mass ratios. For instance, electron corresponds to the Mersenne prime $M_{127} = 2^{127} - 1$.

It should be noticed that the real counterpart of the p-adic inverse temperature $1/T_p$ is naturally defined as

$$(\frac{1}{T_p})_r = (\frac{1}{T_p})_R log(p) \ ,$$

where $log(p)$ factor results from the definition of Boltzmann weights as powers of p rather than power of e. The real counterpart T_r of T_p can be identified as

$$T_r = \frac{1}{nlog(p)} \ . \quad (6.6.1)$$

One might wonder about whether the sign of T_p should be taken as negative since positive exponent of L^0 appears in the Boltzmann weights. The sign is correct; for the opposite sign T_r would be in good approximation equal to $\frac{1}{(p-n)log(p)}$, which is not consistent with the fact that physically temperature decreases when n increases.

As already explained, the new vision about p-adics and cognition forces to modify this early vision by interpreting CP_2 type extremals as cognitive representations of elementary particles rather than genuine elementary particles.

p-Adic mass squared

The thermal expectation of the p-adic mass squared operator is proportional to the thermal expectation of the Virasoro generator L_0:

$$\begin{aligned} M_p^2 &= k\langle L_0\rangle \ , \\ k &= 1 \ . \end{aligned} \quad (6.6.1)$$

The correct choice for the value of the rational number k is $k = 1$ as became clear in the recent reconstruction of the quantum TGD [K51].

The real mass squared M^2 is identified as

$$\begin{aligned} M^2 &= \frac{M_R^2 \pi^2}{l^2} \ , \\ l &\simeq 1.376 \cdot 10^4 \sqrt{G} \ , \end{aligned} \quad (6.6.1)$$

where l is the fundamental p-adic length scale and M_R^2 is the real counterpart of M_p^2 in the canonical identification. $\sqrt{G}$ is Planck length scale.

p-Adic entropy is proportional to p-adic mass squared

The definition of the p-adic entropy involves some number theory. The general definition

$$S = -p_n log(p_n) \ ,$$

in terms of the probabilities p_n of various states does not work as such since the e-based logarithm $log(p_n)$ does not exist p-adically. Since p-adic Boltzmann weights are integer powers of p it is natural to modify somehow the p-based logarithm $log_p(x)$ so that the resulting logarithm $Log_p(x)$ exists for any p-adic number and has the basic property

$$Log_p(xy) = Log_p(x) + Log_p(y) \ ,$$

guaranteeing the additivity of the p-adic entropy for non-interacting systems. The definition satisfying these constraints is

$$Log_p(x = \sum_{n \geq n_0} x_n p^n) \equiv n_0 \quad . \tag{6.6.2}$$

The lowest power in the expansion of x in powers of p fixes the value of the logarithm in the same way as it determines also the norm of the p-adic number. This leads to the definition of p-adic entropy as

$$S_p = -\sum_p p_n Log_p(p_n) \quad . \tag{6.6.3}$$

In p-adic thermodynamics the p-adic probabilities have the general form

$$p_n = \frac{p^{L_0(n)/T_p}}{Z} \quad .$$

Here $L_0(n)$ denotes the eigenvalue of the Virasoro generator L_0, which is integer. The partition function $Z = trace(p^{L_0/T_p})$ has unit p-adic norm if the ground state is massless, so that its p-adic logarithm vanishes in this case: $Log_p(Z) = 0$. This implies $Log_p(p_n) = Log_p(p^{L_0(n)/T_p}) = L_0(n)/T_p$ so that the p-adic entropy reduces to

$$S_p = \frac{1}{T_p}\langle L_0 \rangle \quad , \tag{6.6.4}$$

ane hence that the p-adic mass squared and p-adic entropy are proportional to each other

$$S_p = -\frac{1}{kT_p} M_p^2 \quad . \tag{6.6.5}$$

By noticing that the entropy for Schwartschild black hole is given by

$$S = 4\pi G M^2 \quad , \tag{6.6.6}$$

one finds that in the p-adic context the analog of the Hawking-Bekenstein formula indeed holds as an identity.

The proposed identification of the entropy is in accordance with the formula $dE = TdS$. In the p-adic context E should clearly be replaced by $\langle -L_0 \rangle$ and T by T_p. The differentials do not however make sense since the thermodynamical quantities are now discrete. Since only $\langle -L_0 \rangle$ and T_p appear as variables one could define

$$\langle -L_0 \rangle = T_p S_p \quad .$$

This definition gives $S_p = -\frac{1}{kT_p} M_p^2$ and is in accordance with the standard definition of the Shannon entropy. The definition for the real counterpart of the p-adic entropy is

$$S = log(p) S_R \quad .$$

The inclusion of $log(p)$-factor maximizes the resemblance with the usual Shannon entropy defined in terms of the e-based logarithm and makes it possible to compare the real counterpart of entropy with other kind of entropies.

The real counterparts of entropy and mass squared are not linearly related

Due to the delicacies related to the canonical identification, the real counterparts of entropy and mass squared differ drastically from each other and there is no simple relationship between the two quantities. The reason is that the vacuum expectation of $-L_0$ is of order $-np$ for particles having $T_p = 1$ and, essentially due to the presence of minus sign, one has $S_R(p) = 1$ in an excellent approximation, whereas the real counterpart of M_p^2 is of order n/p. For photon and other (nearly) massless bosons the entropy vanishes or is very small.

The fundamental difference in the thermal properties of fermions and massless bosons should have observable consequences. For instance, the annihilation of fermion-anti-fermion pair to massless particles means a considerable reduction of the p-adic entropy and would not be a favorable process thermodynamically. Thus the second law of thermodynamics would favor the presence of net fermion and anti-fermion number densities. For instance, fermions and anti-fermions could suffer a topological condensation on different space-time sheets to avoid annihilation during early cosmology or anti-fermions could even suffer topological evaporation as suggested in [**?**, **?**]. This in turn would lead to the generation of matter-antimatter asymmetry. It should be noticed that lare entropies are in accordance with the second law of thermodynamics.

Hawking-Bekenstein area formula in elementary particle context

Hawking-Bekenstein formula in the p-adic form $S_p \propto M_p^2$ holds true on basis of the previous considerations although ther are no hopes of deriving the area law from the first principles at this stage. Hawking-Bekenstein formula can be also written in the form

$$S = \frac{A}{4G} \ ,$$

relating black hole entropy to the area of the black hole horizon. One might hope that in the real context a generalization of the area law to the form

$$S = x\frac{A}{4L^2} \ ,$$

where L is some fundamental length scale analogous to the gravitational constant G and x is some numerical constant near unity, would hold true. Since the size of CP_2 defines the fundamental p-adic length scale and replaces $\sqrt{G}$ as a fundamental length scale in TGD, it is conceivable that L is of the order of the CP_2 size $l \sim 10^4\sqrt{G}$. The area in question would be most naturally the area of the elementary particle horizon, where the signature of the induced metric for the topologically condensed CP_2 type extremal changes from Euclidian to Minkowskian. It is well known that l is also the length scale at which the couplings of the standard model become identical and supersymmetry is expected to become manifest. This is what is expected since above cm energy $1/l$ one would have an Euclidian quantum field theory in CP_2.

The radius r of the elementary particle horizon is of order

$$r \simeq \sqrt{log(p)}L \ . \tag{6.6.7}$$

This means that the # contacts connecting the CP_2 type extremal to the background space-time are surrounded by an Euclidian region with a size of order L.

It is interesting to look for the detailed form of the Hawking-Bekenstein law for elementary particles. One obtains the following general relationship

$$\begin{aligned} S &\equiv log(p)S_R = log(p)(\langle\frac{-L_0}{T_p}\rangle)_R == Xlog(p)M_R^2 = X \times log(p)\frac{l^2}{\pi^2}M^2 \ , \\ X &\equiv \frac{M_R^2}{S_R} \ . \end{aligned} \tag{6.6.7}$$

For massive particles $X \sim p$ holds true. Hence the entropy is related by a factor $p \cdot 10^8$ to the corresponding black hole entropy:

$$\begin{aligned} S &= a^2 S_{BH} \ , \\ S_{BH} &= 4\pi G M^2 \\ a &= \sqrt{\frac{log(p)X}{4\pi^3}}\frac{l}{\sqrt{G}} \sim 10^4 \ , \\ l &\simeq 1.376 \cdot 10^4 \sqrt{G} \ . \end{aligned} \quad (6.6.5)$$

6.6.4 P-Adic Length Scale Hypothesis And P-Adic Thermodynamics

The basic assumption of p-adic mass calculations is that physically interesting p-adic primes correspond to prime powers of two:

$$p \simeq 2^k \ , \ k \text{ prime} \ .$$

There are several arguments in favor of this hypothesis but no really convincing argument. The area law however leads to a very attractive, if not even convincing, explanation of the p-adic length scale hypothesis.

The proportionality of the elementary particle horizon radius to $\sqrt{log(p)}$ suggests quite attractive partial explanation for the p-adic length scale hypothesis. The point is that for $p \simeq 2^k$, k prime one has $r \propto L(k)$. Thus, if the numerical constant x is chosen suitably, it is possible to obtain very precisely

$$r(p \simeq 2^k) = L(k) \ .$$

The reason is that the p-adic entropy is in thermal equilibrium very near to its maximum value. The required value of the coefficient x is

$$x = \frac{log(2)}{\pi} \ . \quad (6.6.6)$$

The requirement that r_F (r_B) is as near as possible to the appropriate p-adic length scale $L(k)$ ($L(k)\sqrt{p}$) fixes also the precise value of the p-adic prime $p \simeq 2^k$.

This hypothesis means that the area of the elementary particle horizon is quantized in the multiples of prime k:

$$A = kA_1 \ . \quad (6.6.7)$$

The quantization law for the area has been proposed also in the context of the non-perturbative quantum gravity. A suggestive possibility is that physics is k-adic below the elementary particle horizon and $p \simeq 2^k$-adic above it. The appearance of an additional k-adic length scale suggests that for $p \simeq 2^k$ the degeneracy of the effective space-time surfaces is especially large due to the additional k-adic degeneracy and that the p-adic scattering amplitudes are be especially large for this reason. Hence the favored p-adic primes would emerge purely dynamically.

It must be noticed that k-adic fractality allows also more general primes of type $p \simeq 2^{k^n}$, where k is prime and n is integer. For these primes the radius of the elementary particle horizon is $\sqrt{k^{n-1}}L(k)$ and hence also a natural k-adic length scale. There are very few physically interesting length scales of this type. As the p-adic mass calculations show, the best fit to the neutrino mass squared differences is obtained for $p_\nu \simeq 2^{13^2=169}$ rather than $p \simeq 2^{167}$. The length scale $L(p_\nu)$ is also the natural length scale associated with the double cell layers appearing very frequently in bio-systems ($k = 167$ corresponds to the typical size of a cell)!

6.6.5 Black Hole Entropy As Elementary Particle Entropy?

In TGD Schwartshild metric does not allow a global embedding as a surface in $M^4_+ \times CP_2$. One can however find embeddings, which extend also below the Schwartshild radius. This suggests that particles in the interior of the black hole are topologically condensed below the radius r_s. The problem is whether the single particle entropies are additive as real numbers or as p-adic numbers.

Additivity of real entropies?

Consider first the additivity as real numbers. With this assumption the sum for the real counterparts of the p-adic entropies of various particles gives a lower bound for the black hole entropy:

$$S = \sum_i S(i) = \sum_i k m_i^2 \ .$$

This entropy is by a factor is $10^8 \cdot p$ larger than the corresponding black hole entropy so that black hole-elementary particle analogy does not work at quantitative level. For sufficiently large particle numbers elementary particle entropy becomes smaller than the black hole entropy, which behaves as $(\sum m_i)^2$. In case of protons $p = M_{107} = 2^{107} - 1$ the critical value of N would be roughly $N \sim 10^{32}$, which would mean black hole with a mass of order 100 kilograms.

Additivity of the p-adic entropies?

One can consider also a different definition of the black hole entropy. In p-adic thermodynamics the natural additive quantity for many particle systems is the Virasoro generator L_0 (mass squared essentially) rather than energy. The additivity works quite nicely for the TGD based model of a hadron as a bound state of quarks. Therefore one could consider the possibility that also for black holes the mass squared of elementary particles with same value of p-adic prime p is p-adically additive

$$(m_p^2)_R = (\sum_i m_p^2(i))_R \ \text{ rather than } \ m = \sum m_i \ .$$

Therefore for a black hole containing only particles with single value of the p-adic prime p, the Hawking-Bekenstein formula in the form

$$S_p \propto M_p^2$$

would hold true. For the real counterparts this proportionality does not hold.

When the particle number N exceeds p/n, the mass squared of the system reduces from its upper bound $10^{-4}/\sqrt{G}$ by a factor of order $1/\sqrt{p}$. Thus the mass of, say, the electrons inside black hole, is always below this upper bound irrespective of the number of the electrons!

If particles with several p-adic primes are present inside the black hole then the formula for the black hole entropy reads as

$$S = \sum_p S(p) = \sum_p k(p) M^2(p) \ ,$$

so that the proportionality to the total mass squared does not hold true except approximately (in the case that the mass is in good approximation given by the total mass of a particular particle species).

6.6.6 Why Primes Near Prime Powers Of Two?

The great challenge of TGD is to predict the p-adic prime associated with a given elementary particle. The problem decomposes into the following subproblems.

1. One must understand why there is a definite value of the p-adic prime associated with a given real region of space-time surface (in particular, the space-time time surface describing elementary particle) and how this prime is determined. The new view about p-adicity allows to understand the possibility to label elementary particles by p-adic primes if p-adic–real phase transitions occur already at elementary particle level or if real elementary particle regions are accompanied by p-adic space-time sheets possible providing some kind of a cognitive model of particle. The latter alternative has turned out to be the correct one.

   The great question mark is the correlation of the p-adic prime characterizing the particle with the quantum numbers of the particle: is this correlation due to the intrinsic properties of the particle or perhaps a result of some kind of adaptation at elementary particle length scales. In the latter case sub-cosmologies with quite different elementary particle mass spectra are possible. On the other and, quantum self-organization does not allow too many final state patterns, so that elementary particle mass spectrum could be more or less a constant of Nature.

2. One must understand why quantum evolution by quantum jumps has led to a situation in which elementary particle like surfaces correspond to some preferred primes. It indeed seems that an evolution at elementary particle level is in question (how p-adic evolution follows from simple number theoretic consistency conditions is discussed in the [K39]. It seems that the degeneracy due to the p-adic space-time regions associated with the system must be counted as giving rise to different final states in a quantum jump between quantum histories. If the number $N_d(X^3)$ of the physically equivalent cognitive variants of the space-time surface is especially high, this particular physical state dominates over the other final states of the quantum jump. Highly cognitive systems are winners in the fight for survival. Thus in TGD framework evolution is also, and perhaps basically, evolution of cognition.

3. One should also understand why the primes $p \simeq 2^k$ near prime powers of two are favored physically and to predict the value of k for an elementary particle with given quantum numbers. The analogy between elementary particles and black holes suggests only a partial explanation for the prime powers of 2 and the real explanation should probably involve enhanced cognitive resources for these primes.

In order to formulate the argument supporting p-adic length scale hypothesis one must first describe the general conceptual background.

1. WCW of the 3-surfaces decomposes into regions D_P labelled by infinite p-adic primes. In each quantum jump localization of CH spinor field to single sector D_P must occur if localization in zero modes occurs. Quantum time development corresponds to a sequence of quantum jumps between quantum histories and the value of the infinite-p p-adic prime P characterizing the 3-surface associated with the entire universe increases in a statistical sense. This has natural interpretation as evolution. In a well defined sense the infinite prime characterizing infinitely large universe is a composite of finite p-adic primes characterizing various real regions (space-time sheets) of the space-time. The effective infinite-p p-adic topology associated with this infinite prime is very much like real topology since canonical identification mapping infinite number to its real counterpart just drops the infinitesimals of infinite-p p-adic number. Therefore real physics is an excellent approximation at this level. If the S-matrix is complex rational, the approximation is in fact exact. Note that real topology is quite possible also at the level of WCW and WCW might consist of both real and infinite-P p-adic regions.

2. The requirement that quantum jumps correspond to quantum measurements in the sense of QFT, implies that also localization in zero modes occurs in each quantum jump: localization could occur also in the length scale resolution defined by the p-adic length scale L_p. The strongest hypothesis suggested by the properties of thermodynamical spin glasses is that quantum jump occurs to a state localized around single maximum of the Kähler function.

3. This picture suggests that evolution has occurred already at the elementary particle level and selected preferred p-adic primes characterizing the space-time regions associated with the elementary particles. A crucial question is whether this evolution could have occurred for isolated elementary particles or whether the interaction of the elementary like space-time regions with the surrounding space-time has served as a selective pressure. It might well be that the latter option is the correct one. If this is the case, one can say that the winners in the fight for survival correspond to infinite primes, which are composites of preferred finite primes, perhaps the finite primes given by the p-adic length scale hypothesis.

4. In TGD framework evolution is also evolution of cognition and the most plausible guess is that p-adic non-determinism is what makes cognition possible. Of course, also the classical non-determinism of Kähler action is also present and also important. Perhaps one should call the space-time sheets of finite time duration made possible by this non-determinism

as "sensory space-time sheets" as opposed to p-adic space-time sheets. Certainly this non-determinism should be responsible for volition. In any case, the degenerate space-time sheets are not physically equivalent in this case as they are in case of the p-adic non-determinism. The number $N_d(X^3)$ of the p-adically degenerate and physically equivalent absolute minima $X^4(X^3)$ of Kähler action is the measure for the cognitive resources of the 3-surface. The basic idea is simple: if $N_d(X^3)$ is very large then quantum jumps lead with high probability to some degenerate physically equivalent maximum of the Kähler function associated with given value of p. One can see this also from the point of view of an elementary particle: the high cognitive degeneracy could mean that the particle can adapt to the environment - the surviving elementary particles would be the most intelligent ones! What one should be able to show is that cognitive degeneracy is especially large for some preferred primes so that evolution selects these primes as the most intelligent ones.

About two decades after writing the above lines I would formulate the same idea in terms of hierarchy of Planck constants $h_{eff} = n \times h$. Large h_{eff} assignable to the magnetic body of the particle means high intelligence and large negentropic reources. This can be made concrete by using concrete model for elementary particles as wormhole contacts connected by Kähler magnetic flux tubes carrying monopole flux at both space-time sheets involved. The magnetic flux tubes would correspond to large h_{eff}.

In this conceptual framework one can develop more precise variants for arguments supporting the p-adic length scales hypothesis.

1. The simplest possibility is that single maximum of Kähler function is selected in the quantum jump. In this case the relative rate for quantum jumps to a given physical final state with fixed physical configuration is proportional to the p-adic cognitive degeneracy $N_d(N)$, where N denotes the infinite primes characterizing the interacting space-time surface associated with the final state. N decomposes into a product of infinite primes p and $N_d(N)$ decomposes decomposes into a product $N = \prod_P N_d(P)$ $N_d(N)$ is maximized if $N_d(P)$ is maximzes. The elementary systems for which $N_d(P)$ is especially large are winners.

2. The situation reduces to the level of finite p-adic primes if takes seriously the argument allowing to estimate the value of the gravitational constant. The argument was based on the assumption that P decomposes in a well defined sense into passive primes p_i and active prime p characterizing elementary particle: thus there would be the correspondence $P \leftrightarrow p$. This suggests that it is possible to understand the finite p-adic prime p associated with the elementary particle by restricting the consideration to the 3-surfaces describing topologically condensed elementary particles: that is, CP_2 type extremals glued to a space-time sheet with size of order Compton length. p-Adic cognitive degeneracy $N_d(p)$ should be especially high for p-adic primes predicted by the p-adic length scale hypothesis.

3. The interpretation of p-adic regions as cognitive regions suggests a more concrete explanation for the p-adic length scale hypothesis. The degeneracy due to p-adic non-determinism for the p-adic CP_2 type extremals presumably depends on the value of the p-adic prime characterizing the cognitive version of elementary particle. If real and p-adic space-time sheets accompany each other as adelic vision assumes, one might understand the origin of the p-adic length scale hypothesis. p-Adic primes near prime powers of two are winners because the the degeneracy due to p-adic non-determinism is especially larger for them and allows cognitive representations with high information content. The observed elementary particles would thus dominate in the Universe simply because the thoughts about them are winners in the fight for survival.

   The question why particle correspond to preferred p-adic primes(s) must be also answered and here very general number theoretic explanation has emerged [K109]. Therefore the remaining question is why p-adic length scale hypothesis and why the correlation of p-adic prime with the quantum numbers and topology of the partonic 2-surface. Here the answer about two decades later would be that the value of Planck constant $h_{eff} = n \times h$ measuring the negentropy resources is very large for survivors.

4. The black hole-elementary particle analogy suggests that the primes $p \simeq 2^k$, k prime, are especially interesting since the radius of the elementary particle horizon is the p-adic length scale $L(k)$. This could be understood since k-adicity provides an additional cognitive degeneracy for the absolute minima of Kähler function coming from the region of size $L(k)$ surrounding a topologically condensed elementary particle and any # contact. This enhances the value of $N_d(p)$ further by a multiplicative factor $N_d(k)$ so that $N_d(P)$ becomes especially large.

5. These arguments do not yet tell how to deduce the prime k associated with a given elementary particle. Cognitive resources are measured by a negative on an negentropy type quantity proportional to $N_c = log(N_d(p))$. A natural guess is that N_c is dominated by a term proportional to $log(p)$: $N_c = A(p) + log(p)$. For $p \simeq 2^k$ one has an additional source of cognitive degeneracy which gives $N_c = log(k) + log(p)$ instead of $N_c = log(p)$ and these primes thus correspond to the local maxima of cognitive resources as a function of p. Quite generally, the larger the p, the more probable is its appearance as elementary particle prime (neglecting the constraints coming from, say, the cosmic temperature). Hence it seems that the p-adic evolution of a given elementary particle is frozen to some local maximum of $N_d(p(k))$, with $p(k)$ given by the p-adic length scale hypothesis.

6. Freezing can be understood if the transition probabilities $P(k \rightarrow k_1)$ are so small that further evolution by quantum jumps is impossible. A possible interpretation of the transition $k_i \rightarrow k_j$ is a p-adic phase transition changing the elementary particle horizon from radius L_{k_i} to L_{k_j} so that $P(k_i \rightarrow k_j)$ would describe the probability of this phase transition. For neutrinos the transition probabilities $P(k_i \rightarrow k_j)$ between different sectors allowed by the p-adic length scale hypothesis seem to be largest whereas for higher quark generations they seem to be smallest. Furthermore, k is smaller for higher generations. In particular, $P(k_i \rightarrow k_j)$ seems to be largest for spherical boundary topology. This suggests that the (phase) transition probabilities $P(k_i \rightarrow k_j)$ decrease as a function of the strength of the dominating particle interaction and of the genus of the particle (reflecting itself via the modular contribution to the particle mass increasing as a function of genus).

To sum up, the proposed formula would dictate the evolution of α_s from the evolution of the electro-weak parameters without any need for perturbative computations. Although the formula of proposed kind is encouraged by the strong constraints between classical gauge fields in TGD framework, it should be deduced in a rigorous manner from the basic assumptions of TGD before it can be taken seriously.

6.7 General Vision About Coupling Constant Evolution

Addition: The view about coupling constant evolution has changed radically during 2016-2017 [K35, K38, K10, L25] as the number theoretic vision about TGD as adelic physics and the vision about twistor lift of TGD have co-evolved. Number theoretic vision has extremely powerful consequences and has led to amazingly simple proposals for the scattering amplitudes and coupling constant evolution. The following rather old arguments making guesses about the values of coupling constants are in the light of the new vision obsolete so that this section can be regarded more or less as a curiosity. I have however decided to keep the section.

Zero energy ontology, the construction of M-matrix as time like entanglement coefficients defining Connes tensor product characterizing finite measurement resolution in terms of inclusion of hyper-finite factors of type II_1, the realization that symplectic invariance of N-point functions provides a detailed mechanism eliminating UV divergences, and the understanding of the relationship between super-symplectic and super Kac-Moody symmetries: these are the pieces of the puzzle whose combination making possible a rather concrete vision about coupling constant evolution in TGD Universe and even a rudimentary form of generalized Feynman rules.

p-Adic coupling constant evolution is discrete by p-adic length scale hypothesis justified by zero energy ontology. Discreteness means that continuous mass scale is replaced by mass scales coming as half octaves of CP_2 mass. One key question has been whether it is Kähler coupling strength α_K or gravitational coupling constant, which remains invariant under p-adic coupling constant evolution. Second problem relates to the value of α_K.

The realization that Kähler-Dirac action could be the fundamental variational principle initiated the process, which led to an answer to these and many other questions. The idea that some kind of Dirac determinant gives the vacuum functional identifiable as exponent of Kähler function in turn identifiable as Kähler action S_K for a preferred extremal came first. The basic challenges are to understand the conditions fixing the preferred extremal of Kähler action and how to define the Dirac determinant. After experimentation with several alternatives it became clear that the Kähler-Dirac action contains besides the term defined by Kähler action also a measurement interaction term guaranteeing quantum classical correspondence. An alternative idea inspired by TGD as almost topological QFT vision and quantum holography was that 3-D Chern-Simons action for light-like 3-surfaces at which the induced metric of the space-time surface changes its signature could be enough. This turned out be to not the case.

The most important outcome is a formula for Kähler coupling strength in terms of a calculable and manifestly finite Dirac determinant without any need for zeta function regularization. The formula fixes completely the number theoretic anatomy of Kähler coupling strength and of other gauge coupling strengths. When the formula for the gravitational constant involving Kähler coupling strength and the exponent of Kähler action for CP_2 type vacuum extremal - which remains still a conjecture - is combined with the number theoretical results and with the constraints from the predictions of p-adic mass calculations, one ends up to an identification of Kähler coupling strength as fine structure constant at electron length scale characterized by p-adic prime M_{127}. Also the number theoretic anatomy of the ratio $R^2/\hbar G$, where R is CP_2 size, can be understood to high degree and a relationship between the p-adic evolutions of electromagnetic and color coupling strengths emerges.

6.7.1 General Ideas About Coupling Constant Evolution

Zero energy ontology

In zero energy ontology one replaces positive energy states with zero energy states with positive and negative energy parts of the state at the boundaries of future and past direct light-cones forming a causal diamond. All conserved quantum numbers of the positive and negative energy states are of opposite sign so that these states can be created from vacuum. “Any physical state is creatable from vacuum” becomes thus a basic principle of quantum TGD and together with the notion of quantum jump resolves several philosophical problems (What was the initial state of universe?, What are the values of conserved quantities for Universe, Is theory building completely useless if only single solution of field equations is realized?).

At the level of elementary particle physics positive and negative energy parts of zero energy state are interpreted as initial and final states of a particle reaction so that quantum states become physical events. The proposed realization of Equivelence Principle at quantum level is based on the identification of classical Noether charges in Cartan algebra with the eigenvalues of their quantum counterparts assignable to Kähler-Dirac action. At classical level EP follows at GRT limit obtained by lumping many-sheeted space-time to M^4 with effective metric satisfying Einstein's equations as a reflection of the underlying Poincare invariance.

Does the finiteness of measurement resolution dictate the laws of physics?

The hypothesis that the mere finiteness of measurement resolution could determine the laws of quantum physics [K24] completely belongs to the category of not at all obvious first principles. The basic observation is that the Clifford algebra spanned by the gamma matrices of the “world of classical worlds” represents a von Neumann algebra [A48] known as hyperfinite factor of type II_1 (HFF) [K24, K110, K36]. HFF [A28, A41] is an algebraic fractal having infinite hierarchy of included subalgebras isomorphic to the algebra itself [A2]. The structure of HFF is closely related to several notions of modern theoretical physics such as integrable statistical physical systems [A71], anyons [D2], quantum groups and conformal field theories [A72], and knots and topological quantum field theories [A65, A34].

Zero energy ontology is second key element. In zero energy ontology these inclusions allow an interpretation in terms of a finite measurement resolution: in the standard positive energy ontology this interpretation is not possible. Inclusion hierarchy defines in a natural manner the notion of coupling constant evolution and p-adic length scale hypothesis follows as a prediction. In this framework the extremely heavy machinery of renormalized quantum field theory involving the elimination of infinities is replaced by a precisely defined mathematical framework. More concretely, the included algebra creates states which are equivalent in the measurement resolution used. Zero energy states are associated with causal diamond formed by a pair of future and past directed light-cones having positive and negative energy parts of state at their boundaries. Zero energy state can be modified in a time scale shorter than the time scale of the zero energy state itself.

On can imagine two kinds of measurement resolutions. The element of the included algebra can leave the quantum numbers of the positive and negative energy parts of the state invariant, which means that the action of subalgebra leaves M-matrix invariant. The action of the included algebra can also modify the quantum numbers of the positive and negative energy parts of the state such that the zero energy property is respected. In this case the Hermitian operators subalgebra must commute with M-matrix.

The temporal distance between the tips of light-cones corresponds to the secondary p-adic time scale $T_{p,2} = \sqrt{p}T_p$ by a simple argument based on the observation that light-like randomness of light-like 3-surface is analogous to Brownian motion. This gives the relationship $T_p = L_p^2/Rc$, where R is CP_2 size. The action of the included algebra corresponds to an addition of zero energy parts to either positive or negative energy part of the state and is like addition of quantum fluctuation below the time scale of the measurement resolution. The natural hierarchy of time scales is obtained as $T_n = 2^{-n}T$ since these insertions must belong to either upper or lower half of the causal diamond. This implies that preferred p-adic primes are near powers of 2. For electron the time scale in question is.1 seconds defining the fundamental biorhythm of 10 Hz.

M-matrix representing a generalization of S-matrix and expressible as a product of a positive square root of the density matrix and unitary S-matrix would define the dynamics of quantum theory [K24]. The notion of thermodynamical state would cease to be a theoretical fiction and in a well-defined sense quantum theory could be regarded as a square root of thermodynamics.

How do p-adic coupling constant evolution and p-adic length scale hypothesis emerge?

In zero energy ontology zero energy states have as embedding space correlates causal diamonds for which the distance between the tips of the intersecting future and past directed light-cones comes as integer multiples of a fundamental time scale: $T_n = n \times T_0$. p-Adic length scale hypothesis allows to consider a stronger hypothesis $T_n = 2^n T_0$ and its generalization a slightly more general hypothesis $T_n = p^n T_0$, p prime. It however seems that these scales are dynamically favored but that also other scales are possible.

Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance,

in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around.1 eV would correspond to $L(169) \simeq 5\ \mu$m (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3. For $T_p = pT_0$ the above argument is not enough for p-adic length scale hypothesis and p-adic length scale hypothesis might be seen as an outcome of a process analogous to natural selection. Resonance like effect favoring octaves of a fundamental frequency might be in question. In this case, p would a property of CD and all light-like 3-surfaces inside it and also that corresponding sector of WCW .

6.7.2 The Bosonic Action Defining Kähler Function As The Effective Action Associated With The Induced Spinor Fields

One could *define* the classical action defining Kähler function as the bosonic action giving rise to the divergences of the isometry currents. In this manner bosonic action, especially the value of the Kähler coupling strength, would come out as prediction of the theory containing no free parameters.

Thus the Kähler action S_B of preferred extremal of Käction defining Kähler function could be *defined* by the functional integral over the Grassmann variables for the exponent of the massless Dirac action. Formally the functional integral is defined as

$$\begin{aligned} exp(S_B(X^4)) &= \int exp(S_F) D\Psi D\bar{\Psi} \ , \\ S_F &= \bar{\Psi}\left[\hat{\Gamma}^\alpha D_\alpha^{\rightarrow} - D_\alpha^{\leftarrow}\hat{\Gamma}^\alpha\right]\Psi\sqrt{g} \ . \end{aligned} \tag{6.7.-1}$$

Formally the bosonic effective action is expressible as a logarithm of the fermionic functional determinant resulting from the functional integral over the Grassmann variables

$$\begin{aligned} S_B(X^4) &= log(det(D)) \ , \\ D &= \hat{\Gamma}^\alpha D_\alpha^{\rightarrow} \ . \end{aligned} \tag{6.7.-1}$$

Formula for the Kähler coupling strength

The identification of exponent of Kähler function as Dirac determinant leads to a formula relating Kähler action for the preferred extremal to the Dirac determinant. The eigenvalues are proportional to $1/\alpha_K$ since the matrices $\hat{\Gamma}^\alpha$ have this proportionality. This gives the formula

$$exp(\frac{S_K(X^4(X^3))}{8\pi\alpha_K}) = \prod_i \lambda_i = \frac{\prod_i \lambda_{0,i}}{\alpha_K^N} \ . \tag{6.7.0}$$

Here $\lambda_{0,i}$ corresponds to $\alpha_K = 1$. $S_K = \int J^*J$ is the reduced Kähler action.

For $S_K = 0$, which might correspond to so called massless extremals [K14] one obtains the formula

$$\alpha_K = (\prod_i \lambda_{0,i})^{1/N} \ . \tag{6.7.1}$$

Thus for $S_K = 0$ extremals one has an explicit formula for α_K having interpretation as the geometric mean of the eigenvalues $\lambda_{0,i}$. Several values of α_K are in principle possible.

p-Adicization suggests that $\lambda_{0,i}$ are rational or at most algebraic numbers. This would mean that α_K is N: th root of this kind of number. S_K in turn would be

$$S_K = 8\pi\alpha_K log(\frac{\prod_i \lambda_{0,i}}{\alpha_K^N}) . \quad (6.7.2)$$

so that S_K would be expressible as a product of the transcendental π, N: th root of rational, and logarithm of rational. This result would provide a general answer to the question about number theoretical anatomy of Kähler coupling strength and S_K. Note that S_K makes sense p-adically only if one adds π and its all powers to the extension of p-adic numbers. The exponent of Kähler function however makes sense also p-adically.

6.7.3 A Revised View About Coupling Constant Evolution

The development of the ideas related to number theoretic aspects has been rather tortuous and based on guess work since basic theory has been lacking.

1. The original hypothesis was that Kähler coupling strength is invariant under p-adic coupling constant evolution. Later I gave up this hypothesis and replaced it with the invariance of gravitational coupling since otherwise the prediction would have been that gravitational coupling strength is proportional to p-adic length scale squared. Second first guess was that Kähler coupling strength equals to the value of fine structure constant at electron length scale corresponding to Mersenne prime M_{127}. Later I replaced fine structure constant with electro-weak U(1) coupling strength at this length scale. The recent discussion returns back to the roots in both aspects.

2. The recent discussion relies on the progress made in the understanding of quantum TGD at partonic level [K111]. What comes out is an explicit formula for Kähler couplings strength in terms of Dirac determinant involving only a finite number of eigenvalues of the Kähler-Dirac operator. This formula dictates the number theoretical anatomy of g_K^2 and also of other coupling constants: the most general option is that α_K is a root of rational. The requirement that the rationals involved are simple combined with simple experimental inputs leads to very powerful predictions for the coupling parameters.

3. A further simplification is due to the discreteness of p-adic coupling constant evolution allowing to consider only length scales coming as powers of $\sqrt{2}$. This kind of discretization is necessary also number theoretically since logarithms can be replaced with 2-adic logarithms for powers of 2 giving integers. This raises the question whether $p \simeq 2^k$ should be replaced with 2^k in all formulas as the recent view about quantum TGD suggests.

4. The prediction is that Kähler coupling strength α_K is invariant under p-adic coupling constant evolution and from the constraint coming from electron and top quark masses very near to fine structure constant so that the identification as fine structure constant is natural. Gravitational constant is predicted to be proportional to p-adic length scale squared and corresponds to the largest Mersenne prime (M_{127}), which does not correspond to a completely super-astronomical p-adic length scale. For the parameter R^2/G p-adicization program allows to consider two options: either this constant is of form e^q or 2^q: in both cases q is rational number. $R^2/G = exp(q)$ allows only M_{127} gravitons if number theory is taken completely seriously. $R^2/G = 2^q$ allows all p-adic length scales for gravitons and thus both strong and weak variants of ordinary gravitation.

5. A relationship between electromagnetic and color coupling constant evolutions based on the formula $1/\alpha_{em} + 1/\alpha_s = 1/\alpha_K$ is suggested by the induced gauge field concept, and would mean that the otherwise hard-to-calculate evolution of color coupling strength is fixed completely. The predicted value of α_s at intermediate boson length scale is correct.

It seems fair to conclude that the attempts to understand the implications of p-adicization for coupling constant evolution have begun to bear fruits.

Identifications of Kähler coupling strength and gravitational coupling strength

To construct an expression for gravitational constant one can use the following ingredients.

1. The exponent $exp(2S_K(CP_2))$ defining the value of Kähler function in terms of the Kähler action $S_K(CP_2)$ of CP_2 type extremal representing elementary particle expressible as

$$S_K(CP_2) = \frac{S_{K,R}(CP_2)}{8\pi\alpha_K} = \frac{\pi}{8\alpha_K} . \quad (6.7.3)$$

   Since CP_2 type extremals suffer topological condensation, one expects that the action is modified:

$$S_K(CP_2) \rightarrow a \times S_K(CP_2) . \quad (6.7.4)$$

   $a < 1$ conforms with the idea that a piece of CP_2 type extremal defining a wormhole contact is in question. One must however keep mind open in this respect.

2. The p-adic length scale L_p assignable to the space-time sheet along which gravitational interactions are mediated. Since Mersenne primes seem to characterized elementary bosons and since the Mersenne prime $M_{127} = 2^{127}-1$ defining electron length scale is the largest non-super-astronomical length scale it is natural to guess that M_{127} characterizes these space-time sheets.

1. The formula for the gravitational constant

A long standing basic conjecture has been that gravitational constant satisfies the following formula

$$\begin{aligned} \hbar G &\equiv r\hbar_0 G = L_p^2 \times exp(-2aS_K(CP_2)) , \\ L_p &= \sqrt{p}R . \end{aligned} \quad (6.7.4)$$

Here R is CP_2 radius defined by the length $2\pi R$ of the geodesic circle. What was noticed before is that this relationship allows even constant value of G if a has appropriate dependence on p.

This formula seems to be correct but the argument leading to it was based on two erratic assumptions compensating each other.

1. I assumed that modulus squared for vacuum functional is in question: hence the factor $2a$ in the exponent. The interpretation of zero energy state as a generalized Feynman diagram requires the use of vacuum functional so that the replacement $2a \rightarrow a$ is necessary.

2. Second wrong assumption was that graviton corresponds to CP_2 type vacuum extremal-that is wormhole contact in the recent picture. This does allow graviton to have spin 2. Rather, two wormhole contacts represented by CP_2 vacuum extremals and connected by fluxes associated with various charges at their throats are needed so that graviton is string like object. This saves the factor $2a$ in the exponent.

The highly non-trivial implication to be discussed later is that ordinary coupling constant strengths should be proportional to $exp(-aS_K(CP_2))$.

The basic constraint to the coupling constant evolution comes for the invariance of g_K^2 in p-adic coupling constant evolution:

$$\begin{aligned} g_K^2 &= \frac{a(p,r)\pi^2}{log(pK)} , \\ K &= \frac{R^2}{\hbar G(p)} = \frac{1}{r}\frac{R^2}{\hbar_0 G(p)} \equiv \frac{K_0(p)}{r} . \end{aligned} \quad (6.7.4)$$

2. How to guarantee that g_K^2 is RG invariant and N: th root of rational?

Suppose that g_K^2 is N: th root of rational number and invariant under p-adic coupling constant evolution.

1. The most general manner to guarantee the expressibility of g_K^2 as N: th root of rational is guaranteed for both options by the condition

$$a(p,r) = \frac{g_K^2}{\pi^2} log(\frac{pK_0}{r}) \ . \tag{6.7.5}$$

   That a would depend logarithmically on p and $r = \hbar/\hbar_0$ looks rather natural. Even the invariance of G under p-adic coupling constant evolution can be considered.

2. The condition

$$\frac{r}{p} < K_0(p) \ . \tag{6.7.6}$$

   must hold true to guarantee the condition $a > 0$. Since the value of gravitational Planck constant is very large, also the value of corresponding p-adic prime must very large to guarantee this condition. The condition $a < 1$ is guaranteed by the condition

$$\frac{r}{p} > exp(-\frac{\pi^2}{g_K^2}) \times K_0(p) \ . \tag{6.7.7}$$

   The condition implies that for very large values of p the value of Planck constant must be larger than $\hbar_0$.

3. The two conditions are summarized by the formula

$$K_0(p) \times exp(-\frac{\pi^2}{g_K^2}) < \frac{r}{p} < K_0(p) \tag{6.7.8}$$

   characterizing the allowed interval for r/p. If G does not depend on p, the minimum value for r/p is constant. The factor $exp(-\frac{\pi^2}{g_K^2})$ equals to 1.8×10^{-47} for $\alpha_K = \alpha_{em}$ so that $r > 1$ is required for $p \geq 4.2 \times 10^{-40}$. $M_{127} \sim 10^{38}$ is near the upper bound for p allowing $r = 1$. The constraint on r would be roughly $r \geq 2^{k-131}$ and $p \simeq 2^{131}$ is the first p-adic prime for which $\hbar > 1$ is necessarily. The corresponding p-adic length scale is.1 Angstroms.

   This conclusion need not apply to elementary particles such as neutrinos but only to the space-time sheets mediating gravitational interaction so that in the minimal scenario it would be gravitons which must become dark above this scale. This would bring a new aspect to vision about the role of gravitation in quantum biology and consciousness.

   The upper bound for r behaves roughly as $r < 2.3 \times 10^7 p$. This condition becomes relevant for gravitational Planck constant GM_1M_2/v_0 having gigantic values. For Earth-Sun system and for $v_0 = 2^{-11}$ the condition gives the rough estimate $p > 6 \times 10^{63}$. The corresponding p-adic length scale would be of around $L(215) \sim 40$ meters.

4. p-Adic mass calculations predict the mass of electron as $m_e^2 = (5 + Y_e)2^{-127}/R^2$ where $Y_e \in [0, 1)$ parameterizes the not completely known second order contribution. Top quark mass favors a small value of Y_e (the original experimental estimates for m_t were above the range allowed by TGD but the recent estimates are consistent with small value Y_e [K64]). The range $[0, 1)$ for Y_e restricts $K_0 = R^2/\hbar_0 G$ to the range $[2.3683, 2.5262] \times 10^7$.

5. The best value for the inverse of the fine structure constant is $1/\alpha_{em} = 137.035999070(98)$ and would correspond to $1/g_K^2 = 10.9050$ and to the range $(0.9757, 0.9763)$ for a for $\hbar = \hbar_0$ and $p = M_{127}$. Hence one can seriously consider the possibility that $\alpha_K = \alpha_{em}(M_{127}$ holds true. As a matter fact, this was the original hypothesis but was replaced later with the hypothesis that α_K corresponds to electro-weak $U(1)$ coupling strength in this length scale. The fact that M_{127} defines the largest Mersenne prime, which does not correspond to super-astrophysical length scale might relate to this co-incidence.

To sum up, the recent view about coupling constant evolution differs strongly from previous much more speculative scenarios. It implies that g_K^2 is root of rational number, possibly even rational, and can be assumed to be equal to e^2. Also $R^2/\hbar G$ could be rational. The new element is that G need not be proportional to p and can be even invariant under coupling constant evolution since the parameter a can depend on both p and r. An unexpected constraint relating p and r for space-time sheets mediating gravitation emerges.

Are the color and electromagnetic coupling constant evolutions related?

Classical theory should be also able to say something non-trivial about color coupling strength α_s too at the general level. The basic observations are following.

1. Both classical color YM action and electro-weak U(1) action reduce to Kähler action.
2. Classical color holonomy is Abelian which is consistent also with the fact that the only signature of color that induced spinor fields carry is anomalous color hyper charge identifiable as an electro-weak hyper charge.

Suppose that α_K is a strict RG invariant. One can consider two options.

1. The original idea was that the sum of classical color action and electro-weak U(1) action is RG invariant and thus equals to its asymptotic value obtained for $\alpha_{U(1)} = \alpha_s = 2\alpha_K$. Asymptotically the couplings would approach to a fixed point defined by $2\alpha_K$ rather than to zero as in asymptotically free gauge theories.

   Thus one would have

$$\frac{1}{\alpha_{U(1)}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K} . \qquad (6.7.9)$$

   The relationship between $U(1)$ and em coupling strengths is

$$\begin{aligned} \alpha_{U(1)} &= \frac{\alpha_{em}}{cos^2(\theta_W)} \simeq \frac{1}{104.1867} , \\ sin^2(\theta_W)_{|10\ MeV} &\simeq 0.2397(13) , \\ \alpha_{em}(M_{127}) &= 0.00729735253327 . \end{aligned} \qquad (6.7.8)$$

   Here Weinberg angle corresponds to 10 MeV energy is reasonably near to the value at electron mass scale. The value $sin^2(\theta_W) = 0.2397(13)$ corresponding to 10 MeV mass scale [E3] is used. Note however that the previous argument implying $\alpha_K = \alpha_{em}(M_{127})$ excludes $\alpha = \alpha_{U(1)}(M_{127})$ option.

2. Second option is obtained by replacing $U(1)$ with electromagnetic gauge $U(1)_{em}$.

$$\frac{1}{\alpha_{em}} + \frac{1}{\alpha_s} = \frac{1}{\alpha_K}. \qquad (6.7.9)$$

Possible justifications for this assumption are following. The notion of induced gauge field makes it possible to characterize the dynamics of classical electro-weak gauge fields using only the Kähler part of electro-weak action, and the induced Kähler form appears only in the electromagnetic part of the induced classical gauge field. A further justification is that em and color interactions correspond to unbroken gauge symmetries.

The following arguments are consistent with this conclusion.

1. In TGD framework coupling constant is discrete and comes as powers of $\sqrt{2}$ corresponding to p-adic primes $p \simeq 2^k$. Number theoretic considerations suggest that coupling constants g_i^2 are algebraic or perhaps even rational numbers, and that the logarithm of mass scale appearing as argument of the renormalized coupling constant is replaced with 2-based logarithm of the p-adic length scale so that one would have $g_i^2 = g_i^2(k)$. g_K^2 is predicted to be N: th root of rational but could also reduce to a rational. This would allow rational values for other coupling strengths too. This is possible if $sin(\theta_W)$ and $cos(\theta_W)$ are rational numbers which would mean that Weinberg angle corresponds to a Pythagorean triangle as proposed already earlier. This would mean the formulas $sin(\theta_W) = (r^2-s^2)/(r^2+s^2)$ and $cos(\theta_W) = 2rs(r^2+s^2)$.

2. A very strong prediction is that the beta functions for color and $U(1)$ degrees of freedom are apart from sign identical and the increase of $U(1)$ coupling compensates the decrease of the color coupling. This allows to predict the hard-to-calculate evolution of QCD coupling constant strength completely.

3. $\alpha(M_{127}) = \alpha_K$ implies that M_{127} defines the confinement length scale in which the sign of α_s becomes negative. TGD predicts that also M_{127} copy of QCD should exist and that M_{127} quarks should play a key role in nuclear physics [K93, L2], [L2]. Hence one can argue that color coupling strength indeed diverges at M_{127} (the largest not completely super-astrophysical Mersenne prime) so that one would have $\alpha_K = \alpha(M_{127})$. Therefore the precise knowledge of $\alpha(M_{127})$ in principle fixes the value of parameter $K = R^2/G$ and thus also the second order contribution to the mass of electron.

4. $\alpha_s(M_{89})$ is predicted to be $1/\alpha_s(M_{89}) = 1/\alpha_K - 1/\alpha(M_{89})$. $sin^2(\theta_W) = .23120$, $\alpha_{em}(M_{89}) \simeq 1/127$, and $\alpha_{U(1)} = \alpha_{em}/cos^2(\theta_W)$ give $1/\alpha_{U(1)}(M_{89}) = 97.6374$. $\alpha = \alpha_{em}$ option gives $1/\alpha_s(M_{89}) \simeq 10$, which is consistent with experimental facts. $\alpha = \alpha_{U(1)}$ option gives $\alpha_s(M_{89}) = 0.1572$, which is larger than QCD value. Hence $\alpha = \alpha_{em}$ option is favored.

Can one deduce formulae for gauge couplings?

The improved physical picture behind gravitational constant allows also to consider a general formula for gauge couplings.

1. The natural guess for the general formula would be as

$$g^2(p,r) = kg_K^2 \times exp[-a_g(p,r) \times S_K(CP_2)] \ . \tag{6.7.10}$$

here k is a numerical constant.

2. The condition
$g_K^2 = e^2(M_{127})$ fixes the value of k if it's value does not depend on the character of gauge interaction:

$$k = exp[a_{gr}(M_{127}, r=1) \times S_K(CP_2)] \ . \tag{6.7.11}$$

Hence the general formula reads as

$$g^2(p,r) \quad = \quad g_K^2 \times exp[(-a_g(p,r) + a_{gr}(M_{127}), r = 1)) \times S_K(CP_2)] \ . \tag{6.7.11}$$

The value of $a(M_{127}, r = 1)$ is near to its maximum value so that the exponential factor tends to increase the value of g^2 from e^2. The formula can reproduce α_s and various electro-weak couplings although it is quite possibile that Weinberg angle corresponds to a group theoretic factor not representable in terms of $a_g(p,r)$. The volume of the CP_2 type vacuum extremal would characterize gauge bosons. Analogous formula should apply also in the case of Higgs.

3. α_{em} in very long length scales would correspond to

$$e^2(p \to \infty, r = 1) \quad = \quad e^2 \times exp[(-1 + a(M_{127}), r = 1)) \times S_K(CP_2)] = e^2 x \ , \tag{6.7.11}$$

where x is in the range $[0.6549, 0.6609]$.

Chapter 7

Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

7.1 Introduction

The notion of p-adicization has for a long time been a somewhat obscure attempt to provide a theoretical justification for the successes of the p-adic mass calculations. The reduction of quantum TGD to a generalized number theory and the developments in TGD inspired theory of consciousness have however led to a better understanding what the p-adicization possibly means.

7.1.1 What P-Adic Physics Means?

Contrary to the original expectations finite-p p-adic physics means the physics of the p-adic cognitive representations about real physics rather than "real physics". This forces to update the prejudices about what p-adicization means. The original hypothesis was that p-adicization is a strict one-to-one map from real to p-adic physics and this led to technical problems with symmetries.

The new vision about quantum TGD the notion of the p-adic space-time emerges dynamically and p-adic space-time regions are absolutely "real" and certainly not "p-adicized" in any sense. Furthermore, the new view also encourages the hypothesis that p-adic regions provide cognitive models for the real matter like regions becoming more and more refined in the evolutionary self-organization process by quantum jumps. p-Adic region can serve as a cognitive model for particle itself or for the external world. The model is defined by some cognitive map of real region to its p-adic counterpart. This cognitive map need not be unique. At the level of TGD inspired theory of consciousness the p-adicization becomes modelling of how cognition works.

In this conceptual framework the successes of the p-adic mass calculations can be understood only if p-adic mass calculations provide a model a "cognitive model" of an elementary particle. The successes of the p-adic mass calculations, and also the fact that they rely on the fundamental symmetries of quantum TGD, encourages the idea that one could try to mimic Nature. Thus p-adic physics could be seen as an abstract mimicry for what Nature already does by constructing explicitly p-adic cognitive representations. This new view about p-adic physics allows much more flexibility since p-adicization can be interpreted as a cognitive map mapping real world physics to p-adic physics. In this view p-adicization need not and cannot be a unique procedure.

7.1.2 Number Theoretic Vision Briefly

The number theoretic vision [K95, K96, K94] about the classical dynamics of space-time surfaces is now relatively detailed although it involves unproven conjectures inspired by physical intuition.

1. *Hyper-quaternions and octonions*

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D embedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannian geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a hyper-quaternionic sub-manifold of hyper-octonionic embedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point. One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH--M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \to SU(3)$ satisfying certain integrability conditions guaranteeing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \to G_2$ reducing to maps $HO \to SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces.

Hyper-octonion analytic functions $HO \to HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action correspond to the hyper-quaternion analytic surfaces. This conjecture has several variants. It could be that only asymptotic behavior corresponds to hyper-quaternion analytic function but that that hyper-quaternionicity is general property of absolute minima. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The representation of infinite hyper-octonionic primes as 4-surfaces

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers). Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by hyper-quaternionic 4-surfaces and notion of Kähler calibration. Therefore space-time surface could be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the preferred extremals of Kähler action.

7.1.3 P-Adic Space-Time Sheets As Solutions Of Real Field Equations Continued Algebraically To P-Adic Number Field

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlations for cognition. This requires a generalization of the notion of number by gluing reals and various p-adic number fields together along common rationals. This in turn implies generalization of the notion of embedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

7.1.4 The Notion Of Pinary Cutoff

The notion of pinary cutoff is central for p-adic TGD and it should have some natural definition and interpretation in the new approach. The presence of p-adic pseudo constants implies that there is large number of cognitive representations with varying degrees of faithfulness. Pinary cutoff must serve as a measure for how faithful the p-adic cognitive representation is. Since the cognitive maps are not unique, one cannot even require any universal criterion for the faithfulness of the cognitive map. One can indeed imagine two basic criteria corresponding to self-representations and representations for external world.

1. The subset of rationals common to the real and p-adic space-time surface could define the resolution. In this case, the average distance between common rational points of these two surfaces would serve as a measure for the resolution. Pinary cutoff could be defined as the smallest number of pinary digits in expansions of functions involved above which the resolution does not improve. Physically the optimal resolution would mean that p-adic space-time surface, "cognitive space-time sheet", has a maximal number of intersections with the real space-time surface for which it provides a self-representation. This purely algebraic notion of faithfulness does not respect continuity: two rational points very near in real sense could be arbitrary far from each other with respect to the p-adic norm.

2. One could base the notion of faithfulness on the idea that p-adic space-time sheet provides almost continuous map of the real space-time sheet belonging to the external world by the basic properties of the canonical identification. The real canonical image of the p-adic space-time sheet and real space-time sheet could be compared and some geometric measure for the nearness of these surfaces could define the resolution of the cognitive map and pinary cutoff could be defined in the same way as above.

7.1.5 Program

These ideas lead to a rather well defined p-adicization program. Define precisely the concepts of the p-adic space-time and reduced configuration space, formulate the finite-p p-adic versions of quantum TGD and construct the p-adic variants of TGD. Of course, the aim is not to just construct p-adic version of the real quantum TGD but to understand how real and p-adic quantum TGD:s fuse together to form the full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, probability and unitary concepts to p-adic context. Also new physical thinking and philosophy is needed and this long chapter is devoted to the description of the new elements. Before going to the detailed exposition it is appropriate to give a brief overall view of the basic mathematical tools.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

7.2 P-Adic Numbers And Consciousness

The idea that p-adic physics provides the physics of cognition has become more and more attractive during the 12 years or so that I have spent with p-adic numbers and I feel that it is good to add a summary about these ideas here.

7.2.1 P-Adic Physics As Physics Of Cognition

p-Adic physics began from p-adic mass calculations. The next step in the progress was the idea that p-adic physics serves as a correlate for cognition and this thread gradually led to the recent view requiring the generalization of the number concept.

In what sense space-time surface defomposes to real and p-adic regions?

The original view that space-time surfaces contain genuinely p-adic and possibly even rational-adic regions and that p-adic regions mimic the real regions. This view turned out to be partially wrong. Rather, the adelic view holds true [K109]. Reals and replaced by an infinite Cartesian product of real and various p-adic numbers fields or their extensions corresponding to fixed extension of rationals - all of them are allowed. Same happens at the level of embedding space, space-time, and WCW. Therefore one has separate real and p-adic sectors rather than separate regions. Every system - even elementary particle - has both its real sector and p-adic sectors representing cognitive correlates of existence as indeed suggested by p-adic mass calculations [K51].

p-Adic space-time sheets indeed mimic their real counterparts in the sense that they are p-adic variants of real preferred extremals. This correspondence is induced by the strong form of holography from the correspondence at string world sheets and partonic 2-surfaces induced by the fact that numbers in the extension of rationals can be interpreted as both real and p-adic numbers. The strings world sheets and partonic 2-surfaces, whose defining parameters are in algebraic extension of rationals are continued to preferred extremals in various number fields so that cognitive and sensory representations can be said to result as a consequence. String world sheets themselves can be said to be in the intersection of reality and p-adicities.

Different kinds of cognitive representations

At the level of the space-time surfaces and embedding space p-adicization boils down to the task of finding a map mapping real space-time region to a p-adic space-time region. These regions correspond to definite regions of the rational embedding space so that the map has a clear geometric interpretation at the level of rational physics.

The basic constraint on the map is that both real and p-adic space-time regions satisfy field equations: p-adic field equations make sense even if the integral defining the Kähler action does not exist p-adically. p-Adic nondeterminism makes possible this map when one allows finite pinary cutoff characterizing the resolution of the cognitive representation.

I heave considered several basic types of cognitive representations which might be called self-representations and representations of the external world and the map mediating p-adicization is different for these two maps. It has however turned that the continuation of string world sheets and partonic 2-surfaces in the intersection of reality and various p-adicities to preferred extemals - made possible by strong form of holography implied in turn by strong form of general coordinate invariance - is too elegant manner to define the map between real and p-adic sectors that these options can be safely forgotten. Of course, the notion of finite measurement resolution is still present and is expected to be realized as the assumption that parameters characterizing geometric objects of various dimensions are in algebraic extension of rationals with some cutoff.

1. The correspondence induced by the common rational points respects algebraic structures and defines self-representation. Real and p-adic space-time surfaces have a subset of rational points (defined by the resolution of the cognitive map) as common. The quality of the representation is defined by the resolution of the map and pinary cutoff for the rationals in pinary expansion is a natural measure for the resolution just as decimal cutoff is a natural measure for the resolution of a numerical model.

2. Canonical identification maps rationals to rationals since the periodic pinary expansion of a rational is mapped to a periodic expansion in the canonical identification. The rationals $q = m/n$ for which n is not divisible by p are mapped to rationals with p-adic norm not larger than unity. Canonical identification respects continuity. Real numbers with real norm larger than p are mapped to real numbers with norm smaller than one in canonical identification whereas reals with real norm in the interval $[1, p)$ are mapped to p-adics with p-adic norm

equal to one. Obviously the generalization of the canonical identification can map the world external to a given space-time region into the interior of this region and provides an example of an abstract cognitive representation of the external world. Also now pinary cutoff serves as a natural measure for the quality of the cognitive map.

3. The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity. A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries.

   This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

   It seems that this option, the discovery of which took almost a decade, must be used to relate p-adic transition amplitudes to real ones and vice versa [K57]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

4. The proposal for a map taking p-adic and real preferred extremals to each other utilizing canonical identification was proposed in [K112] and was based on the discretization of space-time surfaces induced by that of the embedding space and in terms of algebraic points assuming pinary cutoffs. Needless to say, the holography based proposal is much more elegant.

A fascinating possibility is that cognitive self-maps and maps of the external world at the level of human brain are basically realized by the holography induced mapping. Obviously canonical identification performed separately for all coordinates is the only possibility if this map is required to be maximally continuous.

p-Adic physics as a mimicry

The success of the p-adic mass calculations suggests that one could apply the idea of p-adic cognitive representation even at the level of quantum TGD to build models which have maximal simplicity and calculational effectiveness. p-Adic mass calculations represent this kind of model: now canonical identification is performed for the p-adic mass squared values and can be interpreted as a map from cognitive representation back to real world.

The basic task is the construction of the cognitive self-map or a cognitive map of external world: the laws of p-adic physics define the cognitive model itself automatically. For the cognitive representations of external world involving some variant of canonical identification mapping the exterior of the embedding space region inside this region. For self-representations situation is much more simpler. In practice, the direct modelling of p-adic physics without explicit construction of the cognitive map could give valuable information about real physics.

In the earlier approach based on phase preserving canonical identification to the mapping of real space-time surface to its p-adic counterpart led to the requirement about existence of unique (almost) embedding space coordinates. In present case the selection of the quaternionic coordinates for the embedding space is unique only apart from quaternion-analytic change of coordinates. This does not seem however pose any problems now. One must also remember that only cognitive representations are in question. These representations are not unique and selection of quaternionic coordinates might be even differentiate between different cognitive representations.

Since infinite primes serve as a bridge between classical and quantum, this map also assigns to a real Fock state associated with infinite prime its p-adic version identifiable as the ground state of a superconformal representation. Thus the map respects quantum symmetries automatically. If the construction of the states of the representation is a completely algebraic process, there are hopes of constructing the p-adic counterpart of S-matrix. If S-matrix is complex rational it can be mapped to its real counterpart. If the localization in zero modes occurs in each quantum jump the predictions of the theory could reduce to the integration in fiber degrees of freedom of CH reducible in turn to purely algebraic expressions making sense also p-adically.

The most recent view about the correspondence between real and p-adic sectors is based on adelic picture. All number fields are present and embedding space-time, space-time, and "world of classical worlds " (WCW) are adeles.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface a cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. String world sheets would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of rationals and p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime. Also the emergence of preferred p-adic primes can be understood: they correspond to so called ramified primes of the extension [K109].

7.2.2 Zero Energy Ontology, Cognition, And Intentionality

Zero Energy Ontology (ZEO) [K25, K61, K24] has become gradually one of the basic building brickes of quantum TGD.

Zero energy ontology classically

In TGD inspired cosmology [K90] the embeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the embeddings of Reissner-Nordström solution [K104] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

As such this view was wrong but served as a stepping stone to ZEO in which the notion of positive energy state is replaced with zero energy state which is pair of states with opposite quantum numbers rather than positive energy state at the limit of vanishing energy.

What is nice in ZEO that one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

ZEO at quantum level

Also the construction of S-matrix [K24] leads to the conclusion that all physical states can be said to possess vanishing conserved quantum numbers. In ZEO zero energy states can be seen as pairs of positive - and negative energy states with opposite quantum numbers identifiable as counterparts of initial and final states of physical events. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

Hyper-finite factors of type II$_1$ and new view about S-matrix

The representation of S-matrix as entanglement coefficients defining a unitary matrix would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II$_1$ [K110]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by WCW gamma matrices (WCW understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II$_1$.

It is now possible to relate the inclusion hierarchies of HFFs to hierarchies of phase transitions reducing quantum criticality and to hierarchy of Planck constants $h_{eff} = n \times h$ labelling phases of dark matter [K36, **?**]

The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [K110]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II$_1$ sense is an open question.

In Zero Energy Ontology (ZEO) which emerged much later than the above text was written the view about quantum measurement theory becomes very detailed and allows to formulate the notion of self quantitatively, understand how the arrow of geometric time emerges, and also to understand sensory perception, cognition, and volition at fundamental level [**?**, K7]. There is also a connection with hyper-finite factors and their inclusions and it reduces to finite cognitive resolution.

How do real and p-adic physics relate?

In zero energy ontology (ZEO) conservation laws do not forbid p-adic-real transitions and one can ask whether this kind of transitions could make sense. My original view was optimistic. During years it has however became clear that the quantum description of the transitions between different number fields poses very strong mathematical challenges suggesting a negative answer to the question of the title. It also seems that one can do without p-adic-to-real transitions if one accepts the adelic view about physics [K109]. All number fields are present and p-adic number fields decribe the correlates of cognition. In this framework even elementary particles have cognitive side as the success of p-adic mass calculations indeed suggests [K51].

The starting point of the original argument was the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly, see **Fig.** `http://tgdtheory.fi/appfigures/book.jpg`, which is also in the appendix of this `http://tgdtheory.fi/appfigures/book.jpg`). At the level of the embedding space this means that p-adic and real space-time sheets intersect only along common rational points of the embedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of embedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

A more detailed view is based on the idea that p-adic space-time sheets indeed define a theory about real space-time sheets. The interaction between real and p-adic number fields would mean that p-adic space-time surfaces define cognitive representations of real space-time surfaces (preferred extremals). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics.

Strong form of holography implied by strong form of General Coordinate Invariance leads to the suggestion that partonic 2-surfaces and string world sheets at which the induced spinor fields are localized in order to have a well-defined em charge (this is only one of the reasons) and having having discrete set as intersection points with partonic 2-surfaces define what might called "space-time genes". Space-time surfaces would be obtained as preferred extremals satisfying certain boundary conditions at string world sheets. Space-time surfaces are defined only modulo transformations of super-symplectic algebra defining its sub-algebra and acting as conformal gauge transformations so that one can talk about conformal gauge equivalences classes of space-time surfaces.

The map assigning to real space-time surface a cognitive representation would be replaced by a correspondence assigning to the string world sheets preferred extremals of Kähler action in various number fields: string world sheets would be indeed like genes. String world sheets would be in the intersection of realities and p-adicities in the sense that the parameters characterizing them would be algebraic numbers associated with the algebraic extension of p-adic numbers in question. It is not clear whether the preferred extremal is possible for all p-adic primes but this would fit nicely with the vision that elementary particles are characterized by p-adic primes. It could be also that the classical non-determinism of Kähler action responsible for the conformal gauge symmetry corresponds to p-adic non-determinism for some particular prime so that the cognitive map is especially good for this prime.

7.3 An Overall View About P-Adicization Of TGD

In this section the basic problems and ideas related to the p-adicization of quantum TGD are discussed. One should define the notions of Riemann geometry and its variants such as Kähler geometry in the p-adic context. The notion of the p-adic space-time surface and its relationship to its real counterpart should be understood. Also the construction of Kähler geometry of "world of classical worlds" (WCW) in p-adic context should be carried out and the notion of WCW spinor fields should be defined in the p-adic context. The crucial technical problems relate to the notion of integral and Fourier analysis, which are the central elements of any physical theory. The basic challenge is to overcome the fact that although the field equations assignable to a given variational principle make sense p-adically, the action defined as an integral over arbitrary space-time surface has no natural p-adic counterpart as such in the generic case. What raises hopes that these challenges could be overcome is the symmetric space property of WCW and the idea of algebraic continuation. If WCW geometry is expressible in terms of rational functions with rational coefficients it allows a generalization to the p-adic context. Also integration can be reduced to Fourier analysis in the case of symmetric spaces.

7.3.1 P-Adic Embedding Space

The construction of both quantum TGD and p-adic QFT limit requires p-adicization of the embedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the embedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost. The essential element is the notion of finite measurement resolution leading to discretization in large and to p-adicization below the resolution scale. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification mapping p-adic mass squared to its real counterpart.

The identification of the fundamental p-adic length scale

The fundamental p-adic length scale corresponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the "radius" R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the "cosmological constant" Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda . \tag{7.3.1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} . \tag{7.3.2}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π: s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

   (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod\ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

(b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R \ ,$$

rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counterpart of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal plane waves. p-Adic plane waves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naïvely expected p, in the expression of the p-adic length scale can be understood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{7.3.2}$$

where m_{kl} is the standard M^4 metric $(1, -1, -1, -1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k: th coordinate axis the expression

$$s = R\sqrt{p} m^k \ . \tag{7.3.3}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{7.3.4}$$

The p-adic distance along the k: th coordinate axis from the origin to the point $m^k = (p-1)(1+p+p^2+...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p}l = \sqrt{p}\pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R\frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{7.3.4}$$

What is remarkable is that the shortest distance in the range $m^k = 1,..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

The two variants of CP_2

As noticed, CP_2 allows two variants based on rational discretization and on the discretiation based on roots of unity. The root of unity option corresponds to the phases associated with $1/(1+r^2) = tan^2(u/2) = (1-cos(u))/(1+cos(u))$ and implies that integrals of spherical harmonics can be reduced to summations when angular resolution $\Delta u = 2\pi/N$ is introduced. In the p-adic context, one can replace distances with trigonometric functions of distances along zig zag curves connecting the points of the discretization. Physically this notion of distance is quite reasonable since distances are often measured using interferometer.

In the case of rational variant of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/columns are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their non-vanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_kX_k) \ ; \ |t_k|_p < 1\} = \{x = 1+iy \ ; \ |y|_p < 1\} \ . \tag{7.3.5}$$

The diagonal elements of the matrices in this group are of the form $1+O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1+\phi_2)\} \ . \tag{7.3.6}$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1+O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the "physical" $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod\ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic sub-manifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1 + r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{7.3.6}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1 + r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \ . \tag{7.3.7}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1 + r^2)^2} \ . \tag{7.3.7}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

7.3.2 Infinite Primes And Cognition

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labelled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_BX/s_F + n_Bs_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labelled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether WCW degrees of freedom and WCW spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labelling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

Precise space-time correlates of cognition

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by WCW spinor for given

3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   (a) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

   (b) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

   (c) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair.

   (d) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

7.3.3 P-Adicization Of Second Quantized Induced Spinor Fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at the 3-dimensional light-like partonic 3-surfaces and the solutions of the Kähler-Dirac equation can be written explicitly [K25, K24] as simple algebraic functions involving powers of the preferred coordinate variables very much like various operators in conformal field theory can be expressed as Laurent series in powers of a complex variable z with operator valued coefficients. This means that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure. The second quantization of these induced spinor fields as free fields is needed to construct WCW geometry and anti-commutation relation between spinor fields are fixed from the requirement that WCW gamma matrices correspond to super-symplectic generators.

The idea about rational physics as the intersection of the physics associated with various number fields inspires the hypothesis that induced spinor fields have only modes labelled by rational valued quantum numbers. Quaternion conformal invariance indeed implies that zero modes are characterized by integers. This means that same oscillator operators can define oscillator operators are universal. Powers of the quaternionic coordinate are indeed well-define in any number field provided the components of quaternion are rational numbers since p-adic quaternions have in this case always inverse.

7.3.4 Should One P-Adicize At The Level Of WCW ?

If Duistermaat-Heckman theorem [A31] holds true in TGD context, one could express WCW functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function defining what might be called reduced WCW CH_{red}. The huge super-conformal symmetries raise the hope that the rest of S-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of S-matrix.

The optimal situation would be that S-matrix elements reduce to algebraic numbers for rational valued incoming momenta and that p-adicization trivializes in the sense that it corresponds only to different interpretations for the embedding space coordinates (interpretation as real or p-adic numbers) appearing in the equations defining the 4-surfaces. For instance, space-time coordinates would correspond to preferred embedding space coordinates and the remaining embedding space coordinates could be rational functions of the latter with algebraic coefficients. Algebraic points in a given extension of rationals would thus be common to real and p-adic surfaces. It could also happen that there are no or very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$.

This picture is probably too simple. The intuitive expectation is that ordinary S-matrix elements are proportional to a factor which in the real case involves an integration over the arguments of an n-point function of a conformal field theory defined at a partonic 2-surface. For p-adic-real transitions the integration should reduce to a sum over the common rational or algebraic points of the p-adic and real surface. Same applies to $p_1 \rightarrow p_2$ type transitions.

If this picture is correct, the p-adicization of WCW would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. If CH_{red} is a discrete subset of CH ultra-metric topology induced from finite-p p-adic norm is indeed natural for it. "Discrete set in CH" need not mean a discrete set in the usual sense and the reduced WCW could be even finite-dimensional continuum. Finite-p p-adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

A basic technical problem is, whether the integral defining the Kähler action appearing in the exponent of Kähler function exists p-adically. Here the hypothesis that the exponent of the Kähler function is identifiable as a Dirac determinant of the Kähler-Dirac operator defined at the light-like partonic 3-surfaces [K111] suggests a solution to the problem. By restricting the generalized eigen values of the Kähler-Dirac operator to an appropriate algebraic extension of rationals one could obtain an algebraic number existing both in the real and p-adic sense if the number of the contributing eigenvalues is finite. The resulting hierarchy of algebraic extensions of R_p would have interpretation as a cognitive hierarchy. If the maxima of Kähler function assignable to the functional integral are such that the number of eigenvalues in a given algebraic extension is finite this hypothesis works.

If Duistermaat-Heckman theorem generalizes, the p-adicization of the entire WCW would be un-necessary and it certainly does not look a good idea in the light of preceding considerations.

1. For a generic 3-surface the number of the eigenvalues in a given algebraic extension of rationals need not be finite so that their product can fail to be an algebraic number.

2. The algebraic continuation of the exponent of the Kähler function from CH_{red} to the entire CH would be analogous to a continuation of a rational valued function from a discrete set to a real or p-adic valued function in a continuous set. It is difficult to see how the continuation could be unique in the p-adic case.

7.4 P-Adic Probabilities

p-Adic Super Virasoro representations necessitate p-adic QM based on the p-adic unitarity and p-adic probability concepts. The concept of a p-adic probability indeed makes sense as shown by [A24]. p-Adic probabilities can be defined as relative frequencies N_i/N in a long series consisting of total number N of observations and N_i outcomes of type i. Probability conservation corresponds to

$$\sum_i N_i = N \ , \tag{7.4.1}$$

and the only difference as compared to the usual probability is that the frequencies are interpreted as p-adic numbers.

The interpretation as p-adic numbers means that the relative frequencies converge to probabilities in a p-adic rather than real sense in the limit of a large number N of observations. If one requires that probabilities are limiting values of the frequency ratios in p-adic sense one must pose restrictions on the possible numbers of the observations N if N is larger than p. For N smaller than p, the situation is similar to the real case. This means that for $p = M_{127} \simeq 10^{38}$, appropriate for the particle physics experiments, p-adic probability differs in no observable manner from the ordinary probability.

If the number of observations is larger than p, the situation changes. If N_1 and N_2 are two numbers of observations they are near to each other in the p-adic sense if they differ by a large power of p. A possible interpretation of this restriction is that the observer at the p: th level of the condensate cannot choose the number of the observations freely. The restrictions to this freedom come from the requirement that the sensible statistical questions in a p-adically conformally invariant world must respect p-adic conformal invariance.

7.4.1 P-Adic Probabilities And P-Adic Fractals

p-Adic probabilities are natural in the statistical description of the fractal structures, which can contain same structural detail with all possible sizes.

1. The concept of a structural detail in a fractal seems to be reasonably well defined concept. The structural detail is clearly fixed by its topology and p-adic conformal invariants associated with it. Clearly, a finite resolution defined by some power of p of the p-adic cutoff scale must be present in the definition. For example, p-adic angles are conformal invariants in the p-adic case, too. The overall size of the detail doesn't matter. Let us therefore assume that it is possible to make a list, possibly infinite, of the structural details appearing in the p-adic fractal.

2. What kind of questions related to the structural details of the p-adic fractal one can ask? The first thing one can ask is how many times i: th structural detail appears in a finite region of the fractal structure: although this number is infinite as a real number it might possess (and probably does so!) finite norm as a p-adic number and provides a useful p-adic invariant of the fractal. If a complete list about the structural details of the fractal is at use one can calculate also the total number of structural details defined as $N = \sum_i N_i$. This means that one can also define p-adic probability for the appearance of i: th structural detail as a relative frequency $p_i = N_i/N$.

3. One can consider conditional probabilities, too. It is natural to ask what is the probability for the occurrence of the structural detail subject to the condition that part of the structural detail is fixed (apart from the p-adic conformal transformations). In order to evaluate these probabilities as relative frequencies one needs to look only for those structural details containing the substructure in question.

4. The evaluation of the p-adic probabilities of occurrence can be done by evaluating the required numbers N_i and N in a given resolution. A better estimate is obtained by increasing the resolution and counting the numbers of the hitherto unobserved structural details. The increase in the resolution greatly increases the number of the observations in case of p-adic fractal and the fluctuations in the values of N_i and N increase with the resolution so that N_i/N has no well defined limit as a real number although one can define the probabilities of occurrence as a resolution dependent concept. In the p-adic sense the increase in the values of N_i and fluctuations are small and the procedure should converge rapidly so that reliable estimates should result with quite a reasonable resolution. Notice that the increase of the fluctuations in the real sense, when resolution is increased is in accordance with the criticality of the system.

5. p-Adic frequencies and probabilities define via the canonical correspondence real valued invariants of the fractal structure.

It must be emphasized that this picture can have practical applications only for small values of p, which could also be important in the macroscopic length scales. In elementary particle physics L_p is of the order of the Compton length associated with the particle and already in the first step CP_2 length scale is achieved and it is questionable whether it makes sense to continue the procedure below the length scale l. In particle physics context the renormalization is related to the change of the reduction of the p-adic length scale L_p in the length scale hierarchy rather than p-adic fractality for a fixed value of p.

The most important application of the p-adic probability in this book is the description of the particle massivation based on p-adic thermodynamics. Instead of energy, Virasoro generator l is thermalized and in the low temperature phase temperature is quantized in the sense that the counterpart of the Boltzmann weight $exp(H/T)$ is $p^{L_0/T}$, where $T = 1/n$ from the requirement that Boltzmann weight exists (L_0 has integer spectrum). The surprising success of the mass calculations shows that p-adic probability theory is much more than a formal possibility.

7.4.2 Relationship Between P-Adic And Real Probabilities

There are uniqueness problems related to the mapping of p-adic probabilities to real ones. These problems find a nice resolution from the requirement that the map respects probability conservation. The implied modification of the original mapping does not change measurably the predictions for the masses of light particles.

How unique the map of p-adic probabilities and mass squared values are mapped to real numbers is?

The mapping of p-adic thermodynamical probabilities and mass squared values to real numbers is not completely unique.

1. Symplectic identification $I : \sum x_n p^n \to \sum x_n p^{-n}$ takes care of this mapping but does not respect the sum of probabilities so that the real images $I(p_n)$ of the probabilities must be normalized. This is a somewhat alarming feature.

2. The modification of the canonical identification mapping rationals by the formula $I(r/s) = I(r)/I(s)$ has appeared naturally in various applications, in particular because it respects unitarity of unitary matrices with rational elements with $r < p, s < p$. In the case of p-adic thermodynamic the formula $I(g(n)p^n/Z) \rightarrow I(g(n)p^n)/I(Z)$ would be very natural although Z need not be rational anymore. For $g(n) < p$ the real counterparts of the p-adic probabilities would sum up to one automatically for this option. One cannot deny that this option is more convincing than the original one. The generalization of this formula to map p-adic mass squared to a real one is obvious.

3. Options 1) and 2) differ dramatically when the $n = 0$ massless ground state has ground state degeneracy $D > 1$. For option 1) the real mass is predicted to be of order CP_2 mass whereas for option 2) it would be by a factor $1/D$ smaller than the minimum mass predicted by the option 1). Thus option 2) would predict a large number of additional exotic states. For those states which are light for option 1), the two options make identical predictions as far as the significant two lowest order terms are considered. Hence this interpretation would not change the predictions of the p-adic mass calculations in this respect. Option 2) is definitely more in accord with the real physics based intuitions and the main role of p-adic thermodynamics would be to guarantee the quantization of the temperature and fix practically uniquely the spectrum of the "Hamiltonian".

Under what conditions the mapping of p-adic ensemble probabilities to real probabilities respects probability conservation?

One can consider also a more general situation. Assume that one has an ensemble consisting of independent elementary events such that the number of events of type i is N_i. The probabilities are given by $p_i = N_i/N$ and $N = \sum N_i$ is the total number of elementary events. Even in the case that N is infinite as a real number it is natural to map the p-adic probabilities to their real counterparts using the rational canonical identification $I(p_i) = I(N_i)/I(N)$. Of course, N_i and N exist as well defined p-adic numbers under very stringent conditions only.

The question is under what conditions this map respects probability conservation. The answer becomes obvious by looking at the pinary expansions of N_i and N. If the integers N_i (possibly infinite as real integers) have pinary expansions having no common pinary digits, the sum of probabilities is conserved in the map. Note that this condition can assign also to a finite ensemble with finite number of a unique value of p.

This means that the selection of a basis for independent events corresponds to a decomposition of the set of integers labelling pinary digits to disjoint sets and brings in mind the selection of orthonormalized basis of quantum states in quantum theory. What is physically highly non-trivial that this "orthogonalization" alone puts strong constraints on probabilities of the allowed elementary events. One can say that the probabilities define distributions of pinary digits analogous to non-negative probability amplitudes in the space of integers labelling pinary digits, and the probabilities of independent events must be orthogonal with respect to the inner product defined by point-wise multiplication in the space of pinary digits.

p-Adic thermodynamics for which Boltzmann weights $g(E)exp(-E/T)$ are replaced by $g(E)p^{E/T}$ such that one has $g(E) < p$ and E/T is integer valued, satisfies this constraint. The quantization of E/T to integer values implies quantization of both T and "energy" spectrum and forces so called super conformal invariance in TGD applications, which is indeed a basic symmetry of the theory.

There are infinitely many ways to choose the elementary events and each choice corresponds to a decomposition of the infinite set of integers n labelling the powers of p to disjoint subsets. These subsets can be also infinite. One can assign to this kind of decomposition a resolution which is the poorer the larger the subsets involved are. p-Adic thermodynamics would represent the situation in which the resolution is maximal since each set contains only single pinary digit. Note the analogy with the basis of completely localized wave functions in a lattice.

How to map p-adic transition probabilities to real ones?

p-Adic variants of TGD, if they exist, give rise to S-matrices and transition probabilities P_{ij}, which are p-adic numbers.

1. The p-adic probabilities defined by rows of S-matrix mapped to real numbers using canonical identification respecting the $q = r/s$ decomposition of rational number or its appropriate generalization should define real probabilities.

2. The simplest example would simple renormalization for the real counterparts of the p-adic probabilities $(P_{ij})_R$ obtained by canonical identification (or more probably its appropriate modification).

$$
\begin{aligned}
P_{ij} &= \sum_{k\geq 0} P_{ij}^k p^k \ , \\
P_{ij} &\rightarrow \sum_{k\geq 0} P_{ij}^k p^{-k} \equiv (P_{ij})_R \ , \\
(P_{ij})_R &\rightarrow \frac{(P_{ij})_R}{\sum_j (P_{ij})_R} \equiv P_{ij}^R \ .
\end{aligned}
\tag{7.4.-1}
$$

   The procedure converges rapidly in powers of p and resembles renormalization procedure of quantum field theories. The procedure automatically divides away one four-momentum delta function from the square of S-matrix element containing the square of delta function with no well defined mathematical meaning. Usually one gets rid of the delta function interpreting it as the inverse of the four-dimensional measurement volume so that transition rate instead of transition probability is obtained. Of course, also now same procedure should work either as a discrete or a continuous version.

3. Probability interpretation would suggest that the real counterparts of p-adic probabilities sum up to unity. This condition is rather strong since it would hold separately for each row and column of the S-matrix.

4. A further condition would be that the real counterparts of the p-adic probabilities for a given prime p are identical with the transition probabilities defined by the real S-matrix for real space-time sheets with effective p-adic topology characterized by p. This condition might allow to deduce all relevant phase information about real and corresponding p-adic S-matrices using as an input only the observable transition probabilities.

What it means that p-adically independent events are not independent in real sense?

A further condition would be that p-adic quantum transitions represent also in the real sense independent elementary events so that the real counterpart for a sum of the p-adic probabilities for a finite number of transitions equals to the sum of corresponding real probabilities. This condition is definitely too strong since only a single transition could correspond to a given p-adic norm of transition probability P_{ij} with i fixed.

The crucial question concerns the physical difference between the real counterpart for the sum of the p-adic transition probabilities and for the sum of the real counterparts of these probabilities, which are in general different:

$$
(\sum_j P_{ij})_R \neq \sum_j (P_{ij})_R \ . \tag{7.4.0}
$$

The suggestion is that p-adic sum of the transition probabilities corresponds to the experimental situation, when one does not monitor individual transitions but using some common experimental signature only looks whether the transition leads to this set of the final states or not. When one looks each transition separately or effectively performs different experiment by considering only one transition channel in each experiment one must use the sum of the real probabilities. More precisely, the choice of the experimental signatures divides the set U of the final states to a disjoint union $U = \cup_i U_i$ and one must define the real counterparts for the transition probabilities P_{iU_k} as

$$\begin{aligned} P_{iU_k} &= \sum_{j\in U_k} P_{ij} \ , \\ P_{iU_k} &\rightarrow (P_{iU_k})_R \ , \\ (P_{iU_k})_R &\rightarrow \frac{(P_{iU_k})_R}{\sum_l (P_{iU_l})_R} \equiv P^R_{iU_k} \ . \end{aligned} \tag{7.4.-2}$$

The assumption means deep a departure from the ordinary probability theory. If p-adic physics is the physics of cognitive systems, there need not be anything mysterious in the dependence of the behavior of system on how it is monitored. At least half-jokingly one might argue that the behavior of an intelligent system indeed depends strongly on whether the boss is nearby or not. The precise definition for the monitoring could be based on the decomposition of the density matrix representing the entangled subsystem into a direct sum over the subspaces associated with the degenerate eigenvalues of the density matrix. This decomposition provides a natural definition for the notions of the monitoring and resolution.

The renormalization procedure is in fact familiar from standard physics. Assume that the labels j correspond to momenta. The division of momentum space to cells of a given size so that the individual momenta inside cells are not monitored separately means that momentum resolution is finite. Therefore one must perform p-adic summation over the cells and define the real probabilities in the proposed manner. p-Adic effects resulting from the difference between p-adic and real summations could be the counterpart of the renormalization effects in QFT. It should be added that similar resolution can be defined also for the initial states by decomposing them into a union of disjoint subsets.

An alternative interpretation for the degenerate eigenvalues has emerged years after writing this. The sub-spaces corresponding to given eigenvalue of density matrix represent entangled states resulting in state function reduction interpreted as measurement of density matrix. This entanglement would be negentropic and represent a rule/concept, whose instances the superposed state pairs are. The information measure would Shannon entropy based on the replacement of the probability appearing as argument of logarithm with its p-adic norm. This entropy would be negative and therefore measure the information associated with the entanglement. This number theoretic entropy characterizes two particle state rather than single particle state and has nothing to do with the ordinary Shannon entropy.

Maybe one could say that finite measurement resolution implies automatically conceptualization and rule building. Abstractions are indeed obtained by dropping out the details.

7.4.3 P-Adic Thermodynamics

The p-adic field theory limit as such is not expected to give a realistic theory at elementary particle physics level. The point is that particles are expected to be either massless or possess mass of order 10^{-4} Planck mass. The p-adic description of particle massivation described in the third part of the book shows that p-adic thermodynamics provides the proper formulation of the problem. What is thermalized is Virasoro generator L_0 (mass squared contribution is not included to L_0 so that states do not have fixed conformal weight). Temperature is quantized purely number theoretically in low temperature limit ($exp(H/kT) \rightarrow p^{L_0/T}$, $T = 1/n$): in fact, partition function does not even exist in high temperature phase. The extremely small mixing of massless states with Planck mass states implies massivation and predictions of the p-adic thermodynamics for the fermionic masses are in excellent agreement with experimental masses. Thermodynamic approach also explains the emergence of the length scale L_p for a given p-adic condensation level and one can develop arguments explaining why primes near prime powers of two are favored.

It should be noticed that rational p-adic temperatures $1/T = k/n$ are possible, if one poses the restriction that thermal probabilities are non-vanishing only for some subalgebra of the Super Virasoro algebra isomorphic to the Super Virasoro algebra itself. The generators L_{kn}, G_{kn}, where k is a positive integer, indeed span this kind of a subalgebra by the fractality of the Super Virasoro algebra and $p^{L_0/T}$ is integer valued with this restriction.

One might apply thermodynamics approach should also in the calculation of S-matrix. What is is needed is thermodynamical expectation value for the transition amplitudes squared over incoming and outgoing states. In this expectation value 3-momenta are fixed and only mass squared varies.

7.4.4 Generalization Of The Notion Of Information

TGD inspired theory of consciousness, in particular the formulation of Negentropy Maximization Principle (NMP) in p-adic context, has forced to rethink the notion of the information concept. In TGD state preparation process is realized as a sequence of self measurements. Each self measurement means a decomposition of the sub-system involved to two unentangled parts. The decomposition is fixed highly uniquely from the requirement that the reduction of the entanglement entropy is maximal.

The additional assumption is that bound state entanglement is stable against self measurement. This assumption is somewhat ad hoc and it would be nice to get rid of it. The only manner to achieve this seems to be a generalized definition of entanglement entropy allowing to assign a negative value of entanglement entropy to the bound state entanglement, so that bound state entanglement would actually carry information, in fact conscious information (experience of understanding). This would be very natural since macro-temporal quantum coherence corresponds to a generation of bound state entanglement, and is indeed crucial for ability to have long lasting non-entropic mental images.

The generalization of the notion of number concept leads immediately to the basic problem. How to generalize the notion of entanglement entropy that it makes sense for a genuinely p-adic entanglement? What about the number-theoretically universal entanglement with entanglement probabilities, which correspond to finite extension of rational numbers? One can also ask whether the generalized notion of information could make sense at the level of the space-time as suggested by quantum-classical correspondence.

In the real context Shannon entropy is defined for an ensemble with probabilities p_n as

$$S = -\sum_n p_n log(p_n) \ . \tag{7.4.-1}$$

As far as theory of consciousness is considered, the basic problem is that Shannon entropy is always non-negative so that as such it does not define a genuine information measure. One could define information as a change of Shannon entropy and this definition is indeed attractive in the sense that quantum jump is the basic element of conscious experience and involves a change. One can however argue that the mere ability to transfer entropy to environment (say by aggressive behavior) is not all that is involved with conscious information, and even less so with the experience of understanding or moment of eureka. One should somehow generalize the Shannon entropy without losing the fundamental additivity property.

p-Adic entropies

The key observation is that in the p-adic context the logarithm function $log(x)$ appearing in the Shannon entropy is not defined if the argument of logarithm has p-adic norm different from 1. Situation changes if one uses an extension of p-adic numbers containing $log(p)$: the conjecture is that this extension is finite-dimensional. One might however argue that Shannon entropy should be well defined even without the extension.

p-Adic thermodynamics inspires a way to achieve this. One can replace $log(x)$ with the logarithm $log_p(|x|_p)$ of the p-adic norm of x, where log_p denotes p-based logarithm. This logarithm is integer valued ($log_p(p^n) = n$), and is interpreted as a p-adic integer. The resulting p-adic entropy

$$\begin{aligned} S_p &= \sum_n p_n k(p_n) \ , \\ k(p_n) &= -log_p(|p_n|) \ . \end{aligned} \tag{7.4.-1}$$

is additive: that is the entropy for two non-interacting systems is the sum of the entropies of composites. Note that this definition differs from Shannon's entropy by the factor $log(p)$. This entropy vanishes identically in the case that the p-adic norms of the probabilities are equal to one. This means that it is possible to have non-entropic entanglement for this entropy.

One can consider a modification of S_p using p-adic logarithm if the extension of the p-adic numbers contains $log(p)$. In this case the entropy is formally identical with the Shannon entropy:

$$S_p = -\sum_n p_n log(p_n) = -\sum_n p_n \left[-k(p_n)log(p) + p^{k_n} log(p_n/p^{k_n}\right] \ . \tag{7.4.0}$$

It seems that this entropy cannot vanish.

One must map the p-adic value entropy to a real number and here canonical identification can be used:

$$\begin{aligned} S_{p,R} &= (S_p)_R \times log(p)) \ , \\ (\sum_n x_n p^n)_R &= \sum_n x_n p^{-n} \ . \end{aligned} \tag{7.4.0}$$

The real counterpart of the p-adic entropy is non-negative.

Number theoretic entropies and bound states

In the case that the probabilities are rational or belong to a finite-dimensional extension of rationals, it is possible to regard them as real numbers or p-adic numbers in some extension of p-adic numbers for any p. The visions that rationals and their finite extensions correspond to islands of order in the seas of chaos of real and p-adic transcendentals suggests that states having entanglement coefficients in finite-dimensional extensions of rational numbers are somehow very special. This is indeed the case. The p-adic entropy entropy $S_p = -\sum_n p_n log_p(|p_n|)log(p)$ can be interpreted in this case as an ordinary rational number in an extension containing $log(p)$.

What makes this entropy so interesting is that it can have also negative values in which case the interpretation as an information measure is natural. In the real context one can fix the value of the value of the prime p by requiring that S_p is maximally negative, so that the information content of the ensemble could be defined as

$$I \equiv Max\{-S_p, \ p \ \text{prime}\} \ . \tag{7.4.1}$$

This information measure is positive when the entanglement probabilities belong to a finite-dimensional extension of rational numbers. Thus kind of entanglement is stable against NMP, and has a natural interpretation as bound state entanglement. The prediction would be that the bound states of real systems form a number theoretical hierarchy according to the prime $p and dimension$ of algebraic extension characterizing the entanglement.

Number theoretically state function reduction and state preparation could be seen as information generating processes projecting the physical states from either real or p-adic sectors of the state space to their intersection. Later an argument that these processes have a purely number theoretical interpretation will be developed based on the generalized notion of unitarity allowing the U-matrix to have matrix elements between the sectors of the state space corresponding to different number fields.

Number theoretic information measures at the space-time level

Quantum classical correspondence suggests that the notion of entropy should have also space-time counterpart. Entropy requires ensemble and both the p-adic non-determinism and the non-determinism of Kähler action allow to define the required ensemble as the ensemble of strictly deterministic regions of the space-time sheet. One can measure various observables at these space-time regions, and the frequencies for the outcomes are rational numbers of form $p_k = n(k)/N$, where N is the number of strictly deterministic regions of the space-time sheet. The number theoretic entropies are well defined and negative if p divides the integer N. Maximum is expected to result for the largest prime power factor of N. This would mean the possibility to assign a unique prime to a given real space-time sheet.

The classical non-determinism resembles p-adic non-determinism in the sense that the space-time sheet obeys effective p-adic topology in some length and time scale range is consistent with this idea since p-adic fractality suggests that N is power of p.

7.5 About P-Adic Quantum Mechanics

An interesting question is whether p-adic quantum mechanics might exist in some sense. The purely formal generalization of the ordinary QM need not be very interesting physically and the following considerations describe p-adic QM as a limiting case of the p-adic field theory limit of TGD to be constructed later. This particular p-adic QM is based on the p-adic Hilbert-space, p-adic unitarity and p-adic probability concepts whereas the physical interpretation is based on the correspondence between the p-adic and real probabilities given by the canonical correspondence. p-Adic QM is expected to apply -if it applies at all- below the p-adic length scale $L_p = \sqrt{p}l$ and above L_p ordinary QM should work, when length scale resolution L_p is used.

Although one can define p-adic Schrödinger equation formally without any difficulty it is not at all obvious whether it emerges from the p-adic QFT limit of TGD. Therefore the following considerations - my first reaction to the question what p-adic quantum theory look like- should be taken as mere warming up exercises perhaps helping to get some familiarity with new concepts. In the next chapter "Negentropy Maximization Principle" a more serious approach starting directly from the condition that real and p-adic approaches must allow fusion to larger coherence whole will be discussed.

7.5.1 P-Adic Modifications Of Ordinary Quantum Mechanics

One can consider several modifications of the ordinary quantum mechanics depending on what kind of p-adicizations one is willing to make.

p-Adicization in dynamical degrees of freedom

The minimal alternative is to replace time- and spatial coordinates with their p-adic counterparts so that the space time is a Cartesian power of R_p. A more radical possibility is to replace the 3-space with a 3-dimensional algebraic extension of the p-adic numbers. This means that space time is replaced with a Cartesian product of R_p and its 3-dimensional extension. The most radical possibility, suggested by the relativistic considerations, is a four-dimensional algebraic extension treating space and time degrees of freedom in an equal position: this alternative is encountered in the formulation of the p-adic field theory limit of TGD.

In practice the formulation of the quantum theory involves an action principle defining the so called classical theory and this is defined by using the integral of the action density. These integrals certainly exists as real quantities and are defined by the Haar measure for the p-adic numbers. Algebraic continuation of real integrals seems to be the only reasonable manner to defined these integrals.

p-Adicization at Hilbert space level

One can imagine essentially two different ways to p-adicize Hilbert space.

1. The first approach, followed in [H1], is to keep Schrö-dinger amplitudes complex. In this case it is better to consider a Cartesian power of R_p instead of an algebraic extension as a coordinate space. The canonical identification allows to replace the expressions of the coordinate and momentum operators via their p-adic counterparts. For example, $x \times \Psi$ is replaced with $x \times_p \Psi$, where p-adic multiplication rule is used. Derivative corresponds to a p-adic derivative. It was the lack of the canonical identification replacement, which forced to give up the straightforward generalization of standard QM in the approach followed in [A26] , [H1]. What this approach effects, is the replacement of the ordinary continuity and

differentiability and concepts with the p-adic differentiability and the approach looks rather reasonable manner to construct a fractal quantum mechanics. This approach however is not applicable in the present context.

2. A more radical approach uses Schrödinger amplitude with values in some complex extension, say a square root allowing extension of the p-adic numbers. p-Adic inner product implies that the ordinary unitarity and probability concepts are replaced with there p-adic counterparts. This approach looks natural for various reasons. The representation theory for the Lie-groups generalizes to p-adic case and the replacement implies certain mathematical elegance since p-analyticity and the realization of the p-adic conformal invariance becomes possible. It will be found that p-adic valued inner product is the natural inner product for the quantized harmonic oscillator and for Super Virasoro representations. The concept of the p-adic probability makes sense as first shown by [A24]. The physical interpretation of the theory is however always in terms of the real numbers and the canonical identification provides the needed tool to map the predictions of the theory to real numbers. That physical observables are always real numbers is suggested by the success of the p-adic mass calculations. p-Adic probabilities can be mapped to real probabilities and in the last chapter of the third part of the book it is shown that this correspondence predicts genuinely novel physical effects.

The p-adic representations of the Super Virasoro algebra to be used are defined in the p-adic Hilbert space and everything is well defined at algebraic level if 4- ($p > 2$) or 8- ($p = 2$) dimensional algebraic extension allowing square roots is used. Unitarity concept generalizes in a straightforward manner to the p-adic context and the elements of the S-matrix should have values in the same extension of the p-adic numbers. The requirement that the squares of S-matrix elements are p-adically real numbers gives strong constraints on the S-matrix elements since the quantities $S(m,n)\bar{S}(m,n)$ in general belong to the 4- (2-) dimensional real subspace $x + \theta y + \sqrt{p}z + \sqrt{p}\theta u$ of the 8- (4-) dimensional extension and p-adic reality implies the conditions: $y = z = .. = u = 0$. Reality conditions can be solved always since the solution involves only square roots of rational functions. What is exciting is that space time and embedding space dimensions for the extension allowing square roots are forced by the quantum mechanical probability concept, by p-adic group theory and by the p-adic Riemannian Geometry.

The existence of the p-adic valued definite integral is crucial concerning the practical construction of the p-adic Quantum Mechanics.

1. In the ordinary wave mechanics the inner product involves an integration over WCW degrees of freedom. This inner product can be generalized to the p-adic integral of $\bar{\Psi}_1\Phi_2$ over the 3-space using p-adic valued integration defined in the first chapter, which works for all analytic functions and also for p-adic counterparts of the plane waves (non-analytic functions).

2. The perturbative formulation QM in terms of the time development operator

$$U(t) = P(exp(i\int exp(\int dt\ V)) \ , \quad (7.5.1)$$

generalizes to the p-adic context. In particular, the concept of the time ordered product $P(...)$ appearing in the definition of the time development operator generalizes since the canonical identification induces ordering for the values of the p-adic time coordinate: $t_1 < t_2$ if $(t_1)_R < (t_2)_R$ holds true. Non-trivialities are related to the p-adic existence of the time development operator: for sufficiently larger values of the time coordinate, the exponent appearing in the time development operator does not exist p-adically and this implies infrared cutoff time and length scale in the p-adic QM.

One can define the action of the time development operator for longer time intervals only if one makes some restrictions on the physical states appearing in the matrix elements. This could explain color confinement number theoretically. For sufficiently long time intervals the color interaction part of the interaction Hamiltonian is so large for colored states that p-adic time development operator fails to exist number theoretically and one must restrict the physical states to be color singlets.

The generalization of the p-adic formula for Riemann integral [K63] suggests an exact formula for the time ordered product. The first guess is that one simply forms the product

$$\begin{aligned} Pexp(i\int_0^t Hdt) &\equiv P\prod_n exp\left[iV(t(n))\Delta t(n)\right] \ , \\ \Delta t(n) &= t_+(n) - t_-(n) = (1+p)p^{m(n)} \ , \end{aligned} \tag{7.5.1}$$

to obtain the value of the time ordered product for time values t having finite number of pinary digits. The product is over all points $t(n)$ having finite number of pinary digits and $m(n)$ is the highest pinary digit in the expansion of $t(n)$ and $t_\pm(n)$ denote the two p-adic images of the real coordinate $t(n)_R$ under canonical identification. $\Delta t(n)$ corresponds to the difference of the p-adic time coordinates, which are mapped to the same value of the real time coordinate in canonical identification so that one can regard the time ordered product as a limiting case in which real time coordinate differences are exactly zero in the time ordered product.

The time ordering of the product is induced by canonical identification from real time ordering. This time development operator is defined for time values with finite number of pinary digits only and defines p-adic pseudo constant. The hope is that the inherent non-determinism of the p-adic differential equations, implied by the existence of the p-adic pseudo constants, makes it possible to continue this function to a p-adically differentiable function of the p-adic time coordinate satisfying the counterpart of the Schrödinger equation for the time development operator.

Not surprisingly, number theoretical problems are encountered also now: the exponential $exp\left[iV(t(n))\Delta t(n)\right]$ need not exist p-adically. The possibility of p-adic pseudo constants suggests that one could simply drop off the troublesome exponentials: this has far reaching physical consequences [K57].

7.5.2 P-Adic Inner Product And Hilbert Spaces

Concerning the physical applications of algebraically extended p-adic numbers the problem is that p-adic norm is not in general bilinear in its arguments and therefore it does not define inner product and angle. One can however consider a generalization of the ordinary complex inner product $\bar{z}z$ to a p-adic valued inner product. It turns out that p-adic quantum mechanics in the sense as it is used in p-adic TGD can be based on this inner product.

The algebraic generalization of the ordinary Hilbert space inner product is bilinear and symmetric, defines p-adic valued norm. The norm can however for non-vanishing states. This inner product leads to p-adic generalization of unitarity and probability concept. The solution of the unitarity condition $\sum_k S_{mk}\bar{S}_{nk} = \delta(m,n)$ involves square root operations and therefore the minimal extension for the Hilbert space is 4-dimensional in $p > 2$ case and 8-dimensional in $p = 2$ case. Of course, extensions of arbitrary dimension are allowed.

The inner product associated with a minimal extension allowing square root near real axis provides a natural generalization of the real and complex Hilbert spaces respectively. Instead of real or complex numbers, a square root allowing algebraic extension extension appears as the multiplier field of the Hilbert space and one can understand the points of Hilbert space as infinite sequences $(Z_1, Z_2, ..., Z_n, ...)$, where Z_i belongs to the extension. The inner product $\sum_k \langle Z_k^1, Z_2^k\rangle$ is completely analogous to the ordinary Hilbert space inner product.

The generalization of the Hilbert space of square integrable functions to a p-adic context is far from trivial since definite integral in in general ill defined procedure. Second problem is posed by the fact that p-adic counterparts of say oscillator operator wave functions do not exist in the entire p-adic variant of the configuration space. Algebraic definition of the inner product by using the rules of Gaussian integration provides a possible solution to the problem.

For Fock space generated by anti-commuting fermionic and commuting bosonic oscillator operators the p-adic counterpart exists naturally and it seems that Fock spaces can be seen as universal Hilbert spaces with rational coefficients identifiable as subspaces of both real Fock space and of all p-adic Fock spaces.

7.5.3 P-Adic Unitarity And P-Adic Cohomology

p-Adic unitarity and probability concepts lead to highly nontrivial conclusions concerning the general structure of the p-adic S-matrix. The most general S-matrix is a product of a complex rational (extended rationals are also possible) unitary S-matrix S_Q and a genuinely p-adic S-matrix S_p which deviates only slightly from unity

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ T &= O(p^0) \ . \end{aligned} \tag{7.5.1}$$

for $p \ mod \ 4=3$ allowing imaginary unit in its four-dimensional algebraic extension. In perturbative context one expects that the p-adic S-matrix differs only slightly from unity. Using the form $S=1+iT$, $T=O(p^0)$ one would obtain in general transition rates of order inverse of Planck mass and theory would have nothing to do with reality. Unitarity requirement implies iterative expansion of T in powers of p and the few lowest powers of p give excellent approximation for the physically most interesting values of p.

The unitarity condition implies that the moduli squared of the matrix T in $S=1+iT$ are of order $O(p^{1/2})$ if one assumes a four-dimensional p-adic extension allowing square root for the ordinary p-adic numbers and one can write

$$\begin{aligned} S &= 1+i\sqrt{p}T \ , \\ i(T-T^{\dagger}) &+ \sqrt{p}TT^{\dagger}=0 \ . \end{aligned} \tag{7.5.1}$$

This expression is completely analogous to the ordinary one since $i\sqrt{p}$ is one of the units of the four-dimensional algebraic extension. Unitarity condition in turn implies a recursive solution of the unitary condition in powers of p:

$$\begin{aligned} T &= \sum_{n\geq 0} T_n p^{n/2} \ , \\ T_n - T_n^{\dagger} &= \frac{1}{i}\sum_{k=0,..,n-1} T_{n-1-k}T_k^{\dagger} \ . \end{aligned} \tag{7.5.1}$$

If algebraic extension is not allowed then the expansion is in powers of p instead of $\sqrt{p}$. Note that the real counterpart of the series converges extremely rapidly for physically interesting primes (such as $M_{127}=2^{127}-1$).

In the p-adic context S-matrix $S=1+T$ satisfies the unitarity conditions

$$T+T^{\dagger} = -TT^{\dagger} \tag{7.5.2}$$

if the conditions

$$\begin{aligned} T &= T^{\dagger} \ , \\ T^2 &= 0 \ . \end{aligned} \tag{7.5.2}$$

defining what might be called p-adic cohomology, are satisfied [L29]. In the real context these conditions are not possible to satisfy as is clear from the fact that the total scattering rate from a given state, which is proportional to T^2_{mm} vanishes.

p-Adic cohomology defines a symmetry analogous to BRST symmetry: if T satisfies unitarity conditions and T_0 satisfies the conditions

$$\begin{aligned} &T_0 = T_0^{\dagger} \ , \qquad\qquad T_0^2 = 0 \ , \\ &\{T_0,T\} = T_0T+TT_0 = 0 \ , \end{aligned} \tag{7.5.3}$$

unitary conditions are satisfies also by the matrix $T_1=T+T_0$. The total scattering rates are same for T and T_1.

7.5.4 The Concept Of Monitoring

The relationship between p-adic and real probabilities involves the hypothesis that real transition probabilities depend on the cognitive resolution. Cognitive resolution is defined by the decomposition of the state space H into direct sum $H = \oplus H_i$ so that the experimental situation cannot differentiate between different states inside H_i. Each resolution defines different real transition probabilities unlike in ordinary quantum mechanics. Physically this means that the arrangement, where each state in H_i is monitored separately differs from the situation, when one only looks whether the state belongs to H_i. One can say that monitoring affects the behavior of a p-adic subsystem. Of course, these exotic effects relate to the physics of cognition rather than real physics.

Standard probability theory, which also lies at the root of the standard quantum theory, predicts that the probability for a certain outcome of experiment does not depend on how the system is monitored. For instance, if system has N outcomes $o_1, o_2, ...o_N$ with probabilities $p_1, ..., p_N$ then the probability that o_1 or o_2 occurs does not depend on whether common signature is used for o_1 and o_2 or whether observer also detects which of these outcomes occurs. The crucial signature of p-adic probability theory is that monitoring affects the behavior of the system.

Physically monitoring is represented by quantum entanglement [K56], and differentiates between two eigen states of the density matrix only provided the eigenvalues of the density matrix are different. If there are several degenerate eigenvalues, quantum jump occurs to any state in the eigen space and one can predict only the total probability for the quantum jump into this eigen space: the real probabilities for jumps into individual states are obtained by dividing total real probability by the degeneracy factor. Hence the p-adic probability for a quantum jump to a given eigenspace of density matrix is p-adic sum of probabilities over the eigen states belonging to this eigenspace:

$$P_i = \frac{(n(i)P(i))_R}{\sum_j (n(j)P(j))_R} \quad .$$

Here n_i are dimensions of various eigenspaces.

If the degeneracy of the eigenvalues is removed by an arbitrary small perturbation, the total probability for the transition to the same subspace of states becomes the sum for the real counterparts of probabilities and one has in good approximation:

$$P^R = \frac{n(i)P(i)_R}{[\sum_{j\neq i}\sum_j (n(j)P(j))_R + n(i)P(i)_R]} \quad .$$

Rather dramatic effects could occur. Suppose that the entanglement probability $P(i)$ is of form $P(i) = np$, $n \in \{0, p-1\}$ and that n is large so that $(np)_R = n/p$ is a considerable fraction of unity. Suppose that this state becomes degenerate with a degeneracy m and $mn > p$ as integer. In this kind of situation modular arithmetics comes into play and $(mnp)_R$ appearing in the real probability $P(1\ or\ 2)$ can become very small. The simplest example is $n = (p+1)/2$: if two states i and j have *very nearly equal but not identical* entanglement probabilities $P(i) = (p+1)p/2 + \epsilon$, $P(j) = (p+1)p/2 - \epsilon$, monitoring distinguishes between them for arbitrary small values of ϵ and the total probability for the quantum jump to this subspace is in a good approximation given by

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= 2\left[(p+1)p/2\right]_R \quad . \end{aligned} \tag{7.5.3}$$

and is rather large. For instance, for Mersenne primes $x \simeq 1/2$ holds true. If the two states become degenerate then one has for the total probability

$$\begin{aligned} P(1\ or\ 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= \frac{1}{p} \quad . \end{aligned} \tag{7.5.3}$$

The order of magnitude for $P(1\ or\ 2)$ is reduced by a factor of order $1/p$!

Since p-adicity is essential for the exotic effects related to monitoring, the exotic phenomena of monitoring should be related to the quantum physics of cognition rather than real quantum physics. A test for quantum TGD would be provided by the study of the dependence of the transition rates of quantum systems on the resolution of monitoring defined by the dimensions of the degenerate eigenspaces of the subsystem density matrix. One could even consider the possibility of measuring the value of the p-adic prime in this manner. The behavior of living systems is known to be sensitive to monitoring and an exciting possibility is that this sensitivity, if it really can be shown to have statistical nature, could be regarded as a direct evidence for TGD inspired theory of consciousness. Note that the mapping of the physical quantities to entanglement probabilities could provide an ideal manner to compare physical quantities with huge accuracy! Perhaps bio-systems have invented this possibility before physicists and this could explain the miraculous accuracy of biochemistry in realizing genetic code. The measurement of the monitoring effect could provide a way to determine the value of p_i for each p-adic region of space-time.

An alternative interpretation for the degenerate eigenvalues has emerged years after writing this. The sub-spaces corresponding to given eigenvalue of density matrix represent entangled states resulting in state function reduction interpreted as measurement of density matrix. This entanglement would be negentropic and represent a rule/concept, whose instances the superposed state pairs are. The information measure would Shannon entropy based on the replacement of the probability appearing as argument of logarithm with its p-adic norm. This entropy would be negative and therefore measure the information associated with the entanglement. This number theoretic entropy characterizes two particle state rather than single particle state and has nothing to do with the ordinary Shannon entropy.

Maybe one could say that finite measurement resolution implies automatically conceptualization and rule building. Abstractions are indeed obtained by dropping out the details.

7.5.5 P-Adic Schrödinger Equation

The emergence of the p-adic infrared cutoff

The experience with the construction of the p-adic counterpart of the standard model shows that p-adic quantum theory involves in practice infrared cutoff length scale in both time and spatial directions. The cutoff length scale comes out purely number theoretically. In the time like direction the cutoff length scale comes out from the exponent of the time ordered integral: p-adic exponent function $exp(x)$ does not exist unless the p-adic norm of the argument is smaller than one and this in turn means that $P(exp(i\int_0^t Vdt))$ does not exist for too larger values of time argument. A more concrete manner to see this is to consider time dependence for the eigenstates of Hamiltonian: the exponent $exp(iEt)$ exists only for $|Et|_p < 1$. The necessity of the spatial cutoff length scale is seen by considering concrete examples. For instance, the p-adic counterparts of the harmonic oscillator Gaussian wave functions are defined only in a finite range of the argument. As far as the definition of exponent function is considered one must keep in mind that the formal exponent function does not have the usual periodicity properties. The definition as a p-adic plane wave gives the needed periodicity properties but also in this case the infrared cutoff is necessary.

One should be able to construct also global solutions of the p-adic Schrödinger equation. The concept of p-adic integration constant might make this possible: by multiplying the solution of thhe Schrödinger equation with a constant depending on a finite number of the pinary digits, one can extend the solution to an arbitrary large region of the space time. What one cannot however avoid is the decomposition of the space time into disjoint quantization volumes.

One of the original motivation to introduce p-adic numbers was to introduce ultraviolet cutoff as a p-adic cutoff but, as the considerations of the second part of the book show, UV divergences are absent in the p-adic case and short distance contributions to the loops are negligibly small so that the mere p-adicization eliminates automatically UV divergences. Rather, it seems that the length scale L_p serves as an infrared cutoff and, if a length scale resolution rougher than L_p is used, ordinary real theory should work. Only in the length scales $L \leq L_p$ should the p-adic field theory and Quantum Mechanics be useful. The applicability of the real QM for length scale resolution $L \geq L_p$ is in accordance with the fact that the real continuity implies p-adic continuity.

Formal p-adicization of the Schrödinger equation

The formal p-adic generalization of the Schrödinger equation is of the following general form

$$\theta \frac{d\Psi}{dt} = H\Psi \ , \qquad (7.5.4)$$

where H is in some sense Hermitian operator. If Schödinger amplitudes are complex values θ can be taken to be imaginary unit i. The same identification is possible if Ψ possesses values in the extension of p-adic allowing square root and the condition $p \ mod \ 4 = 3$ or $p = 2$ guaranteeing that $\sqrt{-1}$ does not exist as an ordinary p-adic number, is satisfied. For $p \ mod \ 4 = 1$ the situation is more complicated since imaginary unit i does not in general belong to the generators of the minimal extension allowing a square root. An open problem is whether one could replace θ appearing in the quadratic extension and define complex conjugation as the operation $\theta \rightarrow -\theta$. The analogy with the ordinary quantum mechanics suggests the form

$$H = -\frac{\nabla^2}{2m} + V \ , \qquad (7.5.4)$$

for the Hamiltonian in $p \ mod \ 4 = 3$ case. In the complex case ∇^2 is obtained by replacing the ordinary derivatives with the p-adic derivatives and V is a p-adically differentiable function of the coordinates typically obtained from a p-analytic function via the canonical identification.

Although the formal p-adicization is possible, it is not at all obvious whether one can get anything physically interesting from the straightforward p-adicization of the Schrödinger equation. The study of the p-adic hydrogen atom shows that formal p-adicization need not have anything to do with physics. For instance, Coulomb potential contains a factor $1/4\pi$ not existing p-adically, the energy eigenvalues depend on π and the straightforward p-adic counterparts of the exponentially decreasing wave functions are not exponentially decreasing functions p-adically and do not even exist for sufficiently large values of the argument r. It seems that a more realistic manner to define the p-adic Schrödinger equation is as limiting case of the p-adic field theory. Of course, it might also be that p-adic Schrödinger equation does not make sense. A more radical solution of the problems is the allowance of finite-dimensional extensions of p-adic numbers allowing also transcendental numbers.

p-Adic harmonic oscillator

The formal treatment of the p-adic oscillator using oscillator operator formalism is completely analogous to that of the ordinary harmonic oscillator. The only natural inner product is the p-adic valued one. That the treatment is correct is suggested by the fact that it is purely algebraic involving only the p-adic counterpart of the oscillator algebra. The matrix elements of the oscillator operators $a^\dagger$ and a involve square roots and they exist provided the minimal extension allowing square roots appears as a coefficient ring of the Hilbert space. If two-dimensional quadratic extension not containing $\sqrt{p}$ is used occupation number must be restricted to the range $[0, p-1]$. If the Hilbert space inner product based on non-degenerate p-adic inner product $Z_c Z + \hat{Z}_c \hat{Z}$ the extension implies a characteristic degeneracy of states with complex amplitudes related to the conjugation $\sqrt{p} \rightarrow -\sqrt{p}$. 2-adic and p-adic cases differ in radical manner since the dimensions of the extension are 4 for $p > 2$ and 8 for $p = 2$. Since the representations of the Kac Moody and Super Virasoro algebras are based on oscillator operators this means that there is deep difference between $p = 2$ and $p > 2$ p-adic conformal field theories.

The p-adic energy eigen values are $E_n = (n + 1/2)\omega_0$ and their real counterparts form a quasi-continuous spectrum in the interval $(2, 4)$ for $p = 2$ and $(1, p)$ for $p > 2$! If p is very large (of order 10^{38} in TGD applications) the small quantum number limit $n < p$ gives the quantum number spectrum of the ordinary quantum mechanics. The occupation numbers $n > p$ have no counterpart in the conventional quantum theory and it seems that the classical theory with a quasi-continuous spectrum but with energy cutoff $p\omega_0$ is obtained at the limit of the arbitrarily large occupation numbers. The limit $p \rightarrow \infty$ gives essentially the classical theory with no upper bound for the energy.

The results suggests the idea that p-adic QM might be somewhere halfway between ordinary QM and classical mechanics. This need not however be the case as the study of the p-adic thermodynamics suggests. p-Adic thermodynamics allows a low temperature phase $exp(E_n/T) \equiv p^{n/T_k}$, $T_k = 1/k$, with quantized value of temperature. In this phase the probabilities for the energy eigenstates E_n, $n = \sum_k n_k p^k$ are extremely small except for the smallest values of n so that low temperature thermodynamics does not allow the effective energy continuum. One might argue that situation changes in the high temperature phase. The problem is that p-adic thermodynamics for the harmonic oscillator allows only formally high temperature phase $T = t_0\omega_0/p^k$, $k = 1, 2, ...$, $|t_0| = 1$. The reason is that Boltzmann weights $exp(-E_n/T) = exp(np^k/t_0)$ have p-adic norm equal to 1 so that the sum of probabilities giving free energy converges only formally. If one accepts the formal definition of the free energy as $exp(F) \equiv 1/(1 - exp(-E_0/T))$ then the real counterpart of the energy spectrum indeed becomes continuum also in the thermodynamic sense.

Consider next what a more concrete treatment using Schrödinger equation gives. The p-adic counterpart of the Schrödinger equation is formally the same as the ordinary Schrödinger equation. Ψ is assumed to have values in a minimal extension of p-adic numbers allowing square root and possessing imaginary unit so that the condition $p \ mod \ 4 = 3$ or $p = 2, 3$ must hold true. For the energy momentum eigenstates the equation reduces to

$$(-\frac{d^2}{dy^2} + y^2)\Psi = 2e\Psi \ , \tag{7.5.5}$$

where the dimensionless variables $y = \sqrt{\omega}x$ and $e = \frac{E}{\omega}$ have been introduced. This transformation makes sense provided ω possesses p-adic square root.

The solution ansatz to this equation can be written in the general form $\Psi = exp(-y^2/2)H_{e-1/2}(y)$, where H is the p-adic counterpart of a Hermite polynomial. The first thing to notice is that vacuum wave function does not converge in a p-adic sense for all values of y. A typical term in series is of the form $X_n = \frac{y^{2n}}{2^n n!}$. In ordinary situation the factors, in particular $n!$, in the numerator imply convergence but in present case the situation is exactly the opposite.

In 2-adic case both the factor 2^n and the factor $n!$ in the denominator cause troubles whereas for $p > 2$ the p-adic norm of 2^n is equal to one. $n!$ gives at worst the power 2^{n-1} to the 2-adic norm. Therefore the 2-adic norm of X_n behaves as $N(X_n) \simeq |y_2|^{2n}2^n2^{n-1}$. The convergence is therefore achieved for $|y|_2 \leq 1/4$ only. For $p > 2$ the convergence is achieved for $|y|_p \leq 1/p$. One can continue the oscillator Gaussian to a globally defined function of y by observing that the scaling $y \rightarrow y/\sqrt{2}$ corresponds to taking a square root of the oscillator Gaussian and this square root exists if minimal quadratic extension allowing square root is used. In the usual situation the function $H_e(y)$ must be polynomial since otherwise it behaves as $exp(y^2)$ and does not converge: this implies the quantization of energy also now.

The inner product, which should orthogonalize the states is the p-adic valued inner product based on the p-adic generalization of the definite integral. The generalizations of the analytic formulas encountered in the real case should hold true also now. The guess motivated by the formal treatment is that p-adic energies are quantized according to the usual formula and classical energies form a continuum below the upper bound $e_R \leq 4$ in 2-adic case and $e_R =\leq p$ in p-adic case. In fact, the mere requirement $|e|_p \leq 1$ implies that energy is quantized according to the formula $e = n + 1/2$ in p-adic case.

p-Adic fractality in the temporal domain

The assumption that p-adic physics gives faithful cognitive representation of the real physics leads to highly nontrivial predictions, the most important prediction being p-adic fractality with long range temporal correlations and microtemporal chaos.

In p-adic context the diagonalization of the Hamiltonian for N-dimensional state space in general requires N-dimensional algebraic extension of p-adic numbers even when the matrix elements of the Hamiltonian are complex rational numbers. TGD as a generalized number theory vision allows all algebraic extensions of p-adic numbers so that this is not a problem. The necessity to decompose p-adic Hamiltonian to a complex rational free part and p-adically small interaction part could provide the fundamental reason for why Hamiltonians have the characteristic decomposition into free and interaction parts. Of course, it might be that Hamiltonian formalism does not make sense in the p-adic context and should be replaced with the approach based on Lagrangian formalism: at least in case of p-adic QFT limit of TGD this approach seems to be more promising. One could also argue that the very fact that p-adic physics provides a cognitive representations of TGD based physics gives a valuable guide to the real physics itself, and that one should try to identify the constraints on real physics from the requirement that its p-adic counterpart exists. The following discussion is motivated by this kind of attitude.

The emergence of various dynamical time scales is a very general phenomenon. For instance, it seems that strong and weak interactions correspond to different time scales in well defined sense and that it is a good approximation to neglect strong interaction in weak time scales and vice versa. p-Adic framework gives hopes of finding a more precise formulation for this heuristics using number theoretical ideas. The basic observation is that the time ordered exponential of a given interaction Hamiltonian exists only over a finite time interval of length $T_p(n) = p^n L_p$. This suggests that one should distinguish between the time developments associated with various p-adic time scales $T_n = p^n L_p/c$: obviously temporal fractality would be in question.

More concretely, the p-adic exponential $exp(iH\Delta t)$ of the free Hamiltonian exists p-adically only if one assumes that Δt is a small rational proportional to a positive power of p: $\Delta t \propto p^n$. Of course, this restriction to the allowed values of Δt might be interpreted as a failure of the cognitive representation rather than a real physical effect. Alternatively, one might argue that the emergence of the p-adic time scales is a real physical effect and that one must define a separate S-matrix for each p-adic time scale $\Delta t \propto p^n$. Thus p-adic S-matrices for time intervals that differ from each other by arbitrarily long real time interval could be essentially identical. This would mean extremely precise fractal long range correlations and chaos in short time scales also at the level of real physics. This is certainly a testable and rather dramatic prediction in sharp contrast with standard physics views. $1/f$ noise could be seen as one manifestation of these long range correlations.

What would distinguish between different times scales would be different decomposition of the Hamiltonian to free and interaction parts to achieve interaction part which is p-adically small in the time scale involved. For instance, it could be possible to understand color confinement in this manner: in quark gluon plasma phase below the length scale L_p many quark states without any constraints on color are the natural state basis whereas above the length scale L_p physical states must be color singlets since otherwise time evolution operator does not exist.

In case of the cognitive representations of the external world canonical identification maps long external time and length scales to short internal time and length scales and vice versa. Thus p-adic fractality of the cognitive dynamics induces at the level of cognitive representation order in short length and time scales and chaos in long length and time scales: this is of course natural since sensory information comes mainly from the nearby spatiotemporal regions of the system. For self-representations there is chaos in short time scales and fractal long range correlations (so that our temptation to see our life as a coherent temporal pattern would not be self deception!). This kind of fractality is of course absolutely essential in order to understand bio-systems as intentional systems able to plan their future behavior. This prediction is about behavioral patterns of cognitive systems and also testable.

One can get a more quantitative grasp on this idea by studying the time development operator associated with a diagonalizable Hamiltonian. If the eigenvalues E_n of the diagonalized Hamiltonian have p-adic norms $|E_n|_p \leq p^{-m}$, the time evolution determined by this Hamiltonian is defined at most over a time interval of length norm $T_p(m) = p^{m-1}L_p$ since for time intervals longer than this the eigenvalues $exp(iE_nt)$ of $exp(iHt)$ do not exists as a p-adic numbers for all energy eigenstates. Thus one must restrict the time evolution to time scale $t \leq p^{m-1}L_p$: this is consistent with a p-adic hierarchy of interaction time scales.

An alternative approach is based on the requirement that the complex phase factors $exp(iET)$ for the eigenstates of the diagonal part of the Hamiltonian are complex rational phases forming a multiplicative group. This means that one can map the phase factors $exp(iET)$ directly to their p-adic counterparts as complex rational numbers. With suitable constraints on the energy spectrum this makes sense if the interaction time T is quantized so that it is proportional to a power of p. The decomposition of the Hamiltonian to free and interacting parts could be done in such a way that the exponential of Hamiltonian decomposes to a product of diagonal part representable as complex rational phases and interaction part which is of higher order in p so that ordinary exponential exists for sufficiently small values of interaction time. This decomposition depends on the p-adic time scale.

How to define time ordered products?

In perturbation theory one must deal with the p-adic counterpart of the time ordered exponential $\prod_n Pexp\left[\int_0^t H_{int}(n)dt\right]$ appearing in the definition of the time development operator. In the case of a non-diagonal, time dependent interaction Hamiltonian the very definition of the p-adic counterpart of the time ordered integral is far from obvious since p-adic numbers do not allow natural ordering. Perhaps the simplest possibility is based on Fourier analysis based on the use of Pythagorean phases. This automatically involves the introduction of a time resolution $\Delta t = q = m/n$ and discretization of the time coordinate. Depending on the p-adic norm of Δt one obtains a hierarchy of S-matrices corresponding to different p-adic fractalities. Time ordering would be naturally induced from the ordering of ordinary integers since only the integer multiples of Δt are involved in the discretized version of integral defined by the inner product for the Pythagorean plane waves. The requirement that all time values have same p-adic norm implies $T = n\Delta t$, $n = 0, ..., p-1$. If one assumes that long range fractal temporal order is present one can also allow time intervals $T = n\delta t + mp^k$ which correspond to arbitrarily long real time intervals.

p-Adic particle stability is not equivalent with real stability

It is natural to require that single hadron states are eigenstates for that part of the total Hamiltonian, which consists of the kinetic part of the Hamiltonian. If this the case, one can require that the effect of $exp(iH_0t)$ is just a multiplication by the factor $exp(iEt)$. The fact that particles are not stable against decay to many-particle states suggests that E must be complex. Generalizing the construction of the p-adic plane waves one could define this prefactor for all values of time even in this case. One can however criticize this approach: the introduction of the decay width as imaginary part of E is is category error since decay width characterizes the statistical aspects of the dynamics associated with quantum jumps rather than the dynamics of the Schrödinger equation.

p-Adic unitarity concept suggests a more elegant description. The truncated S-matrix describing the transitions $H_p \rightarrow H_p$ is unitary despite the fact that the transitions between different sectors are possible. This makes sense because the total p-adic transition probability from H_p to H_q, $q \neq p$, vanishes by generalized unitarity conditions. Generalizing, the p-adic representations of elementary particles and even hadrons would p-adically stable in the sense that the total p-adic decay probability would vanishes for them. One could also say that in absence of monitoring p-adic cognitive representation of particle would be stable. This picture is consistent with the notion of p-adic cohomology reducing unitarity conditions for S-matrix $S = 1 + iT$ to the conditions $T = T^\dagger$ and $T^2 = 0$. Of course, it would apply only at the level of cognitive physics.

7.6 Generalization Of The Notion Of Configuration Space

The number theoretic variants of Shannon entropy make sense for rational and even algebraic entanglement probabilities in finite-dimensional algebraic extensions of rationals and can have negative values so that negentropic entanglement (see **Fig.** `http://tgdtheory.fi/appfigures/cat.jpg` or **Fig. ??** in the appendix of this book) becomes possible. This leads to the vision that life resides in the intersection of real and p-adic worlds for which partonic 2-surfaces- the basic geometric objects- allow a mathematical definition making sense both in real and p-adic sense in preferred coordinates dictated to a high degree by embedding space symmetries. Rational functions with rational or algebraic coefficients provide a basic example of this kind of functions as also algebraic functions. This vision together with Negentropy Maximization Principle leads to an overall view about how the standard physics picture must be modified in TGD framework (see the next chapter [K56]).

The identification of life as something in the intersection means that there should be also physics outside it. In the real context this poses no problems of principle. But should one allow the continuation of the coefficients of rational functions to p-adic integers infinite as real integers? This seems to raise formidable looking challenges.

1. One should be able to formulate the geometry of the world of classical worlds (WCW) in p-adic sense and also construct p-adic counterparts for the integration over WCW. Since no physically acceptable p-adic variant of definite integral does exists, algebraic continuation seems to be the only possible manner to meet this challenge.

2. One must construct the p-adic counterparts of Kähler function or of its exponent (or both), Kähler metric and Kähler form at the level of WCW .

3. Kähler function identified as Kähler action for preferred extremal and defined as integral does not make sense as such in p-adic context and the only manner to define the p-adic variant of Kähler function is by algebraic continuation from the real sector through the intersection of real and p-adic worlds.

7.6.1 Is Algebraic Continuation Between Real And P-Adic Worlds Possible?

It seems that algebraic continuation is the only reasonable manner to tackle these challenges. The following considerations suggests that there are some hopes.

1. Recall that the basic geometric objects can be identified either as light-like 3-surfaces connecting the boundaries of causal diamond (intersection of future and past directed light-cones) or as space-like 3-surfaces at the boundaries of CD. The condition that the identifications are equivalent implies effective 2-dimensionality: the partonic 2-surfaces at the boundaries of causal diamonds (CDs) together with the distribution of four-dimensional tangent planes of space-time surface at the points of the partonic surface, are the basic geometric objects. The tangent space distribution codes for various quantum numbers such as four-momentum so that also these must be rational valued in the common sector. In the following I will just speak about partonic 2-surfaces. It is this space-time 2-surfaces for a given CD, which should be geometrized. 2-dimensionality obviously suggests a connection with algebraic geometry.

2. Number theoretic vision [K96] leads to the conclusion that the space-time sheets are quaternionic in the sense that the Kähler-Dirac gamma matrices assignable to the Kähler action in their octonionic representations span quaternionic (co-quaternionic) and thus associative (co-associative) subspace of complexified octonions at each point of the space-time surface. Quaternionicity would be realized in Minkowskian regions and co-quaternionicity in the space-like regions defining geometrization of Feynman diagrams. This notion is independent of the number field so that the notion of p-adic space-time sheet seems to make sense. Note that also the field equations and criticality condition for the preferred extremals [K111] make sense p-adically as purely algebraic conditions.

3. The representability of WCW as a union of symmetric spaces means an enormous simplification since everything reduces to a single point, most naturally the maximum of Kähler function for given values of zero modes. If this maximum is always an algebraic surface and if the Kähler function or its exponent for it is algebraic number (there is infinity of tunings of zero modes guaranteeing this), the maxima make sense also in suitable algebraic extensions of p-adic numbers. The maxima would obviously define the intersection of real and p-adic worlds.

   One might in fact argue that this is as it must be. What is cognitively representable is in the intersection of realities and p-adicities and mathematician can cognitively represent only these maxima and do perturbation theory around them and hope for a complete integrability.

4. What comes naturally in mind is that only p-adically small deformations of the partonic 2-surfaces in the intersections of the p-adic and real worlds are allowed at the p-adic side. If the exponent of Kähler function exists in some algebraic extension at the common point, its small perturbations can be expanded in powers of p as a functional of the coefficients of rational functions extended to p-adic numbers. Symmetric space structure of WCW raises

the hope that TGD is a completely integrable theory in the sense that the functional integral reduces to the exponent of Kähler action due to the cancellation of metric and Gaussian determinants the n-point functions. One would have effectively free field theory. If this is the case the functional integral would make sense also in p-adic context as algebraic continuation.

Consider now in more detail what the algebraic continuation could mean.

1. Kähler function is not uniquely defined since one can add to it a real part of a holomorphic function of WCW complex coordinates (associated with quantum fluctuating degrees of freedom) without affecting Kähler metric. By a suitable choice of this function algebraicity could be guaranteed for any partonic 2-surface. This symmetry is however much like gauge invariance, which suggests that functional integral expressions for n-point functions involving also normalization factors do not depend on the exponent of Kähler function at maximum. In the perturbative approach to quantum field theory the exponents indeed cancel from n-point functions. This would suggest that the algebraicity of Kähler function is only needed. One should be however be very cautious. The Kähler action for CP_2 type vacuum extremals has a deep meaning in TGD and would have interpretation in terms of a non-perturbative effect. If one allows the introduction of a finite-dimensional non-algebraic extension involving powers of some root of e (e^p exists p-adically) both the exponent of Kähler function and Kähler function exist p-adically if Kähler function is a rational number.

2. WCW Kähler metric can be defined in terms of second partial derivatives of the exponent of Kähler function and is algebraic if Kähler function or its exponent are algebraic functions of the preferred WCW coordinates defined by WCW symmetries. The tangent space distribution at X^2 codes information about quantum numbers - in particular four-momenta- which define a measurement interaction terms in Kähler action and by supersymmetry also in Kähler-Dirac action [K111]: by analogy with thermodynamics these terms are simply Lagrangian multiplier terms equating classical conserved charges of space-time surfaces in the quantum superposition with the quantal counterparts. By holography Kähler function or its exponent is expressible in terms of the data associated with X^2 and its tangent space and should be algebraic function of these data.

3. If Kähler function or its exponent is rational function of the parameters charactering partonic 2-surfaces, the continuation to the p-adic sectors at rational points is in principle possible. If Kähler function is proportional to a positive power of p its exponent exists automatically in p-adic context. For Kähler function this would mean that given partonic 2-surface would correspond to a finite number of primes only. The continuation of the exponent of Kähler function is not however very useful since WCW integral cannot be defined except by algebraic continuation. Exponent function behaves also completely differently in p-adic context than in real context (its p-adic norm equals always to one for instance). p-Adic thermodynamics would in turn suggest that the exponent function should be replaced by a power of p since it has desired convergence properties so that Kähler function divided by $log(p)$ should be rational (allowing roots of p in the algebraic extension).

4. The perturbative approach relies on n-point functions involving WCW Hamiltonians and their super-counterparts at the intersection. One would obtain algebraic expressions for the n-point functions involving also contravariant metric of WCW of as a propagator. If one always works in effectively finite-dimensional space (coefficients of polynomials with finite degree in the definition of partonic 2-surfaces involved and rational valued momenta) one has finite-dimensional space of partonic 2-surraces, and the propagator is an algebraic object as the inverse of the Kähler metric defined by the second derivatives of the Kähler function if K or is exponent is algebraic function. p-Adicization also means the continuation of the momenta to the p-adic sector.

5. WCW Hamiltonians and their super-counterparts are defined as integrals over partonic 2-surface and it is not at all obvious that the result is algebraic number even if these quantities themselves are rational functions even in the partonic 2-surfaces themselves are rational surfaces. The condition for being in the intersection should therefore include also the condition about the algebraic character of these objects.

6. One could of course wonder whether coupling constant renormalization involving logarithmic functions of mass scales and powers of π in QFT context could spoil this nice picture and force to introduce infinite-dimensional transcendental extensions of p-adic numbers. There is indeed the danger that symmetric space property is not enough to avoid infinite perturbation series coming from the expansions of WCW Hamiltonians and their super counterparts. This kind of series would obviously spoil the algebraic character. There are however hopes. First of all, tinite measurement resolution is one of the key aspects of quantum TGD and could boild down to a cutoff for the perturbation series. Secondly, the key idea of quantum criticality is that for the maxima of Kähler function the perturbative corrections sum up to zero since they are coded to the Kähler action itself since the scale of induced metric is proportional to the square of $\hbar$.

If this optimistic picture is correct, the algebraic continuation to p-adic sector would reduce to an algebraic continuation of the expressions for n-point functions and the U-matrix in real sector to the p-adic sector, and would be almost trivial since only continuation in momenta and WCW coordinates parametrizing partonic 2-surfaces representing maxima of Kähler function would be in question. Everything could be computed in the real sector. A practically oriented theoretician might of course have suggested this from the beginning. It must be added that this vision is the latest one and need not completely consistent with all what is represented in the sequel.

7.6.2 P-Adic Counterparts Of WCW Hamiltonians

One must continue the δM^4_+ local CP_2 Hamiltonians appearing in the integrals defining WCW Hamiltonians to various p-adic sectors. CP_2 harmonics are homogeneous polynomials with rational coefficients and do not therefore produce any trouble since normalization factors involve only square roots. The p-adicization of δM^4_+ function basis defining representations of Lorentz group involves more interesting aspects.

p-Adicization of representations of Lorentz group

In the light cone geometry Poincare invariance is strictly speaking broken to Lorentz invariance with respect to the tip of the light cone and at least cosmologically a more natural basis is characterized by the eigenvalues of angular momentum and boost operator in a given direction. The eigenvalue spectrum of the boost operator is continuous without further conditions. One can study these conditions in the realization of the unitary representations of Lorenz group as left translations in the Lorentz group itself by utilizing homogenous functions of four complex variables z^1, z^2, z^3, z^4 satisfying the constraint $z_1z_4 - z_2z_3 = 1$ expressing the fact that they correspond to the homogenous coordinates of the Lorentz group defined by that matrix elements of the SL(2, C) matrix

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} .$$

The function basis consists of

$$\begin{aligned} f^{a_1,a_2,a_3,a_4}(z_1,z_2,z_3,z_4) &= z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4} \ , \\ a_1 = m_1 + i\alpha, \quad & a_2 = m_2 - i\alpha \ , \\ a_3 = m_3 - i\alpha, \quad & a_4 = m_4 + i\alpha \ , \\ m_1 + m_2 = M \ , \quad & m_3 + m_4 = M \ . \end{aligned}$$

The action of Lorentz transformation is given by

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} . \tag{7.6.1}$$

and unimodular ($ad - bc = 1$). Lorentz transformation preserves the imaginary parts $i\alpha$ of the complex degrees $d_i = m \pm i\alpha$ of $z_k^{\pm i\alpha + m_k}$ (as can be seen by using binomial series representations for the transformed coordinates). Also the sums $m_1 + m_2 = M$ and $m_3 + m_4 = M$ are Lorentz invariants. Hence the representation is characterized by the pair (α, M). M corresponds to the minimum angular momentum for the $SU(2)$ decomposition of the representation.

The imaginary parts $i\alpha$ of the complex degrees correspond to the eigen values of Lorentz boost in the direction of the quantization axis of angular momentum. The eigen functions are proportional to the factor

$$\rho_1^{i2\alpha}\rho_2^{-i2\alpha}\rho_3^{-i2\alpha}\rho_4^{i2\alpha}\ per,$$
$$\rho_i = \sqrt{z_i\overline{z}_i}\ .$$

Since one can write $\rho^{i2\alpha} = e^{i2log(\rho)\alpha}$, these are nothing but the logarithmic plane waves. The value set of $\alpha \geq 0$ is continuous in the real context.

The requirement that the logarithmic plane waves are continuable to p-adic number fields and exist p-adically for rational values of $\rho_i = m/n$, quantizes the values of α. This condition is satisfied if the quantities $p^{i2\alpha_i} = e^{i2log(p)\alpha_i}$ exist p-adically for any prime. As shown in [K85], there seems to be no number theoretical obstructions for the simplest hypothesis $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$, with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes. The existence of p^{iy} in a finite-dimensional extension would require that α_i is proportional to π by a coefficient which for a given prime p_1 has sufficiently small p-adic norm so that the exponent can be expanded in powers series.

Obviously p-adicization gives strong quantization conditions. There is also a second possibility. As discussed in the same chapter, the allowance of infinite primes changes the situation. Let $X = \prod p_i$ be the product of all finite primes. $1 + X$ is the simplest infinite prime and the ratio $Y = X/(1 + X)$ equals to 1 in real sense and has p-adic norm $1/p$ for all finite primes. If one allows α to be proportional to a power Y, then the p-adic norm of α can be so small for all primes that the expansion converges without further conditions. Infinite primes will be discussed later in more detail.

Exactly similar exponents (p^{iy}) appear in the partition function decomposition of the Riemann Zeta, and the requirement that these quantities exist in a finite algebraic extension of p-adic numbers for the zeros $z = 1/2 + iy$ of ζ requires that $e^{ilog(p)y}$ is in a finite-dimensional extension involving algebraic numbers and e. One could argue that for the extensions of p-adics the zeros of Zeta define a universal spectrum of the eigen values of the Lorentz boost generator. This might have implications in hadron physics, where the so called rapidity distribution correspond to the distributions of the particles with respect to the variable characterizing finite Lorentz boosts.

Although the realization of the using the functions in Lorentz group differs from the discussed one, the conclusion is same also for them, in particular for the representation realized at the boundary of the light cone which is one of the homogenous spaces associated with Lorentz group.

Function basis of δM^4_+

One can consider two function basis for δM^4_+ and both function basis allow continuation to p-adic values under similar conditions.

1. *Spherical harmonic basis*

The first basis consists of functions $Y^l_m \times (r_M/r_0)^{n/2+i\rho}$, $n = -2, -1, 0, ...$ For $n = -2$ these functions define a unitary representation of Lorentz group. The spherical harmonics Y^l_m require a finite-dimensional algebraic extension of p-adic numbers. Radial part defines a logarithmic wave $exp[i\rho log(r_M/r_0)]$ and the existence of this for finite-dimensional extension of p-adic numbers for rational values ρ and r_M is guaranteed by $log(p) = q_1exp(q_2)/\pi$ ansatz under the conditions already discussed.

2. *Basis consisting of eigen functions of angular momentum and boost*

Another function basis of δM^4_+ defining a non-unitary representation of Lorentz group and of conformal algebra consists of eigen states of rotation generator and Lorentz boost and is given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k\ . \tag{7.6.2}$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2. The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{1+\rho^2)^2} d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n - k)$. This suggests that the condition

$$|m| \leq 2(n - k) \quad (7.6.3)$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3, 1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra. The representations of the Lorentz group are characterized by half-integer valued parameter $l_0 = m/2$ and complex parameter l_1. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [A42].

1. The unitary representations discussed in [A42] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labelled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labelled by three different quantum numbers.

2. The representations of [A42] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labelling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteeing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity in this sense is not possible.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, WCW spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Logarithmic waves and possible connections with number theory and fundamental physics

Logarithmic plane waves labelled by eigenvalues of the scaling momenta appear also in the definition of the Riemann Zeta defined as $\zeta(z) = \sum_n n^{-z}$, n positive integer [K85]. Riemann Zeta is expressible as a product of partition function factors $1/(1+p^{-x-iy})$, p prime and the powers n^{-x-iy} appear as summands in Riemann Zeta. Riemann hypothesis states that the non-trivial zeros of Zeta reside at the line $x = 1/2$. There are indeed intriguing connections. $Log(p)$ corresponds now to the $log(r_M/r_{min})$ and -x-iy corresponds to the scaling momentum $k_1 + ik_2$ so that the special physical role of the conformal weights $k_1 = 1/2 + iy$ corresponds to Riemann hypothesis. The appearance of powers of p in the definition of the Riemann Zeta corresponds to p-adic length scale hypothesis, ($r_M/r_0 = p$ in ζ and corresponds to a secondary p-adic length scale).

The assumption that the logarithmic plane waves are algebraically continuable from the rational points $r_M/r_{min} = m/n$ to p-adic plane waves using a finite-dimensional extension of p-adic numbers leads to the $log(p) = q_1exp(q_2)/\pi$ ansatz. Similar hypothesis is inspired by the hypothesis that Riemann Zeta is a universal function existing simultaneously in all number fields. This inspires several interesting observations.

1. p-adic length scale hypothesis stating that $r_{max}/r_{min} = p^n$ is consistent with the number theoretical universality of the logarithmic waves. The universality of Riemann Zeta inspires the hypothesis that the zeros of Riemann Zeta correspond to rational numbers and to preferred values $k_1 + ik_2$ of the scaling momenta appearing in the logarithmic plane waves. In the recent context the most general hypothesis would be that the allowed momenta k_2 correspond to the linear combinations of the zeros of Riemann Zeta with integer coefficients.

2. Hardmuth Mueller [B1] claims on basis of his observations that gravitational interaction involves logarithmic radial waves for which the nodes come as $r/r_{min} = e^n$. This is true if the the scaling momenta k_2 satisfy the condition $k_2/\pi \in Z$. Perhaps Mueller's logarithmic waves really could be seen as a direct signature of the fundamental symmetries of WCW . In particular, this would require $r_{max}/r_{min} = e^m$.

3. The special role of Golden Mean $\Phi = (1 + \sqrt{5})/2$ in Nature could be understood if also $log(\Phi) = q_1exp(q_2)/\pi$ or more general ansatz holds true. This would imply that the nodes of logarithmic waves can correspond also to the powers of Φ.

   One could of course argue that the number theory at the moment of Big Bang cannot have strong effects on what is observed in laboratory. This might be the case. On the other hand, the non-determinism of the Kähler action however strongly suggests that the construction of the WCW geometry involves all possible light like 3-surfaces of the future light cone so that logarithmic waves would appear in all length scales. Be as it may, it would be amazing if such an abstract mathematical structure as WCW geometry would have direct implications to cosmology and to the physics of living systems.

7.6.3 WCW Integration

Assuming that U-matrix exists simultaneously in all number fields (allowing finite-dimensional extensions of p-adics), the immediate question is whether also the construction procedure of the real S-matrix could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and WCW functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

Does symmetric space structure allow algebraization of WCW integration?

The basic structure is the rational WCW whose points have rational valued coordinates. This space is common to both real and p-adic variants of WCW . Therefore the construction of the generalized WCW as such is not a problem.

The assumption that WCW decomposes into a union of symmetric spaces labeled by zero modes means that the left invariant metric for each space in the union is dictated by isometries. It should be possible to interpret the matrix elements of WCW metric in the basis of properly normalized isometry currents as p-adic numbers in some finite extension of p-adic numbers allowing perhaps also some transcendentals. Note that the Kähler function is proportional to the inverse of Kähler coupling strength α_K which depends on p-adic prime p, and does seem to be a rational number if one takes seriously various arguments leading to the hypothesis $1/\alpha_K = klog(K^2)$, $K^2 = p \times 2 \times 3 \times 5.. \times 23$, and $k = \pi/4$ or $k = 137/107$ for the two alternative options discussed in [K85]. If so then the most general transcendentals required and allowed in the extensions used correspond to roots of polynomials with coefficients in an extension of rationals by e and algebraic numbers. As already discussed, infinite primes might provide the ultimate solution to the problem of continuation.

The continuation of the exponent of Kähler function and of WCW spinor fields to p-adic sectors would require some selection of a subset of points of the rational WCW . On the other hand, the minimum requirement is that it is possible to define WCW integration in the p-adic context. The only manner to achieve this is by defining WCW integration purely algebraically by perturbative expansion. For free field theory Gaussian integrals are in question and one can calculate them trivially. The Gaussian can be regarded as a Kähler function of a flat Kähler manifold having maximal translational and rotational symmetries. Physically infinite number of harmonic oscillators are in question. The origin of the symmetric space is preferred point as far as Kähler function is considered: metric itself is invariant under isometries.

Algebraization of WCW functional integral

WCW is a union of infinite-dimensional symmetric spaces labelled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem [A31] generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each WCW spinor field would define a vertex from which lines representing the propagators defined by the contravariant WCW metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic WCW integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the WCW integrals would exist also in the p-adic sense.

Is the exponential of the Kähler function rational function?

The simplest possibility that one can imagine are that the exponent e^{2K} of Kähler function appearing in WCW inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers.

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the WCW would be possible in the entire WCW . Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also WCW spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

The most obvious manner to realize this idea relies on the restriction of light-like 3-surfaces X_l^3 to those representable in terms of polynomials or rational functions with rational or at most algebraic coefficients serving as natural preferred coordinates of the WCW . This of course requires identification of preferred coordinates also for H. This would lead to a hierarchy of inclusions for sub- WCW s induced by algebraic extensions of rationals.

The presence of cutoffs for the degrees of polynomials involved makes the situation finite-dimensional and give rise to a hierarchy of inclusions also now. These inclusion hierarchies would relate naturally also to hierarchies of inclusions for hyperfinite factors of type II_1 since the spinor spaces associated with these finite-D versions of WCW would be finite-dimensional. Hyper-finiteness means that this kind of cutoff can give arbitrarily precise approximate representation of the infinite-D situation.

This vision is supported by the recent understanding related to the definition of exponent of Kähler function as Dirac determinant [K111]. The number of eigenvalues involved is necessarily finite, and if the eigenvalues of D_{C-S} are algebraic numbers for 3-surfaces X_l^3 for which the coefficients characterizing the rational functions defining X_l^3 are algebraic numbers, the exponent of Kähler function is algebraic number.

The general number theoretical conjectures implied by p-adic physics and physics of cognition support also this conjecture. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad .$$

Coupling constant evolution and number theory

The coupling constant evolution associated with the Kähler action might be at least partially understood number-theoretically.

A given space-time sheet is connected by wormhole contacts to the larger space-time sheets. The induced metric within the wormhole contact has an Euclidian signature so that the wormhole contact is surrounded by elementary particle horizons at which the metric is degenerate so that the horizons are metrically effectively 2-dimensional giving rise to quaternion conformal invariance. Because of the causal horizon it would seem that Kähler coupling strength can depend on the space-time sheet via the p-adic prime characterizing it. If so the exponent of the Kähler function would be simply the product of the exponents for the space-time sheets and one would have finite-dimensional extension as required.

If the exponent of the Kähler function is rational function, also the components of the contravariant Kähler metric are rational functions. This would suggest that one function of the coupling constant evolution is to keep the exponent rational.

From the point of view of p-adicization the ideal situation results if Kähler coupling strength is invariant under the p-adic coupling constant evolution as I believed originally. For a long time it however seemed that this option cannot be realized since the prediction $G = L_p^2 exp(-2S_K(CP_2))$ for the gravitational coupling constant following from dimensional considerations alone implies that

G increases without limit as a function of p-adic length scale if α_K is RG invariant. If one however assumes that bosonic space-time sheets correspond to Mersenne primes, situation changes since M_{127} defining electron length scale is the largest Mersenne prime for which p-adic length scale is not super-astronomical and thus excellent candidate for characterizing gravitonic space-time sheets. There is much stronger motivation for this hypothesis coming from the fact that a nice picture about evolution of electro-weak and color coupling strengths emerges just from the physical interpretation of the fact that classical color action and electro-weak $U(1)$ action are proportional to Kähler action [K110].

The recent progress in the understanding of the definition of the exponent of Kähler function as Dirac determinant [K111] leads to rather detailed picture about the number theoretic anatomy of α_K and other coupling constant strengths as well as the number theoretic anatomy of $R^2/\hbar G$ [L28]. By combining these results with the constraints coming from p-adic mass calculations one ends up to rather strong predictions for α_K and $R^2/\hbar G$.

Consistency check in the case of CP_2

It is interesting to look whether this vision works or fails in a simple finite-dimensional case. For CP_2 the Kähler function is given by $K = -log(1 + r^2)$. This function exists if an extension containing the logarithms of primes is used. $log(1 + x)$, $x = O(p)$ exists as an ordinary p-adic number and a logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ should be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. Also logarithms of roots of integers and their products would exist. The problem is however that the powers of $log(m)$ and $log(p)$ would generate an infinite-dimensional extension since finite-dimensional extension leads to a contradiction as shown in [K85].

The exponent of Kähler function as well as Kähler metric and Kähler form have rational-valued elements for rational values of the standard complex coordinates for CP_2. The exponent of the Kähler function is $1/(1 + r^2)$ and exists as a rational number at 3-spheres of rational valued radius. The negative of the Kähler function has a single maximum at $r = 0$ and vanishes at the coordinate singularity $r \rightarrow \infty$, which corresponds to the geodesic sphere S^2.

If one wants to cognize about geodesic length, areas of geodesic spheres, and about volume of CP_2, π must be introduced to the extension of p-adics and means infinite-dimensional extension by the arguments of [K85]. The introduction of π is not however necessary for introducing of spherical coordinates if one expresses everything in terms of trigonometric functions. For ordinary spherical coordinates this means effectively replacing θ and ϕ by $u = \theta/\pi$ and $v = \phi/2\pi$ as coordinates. By allowing u and v to have a finite number of rational values requires only the introduction of a finite-dimensional algebraic extension in order to define cosines and sines of the angle variables at these values. What seems clear is that the evolution of cognition as the emergence of higher-dimensional extensions corresponds quite concretely to the emergence of finer discretizations.

7.7 How To Realize The Notion Of Finite Measurement Resolution Mathematically?

One of the basic challenges of quantum TGD is to find an elegant realization for the notion of finite measurement resolution. The notion of resolution involves observer in an essential manner and this suggests that cognition is involved. If p-adic physics is indeed physics of cognition, the natural guess is that p-adic physics should provide the primary realization of this notion.

The simplest realization of finite measurement resolution would be just what one would expect it to be except that this realization is most natural in the p-adic context. One can however define this notion also in real context by using canonical identification to map p-adic geometric objets to real ones.

7.7.1 Does Discretization Define An Analog Of Homology Theory?

Discretization in dimension D in terms of pinary cutoff means division of the manifold to cube-like objects. What suggests itself is homology theory defined by the measurement resolution and by the fluxes assigned to the induced Kähler form.

1. One can introduce the decomposition of n-D sub-manifold of the embedding space to n-cubes by $n-1$-planes for which one of the coordinates equals to its pinary cutoff. The construction works in both real and p-adic context. The hyperplanes in turn can be decomposed to $n-1$-cubes by $n-2$-planes assuming that an additional coordinate equals to its pinary cutoff. One can continue this decomposition until one obtains only points as those points for which all coordinates are their own pinary cutoffs. In the case of partonic 2-surfaces these points define in a natural manner the ends of braid strands. Braid strands themselves could correspond to the curves for which two coordinates of a light-like 3-surface are their own pinary cutoffs.

2. The analogy of homology theory defined by the decomposition of the space-time surface to cells of various dimensions is suggestive. In the p-adic context the identification of the boundaries of the regions corresponding to given pinary digits is not possible in purely topological sense since p-adic numbers do not allow well-ordering. One could however identify the boundaries sub-manifolds for which some number of coordinates are equal to their pinary cutoffs or as inverse images of real boundaries. This might allow to formulate homology theory to the p-adic context.

3. The construction is especially interesting for the partonic 2-surfaces. There is hierarchy in the sense that a square like region with given first values of pinary digits decompose to p square like regions labelled by the value $0, ..., p-1$ of the next pinary digit. The lines defining the boundaries of the 2-D square like regions with fixed pinary digits in a given resolution correspond to the situation in which either coordinate equals to its pinary cutoff. These lines define naturally edges of a graph having as its nodes the points for which pinary cutoff for both coordinates equals to the actual point.

4. I have proposed earlier [K19] what I have called symplectic QFT involving a triangulation of the partonic 2-surface. The fluxes of the induced Kähler form over the triangles of the triangulation and the areas of these triangles define symplectic invariants, which are zero modes in the sense that they do not contribute to the line element of WCW although the WCW metric depends on these zero modes as parameters. The physical interpretation is as non-quantum fluctuating classical variables. The triangulation generalizes in an obvious manner to quadrangulation defined by the pinary digits. This quadrangulation is fixed once internal coordinates and measurement accuracy are fixed. If one can identify physically preferred coordinates - say by requiring that coordinates transform in simple manner under isometries - the quadrangulation is highly unique.

5. For 3-surfaces one obtains a decomposition to cube like regions bounded by regions consisting of square like regions and Kähler magnetic fluxes over the squares define symplectic invariants. Also Kähler Chern-Simons invariant for the 3-cube defines an interesting almost symplectic invariant. 4-surface decomposes in a similar manner to 4-cube like regions and now instanton density for the 4-cube reducing to Chern-Simons term at the boundaries of the 4-cube defines symplectic invariant. For 4-surfaces symplectic invariants reduce to Chern-Simons terms over 3-cubes so that in this sense one would have holography. The resulting structure brings in mind lattice gauge theory and effective 2-dimensionality suggests that partonic 2-surfaces are enough.

The simplest realization of this homology theory in p-adic context could be induced by canonical identification from real homology. The homology of p-adic object would the homology of its canonical image.

1. Ordering of the points is essential in homology theory. In p-adic context canonical identification $x = \sum x_n p^n \to \sum x_n p^{-n}$ map to reals induces this ordering and also boundary operation for p-adic homology can be induced. The points of p-adic space would be represented by n-tuples of sequences of pinary digits for n coordinates. p-Adic numbers decompose to disconnected sets characterized by the norm p^{-n} of points in given set. Canonical identification allows to glue these sets together by inducing real topology. The points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ having p-adic norms p^{-n} and p^{-n-1} are mapped to

the same real point p^{-n} under canonical identification and therefore the points p^n and $(p-1)(1+p+p^2+...)p^{n+1}$ can be said to define the endpoints of a continuous interval in the induced topology although they have different p-adic norms. Canonical identification induces real homology to the p-adic realm. This suggests that one should include canonical identification to the boundary operation so that boundary operation would be map from p-adicity to reality.

2. Interior points of p-adic simplices would be p-adic points not equal to their pinary cutoffs defined by the dropping of the pinary digits corresponding p^n, $n > N$. At the boundaries of simplices at least one coordinate would have vanishing pinary digits for p^n, $n > N$. The analogs of $n-1$ simplices would be the p-adic points sets for which one of the coordinates would have vanishing pinary digits for p^n, $n > N$. $n-k$-simplices would correspond to points sets for which k coordinates satisfy this condition. The formal sums and differences of these sets are assumed to make sense and there is natural grading.

3. Could one identify the end points of braid strands in some natural manner in this cohomology? Points with $n \leq N$ pinary digits are closed elements of the cohomology and homologically equivalent with each other if the canonical image of the p-adic geometric object is connected so that there is no manner to identify the ends of braid strands as some special points unless the zeroth homology is non-trivial. In [K6] it was proposed that strand ends correspond to singular points for a covering of sphere or more general Riemann surface. At the singular point the branches of the covering would co-incide.

   The obvious guess is that the singular points are associated with the covering characterized by the value of Planck constant. As a matter fact, the original assumption was that *all* points of the partonic 2-surface are singular in this sense. It would be however enough to make this assumption for the ends of braid strands only. The orbits of braid strands and string world sheet having braid strands as its boundaries would be the singular loci of the covering.

7.7.2 Does The Notion Of Manifold In Finite Measurement Resolution Make Sense?

A modification of the notion of manifold taking into account finite measurement resolution might be useful for the purposes of TGD.

1. The chart pages of the manifold would be characterized by a finite measurement resolution and effectively reduce to discrete point sets. Discretization using a finite pinary cutoff would be the basic notion. Notions like topology, differential structure, complex structure, and metric should be defined only modulo finite measurement resolution. The precise realization of this notion is not quite obvious.

2. Should one assume metric and introduce geodesic coordinates as preferred local coordinates in order to achieve general coordinate invariance? Pinary cutoff would be posed for the geodesic coordinates. Or could one use a subset of geodesic coordinates for $\delta CD \times CP_2$ as preferred coordinates for partonic 2-surfaces? Should one require that isometries leave distances invariant only in the resolution used?

3. A rather natural approach to the notion of manifold is suggested by the p-adic variants of symplectic spaces based on the discretization of angle variables by phases in an algebraic extension of p-adic numbers containing n^{th} root of unity and its powers. One can also assign p-adic continuum to each root of unity [K111]. This approach is natural for compact symmetric Kähler manifolds such as S^2 and CP_2. For instance, CP_2 allows a coordinatization in terms of two pairs (P^k, Q^k) of Darboux coordinates or using two pairs $(\xi^k, \overline{\xi}^k)$, $k = 1, 2$, of complex coordinates. The magnitudes of complex coordinates would be treated in the manner already described and their phases would be described as roots of unity. In the natural quadrangulation defined by the pinary cutoff for $|\xi^k|$ and by roots of unity assigned with their phases, Kähler fluxes would be well-defined within measurement resolution. For light-cone boundary metrically equivalent with S^2 similar coordinatization using complex coordinates $(z, \overline{z})$ is possible. Light-like radial coordinate r would appear only as a parameter in the induced metric and pinary cutoff would apply to it.

7.7.3 Hierachy Of Finite Measurement Resolutions And Hierarchy Of P-Adic Normal Lie Groups

The formulation of quantum TGD is almost completely in terms of various symmetry group and it would be highly desirable to formulate the notion of finite measurement resolution in terms of symmetries.

1. In p-adic context any Lie-algebra gG with p-adic integers as coefficients has a natural grading based on the p-adic norm of the coefficient just like p-adic numbers have grading in terms of their norm. The sub-algebra g_N with the norm of coefficients not larger than p^{-N} is an ideal of the algebra since one has $[g_M, g_N] \subset g_{M+N}$: this has of course direct counterpart at the level of p-adic integers. g_N is a normal sub-algebra in the sense that one has $[g, g_N] \subset g_N$. The standard expansion of the adjoint action $g g_N g^{-1}$ in terms of exponentials and commutators gives that the p-adic Lie group $G_N = exp(tpg_N)$, where t is p-adic integer, is a normal subgroup of $G = exp(tpg)$. If indeed so then also G/G_N is group, and could perhaps be interpreted as a Lie group of symmetries in finite measurement resolution. G_N in turn would represent the degrees of freedom not visible in the measurement resolution used and would have the role of a gauge group.

2. The notion of finite measurement resolution would have rather elegant and universal representation in terms of various symmetries such as isometries of embedding space, Kac-Moody symmetries assignable to light-like wormhole throats, symplectic symmetries of $\delta CD \times CP_2$, the non-local Yangian symmetry [A17] [B18, B14, B15], and also general coordinate transformations. This representation would have a counterpart in real context via canonical identification I in the sense that $A \rightarrow B$ for p-adic geometric objects would correspond to $I(A) \rightarrow I(B)$ for their images under canonical identification. It is rather remarkable that in purely real context this kind of hierarchy of symmetries modulo finite measurement resolution does not exist. The interpretation would be that finite measurement resolution relates to cognition and therefore to p-adic physics.

3. Matrix group G contains only elements of form $g = 1 + O(p^m)$, $m \geq 1$ and does not therefore involve matrices with elements expressible in terms roots of unity. These can be included by writing the elements of the p-adic Lie-group as products of elements of above mentioned G with the elements of a discrete group for which the elements are expressible in terms of roots of unity in an algebraic extension of p-adic numbers. For p-adic prime p p: th roots of unity are natural and suggested strongly by quantum arithmetics [K70].

Chapter 8

Unified Number Theoretical Vision

8.1 Introduction

Octonions, quaternions, quaternionic space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized. This summary involves several corrections to the picture which has been developing for a decade or so.

A brief updated view about $M^8 - H$ duality and twistorialization is in order. There is a beautiful pattern present suggesting that $M^8 - H$ duality makes sense and that $H = M^4 \times CP_2$ is completely unique on number theoretical grounds.

1. $M^8 - H$ duality allows to deduce $M^4 \times CP_2$ via number theoretical compactification. For the option with minimal number of conjectures the associativity/co-associativity of the space-time surfaces in M^8 guarantees that the space-time surfaces in M^8 define space-time surfaces in H. The tangent/normal spaces of quaternionic/hyper-quaternionic surfaces in M^8 contain also an integrable distribution of hyper-complex tangent planes $M^2(x)$.

   An important correction is that associativity/co-associativity does not make sense at the level of H since the spinor structure of H is already complex quaternionic and reducible to the ordinary one by using matrix representations for quaternions. The associativity condition should however have some counterpart at level of H. One could require that the induced gamma matrices at each point could span a *real*-quaternionic sub-space of complexified quaternions for quaternionicity and a purely imaginary quaternionic sub-space for co-quaternionicity. One might hope that it is consistent with - or even better, implies - preferred extremal property. I have not however found a viable definition of quaternionic "reality". On the other hand, it is possible assigne the tangent space M^8 of H with octonion structure and define associativity as in case of M^8.

   $M^8 - H$ duality could generalize to $H - H$ duality in the sense that also the image of the space-time surface under duality map is not only preferred extremal but also associative (co-associative) surface. The duality map $H \rightarrow H$ could be iterated and would define the arrow for the category formed by preferred extremals.

2. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. M^8 allows twistor space for octonionic spinor structure obtained by direct generalization of the standard construction for M^4. $M^4 \times CP_2$ spinors can be regarded as tensor products of quaternionic spinors associated with M^4 and CP_2: this trivial observation forces to challenge the earlier rough vision, which however seems to stand up the challenge.

3. Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing 12-D $T(M^8) = G_2/U(1)\times U(1)$ to the 12-D twistor space $T(H) = CP_3 \times SU^3/U(1)\times U(1)$. The interpretation of the twistor space in the case of M^8 is as the space of choices of quantization axes for the

2-D Cartan algebra of G_2 acting as octonionic automorphisms. For CP_2 one has space for the chocies of quantization axes for the 2-D $SU(3)$ Cartan algebra.

4. It is also possible that the dualities extend to a sequence $M^8 \rightarrow H \rightarrow H...$ by mapping the associative/co-associative tangent space to CP_2 and M^4 point to M^4 point at each step. One has good reasons to expect that this iteration generates fractal as the limiting space-time surface.

5. A fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically 7-sphere. Second analogous structure is the 7-D Lie algebra like structure defined by octonionic analogs of sigma matrices.

The analogy of quaternionicity of M^8 pre-images of preferred extremals and quaternionicity of the tangent space of space-time surfaces in H with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity at the level of M^8 indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program. Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L6].

8.2 Number Theoretic Compactification And $M^8 - H$ Duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally $M^8 - H$ duality was introduced as a number theoretic explanation for $H = M^4 \times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4 \subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M_c^8 using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of $M^8 - H$ duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of $M^8 - H$ duality.

The strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M_c^8 composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M^8_c identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M^8_c must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere K̈hler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3)\times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4 \times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering $M^8 - H$ duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M^8_c = O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under multiplication. One can however however define the notion of associativity for the subspace of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q = q_0 + q^k I_k$. Hyper-quaternions are expressible as $q = q_0 + jq^k I_k$ and form a subspace of complexified quaternions $Q_c = Q \oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O \oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H = q_0 + jq^k I_k + jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4 \subset M^8$ implies unique decomposition $M^8 = M^4 \times E^4$ needed to define $M^8 - H$ duality uniquely. This applies also to M^8_c. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCW s associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M^8_c) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

   The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *embedding space level* and *space-time level.* One must have embedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \rightarrow H \rightarrow H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of Kähler-Dirac gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the Kähler-Dirac gamma matrices need not span the entire tangent space. The space spanned by the Kähler-Dirac gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the Kähler-Dirac gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

   If the space spanned by Kähler-Dirac gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For Kähler-Dirac gamma matrices the notion of co-associativity can produce problems since Kähler-Dirac gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by Kähler-Dirac gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the Kähler-Dirac gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the Kähler-Dirac gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

8.2.1 Basic Idea Behind $M^8 - M^4 \times CP_2$ Duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different ways to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M^8_c. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M_c^8) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \rightarrow S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \rightarrow M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M_c^8: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M_c^8$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a way to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [K14]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [K111] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of Kähler-Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent

plane makes it possible to define $M^8 - H$ duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

8.2.2 Hyper-Octonionic Pauli "Matrices" And The Definition Of Associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [K101, K10]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

   Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

   There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \rightarrow H \rightarrow H$.... This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and Kähler-Dirac gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

8.2.3 Are Kähler And Spinor Structures Necessary In M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the $M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naïve replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1 + 1 + 3 + \overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1 + kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2 + y^2 + z^2 + t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1 + ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of embedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

8.2.4 How Could One Solve Associativity/Co-Associativity Conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \rightarrow H \rightarrow H$... iteration generating new solutions from existing ones.

Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O \oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1 + io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M_c^8 but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M_c^8 (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a, b, c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a way that one of the tangent vectors corresponds to real unit (in the embedding map embedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the

octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition looks very simple - perhaps too simple!- since it involves only first derivatives of the embedding space vectors.

One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of embedding space coordinates (rather than involving embedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$
\begin{aligned}
e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\
A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\
e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\
\Gamma_k &= e^A_k \gamma_A \ .
\end{aligned}
\tag{8.2.-3}
$$

The very naïve idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$
F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{8.2.-2}
$$

One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see `http://tinyurl.com/ya7h3n9z`). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD} x^B y^C z^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [A50] (see **Fig. 8.1**) expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product $e_1 e_2$ (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x) e_1 e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$
e_0 = a \times 1 + b^i e_i
$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

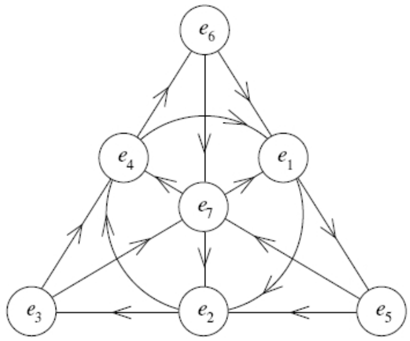

Figure 8.1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

8.2.5 Quaternionicity At The Level Of Embedding Space Quantum Numbers

From the multiplication table of octonions as illustrated by Fano triangle [A50] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2: s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

8.2.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of $M^8 - H$ duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the Kähler-Dirac gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k \gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the Kähler-Dirac gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection Kähler-Dirac gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2 \times S^2 \subset M^4 \times CP_2$. It seems that associativity is satisfied by all known extremals. Hence Kähler-Dirac gamma matrices are flexible enough to realize associativity in H.

3. Kähler-Dirac gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner $M^8 - H$ duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the Kähler-Dirac gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand $M^8 - H$ correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does $M^8 - H$ correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

Minkowskian-Euclidian ↔ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

$M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since WCW degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined

by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K64].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

8.2.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \rightarrow H \rightarrow H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

8.3 Quaternions and TGD

8.3.1 Are Euclidian Regions Of Preferred Extremals Quaternion- Kähler Manifolds?

In blog comments Anonymous gave a link to an article (see `http://tinyurl.com/y7j9hxr8`) about construction of 4-D quaternion-Kähler metrics with an isometry: they are determined by so called $SU(\infty)$ Toda equation. I tried to see whether quaternion-Kähler manifolds could be relevant for TGD.

From Wikipedia (see `http://tinyurl.com/yd8feoev`) one can learn that QK is characterized by its holonomy, which is a subgroup of $Sp(n) \times Sp(1)$: $Sp(n)$ acts as linear symplectic transformations of $2n$-dimensional space (now real). In 4-D case tangent space contains 3-D submanifold identifiable as imaginary quaternions. CP_2 is one example of QK manifold for which the subgroup in question is $SU(2) \times U(1)$ and which has non-vanishing constant curvature: the components of Weyl tensor represent the quaternionic imaginary units. QKs are Einstein manifolds: Einstein tensor is proportional to metric.

What is really interesting from TGD point of view is that twistorial considerations show that one can assign to QK a special kind of twistor space (twistor space in the mildest sense requires only orientability). Wiki tells that if Ricci curvature is positive, this (6-D) twistor space is what is known as projective Fano manifold with a holomorphic contact structure. Fano variety has the nice property that as (complex) line bundle (the twistor space property) it has enough sections to define the embedding of its base space to a projective variety. Fano variety is also complete: this is algebraic geometric analogy of topological property known as compactness.

QK manifolds and twistorial formulation of TGD

How the QKs could relate to the twistorial formulation of TGD?

1. In the twistor formulation of TGD [K101] the space-time surfaces are 4-D base spaces of 6-D twistor spaces in the Cartesian product of 6-D twistor spaces of M^4 and CP_2 - the only twistor spaces with Kähler structure. In TGD framework space-time regions can have either Euclidian or Minkowskian signature of induced metric. The lines of generalized Feynman diagrams are Euclidian.

2. Could the twistor spaces associated with the lines of generalized Feynman diagrams be projective Fano manifolds? Could QK structure characterize Euclidian regions of preferred extremals of Kähler action? Could a generalization to Minkowskian regions exist.

   I have proposed that so called Hamilton-Jacobi structure [K111] characterizes preferred extremals in Minkowskian regions. It could be the natural Minkowskian counterpart for the quaternion Kähler structure, which involves only imaginary quaternions and could make sense also in Minkowski signature. Note that unit sphere of imaginary quaternions defines the sphere serving as fiber of the twistor bundle.

Why it would be natural to have QK that is corresponding twistor space, which is projective contact Fano manifold?

1. QK property looks very strong condition but might be true for the preferred extremals satisfying very strong conditions stating that the classical conformal charges associated with various conformal algebras extending the conformal algebras of string models [K111], [L11]. These conditions would be essentially classical gauge conditions stating that strong form of holography implies by strong form of General Coordinate Invariance (GCI) is realized: that is partonic 2-surfaces and their 4-D tangent space data code for quantum physics.

2. Kähler property makes sense for space-time regions of Euclidian signature and would be natural is these regions can be regarded as small deformations of CP_2 type vacuum extremals with light-like M^4 projection and having the same metric and Kähler form as CP_2 itself.

3. Fano property implies that the 4-D Euclidian space-time region representing line of the Feynman diagram can be imbedded as a sub-manifold to complex projective space CP_n. This would allow to use the powerful machinery of projective geometry in TGD framework. This could also be a space-time correlate for the fact that CP_ns emerge in twistor Grassmann approach expected to generalize to TGD framework.

4. CP_2 allows both projective (trivially) and contact (even symplectic) structures. $\delta M^4_+ \times CP_2$ allows contact structure - I call it loosely symplectic structure. Also 3-D light-like orbits of partonic 2-surfaces allow contact structure. Therefore holomorphic contact structure for the twistor space is natural.

5. Both the holomorphic contact structure and projectivity of CP_2 would be inherited if QK property is true. Contact structures at orbits of partonic 2-surfaces would extend to holomorphic contact structures in the Euclidian regions of space-time surface representing lines of generalized Feynman diagrams. Projectivity of Fano space would be also inherited from CP_2 or its twistor space $SU(3)/U(1)\times U(1)$ (flag manifold identifiable as the space of choices for quantization axes of color isospin and hypercharge).

The article considers a situation in which the QK manifold allows an isometry. Could the isometry (or possibly isometries) for QK be seen as a remnant of color symmetry or rotational symmetries of M^4 factor of embedding space? The only remnant of color symmetry at the level of embedding space spinors is anomalous color hyper charge (color is like orbital angular momentum and associated with spinor harmonic in CP_2 center of mass degrees of freedom). Could the isometry correspond to anomalous hypercharge?

How to choose the quaternionic imaginary units for the space-time surface?

Parallellizability is a very special property of 3-manifolds allowing to choose quaternionic imaginary units: global choice of one of them gives rise to twistor structure.

1. The selection of time coordinate defines a slicing of space-time surface by 3-surfaces. GCI would suggest that a generic slicing gives rise to 3 quaternionic units at each point each 3-surface? The parallelizability of 3-manifolds - a unique property of 3-manifolds - means the possibility to select global coordinate frame as section of the frame bundle: one has 3 sections of tangent bundle whose inner products give rose to the components of the metric (now induced metric) guarantees this. The tri-bein or its dual defined by two-forms obtained by contracting tri-bein vectors with permutation tensor gives the quanternionic imaginary units. The construction depends on 3-metric only and could be carried out also in GRT context. Note however that topology change for 3-manifold might cause some non-trivialities. The metric 2-dimensionality at the light-like orbits of partonic 2-surfaces should not be a problem for a slicing by space-like 3-surfaces. The construction makes sense also for the regions of Minkowskian signature.

2. In fact, any 4-manifold (see `http://tinyurl.com/yb8l34b5`) [A74] allows almost quaternionic as the above slicing argument relying on parallelizibility of 3-manifolds strongly suggests.

3. In zero energy ntology (ZEO)- a purely TGD based feature - there are very natural special slicings. The first one is by linear time-like Minkowski coordinate defined by the direction of the line connecting the tips of the causal diamond (CD). Second one is defined by the light-cone proper time associated with either light-cone in the intersection of future and past directed light-cones defining CD. Neither slicing is global as it is easy to see.

The relationship to quaternionicity conjecture and $M^8 - H$ duality

One of the basic conjectures of TGD is that preferred extremals consist of quaternionic/ co-quaternionic (associative/co-associative) regions [K96]. Second closely related conjecture is $M^8 - H$ duality allowing to map quaternionic/co-quaternionic surfaces of M^8 to those of $M^4 \times CP_2$. Are these conjectures consistent with QK in Euclidian regions and Hamilton-Jacobi property in Minkowskian regions? Consider first the definition of quaternionic and co-quaternionic space-time regions.

1. Quaternionic/associative space-time region (with Minkowskian signature) is defined in terms of induced octonion structure obtained by projecting octonion units defined by vielbein of $H = M^4 \times CP_2$ to space-time surface and demanding that the 4 projections generate quaternionic sub-algebra at each point of space-time.

   If there is also unique complex sub-algebra associated with each point of space-time, one obtains one can assign to the tangent space-of space-time surface a point of CP_2. This allows to realize $M^8 - H$ duality [K96] as the number theoretic analog of spontaneous compactification (but involving no compactification) by assigning to a point of $M^4 = M^4 \times CP_2$ a point of $M^4 \times CP_2$. If the image surface is also quaternionic, this assignment makes sense also for space-time surfaces in H so that $M^8 - H$ duality generalizes to $H - H$ duality allowing to assign to given preferred extremal a hierarchy of extremals by iterating this assignment. One obtains a category with morphisms identifiable as these duality maps.

2. Co-quaternionic/co-associative structure is conjectured for space-time regions of Euclidian signature and 4-D CP_2 projection. In this case normal space of space-time surface is quaternionic/associative. A multiplication of the basis by preferred unit of basis gives rise to a quaternionic tangent space basis so that one can speak of quaternionic structure also in this case.

3. Quaternionicity in this sense requires unique identification of a preferred time coordinate as embedding space coordinate and corresponding slicing by 3-surfaces and is possible only in TGD context. The preferred time direction would correspond to real quaternionic unit.

Preferred time coordinate implies that quaternionic structure in TGD sense is more specific than the QK structure in Euclidian regions.

4. The basis of induced octonionic imaginary unit allows to identify quaternionic imaginary units linearly related to the corresponding units defined by tri-bein vectors. Note that the multiplication of octonionic units is replaced with multiplication of antisymmetric tensors representing them when one assigns to the quaternionic structure potential QK structure. Quaternionic structure does not require Kähler structure and makes sense for both signatures of the induced metric. Hence a consistency with QK and its possible analog in Minkowskian regions is possible.

5. The selection of the preferred imaginary quaternion unit is necessary for $M^8 - H$ correspondence. This selection would also define the twistor structure. For quaternion-Kähler manifold this unit would be covariantly constant and define Kähler form - maybe as the induced Kähler form.

6. Also in Minkowskian regions twistor structure requires a selection of a preferred imaginary quaternion unit. Could the induced Kähler form define the preferred imaginary unit also now? Is the Hamilton-Jacobi structure consistent with this?

   Hamilton-Jacobi structure involves a selection of 2-D complex plane at each point of space-time surface. Could induced Kähler magnetic form for each 3-slice define this plane? It is not necessary to require that 3-D Kähler form is covariantly constant for Minkowskian regions. Indeed, massless extremals representing analogs of photons are characterized by local polarization and momentum direction and carry time-dependent Kähler-electric and -magnetic fields. One can however ask whether monopole flux tubes carry covariantly constant Kähler magnetic field: they are indeed deformations of what I call cosmic strings [K14, K27] for which this condition holds true?

8.3.2 The Notion of Quaternion Analyticity

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals.

Basic idea

One can argue that quaternion analyticity is the more natural option in the sense that the local octonionic embedding space coordinate (or at least M^8 or E^8 coordinate, which is enough if M^8-H duality holds true) would for preferred extremals be expressible in the form

$$o(q) = u(q) + v(q) \times I \ . \tag{8.3.0}$$

Here q is quaternion serving as a coordinate of a quaternionic sub-space of octonions, and I is octonion unit belonging to the complement of the quaternionic sub-space, and multiplies $v(q)$ from *right* so that quaternions and quaternionic differential operators acting from left do not notice these coefficients at all. A stronger condition would be that the coefficients are real. $u(q)$ and $v(q)$ would be quaternionic Taylor- of even Laurent series with coefficients multiplying powers of q from right for the same reason.

The signature of M^4 metric is a problem. I have proposed a complexification of M^8 and M^4 to get rid of the problem by assuming that the embedding space corresponds to surfaces in the space M^8 identified as octonions of form $o_8 = Re(o) + iIm(o)$, where o is imaginary part of ordinary octonion and i is commuting imaginary unit. M^4 would correspond to quaternions of form $q_4 = Re(q) + iIm(q)$. What is important is that powers of q_4 and o_8 belong to this sub-space (as follows from the vanishing of cross product term in the square of octonion/quaternion) so that powers of q_4 (o_8) has imaginary part proportional to $Im(q)$ ($Im(o)$)

I ended up to reconsider the idea of quaternion analyticity after having found two very interesting articles discussing the generalization of Cauchy-Riemann equations. The first article (see http://tinyurl.com/yb8l34b5) [A74] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article (see http://tinyurl.com/y7kww2o2) [A51] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

The first form of Cauchy-Riemann-Fueter conditions

Cauchy-Riemann-Fueter (CRF) conditions generalize Cauchy-Riemann conditions. These conditions are however not unique. Consider first the translationally invariant form of CRF conditions.

1. The translationally invariant form of CRF conditions is $\partial_{\overline{q}} f = 0$ or explicitly

$$\partial_{\overline{q}} f = d_1 f + d_2 f \equiv (\partial_t - \partial_x I)f - (\partial_y J + \partial_z K)f = 0 \ . \tag{8.3.0}$$

   This form is not unique: one can perform SO(3) rotations of the quaternionic imaginary units acting as automorphisms of quaternions. This form does not allow quaternionic Taylor series as a solution. Note that the Taylor coefficients multiplying powers of the coordinate from right are arbitrary quaternions. What looks pathological is that even linear functions of q fail be solve this condition. What is however interesting that in flat space the equation is equivalent with Dirac equation for a pair of Majorana spinors [A74].

   Function $f = t + Ix - Jy - Kz$ is perhaps the simplest solution to the condition. One can define also other variants of $\overline{q}$, in particular the variant $\overline{q} = t + Ix - Jy - Kz$ giving $f = t + Ix + Jy + Kz$ as a solution.

2. The condition allows functions depending on complex coordinate z of some complex-plane only. It also allows functions satisfying two separate analyticity conditions, say $d_1 f = 0$ and $d_2 f = 0$, say

$$\partial_{\overline{u}} f = (\partial_t - \partial_x I)f = 0 \ ,$$
$$\partial_{\overline{v}} f = -(\partial_y J + \partial_z K)f = -J(\partial_y - \partial_z I)f = 0 \ . \tag{8.3.0}$$

   In the latter formula J multiplies from *left*! One has good hopes of obtaining holomorphic functions of two complex coordinates.

   The simplest solution to the conditions is complex value function $f(u = x + iy, v = y + iz)$ of two complex variables. The image of E^4 is 2-dimensional whereas for $f_0 = t + Ix - Jy - Kz$ it is 4-D.

   In Euclidian signature one obtains quaternion valued map if the Taylor coefficients a_{mn} in the series of $f(u, v)$ are quaternions and are taken to the right: $q = f(u, v) = \sum u^m v^n a_{mn}$ to avoid problems from non-commutativity. With this assumption the image would be 4-D in the generic case.

In TGD one must consider Minkowskian signature and it turns out that the situation changes dramatically, and the naïve view about quaternion analyticity must be given up. The experience about extermals of Kähler action suggests a modification of the analyticity properties consistent with the signature but whether one should call this analyticity quaternion analyticity is a matter of taste.

Second form of CRF conditions

Second form of CRF conditions proposed in [A51] is tailored in order to realize the almost obvious manner to realize quaternion analyticity.

1. The ingenious idea is to replace preferred quaternionic imaginary unit by a imaginary unit which is in radial direction: $e_r = (xI + yJ + zK)/r$ and require analyticity with respect to the coordinate $t + e_r r$. The solution to the condition is power series in $t + e_r r = q$ so that one obtains quaternion analyticity.

2. The explicit form of the conditions is

$$(\partial_t - e_r\partial_r)f = (\partial_t - \frac{e_r}{r}r\partial_r)f = 0 \quad . \tag{8.3.0}$$

   This form allows both the desired quaternionic Taylor series and ordinary holomorphic functions of complex variable in one of the 3 complex coordinate planes as general solutions.

3. This form of CRF is neither Lorentz invariant nor translationally invariant but remains invariant under simultaneous scalings of t and r and under time translations. Under rotations of either coordinates or of imaginary units the spatial part transforms like vector so that quaternionic automorphism group $SO(3)$ serves as a moduli space for these operators.

4. The interpretation of the latter solutions inspired by ZEO would be that in Minkowskian regions r corresponds to the light-like radial coordinate of the either boundary of CD, which is part of $\delta M^4_\pm$. The radial scaling operator is that assigned with the light-like radial coordinate of the light-cone boundary. A slicing of CD by surfaces parallel to the $\delta M^4_\pm$ is assumed and implies that the line $r = 0$ connecting the tips of CD is in a special role. The line connecting the tips of CD defines coordinate line of time coordinate. The breaking of rotational invariance corresponds to the selection of a preferred quaternion unit defining the twistor structure and preferred complex sub-space.

   In regions of Euclidian signature r could correspond to the radial Eguchi-Hanson coordinate of CP_2 and $r = 0$ corresponds to a fixed point of $U(2)$ subgroup under which CP_2 complex coordinates transform linearly.

5. Also in this case one can ask whether solutions depending on two complex local coordinates analogous to those for translationally invariant CRF condition are possible. The remain imaginary units would be associated with the surface of sphere allowing complex structure.

Generalization of CRF conditions?

Could the proposed forms of CRF conditions be special cases of much more general CRF conditions as CR conditions are?

1. Ordinary complex analysis suggests that there is an infinite number of choices of the quaternionic coordinates related by the above described quaternion-analytic maps with 4-D images. The form of of the CRF conditions would be different in each of these coordinate systems and would be obtained in a straightforward manner by chain rule.

2. One expects the existence of large number of different quaternion-conformal structures not related by quaternion-analytic transformations analogous to those allowed by higher genus Riemann surfaces and that these conformal equivalence classes of four-manifolds are characterized by a moduli space and the analogs of Teichmüller parameters depending on 3-topology. In TGD framework strong form of holography suggests that these conformal equivalence classes for preferred extremals could reduce to ordinary conformal classes for the partonic 2-surfaces. An attractive possibility is that by conformal gauge symmetries the functional integral over WCW reduces to the integral over the conformal equivalence classes.

3. The quaternion-conformal structures could be characterized by a standard choice of quaternionic coordinates reducing to the choice of a pair of complex coordinates. In these coordinates the general solution to quaternion-analyticity conditions would be of form described for the linear ansatz. The moduli space corresponds to that for complex or hyper-complex structures defined in the space-time region.

Geometric formulation of the CRF conditions

The previous naïve generalization of CRF conditions treats imaginary units without trying to understand their geometric content. This leads to difficulties when when tries to formulate these conditions for maps between quaternionic and hyper-quaternionic spaces using purely algebraic representation of imaginary units since it is not clear how these units relate to each other.

In [A74] the CRF conditions are formulated in terms of the antisymmetric (1, 1) type tensors representing the imaginary units: they exist for almost quaternionic structure. One might hope that this so also for the almost hyper-quaternionic structure needed in Minkowskian signature.

The generalization of CRF conditions is proposed in terms of the Jacobian J of the map mapping tangent space TM to TN and antisymmetric tensors J_u and j_u representing the quaternionic imaginary units of N and M respectively. The generalization of CRF conditions reads as

$$J - \sum_u J_u \circ J \circ j_u = 0 \ . \tag{8.3.1}$$

For $N = M$ it reduces to the translationally invariant algebraic form of the conditions discussed above. These conditions reduce to CR conditions in 2-D case when one has only single J_u. In quaternionic case this form is only replaced with sum over all 3 antisymmetric forms representing quaternionic units.

These conditions are not unique. One can perform an SO(3) rotation (quaternion automorphism) of the imaginary units mediated by matrix Λ^{uv} to obtain

$$J - \Lambda^{uv} J_u \circ J \circ j_v = 0 \ . \tag{8.3.1}$$

The matrix Λ can depend on point so that one has a kind of gauge symmetry. The most general triholomorphic map allows the presence of Λ. Note that these conditions make sense on any coordinates and complex analytic maps generate new forms of these conditions.

In Minkowskian signature one would have 3 forms iJ_u serving as counterparts for iI, iJ, iK. The most natural possibility is that i is represented as algebraic unit and I, J, K as antisymmetry self-dual em fields with E and B constant and parallel to each other and normalize to have unit lengths. Their directions would correspond to 3 orthogonal coordinate axis. The twistor lift forces to introduce the generalization of Kähler form of M^4 and one can introduce all these 3 independent forms as counterpart of hyperquaternionic units: they are introduced also for ordinary twistor structure but one of them is selected as a preferred one. The only change in the conditions is change of sign of the sign of the sum coming from $i^2 = -1$ so that one has

$$J + \sum_u J_u \circ J \circ j_u = 0 \ . \tag{8.3.2}$$

These conditions are therefore formally well-defined also when one maps quaternionic to hyper-quaternionic space or vice versa.

In 2-dimensional hypercomplex case the conditions allow to write hypercomplex map $X - Y = U = f(x - y)$ and $X + Y = V = f(x + y)$. In special case this solutions of massless d'Alembertian in M^2. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. It is however not clear whether one can write a Taylor expansion in hyper-quaternionic coordinate in the similar manner.

Covariant forms of structure constant tensors reduce to octonionic structure constants and this allows to write the conditions explicitly. The index raising of the second index of the structure constants is however needed using the metrics of M and N. This complicates the situation and spoils linearity: in particular, for surfaces induced metric is needed. Whether local SO(3) rotation can eliminate the dependence on induced metric is an interesting question. Since M^4 imaginary units differ only by multiplication by i, Minkowskian structure constants differ only by sign from the Euclidian ones.

In the octonionic case the geometric generalization of CRF conditions does not seem to make sense. By non-associativity of octonion product it is not possible to have a matrix representation for the matrices so that a faithful representation of octonionic imaginary units as antisymmetric 1-1 forms does not make sense. If this representation exists it it must map octonionic associators to zero. Note however that CRF conditions do not involve products of three octonion units so that they make sense as algebraic conditions at least.

Does residue calculus generalize?

CRF conditions allow to generalize Cauchy formula allowing to express value of analytic function in terms of its boundary values [A74]. This would give a concrete realization of the holography in the sense that the physical variables in the interior could be expressed in terms of the data at the light-like partonic orbits and the ends of the space-time surface. Triholomorphic function satisfies d'Alembert/Laplace equations - in induced metric in TGD framework- so that the maximum modulus principle holds true. The general ansatz for a preferred extremals involving Hamilton-Jacobi structure leads to d'Alembert type equations for preferred extremals [K111].

Could the analog of residue calculus exist? Line integral would become 3-D integral reducing to a sum over poles and possible cuts inside the 3-D contour. The space-like 3-surfaces at the ends of CDs could define natural integration contours, and the freedom to choose contour rather freely would reflect General Coordinate Invariance. A possible choice for the integration contour would be the closed 3-surface defined by the union of space-like surfaces at the ends of CD and by the light-like partonic orbits.

Poles and cuts would be in the interior of the space-time surface. Poles have co-dimension 2 and cuts co-dimension 1. Strong form of holography suggests that partonic 2-surfaces and perhaps also string world sheets serve as candidates for poles. Light-like 3-surfaces (partonic orbits) defining the boundaries between Euclidian and Minkowskian regions are singular objects and could serve as cuts. The discontinuity would be due to the change of the signature of the induced metric. There are CDs inside CDs and one can also consider the possibility that sub-CDs define cuts, which in turn reduce to cuts associated with sub-CDs.

8.3.3 Are Preferred Extremals Quaternion-Analytic in Some Sense?

At what level quaternion analyticity could appear in TGD framework? Does it appear only in the formulation of conformal algebras and replace loop algebra with double loop algebra (roughly $z^m \rightarrow u^m v^n$)? Or does it appear in some form also at the level of preferred extremals for which geometric form of quaternionicity is expected to appear - at least at the level of M^8?

Minimalistic view

Before continuing it is good to bring in mind the minimal assumptions and general vision.

1. If $M^8 - H$ duality [K96] holds true, the space-time surface $X^4 \subset M^8 = M^4 \times E^4$ is quaternionic surface in the sense that it have quaternionic tangent space and contains preferred $M^2 \subset M^4$ as part of their tangent space or more generally the 2-D hyper-complex subspaces $M^2(x)$ define and integrable distribution defining 2-D surface.

2. Quaternionicity in geometric sense in M^8 alone *implies* the interpretation as a 4-D surface in $H = M^4 \times CP_2$. There is *no need* to assume quaternionicity in geometric sense in H although it cannot be excluded and would have strong implications [K96]. This one should remember in order to avoid drowning to an inflation of speculations.

It is not at all clear what quaternion analyticity in Minkowskian signature really means or whether it is even possible. The skeptic inside me has a temptation to conclude that the direct extrapolation of quaternion analyticity from Euclidian to Minkowskian signature for space-time surfaces in H is not necessary and might be even impossible. On the other hand, the properties of the known extremals strongly suggest its analog. Quaternion analyticity could however appear at the tangent space level for various generalized conformal algebras transformed to double loop algebras for the proposed realization of the quaternion analyticity.

The naïve generalization of quaternion analyticity to Minkowski signature fails

Quaternion analyticity works nicely in Euclidian signature for maps $E^4 \to E^4$. One can also consider quaternion analytic maps $E^4 \to E^8$ with E^8 regarded as octonionic space of form $E^4 \oplus E^4 J$, where E^4 is quaternionic space and J is octonion unit in the complement of $E^4 \subset E^8$. The maps decompose to sums $f_1 \oplus f_2 J$ where f_i are quaternion analytic maps $E^4 \to E^4$. Consider maps $f : E^4 \to E^8$, whose graph should define Euclidian space-time surface.

1. One can construct octonion valued maps $f(u,v) = f_0 + \sum u^m v^n a_{mn} : E^4 \to E^8$ with E^4 identified as quaternionic sub-space of E^8. Recall that one has $u = t + Iz$, $v = (x + Iy)J$. a_{mn} can be octonions with the proposed definition of the Taylor series. Since each power $u^m v^n$ is analytic function, one still has quaternion analyticity in the proposed sense. The image would be 4-D in the general case.

2. By linearity the solutions obey linear superposition. They can be also multiplied if the product is defined as ordered product in such a way that only the powers $t+ix$ and $y+iz$ are multiplied together at left and coefficients a_{mn} are multiplied together at right. The analogy with quantum non-commutativity is obvious.

Can one generalize this ansatz to Minkowskian signature? One can try to look the ansatz for the embedding $X^4 \subset M^8 = M^4 \times E^4 J$ as sum $f = (f_1, f_2)$ of quaternion analytic maps $f_1 : X^4 \to M^4$ and $f_2 : X^4 \to E^4$. The general quaternion analytic ansatz for $X^4 \subset E^8$ fails due to the non-commutativity of quaternions.

The comparison of 2-dimensional hypercomplex case with 4-D hyperquaternionic case reveals the basic problem.

1. The analogs CR conditions allow to write hypercomplex map $X - Y = U = f(x - y)$ and $X + Y = V = f(x + y)$. In special case this gives the solutions of massless d'Alembertian in M^2 as sum of these solutions. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. The use of hypercomplex numbers and hypercomplex analyticity is equivalent with use of functions $f(x - y)$ or $f(x + y)$.

2. The essential point is that for M^2 regarded as a sub-space of "complexified" complex numbers $z_1 + iz_2$ consisting of points $x + iIy$, the multiplication of numbers of form $x + iIy$ does not lead out of M^2. For M^4 this is not anymore the case since $iI \times iJ = -K$ does not belong to the Minkowskian subspace of complexified quaternions. Hence there are no hopes about the existence of the analog of $f(z) = \sum a_n z^n$. For this reason also non-trivial powers $u^m v^n$ are excluded and one cannot build a Minkowskian generalization of quaternion analytic power series.

3. If one can allow the values of hyper-quaternion analytic functions to be in M_c^4 rather than M^4, there are no problems but if one wants to represents space-time surfaces as graphs of hyper-quaternion analytic maps $f : M^4 \to M^8$ one must pose strong restrictions on allowed functions.

The restrictions on the allowed hyper-quaternion analytic functions look rather obvious for what might be called hyper-quaternion analytic maps $M^4 \to M^4$.

1. Assume a decomposition $M^4 = M^2 \times E^2$ such one has $f = (f_1, f_2)$, where $f_1 : M^2 \to M^2$ is analytic in hyper-complex sense and $f_2 : E^2 \to E^2$ is analytic in complex sense. Both these options are possible. One can write the map as $f(u,v) = f_1(u = t + iIz) + f_2(v = x + Iy)iJ$

and it satisfies the usual conditions $\partial_{\overline{u}} f = 0$ and $\partial_{\overline{v}} f = 0$. Note that iJ is taken to the right so that the differential operators acting from left in the analyticity conditions does not "notice" it.

Linear superpositions of this kind of solutions with real coefficients are possible. One can multiply this kind of solutions if the multiplication is done separately in the Cartesian factors. Also functional composition is possible in the factors.

2. A generalization of the solution ansatz to integrable decompositions $M^4 = M^2(x) \oplus E^2(x)$ is rather plausible. This would mean a foliation of M^4 by pairs of 2-D surfaces. String world sheets and partonic 2-surfaces would be the physical counterpart for these foliations. I have called this kind of foliation Hamilton-Jacobi structure [K8] and it would serve as a generalization of the complex structure to 4-D Minkowskian case. In Euclidian signature it corresponds to ordinary complex structure in 4-D sense.

3. The analogy of double loop Lie algebra replacing powers z^m with $u^m v^n$ does does not however seem to be possible. Could this relate to SH forcing to code data using only functions of u or v and to select either string world sheet or partonic 2-surface (fixing the gauge)?

   On the other hand, the supersymplectic algebra (SSA) and extension of Kac-Moody algebras to light-like orbits of partonic 2-surfaces suggests strongly that functions of form $(t-z)^m v^n$ as basis associated with SSA and SKMAs must be allowed as basis at these 3-D light-like surfaces. These functions generate deformations of boundaries defining symmetries but the corresponding deformations in the interior of the preferred extremals are not expected to be of this form. Double loop algebra would not be lost but would have a nice separable form only at boundaries of CD and at light-like partonic orbits.

What can one conclude?

1. The general experience about the solutions of field equations conforms with this picture coded to the notion of Hamilton-Jacobi structure [K8]. Field equations and purely number theoretic conditions related to Minkowski signature force what might be called number theoretic spontaneous symmetry breaking. This symmetry breaking is analogous to a selection of single imaginary unit defining the analog of Kähler structure for M^4: this imaginary unit defines a new kind of $U(1)$ force in TGD explaining large scale breaking of CP,P, and T [L20]. This kind of selection occurs also for the quaternionic structure of CP_2 [L25].

2. The realistic form of analyticity condition abstractable from the properties known extremals seems to be following. For the Minkowskian space-time surfaces the complex coordinates of H are analytic functions of complex coordinates and of light-like coordinate assignable to space-time surface. These coordinates can be assigned to M^4 and define decomposition $M^4 = M^2 \times E^2$: this decomposition can be local but must be integrable (Hamilton-Jacobi structure). For Euclidian regions with 4-D CP_2 projection complex coordinate of E^2 is complex function of complex coordinates of CP_2 and M^2 light-like coordinate is function of real CP_2 coordinates and second light-like coordinate is constant.

3. The transition to Minkowskian signature by regarding M^4 as sub-space of complex-quaternionic M^4 does not respect the notion of quaternion analyticity in the naïvest sense. Both Euclidian and Minkowskian variants of quaternionic (associative) sub-manifold however makes sense as also co-quaternionic (co-associative) sub-manifold. An attractive hypothesis is that the geometric view about quaternionicity is consistent with the above view about analyticity. The known extremals are consistent with this form of analyticity. Analyticity in this sense should be consistent with the geometric quaternionicity of X^4 in Minkowskian signature and geometric co-quaternionicity in Euclidian signature.

4. The geometric form of quaternionicity (or associativity) requires that the associator $a(bc) - (ab)c$ for any 3 tangent space vectors vanishes. These conditions involve products of 3 partial derivates of embedding space coordinates. For co-associativity this holds true in the normal space. Again one must remember that these conditions might be needed only in M^8 but make sense also for H.

One must be however cautious: quaternionicity (associativity) in M^8 in the geometric sense *need not* imply even the above realistic form of quaternion analyticity condition in M^8 and even less so in H: this however seems to be the case.

Can the known extremals satisfy the realistic form of quaternion-analyticity?

To test the consistency the realistic form of quaternion analyticity, at the level of M^8 or even H, the best thing to do is to look whether quaternion analyticity is possible for the known extremals for the twistor lift of Kähler action.

Twistor lift drops away most vacuum extremals from consideration and leaves only minimal surfaces. The surviving vacuum extremals include CP_2 type extremals with light-like geodesic rather than arbitrary light-like curve as M^4 projection. Vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2 remain in the spectrum only if they are also minimal surfaces: this kind of minimal surfaces are known to exist.

Massless extremals (MEs) with 2-D CP_2 projection remain in the spectrum. Cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2 are extremals of both Kähler action and volume term. One can also check whether Hamilton-Jacobi structure of M^4 and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^4 and Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k \cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ - a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condensate of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure [K8]. The non-geodesic motion of photons with light-velocity and variation of the polarization direction would be due to the interactions with the space-time sheet to which it is topologically condensed.

   Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of the realistic form of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t - x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

   The dependence on the real coordinate $v = t - z$ does not conform with the proposed ansatz since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

   If this picture makes sense, one must modify the ansatz for the preferred extremals with Minkowskian signature. The E^4 and coordinates and perhaps even real CP_2 coordinates can depend on light-like coordinate u.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection survive if they are minimal surfaces. This property should guarantee the realistic form of quaternion analyticity. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals surviving as extremals the M^4 projection is light-like geodesic with $t + z = 0$ with suitable choice of light-like coordinates in M^2. $t - z$ can arbitrary function of CP_2 coordinates. Associativity of the normal space is the only possible option now.

   One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light- like coordinate would vary along random light-like geodesics providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined.

5. In the naïve generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [A51] both forms of the algebraic CRF conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

   Could octonion analytic maps of embedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

Quaternion analyticity and generalized conformal algebras

The realistic quaternionic analyticity should apply at the level of conformal algebras for conformal algebra is replaced with a direct sum of 2-D conformal and hyper-conformal algebra assignable to string world sheets and partonic 2-surfaces. This would conform with SH and the considerations above.

It is however too early to exclude the possibility that the powers z^n of conformal algebras are replaced by $u^m z^n$ ($u = t - z$ and $w = x + iy$) for symmetries restricted to the light-like boundaries of CD and to the light-like orbits of partonic 2-surfaces. This preferred form at boundaries would be essential for reducing degrees of freedom implied by SSA and SKMA gauge conditions. In the interior of space-time surfaces this simple form would be lost.

This would realize the Minkowskian analog of double loop algebras suggested by 4-dimensionality. This option is encouraged by the structure of super-symplectic algebra and generalization of Kac-Moody algebras for light-like orbits of partonic 2-surfaces. Again one must however remember that these algebras should have a realization at the level of M^8 but might not be necessary at the level of H.

1. The basic vision of quantum TGD is that string world sheets are replaced with 4-D surfaces and this forces a generalization of the notion of conformal invariance and one indeed obtains

generalized conformal invariances for both the light-like boundaries of CD and for the light-like 3-surfaces defining partonic orbits as boundaries between Minkowskian and Euclidian space-time regions. One can very roughly say that string string world sheets parameterized by complex coordinate are replaced by space-time surfaces parameterized by two complex coordinates. Quaternion analyticity in the sense discussed would roughly conform with this picture.

2. The recent work with the Yangians [K28] and so called n-structures related to the categorification of TGD [K13] suggest that double loop algebras for which string world sheets are replaced with 4-D complex spaces. Quantum groups and Yangians assignable to Kac-Moody algebras rather than Lie algebras should be also central. Also double quantum groups depending on 2 parameters with so called elliptic R-matrix seem to be important. This physical intuition agrees with the general vision of Russian mathematician Igor Frenkel, who is one of the pioneers of quantum groups. For the article summarizing the work of Frenkel see `http://tinyurl.com/y7eego8c`). This article tells also about the work of Frenkel related to quaternion analyticity, which he sees to be of physical relevance but as a mathematicians is well aware of the fact that the non-commutativity of quaternions poses strong interpretation problems and means the loss of many nice properties of the ordinary analyticity.

3. The twistor lift of TGD suggest similar picture [K38, K10, L25]. The 6-D twistor space of space-time surface would be 6-surface in the product $T(M^4) \times T(CP_2)$ of geometric twistor spaces of M^4 and CP_2 and have induced twistor structure. The detailed analysis of this statement strongly suggests that data given at surfaces with dimension not higher than $D = 2$ should fix the preferred extremals. For the twistor lift action contains besides Kähler action also volume term. Asymptotic solutions are extremals of both Kähler action and minimal surfaces and all non-vacuum extremals of Kähler action are minimal surfaces so that the only change is that vacuum extremals of Kähler action must be restricted to be minimal surfaces.

The article about the work of Igor Frenkel (see `http://tinyurl.com/y7eego8c`) explains the general mathematics inspired vision about 3-levelled hierarchy of symmetries.

1. At the lowest level are Lie algebras. Gauge theories are prime example about this level.

2. At the second level loop algebras and quantum groups (defined as deformations of enveloping algebra of Lie algebra) and also Yangians. Loop algebras correspond to string models and quantum groups to TQFTs formulated at 3-D spaces.

3. At the third level are double loop algebras, quantum variants of loop algebras (also Yangians), and double quantum quantum groups - deformations of Lie algebras for which the R-matrix is elliptic function and depends on 2 complex parameters.

The conjecture of Frenkel (see `http://tinyurl.com/y7eego8c`) based on mathematical intuition is that these levels are actually the only ones. The motivation for this claim is 2-dimensionality making possible braiding and various quantum algebras. The set of poles for the R-matrix forms Abelian group with respect to addition in complex plane and can have rank equal to 0, 1, or (single pole, poles along line, lattice of poles). Higher ranks are impossible in $D = 2$.

In TGD framework physical intuition leads to a similar vision.

1. The dimension $D = 4$ for space-time surface and the choice $H = M^4 \times CP_2$ have both number theoretical and twistorial motivations [K28]. The replacement of point like particle with partonic 2-surface implies that TGD corresponds to the third level since loop algebras are replaced with their double loop analogs. 4-dimensionality makes also possible 2-braids and reconnections giving rise to a new kind of topological physics.

   The double loop group would represent the most dynamical level and its singly and doubly quantized variants correspond to a reduction in degrees of freedom, which one cannot exclude in TGD.

   The interesting additional aspect relates to the adelic physics [L21] implying a hierarchy of physics labelled by extensions of rationals. For cognitive representations the dynamics is discretized [K13]. Light-like 3-surfaces as partonic orbits are part of the picture and Chern-Simons term is naturally associated with them. TGD as almost topological QFT has been one of the guiding ideas in the evolution of TGD.

2. Double loop algebras represent unknown territory of mathematical physics. Igor Frenkel has also considered a possible realization of double loop algebras (see `http://tinyurl.com/y7eego8c`). He starts from the work of Mickelson (by the way, my custos in my thesis defence in 1982!) giving a realization of loop algebras: the idea is clearly motivated by WZW model which is 2-D conformal field theory with action containing a term associated with a 3-ball having 2-sphere as boundary.

   Mickelson starts from a circle represented as a boundary of a disk at which the physical states of CFT are realized. CFT is obtained by gluing together two disks with the boundary circles identified. The sphere in turn can be regarded as a boundary of a ball. The proposal of Frenkel is to complexify all these structures: circle becomes a Riemann surface, disk becomes 4-D manifold possessing complex structure in some sense, and 3-ball becomes 6-D complex manifold in some sense conjectured to be Calabi-Yau manifold.

3. The twistor lift of TGD leads to an analogous proposal. Circle is replaced with partonic 2-surfaces and string world sheets. 2-D complex surface is replaced with space-time region with complex structure or Hamilton-Jacobi structure [K8] and possessing twistor structure. 6-D Calabi-Yau manifold is replaced with the 6-D twistor space of space-time surface (sphere bundle over space-time surface) represented as 6-surface in 12-D Cartesian product $T(H) = T(M^4) \times T(CP_2)$ of the geometric twistor spaces of M^4 and CP_2.

   Twistor structure is induced and this is conjectured to determine the dynamics to be that for the preferred extremals of Kähler action plus volume term. This vision would generalize Penrose's original vision by eliminating gauge fields as primary dynamical variables and replacing there dynamics with the geometrodynamics of space-time surface.

Do isometry currents of preferred extremals satisfy Frobenius integrability conditions?

During the preparation of the book I learned that Agostino Prastaro [A20, A21] has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action of its twistor lift in TGD. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis.

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could the natural topology in the parameter space of Noether charges zero modes of WCW metric) be p-adic and realize adelic physics at the level of WCW? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the 6 lowest dynamical homotopy/homology groups of WCW would be non-trivial. The Kähler structure of WCW suggests that only Π_0, Π_2, and Π_4 are non-trivial.

The interpretation of the analog of Π_1 as deformations of generalized Feynman diagrams with elementary cobordism snipping away a loop as a move leaving scattering amplitude invariant conforms with the number theoretic vision about scattering amplitude as a representation for a sequence of algebraic operation can be always reduced to a tree diagram. TGD would be indeed topological QFT: only the dynamical topology would matter.

A further outcome is an ansatz for generalizing the earlier proposal for preferred extremals and stating that non-vanishing conserved isometry currents satisfy Frobenius integrability conditions so that one could assign global coordinate with their flow lines. This ansatz looks very similar to the CRF conditions stating quaternion analyticity [L8].

Conclusions

To sum up, connections between different conjectures related to the preferred extremals - $M^8 - H$ duality, Hamilton-Jacobi structure, induced twistor space structure, quaternion-Kähler property and its Minkowskian counterpart, and perhaps even quaternion analyticity - albeit not in the naïve form , are clearly emerging. The underlying reason is strong form of GCI forced by the construction of WCW geometry and implying strong from of holography posing extremely powerful quantization conditions on the extremals of Kähler action in ZEO. Without the conformal gauge conditions the mutual inconsistency of these conjectures looks rather infeasible.

8.4 Octo-Twistors And Twistor Space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall $M^8 - H$ duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [K101]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic embedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

8.4.1 Two ways To Twistorialize Embedding Space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1) \times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably,

M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-definec operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4 \times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

   Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are "real" and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and embedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

8.4.2 Octotwistorialization Of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

8.4.3 Octonionicity, $SO(1,7)$, G_2, And Non-Associative Malcev Group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1, 7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \rightarrow G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (`http://tinyurl.com/ybd4lcpy`). The basic observation is that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14 + 14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a, b] = ab - ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}[\gamma_k, \gamma_l] \quad . \tag{8.4.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

This algebra has decomposition $g = h + t$, $[h,t] \subset t$, $[t,t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1,7)$.

8.4.4 Octonionic Spinors In M^8 And Real Complexified-Quaternionic Spinors In H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.

2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

   The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

   One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc) - (ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

8.4.5 What The Replacement Of $SO(7,1)$ Sigma Matrices With Octonionic Sigma Matrices Could Mean?

The basic implication of octonionization is the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4 \times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4 \times CP_2$ but interpret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ \ , \ \ \gamma^i = \gamma^i \otimes \sigma_2 \ \ , \ \ i = 1,..,7 \ \ . \tag{8.4.2}$$

   7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1, ..., 6 \ \ , \ \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ \ . \tag{8.4.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \otimes \sigma_1 \ \ , \ \ \gamma_i = e_i \otimes \sigma_2 \ \ . \tag{8.4.4}$$

   where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = je_i \times \sigma_3 \ \ , \ \ \Sigma_{ij} = jf_{ij}{}^k e_k \otimes 1 \ \ . \tag{8.4.5}$$

   Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A9] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [K15]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the Kähler-Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8?

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://tinyurl.com/y93aerea`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of hyper-complex numbers) such that the second complexified Kähler-Dirac gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

Does this mean that the difference between quarks and leptons becomes visible only at the embedding space level where ground states of super-conformal representations correspond to embedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

Octo-spinors and their relation to ordinary embedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\Psi_{L,i} = e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ,$$
$$\Psi_{q,i} = e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \tag{8.4.5}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{8.4.6}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic Kähler-Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

8.4.6 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K101] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

The case of $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K96] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?

2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.

3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.

4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K25].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?

2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.

3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions

should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

(a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

(b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

(c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

8.5 What Could Be The Origin Of Preferred P-Adic Primes And P-Adic Length Scale Hypothesis?

p-Adic mass calculations [K62] allow to conclude that elementary particles correspond to one or possible several preferred primes assigning p-adic effective topology to the real space-time sheets in discretization in some length scale range. TGD inspired theory of consciousness leads to the identification of p-adic physics as physics of cognition. The recent progress leads to the proposal that quantum TGD is adelic: all p-adic number fields are involved and each gives one particular view about physics. tgdquantum/tgdquantum Adelic approach [K45, K68] plus the view about evolution as emergence of increasingly complex extensions of rationals leads to a possible answer to th question of the title. The algebraic extensions of rationals are characterized by preferred rational primes, namely those which are ramified when expressed in terms of the primes of the extensions. These primes would be natural candidates for preferred p-adic primes. An argument relying on what I call weak form of NMP in turn allows to understand why primes near powers of 2 are preferred: as a matter of fact, also primes near powers of other primes are predicted to be favoured.

8.5.1 Earlier Attempts

How the preferred primes emerge in TGD framework? I have made several attempts to answer this question. As a matter fact, the question has been slightly different: what determines the p-adic prime assigned to elementary particle by p-adic mass calculations [K51]. The recent view assigns to particle entire adele but some p-adic number fields in it are different.

1. Classical non-determinism at space-time level for real space-time sheets could in some length scale range involving rational discretization for space-time surface itself or for parameters characterizing it as a preferred extremal correspond to the non-determinism of p-adic differential equations due to the presence of pseudo constants which have vanishing p-adic derivative. Pseudo- constants are functions depend on finite number of pinary digits of its arguments.

2. The quantum criticality of TGD [?] is suggested to be realized in in terms of infinite hierarchies of super-symplectic symmetry breakings in the sense that only a sub-algebra with

conformal weights which are n-ples of those for the entire algebra act as conformal gauge symmetries [K84]. This might be true for all conformal algebras involved. One has fractal hierarchy since the sub-algebras in question are isomorphic: only the scale of conformal gauge symmetry increases in the phase transition increasing n. The hierarchies correspond to sequences of integers $n(i)$ such tht $n(i)$ divides $n(i+1)$. These hierarchies would very naturally correspond to hierarchies of inclusions of hyper-finite factors and $m(i) = n(i+1)/n(i)$ could correspond to the integer n characterizing the index of inclusion, which has value $n \geq 3$. Possible problem is that $m(i) = 2$ would not correspond to Jones inclusion. Why the scaling by power of two would be different? The natural question is whether the primes dividing $n(i)$ or $m(i)$ could define the preferred primes.

3. Negentropic entanglement corresponds to entanglement for which density matrix is projector [K56]. For n-dimensional projector any prime p dividing n gives rise to negentropic entanglement in the sense that the number theoretic entanglement entropy defined by Shannon formula by replacing p_i in $log(p_i) = log(1/n)$ by its p-adic norm $N_p(1/n)$ is negative if p divides n and maximal for the prime for which the dividing power of prime is largest power-of-prime factor of n. The identification of p-adic primes as factors of n is highly attractive idea. The obvious question is whether n corresponds to the integer characterizing a level in the hierarchy of conformal symmetry breakings.

4. The adelic picture about TGD led to the question whether the notion of unitarity could be generalized. S-matrix would be unitary in adelic sense in the sense that $P_m = (SS^\dagger)_{mm} = 1$ would generalize to adelic context so that one would have product of real norm and p-adic norms of P_m. In the intersection of the realities and p-adicities P_m for reals would be rational and if real and p-adic P_m correspond to the same rational, the condition would be satisfied. The condition that $P_m \leq 1$ seems however natural and forces separate unitary in each sector so that this options seems too tricky.

These are the basic ideas that I have discussed hitherto.

8.5.2 Could Preferred Primes Characterize Algebraic Extensions Of Rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of ramification of primes (see `http://tinyurl.com/hddljlf`) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this language): As one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called integral closure (see `http://tinyurl.com/js6fpvr`) over integers of K decompose to products of prime ideals of L (prime is a more rigorous manner to express primeness).

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K and having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

2. There are two further basic notions related to ramification and characterizing it. Relative discriminant is the ideal divided by all ramified ideals in K and relative different is the ideal of L divided by all ramified P_i:s. Note that te general ideal is analog of integer and these ideas represent the analogous of product of preferred primes P of K and primes P_i of L dividing them.

3. A physical analogy is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, where e_i is the ramification index - the physical analog would be the number of elementary particles of type i in the state (see http://tinyurl.com/h9528pl). Could the ramified rational primes could define the physically preferred primes for a given elementary system?

In TGD framework the extensions of rationals (see http://tinyurl.com/h9528pl) and p-adic number fields (seehttp://tinyurl.com/zq22tvb) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would have gradually proceeded to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. IT would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (see http://tinyurl.com/47kxjz states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

4. Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n :th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified. The physical analog $P = P_0^n$ is Bose-Einstein condensate of n bosons. There is a strong temptation to identify the preferred primes of irreducible polynomials as preferred p-adic primes.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with $\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

5. What does this mean in p-adic context? The identity of the ideals can be stated by saying $P = P_0^n$ for the ideals defined by the primes satisfying the Eisenstein condition. Very loosely one can say that the algebraic extension defined by the root involves n:th root of p-adic prime p. This does not work! Extension would have a number whose n:th power is zero modulo p. On the other hand, the p-adic numbers of the extension modulo p should be finite field but this would not be field anymore since there would exist a number whose n:th power vanishes. The algebraic extension simply does not exist for preferred primes. The physical meaning of this will be considered later.

6. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \to x + 1$ transforms $(x^n - 1)/(x - 1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

8.5.3 A Connection With Langlands Program?

In Langlands program (`http://tinyurl.com/ycej7s43`,`RecentAdvancesinLanglandsprogram`) [A33, A32] the great vision is that the n-dimensional representations of Galois groups G characterizing algebraic extensions of rationals or more general number fields define n-dimensional adelic representations of adelic Lie groups, in particular the adelic linear group $Gl(n, A)$. This would mean that it is possible to reduce these representations to a number theory for adeles. This would be highly relevant in the vision about TGD as a generalized number theory. I have speculated with this possibility earlier (`http://tinyurl.com/y9ee3lk6`) [K45] but the mathematics is so horribly abstract that it takes decade before one can have even hope of building a rough vision.

One can wonder whether the irreducible polynomials could define the preferred extensions K of rationals such that the maximal abelian extensions of the fields K would in turn define the adeles utilized in Langlands program. At least one might hope that everything reduces to the maximally ramified extensions.

At the level of TGD string world sheets with parameters in an extension defined by an irreducible polynomial would define an adele containing various p-adic number fields defined by the primes of the extension. This would define a hierarchy in which the prime ideals of previous level would decompose to those of the higher level. Each irreducible extension of rationals would correspond to some physically preferred p-adic primes.

It should be possible to tell what the preferred character means in terms of the adelic representations. What happens for these representations of Galois group in this case? This is known.

1. For Galois extensions ramification indices are constant: $e(i) = e$ and Galois group acts transitively on ideals P_i dividing P. One obtains an n-dimensional representation of Galois group. Same applies to the subgroup of Galois group G/I where I is subgroup of G leaving P_i invariant. This group is called inertia group. For the maximally ramified case G maps the ideal P_0 in $P = P_0^n$ to itself so that $G = I$ and the action of Galois group is trivial taking P_0 to itself, and one obtains singlet representations.

2. The trivial action of Galois group looks like a technical problem for Langlands program and also for TGD unless the singletness of P_i under G has some physical interpretation. One

possibility is that Galois group acts as like a gauge group and here the hierarchy of sub-algebras of super-symplectic algebra labelled by integers n is highly suggestive. This raises obvious questions. Could the integer n characterizing the sub-algebra of super-symplectic algebra acting as conformal gauge transformations, define the integer defined by the product of ramified primes? P_0^n brings in mind the n conformal equivalence classes which remain invariant under the conformal transformations acting as gauge transformations. . Recalling that relative discriminant is an of K ideal divisible by ramified prime ideals of K, this means that n would correspond to the relative discriminant for $K = Q$. Are the preferred primes those which are "physical" in the sense that one can assign to the states satisfying conformal gauge conditions?

If the Galois group corresponds to gauge symmetries for these primes, it is physically natural that the p-adic algebraic extension does not exists and that p-adic variant of the Galois group is absent. Nothing is lost from cognition since there is nothing to cognize!

8.5.4 What Could Be The Origin Of P-Adic Length Scale Hypothesis?

The argument would explain the existence of preferred p-adic primes. It does not yet explain p-adic length scale hypothesis [K66, K51] stating that p-adic primes near powers of 2 are favored. A possible generalization of this hypothesis is that primes near powers of prime are favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [I2] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present, this is discussed in TGD inspired theory of music harmony and genetic code [K80].

The weak form of NMP might come in rescue here.

1. Entanglement negentropy for a negentropic entanglement [K56] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretic entanglement negentropy. If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real preferred extremal as p-adic preferred extermal (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. Weak form of NMP [K56, K106] suggests a more convincing explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. Weak form of NMP allows free will so that all dimensions $n - k$, $k = 0, 1, ...n - 1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n - k$ is power of prime. For $n = 2^k = M_k + 1$, where M_k is Mersenne prime $n - 1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n = 2^k - r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p = 2$ can be replaced by any prime.

This argument together with the hypothesis that preferred prime is ramified would correlate the character of the irreducible extension and character of super-conformal symmetry breaking. The integer n characterizing super-symplectic conformal sub-algebra acting as gauge algebra would depends on the irreducible algebraic extension of rational involved so that the hierarchy of quantum criticalities would have number theoretical characterization. Ramified primes could appear as divisors of n and n would be essentially a characteristic of ramification known as discriminant. An interesting question is whether only the ramified primes allow the continuation of string world sheet and partonic 2-surface to a 4-D space-time surface. If this is the case, the assumptions behind p-adic mass calculations would have full first principle justification.

8.5.5 A Connection With Infinite Primes?

Infinite primes are one of the mathematical outcomes of TGD [K94]. There are two kinds of infinite primes. There are the analogs of free many particle states consisting of fermions and bosons labelled by primes of the previous level in the hierarchy. They correspond to states of a supersymmetric arithmetic quantum field theory or actually a hierarchy of them obtained by a repeated second quantization of this theory. A connection between infinite primes representing bound statesandirreducible polynomials is highly suggestive.

1. The infinite prime representing free many-particle state decomposes to a sum of infinite part and finite part having no common finite prime divisors so that prime is obtained. The infinite part is obtained from "fermionic vacuum" $X = \prod_k p_k$ by dividing away some fermionic primes p_i and adding their product so that one has $X \rightarrow X/m + m$, where m is square free integer. Also $m = 1$ is allowed and is analogous to fermionic vacuum interpreted as Dirac sea without holes. X is infinite prime and pure many-fermion state physically. One can add bosons by multiplying X with any integers having no common denominators with m and its prime decomposition defines the bosonic contents of the state. One can also multiply m by any integers whose prime factors are prime factors of m.

2. There are also infinite primes, which are analogs of bound states and at the lowest level of the hierarchy they correspond to irreducible polynomials $P(x)$ with integer coefficients. At the second levels the bound states would naturally correspond to irreducible polynomials $P_n(x)$ with coefficients $Q_k(y)$, which are infinite integers at the previous level of the hierarchy.

3. What is remarkable that bound state infinite primes at given level of hierarchy would define maximally ramified algebraic extensions at previous level. One indeed has infinite hierarchy of infinite primes since the infinite primes at given level are infinite primes in the sense that they are not divisible by the primes of the previous level. The formal construction works as such. Infinite primes correspond to polynomials of single variable at the first level, polynomials of two variables at second level, and so on. Could the Langlands program could be generalized from the extensions of rationals to polynomials of complex argument and that one would obtain infinite hierarchy?

4. Infinite integers in turn could correspond to products of irreducible polynomials defining more general extensions. This raises the conjecture that infinite primes for an extension K of rationals could code for the algebraic extensions of K quite generally. If infinite primes correspond to real quantum states they would thus correspond the extensions of rationals to which the parameters appearing in the functions defining partonic 2-surfaces and string world sheets.

   This would support the view that partonic 2-surfaces associated with algebraic extensions defined by infinite integers and thus not irreducible are unstable against decay to partonic 2-surfaces which corresponds to extensions assignable to infinite primes. Infinite composite integer defining intermediate unstable state would decay to its composites. Basic particle physics phenomenology would have number theoretic analog and even more.

5. According to Wikipedia, Eisenstein's criterion (http://tinyurl.com/47kxjz) allows generalization and what comes in mind is that it applies in exactly the same form also at the higher levels of the hierarchy. Primes would be only replaced with prime polynomials and the there would be at least one prime polynomial $Q(y)$ dividing the coefficients of $P_n(x)$ except the highest one such that its square would not divide P_0. Infinite primes would give rise to an infinite hierarchy of functions of many complex variables. At first level zeros of function would give discrete points at partonic 2-surface. At second level one would obtain 2-D surface: partonic 2-surfaces or string world sheet. At the next level one would obtain 4-D surfaces. What about higher levels? Does one obtain higher dimensional objects or something else. The union of n 2-surfaces can be interpreted also as $2n$-dimensional surface and one could think that the hierarchy describes a hierarchy of unions of correlated partonic 2-surfaces. The correlation would be due to the preferred extremal property of Kähler action.

   One can ask whether this hierarchy could allow to generalize number theoretical Langlands to the case of function fields using the notion of prime function assignable to infinite prime. What this hierarchy of polynomials of arbitrary many complex arguments means physically is unclear. Do these polynomials describe many-particle states consisting of partonic 2-surface such that there is a correlation between them as sub-manifolds of the same space-time sheet representing a preferred extremals of Kähler action?

This would suggest strongly the generalization of the notion of p-adicity so that it applies to infinite primes.

1. This looks sensible and maybe even practical! Infinite primes can be mapped to prime polynomials so that the generalized p-adic numbers would be power series in prime polynomial - Taylor expansion in the coordinate variable defined by the infinite prime. Note that infinite primes (irreducible polynomials) would give rise to a hierarchy of preferred coordinate variables. In terms of infinite primes this expansion would require that coefficients are smaller than the infinite prime P used. Are the coefficients lower level primes? Or also infinite integers at the same level smaller than the infinite prime in question? This criterion makes sense since one can calculate the ratios of infinite primes as real numbers.

2. I would guess that the definition of infinite-P p-adicity is not a problem since mathematicians have generalized the number theoretical notions to such a level of abstraction much above of a layman like me. The basic question is how to define p-adic norm for the infinite primes (infinite only in real sense, p-adically they have unit norm for all lower level primes) so that it is finite.

3. There exists an extremely general definition of generalized p-adic number fields (see http://tinyurl.com/y5zreeg). One considers Dedekind domain D, which is a generalization of integers for ordinary number field having the property that ideals factorize uniquely to prime ideals. Now D would contain infinite integers. One introduces the field E of fractions consisting of infinite rationals.

   Consider element e of E and a general fractional ideal eD as counterpart of ordinary rational and decompose it to a ratio of products of powers of ideals defined by prime ideals, now those defined by infinite primes. The general expression for the p-adic norm of x is $x^{-ord(P)}$, where n defines the total number of ideals P appearing in the factorization of a fractional ideal in E: this number can be also negative for rationals. When the residue field is finite (finite field G(p,1) for p-adic numbers), one can take c to the number of its elements ($c = p$ for p-adic numbers.

   Now it seems that this number is not finite since the number of ordinary primes smaller than P is infinite! But this is not a problem since the topology for completion does not depend on the value of c. The simple infinite primes at the first level (free many-particle states) can be mapped to ordinary rationals and q-adic norm suggests itself: could it be that infinite-P p-adicity corresponds to q-adicity discussed by Khrennikov [A19]. Note however that q-adic numbers are not a field.

Finally a loosely related question. Could the transition from infinite primes of K to those of L takes place just by replacing the finite primes appearing in infinite prime with the decompositions?

The resulting entity is infinite prime if the finite and infinite part contain no common prime divisors in L. This is not the case generally if one can have primes P_1 and P_2 of K having common divisors as primes of L: in this case one can include P_1 to the infinite part of infinite prime and P_2 to finite part.

8.6 More About Physical Interpretation Of Algebraic Extensions Of Rationals

The number theoretic vision has begun to show its power. The basic hierarchies of quantum TGD would reduce to a hierarchy of algebraic extensions of rationals and the parameters - such as the degrees of the irreducible polynomials characterizing the extension and the set of ramified primes (see `http://tinyurl.com/hddljlf`) - would characterize quantum criticality and the physics of dark matter as large h_{eff} phases. The value of $h_{eff}/h = n$ would naturally correspond to the order of the Galois group of the extension.

The conjecture is that preferred p-adic primes correspond to ramified primes for extensions of rationals for which especially many number theoretic discretizations of the space-time surfaces allow strong form of holography as an algebraic continuation of string world sheets to space-time surfaces. The generalization of the p-adic length scale hypothesis as a prediction of NMP is another conjecture. What remains to be shown that the primes predicted by generalization p-adic length scale hypothesis indeed are preferred primes in the proposed sense.

By strong form of holography the parameters characterizing string world sheets and partonic 2-surfaces serve as WCW coordinates. By various conformal invariances, one expects that the parameters correspond to conformal moduli, which means a huge simplification of quantum TGD since the mathematical apparatus of superstring theories becomes available and number theoretical vision can be realized. Scattering amplitudes can be constructed for a given algebraic extension and continued to various number fields by continuing the parameters which are conformal moduli and group invariants characterizing incoming particles.

There are many un-answered and even un-asked questions.

1. How the new degrees of freedom assigned to the n-fold covering defined by the space-time surface pop up in the number theoretic picture? How the connection with preferred primes emerges?

2. What are the precise physical correlates of the parameters characterizing the algebraic extension of rationals? Note that the most important extension parameters are the degree of the defining polynomial and ramified primes.

8.6.1 Some Basic Notions

Some basic information about extensions are in order. I emphasize that I am not a specialist.

Basic facts

The algebraic extensions of rationals are determined by roots of polynomials. Polynomials be decomposed to products of irreducible polynomials, which by definition do not contain factors which are polynomials with rational coefficients. These polynomials are characterized by their degree n, which is the most important parameter characterizing the algebraic extension.

One can assign to the extension primes and integers - or more precisely, prime and integer ideals. Integer ideals correspond to roots of monic polynomials $P_n(x) = x^n + ..a_0$ in the extension with integer coefficients. Clearly, for $n = 0$ (trivial extension) one obtains ordinary integers. Primes as such are not a useful concept since roots of unity are possible and primes which differ by a multiplication by a root of unity are equivalent. It is better to speak about prime ideals rather than primes.

Rational prime p can be decomposed to product of powers of primes of extension and if some power is higher than one, the prime is said to be ramified and the exponent is called ramification index. Eisenstein's criterion (see `http://tinyurl.com/47kxjz` states that any polynomial $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_1x + a_0$ for which the coefficients a_i, $i < n$ are divisible by p and

a_0 is not divisible by p^2 allows p as a maximally ramified prime. The corresponding prime ideal is n:th power of the prime ideal of the extensions (roughly n:th root of p). This allows to construct endless variety of algebraic extensions having given primes as ramified primes.

Ramification is analogous to criticality. When the gradient potential function $V(x)$ depending on parameters has multiple roots, the potential function becomes proportional a higher power of $x - x_0$. The appearance of power is analogous to appearance of higher power of prime of extension in ramification. This gives rise to cusp catastrophe. In fact, ramification is expected to be number theoretical correlate for the quantum criticality in TGD framework. What this precisely means at the level of space-time surfaces, is the question.

Galois group as symmetry group of algebraic physics

I have proposed long time ago that Galois group (see `http://tinyurl.com/h9528pl`) acts as fundamental symmetry group of quantum TGD and even made clumsy attempt to make this idea more precise in terms of the notion of number theoretic braid. It seems that this notion is too primitive: the action of Galois group must be realized at more abstract level and WCW provides this level.

First some facts (I am not a number theory professional, as the professional reader might have already noticed!).

1. Galois group acting as automorphisms of the field extension (mapping products to products and sums to sums and preserves norm) characterizes the extension and its elements have maximal order equal to n by algebraic n-dimensionality. For instance, for complex numbers Galois group acs as complex conjugation. Galois group has natural action on prime ideals of extension mapping them to each other and preserving the norm determined by the determinant of the linear map defined by the multiplication with the prime of extension. For instance, for the quadratic extension $Q(\sqrt{5})$ the norm is $N(x + \sqrt{5}y) = x^2 - 5y^2$: not that number theory leads to Minkowkian metric signatures naturally. Prime ideals combine to form orbits of Galois group.

2. Since Galois group leaves the rational prime p invariant, the action must permute the primes of extension in the product representation of p. For ramified primes the points of the orbit of ideal degenerate to single ideal. This means that primes and quite generally, the numbers of extension, define orbits of the Galois group.

Galois group acts in the space of integers or prime ideals of the algebraic extension of rationals and it is also physically attractive to consider the orbits defined by ideals as preferred geometric structures. If the numbers of the extension serve as parameters characterizing string world sheets and partonic 2-surfaces, then the ideals would naturally define subsets of the parameter space in which Galois group would act.

The action of Galois group would leave the space-time surface invariant if the sheets co-incide at ends but permute the sheets. Of course, the space-time sheets permuted by Galois group need not co-incide at ends. In this case the action need not be gauge action and one could have non-trivial representations of the Galois group. In Langlands correspondence these representation relate to the representations of Lie group and something similar might take place in TGD as I have indeed proposed.

The value of effective Planck constant $h_{eff}/h = n$ corresponds to the number of sheets of some kind of covering space defined by the space-time surface. The discretization of the space-time surface identified as a monadic manifold [L17] with embedding space preferred coordinates in extension of rationals defining the adele has Galois group of extension as a group of symmetries permuting the sheets of the covering group. Therefore $n = h_{eff}/h$ would naturally correspond to the dimension of the extension dividing the order of its Galois group. Dark matter in TGD sense would correspond to number theoretic physics.

Remark: Strong form of holography supports also the vision about quaternionic generalization of conformal invariance implying that the adelic space-time surface can be constructed from the data associated with functions of two complex variables, which in turn reduce to functions of single variable.

If this picture is correct, it is possible to talk about quantum amplitudes in the space defined by the numbers of extension and restrict the consideration to prime ideals or more general integer ideals.

1. These number theoretical wave functions are physical if the parameters characterizing the 2-surface belong to this space. One could have purely number theoretical quantal degrees of freedom assignable to the hierarchy of algebraic extensions and these discrete degrees of freedom could be fundamental for living matter and understanding of consciousness.

2. The simplest assumption that Galois group acts as a gauge group when the ends of sheets co-incide at boundaries of CD seems however to destroy hopes about non-trivial number theoretical physics but this need not be the case. Physical intuition suggests that ramification somehow saves the situation and that the non-trivial number theoretic physics could be associated with ramified primes assumed to define preferred p-adic primes.

8.6.2 How New Degrees Of Freedom Emerge For Ramified Primes?

How the new discrete degrees of freedom appear for ramified primes?

1. The space-time surfaces defining singular coverings are n-sheeted in the interior. At the ends of the space-time surface at boundaries of CD however the ends co-incide. This looks very much like a critical phenomenon.

   Hence the idea would be that the end collapse can occur only for the ramified prime ideals of the parameter space - ramification is also a critical phenomenon - and means that some of the sheets or all of them co-incide. Thus the sheets would co-incide at ends only for the preferred p-adic primes and give rise to the singular covering and large h_{eff}. End-collapse would be the essence of criticality! This would occur, when the parameters defining the 2-surfaces are in a ramified prime ideal.

2. Even for the ramified primes there would be n distinct space-time sheets, which are regarded as physically distinct. This would support the view that besides the space-like 3-surfaces at the ends the full 3-surface must include also the light-like portions connecting them so that one obtains a closed 3-surface. The conformal gauge equivalence classes of the light-like portions would give rise to additional degrees of freedom. In space-time interior and for string world sheets they would become visible.

   For ramified primes n distint 3-surfaces would collapse to single one but the n discrete degrees of freedom would be present and particle would obtain them. I have indeed proposed number theoretical second quantization assigning fermionic Clifford algebra to the sheets with n oscillator operators. Note that this option does not require Galois group to act as gauge group in the general case. This number theoretical second quantization might relate to the realization of Boolean algebra suggested by weak form of NMP [K109].

8.6.3 About The Physical Interpretation Of The Parameters Characterizing Algebraic Extension Of Rationals In TGD Framework

It seems that Galois group is naturally associated with the hierarchy $h_{eff}/h = n$ of effective Planck constants defined by the hierarchy of quantum criticalities. n would naturally define the maximal order for the element of Galois group. The analog of singular covering with that of $z^{1/n}$ would suggest that Galois group is very closely related to the conformal symmetries and its action induces permutations of the sheets of the covering of space-time surface.

Without any additional assumptions the values of n and ramified primes are completely independent so that the conjecture that the magnetic flux tube connecting the wormhole contacts associated with elementary particles would not correspond to very large n having the p-adic prime p characterizing particle as factor ($p = M_{127} = 2^{127} - 1$ for electron). This would not induce any catastrophic changes.

TGD based physics could however change the situation and reduce number theoretical degrees of freedom: the intuitive hypothesis that p divides n might hold true after all.

1. The strong form of GCI implies strong form of holography. One implication is that the WCW Kähler metric can be expressed either in terms of Kähler function or as anti-commutators of super-symplectic Noether super-charges defining WCW gamma matrices. This realizes what can be seen as an analog of Ads/CFT correspondence. This duality is much more general. The following argument supports this view.

   (a) Since fermions are localized at string world sheets having ends at partonic 2-surfaces, one expects that also Kähler action can be expressed as an effective stringy action. It is natural to assume that string area action is replaced with the area defined by the effective metric of string world sheet expressible as anti-commutators of Kähler-Dirac gamma matrices defined by contractions of canonical momentum currents with embedding space gamma matrices. It string tension is proportional to h_{eff}^2, string length scales as h_{eff}.

   (b) AdS/CFT analogy inspires the view that strings connecting partonic 2-surfaces serve as correlates for the formation of - at least gravitational - bound states. The distances between string ends would be of the order of Planck length in string models and one can argue that gravitational bound states are not possible in string models and this is the basic reason why one has ended to landscape and multiverse non-sense.

2. In order to obtain reasonable sizes for astrophysical objects (that is sizes larger than Schwartschild radius $r_s = 2GM$) For $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ one obtains reasonable sizes for astrophysical objects. Gravitation would mean quantum coherence in astrophysical length scales.

3. In elementary particle length scales the value of h_{eff} must be such that the geometric size of elementary particle identified as the Minkowski distance between the wormhole contacts defining the length of the magnetic flux tube is of order Compton length - that is p-adic length scale proportional to $\sqrt{p}$. Note that dark physics would be an essential element already at elementary particle level if one accepts this picture also in elementary particle mass scales. This requires more precise specification of what darkness in TGD sense really means.

   One must however distinguish between two options.

   (a) If one assumes $n \simeq \sqrt{p}$, one obtains a large contribution to classical string energy as $\Delta \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/\sqrt{p}$, which is of order particle mass. Dark mass of this size looks un-feasible since p-adic mass calculations assign the mass with the ends wormhole contacts. One must be however very cautious since the interpretations can change.

   (b) Second option allows to understand why the minimal size scale associated with CD characterizing particle correspond to secondary p-adic length scale. The idea is that the string can be thought of as being obtained by a random walk so that the distance between its ends is proportional to the square root of the actual length of the string in the induced metric. This would give that the actual length of string is proportional to p and n is also proportional to p and defines minimal size scale of the CD associated with the particle. The dark contribution to the particle mass would be $\Delta m \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/p$, and completely negligible suggesting that it is not easy to make the dark side of elementary visible.

4. If the latter interpretation is correct, elementary particles would have huge number of hidden degrees of freedom assignable to their CDs. For instance, electron would have $n = 2^{127} - 1 \simeq 10^{38}$ hidden discrete degrees of freedom and would be rather intelligent system - 127 bits is the estimate- and thus far from a point-like idiot of standard physics. Is it a mere accident that the secondary p-adic time scale of electron is .1 seconds - the fundamental biorhythm - and the size scale of the minimal CD is slightly large than the circumference of Earth?

   Note however, that the conservation option assuming that the magnetic flux tubes connecting the wormhole contacts representing elementary particle are in $h_{eff}/h = 1$ phase can be considered as conservative option.

8.7 p-Adicization and adelic physics

This section is devoted to the challenges related to p-adicization and adelization of physics in which the correspondence between real and p-adic numbers via canonical identification serves as the basic building brick. Also the problems associated with p-adic variants of integral, Fourier analysis, Hilbert space, and Riemann geometry should be solved in a way respecting fundamental symmetries and their p-adic variants must be met. The notion of number theoretical universality (NTU) plays a key role here. One should also answer to questions about the origin of preferred primes and p-adic length scale hypothesis.

8.7.1 Challenges

The basic challenges encountered are construction of the p-adic variants of real number based physics, understanding their relationship to real physics, and the fusion of various physics to single coherent whole.

The p-adicization of real physics is not just a straightforward formal generalization of scattering amplitudes of existing theories but requires a deeper understanding of the physics involved. The interpretation of p-adic physics as correlate for cognition and imagination is an important guideline and will be discussed l in more detail in separate section.

Definite integral and Fourier analysis are basic elements of standard physics and their generalization to the p-adic context defines a highly non-trivial challenge. Also the p-adic variants of Riemann geometry and Hilbert space are suggestive. There are however problems.

1. There are problems associated with p-adic definite integral. Riemann sum does not make sense since it approaches zero if the p-adic norm of discretization unit approaches zero. The problems are basically due to the absence of well-orderedness essential for the definition of definite integral and differential forms and their integrals.

   Residue integration might make sense in finite angle resolution. For algebraic extension containing $e^{i\pi/n}$ the number theoretically universal approximation $i\pi = n(e^{i\pi/n} - 1)$ could be used. In twistor approach integrations reduce to multiple residue integrations and since twistor approach generalizes in TGD framework, this approach to integration is very attractive.

   Positivity is a central notion in twistor Grassmannian approach [B17]. Since canonical identification maps p-adic numbers to non-negative real numbers, there is a strong temptation to think that positivity relates to NTU [L10].

2. There are problems with Fourier analysis. The naïve generalization of trigonometric functions by replacing e^{ix} with its p-adic counterpart is not physical. Same applies to e^x. Algebraic extensions are needed to get roots of unity ad e as counterparts of the phases and discretization is necessary and has interpretation in terms of finite resolution for angle/phase and its hyperbolic counterpart.

3. The notion of Hilbert space is problematic. The naïve generalization of Hilbert space norm square $|x|^2 = \sum x_n \overline{x}_n$ for state $(x_1, x_2, ...)$ can vanish p-adically. Also here NTU could help. State would contain as coefficients only roots of e and unity and only the overall factor could be p-adic number. Coefficients could be restricted to the algebraic numbers generating the algebraic extension of rational numbers and would not contain powers of p or even ordinary p-adic numbers expect in the overall normalization factor.

Second challenge relates to the relationship between real and p-adic physics. Canonical identification (CI) $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ or some of its variants should play an important role. CI is expected to map the invariants appearing in scattering amplitudes to their real counterparts.

1. Real and p-adic variants of space-time surfaces should exist and relate to each other somehow. Is this relationship local and involve CI at space-time level or embedding space level? Or is it only a global and non-local assignment of preferred real extremals to their p-adic counterparts? Or is between these extreme options and involves algebraic discretization of the space-time surface weakening the strong form of SH as already proposed? How do real

and p-adic embedding spaces relate to each other and can this relationship induce local correspondence between preferred extremals (PEs) [K8, K14, K10]?

2. NTU in some sense is a highly suggestive approach to these questions and would suggest that canonical identification applies to isometry invariants whereas angles and hyperbolic angles, or rather the corresponding "phases" belonging to an extension of p-adics containing roots of e and roots of unity are mapped to themselves. Note that the roots of e define extensions of rationals, which induce finite dimensional algebraic extensions of p-adic numbers. This would make possible to define embedding space in accordance with NTU. Also the Hilbert space could be defined by requiring that its points correspond to number theoretically universal angles expressible in terms of roots of unity.

3. What about real and p-adic variants of WCW? Are they needed at all? Or could their existence be used as a powerful constraint on real physics? The representability of WCW as a union of infinite-dimensional symmetric spaces labelled by zero modes suggests that the same description applies at the level of WCW and embedding space.

   One cannot circumvent the question about how to generalize functional integral from real WCW to p-adic WCWs. In particular, what is the p-adic variant of the action defining the dynamics of space-time surfaces. In the case of exponent of action the p-adic variant could be defined by assuming algebraic universality: again the roots of e and of unity would be in central role. Also the Kähler structure of WCW implying that Gaussian and metric determinants cancel each other in functional integral, would be absolutely crucial.

   One must remember that the exponents of action for scattering amplitudes for the stationary phase extremal cancel from the path integral representation of scattering amplitudes. Also now this mechanism would allow to get rid of the poorly defined exponent for single minimum. If there is sum over scattering amplitudes assignable to different maxima, normalization sould give ratios of these exponents for different extrema/maxima and only these ratios should belong to the extension of rationals.

   The zero modes of WCW metric are invariants of supersymplectic group so that canonical identification could relate their real and p-adic variants. Zero modes could break NTU and would be behind p-adic thermodynamics and dependence of mass scale on p-adic prime.

The third challenge relates to the fusion of p-adic physics and real physics to a larger structure. Here a generalization of number concept obtained by glueing reals and various p-adics together along an extension of rational numbers inducing the extensions of p-adic numbers is highly suggestive. Adeles associated with the extension of rationals are highly attractive and closely related notion. Real and various p-adic physics would be correlates for sensory and cognitive aspects of the same universal physics rather than separate physics in this framework. One important implication of this view is that real entropy and p-adic negentropies characterize the same entanglement with coefficients in an extension of rationals.

NTU for hyperbolic and ordinary phases is definetely the central idea. How the invariance of angles under conformal transformations does relate to this? Could one perhaps define a discretized version of conformal symmetry preserving the phases defined by the angles between vectors assignable with the tangent spaces of discretized geometric structures and thus respecting NTU? Of should one apply conformal symmetry at Lie algebra level only?

8.7.2 NTU and the correspondence between real and p-adic physics

p-Adic real correspondence is certainly the basic problem of p-adicization and adelization. One can make several general questions about p-adic real correspondence and canonical identification inspired by p-adic mass calculations.

How generally p-adic real correspondence does apply? Could canonical identification for group invariants combined with direct identification of ordinary and hyperbolic phases identified as roots of unity and e apply at WCW and embedding space level having maximally symmetric geometries? Could this make sense even at space-time level as a correspondence induced from embedding space level [L17]? Does canonical identification apply locally for the discretizations of space-time surface or only globally for the parameters characterizing PEs (string world sheets and partonic 2-surfaces by SH), which are general coordinate invariant and Poincare invariant quantities?

The following vision seems to be the most feasible one found hitherto.

1. Preservation of symmetries and continuity compete. Lorenz transformations do not commute with canonical identification. This suggests that canonical identification applies only to Lorentz invariants formed from quantum numbers. This is enough in the case of scattering amplitudes. Canonical identification applies only to isometry invariants at the level of WCW and the phases/exponents of ordinary/hyperbolic angles correspond to numbers in the algebraic extension common to extensions of rationals and various p-adics.

2. Canonical identification applies at the level of momentum space and maps p-adic Lorentz invariants of scattering amplitudes to their real counterparts. Phases of angles and their hyperbolic counterparts should correspond to parameters defining extension and should be mapped as such to their p-adic counterparts.

3. The constraints coming from GCI and symmetries do not allow local correspondence but allow to consider its discretized version at space-time leve induced by the correspondence at the level of embedding space.

   This requires the restriction of isometries and other symmetries to algebraic subgroups defined by the extension of rationals. This would imply reduction of symmetry due to finite cognitive/measurement resolution and should be acceptable. If one wants to realize the ideas about imagination, discretization must be applied also for the space-time interior meaning partial breaking of SH and giving rise to dark matter degrees freedom in TGD sense. SH could apply in real sector for realizable imaginations only. Note that the number of algebraic points of space-time surface is expected to be relatively small.

The correspondence must be considered at the level of embedding space, space-time, and WCW.

1. At the level of embedding space p-adic–real correspondence is induced by points in extension of rationals and is totally discontinuous. This requires that space-time dimension is smaller than embedding space dimension.

2. At space-time level the correspondence involves field equations derivable from a local variational principle make sense also p-adically although the action itself is ill-defined as 4-D integral. The notion of p-adic PE makes sense by strong form of holography applied to 2-surfaces in the intersection. p-Adically however only the vanishing of Noether currents for a sub-algebra of the super-symplectic algebra might make sense. This condition is stronger than the vanishing of Noether charges defined by 3-D integrals.

3. Correspondence at the level of WCW can make sense and reduces to that for string world sheets and partonic 2-surfaces by SH. Real and p-adic 4-surfaces would be obtained by algebraic continuation as PEs from 2-surfaces by assuming that the space-time surface contains subset of points of embedding space belonging to the extension of rationals. p-Adic pseudo constants make p-adic continuation easy. Real continuation need not exist always. p-Adic WCW would be considerably larger than real WCW and make possible a predictive quantum theory of imagination and cognition.

   What I have called intersection of realities and p-adicities can be identified as the set of 2-surfaces plus algebraic discretization of space-time interior. Also the values of induced spinor fields at the points of discretization must be given. The parameters characterizing the extremals (say coefficients of polynomials) - WCW coordinates - would be in extension of rationals inducing a finite-D extension of p-adic number fields.

   The hierarchy of algebraic extensions induces an evolutionary hierarchy of adeles. The interpretation could be as a mathematical correlate for cosmic evolution realized at the level of the core of WCW defined by the intersection? 2-surfaces could be called space-time genes.

4. Also the p-adic variant Kähler action or at least the exponent of Kähler action defining vacuum functional should be obtainable by algebraic continuation. The weakest condition states that the ratios of action exponents for the maxima of Kähler function to the sum of action exponents for maxima belong to the extension. Without this condition the hopes of satisfying NTU seem rather meager.

8.7.3 NTU at space-time level

What about NTU at space-time level? NTU requires a correspondence between real and p-adic numbers and the details of this corresponds have been a long standing problem.

1. The recent view about the correspondence between real PEs to their p-adic counterparts does not demand discrete local correspondence assumed in the earlier proposal [K112]. The most abstract approach would give up the local correspondence at space-time level altogether, and restrict the preferred coordinates of WCW (having maximal group of isometries) to numbers in the extension of rationals considered. WCW would be discretized.

   Intuitively a more realistic view is a correspondence at space-time level in the sense that real and p-adic space-time sheets intersect at points belonging to the extension of rationals and defining "cognitive representations". Only some p-adic space-time surfaces would have real counterpart.

2. The strongest form of NTU would require that the allowed points of embedding space belonging an extension of rationals are mapped as such to corresponding extensions of p-adic number fields (no canonical identification). At embedding space level this correspondence would be extremely discontinuous. The "spines" of space-time surfaces would however contain only a subset of points of extension, and a natural resolution length scale could emerge and prevent the fluctuation. This could be also seen as a reason for why space-times surfaces must be 4-D. The fact that the curve $x^n + y^n = z^n$ has no rational points for $n > 2$, raises the hope that the resolution scale could emerge spontaneously.

3. The notion of monadic geometry discussed in detail in [L17] would realize this idea. Define first a number theoretic discretization of embedding space in terms of points, whose coordinates in group theoretically preferred coordinate system belong to the extension of rationals considered. One can say that these algebraic points are in the intersection of reality and various p-adicities. Overlapping open sets assigned with this discretization define in the real sector a covering by open sets. In p-adic sector compact-open-topology allows to assign with each point 8^{th} Cartesian power of algebraic extension of p-adic numbers. These compact open sets define analogs for the monads of Leibniz and p-adic variants of field equations make sense inside them.

   The monadic manifold structure of H is induced to space-time surfaces containing discrete subset of points in the algebraic discretization with field equations defining a continuation to space-time surface in given number field, and unique only in finite measurement resolution. This approach would resolve the tension between continuity and symmetries in p-adic–real correspondence: isometry groups would be replaced by their sub-groups with parameters in extension of rationals considered and acting in the intersection of reality and p-adicities.

   The Galois group of extension acts non-trivially on the "spines" of space-time surfaces. Hence the number theoretical symmetries act as physical symmetries and define the orbit of given space-time surface as a kind of covering space. The coverings assigned to the hierarchy of Planck constants would naturally correspond to Galois coverings and dark matter would represent number theoretical physics.

   This would give rise to a kind of algebraic hierarchy of adelic 4-surfaces identifiable as evolutionary hierarchy: the higher the dimension of the extension, the higher the evolutionary level.

8.7.4 NTU and WCW

p-Adic–real correspondence at the level of WCW

It has not been obvious whether one should perform p-adicization and adelization at the level of WCW. Minimalist could argue that scattering amplitudes are all we want and that their p-adicization and adelization by algebraic continuation can be tolerated only if it can give powerful enough constraints on the amplitudes.

1. The anti-commutations for fermionic oscillator operators are number theoretically universal. Supersymmetry suggests that also WCW bosonic degrees of freedom satisfy NTU. This could mean that the coordinates of p-adic WCW consist of super-symplectic invariants mappable by canonical identification to their real counterparts plus phases and their hyperbolic counterparts expressible as genuinely algebraic numbers common to all number fields. This kind of coordinates are naturally assignable to symmetric spaces [L17].

2. Kähler structure should be mapped from p-adic to real sector and vice versa. Vacuum functional identified as exponent of action should be NTU. Algebraic continuation defined by SH involves p-adic pseudo constants. All p-adic continuations by SH should correspond to the same value of exponent of action obtained by algebraic continuation from its real value. The degeneracy associated with p-adic pseudo-constants would be analogous to gauge invariance - imagination in TGD inspired theory of consciousness.

3. Ist it possible have NTU for WCW functional integration? Or is it enough to realize NTU for scattering amplitudes only. What seems clear that functional integral must reduce to a discrete sum. Physical intuition suggests a sum over maxima of Kähler function forming a subset of PEs representing stationary points. One cannot even exclude the possibility that the set of PEs is discrete and that one can sum over all of them.

   Restriction to maximum/stationary phase approximation gives rise to sum over exponents multiplied with Gaussian determinants. The determinant of Kähler metric however cancels the Gaussian determinants, and one obtains only a sum over the exponents of action.

   The breaking of strong NTU could happen: consider only p-adic mass calculations. This breaking is however associated with the parts of quantum states assignable to the boundaries of CD, not with the vacuum functional.

NTU for functional integral

Number theoretical vision relies on NTU. In fermionic sector NTU is necessary: one cannot speak about real and p-adic fermions as separate entities and fermionic anti-commutation relations are indeed number theoretically universal.

What about NTU in case of functional integral? There are two opposite views.

1. One can define p-adic variants of field equations without difficulties if preferred extremals are minimal surface extremals of Kähler action so that coupling constants do not appear in the solutions. If the extremal property is determined solely by the analyticity properties as it is for various conjectures, it makes sense independent of number field. Therefore there would be no need to continue the functional integral to p-adic sectors. This in accordance with the philosophy that thought cannot be put in scale. This would be also the option favored by pragmatist.

2. Consciousness theorist might argue that also cognition and imagination allow quantum description. The supersymmetry NTU should apply also to functional integral over WCW (more precisely, its sector defined by CD) involved with the definition of scattering amplitudes.

1. Key observations

The general vision involves some crucial observations.

1. Only the expressions for the scatterings amplitudes should should satisfy NTU. This does not require that the functional integral satisfies NTU.

2. Since the Gaussian and metric determinants cancel in WCW Kähler metric the contributions form maxima are proportional to action exponentials $exp(S_k)$ divided by the $\sum_k exp(S_k)$. Loops vanish by quantum criticality.

3. Scattering amplitudes can be defined as sums over the contributions from the maxima, which would have also stationary phase by the double extremal property made possible by the complex value of α_K. These contributions are normalized by the vacuum amplitude.

   It is enough to require NTU for $X_i = exp(S_i)/\sum_k exp(S_k)$. This requires that $S_k - S_l$ has form $q_1 + q_2 i\pi + q_3 log(n)$. The condition brings in mind homology theory without boundary operation defined by the difference $S_k - S_l$. NTU for both S_k and $exp(S_k)$ would only values of general form $S_k = q_1 + q_2 i\pi + q_3 log(n)$ for S_k and this looks quite too strong a condition.

4. If it is possible to express the 4-D exponentials as single 2-D exponential associated with union of string world sheets, vacuum functional disappears completely from consideration! There is only a sum over discretization with the same effective action and one obtains purely combinatorial expression.

2. What does one mean with functional integral?

The definition of functional integral in WCW is one of the key technical problems of quantum TGD [K109]. NTU states that the integral should be defined simultaneously in all number fields in the intersection of real and p-adic worlds defined by string world sheets and partonic 2-surfaces with WCW coordinates in algebraic extension of rationals and allowing by strong holography continuation to 4-D space-time surface. NTU is powerful constraint and could help in this respect.

1. Path integral is not in question. Rather, the functional integral is analogous to Wiener integral and perhaps allows identification as a genuine integral in the real sector. In p-adic sectors algebraic continuation should give the integral and here number theoretical universality gives excellent hopes. The integral would have exactly the same form in real and p-adic sector and expressible solely in terms of algebraic numbers characterizing algebraic extension and finite roots of e and roots of unity $U_n = exp(i2\pi/n)$ in algebraic extension of p-adic numbers.

   Since vacuum functional $exp(S)$ is exponential of complex action S, the natural idea is that only rational powers e^q and roots of unity and phases $exp(i2\pi q)$ are involved and there is no dependence on p-adic prime p! This is true in the integer part of q is smaller than p so that one does not obtain e^{kp}, which is ordinary p-adic number and would spoil the number theoretic universality. This condition is not possible to satisfy for all values of p unless the value of Kähler function is smaller than 2. One might consider the possibility that the allow primes are above some minimum value.

   The minimal solution to NTU conditions is that the ratios of action exponentials for maxima of Kähler function to the sum of these exponentials belong to the extension of rationals considered.

2. What does one mean with functional integral? TGD is expected to be an integrable in some sense. In integrable QFTs functional integral reduces to a sum over stationary points of the action: typically only single point contributes - at least in good approximation.

   For real α_K and Λ vacuum functional decomposes to a product of exponents of real contribution from Euclidian regions ($\sqrt{g_4}$ real) and imaginary contribution Minkowskian regions ($\sqrt{g_4}$ imaginary). There would be no exchange of momentum between Minkowskian and Euclidian regions. For complex values of α_K [K35] situation changes and Kähler function as real part of action receives contributions from both Euclidian and Minkowskian regions. The imaginary part of action has interpretation as analog of Morse function and action as it appears in QFTs. Now saddle points must be considered.

   PEs satisfy extremely strong conditions [K8, K10]. All classical Noether charges for a sub-algebra associated with super-symplectic algebra and isomorphic to the algebra itself vanish at both ends of CD. The conformal weights of this algebra are $n > 0$-ples of those for the entire algebra. What is fascinating that the condition that the preferred extremals are minimal surface extremals of Kähler action could solve these conditions and guarantee also NTU at the level of space-time surfaces. Supersymplectic boundary conditions at the ends of CD would however pose number theoretic conditions on the coupling parameters. In p-adic case the conditions should reduce to purely local conditions since p-adic charges are not well-defined as integrals.

3. In TGD framework one is constructing zero energy states rather calculating the matrix elements of S-matrix in terms of path integral. This gives certain liberties but a natural expectation is that functional integral as a formal tool at least is involved.

   Could one *define* the functional integral as a discrete sum of contributions of standard form for the preferred extremals, which correspond to maxima in Euclidian regions and associated stationary phase points in Minkowskian regions? Could one assume that WCW spinor field is concentrated along single maximum/stationary point.

   Quantum classical correspondence suggests that in Cartan algebra isometry charges are equal to the quantal charges for quantum states expressible in number theoretically universal manner in terms of fermionic oscillator operators or WCW gamma matrices? Even stronger condition would be that classical correlation functions are identical with quantal ones for allowed space-time surfaces in the quantum superposition. Could the reduction to a discrete sum be interpreted in terms of a finite measurement resolution?

4. In QFT Gaussian determinants produce problems because they are often poorly defined. In the recent case they could also spoil the NTU based on the exceptional properties of e. In the recent case however Gaussian determinant and metric determinant for Kähler metric cancel each other and this problem disappears. One could obtain just a sum over products of roots of e and roots of unity. Thus also Kähler structure seems to be crucial for the dream about NTU.

8.7.5 Breaking of NTU at the level of scattering amplitudes

NTU in strong sense could be broken at the level of scattering amplitudes. At space-time level the breaking does not look natural in the recent framework. Consider only p-adic mass calculations predicting that mass scale depends on p-adic prime. Also for the action strong form of NTU might fail for small p-adic primes since the value of the real part of action would be larger than than p. Should one allow this? What does one actually mean with NTU in the case of action?

Canonical identification is an important element of p-adic mass calculations and might also be needed to map p-adic variants of scattering amplitudes to their real counterparts. The breaking of NTU would take place, when the canonical real valued image of the p-adic scattering amplitude differs from the real scattering amplitude. The interpretation would be in terms of finite measurement resolution. By the finite measurement/cognitive resolution characterized by p one cannot detect the difference.

The simplest form of the canonical identification is $x = \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$. Product xy and sum $x + y$ do not in general map to product and sum in canonical identification. The interpretation would be in terms of a finite measurement resolution: $(xy)_R = x_R y_R$ and $(x+y)_R = x_R + y_R$ only modulo finite measurement resolution. p-Adic scattering amplitudes are obtained by algebraic continuation from the intersection by replacing algebraic number valued parameters (such as momenta) by general p-adic numbers. The real images of these amplitudes under canonical identification are in general not identical with real scattering amplitudes the interpretation being in terms of a finite measurement resolution.

In p-adic thermodynamics NTU in the strong sense fails since thermal masses depend on p-adic mass scale. NTU can be broken by the fermionic matrix elements in the functional integral so that the real scattering amplitudes differ from the canonical images of the p-adic scattering amplitudes. For instance, the elementary particle vacuum functionals in the space of Teichmueller parameters for the partonic 2-surfaces and string world sheets should break NTU [K23].

8.7.6 NTU and the spectrum of Kähler coupling strength

During years I have made several attempts to understand coupling evolution in TGD framework. The most convincing proposal has emerged rather recently and relates the spectrum of $1/\alpha_K$ to that for the zeros of Riemann zeta [K35] and to the evolution of the electroweak U(1) couplings strength.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K43]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.

   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K113] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking.

   The problem is that the dynamics in Minkowskian and Euclidian regions decouple completely and if Euclidian regions serve as space-time correlates for physical objects, there would be no exchanges of classical charges between physical objects. Should one conclude that α_K must be complex?

2. p-Adic mass calculations for 2 decades ago [K51] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity [L28]. Needless to say these attempts were premature and a hoc.

3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [**?**] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

   Quite recently the number theoretic interpretation of coupling constant evolution [K109] [L9] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K104]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets

add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.

   Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.

   The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s = 2$. The trivial poles for $s = 2n$, $n = 1, 2, ..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff} = n \times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s = 2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?

   It turns out that at p-adic length scale $k = 131$ ($p \simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K109]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k = 127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w = w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `http://tinyurl.com/gwjs85b`) with real coefficients (element of $GL(2, R)$) so that one as $\zeta_F((as + b)/(cs + d))$. Rather general arguments force it to be and element of $GL(2, Q)$, $GL(2, Z)$ or maybe even $SL(2, Z)$ ($ad - bc = 1$) satisfying additional constraints. Since TGD predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2, Z)$ and by a scaling factor K.

   Could one understand the general qualitative features of color and weak coupling contant evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs + d$ and color confinement with the zero of $as + b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as + b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as+b/)(cs+d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p \simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a, b, c, d). In the sequel this vision is discussed in more detail.

8.7.7 Generalization of Riemann zeta to Dedekind zeta and adelic physics

8.7.8 Generalization of Riemann zeta to Dedekind zeta and adelic physics

A further insight to adelic physics comes from the possible physical interpretation of the L-functions appearing also in Langlands program [K46]. The most important L-function would be generalization of Riemann zeta to extension of rationals. I have proposed several roles for ζ, which would be the simplest L-function assignable to rational primes, and for its zeros.

1. Riemann zeta itself could be identifiable as an analog of partition function for a system with energies given by logarithms of prime. One can define also the fermionic counterpart of ζ as ζ_F. In ZEO this function could be regarded as complex square root of thermodynamical partition function in accordance with the interpretation of quantum theory as complex square root of thermodynamics.

2. The zeros of zeta could define the conformal weights for the generators of super-symplectic algebra so that the number of generators would be infinite. The rough idea - certainly not correct as such except at the limit of infinitely large CD - is that the scaling operator $L_0 = r_M d/dr_M$, where r_M is light-like coordinate of light-cone boundary (containing upper or lower boundary of the causal diamond (CD)), has as eigenfunctions the functions $(r_M/r_0)^{s_n}$ $s_n = 1/2 + iy_n$, where s_n is the radial conformal weight identified as complex zero of ζ. Periodic boundary conditions for CD do not allow all possible zeros as conformal weights so that for given CD only finite subset corresponds to generators of the supersymplectic algebra. Conformal confinement would hold true in the sense that the sum $\sum_n s_n$ for physical states would be integer. Roots and their conjugates should appear as pairs in physical states.

3. On basis of numerical evidence Dyson [A47] (`http://tinyurl.com/hjbfsuv`) has conjectured that the Fourier transform for the set formed by zeros of zeta consists of primes so that one could regard zeros as one-dimensional quasi-crystal. This hypothesis makes sense if the zeros of zeta decompose into disjoint sets such that each set corresponds to its own prime (and its powers) and one has $p^{iy} = U_{m/n} = exp(i2\pi m/n)$ (see the appendix of [L9]). This hypothesis is also motivated by number theoretical universality [K109, L23].

4. I have considered the possibility [K35] that the discrete values for the inverse of the electro-weak $U(1)$ coupling constant for a gauge field assignable to the Kähler form of CP_2 assignable to p-adic coupling constant evolution corresponds to poles of the fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ coming from $s_n/2$ (denominator) and pole at $s = 1$ (numerator) zeros of zeta assignable to rational primes. Note that also odd negative integers at real axis would be poles.

   It is also possible to consider scaling of the argument of $\zeta_F(s)$. More general coupling constant evolutions could correspond to $\zeta_F(m(s))$, where $m(s) = (as+b)/(cs+d)$ is Möbius transformation performed for the argument mapping upper complex plane to itself so that a, b, c, d are real and also rational by number theoretical universality.

Suppose for a moment that more precise formulations of these physics inspired conjectures make sense and even that their generalization for extensions K/Q of rationals holds true. This would solve a big part of adelic physics! Not surprisingly, the generalization of zeta function was proposed already by Dedekind (see `http://tinyurl.com/yarwbo6h`).

1. The definition of Dedekind zeta function ζ_K relies on the product representation and analytic continuation allows to deduce ζ_K elsewhere. One has a product over prime ideals of K/Q of

rationals with the factors $1/(1-p^{-s})$ associated with the ordinary primes in Riemann zeta replaced with the factors $X(P) = 1/(1-N_{K/Q}(P)^{-s})$, where P is prime for the integers $O(K)$ of extension and $N_{K/Q}(P)$ is the norm of P in the extension. In the region $s > 1$ where the product converges, ζ_K is non-vanishing and $s = 1$ is a pole of ζ_K. The functional identifies of ζ hold true for ζ_K as well. Riemann hypothesis is generalized for ζ_K.

2. It is possible to understand ζ_K in terms of a physical picture. By the results of `http://tinyurl.com/yckfjgpk` one has $N_{K/Q}(P) = p^r$, $r > 0$ integer. This implies that one can arrange in ζ_K all primes P for which the norm is power or given p in the same group. The prime ideals p of ordinary integers decompose to products of prime ideals P of the extension: one has $p = \prod_{r=1}^{g} P_r^{e_r}$, where e_r is so called ramification index. One can say that each factor of ζ decomposes to a product of factors associated with corresponding primes P with norm power of p. In the language of physics, the particle state represented by p decomposes in improved resolution to a product of many-particle states consisting of e_r particles in state P_r, very much like hadron decomposes to quarks.

   The norms of $N_{K/Q}(P_r) = p^{d_r}$ satisfy the condition $\sum_{r=1}^{g} d_r e_r = n$. Mathematician would say that the prime ideals of Q modulo p decompose in n-dimensional extension K to products of prime power ideals $P_r^{e_r}$ and that P_r corresponds to a finite field $G(p, d_r)$ with algebraic dimension d_r. The formula $\sum_{r=1}^{g} d_r e_r = n$ reflects the fact the dimension n of extension is same independent of p even when one has $g < n$ and ramification occurs.

   Physicist would say that the number of degrees of freedom is n and is preserved although one has only $g < n$ different particle types with e_r particles having d_r internal degrees of freedom. The factor replacing $1/(1-p^{-s})$ for the general prime p is given by $\prod_{r=1}^{g} 1/(1-p^{-e_r d_r s})$.

3. There are only finite number of ramified primes p having $e_r > 1$ for some r and they correspond to primes dividing the so called discriminant D of the irreducible polynomial P defining the extension. $D \ mod \ p$ obviously vanishes if D is divisible by p. For second order polynomials $P = x^2 + bx + c$ equals to the familiar $D = b^2 - 4c$ and in this case the two roots indeed co-incide. For quadratic extensions with $D = b^2 - 4c > 0$ the ramified primes divide D.

   Remark: Resultant $R(P, Q)$ and discriminant $D(P) = R(P, dP/dx)$ are elegant tools used by number theorists to study extensions of rationals defined by irreducible polynomials (see `http://tinyurl.com/oyumsnk` and `http://tinyurl.com/p67rdgb`). From Wikipedia articles one finds an elegant proof for the facts that $R(P, Q)$ is proportional to the product of differences of the roots of P and Q, and D to the product of squares for the differences of distinct roots. $R(P, Q) = 0$ tells that two polynomials have a common root. $D \ mod \ p = 0$ tells that polynomial and its derivative have a common root so that there is a degenerate root modulo p and the prime is indeed ramified. For modulo p reduction of P the vanishing of $D(P) \ mod \ p$ follows if D is divisible by p. There exists clearly only a finite number of primes of this kind.

   Most primes are unramified and one has maximum number n of factors in the decomposition and $e_r = 1$: maximum splitting of p occurs. The factor $1/(1-p^{-s})$ is replaced with its n:th power $1/(1-p^{-s})^n$. The geometric interpretation is that space-time sheet is replaced with n-fold covering and each sheet gives one factor in the power. It is also possible to have a situation in which no splitting occurs and one as $e_r = 1$ for one prime $P_r = p$. The factor is in this case equal to $1/(1-p^{-ns})$.

From Wikipedia (see `http://tinyurl.com/yckfjgpk`) one learns that for Galois extensions L/K the ratio ζ_L/ζ_K is so called Artin L-function of the regular representation (group algebra) of Galois group factorizing in terms of irreps of $Gal(L/K)$ is *holomorphic* (no poles!) so that ζ_L must have also the zeros of ζ_K. This holds in the special case $K = Q$. Therefore extension of rationals can only bring new zeros but no new poles!

1. This result is quite far reaching if one accepts the hypothesis about super-symplectic conformal weights as zeros of ζ_K and the conjecture about coupling constant evolution. In the case of $\zeta_{F,K}$ this means new poles meaning new conformal weights due to increased complexity

and a modification of the conjecture for the coupling constant evolution due to new primes in extension. The outcome looks physically sensible.

2. Quadratic field $Q(\sqrt{m})$ serves as example. Quite generally, the factorization of rational primes to the primes of extension corresponds to the factorization of the minimal polynomial for the generating element θ for the integers of extension and one has $p = P_i^{e_i}$, where e_i is ramification index. The norm of p factorizes to the produce of norms of $P_i^{e_i}$.

   Rational prime can either remain prime in which case $x^2 - m$ does not factorize mod p, split when $x^2 - m$ factorizes mod P, or ramify when it divides the discriminant of $x^2 - m = 4m$. From this it is clear that for unramfied primes the factors in ζ are replaced by either $1/(1-p^{-s})^2$ or $1/(1-p^{-2s}) = 1/(1-p^{-s})(1+p^{-s})$. For a finite number of unramified primes one can have something different.

   For Gaussian primes with $m = -1$ one has $e_r = 1$ for $p \ mod\ 4 = 3$ and $e_r = 2$ for $p = \ mod\ 4 = 1$. z_K therefore decomposes into two factors corresponding to primes $p\ mod\ 4 = 3$ and $p\ mod\ 4 = 3$. One can extract out Riemann zeta and the remaining factor

$$\prod_{p\ mod\ 4=3} \frac{1}{(1-p^{-s})} \times \prod_{p\ mod\ 4=1} \frac{1}{(1+p^{-s})}$$

   should be holomorphic and without poles but having possibly additional zeros at critical line. That ζ_K should possess also the poles of ζ as poles looks therefore highly non-trivial.

8.7.9 Other applications of NTU

NTU in the strongest form says that all numbers involved at "basic level" (whatever this means!) of adelic TGD are products of roots of unity and of power of a root of e. This is extremely powerful physics inspired conjecture with a wide range of possible mathematical applications.

1. For instance, vacuum functional defined as an exponent of action for preferred externals would be number of this kind. One could define functional integral as adelic operation in all number fields: essentially as sum of exponents of action for stationary preferred extremals since Gaussian and metric determinants potentially spoiling NTU would cancel each other leaving only the exponent.

2. The implications of NTU for the zeros of Riemann zeta [L9] will be discussed in more detail in the Appendix. Suffice it to say that the observations about Fourier transform for the distribution of loci of non-trivial zeros of zeta together with NTU leads to explicit proposal for the algebraic for of zeros of zeta. The testable proposal is that zeros decompose to disjoint classes $C(p)$ labelled by primes p and the condition that p^{iy} is root of unity in given class $C(p)$.

3. NTU generalises to all Lie groups. Exponents $exp(in_iJ_i/n)$ of lie-algebra generators define generalisations of number theoretically universal group elements and generate a discrete subgroup of compact Lie group. Also hyperbolic "phases" based on the roots $e^{m/n}$ are possible and make possible discretized NTU versions of all Lie-groups expected to play a key role in adelization of TGD.

   NTU generalises also to quaternions and octonions and allows to define them as number theoretically universal entities. Note that ordinary p-adic variants of quaternions and octonions do not give rise to a number field: inverse of quaternion can have vanishing p-adic variant of norm squared satisfying $\sum_n x_n^2 = 0$.

   NTU allows to define also the notion of Hilbert space as an adelic notion. The exponents of angles characterising unit vector of Hilbert space would correspond to roots of unity.

8.7.10 Going to the roots of p-adicity

The basic questions raised by the p-adic mass calculations concern the origin of preferred p-adic primes and of p-adic length scale hypothesis. One can also ask whether there might be a natural origin for p-adicity at the level of WCW.

Preferred primes as ramified primes for extensions of rationals?

Preferred primes as ramified primes for extensions of rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and to me the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of *ramification of primes* (`http://tinyurl.com/hddljlf`) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Ramification is completely analogous to the degeneracy of some roots of polynomial and corresponds to criticality if the polynomial corresponds to criticality (catastrophe theory of Thom is one application). Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this sloppy language of physicist): as one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called *integral closure* (`http://tinyurl.com/js6fpvr`) over integers of K decompose to products of prime ideals of L (prime ideal is a more rigorous manner to express primeness). Note that the general ideal is analog of integer.

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

   Prime ideals of K can be decomposed to products of prime ideals of L: $P = \prod P_i^{e_i}$, where e_i is the ramification index. If $e_i > 1$ is true for some i, *ramification* occurs. P_i:s in question are like co-inciding roots of polynomial, which for in thermodynamics and Thom's catastrophe theory corresponds to criticality. Ramification could therefore be a natural aspect of quantum criticality and ramified primes P are good candidates for preferred primes for a given extension of rationals. Note that the ramification make sense also for extensions of given extension of rationals.

2. A physical analogy for the decomposition of ideals to ideals of extension is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, the physical analog would be the number of elementary particles of type i in the state (`http://tinyurl.com/h9528pl`). Unramified prime P would be analogous a state with e fermions. Maximally ramified prime would be analogous to Bose-Einstein condensate of e bosons. General ramified prime would be analogous to an e-particle state containing both fermions and condensed bosons. This is of course just a formal analogy.

3. There are two further basic notions related to ramification and characterizing it. *Relative discriminant* is the ideal divided by all ramified ideals in K (integer of K having no ramified prime factors) and relative different for P is the ideal of L divided by all ramified P_i:s (product of prime factors of P in L). These ideals represent the analogs of product of preferred primes P of K and primes P_i of L dividing them. These two integers ideals would characterize the ramification.

In TGD framework the extensions of rationals (`http://tinyurl.com/h9528pl`) and p-adic number fields (`http://tinyurl.com/zq22tvb`) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would gradually proceed to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

How could ramification relate to p-adic and adelic physics and could it explain preferred primes?

1. Ramified p-adic prime $P = P_i^e$ would be replaced with its e:th root P_i in p-adicization. Same would apply to general ramified primes. Each un-ramified prime of K is replaced with $e = K : L$ primes of L and ramified primes P with $\#\{P_i\} < e$ primes of L: the increase of algebraic dimension is smaller. An interesting question relates to p-adic length scale. What happens to p-adic length scales. Is p-adic prime effectively replaced with e:th root of p-adic prime: $L_p \propto p^{1/2}L_1 \rightarrow p^{1/2e}L_1$? The only physical option is that the p-adic temperature for P would be scaled down $T_p = 1/n \rightarrow 1/ne$ for its e:th root (for fermions serving as fundamental particles in TGD one actually has $T_p = 1$). Could the lower temperature state be more stable and select the preferred primes as maximimally ramified ones? What about general ramified primes?

2. This need not be the whole story. Some algebraic extensions would be more favored than others and p-adic view about realizable imaginations could be involved. p-Adic pseudo constants are expected to allow p-adic continuations of string world sheets and partonic 2-surfaces to 4-D preferred extremals with number theoretic discretization. For real continuations the situation is more difficult. For preferred extensions - and therefore for corresponding ramified primes - the number of real continuations - realizable imaginations - would be especially large.

   The challenge would be to understand why primes near powers of 2 and possibly also of other small primes would be favored. Why for them the number of realizable imaginations would be especially large so that they would be winners in number theoretical fight for survival?

Can one make this picture more concrete? What kind of algebraic extensions could be considered?

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. It would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (`http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

   Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n:th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with

$\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

4. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \to x + 1$ transforms $(x^n - 1)/(x - 1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

The origin of p-adic length scale hypothesis?

p-Adic length scale hypothesis emerged from p-adic length scale hypothesis. A possible generalization of this hypothesis is that p-adic primes near powers of prime are physically favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [I2] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present: this is discussed in TGD inspired theory of music harmony and genetic code [K80]. See also [L19, L14].

One explanation would be that for preferred primes the number of p-adic space-time sheets representable also as real space-time sheets is maximal. Imagined worlds would be maximally realizable. Preferred p-adic primes would correspond to ramified primes for extensions with the property that the number of realizable imaginations is especially large for them. Why primes satisfying p-adic length scale hypothesis or its generalization would appear as ramified primes for extensions, which are winners in number theoretical evolution?

Also the weak form of NMP (WNMP) applying also to the purely number theoretic form of NMP [K56] might come in rescue here.

1. Entanglement negentropy for a NE [K56] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretical entanglement negentropy (NEN). If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured, which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real PE as p-adic PE (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. WNMP [K56, K106] suggests a more feasible explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. WNMP allows "free will" so that all dimensions $n-k$, $k=0,1,...n-1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n-k$ is power of prime. For $n=2^k=M_k+1$, where M_k is Mersenne prime $n-1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n=2^k-r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p=2$ can be replaced by any prime.

8.8 What could be the role of complexity theory in TGD?

I have many times wondered what could be the role of chaos theory or better in TGD. In fact, I would prefer to talk about complexity theory since the chaos in the sense as it is used is only apparent and very different from thermodynamical chaos.

Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a nice summary about the history of chaos theory and I repeat only some main points here. Chaos theory has roots already at the end of 18the century by the works of Poincare (non-periodic orbits in 3-body system) and Hadamard (free particle gliding frictionlessly on surface of constant negative curvature, "Hadamard billiard". In this case all trajectories are unstable diverging exponentially from each other: this is characterized by positive Lyapunov exponent.

Chaos theory got is start from ergodic theory (see `http://tinyurl.com/pfcrz4c`) studying dynamical systems with the original motivation coming from statistical physics. For instance, spin glasses are a representative example of non-ergodic system in which the trajectory of point does not go arbitrary near to every point. The study of non-linear differential equations George David Birkhoff, Andrey Nikolaevich Kolmogorov, Mary Lucy Cartwright and John Edensor Littlewood, and Stephen Smale provides was purely mathematical study of chaotic systems. Smale discovered strange attractor at which periodic orbits form a dense set. Chaos theory was formalized around 1950. At this time it was also discovered that finite-D linear systems do not allow chaos.

The emergence of computers meant breakthrough. Much of chaos theory involves repeated iteration of simple mathematical formulas. Edward Lorentz was a pioneer of chaos theory working with weather prediction and accidentally discovered initial value sensitivity. Benard Mandelbrot discovered fractality and Mitchell Feigenbaum the universality of chaos for iteration of functions of real variable.

Chaotic systems are as far from integrable systems as one could imagine: all orbits are cycles in integrable Hamiltonian dynamics. There are good reasons to suspect that TGD Universe is completely integrable classically. Chaos theory however describes also the emergence of complexity through phase transition like steps - period n-tupling and most importantly by period doubling for iteration of maps.

Chaotic (or actually extremely complex and only apparently chaotic) systems seem to be the diametrical opposite of completely integrable systems about which TGD is a possible example. There is however also something common: in completely integrable classical systems all orbits are cyclic and in chaotic systems they form a dense set in the space of orbits. Furthermore, in chaotic systems the approach to chaos occurs via steps as a control parameter is changed. Same would take place in adelic TGD fusing the descriptions of matter and cognition.

In TGD Universe the hierarchy of extensions of rationals inducing finite-dimensional extension of p-adic number fields defines a hierarchy of adelic physics and provides a natural correlate for evolution. Galois groups and ramified primes appear as characterizers of the extensions. The sequences of Galois groups could characterize an evolution by phase transitions increasing the dimension of the extension associated with the coordinates of "world of classical worlds" (WCW) in turn inducing the extension used at space-time and Hilbert space level. WCW decomposes to sectors characterized by Galois groups G_3 of extensions associated with the 3-surfaces at the ends of space-time surface at boundaries of causal diamond (CD) and G_4 characterizing the space-time surface itself. G_3 (G_4) acts on the discretization and induces a covering structure of the 3-surface (space-time surface). If the state function reduction to the opposite boundary of CD involves localization into a sector with fixed G_3, evolution is indeed mapped to a sequence of G_3s.

Also the cognitive representation defined by the intersection of real and p-adic surfaces with coordinates of points in an extension of rationals evolve. The number of points in this representation becomes increasingly complex during evolution. Fermions at partonic 2-surfaces connected by fermionic strings define a tensor network, which also evolves since the number of fermions can change.

The points of space-time surface invariant under non-trivial subgroup of Galois group define singularities of the covering, and the positions of fermions at partonic surfaces could correspond to these singularities - maybe even the maximal ones, in which case the singular points would be rational. There is a temptation to interpret the p-adic prime characterizing elementary particle as a ramified prime of extension having a decomposition similar to that of singularity so that category theoretic view suggests itself.

One also ends up to ask how the number theoretic evolution could select preferred p-adic primes satisfying the p-adic length scale hypothesis as a survivors in number theoretic evolution, and ends up to a vision bringing strongly in mind the notion of conserved genes as analogy for conservation of ramified primes in extensions of extension. $h_{eff}/h = n$ has natural interpretation as divisor of the order of Galois group of extension. The generalization of $\hbar_{gr} = GMm/v_0 = \hbar_{eff}$ hypothesis to other interactions is discussed in terms of number theoretic evolution as increase of G_3, and one ends up to surprisingly concrete vision for what might happen in the transition from prokaryotes to eukaryotes.

8.8.1 Basic notions of chaos theory

It is good to begin by summarizing the basic concepts of chaos theory. Again Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a more detailed representation and references. Citing Wikipedia freely: Within the apparent randomness of chaotic complex systems there are patterns, constant feedback loops, repetition, self-similarity, fractals, self-organization and there is sensitivity to initial conditions (butterfly effect) implying the loss of predictability although chaotic systems as such are deterministic.

Basic prerequisites for chaotic dynamics

Wikipedia article lists three basic conditions for chaotic dynamics. Dynamics must **a)** be sensitive to initial conditions, **b)** allow topological mixing, **c)** have dense set of periodic orbits.

1. Sensitivity to initial conditions.

   Mathematical formulation for the sensitivity to initial conditions can be formulated by perturbation theory for differential equations. The rate of separation of images of points initially near to each other increases exponentially as $exp(\lambda t)$ in initial value sensitive situation and the approximation fails soon. Lyapunov exponent λ characterizes the time evolution of the difference. In multi-dimensional case there are several Lyapunov exponents but the largest one is often enough to characterize the situation.

2. Topological mixing (transitivity).

   This notion corresponds to everyday intuition about mixing. For instance, the flow defined by a vector field mixes the marker completely with the fluid. Iteration of simple scaling operation is initial value sensitive but does not cause topological mixing. In 1-D case all points larger than one approach to infinity and smaller than 1 to zero so that the behavior is extremely simple.

3. Dense set of periodic orbits.

   Periodic orbits should form a dense set in the space of orbits: every point of space is approached arbitrarily closely by a periodic orbit. In completely integrable system all orbits would be periodic orbits so that the difference of these systems is very delicate and one can wonder whether the conditions a) and b) follow from this delicate difference. One can also ask whether there might be a deep connection between completely integrable and chaotic systems.

Sharkovkii's theorem states that any 1-D system with dynamics determined by iteration of a continuous function of real argument exhibits a regular cycle of period 3 exhibits all other cycles. This theorem can be generalized further (see `http://tinyurl.com/l7q3rah`). Introduce Sharkovskii ordering of integers as union of sets consisting of odd integers multiplied by powers of 2. The generalization of the theorem states that if n is a peri89od and precedes k in Sharkovskii ordering then k is prime period (it is not a multiple of smaller period).

The theorem holds true for reals but not for periodic functions at circle which are encountered for iterations defined by powers of cyclic group elements. The discrete subgroup of hyperbolic subgroups of Lie groups do not have not cycles at all.

Strange attractors and Julia sets

Logistic map $x \rightarrow kx(1-x)$ is chaotic everywhere but many systems are chaotic only in a subset of phase space. An interesting situation arises when the chaotic behavior takes place at attractor, since all initial positions in the basic of the attractor lead to the attractor and to a chaotic behavior. Lorentz attractor is a well-known example of strange attractor (see Wikipedia article for illustration). It contains dense sets of both periodic and aperiodic orbits.

Julia set (see `http://tinyurl.com/l8jl5ne`) is the boundary of the basin of attraction in chaotic systems defined by iteration of a rational function of complex argument mapping complex plane to itself. Both Julia sets and strange attractors have a fractal structure.

Strange attractors can appear only in spaces with dimension $D \geq 3$. Poincare-Bendixon theorem states that 2-D differential equations on Euclidian plane have very regular behavior. In non-Euclidian geometry situation changes and the hyperbolic character of the geometry implying initial value sensitivity of geodesic motion is the reason for this. Also infinite-D linear systems can exhibit chaotic behavior.

8.8.2 How to assign chaos/complexity theory with TGD?

Completely integrable systems can be seen as a diametric opposite of chaotic systems. If classical TGD indeed represents a completely integrable system meaning that space-time surfaces as preferred extremals can be constructed explicitly, one might think that chaos theory need not have much to do with classical TGD. Chaos is however the end product of transitions making the system more complex, and it might well be that the understanding about the emergence of complexity in chaotic systems could help to develop the vision about emergence of complexity in TGD. Note also that periodic orbit are dense in chaotic systems so that diametrical opposites might actually meet.

The most relevant TGD based ingredients used in the sequel are following: WCW [K84]; strong form of holography (SH) [K111], quantum classical correspondence (QCC), zero energy ontology (ZEO) [K61], dark matter as hierarchy of phases with effective Planck constant $h_{eff}/h = n$ [K36, **?**, K72], p-Adic physics as physics of cognition [K65, K56, K76] [L26], adelic physics [L26] fusing the physics of matter and cognition by integrating reals and extensions of various p-adic number fields induced by an extension of rationals to a larger structure, and the notions of adelic manifold and associated cognitive representation [L17] , Negentropy Maximization Principle (NMP) [K56] satisfied automatically in statistical sense in adelic physics [L26].

Complexity in TGD

Complexity is often taken to mean computational complexity for classical computations. Complexity as it is understood in the sequel relates very closely cognition. Too complex looks chaotic since our cognitive abilities do not allow to discern too complex patterns. Hence complexity theory should characterize cognitive representations whatever they are.

Number theoretic vision about TGD serves as the guideline here.

1. In adelic TGD [K47] cognitive representations correspond to the intersections of real space-time surfaces and their p-adic variants obeying same field equations and representing correlates for cognition. In these intersections the coordinates of points belong to an extension of rationals defining adele [L17].

   One ends up with a generalization of the notion of manifold to adelic manifold. Intersection defines a common discrete spine consisting of points with coordinates in the extension of rationals defining the adele. These poins are shared by the real and p-adic variants of the adelic manifold. I have called this manifold also monadic manifold since there is strong resemblance with the ideas of Leibniz. In real sector this manifold differs from ordinary manifold in that the open sets are labelled by a discrete set of points in the intersection.

   In TGD framework it is essential that the spine of the space-time surface consists of points of embedding space for which it is convenient to use preferred coordinates.

2. Complexity corresponds roughly to the dimension of extension of rationals defining the adeles. p-Adic differential equations are non-deterministic due to the existence of p-adic pseudo constants depending on finite number of p-adic digits of the p-adic number. This non-determinism is identified as a correlate for imagination. p-Adic variants of space-time surfaces are not uniquely determined this means finite cognitive resolution.

   By SH [K108] the data associated with string world sheets, partonic 2-surfaces, and discretization allow to construct space-time surfaces as preferred extremals of the action principle defining classical TGD and to find the Kähler function for WCW geometry. It is quite well possible that the data allowing to construct p-adic space-time surfaces does not allow continuation to a preferred extremal: all imaginations are not realizable!

   The algebraic dimension of the extension could be relevant for the ability of mathematical cognition to imagine spaces with dimension higher than that for the real 3-space. Besides the extensions of p-adics induced by algebraic extensions of rationals also those induced by some root of e are algebraically finite-dimensional. One can imagine also other extensions involving transcendentals in real sense but it is not clear whether there are finite dimensional extensions among them. The finiteness of cognition suggests that only these extensions can be allowed. All imaginations are not realizable!

3. Extension is characterized partially by Galois group (see `http://tinyurl.com/mrvqhz2`) acting as automorphisms meaning that Galois group permutes the roots of the n:th order polynomials defining extensions of rationals via their non-rational roots. So called ramified primes (see `http://tinyurl.com/m32nvcz` and `http://tinyurl.com/oh7tgsw`) provide additional characteristics.

   Iteration cycles appearing in complexity theory for iteration of functions and repeated action of an element Galois group defining a finite Abelian group are mathematically similar notions. Now only cycles are present whereas chaotic systems have aperiodic orbits. The cyclic subgroups of Galois group do not seem to have an natural realization as iterative dynamics except in quantum sense meaning that cyclic orbits are replaced with wave functions labelled by number theoretic integer valued "momenta" for the action of the analog of Cartan subgroup as maximal commutative subgroup for the Galois group. The maximal Abelian Galois group is analog of Cartan subgroup for Galois group of algebraic numbers and states are in its irreducible representations.

   Remark: What is interesting that for polynomials with order larger than 4, one cannot write closed analytic expressions for the roots of the polynomials. This obviously means a fundamental limitation on symbolic cognitive representations provided by explicit formulas. The realization of was a huge step in the evolution of mathematics. Could also the emergence of Galois groups with order larger at space-time level than 5 have meant cognitive revolution - probably at much lower level in the hierarchy? Could this relate also to the fact that space-time dimension is $D = 4$ and thus imaginable using 4-D algebraic extension of rationals?

   A possible measure for the cognitive complexity is the dimension of the Galois group of the extension. One can speak also about the complexity of the Galois group itself - the non-Abelianity of Galois group brings in additional complexity. The number of generators and number of relations between them serve as a measure for complexity of Galois group.

Extension of rationals is also characterized by so called ramified primes and should have a profound physical meaning. p-Adic length scale hypothesis states that physically preferred primes are near powers of 2 and perhaps also other small primes. Could they correspond to ramified primes. Why just these ramified primes would be survivors in the number theoretic evolution, is the fascinating question to be addressed later.

4. The increase of the dimension of extension or complexity of its Galois group corresponds naturally to evolution interpreted as emergence of algebraic complexity and evolutionary paths could be seen as sequences of inclusions for Galois groups. Chaos would correspond to the limit when the extension of rationals approaches to infinite sub-field of algebraic numbers - say maximal Abelian extension of rationals - so that the number of points in the cognitive representation becomes infinite.

   The Galois group of algebraic numbers - the magic Absolute Group - would characterize this limit as a kind of never achievable mathematical enlightenment. A more practical definition would be that external system is experienced as complex, when its number theoretical complexity exceeds that of the conscious observer so that it is impossible to form a faithful cognitive representation about the system. Note that these cognitive representations could be formulated as homomorphisms between Galois groups. This would suggest a rather nice category theoretical picture about cognitive representations in the self hierarchy.

5. Galois group acts on the cognitive representation associated with the space-time sheet and in general gives n-fold covering of the space-time sheet: n is naturally the dimension of the extension and thus a divisor of the order of Galois group since Galois group acts on the discretization and implies n-sheeted structure for it and therefore also for the space-time surface.

   The value of the effective Planck constant assigned with dark matter as phases of ordinary matter $h_{eff}/h = n$ was identified from very beginning as number of sheets for some kind of covering space of embedding space. n would correspond to a divisor for the order of Galois group for discretized embedding space consisting of points with coordinates in extension of rational. The increase of h_{eff} corresponds to the emergence of also cognitive complexity. Physically it is accompanied by the emergence of quantum coherence and non-locality in increasingly long scales.

General vision about evolution as emergence of complexity

Evolution would mean emergence of number theoretical complexity. Evolutionary paths would naturally correspond to sequences of inclusions (note that recent view allows also temporary "de-evolutions" but in statistical sense evolution occurs). There are infinitely many evolutionary pathways of this kind.

There is a strong resemblance with the inclusion sequences of hyper-finite factors of type II_1 (HHFs) for von Neumann algebras [K110] also playing a central role in TGD and assignable to a fractal hierarchy of isomorphic sub-algebras of super-symplectic algebra associated with the isometries of WCW and related Kac-Moody algebras. It is difficult to believe that this could be an accident.

Evolution must mean a discrete time evolution of some kind - most naturally by non-deterministic quantum version of discrete dynamics, which can be deterministic only in statistical sense. By QCC this evolution should have classical correlates at space-time level. ZEO and TGD inspired theory of consciousness, which can be regarded as a generalization of quantum measurement theory in ZEO, is essential in attempts to concretize this intuition.

1. Galois group codes for the complexity and evolution means the emergence of increasingly complex Galois groups assignable to spacetime surface in a sector of WCW for which WCW coordinates are in corresponding extension of rationals. One can say that evolution defines a path in the space of sectors of WCW characterized by Galois groups. Although the space-time dynamics is expected to be integrable, the notion of complexity still has meaning, and ultimate chaos would emerge at the limit of algebraic numbers as extension of rationals.

2. One can assign Galois group G_5 to space-time surface. Suppose that one an assign Galois groups $G_3 \subset G_4$ with the 3-surfaces at the ends of space-time surfaces at boundaries of CD. This point will be discused below in more detail.

3. At quantum level conscious entities - selves - correspond to sequences off steps consisting of unitary evolution followed by a localization in the moduli space of CD. State function reduction to the opposite boundary of CD means death of self and re-incarnation of self with opposite arrow of time: also this means localization to a definite sector of WCW [L26, L24]. The sequence of pairs of selves and their time reversals associated with the opposite boundaries of CD (, which itself increases in size) defines a candidate for the non-deterministic quantum analog of iteration in complexity theory.

4. There is a temptation to assume that for the passive boundary of CD all 3-surfaces in quantum superposition have same G_3 - the G_3 that emerged in the first state function reduction to the passive boundary when this self was born. G_3 so would be automatically measured observable and sequence of reductions would define a sequence of G_3s analogous to iteration sequence and also to evolution.

   But can one assume that G_3 is measured automatically in the re-incarnation of self as its time-reversal [K7, K47]? Could only some charateristics of G_3 - say order $n = h_{eff}/h$ - be measured? Also ramified primes characteristize extensions and their measurement is also possible and proposed to characterize elementary particles: they do not fix G_3. These uncertainties are not relevant for the general vision.

5. For the active boundary one would have a superposition of 3-surfaces with different Galois groups and the sequence of the steps consisting of unitary evolution followed by a localization in the moduli space of CDs including also a localization in clock time determined by distance between the tips of CD. Also this would give to quantal discrete dynamics. Also now one can wonder whether Galois group is measured or not. If not, one would have a dispersion like process in the space of Galois groups labelling sectors of WCW.

6. Also the evolution of the tensor net defined by fermionic strings connecting the positions of fermions at partonic 2-surfaces would define a discrete dynamics in the space of these networks both at classical and quantum level [L15]. The dynamics of many-fermion states would determine this evolution.

In the sequel this picture is discussed in more detail.

How can one assign an extension of rationals to WCW, embedding space, and a region of space-time surface?

What fixes the extension used at both WCW level, embedding space level, and space-time level? The natural assumption is that the extension used for WCW coordinates induces the extension used at embedding space level and space-time level. At the level of space-time surfaces WCW coordinates appear as moduli (parameters) characterizing preferred extremals and would have values in an extension of rationals characterizing the adele by inducing the extensions of p-adic sectors.

1. The simplest option is that the extension is dictated by WCW. Preferred WCW coordinates - made possible by maximal isometries and fixed apart from the isometries of WCW - are in the extension: this makes the space of allowed 3-surfaces discrete. This in turn induces a constraint on space-time surfaces: WCW coordinates define parameters characterizing the space-time surface as a preferred extremal. One could use also other coordinates of WCW but these would not be optimal as cognitive representations.

   This applies also at the level of embedding space. Contrary to what I first thought, it is not actually absolutely necessary to use preferred space-time coordinates (subset of embedding space coordinates) since cognitive representation depends on coordinates in finite measurement resolution: consider only spherical and Cartesian coordinates with given resolution defining different discretizations. The preferred coordinates would be preferred because they are cognitively optimal.

2. Real embedding space is replaced with a discrete set of points of H with preferred coordinates in an extension of rationals. The direct identification of the points of extension as real numbers with p-adic numbers is extremely discontinuous although it would respect algebraic symmetries. The situation is saved by the lower dimensionality of space-time surfaces for which the set of points with coordinates in extension is discrete and even finite in the generic case. The surface $x^n + y^n = z^n$ has only one rational point for $n > 2$! $D = 4 < 8$ for space-time surfaces automatically brings in finite measurement resolution and cognitive resolution induced directly from the restriction on WCW parameters.

   SH has as data the intersection plus string world sheets (SH). String world sheets are in the intersection of reality and p-adicities defined by rational functions with coefficients of polynomials in extension, and makes sense both in real and p-adic sense. To these initial data one can assign as a preferred extremal of Kähler action a smooth p-adic space-time surface such that each point is contained in an open set consisting of points with p-adic coordinates having norm smaller than some power of p. This extremal is not unique in the p-adic sectors. In real sector it might not exist at all as already discussed.

3. 3-surface is seen as pair of 3-surfaces assigned to the ends of the space-time surface at boundaries of CD. WCW coordinates parameterize this pair and correspond to extension in 4-D sense. These parameters are expected to decompose to sets of parameters characterizing the 3-D members of pair and parameters characterizing the connecting space-time surface unless it is unique. If so, one can assign to the initial and final 3-surfaces subsets of WCW coordinates.

   The extensions associated with the ends of CD would be extensions in 3-D sense and sub-extensions of the extension in 4-D sense. Hence one can say that classical space-time evolution connecting initial and final 3-surfaces can modify the extension, its Galois group, and therefore also $h_{eff}/h = n$. This would be the classical view a about number theoretic evolution and also about quantum critical fluctuation changing the value of $h_{eff}/h = n$.

4. The extension of rationals for WCW coordinates induces the cognitive representation posing constraints of p-adic space-time surfaces. Adelic sub-WCW consisting of preferred extremals inside given CD decomposes to sectors characterized by an extension of rationals and evolution should correspond number theoretically to a path in the space of WCW sectors.

   This is a restriction on p-adic space-time sheets and thus cognition: the larger the number of points in the intersection, the more precise the cognitive representation is. The increase of the dimension of extension implies that the number of points of cognitive representation increases and it becomes more precise. The cognitive abilities of the system evolve. p-Adic pseudo constants allow imagination but also make the representation imprecise in scales below that defined by the cognitive representation. The continuation to smooth p-adic surface would however explain the highly non-trivial fact that we automatically tend to associate continuous structures with discrete data.

5. The fermions at partonic 2-surfaces are at positions for which preferred space-time coordinates are in extension and can be said to actualize the cognitive representation. It turns out that these positions could naturally correspond to the singularities of the space-time surfaces as n-fold covering in the sense that the dimension of the orbit of Galois group would be reduced at these points.

Can one assign the analog of discrete dynamics to TGD at fundamental level?

Could one assign a discrete symbolic dynamics to classical and quantum TGD?

At classical level the dynamics would correspond to space-time surface connecting the boundaries of CD and 3-surfaces at them. As already explained, the WCW coordinates characterizing space-time surface as a preferred extremal correspond to what might be called Galois group in 4-D sense. These coordinates decompose to coordinates characterizing the coordinates at the 3-surfaces at the ends of of space-time at boundaries of CD in extensions characterized by Galois groups in 3-D sense - the initial and final Galois group. The classical evolutionary step would be a step leading from the initial fo final Galois group serving as classical correlate for quantum evolution.

What about quantum level?

1. One expects that zero energy state in general is a superposition of space-time surfaces with different Galois groups in 4-D sense, G_4. The Galois groups in 3-D sense - G_3 - assignable to the ends of space-time surface would be sub-groups of G_4. If the first state function reduction to the opposite boundary of CD involves a localization to a sector of WCW having same G_3 at passive boundary for all 3-surfaces in the superposition.

   Subsequent reductions at opposite boundaries would define evolutionary pathway in the space of Galois groups G_3 leading in statistical sense to the increase of complexity.

2. The original vision was that Negentropy Maximization Principle (NMP) [K56] is needed as a separate principle to guarantee evolution but adelic physics implies it in statistical sense automatically [L26]. There is infinite number of extensions more complex than given one and only finite number of them less complex.

3. At quantum level the basic notion is self. It corresponds to a discrete sequence steps consisting of unitary evolution followed by a localization in the moduli space of CDs. This would correspond to a dispersion in WCW to sectors characterized by different Galois groups G_4 and G_3 associated with the 3-surface at active boundary. As explained, the state function reduction to the opposite boundary of CD analogous to a halting of quantum computation would correspond to a localization to a sector with definite Galois group G_3.

4. These time discrete time evolutions are non-deterministic unlike the dynamical evolutions studied in chaos theory defined by differential equations or iteration of function. The sequence of unitary time evolutions involving localization in the moduli of CD would however give rise to a quantum analog of iteration and one can ask whether the quantum counterparts for the notions of cycle, super-stable cycle etc... could make sense for the quantum superpositions of 4-surfaces involved. One expects dispersion in the space of Galois groups so that this idea does not look promising. One can also wonder if the sequence of unitary transformations could lead to some kind of asymptotic self-organization pattern before the first state function reduction to the opposite boundary of CD.

It is natural to consider also the evolution of the cognitive representation itself both at the space-time level and forced by the change of the many-fermion state and at quantum level.

1. For a given preferred extremal cognitive representation defines a discrete set of points in an extension of rationals and the number of points in the extension increases as it grows. The positions of fermions at partonic 2-surfaces define the nodes of a graph with strings connecting fermions at different partonic 2-surfaces serving as edges. Evolution of fermionic state changes the topology of this network by adding vertices and changing the connection.

   One can assign a complexity theory to these graphs. A connection with tensor nets [L15] emerging in the description of quantum complexity is highly suggestive. The nodes of the tensor net would correspond to fermions at partonic 2-surfaces. As the number of fermions increases, the complexity of this network increases and also the space-time surface itself becomes more complex. The maximum number of fermions increases with the dimension of extension.

   An interesting proposal is that fermion lines are accompanied by magnetic flux tubes taking the role of wormholes in ER-EPR correspondence (see `http://tinyurl.com/hzqlo6r`), which emerged more than half decade after its TGD analog. The discrete evolution of many-fermion state in state function reductions in the fermionic sector induces the evolution of this network.

2. In the case of graphs one can speak about various kinds of cycles, in particular Hamiltonian cycles going through all points of graph and having no self-intersections. Interestingly, Hamiltonian cycles for icosahedron (here the isometry group of icosahedron is involved as an additional structure) lead to a vision about genetic code and music harmonies [L7].

3. An interesting question concerns the extensions of rationals having as Galois group the isometry groups of Platonic solids: they probably exist. One can also consider the counterparts of Galois groups as discrete subgroups of the Galois group $SO(3)$ of quaternions. They

emerge naturally for algebraic discretizations of M^4 regarded as a subspace of complexified quaternions with time axis identified as the real axis for quaternions (for $M^8 - H$ correspondence [K96, K109] see http://tinyurl.com/mdvazmr). Platonic solids correspond to *finite* discretizations with finite isometry groups belonging to a hierarchy of finite discrete subgroups of $SO(3)$ labelling the hierarchy of inclusions of HFFs: a connection between HFFs and quaternions is suggestive. For HFFs Platonic solids are in unique role in the sense that only for them the action of $SO(3)$ is genuinely 3-D. In Mac Kay correspondence they correspond to exceptional groups.

For this generalization evolution would correspond to evolution in the space of Galois groups for finite-dimensional extensions of rational valued quaternions. p-Adic quaternions do not however form a field since p-adic quaternion can have vanishing norm squared.

4. The wave functions in the Galois group G reduce to wave functions in its coset space G/H if they are invariant under subgroup H. One can also perform the analog of second quantization for fermions in Galois group labelling the space-time sheets (or those of 3-space). In the model of harmony based on Hamilton's cycles the notes of 12-note scale would correspond to vertices of icosahedron obtained as coset space of I/Z_5, where I is icosahedral group with 60 elements. 3-chords of the harmony for a given Hamiltonian cycle would correspond to faces, which are triangles. Single particle fermion states localized at vertices (points of coset space) would correspond to notes of the scale and 3-fermion states localized at vertices of triangle to allowed 3-chords. The observation that one can understand the degeneracies of vertebrate genetic code by introducing besides icosahedron also tetrahedron suggests that both music and genetic code could relate directly to cognition described number theoretically.

5. It is also known that graphs can be identified as representations for Boolean statements (see http://tinyurl.com/myrkhny). Many-fermion states represent in TGD framework quantum Boolean statements with fermion number taking the role of bit. Could it be that this graphs indeed represent entanglement many-fermion states having interpretation as quantum Boolean statements?

Can one imagine a quantum counterpart of iteration cycle? The space-time sheets can be seen as covering spaces with the number of sheets equal to the order $n = h_{eff}/h$ of Galois group. This gives additional discrete degrees of freedom and one could have wave functions in Galois group and also in its cyclic subgroup. These might serve as quantum counterparts for iteration cycles. An open question is whether n is always accompanied by $1/n$ fractionization of quantum numbers so that dark elementary particles would have same quantum numbers as ordinary ones but could be said to decompose to n pieces corresponding to sheets of covering.

One can also imagine that the cycles appear in the statistical description. At this limit one obtains deterministic kinetic equations and by their non-linearity one expects that they exhibit chaotic behavior in the usual sense.

Why would primes near powers of two (or small primes) be important?

p-Adic length scale hypothesis states that physically preferred p-adic primes come as primes near prime powers of two and possibly also other small primes. Does this have some analog to complexity theory, period doubling, and with the super-stability associated with period doublings?

Also ramified primes characterize the extension of rationals and would define naturally preferred primes for a given extension.

1. Any rational prime p can be decomposes to a product of powers P_i^k of primes P_i of extension given by $p = \prod_i P_i^{k_i}$, $\sum k_i = n$. If one has $k_i \neq 1$ for some i, one has ramified prime. Prime p is Galois invariant but ramified prime decomposes to lower-dimensional orbits of Galois group formed by a subset of $P_i^{k_i}$ with the same index k_i . One might say that ramified primes are more structured and informative than un-ramified ones. This could mean also representative capacity.

2. Ramification has as its analog criticality leading to the degenerate roots of a polynomial or the lowering of the rank of the matrix defined by the second derivatives of potential

function depending on parameters. The graph of potential function in the space defined by its arguments and parameters if n-sheeted singular covering of this space since the potential has several extrema for given parameters. At boundaries of the n-sheeted structure some sheets degenerate and the dimension is reduced locally . Cusp catastrophe with 3-sheets in catastrophe region is standard example about this.

Ramification also brings in mind super-stability of n-cycle for the iteration of functions meaning that the derivative of n:th iterate $f(f(...)(x) \equiv f^{n)}(x)$ vanishes. Superstability occurs for the iteration of function $f = ax + bx^2$ for $a = 0$.

3. I have considered the possibility that the n-sheeted coverings of the space-time surface are singular in that the sheet co-incide at the ends of space-time surface or at some partonic 2-surfaces. One can also consider the possibility that only some sheets or partonic 2-surfaces co-incide.

   The extreme option is that the singularities occur only at the points representing fermions at partonic 2-surfaces. Fermions could in this case correspond to different ramified primes. The graph of $w = z^{1/2}$ defining 2-fold covering of complex plane with singularity at origin gives an idea about what would be involved. This option looks the most attractive one and conforms with the idea that singularities of the coverings in general correspond to isolated points. It also conforms with the hypothesis that fermions are labelled by p-adic primes and the connection between ramifications and Galois singularities could justify this hypothesis.

4. Category theorists love structural similarities and might ask whether there might be a morphism mapping these singularities of the space-time surfaces as Galois coverings to the ramified primes so that sheets would correspond to primes of extension appearing in the decomposition of prime to primes of extension.

   Could the singularities of the covering correspond to the ramification of primes of extension? Could this degeneracy for given extension be coded by a ramified prime? Could quantum criticality of TGD favour ramified primes and singularities at the locations of fermions at partonic 2-surfaces?

   Could the fundamental fermions at the partonic surfaces be quite generally localize at the singularities of the covering space serving as markings for them? This also conforms with the assumption that fermions with standard value of Planck constants corresponds to 2-sheeted coverings.

5. What could the ramification for a point of cognitive representation mean algebraically? The covering orbit of point is obtained as orbit of Galois group. For maximal singularity the Galois orbit reduces to single point so that the point is rational. Maximally ramified fermions would be located at rational points of extension. For non-maximal ramifications the number of sheets would be reduced but there would be several of them and one can ask whether only maximally ramified primes are realized. Could this relate at the deeper level to the fact that only rational numbers can be represented in computers exactly.

6. Can one imagine a physical correlate for the singular points of the space-time sheets at the ends of the space-time surface? Quantum criticality as analogy of criticality associated with super-stable cycles in chaos theory could be in question. Could the fusion of the space-time sheets correspond to a phenomenon analogous to Bose-Einstein condensation? Most naturally the condensate would correspond to a fractionization of fermion number allowing to put n fermions to point with same M^4 projection? The largest condensate would correspond to a maximal ramification $p = P_i^n$.

Why ramified primes would tend to be primes near powers of two or of small prime? The attempt to answer this question forces to ask what it means to be a survivor in number theoretical evolution. One can imagine two kinds of explanations.

1. Some extensions are winners in the number theoretic evolution, and also the ramified primes assignable to them. These extensions would be especially stable against further evolution

representing analogs of evolutionary fossils. As proposed earlier, they could also allow exceptionally large cognitive representations that is large number of points of real space-time surface in extension.

2. Certain primes as ramified primes are winners in the sense the further extensions conserve the property of being ramified.

   (a) The first possibility is that further evolution could preserve these ramified primes and only add new ramified primes. The preferred primes would be like genes, which are conserved during biological evolution. What kind of extensions of existing extension preserve the already existing ramified primes. One could naïvely think that extension of an extension replaces P_i in the extension for $P_i = Q_{ik}^{k_i}$ so that the ramified primes would remain ramified primes.

   (b) Surviving ramified primes could be associated with a exceptionally large number of extensions and thus with their Galois groups. In other words, some primes would have strong tendency to ramify. They would be at criticality with respect to ramification. They would be critical in the sense that multiple roots appear.

   Can one find any support for this purely TGD inspired conjecture from literature? I am not a number theorist so that I can only go to web and search and try to understand what I found. Web search led to a thesis (see `http://tinyurl.com/mkhrssy`) studying Galois group with prescribed ramified primes.

   The thesis contained the statement that not every finite group can appear as Galois group with prescribed ramification. The second statement was that as the number and size of ramified primes increases more Galois groups are possible for given predetermined ramified primes. This would conform with the conjecture. The number and size of ramified primes would be a measure for complexity of the system, and both would increase with the size of the system.

   (c) Of course, both mechanisms could be involved.

Why ramified primes near powers of 2 would be winners? Do they correspond to ramified primes associated with especially many extension and are they conserved in evolution by subsequent extensions of Galois group. But why? This brings in mind the fact that $n = 2^k$-cycles becomes super-stable and thus critical at certain critical value of the control parameter. Note also that ramified primes are analogous to prime cycles in iteration. Analogy with the evolution of genome is also strongly suggestive.

$h_{eff}/h = n$ hypothesis and Galois groups

The natural hypothesis is that $h_{eff}/h = n$ equals to dimension of the extension of rationals in the case that it gives the number of sheets of the covering assignable to the space-time surfaces. The stronger hypothesis is that $h_{eff}/h = n$ is associated with flux tubes and is proportional to the quantum numbers associated with the ends.

1. The basic idea is that Mother Nature is theoretician friendly. As perturbation theory breaks down, the interaction strength expressible as a product of appropriate charges divided by Planck constant, is reduced in the phase transition $\hbar \rightarrow \hbar_{eff}$.

2. In the case of gravitation $GMm \rightarrow= GMm(h/h_{eff})$. Equivalence Principle is satisfied if one has $\hbar_{eff} = \hbar_{gr} = GMm/v_0$, where v_0 is parameter with dimensions of velocity and of the order of some rotation velocity associated with the system. If the masses move with relativistic velocities the interaction strength is proportional to the inner product of four-momenta and therefore to Lorentz boost factors for energies in the rest system of the entire system. In this case one must assume quantization of energies to satisfy the constraint or a compensating reduction of v_0. Interactions strength becomes equal to $\beta_0 = v_0/c$ having no dependence on the masses: this brings in mind the universality associated with quantum criticality.

3. The hypothesis applies to all interactions. For electromagnetism one would have the replacements $Z_1Z_2\alpha \rightarrow Z_1Z_2\alpha(h/h_{em})$ and $\hbar_{em} = Z_1Z_2\alpha/\beta_0$ giving Universal interaction strength. In the case of color interactions the phase transition would lead to the emergence of hadron and it could be that inside hadrons the valence quark have $h_{eff}/h = n > 1$. In this case one could consider a generalization in which the product of masses is replaced with the inner product of four-momenta. In this case quantization of energy at either or both ends is required. For astrophysical energies one would have effective energy continuum.

This hypothesis suggests the interpretation of $h_{eff}/h = n$ as either the dimension of the extension or the order of its Galois group. If the extensions have dimensions n_1 and n_2, then the composite system would be n_2-dimensional extension of n_1-dimensional extension and have dimension $n_1 \times n_2$. This could be also true for the orders of Galois groups. This would be the case if Galois group of the entire system is free group generated by the G_1 and G_2. One just takes all products of elements of G_1 and G_2 and assumes that they commute to get $G_1 \times G_2$.

Consider gravitation as example.

1. The dimension of the extension should coincide with $\hbar_{eff}/\hbar = n = \hbar_{gr}/\hbar = GMm/v_0\hbar$. The transition occurs only if the value of $\hbar_{gr}/\hbar$ is larger than one. One can say that the dimension of the extension is proportional the product of masses using as unit Planck mass. Rather large extensions are involved and the number of sheets in the Galois covering is huge.

   Note that it is difficult to say how larger Planck constants are actually involved since by gravitational binding the classical gravitational forces are additive and by Equivalence principle same potential is obtained as sum of potentials for splitting of masses into pieces. Also the gravitational Compton length $\lambda_{gr} = GM/v_0$ for m does not depend on m at all so that all particles have same $\lambda_{gr} = GM/v_0$ irrespective of mass (note that v_0 is expressed using units with $c = 1$).

   The maximally incoherent situation would correspond to ordinary Planck constant and the usual view about gravitational interaction between particles. The extreme quantum coherence would mean that both M and m behave as single quantum unit. In many-sheeted space-time this could be understood in terms of a picture based on flux tubes. The interpretation for the degree of coherence is discussed in terms of flux tube connections mediating gravitational flux is discussed in [?].

2. h_{gr}/h would be the dimension of the extension, and there is a temptation to associate with the product of masses the product $n = n_1n_2$ of dimensions n_i associated masses M and m at least in some situations.

   The problem is that the dimension of the extension associated with m would be smaller than 1 for masses $m < m_P/\sqrt{\beta_0}$. Planck mass is about 1.3×10^{19} proton masses and corresponds to a blob of water with size scale 10^{-4} meters - size scale of a large neuron so that only above these scale gravitational quantum coherence would be possible. For $v_0 < 1$ it would seem that even in the case of large neurons one must have more than one neurons. Maybe pyramidal neurons could satisfy the mass constraint and would represent higher level of conscious as compared to other neurons and cells. The giant neurons discovered by the group led by Christof Koch in the brain of of mouse having axonal connections distributed over the entire brain might fulfil the constraint (see `http://tinyurl.com/gvwggsc`).

3. It is difficult to avoid the idea that macroscopic quantum gravitational coherence for multicellular objects with mass at least that for the largest neurons could be involved with biology. Multicellular systems can have mass above this threshold for some critical cell number. This might explain the dramatic evolutionary step distinguishing between prokaryotes (mono-cellulars consisting of Archaea and bacteria including also cellular organelles and cells with sub-critical size) and eukaryotes (multi-cellulars).

4. I have proposed an explanation of the fountain effect appearing in super-fluidity and apparently defying the law of gravity. In this case m was assumed to be the mass of ^4He atom in case of super-fluidity to explain fountain effect [?]. The above arguments however allow to ask whether anything changes if one allows the blobs of superfluid to have masses coming as

a multiple of $m_P/\sqrt{\beta_0}$. One could check whether fountain effect is possible for super-fluid volumes with mass below $m_P/\sqrt{\beta_0}$.

What about h_{em}? In the case of super-conductivity the interpretation of h_{em}/h as product of orders of Galois groups would allow to estimate the number $N = Q/2e$ of Cooper pairs of a minimal blob of super-conducting matter from the condition that the order of its Galois group is larger than integer. The number $N = Q/2e$ is such that one has $2N\sqrt{\alpha/\beta_0} = n$. The condition is satisfied if one has $\alpha/\beta_0 = q^2$, with $q = k/2l$ such that N is divisible by l. The number of Cooper pairs would be quantized as multiples of l. What is clear that em interaction would correspond to a lower level of cognitive consciousness and that the step to gravitation dominated cognition would be huge if the dark gravitational interaction with size of astrophysical systems is involved [K72]. Many-sheeted space-time allows this in principle.

These arguments support the view that quantum information theory indeed closely relates not only to gravitation but also other interactions. Speculations revolving around blackhole, entropy, and holography, and emergence of space would be replaced with the number theoretic vision about cognition providing information theoretic interpretation of basic interactions in terms of entangled tensor networks [L15]. Negentropic entanglement would have magnetic flux tubes (and fermionic strings at them) as topological correlates. The increase of the complexity of quantum states could occur by the "fusion" of Galois groups associated with various nodes of this network as macroscopic quantum states are formed. Galois groups and their representations would define the basic information theoretic concepts. The emergence of gravitational quantum coherence identified as the emergence of multi-cellulars would mean a major step in biological evolution.

8.9 Why The Non-trivial Zeros Of Riemann Zeta Should Reside At Critical Line?

The following argument shows that the troublesome looking "1/2" in the non-trivial zeros of Riemann zeta can be understood as being necessary to allow a hermitian realization of the radial scaling generator rd/dr at light-cone boundary, which in the radial light-like radial direction corresponds to half-line R^+. Its presence allows unitary inner product and reduces the situation to that for ordinary plane waves on real axis. For preferred extremals strong form of holography poses extremely strong conditions expected to reduce the scaling momenta $s = 1/2 + iy$ to the zeros of zeta at critical line. RH could be also seen as a necessary condition for the existence of super-symplectic representations and thus for the existence of the "World of Classical Worlds" as a mathematically well-defined object. We can thank the correctness of Riemann's hypothesis for our existence!

8.9.1 What Is The Origin Of The Troublesome 1/2 In Non-trivial Zeros Of Zeta?

Riemann Hypothess (RH) states that the non-trivial (critical) zeros of zeta lie at critical line $s = 1/2$. It would be interesting to know how many physical justifications for why this should be the case has been proposed during years. Probably this number is finite, but very large it certainly is. In Zero Energy Ontology (ZEO) forming one of the cornerstones of the ontology of quantum TGD, the following justification emerges naturally.

1. The "World of Classical Worlds" (WCW) consisting of space-time surfaces having ends at the boundaries of causal diamond (CD), the intersection of future and past directed light-cones times CP_2 (recall that CDs form a fractal hierarchy). WCW thus decomposes to sub-WCWs and conscious experience for the self associated with CD is only about space-time surfaces in the interior of CD: this is a trong restriction to epistemology, would philosopher say.

   Also the light-like orbits of the partonic 2-surfaces define boundary like entities but as surfaces at which the signature of the induced metric changes from Euclidian to Minkowskian. By holography either kinds of 3-surfaces can be taken as basic objects, and if one accepts strong form of holography, partonic 2-surfaces defined by their intersections plus string world sheets become the basic entities.

2. One must construct tangent space basis for WCW if one wants to define WCW Kähler metric and gamma matrices. Tangent space consists of allowed deformations of 3-surfaces at the ends of space-time surface at boundaries of CD, and also at light-like parton orbits extended by field equations to deformations of the entire space-time surface. By strong form of holography only very few deformations are allowed since they must respect the vanishing of the elements of a sub-algebra of the classical symplectic charges isomorphic with the entire algebra. One has almost 2-dimensionality: most deformations lead outside WCW and have zero norm in WCW metric.

3. One can express the deformations of the space-like 3-surface at the ends of space-time using a suitable function basis. For CP_2 degrees of freedom color partial waves with well defined color quantum numbers are natural. For light-cone boundary $S^2 \times R^+$, where R^+ corresponds to the light-like radial coordinate, spherical harmonics with well defined spin are natural choice for S^2 and for R^+ analogs of plane waves are natural. By scaling invariance in the light-like radial direction they look like plane waves $\psi_s(r) = r^s = exp(us)$, $u = log(r/r_0)$, $s = x + iy$. Clearly, u is the natural coordinate since it replaces R^+ with R natural for ordinary plane waves.

4. One can understand why $Re[s] = 1/2$ is the only possible option by using a simple argument. One has super-symplectic symmetry and conformal invariance extended from 2-D Riemann surface to metrically 2-dimensional light-cone boundary. The natural scaling invariant integration measure defining inner product for plane waves in R^+ is $du = dr/r = dlog(r/r_0)$ with u varying from $-\infty$ to $+\infty$ so that R^+ is effectively replaced with R. The inner product must be same as for the ordinary plane waves and indeed is for $\psi_s(r)$ with $s = 1/2 + iy$ since the inner product reads as

$$\langle s_1, s_2\rangle \equiv \int_0^\infty \overline{\psi_{s_1}}\psi_{s_2}dr = \int_0^\infty exp(i(y_1-y_2)r^{-x_1-x_2}dr \quad .$$

For $x_1 + x_2 = 1$ one obtains standard delta function normalization for ordinary plane waves:

$$\langle s_1, s_2\rangle \int \int_{-\infty}^\infty exp[i(y_1-y_2)u]du \propto \delta(y_1-y_2) \quad .$$

If one requires that this holds true for all pairs (s_1, s_2), one obtains $x_i = 1/2$ for all s_i. Preferred extremal condition gives extremely powerful additional constraints and leads to a quantisation of $s = -x - iy$: the first guess is that non-trivial zeros of zeta are obtained: $s = 1/2 + iy$. This identification would be natural by generalised conformal invariance. Thus RH is physically extremely well motivated but this of course does not prove it.

5. The presence of the real part $Re[s] = 1/2$ in the eigenvalues of scaling operator apparently breaks hermiticity of the scaling operator. There is however a compensating breaking of hermiticity coming from the fact that real axis is replaced with half-line and origin is pathological. What happens that real part 1/2 effectively replaces half-line with real axis and obtains standard plane wave basis. Note also that the integration measure becomes scaling invariant - something very essential for the representations of super-symplectic algebra. For $Re[s] = 1/2$ the hermicity conditions for the scaling generator rd/dr in R^+ reduce to those for the translation generator d/du in R.

8.9.2 Relation To Number Theoretical Universality And Existence Of WCW

This relates also to the number theoretical universality and mathematical existence of WCW in an interesting manner.

1. If one assumes that p-adic primes p correspond to zeros $s = 1/2+y$ of zeta in 1-1 manner in the sense that $p^{iy(p)}$ is root of unity existing in all number fields (algebraic extension of p-adics) one obtains that the plane wave exists for p at points $r = p^n$. p-Adically wave function

is discretized to a delta function distribution concentrated at $(r/r_0) = p^n$- a logarithmic lattice. This can be seen as space-time correlate for p-adicity for light-like momenta to be distinguished from that for massive states where length scales come as powers of $p^{1/2}$. Something very similar is obtained from the Fourier transform of the distribution of zeros at critical line (Dyson's argument), which led to a the TGD inspired vision about number theoretical universality [L9] (see `http://tinyurl.com/y7gl4huo`).

2. My article "Strategy for Proving Riemann Hypothesis" (`http://tinyurl.com/yd7k46ar`) [L1] written for 12 years ago ((for a slightly improved version see `http://tinyurl.com/ydcfkxwr`) relies on coherent states instead of eigenstates of Hamiltonian. The above approach in turn absorbs the problematic 1/2 to the integration measure at light cone boundary and conformal invariance is also now central.

3. Quite generally, I believe that conformal invariance in the extended form applying at metrically 2-D light-cone boundary (and at light-like orbits of partonic 2-surfaces) could be central for understanding why physics requires RH and maybe even for proving RH assuming it is provable at all in existing standard axiomatic system. For instance, the number of generating elements of the extended supersymplectic algebra is infinite (rather than finite as for ordinary conformal algebras) and generators are labelled by conformal weights defined by zeros of zeta (perhaps also the trivial conformal weights). $s = 1/2 + iy$ guarantees that the real parts of conformal weights for all states are integers. By conformal confinement the sum of ys vanishes for physical states. If some weight is not at critical line the situation changes. One obtains as net conformal weights all multiples of x shifted by all half odd integer values. And of course, the realisation as plane waves at boundary of light-cone fails and the resulting loss of unitary makes things too pathological and the mathematical existence of WCW is threatened.

4. The existence of non-trivial zeros outside the critical line could thus spoil the representations of super-symplectic algebra and destroy WCW geometry. RH would be crucial for the mathematical existence of the physical world! And the physical worlds exist only as mathematical objects in TGD based ontology: there are no physical realities behind the mathematical objects (WCW spinor fields) representing the quantum states. TGD inspired theory of consciousness tells that quantum jumps between the zero energy states give rise to conscious experience, and this is in principle all that is needed to understand what we experience.

8.10 Why Mersenne primes are so special?

Mersenne primes are central in TGD based world view. p-Adic thermodynamics combined with p-adic length scale hypothesis stating that primes near powers of two are physically preferred provides a nice understanding of elementary particle mass spectrum. Mersenne primes $M_k = 2^k - 1$, where also k must be prime, seem to be preferred. Mersenne prime labels hadronic mass scale (there is now evidence from LHC for two new hadronic physics labelled by Mersenne and Gaussian Mersenne), and weak mass scale. Also electron and tau lepton are labelled by Mersenne prime. Also Gaussian Mersennes $M_{G,k} = (1 + i)^k - 1$ seem to be important. Muon is labelled by Gaussian Mersenne and the range of length scales between cell membrane thickness and size of cell nucleus contains 4 Gaussian Mersennes!

What gives Mersenne primes so special physical status? I have considered this problem many times during years. The key idea is that natural selection is realized in much more general sense than usually thought, and has chosen them and corresponding p-adic length scales. Particles characterized by p-adic length scales should be stable in some well-defined sense.

Since evolution in TGD corresponds to generation of information, the obvious guess is that Mersenne primes are information theoretically special. Could the fact that $2^k - 1$ represents almost k bits be of significance? Or could Mersenne primes characterize systems, which are information theoretically especially stable? In the following a more refined TGD inspired quantum information theoretic argument based on stability of entanglement against state function reduction, which would be fundamental process governed by Negentropy Maximization Principle (NMP) and requiring no human observer, will be discussed.

8.10.1 How to achieve stability against state function reductions?

TGD provides actually several ideas about how to achieve stability against state function reductions. This stability would be of course marvellous fact from the point of view of quantum computation since it would make possible stable quantum information storage. Also living systems could apply this kind of storage mechanism.

1. p-Adic physics leads to the notion of negentropic entanglement (NE) for which number theoretic entanglement entropy is negative and thus measures genuine, possibly conscious information assignable to entanglement (ordinary entanglement entropy measures the lack of information about the state of either entangled system). NMP favors the generation of NE. NE can be however transferred from system to another (stolen using less diplomatically correct expression!), and this kind of transfer is associated with metabolism. This kind of transfer would be the most fundamental crime: biology would be basically criminal activity! Religious thinker might talk about original sin.

   In living matter NE would make possible information storage. In fact, TGD inspired theory of consciousness constructed as a generalization of quantum measurement theory in Zero Energy Ontology (ZEO) identifies the permanent self of living system (replaced with a more negentropic one in biological death, which is also a reincarnation) as the boundary of CD, which is not affected in subsequent state function reductions and carries NE. The changing part of self - sensory input and cognition - can be assigned with opposite changing boundary of CD.

2. Also number theoretic stability can be considered. Suppose that one can assign to the system some extension of algebraic numbers characterizing the WCW coordinates ("world of classical worlds") parametrizing the space-time surface (by strong form of holography (SH) the string world sheets and partonic 2-surfaces continuable to 4-D preferred extremal) associated with it.

   This extension of rationals and corresponding algebraic extensions of p-adic numbers would define the number fields defining the coefficient fields of Hilbert spaces. Assume that you have an entangled system with entanglement coefficients in this number field. Suppose you want to diagonalize the corresponding density matrix. The eigenvalues belong in general case to a larger algebraic extension since they correspond to roots of a characteristic polynomials assignable to the density matrix. Could one say, that this kind of entanglement is stable (at least to some degree) against state function reduction since it means going to an eigenstate which does not belong to the extension used? Reader can decide!

3. Hilbert spaces are like natural numbers with respect to direct sum and tensor product. The dimension of the tensor product is product mn of the dimensions of the tensor factors. Hilbert space with dimension n can be decomposed to a tensor product of prime Hilbert spaces with dimensions which are prime factors of n. In TGD Universe state function reduction is a dynamical process, which implies that the states in state spaces with prime valued dimension are stable against state function reduction since one cannot even speak about tensor product decomposition, entanglement, or reduction of entanglement. These state spaces are quantum indecomposable and would be thus ideal for the storage of quantum information!

   Interestingly, the system consisting of k qubits have Hilbert space dimension $D = 2^k$ and is thus maximally unstable against decomposition to $D = 2$-dimensional tensor factors! In TGD Universe NE might save the situation. Could one imagine a situation in which Hilbert space with dimension $M_k = 2^k - 1$ stores the information stably? When information is processed this state space would be mapped isometrically to 2^k-dimensional state space making possible quantum computations using qubits. The outcome of state function reduction halting the computation would be mapped isometrically back to M_k-D space. Note that isometric maps generalizing unitary transformations are an essential element in the proposal for the tensor net realization of holography and error correcting codes [L15]. Can one imagine any concrete realization for this idea? This question be considered in the sequel.

8.10.2 How to realize $M_k = 2^k - 1$-dimensional Hilbert space physically?

One can imagine at least three physical realizations of $M_k = 2^k - 1$-dimensional Hilbert space.

1. The set with k elements has 2^k subsets. One of them is empty set and cannot be physically realized. Here the reader might of course argue that if they are realized as empty boxes, one can realize them. If empty set has no physical realization, the wave functions in the set of non-empty subsets with $2^k - 1$ elements define $2^k - 1$-dimensional Hilbert space. If $2^k - 1$ is Mersenne prime, this state state space is stable against state function reductions since one cannot even speak about entanglement!

   To make quantum computation possible one must map this state space to 2^k dimensional state space by isometric embedding. This is possible by just adding a new element to the set and considering only wave functions in the set of subsets containing this new element. Now also the empty set is mapped to a set containing only this new element and thus belongs to the state space. One has 2^k dimensions and quantum computations are possible. When the computation halts, one just removes this new element from the system, and the data are stored stably!

2. Second realization relies on k bits represented as spins such that $2^k - 1$ is Mersenne prime. Suppose that the ground state is spontaneously magnetized state with $k + l$ parallel spins, with the l spins in the direction of spontaneous magnetization and stabilizing it. $l > 1$ is probably needed to stabilize the direction of magnetization: $l \leq k$ suggests itself as the first guess. Here thermodynamics and a model for spin-spin interaction would give a better estimate.

   The state with the k spins in direction opposite to that for l spins would be analogous to empty set. Spontaneous magnetization disappears, when a sufficient number of spins is in direction opposite to that of magnetization. Suppose that k corresponds to a critical number of spins in the sense that spontaneous magnetization occurs for this number of parallel spins. Quantum superpositions of $2^k - 1$ states for k spins would be stable against state function reduction also now.

   The transformation of the data to a processable form would require an addition of $m \geq 1$ spins in the direction of the magnetization to guarantee that the state with all k spins in direction opposite to the spontaneous magnetization does not induce spontaneous magnetization in opposite direction. Note that these additional stabilizing spins are classical and their direction could be kept fixed by a repeated state function reduction (Zeno effect). One would clearly have a critical system.

3. Third realization is suggested by TGD inspired view about Boolean consciousness. Boolean logic is represented by the Fock state basis of many-fermion states. Each fermion mode defines one bit: fermion in given mode is present or not. One obtains 2^k states. These states have different fermion numbers and in ordinary positive energy ontology their realization is not possible.

   In ZEO situation changes. Fermionic zero energy states are superpositions of pairs of states at opposite boundaries of CD such that the total quantum numbers are opposite. This applies to fermion number too. This allows to have time-like entanglement in which one has superposition of states for which fermion numbers at given boundary are different. This kind of states might be realized for super-conductors to which one at least formally assigns coherent state of Cooper pairs having ill-defined fermion number.

   Now the non-realizable state would correspond to fermion vacuum analogous to empty set. Reader can of course argue that the bosonic degrees of freedom assignable to the space-time surface are still present. I defend this idea by saying that the purely bosonic state might be unstable or maybe even non-realizable as vacuum state and remind that also bosons in TGD framework consists of pairs of fundamental fermions.

   If this state is effectively decoupled from the rest of the Universe, one has $2^k - 1$-dimensional state space and states are stable against state function reduction. Information processing becomes possible by adding some positive energy fermions and corresponding negative energy fermions at the opposite boundaries of CD. Note that the added fermions do not have time-like quantum entanglement and do not change spin direction during time evolution.

The proposal is that Boolean consciousness is realized in this manner and zero energy states represents quantum Boolean thoughts as superposition of pairs $(b_1 \otimes b_2)$ of positive and negative energy states and having identification as Boolean statements $b_1 \rightarrow b_2$. The mechanism would allow both storage of thoughts as memories and their processing by introducing the additional fermion.

8.10.3 Why Mersenne primes would be so special?

Returning to the original question "Why Mersenne primes are so special?". A possible explanation is that elementary particle or hadron characterized by a p-adic length scale $p = M_k = 2^k - 1$ both stores and processes information with maximal effectiveness. This would not be surprising if p-adic physics defines the physical correlates of cognition assumed to be universal rather than being restricted to human brain.

In adelic physics p-dimensional Hilbert space could be naturally associated with the p-adic adelic sector of the system. Information storage could take place in $p = M_k = 2^k - 1$ phase and information processing (cognition) would take place in 2^k-dimensional state space. This state space would be reached in a phase transition $p = 2^k - 1 \rightarrow 2$ changing effective p-adic topology in real sector and genuine p-adic topology in p-adic sector and replacing padic length scale $\propto \sqrt{p} \simeq 2^{k/2}$ with k-nary 2-adic length scale $\propto 2^{k/2}$.

Electron is characterized by the largest not completely super-astrophysical Mersenne prime M_{127} and corresponds to $k = 127$ bits. Intriguingly, the secondary p-adic time scale of electron corresponds to .1 seconds defining the fundamental biorhythm of 10 Hz.

This proposal suffers from deficiencies. It does not explain why Gaussian Mersennes are also special. Gaussian Mersennes correspond ordinary primes near power of 2 but not so near as Mersenne primes do. Neither does it explain why also more general primes $p \simeq 2^k$ seem to be preferred. Furthermore, p-adic length scale hypothesis generalizes and states that primes near powers of at least small primes q: $p \simeq q^k$ are special at least number theoretically. For instance, $q = 3$ seems to be important for music experience and also $q = 5$ might be important (Golden Mean)

Could it be that the proposed model relying on criticality generalizes. There would be $p < 2^k$-dimensional state space allowing isometric embedding to 2^k-dimensional space such that the bit configurations orthogonal to the image would be unstable in some sense. Say against a phase transition changing the direction of magnetization. One can imagine the variants of above described mechanism also now. For $q > 2$ one should consider pinary digits instead of bits but the same arguments would apply (except in the case of Boolean logic).

8.10.4 Brain and Mersenne integers

I received a link to an interesting the article "Brain Computation Is Organized via Power-of-Two-Based Permutation Logic" by Kun Xie, Grace E. Fox, Jun Liu, Cheng Lyu, Jason C. Lee, Hui Kuang, Stephanie Jacobs, Meng Li, Tianming Liu, Sen Song and Joe Z. Tsien in Frontiers in Systems Neuroscience [?]see `http://tinyurl.com/zfymqrq`).

The proposed model is about how brain classifies neuronal inputs. The following represents my attempt to understand the model of the article.

1. One can consider a situation in which one has n inputs identifiable as bits: bit could correspond to neuron firing or not. The question is however to classify various input combinations. The obvious criterion is how many bits are equal to 1 (corresponding neuron fires). The input combinations in the same class have same number of firing neurons and the number of subsets with k elements is given by the binomial coefficient $B(n,k) = n!/k!(n-k)!$. There are clearly $n-1$ different classes in the classification since no neurons firing is not a possible observation. The conceptualization would tell how many neurons fire but would not specify which of them.

2. To represent these bit combinations one needs $2^n - 1$ neuron groups acting as unit representing one particular firing combination. These subsets with k elements would be mapped to neuron cliques with k firing neutrons. For given input individual firing neurons ($k = 1$) would represent features, lowest level information. The n cliques with $k = 2$ neurons would represent a more general classification of input. One obtains $M_n = 2^n - 1$ combinations of firing neurons since the situations in which no neurons are firing is not counted as an input.

3. If all neurons are firing then all the however level cliques are also activated. Set theoretically the subsets of set partially ordered by the number of elements form an inclusion hierarchy, which in Boolean algebra corresponds to the hierarchy of implications in opposite direction. The clique with all neurons firing correspond to the most general statement implying all the lower level statements. At k:th level of hierarchy the statements are inconsistent so that one has $B(n, k)$ disjoint classes.

The $M_n = 2^n - 1$ (Mersenne number) labelling the algorithm is more than familiar to me.

1. For instance, electron's p-adic prime corresponds to Mersenne prime $M_{127} = 2^{127} - 1$, the largest not completely super-astrophysical Mersenne prime for which the mass of particle would be extremely small. Hadron physics corresponds to M_{107} and M_{89} to weak bosons and possible scaled up variant of hadron physics with mass scale scaled up by a factor 512 ($= 2^{(107-89)/2}$). Also Gaussian Mersennes seem to be physically important: for instance, muon and also nuclear physics corresponds to $M_{G,n} = (1 + i)^n - 1$, $n = 113$.

2. In biology the Mersenne prime $M_7 = 2^7 - 1$ is especially interesting. The number of statements in Boolean algebra of 7 bits is 128 and the number of statements that are consistent with given atomic statement (one bit fixed) is $2^6 = 64$. This is the number of genetic codons which suggests that the letters of code represent 2 bits. As a matter of fact, the so called Combinatorial Hierarchy $M(n) = M_{M(n-1)}$ consists of Mersenne primes $n = 3, 7, 127, 2^{127} - 1$ and would have an interpretation as a hierarchy of statements about statements about ... It is now known whether the hierarchy continues beyond M_{127} and what it means if it does not continue. One can ask whether M_{127} defines a higher level code - memetic code as I have called it - and realizable in terms of DNA codon sequences of 21 codons [L14] (see `http://tinyurl.com/jukyq6y`).

3. The Gaussian Mersennes $M_{G,n}$ $n = 151, 157, 163, 167$, can be regarded as a number theoretical miracles since the these primes are so near to each other. They correspond to p-adic length scales varying between cell membrane thickness 10 nm and cell nucleus size 2.5 μm and should be of fundamental importance in biology. I have proposed that p-adically scaled down variants of hadron physics and perhaps also weak interaction physics are associated with them.

I have made attempts to understand why Mersenne primes M_n and more generally primes near powers of 2 seem to be so important physically in TGD Universe.

1. The states formed from n fermions form a Boolean algebra with 2^n elements, but one of the elements is vacuum state and could be argued to be non-realizable. Hence Mersenne number $M_n = 2^n - 1$. The realization as algebra of subsets contains empty set, which is also physically non-realizable. Mersenne primes are especially interesting as sine the reduction of statements to prime nearest to M_n corresponds to the number $M_n - 1$ of physically representable Boolean statements.

2. Quantum information theory suggests itself as explanation for the importance of Mersenne primes since M_n would correspond the number of physically representable Boolean statements of a Boolean algebra with n-elements. The prime $p \leq M_n$ could represent the number of elements of Boolean algebra representable p-adically [L19] (see `http://tinyurl.com/gp9mspa`).

3. In TGD Fermion Fock states basis has interpretation as elements of quantum Boolean algebra and fermionic zero energy states in ZEO expressible as superpositions of pairs of states with

same net fermion numbers can be interpreted as logical implications. WCW spinor structure would define quantum Boolean logic as "square root of Kähler geometry". This Boolean algebra would be infinite-dimensional and the above classification for the abstractness of concept by the number of elements in subset would correspond to similar classification by fermion number. One could say that bosonic degrees of freedom (the geometry of 3-surfaces) represent sensory world and spinor structure (many-fermion states) represent that logical thought in quantum sense.

4. Fermion number conservation would seem to represent an obstacle but in ZEO it can circumvented since zero energy states can be superpositions of pair of states with opposite fermion number F at opposite boundaries of causal diamond (CD) in such a way that F varies. In state function reduction however localization to single value of F is expected to happen usually. If superconductors carry coherent states of Cooper pairs, fermion number for them is ill defined and this makes sense in ZEO but not in standard ontology unless one gives up the super-selection rule that fermion number of quantum states is well-defined.

One can of course ask whether primes n defining Mersenne primes (see `http://tinyurl.com/l3lxe2n`) could define preferred numbers of inputs for subsystems of neurons. This would predict $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257, ..$ define favoured numbers of inputs. $n = 127$ would correspond to memetic code.

8.11 Number Theoretical Feats and TGD Inspired Theory of Consciousness

Number theoretical feats of some mathematicians like Ramanujan remain a mystery for those believing that brain is a classical computer. Also the ability of idiot savants - lacking even the idea about what prime is - to factorize integers to primes challenges the idea that an algorithm is involved. In this article I discuss ideas about how various arithmetical feats such as partitioning integer to a sum of integers and to a product of prime factors might take place. The ideas are inspired by the number theoretic vision about TGD suggesting that basic arithmetics might be realized as naturally occurring processes at quantum level and the outcomes might be "sensorily perceived". One can also ask whether zero energy ontology (ZEO) could allow to perform quantum computations in polynomial instead of exponential time.

The indian mathematician Srinivasa Ramanujan is perhaps the most well-known example about a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess. A possible TGD based explanation of this feat relies on the idea that in zero energy ontology (ZEO) quantum computation like activity could consist of steps consisting quantum computation and its time reversal with long-lasting part of each step performed in reverse time direction at opposite boundary of causal diamond so that the net time used would be short at second boundary.

The adelic picture about state function reduction in ZEO suggests that it might be possible to have direct sensory experience about prime factorization of integers [L18]. What about partitions of integers to sums of primes? For years ago I proposed that symplectic QFT is an essential part of TGD. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian Wilson line. I could not specify at that time how to select these polygons. A very natural manner to fix the vertices of polygon (or polygons) is to assume that they correspond ends of fermion lines which appear as boundaries of string world sheets. The polygons would be fixed rather uniquely by requiring that fermions reside at their vertices.

The number 1 is the only prime for addition so that the analog of prime factorization for sum is not of much use. Polygons with $n = 3, 4, 5$ vertices are special in that one cannot decompose them to non-degenerate polygons. Non-degenerate polygons also represent integers $n > 2$. This inspires the idea about numbers $\{3,4,5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. These polygons could be associated many-fermion states with negentropic entanglement (NE) - this notion relate to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define conscious representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition.

8.11.1 How Ramanujan did it?

Lubos Motl wrote recently a blog posting (http://tinyurl.com/zduu72p) about $P \neq NP$ computer in the theory of computation based on Turing's work. This unproven conjecture relies on a classical model of computation developed by formulating mathematically what the women doing the hard computational work in offices at the time of Turing did. Turing's model is extremely beautiful mathematical abstraction of something very every-daily but does not involve fundamental physics in any manner so that it must be taken with caution. The basic notions include those of algorithm and recursive function, and the mathematics used in the model is mathematics of integers. Nothing is assumed about what conscious computation is and it is somewhat ironic that this model has been taken by strong AI people as a model of consciousness!

1. A canonical model for classical computation is in terms of Turing machine, which has bit sequence as inputs and transforms them to outputs and each step changes its internal state. A more concrete model is in terms of a network of gates representing basic operations for the incoming bits: from this basic functions one constructs all recursive functions. The computer and program actualize the algorithm represented as a computer program and eventually halts - at least one can hope that it does so. Assuming that the elementary operations require some minimum time, one can estimate the number of steps required and get an estimate for the dependence of the computation time as function of the size of computation.

2. If the time required by a computation, whose size is characterized by the number N of relevant bits, can be carried in time proportional to some power of N and is thus polynomial, one says that computation is in class P. Non-polynomial computation in class NP would correspond to a computation time increasing with N faster than any power of N, say exponentially. Donald Knuth, whose name is familiar for everyone using Latex to produce mathematical text, believes on $P = NP$ in the framework of classical computation. Lubos in turn thinks that the Turing model is probably too primitive and that quantum physics based model is needed and this might allow $P = NP$.

What about quantum computation as we understand it in the recent quantum physics: can it achieve $P = NP$?

1. Quantum computation is often compared to a superposition of classical computations and this might encourage to think that this could make it much more effective but this does not seem to be the case. Note however that the amount of information represents by N qubits is however exponentially larger than that represented by N classical bits since entanglement is possible. The prevailing wisdom seems to be that in some situations quantum computation can be faster than the classical one but that if $P = NP$ holds true for classical computation, it holds true also for quantum computations. Presumably because the model of quantum computation begins from the classical model and only (quantum computer scientists must experience this statement as an insult - apologies!) replaces bits with qubits.

2. In quantum computer one replaces bits with entangled qubits and gates with quantum gates and computation corresponds to a unitary time evolution with respect to a discretized time parameter constructed in terms of fundamental simple building bricks. So called tensor networks realize the idea of local unitary in a nice manner and has been proposed to defined error correcting quantum codes. State function reduction halts the computation. The outcome is non-deterministic but one can perform large number of computations and deduce from the distribution of outcomes the results of computation.

What about conscious computations? Or more generally, conscious information processing. Could it proceed faster than computation in these sense of Turing? To answer this question one must first try to understand what conscious information processing might be. TGD inspired theory of consciousnesss provides one a possible answer to the question involving not only quantum physics but also new quantum physics.

1. In TGD framework Zero energy ontology (ZEO) replaces ordinary positive energy ontology and forces to generalize the theory of quantum measurement. This brings in several new elements. In particular, state function reductions can occur at both boundaries of causal diamond (CD), which is intersection of future and past direct light-cones and defines a geometric correlate for self. Selves for a fractal hierarchy - CDs within CDs and maybe also overlapping. Negentropy Maximization Principle (NMP) is the basic variational principle of consciousness and tells that the state function reductions generate maximum amount of conscious information. The notion of negentropic entanglement (NE) involving p-adic physics as physics of cognition and hierarchy of Planck constants assigned with dark matter are also central elements.

2. NMP allows a sequence of state function reductions to occur at given boundary of diamond-like CD - call it passive boundary. The state function reduction sequence leaving everything unchanged at the passive boundary of CD defines self as a generalized Zeno effect. Each step shifts the opposite - active - boundary of CD "upwards" and increases its distance from the passive boundary. Also the states at it change and one has the counterpart of unitary time evolution. The shifting of the active boundary gives rise to the experienced time flow and sensory input generating cognitive mental images - the "Maya" aspect of conscious experienced. Passive boundary corresponds to permanent unchanging "Self".

3. Eventually NMP forces the first reduction to the opposite boundary to occur. Self dies and reincarnates as a time reversed self. The opposite boundary of CD would be now shifting "downwards" and increasing CD size further. At the next reduction to opposite boundary re-incarnation of self in the geometric future of the original self would occur. This would be re-incarnation in the sense of Eastern philosophies. It would make sense to wonder whose incarnation in geometric past I might represent!

Could this allow to perform fast quantal computations by decomposing the computation to a sequence in which one proceeds in both directions of time? Could the incredible feats of some "human computers" rely on this quantum mechanism (see `http://tinyurl.com/hk5baty`). The indian mathematician Srinivasa Ramanujan (see `http://tinyurl.com/l42q7a2`) is the most well-known example of a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess.

Might it be possible in ZEO to perform quantally computations requiring classically non-polynomial time much faster - even in polynomial time? If this were the case, one might at least try to understand how Ramanujan did it although higher levels selves might be involved also (did his Goddess do the job?).

1. Quantal computation would correspond to a state function reduction sequence at fixed boundary of CD defining a mathematical mental image as sub-self. In the first reduction to the opposite boundary of CD sub-self representing mathematical mental image would die and quantum computation would halt. A new computation at opposite boundary proceeding to opposite direction of geometric time would begin and define a time-reversed mathematical mental image. This sequence of reincarnations of sub-self as its time reversal could give rise to a sequence of quantum computation like processes taking less time than usually since one half of computations would take place at the opposite boundary to opposite time direction (the size of CD increases as the boundary shifts).

2. If the average computation time is same at both boundaries, the computation time would be only halved. Not very impressive. However, if the mental images at second boundary

- call it A - are short-lived and the selves at opposite boundary B are very long-lived and represent very long computations, the process could be very fast from the point of view of A! Could one overcome the $P \neq NP$ constraint by performing computations during time-reversed re-incarnations?! Short living mental images at this boundary and very long-lived mental images at the opposite boundary - could this be the secret of Ramanujan?

3. Was the Goddess of Ramanujan - self at higher level of self-hierarchy - nothing but a time reversal for some mathematical mental image of Ramanujan (Brahman=Atman!), representing very long quantal computations! We have night-day cycle of personal consciousness and it could correspond to a sequence of re-incarnations at some level of our personal self-hierarchy. Ramanujan tells that he met his Goddess in dreams. Was his Goddess the time reversal of that part of Ramanujan, which was unconscious when Ramanujan slept? Intriguingly, Ramanujan was rather short-lived himself - he died at the age of 32! In fact, many geniuses have been rather short-lived.

4. Why the alter ego of Ramanujan was Goddess? Jung intuited that our psyche has two aspects: anima and animus. Do they quite universally correspond to self and its time reversal? Do our mental images have gender?! Could our self-hierarchy be a hierarchical collection of anima and animi so that gender would be something much deeper than biological sex! And what about Yin-Yang duality of Chinese philosophy and the ka as the shadow of persona in the mythology of ancient Egypt?

8.11.2 Symplectic QFT, $\{3,4,5\}$ as Additive Primes, and Arithmetic Consciousness

For years ago I proposed that symplectic QFT is an essential part of TGD [K19, K96]. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian Wilson line. I could not specify at that time how to select these polygons in the case of partonic 2-surfaces.

The recent proposal of Maldacena and Arkani-Hamed [B26] (see `http://tinyurl.com/ych26gcm`) that CMB might contain signature of inflationary cosmology as triangles and polygons for which the magnitude of n-point correlation function is enhanced led to a progress in this respect. In the proposal of Maldacena and Arkani-Hamed the polygons are defined by momentum conservation. Now the polygons would be fixed rather uniquely by requiring that fermions reside at their vertices and momentum conservation is not involved.

This inspires the idea about numbers $\{3,4,5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. Geometrically one could speak of prime polygons not decomposable to lower non-degenerate polygons. These polygons are different from those of Maldacena and Arkani-Hamed and would be associated many-fermion states with negentropic entanglement (NE) - this notion relates to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition. I have already earlier considered a possible counterpart for conscious prime factorization in the adelic framework [L18].

Basic ideas of TGD inspired theory of conscious very briefly

Negentropy Maximization Principle (NMP) is the variational principle of consciousness in TGD framework. It says that negentropy gain in state function reduction (quantum jump re-creating Universe) is maximal. State function reduction is basically quantum measurement in standard QM and sensory qualia (for instance) could be perhaps understood as quantum numbers of state resulting in state function reduction. NMP poses conditions on whether this reduction can occur. In standard ontology it would occur always when the state is entangled: reduction would destroy the entanglement and minimize entanglement entropy. When cognition is brought in, the situation changes.

The first challenge is to define what negentropic entanglement (NE) and negentropy could mean.

1. In real physics without cognition one does not have any definition of negentropy: on must define negentropy as reduction of entropy resulting as conscious entity gains information. This kind of definition is circular in consciousness theory.

2. In p-adic physics one can define number theoretic entanglement entropy with same basic properties as ordinary Shannon entropy. For some p-adic number fields this entropy can be negative and this motivates an interpretation as conscious information related to entanglement - rather to the ignorance of external observer about entangled state. The prerequisite is that the entanglement probabilities belong to an an extension of rationals inducing a finite-dimensional extension of rationals. Algebraic extensions are such extensions as also those generate by a root of e (e^p is p-adic number in Q_p).

A crucial step is to fuse together sensory and cognitive worlds as different aspects of existence.

1. One must replace real universe with adelic one so that one has real space-time surfaces and their p-adic variants for various primes p satisfying identical field equations. These are related by strong form of holography (SH) in which 2-D surfaces (string world sheets and partonic 2-surfaces) serve as "space-time genes" and obey equations which make sense both p-adically in real sense so that one can identify them as points of "world of classical worlds" (WCW).

2. One can say that these 2-surfaces belong to intersection of realities and p-adicities - intersection of sensory and cognitive. This demands that the parameters appearing in the equations for 2-surface belong algebraic extension of rational numbers: the interpretation is that this hierarchy of extensions corresponds to evolutionary hierarchy. This also explains imagination in terms of the p-adic space-time surfaces which are not so unique as the real one because of inherent non-determinism of p-adic differental equations. What can be imagined cannot be necessarily realized. You can continued p-adic 2-surface to 4-D surface but not to real one.

There is also second key assumption involved.

1. Hilbert space of quantum states is *same* for real and p-adic sectors of adelic world: for instance, tensor product would lead to total nonsense since there would be both real and p-adic fermions. This means same quantum state and same entanglement but seen from sensory and various cognitive perspectives. This is the basic idea of adelicity: the p-adic norms of rational number characterize the norm of rational number. Now various p-adic conscious experiences characterize the quantum state.

2. Real perspective sees entanglement always as entropic. For some finite number number of primes p p-adic entanglement is however negentropic. For instance, for entanglement probabilities $p_i = 1/N$, the primes appearing as factors of N are such information carrying primes. The presence of these primes can make the entanglement stable. The total entropy equal to the sum of negative real negentropy + various p-adic negentropies can be positive and cannot be reduced in the reduction so that reduction does not occur at all! Entanglement is stabilized by cognition and the randomness of state function reduction tamed: matter has power over matter!

3. There is analogy with the reductionism-holism dichotomy. Real number based view is reductionistic: information is obtained when the entangled state is split into un-entangled part. p-Adic number based view is holistic: information is inthe negentropic entanglement and can be seen as abstraction or rule. The superposition of state pairs represents a rule with state pairs (a_i, b_i) representing the instance of the rule A↔B. Maximal entanglement defined by entanglement probabilities $p_i = 1/N$ makes clear the profound distinction between these views. In real sector the negentropy is negative and smallest possible. In p-adic sector the negentropy is maximum for p-adic primes appearing as factors of N and total negentropy as their sum is large. NE allows to select unique state basis if the probabilities p_i are different.

   For $p_i = 1/N$ one can choose any unitary related state basis since unit matrix is invariant under unitary transformations. From the real point of view the ignorance is maximal and entanglement entropy is indeed maximal. For instance, in case of Schrödinger cat one could choose the cat's state basis to be any superposition of dead and alive cat and a state orthogonal to it. From p-adic view information is maximal. The reports of meditators, in particular Zen buddhists, support this interpretation. In "enlightened state" all discriminations disappear: it does not make sense to speak about dead or alive cat or anything between these two options. The state contains information about entire state - not about its parts. It is not information expressible using language relying on making of distinctions but silent wisdom.

How do polygons emerge in TGD framework?

The duality defined by strong form of holography (SH) has 2 sides. Space-time side (bulk) and boundary side (string world sheets and partonic 2-surfaces). 2-D half of SH would suggest a description based on string world sheets and partonic 2-surfaces. This description should be especially simple for the quantum states realized as spinor fields in WCW ("world of classical worlds"). The spinors (as opposed to spinor fields) are now fermionic Fock states assignable to space-time surface defining a point of WCW. TGD extends ordinary 2-D conformal invariance to super-symplectic symmetry applying at the boundary of light-cone: note that given boundary of causal diamond (CD) is contained by light-cone boundary.

1. The correlation functions at embedding space level for fundamental objects, which are fermions at partonic 2-surfaces could be calculated by applying super-symplectic invariance having conformal structure. I have made rather concrete proposals in this respect. For instance, I have suggested that the conformal weights for the generators of supersymplectic algebra are given by poles of fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ and thus include zeros of zeta scaled down by factor 1/2 [K35]. A related proposal is conformal confinement guaranteeing the reality of net conformal weights.

2. The conformally invariant correlation functions are those of super-symplectic CFT at light-cone boundary or its extension to CD. There would be the analog of conformal invariance associated with the light-like radial coordinate r_M and symplectic invariance associated with CP_2 and sphere S^2 localized with respect to r_M analogous to the complex coordinate in ordinary conformal invariance and naturally continued to hypercomplex coordinate at string world sheets carrying the fermionic modes and together with partonic 2-surfaces defining the boundary part of SH.

Symplectic invariants emerge in the following manner. Positive and negative energy parts of zero energy states would also depend on zero modes defined by super-symplectic invariants and this brings in polygons. Polygons emerge also from four-momentum conservation. These of course are also now present and involve the product of Lorentz group and color group assignable to CD near its either boundary. It seems that the extension of Poincare translations to Kac-Moody type symmetry allows to have full Poincare invariance (in its interior CD looks locally like $M^4 \times CP_2$).

1. One can define the symplectic invariants as magnetic fluxes associated with S^2 and CP_2 Kähler forms. For string world sheets one would obtain non-integrable phase factors. The vertices of polygons defined by string world sheets would correspond to the intersections of the string world sheets with partonic 2-surfaces at the boundaries of CD and at partonic 2-surfaces defining generalized vertices at which 3 light-like 3-surfaces meet along their ends.

2. Any polygon at partonic 2-surface would also allow to define such invariants. A physically natural assumption is that the vertices of these polygons are realized physically by adding fermions or antifermions at them. Kähler fluxes can be expressed in terms of non-integrable phase factors associated with the edges. This assumption would give the desired connection with quantum physics and fix highly uniquely but not completely the invariants appearing in physical states.

The correlated polygons would be thus naturally associated with fundamental fermions and a better analogy would be negentropically entangled n-fermion state rather than corresponding to maximum of the modulus of n-point correlation function. Hierarchy of Planck constants makes these states possible even in cosmological scales. The point would be that negentropic entanglement assignable to the p-adic sectors of WCW would be in key role.

Symplectic invariants and Abelian non-integrable phase factors

Consider now the polygons assignable to many-fermion states at partonic 2-surfaces.

1. The polygon associated with a given set of vertices defined by the position of fermions is far from unique and different polygons correspond to different physical situations. Certainly one must require that the geodesic polygon is not self-intersecting and defines a polygon or set of polygons.

2. Geometrically the polygon is not unique unless it is convex. For instance, one can take regular n-gon and add one vertex to its interior. The polygon can be also constructed in several ways. From this one obtains a non-convex $n+1$-gon in $n+1$ ways.

3. Given polygon is analogous with Hamiltonian cycle connecting all points of given graph. Now one does not have graph structure with edges and vertices unless one defines it by nearest neighbor property. Platonic solids provide an example of this kind of situation. Hamiltonian cycles [A5, A16] are key element in the TGD inspired model for music harmony leading also to a model of genetic code [K80] [L7].

4. One should somehow fix the edges of the polygon. For string world sheets the edges would be boundaries of string world sheet. For partonic 2-surfaces the simplest option is that the edges are geodesic lines and thus have shortest possible length. This would bring in metric so that the idea about TGD as almost topological QFT would be realized.

One can distinguish between two cases: single polygon or several polygons.

1. One has maximal entanglement between fundamental fermions, when the vertices define single polygon. One can however have several polygons for a given set of vertices and in this case the coherence is reduced. Minimal correlations correspond to maximal number of 3-gons and minimal number of 4-gons and 5-gons.

2. For large $h_{eff} = n \times h$ the partonic 2-surfaces can have macroscopic and even astrophysical size and one can consider assigning many-fermion states with them. For instance, anyonic states could be interpreted in this manner. In this case it would be natural to consider various decompositions of the state to polygons representing entangled fermions.

The definition of symplectic invariant depends on whether one has single polygon or several polygons.

1. In the case that there are several polygons not containing polygons inside them (if this the case,then the complement of polygon must satisfy the condition) one can uniquely identify the interior of each polygon and assign a flux with it. Non-integrable phase factor is well-defined now. If there is only single polygon then also the complement of polygon could define the flux. Polygon and its complement define fluxes Φ and $\Phi_{tot} - \Phi$.

2. If partonic 2-surface carries monopole Kähler charge Φ_{tot} is essentially $n\pi$, where n is magnetic monopole flux through the partonic 2-surface. This is half integer - not integer: this is key feature of TGD and forces the coupling of Kähler gauge potential to the spinors leading to the quantum number spectrum of standard model. The exponent can be equal to -1 for half-odd integer.

   This problem disappears if both throats of the wormhole contact connecting the space-time sheets with Minkowski signature give their contribution so that two minus-signs give one plus sign. Elementary particles necessarily consist of wormhole contacts through which monopole flux flows and runs along second space-time sheet to another contact and returns along second space-time sheet so that closed monopole flux tube is obtained. The function of the flux must be single valued. This demands that it must reduce to the cosine of the integer multiple of the flux and identifiable as as the real part of the integer power of magnetic flux through the polygon.

The number theoretically deepest point is geometrically completely trivial.

1. Only $n > 2$-gons are non-degenerate and 3-, 4- and 5-gons are prime polygons in the sense that they cannot be sliced to lower polygons. Already 6-gon decomposes to 2 triangles.

2. One can wonder whether the appearance of 3 prime polygons might relate to family replication phenomenon for which TGD suggests an explanation in terms of genus of the partonic 2-surface [K23]. This does not seem to be the case. There is however other three special integers: namely 0,1, and 2.

   The connection with family replication phenomenon could be following. When the number of handles at the parton surface exceeds 2, the system forms entangled/bound states describable in terms of polygons with handles at vertices. This would be kind of phase transition. Fundamental fermion families with handle number 0,1,2 would be analogous to integers 0,1,2 and the anyonic many-handle states with NE would be analogous to partitions of integers $n > 2$ represented by the prime polygons. They would correspond to the emergence of p-adic cognition. One could not assign NE and cognition with elementary particles but only to more complex objects such as anyonic states associated with large partonic 2-surfaces (perhaps large because they have large Planck constant $h_{eff} = n \times h$) [K73].

Integers $(3, 4, 5)$ as "additive primes" for integers $n \geq 3$: a connection with arithmetic consciousness

The above observations encourage a more detailed study of the decomposition of polygons to smaller polygons as a geometric representation for the partition of integers to a sum of smaller integers. The idea about integers $(3, 4, 5)$ as "additive primes" represented by prime polygons is especially attractive. This leads to a conjecture about NE associated with polygons as quantum correlates of arithmetic consciousness.

1. Motivations

The key idea is to look whether the notion of divisibility and primeness could have practical value in additive arithmetics. 1 is the only prime for addition in general case. $n = 1 + 1 + ...$ is analogous to p^n and all integers are "additive powers" of 1.

What happens if one considers integers $n \geq 3$? The basic motivation is that $n \geq 3$ is represented as a non-degenerate n-gon for $n \geq 3$. Therefore geometric representation of these primes is used in the following. One cannot split triangles from 4-gon and 5-gon. But already for 6-gon one can and obtains 2 triangles. Thus $\{3, 4, 5\}$ would be the additive primes for $n \geq 3$ represented as prime polygons.

The n-gons with $n \in \{3, 4, 5\}$ appear as faces of the Platonic solids! The inclusions of von Neumann algebras known as hyperfinite factors of type II_1 central in TGDs correspond to quantum phases $exp(\pi/n)$ $n = 3, 4, 5....$ Platonic solids correspond to particular finite subgroups of 3-D rotation group, which are in one-one correspondence with simply laced Lie-groups (ADE). There is also a direct connection with the classification of $\mathcal{N} = 2$ super-conformal theories, which seem to be relevant for TGD.

I cannot resist the temptation to mention also a personal reminiscence about a long lasting altered state of consciousness about 3 decades ago. I called it Great Experience and it boosted among other things serious work in order to understand consciousness in terms of quantum physics. One of the mathematical visions was that number 3 is in some sense fundamental for physics and mathematics. I also precognized infinite primes and much later indeed discovered them. I have repeatedly returned to the precognition about number 3 but found no really convincing reason for its unique role although it pops up again and again in physics and mathematics: 3 particle families, 3 colors for quarks, 3 spatial dimensions, 3 quaternionic imaginaryunits, triality for octonions, to say nothing about the role of trinity in mystics and religions. The following provides the first argument for the special role of number 3 that I can take seriously.

2. Partition of integer to additive primes

The problem is to find a partition of an integer to additive primes $3, 4, 5$. The problem can be solved using a representation in terms of $n > 2$-gons as a geometrical visualization. Some general aspects of the representation.

1. The detailed shape of n-gons in the geometric representation of partitions does not matter: they just represent geometrically a partition of integer to a sum. The partition can be regarded as a dynamical process. n-gons splits to smaller n-gons producing a representation for a partition $n = \sum_i n_i$. What this means is easiest to grasp by imagining how polygon can be decomposed to smaller ones. Interestingly, the decompositions of polytopes to smaller ones - triangulations - appear also in Grassmannian twistor approach to $\mathcal{N} = 4$ super Yang Mills theory.

2. For a given partition the decomposition to n-gons is not unique. For instance, integer 12 can be represented by 3 4-gons or 4 3-gons. Integers $n \in \{3, 4, 5\}$ are special and partitions to these n-gons are in some sense maximal leading to a maximal decoherence as quantum physicist might say.

   The partitions are not unique and there is large number of partitions involving 3-gons,4-gons, 5-gons. The reason is that one can split from n-gons any n_1-gon with $n_1 < n$ except for $n = 3, 4, 5$.

3. The daydream of non-mathematician not knowing that everything has been very probably done for aeons ago is that one could chose n_1 to be indivisible by 4 and 5, n_2 indivisible by 3 and 5 and n_3 indivisible by 3 and 4 so that one might even hope for having a unique partition. For instance, double modding by 4 and 5 would reduce to double modding of $n_1 \times 3$ giving a non-vanishing result, and one might hope that n_1, n_2 and n_3 could be determined from the double modded values of n_i uniquely. Note that for $n_i \in \{1, 2\}$ the number $n = 24 = 2 \times 3 + 2 \times 4 + 2 \times 5$ playing key role in string model related mathematics is the largest integer having this kind of representation. One should numerically check whether any general orbit characterized by the above formulas contains a point satisfying the additional number theoretic conditions.

   Therefore the task is to find partitions satisfying these indivisibility conditions. It is however reasonable to consider first general partitions.

4. By linearity the task of finding general partitions (forgetting divisibility conditions) is analogous to that of finding of solutions of non-homogenous linear equations. Suppose that one has found a partition

   $$n = n_1 \times 3 + n_2 \times 4 + n_3 \times 5 \leftrightarrow (n_1, n_2, n_3) \ . \tag{8.11.1}$$

   This serves as the analog for the special solution of non-homogenous equation. One obtains a general solutions of equation as the sum $(n_1 + k_1, n_2 + k_1, n_3 + k_3)$ of the special solution and general solution of homogenous equation

   $$k_1 \times 3 + k_2 \times 4 + k_3 \times 5 = 0 \ . \tag{8.11.2}$$

   This is equation of plane in N^3 - 3-D integer lattice.

   Using $4 = 3 + 1$ and $5 = 3 + 2$ this gives equations

   $$k_2 + 2 \times k_3 = 3 \times m \ , \quad k_1 - k_3 + 4 \times m = 0 \ , \quad m = 0, 1, 2, ... \tag{8.11.3}$$

5. There is periodicity of $3 \times 4 \times 5 = 60$. If (k_1, k_2, k_3, m) is allowed deformation, one obtains a new one with same divisibility properties as the original one as $(k_1 + 60, k_2 - 120, k_3 + 60, m)$. If one does not require divisibility properties for all solutions, one obtains much larger set of solutions. For instance $(k_1, k_2, k_3) = m \times (1, -2, 1)$ defines a line in the plane containing the solutions. Also other elementary moves than (1,-2,1) are possible.

One can identify very simple partitions deserving to be called standard partitions and involve mostly triangles and minimal number of 4- and 5-gons. The physical interpretation is that the coherence is minimal for them since mostly the quantum coherent negentropically entangled units are minimal triangles.

1. One starts from n vertices and constructs n-gon. For number theoretic purposes the shape does not matter and the polygon can be chosen to be convex. One slices from it 3-gons one by one so that eventually one is left with $k \equiv n \ mod \ 3 == 0, 1$ or 2 vertices. For $k = 0$ no further operations are needed. For $k = 1$ *resp.* $k = 2$ one combines one of the triangles and edge associated with 1 *resp.* 2 vertices to 4-gon *resp.* 5-gon and is done. The outcome is one of the partitions

$$n = n_1 \times 3 \ , \quad n = n_1 \times 3 + 4, n = n_1 \times 3 + 5 \tag{8.11.4}$$

   These partitions are very simple, and one can easily calculate similar partitions for products and powers. It is easy to write a computer program for the products and powers of integers in terms of these partitions.

2. There is however a uniqueness problem. If n_1 is divisible by 4 or 5 - $n_1 = 4 \times m_1$ or $n_1 = 5 \times m_1$ - one can interpret $n_1 \times 3$ as a collection of m_1 4-gons or 5-gons. Thus the geometric representation of the partition is not unique. Similar uniqueness condition must apply to n_2 and n_3 and is trivially true in above partitions.

   To overcome this problem one can pose a further requirement. If one wants n_1 to be indivisible by 4 and 5 one can transform 2 or or 4 triangles and existing 4-gon or 5-gon or 3 or 6 triangles to 4-gons and 5-gons.

   (a) Suppose $n = n_1 \times 3 + 4$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 2$ is not and 4-gon and 2 3-gons can be transformed to 2 5-gons: $(n_1, 1, 0) \rightarrow (n_1 - 2, 0, 2)$. If $n_1 - 2$ is divisible by 5, $n_1 - 3$ is not divisible by either 4 or 5 and 3 triangles can be transformed to 4-gon and 5-gon: $(n_1, 1, 0) \rightarrow (n_1 - 3, 2, 1)$.

   (b) Suppose $n = n_1 \times 3 + 5$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 1$ is not and triangle and 5-gon can be transformed to 2 4-gons: $(n_1, 0, 1) \rightarrow (n_1 - 1, 2, 0)$. If $n_1 - 1$ is divisible by 4 or 5, $n_1 - 3$ is not and 3 triangles and 5-gon can be transformed to 2 5-gons and 4-gon: $(n_1, 0, 1) \rightarrow (n_1 - 3, 1, 2)$.

   (c) For $n = n_1 \times 3$ divisible by 4 or 5 or both one can remove only $m \times 3$ triangles, $m \in \{1, 2\}$ since only in these case the resulting $m \times 3$ (9 or 18) vertices can partitioned to a union of 4-gon and 5-gon or of 2 4-gons and 2 5-gons: $(n_1, 0, 0) \rightarrow (n_1 - 3, 1, 1)$ or $(n_1, 0, 0) \rightarrow (n_1 - 6, 2, 2)$.

   These transformations seem to be the minimal transformations allowing to achieve indivisibility by starting from the partition with maximum number of triangles and minimal coherence.

Some further remarks about the partitions satisfying the divisibility conditions are in order.

1. The multiplication of n with partition (n_1, n_2, n_3) satisfying indivisibility conditions by an integer m not divisible by $k \in \{3, 4, 5\}$ gives integer with partition $m \times (n_1, n_2, n_3)$. Note also that if n is not divisible by $k \in \{3, 4, 5\}$ the powers of n, n^k has partition $n^{k-1} \times (n_1, n_2, n_3)$ and this could help to solve Diophantine equations.

2. Concerning the uniqueness of the partition satisfying the indivisibility conditions, the answer is negative. $8 = 3 + 5 = 4 + 4$ is the simplest counter example. Also the m-multiples of 8 such that m is indivisible by 2,3,4,5 serve as counter examples. 60-periodicity implies that for sufficiently large values of n the indivisibility conditions do not fix the partition uniquely. (n_1, n_2, n_3) can be replaced with $(n_1 + 60 + n_2 - 120, n_3 + 60)$ without affecting divisibility properties.

3. Intriguing observations related to 60-periodicity

60-periodicity seems to have deep connections with both music consciousness and genetic code if the TGD inspired model of genetic code is taken seriously code [K80] [L7].

1. The TGD inspired model for musical harmony and genetic involves icosahedron with 20 triangular faces and tetrahedron with 4 triangular faces. The 12 vertices of icosahedron correspond to the 12 notes. The model leads to the number 60. One can say that there are 60 +4 DNA codons and each 20 codon group is 60=20+20+20 corresponds to a subset of aminoacids and 20 DNAs assignable to the triangles of icosahedron and representing also 3-chords of the associated harmony. The remaining 4 DNAs are associated with tetrahedron.

   Geometrically the identification of harmonies is reduced to the construction of Hamiltonian cycles - closed isometrically non-equivalent non-self-intersecting paths at icosahedron going through all 12 vertices. The symmetries of the Hamiltonian cycles defined by subgroups of the icosahedral isometry group provide a classification of harmonies and suggest that also genetic code carries additional information assignable to what I call bio-harmony perhaps related to the expression of emotions - even at the level of biomolecules - in terms of "music" defined as sequences 3-chords realized in terms of triplets of dark photons (or notes) in 1-1 correspondence with DNA codons in given harmony.

2. Also the structure of time units and angle units involves number 60. Hour consists of 60 minutes, which consists of 60 seconds. Could this accident somehow reflect fundamental aspects of cognition? Could we be performing sub-conscious additive arithmetics using partitions of n-gons? Could it be possible to "see" the partitions if they correspond to NE?

4. Could additive primes be useful in Diophantine mathematics?

The natural question is whether it could be number theoretically practical to use "additive primes" $\{3, 4, 5\}$ in the construction of natural numbers $n \geq 3$ rather than number 1 and successor axiom. This might even provide a practical tool for solving Diophantine equations (it might well be that mathematicians have long ago discovered the additive primes).

The most famous Diophantine equation is $x^n + y^n = z^n$ and Fermat's theorem - proved by Wiles - states that for $n > 2$ it has no solutions. Non-mathematician can naïvely ask whether the proposed partition to additive primes could provide an elementary proof for Fermat's theorem and continue to test the patience of a real mathematician by wondering whether the parition for a sum of powers $n > 2$ could be always different from that for single power $n > 2$ perhaps because of some other constraints on the integers involved?

5. Could one identify quantum physical correlates for arithmetic consciousness?

Even animals and idiot savants can do arithmetics. How this is possible? Could one imagine physical correlates for arithmetic consciousness for which product and addition are the fundamental aspects? Is elementary arithmetic cognition universal and analogous to direct sensory experience. Could it reduce at quantum level to a kind of quantum measurement process quite generally giving rise to mental images as outcomes of quantum measurement by repeated state function reduction lasting as long as the corresponding sub-self (mental image) lives?

Consider a partition of integer to a product of primes first. I have proposed a general model for how partition of integer to primes could be experienced directly [L18]. For negentropically entangled state with maximal possible negentropy having entanglement probabilities $p_i = 1/N$, the negentropic primes are factors of N and they could be directly "seen" as negentropic p-adic factors in the adelic decomposition (reals and extensions of various p-adic number fields defined by extension of rationals defined the factors of adele and space-time surfaces as preferred extremals of Kähler action decompose to real and p-adic sectors).

What about additive arithmetics?

1. The physical motivation for n-gons is provided symplectic QFT [K19, K96], which is one aspect of TGD forced by super symplectic conformal invariance having structure of conformal symmetry. Symplectic QFT would be analogous to conformal QFT. The key challenge is to identify symplectic invariants on which the positive and negative energy parts of zero energy

states can depend. The magnetic flux through a given area of 2-surface is key invariant of this kind. String world sheet and partonic 2-surfaces are possible identifications for the surface containing the polygon.

Both the Kähler form associated with the light-cone boundary, which is metrically sphere with constant radius r_M (defining light-like radial coordinate) and the induced Kähler form of CP_2 define these kind of fluxes.

2. One can assign fluxes with string world sheets. In this case one has analog of magnetic flux but over a surface with metric signature (1,-1). Fluxes can be also assigned as magnetic fluxes with partonic 2-surfaces at which fundamental fermions can be said to reside. n fermions defining the vertices at partonic 2-surface define naturally an n-gon or several of them. The interpretation would be as Abelian Wilson loop or equvalently non-integrable phase factor.

3. The polygons are not completely unique but this reflect the possibility of several physical states. n-gon could correspond to NE. The imaginary exponent of Kähler magnetic flux Φ through n-gon is symplectic invariant defining a non-integrable phase factor and defines a multiplicative factor of wave function. When the state decomposes to several polygons, one can uniquely identify the interior of the polygon and thus also the non-integrable phase factor.

   There is however non-uniqueness, when one has only single n-gon since also the complement of n-gon at partonic 2-surface containing now now polygons defines n-gon and the corresponding flux is $\Phi_{tot} - \Phi$. The flux Φ_{tot} is quantized and equal to the integer valued magnetic charge times 2π. The total flux disappears in the imaginary exponent and the non-integrable phase factor for the complementary polygon reduces to complex conjugate of that for polygon. Uniqueness allows only the cosine for an integer multiple of the flux.

   The non-integrable phase factor assignable to fermionic polygon would give rise to a correlation between fermions in zero modes invariant under symplectic group. The correlations defined by the n-gons at partonic 2-surfaces would be analogous to that in momentum space implied by the momentum conservation forcing the momenta to form a closed polygon but having totally different origin.

Could it be that the wave functions representing collections of n-gons representing partition of integer to a sum could be experienced directly by people capable of perplexing mathematical feats. The partition to a sum would correspond to a geometric partition of polygon representing partition of positive integer $n \geq 3$ to a sum of integers. Quantum physically it would correspond to NE as a representation of integer.

This might explain number theoretic miracles related to addition of integers in terms of direct “seeing”. The arithmetic feats could be dynamical quantum processes in which polygons would decompose to smaller polygons, which would be directly “seen”. This would require at least two representations: the original polygon and the decomposed polygon resulting in the state function reduction to the opposite boundary of CD. An ensemble of arithmetic sub-selves would seem to be needed. NMP does not seem to favour this kind of partition since negentropy is reduced but if its time reversal occurs in geometric time direction opposite to that of self it might look like partition for the self having sub-self as mental image.

8.12 p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework

In TGD framework p-adicization and adelization are carried out at all levels of geometry: embedding space, space-time and WCW. Adelization at the level of state spaces requires that it is common from all sectors of the adele and has as coefficient field an extension of rationals allowing both real and p-adic interpretations: the sectors of adele give only different views about the same quantum state.

In the sequel the recent view about the p-adic variants of embedding space, space-time and WCW is discussed. The notion of finite measurement resolution reducing to number theoretic existence in p-adic sense is the fundamental notion. p-Adic geometries replace discrete points of discretization with p-adic analogs of monads of Leibniz making possible to construct differential calculus and formulate p-adic variants of field equations allowing to construct p-adic cognitive representations for real space-time surfaces.

This leads to a beautiful construction for the hierarchy of p-adic variants of embedding space inducing in turn the construction of p-adic variants of space-time surfaces. Number theoretical existence reduces to conditions demanding that all ordinary (hyperbolic) phases assignable to (hyperbolic) angles are expressible in terms of roots of unity (roots of e).

For $SU(2)$ one obtains as a special case Platonic solids and regular polygons as preferred p-adic geometries assignable also to the inclusions of hyperfinite factors [K110, K37]. Platonic solids represent idealized geometric objects ofthe p-adic world serving as a correlate for cognition as contrast to the geometric objects of the sensory world relying on real continuum.

In the case of causal diamonds (CDs) - the construction leads to the discrete variants of Lorentz group $SO(1,3)$ and hyperbolic spaces $SO(1,3)/SO(3)$. The construction gives not only the p-adicizable discrete subgroups of $SU(2)$ and $SU(3)$ but applies iteratively for all classical Lie groups meaning that the counterparts of Platonic solids are countered also for their p-adic coset spaces. Even the p-adic variants of WCW might be constructed if the general recipe for the construction of finite-dimensional symplectic groups applies also to the symplectic group assignable to $\Delta CD \times CP_2$.

The emergence of Platonic solids is very remarkable also from the point of view of TGD inspired theory of consciousness and quantum biology. For a couple of years ago I developed a model of music harmony [K80] [L7] relying on the geometries of icosahedron and tetrahedron. The basic observation is that 12-note scale can be represented as a closed curve connecting nearest number points (Hamiltonian cycle) at icosahedron going through all 12 vertices without self intersections. Icosahedron has also 20 triangles as faces. The idea is that the faces represent 3-chords for a given harmony characterized by Hamiltonian cycle. Also the interpretation terms of 20 amino-acids identifiable and genetic code with 3-chords identifiable as DNA codons consisting of three letters is highly suggestive.

One ends up with a model of music harmony predicting correctly the numbers of DNA codons coding for a given amino-acid. This however requires the inclusion of also tetrahedron. Why icosahedron should relate to music experience and genetic code? Icosahedral geometry and its dodecahedral dual as well as tetrahedral geometry appear frequently in molecular biology but its appearance as a preferred p-adic geometry is what provides an intuitive justification for the model of genetic code. Music experience involves both emotion and cognition. Musical notes could code for the points of p-adic geometries of the cognitive world. The model of harmony in fact generalizes. One can assign Hamiltonian cycles to any graph in any dimension and assign chords and harmonies with them. Hence one can ask whether music experience could be a form of p-adic geometric cognition in much more general sense.

The geometries of biomolecules brings strongly in mind the geometry p-adic space-time sheets. p-Adic space-time sheets can be regarded as collections of p-adic monad like objects at algebraic space-time points common to real and p-adic space-time sheets. Monad corresponds to p-adic units with norm smaller than unit. The collections of algebraic points defining the positions of monads and also intersections with real space-time sheets are highly symmetric and determined by the discrete p-adicizable subgroups of Lorentz group and color group. When the subgroup of the rotation group is finite one obtains polygons and Platonic solids. Bio-molecules typically consists of this kind of structures - such as regular hexagons and pentagons - and could be seen as cognitive representations of these geometries often called sacred! I have proposed this idea long time ago and the discovery of the recipe for the construction of p-adic geometries gave a justification for this idea.

8.12.1 p-Adic variants of causal diamonds

To construct p-adic variants of space-time surfaces one must construct p-adic variants of the embedding space. The assumption that the p-adic geometry for the embedding space induces p-adic geometry for sub-manifolds implies a huge simplification in the definition of p-adic variants of preferred extremals. The natural guess is that real and p-adic space-time surfaces gave algebraic points as common: so that the first challenge is to pick the algebraic points of the real space-time surface. To define p-adic space-time surface one needs field equations and the notion of p-adic continuum and by assigning to each algebraic point a p-adic continuum to make it monad, one can solve p-adic field equations inside these monads.

The idea of finite measurement resolution suggests that the solutions of p-adic field equations inside monads are arbitrary. Whether this is consistent with the idea that same solutions of field equations can be interpreted either p-adically or in real sense is not quite clear. This would be guaranteed if the p-adic solution has same formal representation as the real solution in the vicinity of given discrete point - say in terms of polynomials with rational coefficients and coordinate variables which vanish for the algebraic point.

Real and p-adic space-time surfaces would intersect at points common to all number fields for given adele: cognition and sensory worlds intersect not only at the level of WCW but also at the level of space-time. I had already considered giving up the latter assumption but it seems to be necessary at least for string world sheets and partonic 2-surfaces if not for entire space-time surfaces.

General recipe

The recipe would be following.

1. One starts from a discrete variant of $CD \times CP_2$ defined by an appropriate discrete symmetry groups and their subgroups using coset space construction. This discretization consists of points in finite-dimensional extension of p-adics induced by an extension of rationals. These points are assumed to be in the intersection of reality and p-adicities at space-time level - that is common for real and p-adic space-time surfaces. Cognitive representations in the real world are thus discrete and induced by the intersection. This is the original idea which I was ready to give up as the vision about discretization at WCW level allowing to solve all problems related to symmetries emerged. At space-time level the p-adic discretization reduces symmetry groups to their discrete subgroups: cognitive representations unavoidably break the symmetries. What is important the distance between discrete p-adic points labelling monads is naturally their real distance. This fixes metrically real-p-adic/sensory-cognitive correspondence.

2. One replaces each point of this discrete variant $CD \times CP_2$ with p-adic continuum defined by an algebraic extension of p-adics for the adele considered so that differentiation and therefore also p-adic field equations make sense. The continuum for given discrete point of $CD_d \times CP_{2,d}$ defines kind of Leibnizian monad representing field equations p-adically. The solution decomposes to p-adically differentiable pieces and the global solution of field equations makes sense since it can be interpreted in terms of pseudo-constants. p-Adicization means discretization but with discrete points replaced with p-adic monads preserving also the information about local behavior. The loss of well-ordering inside p-adic monad reflects its loss due to the finiteness of measurement resolution.

3. The distances between monads correspond to their distances for real variant of $CD \times CP_2$. Are there natural restrictions on the p-adic sizes of monads? Since p-adic units are in question that size in suitable units is $p^{-N} < 1$. It would look natural that the p-adic size of the is smaller than the distance to the nearest monad. The denser the discretization is, the larger the value of N would be. The size of the monad decreases at least like $1/p$ and for large primes assignable to elementary particles ($M_{127} = 2^{127} - 1$) is rather small. The discretizations of the subgroups share the properties of the group invariant geometry of groups so that they are to form a regular lattice like structure with constant distance to nearest neighbors. At the embedding level therefore p-adic geometries are extremely symmetric. At the level of space-time geometries only a subset of algebraic points is picked and the symmetry tends to be lost.

CD degrees of freedom

Consider first CD degrees of freedom.

1. For M^4 one has 4 linear coordinates. Should one p-adicize these or should one discretize CDs defined as intersections of future and past directed light-cones and strongly suggested

by ZEO. CD seems to represent the more natural option. The construction of a given CD suggests that one should replace the usual representation of manifold as a union of overlapping regions with intersection of two light-cones with coordinates related in the intersection as in the case of ordinary manifold: $\cup \rightarrow \cap$.

2. For a given light-cone one must introduce light-cone proper time a, hyperbolic angle η and two angle coordinates (θ, ϕ). Light-cone proper time a is Lorentz invariant and corresponds naturally to an ordinary p-adic number of more generally to a p-adic number in algebraic extension which does not involve phases.

   The two angle coordinates (θ, ϕ) parameterizing S^2 can be represented in terms of phases and discretized. The hyperbolic coordinate can be also discretized since e^p exists p-adically, and one obtains a finite-dimensional extension of p-adic numbers by adding roots of e and its powers. e is completely exceptional in that it is p-adically an algebraic number.

3. This procedure gives a discretization in angle coordinates. By replacing each discrete value of angle by p-adic continuum one obtains also now the monad structure. The replacement with continuum means the replacement

$$U_{m,n} \equiv exp(i2\pi m/n) \rightarrow U_{m,n} \times exp(i\phi) \ , \tag{8.12.1}$$

   where ϕ is p-adic number with norm $p^{-N} < 1$ It can also belong to an algebraic extension of p-adic numbers. Building the monad is like replacing in finite measurement the representative point of measurement resolution interval with the entire interval. By finite measurement resolution one cannot fix the order inside the interval. Note that one obtains a hierarchy of subgroups depending on the upper bound p^{-n} for the modulus. For $p \ mod \ 4 = 1$ imaginary unit exist as ordinary p-adic number and for $p \ mod \ 4 = 3$ in an extension including $\sqrt{-1}$.

4. For the hyperbolic angle one has

$$E_{m,n} \equiv exp(m/n) \rightarrow E_{m,n} \times exp(\eta) \tag{8.12.2}$$

   with the ordinary p-adic number η having norm $p^{-N} < 1$. Lorentz symmetry is broken to a discrete subgroup: this could be interpreted in terms of finite cognitive resolution. Since e^p is p-adic number also hyperbolic angle has finite number of values and one has compactness in well-defined sense although in real context one has non-compactness.

   In cosmology this discretization means quantization of redshift and thus recession velocities. A concise manner to express the discretization to say that the cosmic time constant hyperboloids are discrete variants of Lobatchevski spaces $SO(3,1)/SO(3)$. The spaces appear naturally in TGD inspired cosmology.

5. The coordinate transformation relating the coordinates in the two intersecting coordinate patches maps hyperbolic and ordinary phases to each other as such. Light-cone proper time coordinates are related in more complex manner. $a_+^2 = t^2 - r^2$ and $a_-^2 = (t-T)^2 - r^2$ are related by $a_+^2 - a_-^2 = 2tT - T^2 = 2a_+ cosh(\eta)T - T^2$.

   This leads to a problem unless one allows a_+ and a_- to belong to an algebraic extension containing the roots of e making possible to define hyperbolic angle. The coordinates $a_\pm$ can also belong to a larger extension of p-adic numbers. The expectation is that one obtains an infinite hierarchy of algebraic extensions of rationals involving besides the phases also other non-Abelian extension parameters. It would seem that the Abelian extension for phases and the extension for a must factorize somehow. Note also that the expression of a_+ in terms of a_- given by

$$a_+ = -cosh(\eta)T \pm \sqrt{sinh^2(\eta)T^2 + a_-^2} \ . \tag{8.12.3}$$

This expression makes sense p-adically for all values of a_- if one can expand the square root as a converning power series with respect to a_-. This is true if $a_-/sinh(\eta)T$ has p-adic norm smaller than 1.

6. What about the boundary of CD which corresponds to a coordinate singularity? It seems that this must be treated separately. The boundary has topology $S^2 \times R_+$ and S^2 can be p-adicized as already explained. The light-like radial coordinate $r = asinh(\eta)$ vanishes identically for finite values of $sinh(\eta)$. Should one regard r as ordinary p-adic number? Or should one think that entire light-one boundary corresponds to single point $r = 0$? The discretization of r is in powers of a roots of e is very natural so that each power $E_{m,n}$ corresponds to a p-adic monad. If now powers $E_{m,n}$ are involved, one obtains just the monad at $r = 0$.

   The construction of quantum TGD leads to the introduction of powers $exp(log(r/r_0)s)$, where s is zero of Riemann Zeta [K35]. These make sense p-adically if $u = log(r/r_0)$ has p-adic norm smaller than unity and s makes sense p-adically. The latter condition demanding that the zeros are algebraic numbers is quite strong.

8.12.2 Construction for $SU(2)$, $SU(3)$, and classical Lie groups

In the following the detailed construction for $SU(2)$, $SU(3)$, and classical Lie groups will be sketched.

Subgroups of $SU(2)$ having p-adic counterparts

In the case $U(1)$ the subgroups defined by roots of unity reduce to a finite group Z_n. What can one say about p-adicizable discrete subgroups of $SU(2)$?

1. To see what happens in the case of $SU(2)$ one can write $SU(2)$ element explicitly in quaternionic matrix representation

$$(\theta, n) \equiv cos(\theta)Id + sin(\theta)\sum_i n_i I_i \quad . \tag{8.12.4}$$

   Here Id is quaternionic real unit and I_i are quaternionic imaginary units. $n = (n_1, n_2, n_2)$ is a unit vector representable as $(cos(\phi), sin(\phi)cos(\psi), sin(\phi)sin(\psi))$. This representation exists p-adically if the phases $exp(i\theta)$, $exp(i(\phi)$ and $exp(i\psi)$ exist p-adically so that they must be roots of unity.

   The geometric interpretation is that n defines the direction of rotation axis and θ defines the rotation angle.

2. This representation is not the most general one in p-adic context. Suppose that one has two elements of this kind characterized by (θ_i, n_i)such that the rotation axes are different. From the multiplication table of quaternions one has for the product (θ_{12}, n_{12}) of these

$$cos(\theta_{12}) \quad = \quad cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)n_1 \cdot n_2 \quad . \tag{8.12.5}$$

   This makes sense p-adically if the inner product $cos(\chi) \equiv n_1 \cdot n_2$ corresponds to root of unity in the extension of rationals used. Therefore the angle between the rotation axes is number theoretically quantized in order that p-adicization works.

   One can solve θ_{12} from the above equation in real context but in the general case it does not correspond to $U_{m,n}$. This is not however a problem from p-adic point of view. The reduction to a root of unity is true only in some special cases. For $n_1 = n_2$ the group generated by the products reduces a discrete $Z_n \subset U(1)$ generated by a root of unity. If n_1 and n_2 are orthogonal the angle between rotation axes corresponds trivially to a root of unity. In this case one has the isometries of cube. For other Platonic solids the angles between rotation axes associated with various $U(1)$ subgroups generating the entire sub-group are fixed by their geometries. The rotation angles correspond to $n = 3$ for tetrahedron and icosahedron and $n = 5$ dodecahedron and for $n = 3$. There is also duality between cube and octahedron and icosahedron and dodecahedron.

3. Platonic solids can be geometrically seen as discretized variants of $SU(2)$ and it seems that they correspond to finite discrete subgroups of $SU(2)$ defining $SU(2)_d$. Platonic sub-groups appear in the hierarchy of Jones inclusions. The other finite subgroups of $SU(2)$ appearing in this hierarchy act on polygons of plane and being generated by Z_n and rotations around the axes of plane and would naturally correspond to discrete U(1) sub-groups of $SU(2)$ and in a well-defined sense to a degenerate situation. By Mc-Kay correspondence all these groups correspond to ADE type Lie groups. These subgroups define finite discretizations of $SU(2)$ and S^2. p-Adicization would lead directly to the hierarchy of inclusions assigned also with the hierarchy of sub-algebras of super-symplectic algebra characterized by the hierarchy of Planck constants.

4. There are also p-adicizable discrete subgroups, which are infinite. By taking two rotations with angles which correspond to root of unity with rotation axes, whose mutual angle corresponds to root of unity one can generate an infinite discrete subgroup of $SU(2)$ existing in p-adic sense. More general discrete U(1) subgroups are obtained by taking n rotation axes with mutual angles corresponding to roots of unity and generating the subgroup from these. In case of Platonic solids this gives a finite subgroup.

Construction of p-adicizable discrete subgroups of CP_2

The construction of p-adic CP_2 proceeds along similar lines.

1. In the original ultra-naïve approach the local p-adic metric of CP_2 is obtained by a purely formal replacement of the ordinary metric of CP_2 with its p-adic counterpart and it defines the CP_2 contribution to induced metric. This makes sense since Kähler function is rational function and components of CP_2 metric and spinor connection are rational functions. This allows to formulate p-adic variants of field equations. This description is however only local. It says nothing about global aspects of CP_2 related to the introduction of algebraic extension of p-adic numbers.

   One should be able to realize the angle coordinates of CP_2 in a physically acceptable manner. The coordinates of CP_2 can be expressed by compactness in terms of trigonometric functions, which suggests a realization of them as phases for the roots of unity. The number of points depends on the Abelian extension of rationals inducing that of p-adics which is chosen. This gives however only discrete version of p-adic CP_2 serving as a kind of spine. Also the flesh replacing points with monads is needed.

2. A more profound approach constructs the algebraic variants of CP_2 as discrete versions of the coset space $CP_2 = SU(3)/U(2)$. One restricts the consideration to an algebraic subgroup of $SU(3)_d$ with elements, which are 3×3 matrices with components, which are algebraic numbers in the extension of rationals. Since they are expressible in terms of phases one can express them in terms of roots of unity. In the same manner one identifies $U(2)_d \subset SU(3)_d$. $CP_{2,d}$ is the coset space $SU(3)_D/U(2)_d$ of these. The representative of a given coset is a point in the coset and expressible in terms of roots of unity.

3. The construction of the p-adicizable subgroups of $SU(3)$ suggests a generalization. Since $SU(3)$ is 8-D and Cartan algebra is 2-D the coset space is 6-dimensional flag-manifold $F = SU(3)/U(1)\times U(1)$ with coset consisting of elements related by automorphism $g \equiv hgh^{-1}$. F defines the twistor space of CP_2 characerizing the choices for the quantization axes of color quantum numbers. The points of F should be expressible in terms of phase angles analogous to the angle defining rotation axis in the case of SU(2).

   In the case of $SU(2)$ n U(1) subgroups with specified rotation axes with p-adically existing mutual angles are considered. The construction as such generates only $SU(2)_d$ subgroup which can be trivially extended to $U(2)_d$. The challenge is to proceed further.

Cartan decomposition of the Lie algebra (see http://tinyurl.com/y7cjbm4c) seems to provide a solution to the problem. In the case of $SU(3)$ it corresponds to the decomposition to U(2) sub-algebra and its complement. One could use the decomposition $G = KAK$ where K is maximal compact subgroup. A is exponentiation of the maximal Abelian subalgebra, which is 3-dimensional for CP_2. By Abelianity the p-adicization of A in terms of roots of unity simple. The image of A in G/K is totally geodesic sub-manifold. In the recent case one has $G/Ki = CP_2$ so that the image of A is geodesic sphere S^2. This decomposition implies the representation using roots of unity. The construction of discrete p-adicizable subgroups of $SU(n)$ for $n > 3$ would continue iteratively.

4. Since the construction starts from $SU(2)$, $U(1)$, and Abelian groups, and proceeds iteratively it seems that Platonic solids have counterparts for all classical Lie groups containing $SU(2)$. Also level p-adicizable discrete coset spaces have analogous of Platonic solids.

The results imply that $CD \times CP_2$ is replaced by a discrete set of p-adic monads at a given level of hierarchy corresponding to the finite cognitive resolution.

Generalization to other groups

The above argument demonstrates that p-adicization works iteratively for $SU(n)$ and thus for $U(n)$. For finite-dimensional symplectic group $Sp(n, R)$ the maximal compact sub-group is $U(n)$ so that KAK construction should work also now. $SO(n)$ can be regarded as subgroup of $SU(n)$ so that the p-adiced discretrized variants of maximal compact subgroups should be constructible and KAK give the groups. The inspection of the table of the Wikipedia article (see http://tinyurl.com/j44639q) encourages the conjecture that the construction of $SU(n)$ and $U(n)$ generalizes to all classical Lie groups.

This construction could simplify enormously also the p-adicization of WCW and the theory would discretize even in non-compact degrees of freedom. The non-zero modes of WCW correspond to the symplectic group for $\delta M^4 \times CP_2$, and one might hope that the p-adicization works also at the limit of infinite-dimensional symplectic group with $U(\infty)$ taking the role of K.

8.13 Some layman considerations related to the fundamentals of mathematics

I am not a mathematician and therefore should refrain from consideration of anything related to fundamentals of mathematics. In the discussions with Santeri Satama I could not avoid the temptation to break this rule. I however feel that I must confess my sins and in the following I will do this.

1. Gödel's problematics is shown to have a topological analog in real topology, which however disappears in p-adic topology which raises the question whether the replacement of the arithmetics of natural numbers with that of p-adic integers could allow to avoid Gödel's problematics.

2. Number theory looks from the point of view of TGD more fundamental than set theory and inspires the question whether the notion of algebraic number could emerge naturally from TGD. There are two ways to understand the emergence of algebraic numbers: the hierarchy of infinite primes in which ordinary primes are starting point and the arithmetics of Hilbert spaces with tensor product and direct sum replacing the usual arithmetic operations. Extensions of rationals give also rise to cognitive variants of n-D spaces.

3. The notion of empty set looks artificial from the point of view of physicist and a possible cure is to take arithmetics as a model. Natural numbers would be analogous to nonempty sets and integers would correspond to pairs of sets (A, B), $A \subset B$ or $B \subset A$ with equivalence $A, B) \equiv (A \cup C, B \cup C)$. Empty set would correspond to pairs (A, A). In quantum context the generalization of the notion of being member of set $a \in A$ suggests a generalization: being an element in set would generalize to being single particle state which in general is de-localized

to the set. Subsets would correspond to many-particle states. The basic operation would be addition or removal of element represented in terms of oscillator operator. The order of elements of set does not matter: this would generalize to bosonic and fermionic many particle states and even braid statistics can be considered. In bosonic case one can have multiple points - kind of Bose-Einstein condensate.

4. One can also start from finite-D Hilbert space and identify set as the collection of labels for the states. In infinite-D case there are two cases corresponding to separable and non-separable Hilbert spaces. The condition that the norm of the state is finite without infinite normalization constants forces selection of de-localized discrete basis in the case of a continuous set like reals. This inspires the question whether the axiom of choice should be given up. One possibility is that one can have only states localized to finite or at least discrete set of points which correspond points with coordinates in an extension of rationals.

8.13.1 Geometric analog for Gödel's problematics

Godel's problematics involves statements which cannot be proved to be true or false or are simultaneously true and false. This problematics has also a purely geometric analog in terms of set theoretic representation of Boolean algebras when real topology is used but not when p-adic topology is used.

The natural idea is that Boolean algebra is realized in terms of open sets such that the negation of statement corresponds to the complement of the set. In p-adic topologies open sets are simultaneously also closed and there are no boundaries: this makes them and - more generally Stone spaces - ideal for realizing Boolean algebra set theoretically. In real topology the complement of open set is closed and therefore not open and one has a problem.

Could one circumvent the problem somehow?

1. If one replaces open sets with their closures (the closure of open set includes also its boundary, which does not belong to the open set) and closed complements of open sets, the analog of Boolean algebra would consist of closed sets. Closure of an open set and the closure of its open complement - stament and its negation - share the common boundary. Statement and its negation would be simultaneously true at the boundary. This strange situation reminds of Russell's paradox but in geometric form.

2. If one replaces the closed complements of open sets with their open interiors, one has only open sets. Now the sphere would represent statement about which one cannot say whether it is true or false. This would look like Gödelian sentence but represented geometrically.

   This leads to an already familiar conclusion: p-adic topology is natural for the geometric correlates of cognition, in particular Boolean cognition. Real topology is natural for the geometric correlates of sensory experience.

3. Gödelian problematics is encountered already for arithmetics of natural numbers although naturals have no boundary in the discrete topology. Discrete topology does not however allow well-ordering of natural numbers crucial for the definition of natural number. In the induced real topology one can order them and can speak of boundaries of subsets of naturals. The ordering of natural numbers by size reflects the ordering of reals: it is very difficult to think about discrete without implicitly bringing in the continuum.

   For p-adic integers the induced topology is p-adic. Is Gödelian problematics is absent in p-adic Boolean logic in which set and its complement are both open and closed. If this view is correct, p-adic integers might replace naturals in the axiomatics of arithmetics. The new element would be that most p-adic integers are of infinite size in real sense. One has a natural division of them to cognitively representable ones finite also in real sense and non-representable ones infinite in real sense. Note however that rationals have periodic pinary expansion and can be represented as pairs of finite natural numbers.

In algebraic geometry Zariski topology in which closed sets correspond to algebraic surfaces of various dimensions, is natural. Open sets correspond to their complements and are of same dimension as the embedding space. Also now one encounters asymmetry. Could one say that algebraic surfaces characterize "representable" (="geometrically provable"?) statements as elements of Boolean algebra and their complements the non-representable ones? 4-D space-time (as possibly associative/co-associative) algebraic variety in 8-D octonionic space would be example of representable statement. Finite unions and intersections of algebraic surfaces would form the set of representable statements. This new-to-me notion of representability is somehow analogous to provability or demonstrability.

8.13.2 Number theory from quantum theory

Could one define or at least represent the notion of number using the notions of quantum physics? A natural starting point is hierarchy of extensions of rationals defining hierarchy of adeles. Could one obtain rationals and their extensions from simplest possible quantum theory in which one just constructs many particle states by adding or removing particles using creation and annihilation operators?

How to obtain rationals and their extensions?

Rationals and their extensions are fundamental in TGD. Can one have quantal construction for them?

1. One should construct rationals first. Suppose one starts from the notion of finite prime as something God-given. At the first step one constructs infinite primes as analogs for many-particle states in super-symmetric arithmetic quantum field theory [K94]. Ordinary primes label states of fermions and bosons. Infinite primes as the analogs of free many-particle states correspond to rationals in a natural manner.

2. One obtains also analogs of bound states which are mappable to irreducible polynomials, whose roots define algebraic numbers. This would give hierarchy of algebraic extensions of rationals. At higher levels of the hierarchy one obtains also analogs of prime polynomials with number of variables larger than 1. One might say that algebraic geometry has quantal representation. This might be very relevant for the physical representability of basic mathematical structures.

Arithmetics of Hilbert spaces

The notions of prime and divisibility and even basic arithmetics emerge also from the tensor product and direct sum for Hilbert spaces. Hilbert spaces with prime dimension do not decompose to tensor products of lower-dimensional Hilbert spaces. One can even perform a formal generalization of the dimension of Hilbert space so that it becomes rational and even algebraic number.

For some years ago I indeed played with this thought but at that time I did not have in mind reduction of number theory to the arithemetics of Hilbert spaces. If this really makes sense, numbers could be replaced by Hilbert spaces with product and sum identified as tensor product and direct sum!

Finite-dimensional Hilbert space represent the analogs of natural numbers. The analogs of integers could be defined as pairs (m, n) of Hilbert spaces with spaces (m, n) and $(m + r, n + r)$ identified (this space would have dimension $m - n$. This identification would hold true also at the level of states. Hilbert spaces with negative dimension would correspond to pairs with $(m-n) < 0$: the canonical representives for m and $-m$ would be $(m, 0)$ and $(0, m)$. Rationals can be defined as pairs (m, n) of Hilbert spaces with pairs (m, n) and (km, kn) identified. These identifications would give rise to kind of gauge conditions and canonical representatives for m and $1/m$ are $(m, 1)$ and $(1, m)$.

What about Hilbert spaces for which the dimension is algebraic number? Algebraic numbers allow a description in terms of partial fractions and Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq` and `http://tinyurl.com/yc6hhboo`) containing given rational number once. S-B tree allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. Algebraic number could be seen as a periodic partial fraction defining an infinite path in S-B tree. Each node along this path would correspond to a rational having Hilbert space analog. Hilbert space with algebraic dimension would correspond to this kind of path in the space of Hilbert spaces with rational dimension. Transcendentals allow identification as non-pediodic partial fraction and could correspond to non-periodic paths so that also they could have Hilbert spaces counterparts.

How to obtain the analogs higher-D spaces?

Algebraic extensions of rationals allow cognitive realization of spaces with arbitrary dimension identified as algebraic dimension of extension of rationals.

1. One can obtain n-dimensional spaces (in algebraic sense) with integer valued coordinates from n-D extensions of rationals. Now the n-tuples defining numbers of extension and differing by permutations are not equivalent so that one obtains n-D space rather than n-D space divided by permutation group S_n. This is enough at the level of cognitive representations and could explain why we are able to imagine spaces of arbitrary dimension although we cannot represent them cognitively.

2. One obtains also Galois group and orbits of set A of points of extension under Galois group as $G(A)$. One obtains also discrete coset spaces G/H and alike. These do not have any direct analog in the set theory. The hierarchy of Galois groups would bring in discrete group theory automatically. The basic machinery of quantum theory emerges elegantly from number theoretic vision.

3. In octonionic approach to quantum TGD one obtains also hierarchy of extensions of rationals since space-time surface correspond zero loci for RE or IM for octonionic polynomials obtained by algebraic continuation from real polynomials with coeffficients in extension of rationals [**?**].

8.13.3 Could quantum set theory make sense?

In the following my view point is that of quantum physicist fascinated by number theory and willing to reduce set theory to what could be called called quantum set theory. It would follow from physics as generalised number theory (adelic physics) and have ordinary set theory as classical correlate.

1. From the point of quantum physics set theory and the notion of number based on set theory look somewhat artificial constructs. Nonempty set is a natural concept but empty set and set having empty set as element used as basic building brick in the construction of natural numbers looks weird to me.

2. From TGD point of view it would seem that number theory plus some basic pieces of quantum theory might be more fundamental than set theory. Could set theory emerge as a classical correlate for quantum number theory already considered and could quantal set theory make sense?

Quantum set theory

What quantum set theory could mean? Suppose that number theory-quantum theory connection really works. What about set theory? Or perhaps its quantum counterpart having ordinary set theory as a classical correlate?

1. A purely quantal input to the notion of set would be replacement of points delocalized states in the set. A generic single particle quantum state as analog of element of set would not be localized to a single element of set. The condition that the state has finite norm implies in the case of continuous set like reals that one cannot have completely localized states. This would give quantal limitation to the axiom of choice. One can have any discrete basis of state functions in the set but one cannot pick up just one point since this state would have infinite norm.

   The idea about allowing only say rationals is not needed since there is infinite number of different choices of basis. Finite measurement resolution is however unvoidable. An alternative option is restriction of the domains of wave functions to a discrete set of points. This set can be chosen in very many ways and points with coordinates in extension of rationals are very natural and would define cognitive representation.

2. One can construct also the analogs of subsets as many-particle states. The basic operation would be addition/removal of a particle from quantum state represented by the action of creation/annihilation operator.

   Bosonic states would be invariant under permutations of single particle states just like set is the equivalence class for a collection of elements $(a_1, ..., a_n)$ such that any two permutations are equivalent. Quantum set theory would however bring in something new: the possibility of both bosonic and fermionic statistics. Permutation would change the state by phase factor -1. One would have fermionic and bosonic sets. For bosonic sets one could have multiplet elements ("Bose-Einstein condensation"): in the theory of surfaces this could allow multiple copies of the same surface. Even braid statistics is possible. The phase factor in permutation could be complex. Even non-commutative statistics can be considered.

   Many particle states formed from particles, which are not identical are also possible and now the different particle types can be ordered. On obtains n-ples decomposing to ordered K-ple of n_i-ples, which are consist of identical particles and are quantum sets. One could talk about K-sets as a generalization of set as analogs of classical sets with K-colored elements. Group theory would enter into the picture via permutation groups and braid groups would bring in braid statistics. Braids strands would have K colors.

How to obtain classical set theory?

How could one obtain classical set theory?

1. Many-particle states represented algebraically are detected in lab as sets: this is quantum classical correspondence. This remains to me one of the really mysterious looking aspects in the interpretation of quantum field theory. For some reason it is usually not mentioned at all in popularizations. The reason is probably that popularization deals typically with wave mechanics but not quantum field theory unless it is about Higgs mechanism, which is the weakest part of quantum field theory!

2. From the point of quantum theory empty set would correspond to vacuum. It is not observable as such. Could the situation change in the presence of second state representing the environment? Could the fundamental sets be always *non-empty* and correspond to states with non-vanishing particle number. Natural numbers would correspond to eigenvalues of an observable telling the cardinality of set. Could representable sets be like natural numbers?

3. Usually integers are identified as pairs of natural numbers (m, n) such that integer corresponds to $m - n$. Could the set theoretic analog of integer be a pair (A, B) of sets such that A is subset of B or vice versa? Note that this does not allow pairs with disjoint members. (A, A) would correspond to empty set. This would give rise to sets (A, B) and their "antisets" (B, A) as analogs of positive and negative integers.

   One can argue that antisets are not physically realizable. Sets and antisets would have as analogs two quantizations in which the roles of oscillator operators and their hermitian conjugates are changed. The operators annihilating the ground state are called annilation operators. Only either of these realization is possible but not both simultaneously.

   In ZEO one can ask whether these two options correspond to positive and negative energy parts of zero energy states or to the states with state function reduction at either boundary of CD identified as correlates for conscious entities with opposite arrows of geometric time (generalized Zeno effect).

4. The cardinality of set, the number of elements in the set, could correspond to eigenvalue of observable measuring particle number. Many-particle states consisting of bosons or fermions would be analogs for sets since the ordering does not matter. Also braid statistics would be possible.

What about cardinality as a p-adic integer? In p-adic context one can assign to integer m, integer $-m$ as $m \times (p-1) \times (1+p+p^2+...)$. This is infinite as real integer but finite as p-adic integer. Could one say that the antiset of m-element as analog of negative integer has cardinality $-m = m(p-1)(1+p+p^2+..)$. This number does not have cognitive representation since it is not finite as real number but is cognizable.

One could argue that negative numbers are cognizable but not cognitively representable as cardinality of set? This representation must be distinguished from cognitive representations as a point of embedding space with coordinates in extension of rationals. Could one say that antisets and empty set as its own antiset can be cognized but cannot be cognitively represented?

Nasty mathematician would ask whether I can really start from Hilbert space of state functions and deduce from this the underlying set. The elements of set itself should emerge from this as analogs of completely localized single particle states labelled by points of set. In the case of finite-dimensional Hilbert space this is trivial. The number of points in the set would be equal to the dimension of Hilbert space. In the case of infinite-D Hilbert space the set would have infinite number of points.

Here one has two views about infinite set. One has both separable (infinite-D in discrete sense: particle in box with discrete momentum spectrum) and non-separable (infinite-D in real sense: free particle with continuous momentum spectrum) Hilbert spaces. In the latter case the completely localized single particle states would be represented by delta functions divided by infinite normalization factors. They are routinely used in Dirac's bra-ket formalism but problems emerge in quantum field theory.

A possible solution is that one weakens the axiom of choice and accepts that only discrete points set (possibly finite) are cognitively representable and one has wave functions localized to discrete set of points. A stronger assumption is that these points have coordinates in extension of rationals so that one obtains number theoretical universality and adeles. This is TGD view and conforms also with the identification of hyper-finite factors of type II_1 as basic algebraic objects in TGD based quantum theory as opposed to wave mechanics (type I) and quantum field theory (type III). They are infinite-D but allow excellent approximation as finite-D objects.

This picture could relate to the notion of non-commutative geometry, where set emerges as spectrum of algebra: the points of spectrum label the ideals of the integer elements of algebra.

8.14 Abelian Class Field Theory And TGD

The context leading to the discovery of adeles (`http://tinyurl.com/64pgerm`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \mod 4 = 1$ they decompose as $p = (a+ib)(a-ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1+i)(1-i) = -i(1+i)^2 = i(1-i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (see `http://tinyurl.com/y8aefmg2`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

8.14.1 Adeles And Ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://tinyurl.com/y8aefmg2`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [K45, A33, A32]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where embedding space is replaced with Cartesian product of real embedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://tinyurl.com/64pgerm`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely

among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod'_p Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://tinyurl.com/yc3yrcxx`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

8.14.2 Questions About Adeles, Ideles And Quantum TGD

The intriguing general result of class field theory (`http://tinyurl.com/y8aefmg2`) is that the the maximal Abelian extension for rationals is homomorphic with the multiplicative group of ideles. This correspondence plays a key role in Langlands correspondence.

Does this mean that it is not absolutely necessary to introduce p-adic numbers? This is actually not so. The Galois group of the maximal abelian extension is rather complex objects (absolute Galois group, AGG, defines as the Galois group of algebraic numbers is even more complex!). The ring $\hat{Z}$ of adeles defining the group of ideles as its invertible elements homeomorphic to the Galois group of maximal Abelian extension is profinite group (`http://tinyurl.com/y9d8vro7`). This means that it is totally disconnected space as also p-adic integers and numbers are. What is intriguing that p-dic integers are however a continuous structure in the sense that differential calculus is possible. A concrete example is provided by 2-adic units consisting of bit sequences which can have literally infinite non-vanishing bits. This space is formally discrete but one can construct differential calculus since the situation is not democratic. The higher the pinary digit in the expansion is, the less significant it is, and p-adic norm approaching to zero expresses the reduction of the insignificance.

1. Could TGD based physics reduce to a representation theory for the Galois groups of quaternions and octonions?

Number theoretical vision about TGD raises questions about whether adeles and ideles could be helpful in the formulation of TGD. I have already earlier considered the idea that quantum TGD could reduce to a representation theory of appropriate Galois groups. I proceed to make questions.

1. Could real physics and various p-adic physics on one hand, and number theoretic physics based on maximal Abelian extension of rational octonions and quaternions on one hand, define equivalent formulations of physics?

2. Besides various p-adic physics all classical number fields (reals, complex numbers, quaternions, and octonions) are central in the number theoretical vision about TGD. The technical problem is that p-adic quaternions and octonions exist only as a ring unless one poses some additional conditions. Is it possible to pose such conditions so that one could define what might be called quaternionic and octonionic adeles and ideles?

   It will be found that this is the case: p-adic quaternions/octonions would be products of rational quaternions/octonions with a p-adic unit. This definition applies also to algebraic extensions of rationals and makes it possible to define the notion of derivative for corresponding adeles. Furthermore, the rational quaternions define non-commutative automorphisms of quaternions and rational octonions at least formally define a non-associative analog of group of octonionic automorphisms [K96, K109].

3. I have already earlier considered the idea about Galois group as the ultimate symmetry group of physics. The representations of Galois group of maximal Abelian extension (or even that for algebraic numbers) would define the quantum states. The representation space could be group algebra of the Galois group and in Abelian case equivalently the group algebra of ideles or adeles. One would have wave functions in the space of ideles.

The Galois group of maximal Abelian extension would be the Cartan subgroup of the absolute Galois group of algebraic numbers associated with given extension of rationals and it would be natural to classify the quantum states by the corresponding quantum numbers (number theoretic observables).

If octonionic and quaternionic (associative) adeles make sense, the associativity condition would reduce the analogs of wave functions to those at 4-dimensional associative sub-manifolds of octonionic adeles identifiable as space-time surfaces so that also space-time physics in various number fields would result as representations of Galois group in the maximal Abelian Galois group of rational octonions/quaternions. TGD would reduce to classical number theory! One can hope that WCW spinor fields assignable to the associative and co-associative space-time surfaces provide the adelic representations for super-conformal algebras replacing symmetries for point like objects.

This of course involves huge challenges: one should find an adelic formulation for WCW in terms octonionic and quaternionic adeles, similar formulation for WCW spinor fields in terms of adelic induced spinor fields or their octonionic variants is needed. Also zero energy ontology, causal diamonds, light-like 3-surfaces at which the signature of the induced metric changes, space-like 3-surfaces and partonic 2-surfaces at the boundaries of CDs, M^8-H duality, possible representation of space-time surfaces in terms of of O_c-real analytic functions (O_c denotes for complexified octonions), etc. should be generalized to adelic framework.

4. Absolute Galois group is the Galois group of the maximal algebraic extension and as such a poorly defined concept. One can however consider the hierarchy of all finite-dimensional algebraic extensions (including non-Abelian ones) and maximal Abelian extensions associated with these and obtain in this manner a hierarchy of physics defined as representations of these Galois groups homomorphic with the corresponding idele groups.

5. In this approach the symmetries of the theory would have automatically adelic representations and one might hope about connection with Langlands program [K45], [K45, A33, A32].

2. Adelic variant of space-time dynamics and spinorial dynamics?

As an innocent novice I can continue to pose stupid questions. Now about adelic variant of the space-time dynamics based on the generalization of Kähler action discussed already earlier but without mentioning adeles ([K112]).

1. Could one think that adeles or ideles could extend reals in the formulation of the theory: note that reals are included as Cartesian factor to adeles. Could one speak about adelic space-time surfaces endowed with adelic coordinates? Could one formulate variational principle in terms of adeles so that exponent of action would be product of actions exponents associated with various factors with Neper number replaced by p for Z_p. The minimal interpretation would be that in adelic picture one collects under the same umbrella real physics and various p-adic physics.

2. Number theoretic vision suggests that 4: th/8: th Cartesian powers of adeles have interpretation as adelic variants of quaternions/ octonions. If so, one can ask whether adelic quaternions and octonions could have some number theoretical meaning. Adelic quaternions and octonions are not number fields without additional assumptions since the moduli squared for a p-adic analog of quaternion and octonion can vanish so that the inverse fails to exist at the light-cone boundary which is 17-dimensional for complexified octonions and 7-dimensional for complexified quaternions. The reason is that norm squared is difference $N(o_1)-N(o_2)$ for $o_1 \oplus io_2$. This allows to define differential calculus for Taylor series and one can consider even rational functions. Hence the restriction is not fatal.

   If one can pose a condition guaranteeing the existence of inverse for octonionic adel, one could define the multiplicative group of ideles for quaternions. For octonions one would obtain non-associative analog of the multiplicative group. If this kind of structures exist then four-dimensional associative/co-associative sub-manifolds in the space of non-associative ideles define associative/co-associative adeles in which ideles act. It is easy to find that octonionic ideles form 1-dimensional objects so that one must accept octonions with arbitrary real or p-adic components.

3. What about equations for space-time surfaces. Do field equations reduce to separate field equations for each factor? Can one pose as an additional condition the constraint that p-adic surfaces provide in some sense cognitive representations of real space-time surfaces: this idea is formulated more precisely in terms of p-adic manifold concept [K112] (see the appendix of the book). Or is this correspondence an outcome of evolution?

   Physical intuition would suggest that in most p-adic factors space-time surface corresponds to a point, or at least to a vacuum extremal. One can consider also the possibility that same algebraic equation describes the surface in various factors of the adele. Could this hold true in the intersection of real and p-adic worlds for which rationals appear in the polynomials defining the preferred extremals.

4. To define field equations one must have the notion of derivative. Derivative is an operation involving division and can be tricky since adeles are not number field. The above argument suggests this is not actually a problem. Of course, if one can guarantee that the p-adic variants of octonions and quaternions are number fields, there are good hopes about well-defined derivative. Derivative as limiting value $df/dx = \lim(f(x+dx) - f(x))/dx$ for a function decomposing to Cartesian product of real function $f(x)$ and p-adic valued functions $f_p(x_p)$ would require that $f_p(x)$ is non-constant only for a finite number of primes: this is in accordance with the physical picture that only finite number of p-adic primes are active and define "cognitive representations" of real space-time surface. The second condition is that dx is proportional to product $dx \times \prod dx_p$ of differentials dx and dx_p, which are rational numbers. dx goes to xero as a real number but not p-adically for any of the primes involved. dx_p in turn goes to zero p-adically only for Q_p.

5. The idea about rationals as points common to all number fields is central in number theoretical vision. This vision is realized for adeles in the minimal sense that the action of rationals is well-defined in all Cartesian factors of the adeles. Number theoretical vision allows also to talk about common rational points of real and various p-adic space-time surfaces in preferred coordinate choices made possible by symmetries of the embedding space, and one ends up to the vision about life as something residing in the intersection of real and p-adic number fields. It is not clear whether and how adeles could allow to formulate this idea.

6. For adelic variants of embedding space spinors Cartesian product of real and p-adc variants of embedding spaces is mapped to their tensor product. This gives justification for the physical vision that various p-adic physics appear as tensor factors. Does this mean that the generalized induced spinors are infinite tensor products of real and various p-adic spinors and Clifford algebra generated by induced gamma matrices is obtained by tensor product construction? Does the generalization of massless Dirac equation reduce to a sum of d'Alembertians for the factors? Does each of them annihilate the appropriate spinor? If only finite number of Cartesian factors corresponds to a space-time surface which is not vacuum extremal vanishing induced Kähler form, Kähler Dirac equation is non-trivial only in finite number of adelic factors.

3. Objections leading to the identification of octonionic adeles and ideles

The basic idea is that appropriately defined invertible quaternionic/octonionic adeles can be regarded as elements of Galois group assignable to quaternions/octonions. The best manner to proceed is to invent objections against this idea.

1. The first objection is that p-adic quaternions and octonions do not make sense since p-adic variants of quaternions and octonions do not exist in general. The reason is that the p-adic norm squared $\sum x_i^2$ for p-adic variant of quaternion, octonion, or even complex number can vanish so that its inverse does not exist.

2. Second objection is that automorphisms of the ring of quaternions (octonions) in the maximal Abelian extension are products of transformations of the subgroup of $SO(3)$ (G_2) represented

by matrices with elements in the extension and in the Galois group of the extension itself. Ideles separate out as 1-dimensional Cartesian factor from this group so that one does not obtain 4-field (8-fold) Cartesian power of this Galois group.

One can define quaternionic/octonionic ideles in terms of rational quaternions/octonions multiplied by p-adic number. For adeles this condition produces non-sensical results.

1. This condition indeed allows to construct the inverse of p-adic quaternion/octonion as a product of inverses for rational quaternion/octonion and p-adic number. The reason is that the solutions to $\sum x_i^2 = 0$ involve always p-adic numbers with an infinite number of pinary digits - at least one and the identification excludes this possibility. The ideles form also a group as required.

2. One can interpret also the quaternionicity/octonionicity in terms of Galois group. The 7-dimensional non-associative counterparts for octonionic automorphisms act as transformations $x \to gxg^{-1}$. Therefore octonions represent this group like structure and the p-adic octonions would have interpretation as combination of octonionic automorphisms with those of rationals.

3. One cannot assign to ideles 4-D idelic surfaces. The reason is that the non-constant part of all 8-coordinates is proportional to the same p-adic valued function of space-time point so that space-time surface would be a disjoint union of effectively 1-dimensional structures labelled by a subset of rational points of M^8. Induced metric would be 1-dimensional and induced Kähler and spinor curvature would vanish identically.

4. One must allow p-adic octonions to have arbitrary p-adic components. The action of ideles representing Galois group on these surfaces is well-defined. Number field property is lost but this feature comes in play as poles only when one considers rational functions. Already the Minkowskian signature forces to consider complexified octonions and quaternions leading to the loss of field property. It would not be surprising if p-adic poles would be associated with the light-like orbits of partonic 2-surfaces. Both p-adic and Minkowskian poles might therefore be highly relevant physically and analogous to the poles of ordinary analytic functions. For instance, n-point functions could have poles at the light-like boundaries of causal diamonds and at light-like partonic orbits and explain their special physical role.

   The action of ideles in the quaternionic tangent space of space-time surface would be analogous to the action of of adelic linear group $Gl_n(A)$ in n-dimensional space.

5. Adelic variants of octonions would be Cartesian products of ordinary and various p-adic octonions and would define a ring. Quaternionic 4-surfaces would define associative local sub-rings of octonion-adelic ring.

Chapter 9

Philosophy of Adelic Physics

9.1 Introduction

I have developed during last 39 years a proposal for unifying fundamental interactions which I call "Topological Geometrodynamics" (TGD). During last twenty years TGD has expanded to a theory of consciousness and quantum biology and also p-adic and adelic physics have emerged as one thread in the number theoretical vision about TGD.

Since Quantum TGD and physical arguments have served as basic guidelines in the development of p-adic ideas, the best way to introduce the subject of p-adic physics, is by describing first TGD briefly.

In this article I will consider the p-adic aspects of TGD - the first thread of the number theoretic vision - as I see them at this moment.

1. I will describe p-adic mass calculations based on p-adic generalization of thermodynamics and super-conformal invariance [K51, K23] with number theoretical existence constrains leading to highly non-trivial and successful physical predictions. Here the notion of canonical identification mapping p-adic mass squared to real mass squared emerges and is expected to be key player of adelic physics and allow to map various invariants from p-adics to reals and vice versa.

2. I will propose the formulation of p-adicization of real physics and adelization meaning the fusion of real physics and various p-adic physics to single coherent whole by a generalization of number concept fusing reals and p-adics to larger structure having algebraic extension of rationals as a kind of intersection.

   The existence of p-adic variants of definite integral, Fourier analysis, Hilbert space, and Riemann geometry is far form obvious, and various constraints lead to the idea of NTU and finite measurement resolution realized in terms of number theory. Maybe the only way to overcome the problems relies on the idea that various angles and their hyperbolic analogs are replaced with their exponentials and identified as roots of unity and roots of e existing in finite-dimensional algebraic extension of p-adic numbers. Only group invariants - typically squares of distances and norms - are mapped by canonical identification from p-adic to real realm and various phases are mapped to themselves as number theoretically universal entities.

   Another challenge is the correspondence between real and p-adic physics at various levels: space-time level, embedding space level, and WCW level. Here the enormous symmetries of WCW and those of embedding space are in crucial role. Strong form of holography (SH) allows a correspondence between real and p-adic space-time surfaces induced by algebraic continuation from string world sheets and partonic 2-surface, which can be said to be common to real and p-adic space-time surfaces.

3. In the last section I will describe the role of p-adic physics in TGD inspired theory of consciousness. The key notion is Negentropic entanglement (NE) characterized in terms of number theoretic entanglement negentropy (NEN). Negentropy Maximization Principle (NMP) would force the growth of NE. The interpretation would be in terms of evolution as increase

of negentropy resources - Akashic records as one might poetically say. The newest finding is that NMP in statistical sense follows from the mere fact that the dimension of extension of rationals defining adeles increases unavoidably in statistical sense - separate NMP would not be necessary.

In the sequel I will use some shorthand notations for key principles and key notions. Quantum Field Theory (QFT); Relativity Principle (RP); Equivalence Principle (EP); General Coordinate Invariance (GCI); World of Classical Worlds (WCW); Strong Form of GCI (SGCI); Strong Form of Holography (SH); Preferred Extremal (PE); Zero Energy Ontology (ZEO); Quantum Criticality (QC); Hyper-finite Factor of Type II_1 (HFF); Number Theoretical Universality (NTU); Canonical Identification (CI); Negentropy Maximization Principle (NMP); Negentropic entanglement (NE); Number Theoretical Entanglement Negentropy (NEN); are the most often occurring acronyms.

9.2 TGD briefly

This section gives a brief summary of classical and quantum TGD, which to my opinion is necessary for understanding the number theoretic vision.

9.2.1 Space-time as 4-surface

TGD forces a new view about space-time as 4-surface of 8-D imbedding space. This view is extremely simple locally but by its many-sheetedness topologically much more complex than GRT space-time.

Energy problem of GRT as starting point

The physical motivation for TGD was what I have christened the energy problem of General Relativity [K113, K18].

1. The notion of energy is ill-defined because the basic symmetries of empty space-time are lost in the presence of gravity. The presence of matter curves empty Minkowski space M^4 so that its isometries realized as transformations leaving the distances between points and thus shapes of 4-D objects invariant are lost. Noether's theorem states that symmetries and conservation laws correspond to each other. Hence conservation laws are lost and conserved quantities are ill-defined. Usually this is not seen a practical problem since gravitation is so weak interaction.

2. The proposed way out of the problem is based on the assumption that space-times are imbeddable as 4-surfaces to some 8-dimensional space $H = M^4 \times S$ by replacing the points of 4-D empty Minkowski space with 4-D very small internal space S. The space S is unique from the requirement that the theory has the symmetries of standard model: $S = CP_2$, where CP_2 is complex projective space with 4 real dimensions [K113]. Isometries of space-time are replaced with those of imbedding space. Noether's theorem predicts the classical conserved charges for given general coordinate invariant (GCI) action principle.

Also now the curvature of space-time codes for gravitation. Equivalence Principle (EP) and General Coordinate Invariance (GCI) of GRT augmented with Relativity Principle (RT) of SRT remain the basic principles. Now however the number of solutions to field equations - preferred extremals (PEs) - is dramatically smaller than in Einstein's theory [K8, K14].

1. An unexpected bonus was geometrization classical fields of standard model for $S = CP_2$. Also the space-time counterparts for field quanta emerge naturally but this requires a profound generalization of the notion of space-time: the topological inhomogenities of space-time surface are identified as particles. This means a further huge reduction for dynamical field like variables at the level of single space-time sheet. By general coordinate invariance (GCI) only four imbedding space coordinates appear as variables analogous to classical fields: in a typical GUT their number is hundreds.

2. CP_2 also codes for the standard model quantum numbers in its geometry in the sense that electromagnetic charge and weak isospin emerge from CP_2 geometry: the corresponding symmetries are not isometries so that electroweak symmetry breaking is coded already at this level. Color quantum numbers correspond to the isometries of CP_2 defining an unbroken symmetry: this also conforms with empirical facts. The color of TGD however differs from that in standard model in several aspects and LHC has began to exhibit these differences via the unexpected behavior of what was believed to be quark gluon plasma [K58]. The conservation of baryon and lepton numbers follows as a prediction. Leptons and quarks correspond to opposite chiralities for imbedding space spinors.

3. What remains to be explained in standard model is family replication phenomenon for leptons and quarks. Both quarks and leptons appear as three families identical apart from having different masses. The conjecture was is that fermion families correspond to different topologies for 2-D surfaces characterized by genus telling the number g (genus) of handles attached to sphere to obtain the surface: sphere, torus, ... The 2-surfaces are identified as "partonic 2-surfaces" whose orbits are light-like 3-surface at which the signature of the induced metric of space-time surface transforms from Minkowskian to Euclidian. The partonic orbits replace the lines of Feynman diagrams in TGD Universe in accordance with the replacement of point-like particle with 2-surface.

   Only the three lowest genera are observed experimentally. A possible explanation is in terms of conformal symmetries: the genera $g \leq 2$ allow always Z_2 as a subgroup of conformal symmetries (hyper-ellipticity) whereas higher genera in general do not. The handles of partonic 2-surfaces could form analogs of unbound many-particle states for $g > 2$ with a continuous spectrum of mass squared but for $g = 2$ form a bound state by hyper-ellipticity [K23].

4. Later further arguments in favor of$H = M^4 \times CP_2$ have emerged. One of them relates to twistorialization and twistor lift of TGD [K101, K38, K10]. 4-D Minkowski space is unique space-time with Minkowskian signature of metric in the sense that it allows twistor structure. This is a problem in attempts to introduce twistors to General Relativity Theory (GRT) and a serious obstacle in the quantization based on twistor Grassmann approach, which has demonstrate its enormous power in the quantization of gauge theories. In TGD framework one can ask whether one could lift also the twistor structure to the level of H. M^4 has twistor structure and so does also CP_2: which is the only Euclidian 4-manifold allowing twistor space, which is also a Kähler manifold! This led to the notion of twistor lift of TGD inducing rather recent breakthrough in the understanding of TGD.

TGD can be also seen as a generalization of hadronic string model - not yet superstring model since this model became fashionable two years after the thesis about TGD [K2]. Later it has become clear that string like objects, which look like strings but are actually 3-D are basic stuff of TGD Universe and appear in all scales [K27, K8]. Also strictly 2-D string world sheets popped up in the formulation of quantum TGD (analogy with branes) [**?**]o that one can say that string model in 4-D space-time is part of TGD.

Concluding, TGD generalizes standard model symmetries and provides an incredibly simple proposal for a dynamics: only 4 classical field variables and in fermionic sector only quark and lepton like spinor fields. The basic objection against TGD looks rather obvious in the light of after-wisdom. One loses linear superposition of fields, which holds in good approximation in ordinary field theories, which are almost linear. The solution of the problem to be discussed later relies on the notion many-sheeted space-time [K18].

Many-sheeted space-time

The replacement of the abstract manifold geometry of general relativity with the geometry of 4-surfaces brings in the shape of surface as seen from the perspective of 8-D space-time as additional degrees of freedom giving excellent hopes of realizing the dream of Einstein about geometrization of fundamental interactions.

The work with the generic solutions of the field equations assignable to almost any variational principle satisfying GCI led soon to the realization that the topological structure of space-time in this framework is much more richer than in GRT.

1. Space-time decomposes into space-time sheets of finite size. This led to the identification of physical objects that we perceive around us as space-time sheets. The original identification of space-time sheet was as a surface of in H with outer boundary. For instance, the outer boundary of the table would be where that particular space-time sheet ends (what "ends" means is not however quite obvious!). We would directly see the complex topology of many-sheeted space-time! Besides sheets also string like objects and elementary particle like objects appear so that TGD can be regarded also as a generalization of string models obtained by replacing strings with 3-D surfaces.

   It turned that boundaries are probably excluded by boundary conditions. Rather, two sheets with boundaries must be glued along their boundaries together to get double covering. Sphere can be seen as simplest example of this kind of covering: northern and southern hemispheres are glued along equator together.

2. The original vision was that elementary particles are topological inhomogenities glued to these space-time sheets using topological sum contacts. This means drilling a hole to both sheets and connecting with a very short cylinder. 2-dimensional illustration should give the idea. In this conceptual framework material structures and shapes would not be due to some mysterious substance in slightly curved space-time but reduce to space-time topology just as energy- momentum currents reduce to space-time curvature in GRT.

   This view has gradually evolved to much more detailed picture. Elementary particles have wormhole contacts as basic building bricks. Wormhole contact is very small region with *Euclidian (!)* signature of the induced metric connecting two Minkowskian space-time sheets with light-like boundaries carrying spinor fields and there particle quantum numbers. Wormhole contact carries magnetic monopole flux through it and there must be second wormhole contact in order to have closed lines of magnetic flux. Particle world lines are replaced with 3-D light-like surfaces - orbits of partonic 2-surfaces - at which the signature of the induced metric changes.

   One might describe particle as a pair of magnetic monopoles with opposite charges. With some natural assumptions the explanation for the family replication phenomenon in terms of the genus g of the partonic 2-surface is not affected. Bosons emerge as fermion anti-fermion pairs with fermion and anti-fermion at the opposite throats of the wormhole contact. In principle family replication phenomenon should have bosonic analog. This picture assigns to particles strings connecting the two wormhole throats at each space-time sheet so that string model mathematics becomes part of TGD.

The notion of classical field differs in TGD framework in many respects from that in Maxwellian theory.

1. In TGD framework fields do not obey linear superposition and all classical fields are expressible in terms of four imbedding space coordinates in given region of space-time surface. Superposition for classical fields is replaced with *superposition of their effects* [K97, K113] - in full accordance with operationalism. Particle can topologically condense simultaneously to several space-time sheets by generating topological sum contacts (not stable like the wormhole contacts carrying magnetic monopole flux and defining building bricks of particles). Particle "experiences" the superposition of the effects of the classical fields at various space-time sheets rather than the superposition of the fields.

   It is also natural to expect that at macroscopic length scales the physics of classical fields (to be distinguished from that for field quanta) can be explained using only four primary field like variables. Electromagnetic gauge potential has only four components and classical electromagnetc fields give and excellent description of physics. This relates directly to electroweak symmetry breaking in color confinement which in standard model imply the effective absence of weak and color gauge fields in macroscopic scales. TGD however predicts that copies of hadronic physics and electroweak physics could exist in arbitrary long scales [K57] and there are indications that just this makes living matter so different as compared to inanimate matter.

2. The notion of induced gauge field means that one induces electroweak gauge potentials defining so called spinor connection at space-time surface (induction of bundle structure). Induction boils down locally to a projection of the imbedding space vectors representing the spinor connection. The classical fields at the imbedding space level are non-dynamical and fixed and extremely simple: one can say that one has generalization of constant electric field and magnetic fields in CP_2. The dynamics of the 3-surface however implies that induced fields can form arbitrarily complex field patterns. This is essentially dynamics of shadows.

   Induced gauge fields are not equivalent with ordinary free gauge fields. For instance, the attempt to represent constant magnetic or electric field as a space-time time surface has a limited success. Only a finite portion of space-time carrying this field allows realization as 4-surface. I call this topological field quantization. The magnetization of electric and magnetic fluxes is the outcome. Also gravitational field patterns allowing imbedding are very restricted: one implication is that topological with over-critical mass density are not globally imbeddable. This would explain why the mass density in cosmology can be at most critical. This solves one of the mysteries of GRT based cosmology [K90].

   Quite generally, the field patterns are extremely restricted: not only due to imbeddability constraint but also due to the fact that by SH only very restricted set of space-time surfaces can appear solutions of field equations: I speak of preferred extremals (PEs) [K8, K14, K18]. One might speak about archetypes at the level of physics: they are in quite strict sense analogies of Bohr orbits in atomic physics: this is implies by the realization of GCI. This kind of simplicity does not conform with what we observed. The way out is many-sheeted space-time. Although fields do not superpose, particles experience the superposition of effects from the archetypal field configurations (superposition is replaced with set theoretic union).

3. The important implication is that one can assign to each material system a field identity since electromagnetic and other fields decompose to topological field quanta. Examples are magnetic and electric flux tubes and flux sheets and topological light rays representing light propagating along tube like structure without dispersion and dissipation making em ideal tool for communications [K69]. One can speak about field body or magnetic body of the system.

   Field body indeed becomes the key notion distinguishing TGD inspired model of quantum biology from competitors but having applications also in particle physics since also leptons and quarks possess field bodies. The is evidence for the Lamb shift anomaly of muonic hydrogen [C1] and the color magnetic body of u quark whose size is somewhat larger than the Bohr radius could explain the anomaly [K58]. The magnetic flux tubes of magnetic body carry monopole fluxes existing without generating currents. In cosmology the flux tubes assignable to the remnants of cosmic strings make possible long range magnetic fields in all scales impossible in standard cosmology. Also super-conductivity is proposed to rely on dark $h_{eff} = n \times h$ Cooper pairs at pairs of flux tubes carrying monopole flux.

GRT and gauge theory limit of TGD is obtained as an approximation.

1. GRT/gauge theory type description is an approximation obtained by lumping together the space-time sheets to single region of M^4, with gravitational fields and gauge potentials as sums of corresponding induced field quantities at space-time surface geometrized in terms of geometry of H. Gravitational field corresponds to the deviation of the induced metric from Minkowski metric using M^4 coordinates for space-time surface so that the description applies only in long length scale limit.

   Space-time surface has both Minkowskian and Euclidian regions. Euclidian regions are identified in terms of what I call generalized scattering/twistor diagrams. The 3-D boundaries between Euclidian and Minkowskian regions have degenerate induced 4-metric and I call them light-like orbits of partonic 2-surfaces or light-like wormhole throats analogous to blackhole horizons. The interiors of blackholes are replaced with the Euclidian regions and every physical system is characterized by this kind of region.

   Lumping of sheets together implies that global conservation laws cannot hold exactly true for the resulting GRT type space-time. Equivalence Principle (EP) as Einstein's equations stating conservation laws locally follows as a local remnant of Poincare invariance.

2. Euclidian regions are identified as slightly deformed pieces of CP_2 connecting two Minkowskian space-time regions. Partonic 2-surfaces defining their boundaries are connected to each other by magnetic flux tubes carrying monopole flux.

   Wormhole contacts connect two Minkowskian space-time sheets already at elementary particle level, and appear in pairs by the conservation of the monopole flux. Flux tube can be visualized as a highly flattened square traversing along and between the space-time sheets involved. Flux tubes are accompanied by fermionic strings carrying fermion number. Fermionic strings give rise to string world sheets carrying vanishing induced em charged weak fields (otherwise em charge would not be well-defined for spinor modes). String theory in space-time surface becomes part of TGD. Fermions at the ends of strings can get entangled and entanglement can carry information.

3. Strong form of GCI (SGCI) states that light-like orbits of partonic 2-surfaces on one hand and space-like 3-surfaces at the ends of causal diamonds on the other hand provide equivalent descriptions of physics. The outcome is that partonic 2-surfaces and string world sheets at the ends of CD can be regarded as basic dynamical objects.

   Strong form of holography (SH) states the correspondence between quantum description based on these 2-surfaces and 4-D classical space-time description, and generalizes AdS/CFT correspondence. One has huge super-symplectic symmetry algebra acting as isometries of WCW with conformal structure [K26, K84, K111], conformal algebra of light-cone boundary extending the ordinary conformal algebra, and ordinary Kac-Moody and conformal symmetries of string world sheets. This explains why 10-D space-time can be replaced with ordinary space-time and 4-D Minkowski space can be replaced with partonic 2-surfaces and string world sheets. This holography looks very much like the one we are accustomed with!

9.2.2 Zero energy ontology (ZEO)

In standard ontology of quantum physics physical states are assumed to have positive energy. In zero energy ontology (ZEO) [K61] physical states decompose to pairs of positive and negative energy states such that the net values of the conserved quantum numbers vanish. The interpretation of these states in ordinary ontology would be as transitions between initial and final states, physical events.

ZEO and positive energy ontology

ZEO is consistent with the crossing symmetry of QFTs meaning that the final states of the quantum scattering event can be described formally as negative energy states. As long as one can restrict the consideration to either positive or negative energy part of the state ZEO is consistent with positive energy ontology. This is the case when the observer characterized by a particular CD studies the physics in the time scale of much larger CD containing observer's CD as a sub-CD. When the time scale sub-CD of the studied system is much shorter that the time scale of sub-CD characterizing the observer, the interpretation of states associated with sub-CD is in terms of quantum fluctuations.

ZEO solves the problem, which emerges in any theory assuming symmetries giving rise to conservation laws. The problem is that the theory itself is not able to characterize the values of conserved quantum numbers of the initial state of say cosmology. In ZEO this problem disappears since in principle any zero energy state is obtained from any other state by a sequence of quantum jumps without breaking of conservation laws. The fact that energy is not conserved in GRT based cosmologies can be also understood since each CD is characterized by its own conserved quantities. As a matter fact, one must speak about average values of conserved quantities since one can have a quantum superposition of zero energy states with the quantum numbers of the positive energy part varying over some range.

At the level of principle the implications are quite dramatic. In quantum jump as recreation replacing the quantum Universe with a new one it is possible to create entire sub-universes from vacuum without breaking the fundamental conservation laws. From the point of view of consciousness theory the important implication is that "free will" is consistent with the laws of physics. This makes obsolete the basic arguments in favor of materialistic and deterministic world view.

Zero energy states are located inside causal diamond (CD)

By quantum classical correspondence zero energy states must have space-time and imbedding space correlates.

1. Positive and negative energy parts of zero energy state reside at future and past light-like boundaries of causal diamond (CD) identified as intersection of future and past directed light-cones and visualizable as double cone. The analog of CD in cosmology is big bang followed by big crunch. Penrose diagrams provide an excellent 2-D visualization of the notion. CDs form a fractal hierarchy containing CDs within CDs. Disjoint CDs are possible and CDs could also intersect.

   The interpretation of CD in TGD inspired theory of consciousness is as an imbedding space correlate for perceptive field of conscious entity: the contents of conscious experience is about the region defined by CD. At the level of space-time sheets the experience come from space-time sheets in the interior of CD. Whether the sheets can be assumed to continue outside CD is still unclear.

   Quantum measurement theory must be modified in ZEO since state function reduction can happen at both boundaries of CD and the reduced states at opposite boundaries are related by time reversal. One can also have quantum superposition of CDs changing between reductions at active boundary followed by localization in the moduli space of CDs with the tip of passive boundary fixed. Quantum measurement theory generalizes to a theory of consciousness with continuous entity identified as a sequence of state function reductions at active (changing) boundary of CD [K7].

2. By number theoretical universality (NTU) the temporal distances between the tips of the intersecting light-cones are assumed to come as integer multiples $T = m\times T_0$ of a fundamental time scale T_0 defined by CP_2 size R as $T_0 = R/c$. p-Adic length scale hypothesis [K63, K109] motivates the stonger hypothesis that the distances tend to come as as octaves of T_0: $T = 2^n T_0$. One prediction is that in the case of electron this time scale is .1 seconds defining the fundamental biorhythm. Also in the case u and d quarks the time scales correspond to biologically important time scales given by 10 ms for u quark and by and 2.5 ms for d quark [K9] . This means a direct coupling between microscopic and macroscopic scales.

9.2.3 Quantum physics as physics of classical spinor fields in WCW

The notions of Kähler geometry of "World of Classical Worlds" (WCW) and WCW spinor structure are inspired by the vision about the geometrization of the entire quantum theory.

Motivations for WCW

The notion of "World of Classical Worlds" (WCW) [K43, K26, K84] was forced by the failure of both path integral approach and canonical quantization in TGD framework. The idea is that the Kähler function defining WCW Kähler geometry is determined by the real part of an action S determining space-time dynamics and receiving contributions from both Minkowskian and Euclidian regions of space-time surface X^4 (note that $\sqrt{g_4}$ is proportional to imaginary unit in Minkoskian regions).

1. If S is space-time volume both canonical quantization and path integral would make sense at least formally since in principle one could solve the time derivatives of four imbedding space coordinates as functions of canonical momentum densities (general coordinate invariance allows to eliminate four coordinates). The calculation of path integral is however more or less hopeless challenge in practice.

2. A mere space-time volume as action is however not physically attractive. This was thought to leave under consideration only Kähler action S_K - Maxwell action for the induced Kähler form expressible in terms of gauge potential defined by the induced Kähler gauge potential of CP_2. This action has however a huge vacuum degeneracy. Any space-time surface with at most 2-D CP_2 projection, which is Lagrangian sub-manifold of CP_2, is vacuum extremal. Symplectic transformations acting like U(1) gauge transformations generate new vacuum

extremals. They however fail to act as symmetries of non-vacuum extremals so that gauge invariance is not in question: the deviation of the induced metric from flat metric is the reason for the failure. This degeneracy is assumed to give rise to what might be called 4-D spin glass degeneracy meaning that the landscape for the maxima of Kähler function is fractal.

3. Canonical quantization fails because by the extreme non-linearity of the action principle making it is impossible to solve time derivatives explicitly in terms of canonical momentum densities. The problem is especially acute for the canonical imbedding of empty Minkowski space to $M^4 \times CP_2$. The action is vanishing up to fourth order in imbedding space coordinates so that canonical momentum densities vanish identically and there is no hope of defining propagator in path integral approach. The mechanical analog would be criticality around which the potential reduces to $V \propto x^4$. Quantum criticality is indeed a basic aspect of TGD Universe.

The hope held for a long time was that WCW geometry allowing to get rid of path integral would solve the problems. One could however worry about vacuum degeneracy implying that WCW metric becomes extremely degenerate for vacuum extremals and also holography becomes extremely non-unique for them. Also the expected feailure of perturbative approach around M^4 is troublesome.

WCW and twistor lift of TGD

During last year this picture has indeed changed thanks to what might be called twistor lift of TGD [K101, K38, K10] inspired by twistor Grassmann approach to supersymmetric gauge theories [B16]. Remarkably, twistor lift would provide automatically the fundamental couplings of standard model and GRT and also the scale assigned to GUTs as CP_2 radius. PEs would be both extremals of Kähler action and minimal surfaces.

1. The basic observation is E^4, and its Euclidian compactification S^4 and CP_2 are completely unique in that they allow twistor space with Kähler structure [A45]. This was discovered by Hitchin at roughly the same time as I discovered TGD! This generalizes to M^4 having a generalization of ordinary Kähler structure to what I have called Hamilton-Jacobi structure by decomposition $M^4 = M^2 \times E^2$, where M^2 allows hypercomplex structure [K101, K38]. One can consider also integral distributions of tangent decompositions $M^4 = M^2(x) \times E^2(x)$, depending on position. The twistor space has a double fibration by S^2 with base spaces identifiable as M^4 and conformal compactification of M^4 for which metric is defined only up to conformal scaling. The first fibration $M^4 \times S^2$ with a well-defined metric would correspond to the classical TGD.

2. Both Newton's constant G and cosmological constant Λ emerge from twistor lift in M^4 factor. The radius of S^2 is identified in terms of Planck length $l_P = \sqrt{G}$. For CP_2 factor, the radius corresponds to the radius of CP_2 geodesic sphere. 4-D Kähler action can be lifted to 6-D Kähler action only for $M^4 \times CP_2$ so that TGD would be completely unique both mathematically and physically. The twistor space of CP_2 is flag-manifold $SU(3)/U(1) \times U(1)$ having interpretation as the space for the choices of quantization axis of color isospin and hypercharge. This choice could correspond to a selection of Eguchi-Hanson complex coordinates for CP_2 by fixing their phase angles in which isospin and hypercharge rotations induce shifts.

3. The physically motivated conjecture is that the PEs can be lifted to their 6-D twistor bundles with S^2 serving as a fiber, that one induce the twistor structure and the outcome is equal to the twistor structure of space-time surface, and that this condition is at least part of the PE property. This would correspond to the solution of massless wave equations in terms of twistors in the original twistor approach of Penrose [B28]. The analog of spontaneous compactification would lead to 4-D action equal to Kähler action plus volume term. One could of course postulate this action directly without mentioning twistors at all.

   The coefficient of the volume term would correspond to dark energy density characterized by cosmological constant Λ being extremely small in cosmological scales. It removes vacuum degeneracy although the situation remains highly non-perturbative. This can be combined with the earlier conjecture that cosmological constant Λ behaves as $\Lambda \propto 1/p$ under p-adic coupling constant evolution so that Λ would be large in primordial cosmology.

4. The generic extremals of space-time action would depend on coupling parameters, which does not fit with the number theoretic vision inspiring speculations that space-time surface can be seen as quaternionic sub-manifolds of 8-D octonionic space-time [K96], satisfying quaternion analyticity [K38], or a 4-D generalization of holomorphy. By SH the extremals are however "preferred". What could this imply?

   Intriguingly, all known non-vacuum extremals and also CP_2 type vacuum extremals having null-geodesic as M^4 projection are extremals of both Kähler action and volume term separately! The dynamics for volume term and Kähler action effectively decouple and coupling constants do not appear at all in field equations. The twistor lift would only select minimal surface amongst vacuum extremals, modify the Kähler function of WCW identifiable as exponent for the real part of action, and provide a profound mathematical and physical motivation for cosmological constant Λ remaining mysterious GRT framework. One could even hope that preferred extremals are nothing but minimal surface extremals of Kähler action with the vanishing conditions for some sub-algebra of super-symplectic algebra satisfied automatically!

   The analog of decoupling of Kähler action and volume term should take place also for induced spinors. This is expected if mere analyticity properties make spinor modes solutions of modified Dirac equations. This is true in 2-D case Hamilton-Jacobi structure should guarantee this in 4-D case [K111, K38].

   PEs depend on coupling parameters only via boundary conditions stating the vanishing of Noether charges for a sub-algebra of super-symplectic algebra and its commutator with entire algebra. Also the conservation conditions at 3-D light-like surfaces at which the signature of metric changes imply dependence on coupling parameters. These conditions allow the transfer of classical charges between Minkowskian and Euclidian regions necessary to understand momentum exchange between particles and environment classically only if Kähler couplings strength is complex - otherwise there is no exchange of conserved quantities since their real *resp.* imaginary at the two sides [K35]. Interestingly, also in twistor Grassmann approach the massless poles in propagators are complex.

   This picture conforms with the conjecture that discrete p-adic evolution of the Kähler coupling strength in subset of primes near prime powers of two corresponds to complex zeros of zeta [K35]. This conforms also with the conjectured discreteness of p-adic coupling constant evolution by phase transitions changing the values of coupling parameters. One implication is that all loop corrections in functional integral vanish.

5. In path integral approach quantum TGD would be extremely non-perturbative around extremals for which Kähler action vanishes. Same is true also in WCW approach. The cure would be provided by the hierarchy of Planck constants $h_{eff}/h = n$, which effectively scales Λ down to Λ/n. n would be the number sheets of the M^4 covering defined by the space-time surface: the action of Galois group for the number theoretic discretization of space-time surface could give rise to this covering. The finiteness of the volume term in turn forces ZEO: the volume of space-time surface is indeed finite due to the finite size of CD.

Consider now the delicacies of this picture.

1. Should assign also to M^4 the analog of symplectic structure giving an additional contribution to the induced Kähler form? The symmetry between M^4 and CP_2 suggests this, and this term could be highly relevant for the understanding of the observed CP breaking and matter antimatter asymmetry [L27]. Poincare invariance is not lost since the needed moduli space for M^4 Kähler forms would be the moduli space of CDs forced by ZEO in any case, and M^4 Kähler form would serve as the correlate for fixing rest system and spin quantization axis in quantum measurement.

2. Also induced spinor fields are present. The well-definedness of electro-magnetic charge for the spinor modes forces in the generic case the localization of the modes of induced spinor fields at string world sheets (and possibly to partonic 2-surfaces) at which the induced charged weak gauge fields and possibly also neutral Z^0 gauge field vanish. The analogy with branes and super-symmetry force to consider two options.

   Option I: The *fundamental* action principle for space-time surfaces contains besides 4-D action also 2-D action assignable to string world sheets, whose topological part (magnetic flux) gives rise to a coupling term to Kähler gauge potentials assignable to the 1-D boundaries of string world sheets containing also geodesic length part. Super-symplectic symmetry demands that modified Dirac action has 1-, 2-, and 4-D parts: spinor modes would exist at both string boundaries, string world sheets, and space-time interior. A possible interpretation for the interior modes would be as generators of space-time super-symmetries [K87].

   This option is not quite in the spirit of SH and string tension appears as an additional parameter. Also the conservation of em charge forces 2-D string world sheets carrying vanishing induced W fields and this is in conflict with the existence of 4-D spinor modes unless they satisfy the same condition. This looks strange.

   Option II: Stringy action and its fermionic counterpart are effective actions only and justified by SH. In this case there are no problems of interpretation. SH requires only that the induced spinor fields at string world sheets determine them in the interior much like the values of analytic function at curve determine it in an open set of complex plane. At the level of quantum theory the scattering amplitudes should be determined by the data at string world sheets. If induced W fields at string world sheets are vanishing, the mixing of different charge states in the interior of X^4 would not make itself visible at the level of scattering amplitudes! In this case 4-D spinor modes do not define space-time super-symmetries.

3. Why the string world sheets coding for effective action should carry vanishing weak gauge fields? If M^4 has the analog of Kähler structure [L27], one can speak about Lagrangian sub-manifolds in the sense that the sum of the symplectic forms of M^4 and CP_2 projected to Lagrangian sub-manifold vanishes. Could the induced spinor fields for effective action be localized to generalized Lagrangian sub-manifolds? This would allow both string world sheets and 4-D space-time surfaces but SH would select 2-D Lagrangian manifolds. At the level of effective action the theory would be incredibly simple.

   Induced spinor fields at string world sheets could obey the "dynamics of avoidance" in the sense that *both* the induced weak gauge fields W, Z^0 and induced Kähler form (to achieve this U(1) gauge potential must be sum of M^4 and CP_2 parts) would vanish for the regions carrying induced spinor fields. They would coupleonly to the *induced em field (!)* given by the R_{12} part of CP_2 spinor curvature [K15] for $D = 2, 4$. For $D = 1$ at boundaries of string world sheets the coupling to gauge potentials would be non-trivial since gauge potentials need *not* vanish there. Spinorial dynamics would be extremely simple and would conform with the vision about symmetry breaking of weak group to electromagnetic gauge group.

   The projections of canonical currents of Kähler action to string world sheets would vanish, and the projections of the 4-D modified gamma matrices would define just the induced 2-D metric. If the induced metric of space-time surface reduces to an orthogonal direct sum of string world sheet metric and metric acting in normal space, the flow defined by 4-D canonical momentum currents is parallel to string world sheet. These conditions could define the "boundary" conditions at string world sheets for SH.

This admittedly speculative picture has revolutionized the understanding of both classical and quantum TGD during last year. [K38, K10, K18]. In particular, the construction of single-sheeted PEs as minimal surfaces allows a kind of lego like engineering of more complex PEs [L13]. The minimal surface equations generalize Laplace equation of Newton's gravitational theory to non-linear massless d'Alembert equation with gravitational self-coupling. One obtains the analog of Schwartschild solution and radiative solutions describing also gravitational radiation [K18]. An open question is whether classical theory makes sense if also the analog of Kähler form in M^4 is allowed.

Identification of WCW

The notion of WCW [K43, K26, K84] was inspired by the super-space approach of Wheeler in which 3-geometries are the basic geometric entities.

1. In TGD framework 3-surfaces take this role. Einstein's program for geometrizing classical physics is generalized to a geometrization of entire quantum physics. Hermitian conjugation corresponds to complex conjugation in infinite-dimensional context so that WCW must have Kähler geometry. The geometrization of fermionic statistics/oscillator operators is in terms of gamma matrices of WCW expressible as linear combinations of oscillator operators for second quantized induced spinor field. Formally purely classical spinor modes of WCW represent many fermion states as functionals of 3-surface. One can also interpret gamma matrices as generators of super-conformal symmetries in accordance with the fact that also SUSY involves Clifford algebra.

   In ZEO the entanglement coefficients between positive and negative energy parts of zero energy states determine the S-matrix so that S-matrix would be coded by the modes of WCW spinor fields. Twistor approach to TGD [K38] suggests that the S-matrix reduces completely to the symmetries defined by the multi-local (locus corresponds to partonic 2-surface) generators of the Yangian associated with the super-symplectic algebra.

2. ZEO forces to identify 3-surfaces as pairs of 3-surfaces with members at the opposite boundaries of CD. SH reduces them to a collection of partonic 2-surfaces at boundaries of CD plus number theoretic discretization in space-time interior. Basic geometric objects are pairs of initial and final states (coordinates for both in mechanical analogy) rather than initial states with initial value conditions (coordinates and velocities in mechanical analogy) and initial value problem transforms to boundary value problem. Processes rather than states become the basic elements of ontology: this has far reaching consequences in biology and neuroscience.

3. The realization of GCI requires that the definition of WCW Kähler function assigns to a "physically" 3-surface a unique 4-surface for 4-D general coordinate transformations to act: "physically" could mean "apart from transformations acting as gauge transformations" not affecting the action and conserved classical charges. The outcome is holography.

4. Strong form of holography (SH) would emerge as follows. The condition that light-like 3-surfaces defining boundaries between Euclidian and Minkowskian regions are basic geometric entities equivalent with pairs of space-like 3-surfaces at the ends of given causal diamond CD implies SH: partonic 2-surfaces and their 4-D tangent space data should code the physics. One could also speak about almost/effective 2-dimensionality. Tangent space data could in turn be coded by string world sheets. Number theoretical discretitization of space-time interior with preferred coordinates in the extension of rationals could give meaning for "almost".

5. Kähler metric is expressible both in terms of second derivatives of Kähler function K [K43] and as anticommutators of WCW gamma matrices expressible as linear combinations of fermionic oscillator operators. This suggests a close relationship between space-time dynamics and spinor dynamics.

   Super-symplectic symmetry between the action defining space-time surfaces (Kähler action plus volume term) and modified Dirac action would realize this relationship. This is achieved if the modified gamma matrices are defined by the canonical momentum currents of 2-D action associated with string world sheets. These currents are parallel to the string world sheets. This implies the analog of AdS/CFT correspondence requiring only that induced spinor modes at string world sheets determine them in space-time interior (this is like analytic continuation). The localization of spinor modes at string world sheets is *not* required as I believed first.

The geometry of loop spaces developed by Freed [A30] serves as a model in the construction of WCW Kähler geometry [K84].

1. The existence of loop space Riemann connection requires maximal isometry group identifiable as Kac-Moody group so that Killing vector fields span the entire tangent space of the loop space.

2. In TGD framework the properties of Kähler action lead to the idea that WCW is union of homogenous or even symmetric spaces of symplectic algebra acting at the boundary of $\delta CD \subset \delta CD_+ \cup \delta CD_-$, $\delta CD_\pm \subset \delta M^4_\pm \times CP_2$. ZEO requires that the conserved quantum numbers for physical states are opposite for the positive and negative energy parts of the states at the two opposite boundary parts of CD. The symmetric spaces G/H in the union are labelled by zero modes, which do not appear in the line element as differentials but only as parameters of the metric. Conserved Noether charges of isometries and symplectic invariants of examples of zero modes as also the super-symplectic Noether charges invariant under complex conjugation of WCW coordinates.

3. Homogenous spaces of the symplectic group G are obtained by dividing by a subgroup H. An especially attractive option is suggested by the fractal structure of the symplectic algebra containing an infinite hierarchy of sub-algebras G_n for which conformal weights are $n > 0$-multiples of those for G. For this option $H = G_n$ is isomorphic to G and one could have infinite hierarchies of inclusions analogous to the hierarchy of inclusions of hyperfinite factors of type II_1 (HFFs). PE property requires almost 2-dimensionality and elimination of huge number of degrees of freedom. The natural condition is that the Noether charges of G_n vanish at the ends of CD. A stronger condition is that also the Noether charges for $[G, G_n]$ vanish. This implies effective normal algebra property and G/G_n acts effectively like group.

   The inclusion of HFFs would define measurement resolution with included factor acting like gauge algebra. Measurement resolution would be naturally determined by the number theoretic discretization of the space-time surface so that physics as geometry and number theory visions would meet each other.

4. This inclusion hierarchy can be identified in terms of quantum criticality (QC). The transitions $n \to kn$ increasing the value of $n > 0$ reduce QC since pure gauge symmetries are reduced, and new physical super-symplectic degrees of freedom emerge. QC also requires that Kähler couplings strength analogous to temperature is analogous to critical temperature so that the quantum theory is uniquely defined if their is only one critical temperature. Spectrum for α_K seems more plausible and the possibility that Kähler coupling strength depends on the level of the number theoretical hierarchy defined by the allowed extensions of rationals can be considered [K35].

WCW spinor structure

The basic idea is geometrization of quantum states by identifying them as modes of WCW spinor fields [K111, K84]. This requires definition of WCW spinors and WCW spinor structure, WCW gamma matrices and Dirac operator, etc..

The starting point is the definition of WCW gamma matrices using a representation analogous to the usual vielbein representation as linear combinations of flat space gamma matrices. The conceptual leap is the observation that there is no need to assume that the counterparts of flat space gamma matrices have vectorial quantum numbers. Instead, they are identified as fermionic oscillator operators for second quantized free induced spinor fields at space-time surface.

This allows geometrization of the fermionic statistics since WVW spinors for a given 3-surface are analogous to fermionic Fock states. One can also say that spinor structure follows as a square root of metric and also that the spinor basis defines a geometric correlate of Boolean mind [K22]. The dependence of WCW spinor field on 3-surface represents the bosonic degrees of freedom not reducible to many-fermion states. For instance, most of hadron mass would be associated with these degrees of freedom.

Quantum TGD involves Dirac equations at space-time level, imbedding space level, and level of WCW. The dynamics of the induced spinor fields is related by super-symmetry to the action defining space-time surfaces as preferred extremals. [K111, K84].

1. The gamma matrices in the equation - modified gamma matrices - are determined by contractions of the canonical momentum currents of Kähler action with the imbedding space

gamma matrices. The localization at string world sheets for which only induced neutral weak fields or only em field are non-vanishing is accompanied by the integrability condition that various conserved currents run along string world sheets: one can speak of sub-flow. I

2. Modified Dirac equation can be solved exactly just like in the case of string models using holomorphy and the properties of complexified modified gamma matrices. This is expected to be true also in 4-D case by Hamilton-Jacobi structure. If the dynamics of avoidance is realized the modified Dirac equation would be essentially free Dirac equation and holomorphy would allow to solve it.

At the level of WCW one obtains also the analog of massless Dirac equation as the analog of super Virasoro conditions of Super Virasoro algebra.

1. The fermionic counterparts of super-conformal gauge conditions assignable with sub-algebra G_n of supersymplectic conformal symmetry associated with the both light-cone boundary (light-like radial coordinate), with conformal symmetries of light-cone boundary, and with string world sheets.

2. The ground states of supersymplectic representations satisfy massless imbedding space Dirac equation in imbedding space so that Dirac equations in WCW, in imbedding space, and at string world sheets are involved. In twistorialization also massless M^8 Dirac equation emerges in the tangent space M^8 of imbedding space assignable to the partonic 2-surfaces and generalizes the 4-D light-likeness with its 8-D counterpart applying to states with M^4 mass. Here octonionic representation of imbedding space gamma matrices emerges naturally and allows to speak about 8-D analogs of Pauli's sigma matrices [K101].

9.2.4 Quantum criticality, measurement resolution, and hierarchy of Planck constants

The notions of quantum criticality (QC), finite measurement resolution, and hierarchy of Planck constants proposed to give rise to dark matter as phases of ordinary matter are central for TGD [**?**, K110, K36].

These notions relate closely to the strong form of holography (SH) implied by strong form of general coordinate invariance (SGCI). In adelic physics all this would relate closely to the hierarchy of extensions of rationals serving as a correlate for number theoretical evolution.

Finite measurement resolution and fractal inclusion hierarchy of super-symplectic algebras

The fractal hierarchy of isomorphic sub-algebras of supersymplectic algebra - call it g - defines an excellent candidate for the realization of finite the measurement resolution. Similar hierarchies can be assigned also for the extended super-conformal algebra assignable with light-like boundaries of CD and with Kac-Moody and conformal algebras assignable to string world sheets.

An interesting possibility is that the the conformal weights assignable to infinitesimal scaling operator of the light-like radial coordinate of light-cone boundary correspond to zeros of Riemann zeta [K109] [L9]. A kind of dual spectrum would correspond to conformal weights that correspond to logarithms for powers of primes. One can identify the conformal weight as negative of the pole of fermionic zeta $z_F = \zeta(s)/\zeta(2s)$ natural in TGD framework. The real part of conformal weight for the generators is $h_R = -1/4$ for "non-trivial" poles and positive integer $h = n > 0$ for "trivial" poles. There is also a pole for $h = -1$. Hence one obtains tachyonic ground states, which must be assumed also in p-adic mass calculations [K51].

Also the generators of Yangian algebra [K101] integrating the algebras assignable to various partonic 2-surfaces to a multi-local algebra are labelled by a non-negative integer n analogous to conformal weight and telling the number of partonic 2-surfaces involved with the action of the generator. Also this algebra has similar fractal hierarchy of sub-algebras so that the considerations that follow might apply also to it. Now that number of partonic 2-surface would play the role of measurement resolution.

As noticed, there are also other algebras, which allow conformal hierarchy if one can restrict the conformal weights to be non-negative. The first of them generates generalized conformal transformations of light-cone boundary depending on light-like radial coordinate as parameter: also now radial conformal weights for generators can have zeros of zeta as spectrum. As a special case one obtains infinite-dimensional group of isometries of light-cone boundary. Second one corresponds to ordinary conformal and Kac-Moody symmetries for induced spinor fields acting on string world sheets. Also here similar hierarchy of sub-algebras can be considered. In the following argument one restricts to super-symplectic algebra assumed to act as isometries of WCW.

Consider now how the finite measurement resolution could be realized as an infinite hierarchy of super-symplectic gauge symmetry breakings. The physical picture relies on quantum criticality of TGD Universe. The levels of the hierarchy labelled by positive integer n and a ball at the top of ball at... serves as a convenient metaphor.

1. The sub-algebra g_n for which conformal weights of generators (whose commutators give the sub-algebra) are positive integer multiples for those of the entire algebra g defines the algebra acting as pure gauge algebra defining a sub-group of symplectic group. The action of g_n as gauge algebra would mean that it affects on degrees of freedom below the measurement resolution. One can assign to this algebra a coset space G/G_n of the entire symplectic group G and of subgroup G_n. This coset space would describe the dynamical degrees of freedom. If the subgroup were a normal subgroup, the coset space would be a group. This is not the case now since the commutator $[g, g_n]$ of the entire algebra with the sub-algebra does not belong to g_n.

   However, if one poses stronger - physically very attractive - gauge conditions stating that not only g_n but also the commutator algebra $[g, g_n]$ annihilates the physical states and that corresponding classical Noether charges vanish, one obtains effectively a normal subgroup and one has good hopes that coset space acts effectively as group, which is finite-dimensional as far as conformal weights are considered.

2. $n > 0$ is essential for obtaining effective normal algebra property. Without this assumption the commutator $[g, g_n]$ would be entire g. If the spectrum of supersymplectic conformal weights is integer valued it is not obvious why one should pose the restriction $n \geq 1$.

3. In this framework pure conformal invariance could reduce to a finite-dimensional gauge symmetry. A possible interpretation would be in terms of Mc-Kay correspondence [A55] assigning to the inclusions of HFFs labelled by integer $n \geq 3$ a hierarchy of simply laced Lie-groups. Since the included algebra would naturally correspond to degrees of freedom not visible in the resolution used, the interpretation as a dynamical gauge group is suggestive. The dynamical gauge group could correspond to n-dimensional Cartan algebra acting in conformal degrees of freedom identifiable as a simply laced Lie group. This would assign a infinite hierarchy of dynamical gauge symmetries to the broken conformal gauge invariance acting as symmetries of dark matter. This still leaves infinite number of degrees of freedom assignable to the imbedding space Hamiltonians and spectrum generated by zeros of zeta but this might have interpretation in terms of gauging so that additional vanishing conditions for Noether charges are suggestive.

Dark matter as large phases with large gravitational Planck constant $h_{eff} = h_{gr}$

D. Da Rocha and Laurent Nottale [E1] have proposed that Schrödinger equation with Planck constant $\hbar$ replaced with what might be called gravitational Planck constant $\hbar_{gr} = \frac{GmM}{v_0}$ ($\hbar = c = 1$). v_0 is a velocity parameter having the value $v_0 = 144.7 \pm .7$ km/s giving $v_0/c = 4.6 \times 10^{-4}$. This is rather near to the peak orbital velocity of stars in galactic halos. Also subharmonics and harmonics of v_0 seem to appear. The support for the hypothesis coming from empirical data is impressive [K89, K71].

1. The proposal is that a Schrödinger equation results from a fractal hydrodynamics. Many-sheeted space-time however suggests that astrophysical systems are at some levels of the hierarchy of space-time sheets macroscopic quantum systems and that only the generalizations of Bohr orbits are involved. The space-time sheets in question would carry dark matter.

2. Nottale's hypothesis would predict a gigantic value of $\hbar_{gr}$. Equivalence Principle and the independence of gravitational Compton length $\Lambda_{gr} = \hbar_{gr}/m = GM/v_0 = 2r_S/v_0$ (typically astrophysical scale) on mass m implies however that one can restrict the values of mass m to masses of microscopic objects so that $\hbar_{gr}$ would be much smaller. Large $\hbar_{gr}$ could provide a solution of the black hole collapse (IR catastrophe) problem encountered at the classical level. The resolution of the problem inspired by TGD inspired theory of living matter is that it is the dark matter at larger space-time sheets, which is quantum coherent in the required time scale [K89].

   One could criticize the hypothesis since it treats the masses M and m asymmetrically: this is only apparently true [**?**].

3. It is natural to assign the values of Planck constants postulated by Nottale to the space-time sheets mediating gravitational interaction and identifiable as magnetic flux tubes (quanta). The cross section of the flux tube corresponds to a sphere $S^2_i \subset CP_2$, $i = I, II$ [K10]. S^2_I is homologically non-trivial carrying Kähler magnetic monopole flux. S^2_{II} is homologically trivial carrying vanishing Kähler magnetic flux but non-vanishing electro-weak flux [K10].

   The flux tubes of type I have both Kähler magnetic energy and dark energy due to the volume action. Flux tubes of type II would have only the volume energy. Both flux tubes could be remnants of cosmic string phase of primordial cosmology. The energy of these flux quanta would be correlated for galactic dark matter and volume action and also magnetic tension would give rise to negative "pressure" forcing accelerated cosmological expansion. This leads to a rather detailed vision about the evolution of stars and galaxies identified as bubbles of ordinary and dark matter inside flux tubes identifiable also as dark energy.

4. Both theoretical consistency and certain experimental findings from astrophysics [E2, E4] and biology [K21, K12] suggest the identification $h_{eff} = n \times h = h_{gr}$. The large value of h_{gr} can be seen as a manner to reduce the string tension of fermionic strings so that gravitational (in fact all!) bound states can be described in terms of strings connecting the partonic 2-surfaces defining particles (analogous to AdS/CFT description) [K84]. The values $h_{eff}/h = n$ can be interpreted in terms of a hierarchy of breakings of super-conformal symmetry in which the super-conformal generators act as gauge symmetries only for a sub-algebras with conformal weights coming as multiples of n. Macroscopic quantum coherence in astrophysical scales is implied. If also modified Dirac action is present, part of the interior degrees of freedom associated with the fermionic part of conformal algebra become physical.

   Fermionic oscillator operators could generate super-symmetries and sparticles could correspond to dark matter with $h_{eff}/h = n > 1$. One implication would be that at least part if not all gravitons would be dark and be observed only through their decays to an ordinary high frequency graviton ($E = hf_{high} = h_{eff}f_{low}$) or to a bunch of n low energy gravitons.

Hierarchies of quantum criticalities, Planck constants, and dark matters

Quantum criticality is one of the corner stone assumptions of TGD. In the original approach the value of Kähler coupling strength α_K together with CP_2 radius R fixed quantum TGD and is analogous to critical temperature. Twistor lift [K10] brings in additional coupling constant Λ obeying p-adic coupling constant evolution and Planck length l_G, which like CP_2 radius would not obey coupling constant evolution (as also G). The values of these parameters should be fixed by quantum criticality. What else does quantum criticality mean is however far from obvious, and I have pondered the notion repeatedly both from the point of view of mathematical description and phenomenology [K43, K111, K84].

1. Criticality is characterized by long range correlations and sensitivity to external perturbations and living systems define an excellent example of critical systems - even in the scale of populations since without sensitivity and long range correlations cultural evolution and society would not be possible. For a physicist with the conceptual tools of existing theoretical physics the recent information society in which the actions of people at different side of globe are highly correlated, should look like a miracle.

2. The hierarchy of Planck constants with dark matter identified as phases of ordinary matter with non-standard value $h_{eff} = n \times h$ of Planck constant is one of the "almost-predictions" of TGD is definitely something essentially new physics. The phase transition transforming ordinary matter to dark matter in this sense generates long range quantal correlations and even macroscopic quantum coherence.

   Finding of a universal mechanism generating dark matter have been a key challenge during last ten years. Could quantum criticality having classical or perhaps even thermodynamical criticality as its correlate be always accompanied by the generation of dark matter? If this were the case, the recipe would be stupifyingly simple: create a critical system! Dark matter would be everywhere and we would have observed its effects for centuries! Magnetic flux tubes (possibly carrying monopole flux) define the space-time correlates for long range correlations at criticality and would carry the dark matter. They are indeed key players in TGD inspired quantum biology.

3. Change of symmetry is assigned with criticality as also conformal symmetry (in 2-D case). In TGD framework conformal symmetry is extended and infinite hierarchy of breakings of conformal symmetry so that a sub-algebras of various conformal algebras with conformal weights coming as integer multiples of integer n defining h_{eff} would occur.

4. Phase separation is what typically occurs at criticality and one should understand also this. The strengthening of this hypothesis with the assumption $h_{eff} = h_{gr}$, where $h_{gr} = GMm/v_0$ is is the gravitational Planck constant originally introduced by Nottale [K72, ?]. In the formula v_0 has dimensions of velocity, and will be proposed to be determined by a condition relating the size of the system with mass M to the radius within which the wave function of particle m with $h_{eff} = h_{gr}$ is localized in the gravitational field of M.

   The condition $h_{eff} = h_{gr}$ implies that the integer n in h_{eff} is proportional to the mass of the particle. The implication is that particles with different masses reside at flux tubes with different Planck constant and separation of phases indeed occurs.

5. What is remarkable is that neither gravitational Compton length nor cyclotron energy spectrum depends on the mass of the particle. This universality could play key role in living matter. One can assign Planck constant also to other interactions such as electromagnetic interaction so that one would have $h_{em} = Z_1Z_2e^2/v_0$. The phase transition could take place when the perturbation series based on the coupling strength $\alpha = Z_1Z_2e^2/\hbar$ ceases to converge. In the new phase perturbation series would converge since the coupling strength is proportional to $1/h_{eff}$. Hence criticality and separation into phases serve as criteria as one tries to see whether the earlier proposals for the mechanisms giving rise to large h_{eff} phases make sense. One can also check whether the systems to which large h_{eff} has been assigned are indeed critical.

One example of criticality is super-fluidity. Superfluids exhibit rather mysterious looking effects such as fountain effect [D3] and what looks like quantum coherence of superfluid containers, which should be classically isolated. These findings serve as a motivation for the proposal that genuine superfluid portion of superfluid corresponds to a large h_{eff} phase near criticality at least and that also in other phase transition like phenomena a phase transition to dark phase occurs near the vicinity [?].

But how does quantum criticality relate to number theory and adelic physics? $h_{eff}/h = n$ has been identified as the number of sheets of space-time surface identified as a covering space of some kind. Number theoretic discretization defining the "spine" for a monadic space-time surface [L17] defines also a covering space with Galois group for an extension of rationals acting as covering group. Could n be identifiable as the order for a sub-group of Galois group? If this is the case, the proposed rule for h_{eff} changing phase transitions stating that the reduction of n occurs to its factor would translate to spontaneous symmetry breaking for Galois group and spontaneous - symmetry breakings indeed accompany phase transitions.

TGD variant of AdS/CFT duality

AdS/CFT duality [B23] has provided a powerful approach in the attempts to understand the non-perturbative aspects of super-string theories. The duality states that conformal field theory in

n-dimensional Minkowski space M^n identifiable as a boundary of $n+1$-dimensional space AdS_{n+1} is dual to a string theory in $AdS_{n+1}\times S^{9-n}$.

As a mathematical discovery AdS/CFT duality is extremely interesting but it seems that it need not have much to do with physics as such. From TGD point of view the reason is obvious: the notion of conformal invariance is quite too limited. In TGD framework conformal invariance is extended to a super-symplectic symmetry in $\delta M^4_\pm\times CP_2$, whose Lie-algebra has the structure of conformal algebra. Also ordinary super-conformal symmetries associated with string world sheets are present as well as generalization of 2-D conformal symmetries to their analogs at light-cone boundary and light-like orbits of partonic 2-surfaces. In this framework AdS/CFT duality is expected to be modified.

The matrix elements $G_{K\overline{L}}$ of Kähler metric of WCW can be expressed in two manners. As contractions of the derivatives $\partial_K\partial_{\overline{L}}K$ of the Kähler function of WCW with isometry generators or as anticommutators $\{\Gamma_K,\Gamma_{\overline{L}}\}$ of WCW gamma matrices identified as supersymplectic Noether super charges assignable to fermionic strings connecting partonic 2-surfaces. Kähler function is identified as real part of the action: if coupling parameters are real it reduces to the action for the Euclidian space-time regions with 4-D CP_2 projection and otherwise contains contributions from both Minkowskian and Euclidian regions. The action defines the modified gamma matrices appearing in modified Dirac action as contractions of canonical momentum currents with imbedding space gamma matrices.

This observation suggests that there is a super-symmetry between action and modified Dirac action. The problem is that induced spinor fields naive of SH and also well-definedness of em charge demand the localization of induced spinor modes at 2-D string world sheets. This simply cannot be true. On the other hand, SH only requires that the data about induced spinor fields and space-time surface at the string world sheets is enough to construct the modes in space-time interior.

This leaves two options if one assumes that SH is exact (recall however that the number theoretic interpretation for the hierarchy of Planck constants suggests that the number-theoretic spin of monadic space-time surface represents additional discrete data needed besides that assignable to string world sheets to describe dark matter). As found in the section 9.2.3, there are two options.

Option I: The analog of brane hierarchy is realized at the level of fundamental action. There is a separate fundamental 2-D action assignable with string world sheets - area and topological magnetic flux term - as also world line action assignable to the boundaries of string world sheets. By previous argument string tension should be determined by the value of the cosmological constant Λ obeying -adic coupling constant evolution rather than by G: otherwise there is no hope about gravitationally bound states above Planck scale. String tension would appear as an additional fundamental coupling parameter (perhaps fixed by quantum criticality). This option does not quite conform with the spirit of SH.

Option II: 4-D space-time action and corresponding modified Dirac action defining fundamental actions are expressible as effective actions assignable to string world sheets and their boundaries. String world sheet effective action action could be expressible as string area for the effective metric defined by the anti-commutators of modified gamma matrices at string world sheet. If the sum of the induced Kähler forms of M^4 and CP_2 vanishes at string world sheets the effective metric would be the induced 2-D metric: this together with the observed CP breaking could provide a justification for the introduction of the analog of Kähler form in M^4. String tension would be dynamical rather than determined by l_P and depend on Λ, l_P, R and α_K. This representation of Kähler action would be one aspect of the analog of AdS/CFT duality in TGD framework.

Both options would allow to understand how strings connecting partonic 2-surfaces give rise to the formation of gravitationally bound states. Bound states of macroscopic size are possible only if one allows hierarchy of Planck constants and this is required also by the (extremely) small value of Λ (in cosmic scales).

Consider the concerete realizations for this vision.

1. SGCI requires effective 2-dimensionality. In given UV and IR resolutions partonic 2-surfaces and string world sheets are assignable to a finite hierarchy of CDs inside CDs with given CD characterized by a discrete scale coming as an integer multiple of a fundamental scale (essentially CP_2 size). Λ would closely relate to the size scale of CD. String world sheets have boundaries consisting of either light-like curves in induced metric at light-like wormhole throats and space-like curves at the ends of CD whose M^4 projections are light-like. These

braids carrying fermionic quantum numbers intersect partonic 2-surfaces at discrete points.

2. This implies a rather concrete analogy with $AdS_5 \times S_5$ duality, which describes gluons as open strings. In zero energy ontology (ZEO) string world sheets are indeed a fundamental notion and the natural conjecture is that these surfaces are minimal surfaces, whose area by quantum classical correspondence depends on the quantum numbers of the external particles.

String tension of gravitational flux tubes

For Planckian cosmic strings only quantum gravitational bound states of length of order Planck length are possible. There must be a mechanism reducing the string tension. The *effective* string tension assignable to magnetic flux tubes must be inversely proportional to $1/h_{eff}^2$, $h_{eff} = n \times h = h_{gr} = 2\pi GMm/v_0$ in order to obtain gravitationally bound states in macroscopic length scales identified as structures for which partonic 2-surfaces are connected by flux tubes accompanied by fermionic strings.

The reason is that the size scale of (quantum) gravitationally bound states of masses M and m is given by grvitational Compton length $\Lambda_{gr} = GM/v_0$ [K89, K72, **?**] assignable to the gravitational flux tubes connecting the masses M and m. If the string tension is of order Λ_{gr}^2 this is achieved since the typical length of string would be Λ_{gr}. Gravitational string tension must be therefore of order $T_{gr} \sim 1/\Lambda_{gr}^2$. How could this be achieved? One can imagine several options and here only the option based on the assumptions

1. Twistor lift makes sense.

2. Fundamental action is 4-D for both space-time and fermionic degrees of freedom and 2-D string world sheet action is an effective action realizing SH. Note effective action makes also possible braid statistics, which does not make sense at fundamental level.

3. Also M^4 carries the analog of Kähler form and the sum of induced Kähler forms from M^4 and CP_2 vanishes at string world sheets and also weak gauge fields vanishes at string world sheets leaving only em field.

is considered since it avoids all the objections that I have been able to invent.

For the twistor lift of TGD [K10] predicting cosmological constant Λ depending on p-adic length scale $\Lambda \propto 1/p$ the gravitational strings would be naturally homologically trivial cosmic strings. These vacuum extremals of Kähler action transform to minimal surface extremals with string tension given by $\rho_{vac}S$, where ρ_{vac} the density of dark energy assignable to the volume term of the action and S the transverse area of the flux tube. One should have $\rho_{vac}S = 8\pi\Lambda S/G = 1/\Lambda_{gr}^2$ so that one would have

$$8\pi\Lambda S = \frac{G}{\Lambda_{gr}^2} \quad .$$

Λ for flux tubes (characterizing the size of CDs containing them) would depend on the gravitational coupling Mm.

9.2.5 Number theoretical vision

Physics as infinite-D spinor geometry of WCW and physics as generalized number theory are the two basic vision about TGD. The number theoretical vision involves three threads [K95, K96, K94].

1. The first thread [K95] involves the notion of number theoretical universality NTU: quantum TGD should make sense in both real and p-adic number fields (and their algebraic extensions induced by extensions of rationals). p-Adic number fields are needed to understand the space-time correlates of cognition and intentionality [K63, K39, K65].

   p-Adic mass calcuations lead to the notion of a p-adic length scale hierarchy quantifying the notion of the many-sheeted space-time [K63, K39]. One of the first applications was the calculation of elementary particle masses [K51]. The basic predictions are only weakly model independent since only p-adic thermodynamics for Super Virasoro algebra are involved. Not only the fundamental mass scales would reduce to number theory but also particle masses are predicted correctly under rather mild assumptions and are exponentially sensitive to the p-adic length scale predicted by p-adic length scale hypothesis. Also predictions such as the possibility of neutrinos to have several mass scales were made on the basis of number theoretical arguments and have found experimental support [K51, K23].

2. Second thread [K96] is inspired by the dimensions $D = 1, 2, 4, 8$ of the basic objects of TGD and assumes that classical number fields are in a crucial role in TGD. 8-D imbedding space would have octonionic structure and space-time surfaces would have associative (quaternionic) tangent space or normal space. String world sheets could correspond to commutative surfaces. Also the notion of $M^8 - H$-duality is part of this thread and states that quaternionic 4-surfaces of M^8 containing preferred M^2 in its tangent space can be mapped to PEs in H by assigning to the tangent space CP_2 point parametrizing it. M^2 could be replaced by integrable distribution of $M^2(x)$. If PEs are also quaternionic one has also $H - H$ duality allowing to iterate the map so that PEs form a category. Also quaternion analyticity of PEs is a highly attractive hypothesis [K101]. For instance, it might be possible to interpret string world sheets and partonic 2-surfaces appearing in strong form of holography (SH) as co-dimension 2 surfaces analogous to poles of analytic function in complex plane. Light-like 3-surfaces might be seen as analogs of cuts. The coding of analytic function by its singularities could be seen as analog of SH.

3. The third thread [K94] corresponds to infinite primes and leads to several speculations. The construction of infinite primes is structurally analogous to a repeated second quantization of a supersymmetric arithmetic quantum field theory with free particle states characterized by primes. The many-sheeted structure of TGD space-time could reflect directly the structure of infinite prime coding it. Space-time point would become infinitely structured in various p-adic senses but not in real sense (that is cognitively) so that the vision of Leibniz about monads reflecting the external world in their structure is realized in terms of algebraic holography. Space-time becomes algebraic hologram and realizes also Brahman=Atman idea of Eastern philosophies.

9.3 p-Adic mass calculations and p-adic thermodynamics

p-Adic mass calculations carried for the first time around 1995 were the stimulus eventually leading to the number theoretical vision as a kind dual for the geometric vision about TGD. In this secton I will roughly describe the calculations [K23, K51] and the questions and challenges raised by them.

9.3.1 p-Adic numbers

Like real numbers, p-adic numbers (`http://tinyurl.com/hmgqtoh`) can be regarded as completions of the rational numbers to a larger number field [K39]. Each prime p defines a p-adic number field allowing the counterparts of the usual arithmetic operations.

1. The basic difference between real and p-adic numbers is that p-adic topology is ultra-metric. Ultrametricity means that the distance function $d(x, y)$ (the counterpart of $|x-y|$ in the real context) satisfies the inequality

$$d(x, z) \leq Max\{d(x, y), d(y, z)\} \quad ,$$

(Max(a, b) denotes maximum of a and b) rather than the usual triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) \quad .$$

2. The topology defined by p-adic numbers is compact-open. Hence the generalization of manifold obtained by gluing together n-balls fails because smallest open n-balls are just points and one has totally disconnected topology.

3. p-Adic numbers are not well-ordered like real numbers. Therefore one cannot assign orientation to the p-adic number line. This in turn leads to difficulties with attempts to define definite integrals and the notion of differential form although indefinite integral is well-defined. These difficulties serve as important guidelines in the attempts to understand what p-adic physics is and also how to fuse real and various p-adic physics to a larger structure.

4. p-Adic numbers allow an expansion in powers of p analogous to the decimal expansion

$$x = \sum_{n \geq 0} x_n p^n \quad ,$$

and the number of terms in the expansion can be infinite so that p-adic number need not be finite as a real number. The norm of the p-adic number (counterpart of $|x|$ for real numbers) is defined as

$$N_p(x) = \sum_{n \geq 0} x_n p^n = p^{-n_0} \quad ,$$

and depends only very weakly on p-adic number. The ultra-metric distance function can be defined as $d_p(x,y) = N_p(x-y)$.

5. p-Adic numbers allow a generalization of the differential calculus. The basic rules of the p-adic differential calculus are the same as those of the ordinary differential calculus. There is however one important new element: the set of the functions having vanishing p-adic derivative consists of so called pseudo constants, which are analogs of real valued piecewise constant functions. In the real case only constant functions have vanishing derivative. This implies that p-adic differential equations are non-deterministic. This non-determinism is identified as a counterpart of the non-determinism of cognition and imagination [K65].

9.3.2 Model of elementary particle

p-Adic mass calculations [K23, K51] rely heavily on a topological model for elementary particle and it is appropriate to describe it before going to the summary of calculations.

Family replication phenomenon topologically

One of the basic ideas of TGD approach to particle physics has been genus-generation correspondence: boundary components of the 3-surface should be carriers of elementary particle numbers and the observed particle families should correspond to various boundary topologies.

With the advent of zero energy ontology (ZEO) this picture has changed somewhat.

1. The wormhole throats identified as light-like 3-surfaces at with the induced metric of the space-time surface changes its signature from Minkowskian to Euclidian correspond to the light-like orbits of partonic 2-surfaces. One cannot of course exclude the possibility that also boundary components allow to satisfy boundary conditions without assuming vacuum extremal property of nearby space-time surface.

   The intersections of the wormhole throats with the light-like boundaries of causal diamonds (CDs) identified as intersections of future and past directed light cones ($CD \times CP_2$ is actually in question but I will speak about CDs) define special partonic 2-surfaces and the conformal moduli of these partonic 2-surfaces appear in the elementary particle vacuum functionals [K23] naturally. A modification of the original simple picture came from the proposed identification of physical particles as bound states of two wormhole contacts connected by tubes carrying monopole fluxes.

2. For generalized scattering diagrams stringy trouser vertices are replaced with vertices at which the ends of light-like wormhole throats meet. This vertex is the analog of 3-vertex for Feynman diagrams in particle physics lengths scales and for the biological replication (DNA and even cell) in macroscopic length scales.

In this picture the interpretation of the analog of trouser vertex is in terms of propagation of same particle along two different paths. This interpretation is mathematically natural since vertices correspond to 2-manifolds rather than singular 2-manifolds, which are just splitting to two disjoint components. Second complication comes from the weak form of electric-magnetic duality forcing to identify physical particles as weak strings with magnetic monopoles at their ends and one should understand also the possible complications caused by this generalization.

These modifications force to consider several options concerning the identification of light fermions and bosons and one can end up with a unique identification only by making some assumptions. Masslessness of all wormhole throats - also those appearing in internal lines - and dynamical $SU(3)$ symmetry for particle generations are attractive general enough assumptions of this kind. Bosons and their possible spartners would correspond to wormhole contacts with fermion and anti-fermion at the throats of the contact. The expectation was tht free fermions and their possible spartners correspond to CP_2 type vacuum extremals with single wormhole throat. It however turned however that dynamical $SU(3)$ symmetry forces to identify massive (and possibly topologically condensed) fermions as pairs of (g, g) type wormhole contacts. The existence of higher boson families would mean breaking of quark and lepton universality and there are indications for this kind of anomaly [K57] .

The notion of elementary particle vacuum functional

Obviously one must know something about the dependence of the elementary particle state functionals on the geometric properties of the boundary component and in the sequel an attempt to construct what might be called elementary particle vacuum functionals (EPVFs), is made. The basic assumptions underlying the construction are the following ones [K23].

1. EPVFs depend on the geometric properties of the two-surface X^2 representing elementary particle.
2. EPVFs possess extended Diff invariance: all 2-surfaces on the orbit of the 2-surface X^2 correspond to the same value of the vacuum functional. This condition is satisfied if vacuum functionals have as their argument, not X^2 as such, but some 2- surface Y^2 belonging to the unique orbit of X^2 (determined by the principle selecting PE as a generalized Bohr orbit [K43, K8, K14]) and determined in general coordinate invariant manner.
3. ZEO allows to select uniquely the partonic 2-surface as the intersection of the wormhole throat at which the signature of the induced 4-metric changes with either the upper or lower boundary of $CD \times CP_2$. This is essential since otherwise one one could not specify the vacuum functional uniquely.
4. Vacuum functionals possess conformal invariance and therefore for a given genus depend on a finite number of variables specifying the conformal equivalence class of Y^2.
5. Vacuum functionals satisfy the cluster decomposition property: when the surface Y^2 degenerates to a union of two disjoint surfaces (particle decay in string model inspired picture), vacuum functional decomposes into a product of the vacuum functionals associated with disjoint surfaces.
6. EPVFs are stable against the decay $g \rightarrow g_1 + g_2$ and one particle decay $g \rightarrow g - 1$. This process corresponds to genuine particle decay only for stringy diagrams. For generalized scattering diagrams the interpretation is in terms of propagation along two different paths simultaneously.

In [K23] the construction of EPVFs is described in detail. This requires some basic concepts related to the description of the space of the conformal equivalence classes of Riemann surfaces and the concept of hyper-ellipticity. Since theta functions will play a central role in the construction of the vacuum functionals, also their basic properties are needed. Also possible explanations for the experimental absence of the higher fermion families are considered. Concerning p-adic mass calculations, the key question is how to construct p-adic variants of EPVFs.

9.3.3 p-Adic mass calculations

p-Adic thermodynamics

Consider first the basic ideas of p-adic thermodynamics.

1. p-Adic valued mass squared is identified as as thermal mass in p-adic thermodynamics. Boltzmann weights $exp(-E/T)$ do not make sense if one just replaces exponent function with the p-adic variant of its Taylor series. The reason is that $exp(x)$ has p-adic norm equal to 1 for all acceptable values of the argument x (having p-adic norm smaller than one) so that partition function does not have the usual exponential convergence property. Nothing however prevents from consider Boltzmann weights as powers p^n making sense for integer values of n. Here the p-adic norm approaches zero for $n \to +\infty$: thus the correspondences $e^{-E/T} \leftrightarrow p^{E/T_p}$.

   The values of E/T_p must be quantized to integers. This is guaranteed if E is integer valued in suitable unit of energy and $1/T_p$ has integer valued spectrum using same unit for T_p. Super-conformal invariance guarantees integer valued spectrum of E, which in the recent case corresponds to mass squared. These number theoretical conditions are very powerful and lead to the quantization of also thermal mass squared for given p-adic prime p.

2. The p-adic mass squared is mapped to real number by canonical identification $I : \sum x_n p^n \to \sum x_n p^{-n}$ or its variant for rationals. Canonical identification is continuous and maps powers of p^n to their inverses. One modification of canonical identification maps rationals m/n in their representation in which m and n have no common divisors to $I(m)/I(n)$. The predictions of calculations depend in some cases on which variant one uses but rational option looks the most reasonable choice.

3. p-Adic length scale hypothesis states that preferred p-adic primes correspond to powers of 2: $p \simeq 2^k$, but smaller than 2^k. The values of k form with $p = 2^k - 1$ is prime - Mersenne prime - are especially favored. The nearer the prime p to 2^k, the more favored p is physically. One justification for the hypothesis is that preferred primes have been selected by an evolutionary process.

4. It turns out that p-adic temperature is $T_p = 1$ for fermions. For gauge bosons $T_p \leq 1/2$ seems to be necessary assumption for gauge bosons implying that the contribution to mass squared is very small so that super-symplectic contribution assignable to the wormhole magnetic flux tube dominates for weak bosons. For canonical identification $m/n \to I(m)/I(n)$ second order contribution to fermionic mass squared is very small.

5. The large values of p-adic prime p guarantee that the p-adic thermodynamics convervges extremely rapidly. For $m/n \to I(m)/I(n)$ already the second order contribution is extremely small since the expansion for the real mass squared is in terms of $1/p$ and for electron with $p = M_{127}$ one has $p \sim 10^{38}$. Hence the calculations are essentially exact and errors are those of the model. It is quite possible that calculations could be done exactly using exact expressions for the super-symplectic partition functions generalized to p-adic context. The success of the p-adic mass calculations is especially remarkable because p-adic length scale hypothesis $p \simeq 2^k$ predicts exponential sensitivity of the particle mass scale on k.

Symmetries

The number theoretical existence of p-adic thermodynamics requires powerful symmetries to guarantee integer valued spectrum for the thermalized contribution to the mass squared.

1. Super-conformal symmetry with integer valued conformal weights for Virasoro scaling generator L_0 is essential because it predicts in string models that mass squared is apart from ground state contribution integer valued in suitable units. In TGD framework fermionic string world sheets are characterized by super-conformal symmetry. This gives the p-adic thermodynamics assumed in the calculations. One could however assign Super Virasoro algebra also to super-symplectic algebra having its analog as sub-algebra with positive integer conformal weights. Same applies to the extended conformal algebra of light-cone boundary.

2. TGD however predicts also generalization of conformal symmetry associated with light-cone boundary involving ordinary complex conformal weights and the conformal weight associated with the light-like radial coordinate. For the latter conformal weights for the generators of supersymmetry might be given by $h = -s_n/2$. s_n zero of zeta or pole $h = -s = -1$ of zeta.

   Also super-symplectic symmetries would have similar radial spectrum of conformal weights. Conformal confinement requiring that the conformal weights of states are real implies that the spectrum of conformal weights for physical states consists of non-negative integers as for ordinary superconformal invariance.

   It is not clear whether thermalization occurs in these degrees of freedom except perhaps for trivial conformal weights. These degrees of freedom need not therefore contribute to thermal masses of leptons and quarks but would give dominating contribution to hadron masses and weak boson masses. The negative conformal weights predicted by $h = -s/2$ hypothesis predicts that ground state weight is negative for super-symplectic representations and must be compensated for massless states.

   The assumption that ground state conformal weight is negative and thus tachyonic is essential in case of p-adic mass calculations [K51], and only for massless particles (graviton, photon, gluons) it vanishes or is of order $O(1/p)$. This could be achieved if the ground state of super-symplectic representation has $h = 0$.

3. Modular invariance [K23] assignable to partonic 2-surfaces is a further assumption similar to that made made in string models. This invariance means that for a given genus the dynamical degrees of freedom of the partonic 2-surface correspond to finite-dimensional space of Teichmueller parameters. For genus $g = 0$ this space is trivial.

   Also modular invariance for string world sheets can be considered. By SH the information needed in mass calculations should be assignable to partonic 2-surfaces: the assumption is that one can assign this information to single partonic 2-surface. Stringy contribution would be seen only in scattering amplitudes.

   This might be true only effectively: the recent view about elementary particles is that they are pairs of wormhole contacts connected by flux tubes defining a closed monopole flux and wormhole throats of contact have same genus for light states. Furthermore the quantum numbers of particle are associated with single throat for fermions and with opposite throats of single contact for bosons. The second wormhole contact would carry neutralizing weak charges to realize the finite range of weak interactions as "weak confinement".

   The number of genera is infinite and one must understand why only three quark and lepton generations are observed. An attractive explanation is in terms of symmetry. For the three lowest genera the partonic 2-surfaces are always hyper-elliptic and have thus global conformal Z_2 symmetry. For higher genera this is not true always and EPVFs constructed from the assumption of modular invariance vanish for the hyper-elliptic surfaces. This suggests that the higher genera are very massive or can be interpreted as many-particle states of handles, which are not bound states but have continuous mass squared.

Contributions to mass squared

There are several contributions to the p-adic thermal mass squared come from the degrees of freedom, which are thermalized.

Super-conformal degrees of freedom associated with string world sheets are certainly thermalized. p-Adic mass calculations strongly suggest that the number of super-conformal tensor factors is $N = 5$ but also $N = 4$ and $N = 6$ can be considered marginally.

I have considered several identifications of tensor factors and not found a compelling alternative. If one assumes that super-symplectic degrees of freedom do not contribute to the thermal mass, string world sheets should explain masses of elementary fermions. Here charged lepton masses are the test bench. One other hand, if super-symplectic degrees of freedom contribute one obtains additional tensor factor assignable to $h = -s/2$, s trivial zero of zeta). Only one tensor factor emerges since Hamiltonians correspond to the products of functions of the coordinates of light-cone boundary and CP_2).

1. $SU(2)_L \times U(1)$ gives 2 tensor factors. $SU(3)$ gives 1 tensor factor. The two transversal degrees of freedom for string world sheet suggest 2 degrees of freedom corresponding to Abelian group E^2. Rotations however transforms these degrees to each other so that 1 tensor factor should emerge. This gives 4 tensor factors. Could it correspond to the degrees of freedom parallel to string at its end assignable to wormhole throat? Could normal vibrations of partonic 2-surface? This would $N = 5$ tensor factors. Another possibility is that the fifth tensor factor comes from super-symplectic Super-Virasoro algebra defined by trivial conformal weights.

2. Super-symplectic contributions need not be present for ordinary elementary fermions. For weak bosons they could give string tension assignable to the magnetic flux tube connecting the wormhole contacts. It is not clear whether this contribution is thermalized. This contribution might be present only for the phases with $h_{eff} = n \times h$. This contribution would dominate in hadron masses.

3. Color degrees of freedom contribute to the ground state mass squared since ground state corresponds to an imbedding space spinor mode massless in 8-D sense. The mass squared contribution corresponds to an eigenvalue of CP_2 spinor d'Alembertian. Its eigenvalues correspond to color multiplets and only the covariantly constant right handed neutrino is color singlet. For the other modes the color representation is non-trivial and depends on weak quantum numbers of the fermion. The construction of the massless state from a tachyonic ground state with conformal weight $h_{vac} = -3$ must involve colored super-Kac Moody generators compensating for the anomalous color charge so that one obtains color single for leptons and color triplet for quarks as massless state.

4. Modular degrees of freedom give a contribution depending on the genus g of the partonic 2-surface. This contribution is estimated by considering p-adic variants of elementary particle vacuum functionals Ω_{vac} [K51] expressible as products of theta functions with the structure of partition function. Theta functions are expressible as sums of exponent functions $exp(X)$ with X defined as a contraction of the matrix Ω_{ij} defined by Teichmueller parameters between integer valued vectors.

   In ZEO the interpretation of Ω_{vac} is as a complex square root of partition functional (quantum theory as complex square root of thermodynamics in ZEO). The integral of $|\Omega|^2$ over allowed moduli has interpretation as partition function. The exponential $exp(Re(X)) = p^{Re(X)/log(p)}$ has interpretation as an exponential of "Hamiltonian" defined by the vacuum conformal weight defined by moduli. $T = log(p)$ is identified as p-adic temperature as in ordinary p-adic thermodynamics.

   NTU requires that the integration over the moduli parameters reduces to a sum over number theoretically universal moduli parameters. The exponents $exp(X)$ must exist p-adically. PE property alone could guarantee this. The exponentials appearing in theta functions should reduce to products $p^k p^{iy} = exp(k/log(p))p^{iy}$ with k is integer and p^{iy} a root of unity. The vacuum expectation value of $Re(X)$ contributing to the mass squared is obtained from the standard formula as logarithmic temperature derivative of the "integral" $\int |\Omega_{vac}|^2$. The formula is same as for the Super-Virasoro contributions apart from the integration reducing to a sum.

   The considerations of the section 9.4.2 [L9] suggest that for given p-adic prime p the exponent $k + iy$ corresponds to a linear combinations of poles of fermionic zeta $z_F(s) = \zeta(s)/\zeta(2s)$ in the class $C(p)$ with non-negative integer coefficients. This class corresponds essentially to the conformal weights of a fractal sub-algebra of super-symplectic algebra. It could give rise also to the complex values of action so that Riemann zeta would define the core of TGD.

   The general dependence of the contribution of genus g to mass squared on g follows from the functional form of EPVF as a product theta functions serving as building brick partition functions apart from overall multiplicative constant and gives a nice agreement with the observed charged lepton mass ratios. The basic feature of the formula is exponential dependence on g.

5. The super-symplectic stringy contribution assignable to the magnetic flux tube dominates for weak bosons and is analogous to the stringy contribution to the hadron masses.

p-Adic mass calculations leave open several questions. What is the precise origin of preferred p-adic primes and of p-adic length scale hypothesis? How to understand the preferred number $N = 5$ of Super-Kac-Moody tensor factors? How to calculate the contribution of super-symplectic degrees of freedom - are they thermalized? Why only 3 lowest genera are light and what are the masses of the predicted bosonic higher genera implying breaking of fermion universality.

9.3.4 p-Adic length scale hypothesis

p-Adic length scale hypothesis [K1, K63] has served as a basic hypothesis of p-adic TGD for several years. This hypothesis states that the scales $L_p = \sqrt{p}l$, $l = 1.376 \cdot 10^4\sqrt{G}$ are fundamental length scale at p-adic condensate level p. The original interpretation of the hypothesis was following:

1. Above the length scale L_p p-adicity sets on and effective course grained space-time or imbedding space topology is p-adic rather than ordinary real topology. Imbedding space topology seems to be more appropriate identification.
2. The length scale L_p serves as a p-adic length scale cutoff for the quantum field theory description of particles. This means that space-time begins to look like Minkowski space so that the QFT $M^4 \rightarrow CP_2$ becomes a realistic approximation. Below this length scale string like objects and other particle like 3-surfaces are important.
3. It is un-natural to assume that just single p-adic field would be chosen from the infinite number of possibilities. Rather, there is an infinite number of cutoff length scales. To each prime p there corresponds a cutoff length scale L_p above which p-adic quantum field theory $M^4 \rightarrow CP_2$ makes sense and one has a hierarchy of p-adic QFTs. These different p-adic field theories correspond to different hierarchically levels possibly present in the topological condensate. Hierarchical ordering $< p_1 < p_2 < ...$ means that only the surface $p_1 < p_2$ can condense on the surface p_2. The condensed surface can in practice be regarded as a point like particle at level p_2 described by the p-adic conformal field theory below length scale L_{p_2}.

The recent view inspired by adelic physics is that preferred p-adic primes correspond to so called ramified primes for the algebraic extension of rationals defining the adele [K109]. Weak form of Negentropy Maximization Principle (WNMP) [K56] in turn allows to conclude that the length scales corresponding to powers of primes are preferred. Therefore p-adic length scale hypothesis generalizes. There is evidence for 3-adic time scales in biology [I1, I2] and 3-adic time scales can be also assigned with Pythagorean scale in geometric theory of harmony [K80] [L7].

9.3.5 Mersenne primes and Gaussian Mersennes are special

Mersenne primes and their complex counterparts Gaussian Mersennes pop up in p-adic mass calculations and both elementary particle physics, biology [K77], and astrophysics and cosmology [K54] provide support for them.

Mersenne primes

One can also consider the milder requirement that the exponent $\lambda = 2^{\epsilon L_0}$ represents trivial scaling represented by unit in good approximation for some p-adic topology. Not surprisingly, this is the case for $L_0 = mp^k$ since by Fermat's theorem $a^p \ mod \ p = 1$ for any integer a, in particular $a = 2$. This is also the case for $L_0 = mk$ such that $2^k \ mod \ p = 1$ for p prime. This occurs if $2^k - 1$ is Mersenne prime: in this case one has $2^{L_0} = 1$ modulo p so that the sizes of the fractal sub-algebras are exponentially larger than the sizes of $L_0 \propto p^n$ algebras. Note that all scalings a^{L_0} are near to unity for $L_0 = p^n$ whereas now only $a = 2$ gives scalings near unity for Mersenne primes. Perhaps this extended fractality provides the fundamental explanation for the special importance of Mersenne primes.

In this case integrated scalings 2^{L_0} leave the states almost invariant so that even a stronger form of the breaking of the exact conformal invariance would be in question in the super-symplectic case. The representation would be defined by the generators for which conformal weights are odd multiples of n ($M_n = 2^n - 1$) and L_{-kn}, $k > 0$ would generate zero norm states only in order $O(1/M_n)$.

Especially interesting is the hierarchy of primes defined by the so called Combinatorial Hierarchy resulting from TGD based model for abstraction process. The primes are given by $2, 3, 7 = 2^3 - 1, 127 = 2^7 - 1, 2^{127} - 1, ..$: $L_0 = n \times 127$ would correspond to M_{127}-adicity crucial for the memetic code.

Gaussian Mersennes are also special

If one allows also Gaussian primes then the notion of Mersenne prime generalizes: Gaussian Mersennes are of form $(1 \pm i)^n - 1$. In this case one could replace the scaling operations by scaling combined with a twist of $\pi/4$ around some symmetry axis: $1 + i = \sqrt{2}exp(i\pi/4)$ and generalized p-adic fractality would mean that for certain values of n the exponentiated operation consisting of n basic operations would be very near to unity.

1. The integers k associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79, 113$. $k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon. Thus all known charged leptons, rather than only e and τ, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

2. The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 μm. This observation also suggests that cellular organisms have evolved to their present form through four basic evolutionary stages. This also encourages to think that $\sqrt{2}exp(i\pi/4)$ operation giving rise to logarithmic spirals abundant in living matter is fundamental dynamical symmetry in bio-matter.

   Logarithmic spiral provides the simplest model for biological growth as a repetition of the basic operation $\sqrt{2}exp(i\pi/4)$. The naive interpretation would be that growth processes consist of $k = 151, 157, 163, 167$ steps involving scaling by $\sqrt{2}$. This however requires the strange looking assumption that growth starts from a structure of size of order CP_2 length. Perhaps this exotic growth process is associated with pair of MEs or magnetic flux tubes of opposite time orientation and energy emergenging CP_2 sized region in a mini big bang type process and that the resulting structure serves as a template for the biological growth.

3. $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 min. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about 2.6×10^6 light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely superastronomical time and length scale.

9.3.6 Questions

The proposed picture leaves open several questions.

1. Could the descriptions by both real and p-adic thermodynamics be possible? Could they be equivalent (possibly in finite measurement resolution) as is suggested by NTU? The consistency of these descriptions would imply temperature quantization and p-adic length scale hypothesis not possible in purely real context.

2. What could the extension of conformal symmetry to supersymplectic symmetry mean? One possible view is that super-symplectic symmetries correspond to dark degrees of freedom and that only the super-symplectic ground states with negative conformal weights affect the p-adic thermodynamics, which applies only to fermionic degrees of freedom at string world sheets. Super-symplectic degrees of freedom would give the dominant contribution to hadron masses and could contribute also to weak gauge boson masses. $N = 5$ for the needed number

of tensor factors is however a strong constraint and perhaps most naturally obtained when also the super-symplectic Virasoro associated with the trivial zeros of zeta is thermalized.

3. What happens in dark sectors. Preferred extremal property is proposed to mean that the states are annihilated by super-symplectic sub-algebra isomorphic to the original algebra and its commutator with the entire algebra. The conjecture is that this gives rise to Kac-Moody algebras as dynamical symmetries - maybe ADE type algebras, whose Dynkin diagrams characterize the inclusion of HFFs. Does this give an additional tensor factor to super-Virasoro algebra?

4. Superconformal symmetry true in the sense that Super Virasoro conditions hold true. Partition function however depends on mass squared only rather than the entire scaling generator L_0 as thought erratically in the first formulation of p-adic calculation. This does not mean breaking of conformal invariance. Super Virasoro conditions hold true although partition function is for the vibrational part of L_0 determining the mass squared spectrum.

9.4 p-Adicization and adelic physics

This section is devoted to the challenges related to p-adicization and adelization of physics in which the correspondence between real and p-adic numbers via canonical identification serves as the basic building brick. Also the problems associated with p-adic variants of integral, Fourier analysis, Hilbert space, and Riemann geometry should be solved in a way respecting fundamental symmetries and their p-adic variants must be met. The notion of number theoretical universality (NTU) plays a key role here. One should also answer to questions about the origin of preferred primes and p-adic length scale hypothesis.

9.4.1 Challenges

The basic challenges encountered are construction of the p-adic variants of real number based physics, understanding their relationship to real physics, and the fusion of various physics to single coherent whole.

The p-adicization of real physics is not just a straightforward formal generalization of scattering amplitudes of existing theories but requires a deeper understanding of the physics involved. The interpretation of p-adic physics as correlate for cognition and imagination is an important guideline and will be discussed l in more detail in separate section.

Definite integral and Fourier analysis are basic elements of standard physics and their generalization to the p-adic context defines a highly non-trivial challenge. Also the p-adic variants of Riemann geometry and Hilbert space are suggestive. There are however problems.

1. There are problems associated with p-adic definite integral. Riemann sum does not make sense since it approaches zero if the p-adic norm of discretization unit approaches zero. The problems are basically due to the absence of well-orderedness essential for the definition of definite integral and differential forms and their integrals.

   Residue integration might make sense in finite angle resolution. For algebraic extension containing $e^{i\pi/n}$ the number theoretically universal approximation $i\pi = n(e^{i\pi/n} - 1)$ could be used. In twistor approach integrations reduce to multiple residue integrations and since twistor approach generalizes in TGD framework, this approach to integration is very attractive.

   Positivity is a central notion in twistor Grassmannian approach [B17]. Since canonical identification maps p-adic numbers to non-negative real numbers, there is a strong temptation to think that positivity relates to NTU [L10].

2. There are problems with Fourier analysis. The naive generalization of trigonometric functions by replacing e^{ix} with its p-adic counterpart is not physical. Same applies to e^x. Algebraic extensions are needed to get roots of unity ad e as counterparts of the phases and discretization is necessary and has interpretation in terms of finite resolution for angle/phase and its hyperbolic counterpart.

3. The notion of Hilbert space is problematic. The naive generalization of Hilbert space norm square $|x|^2 = \sum x_n \overline{x}_n$ for state $(x_1, x_2, ...)$ can vanish p-adically. Also here NTU could help. State would contain as coefficients only roots of e and unity and only the overall factor could be p-adic number. Coefficients could be restricted to the algebraic numbers generating the algebraic extension of rational numbers and would not contain powers of p or even ordinary p-adic numbers expect in the overall normalization factor.

Second challenge relates to the relationship between real and p-adic physics. Canonical identification (CI) $\sum x_n p^n \to \sum x_n p^{-n}$ or some of its variants should play an important role. CI is expected to map the invariants appearing in scattering amplitudes to their real counterparts.

1. Real and p-adic variants of space-time surfaces should exist and relate to each other somehow. Is this relationship local and involve CI at space-time level or imbedding space level? Or is it only a global and non-local assignment of preferred real extremals to their p-adic counterparts? Or is between these extreme options and involves algebraic discretization of the space-time surface weakening the strong form of SH as already proposed? How do real and p-adic imbedding spaces relate to each other and can this relationship induce local correspondence between preferred extremals (PEs) [K8, K14, K10]?

2. NTU in some sense is a highly suggestive approach to these questions and would suggest that canonical identification applies to isometry invariants whereas angles and hyperbolic angles, or rather the corresponding "phases" belonging to an extension of p-adics containing roots of e and roots of unity are mapped to themselves. Note that the roots of e define extensions of rationals, which induce finite dimensional algebraic extensions of p-adic numbers. This would make possible to define imbedding space in accordance with NTU. Also the Hilbert space could be defined by requiring that its points correspond to number theoretically universal angles expressible in terms of roots of unity.

3. What about real and p-adic variants of WCW? Are they needed at all? Or could their existence be used as a powerful constraint on real physics? The representability of WCW as a union of infinite-dimensional symmetric spaces labelled by zero modes suggests that the same description applies at the level of WCW and imbedding space.

   One cannot circumvent the question about how to generalize functional integral from real WCW to p-adic WCWs. In particular, what is the p-adic variant of the action defining the dynamics of space-time surfaces. In the case of exponent of action the p-adic variant could be defined by assuming algebraic universality: again the roots of e and of unity would be in central role. Also the Kähler structure of WCW implying that Gaussian and metric determinants cancel each other in functional integral, would be absolutely crucial.

   One must remember that the exponents of action for scattering amplitudes for the stationary phase extremal cancel from the path integral representation of scattering amplitudes. Also now this mechanism would allow to get rid of the poorly defined exponent for single minimum. If there is sum over scattering amplitudes assignable to different maxima, normalization sould give ratios of these exponents for different extrema/maxima and only these ratios should belong to the extension of rationals.

   The zero modes of WCW metric are invariants of supersymplectic group so that canonical identification could relate their real and p-adic variants. Zero modes could break NTU and would be behind p-adic thermodynamics and dependence of mass scale on p-adic prime.

The third challenge relates to the fusion of p-adic physics and real physics to a larger structure. Here a generalization of number concept obtained by glueing reals and various p-adics together along an extension of rational numbers inducing the extensions of p-adic numbers is highly suggestive. Adeles associated with the extension of rationals are highly attractive and closely related notion. Real and various p-adic physics would be correlates for sensory and cognitive aspects of the same universal physics rather than separate physics in this framework. One important implication of this view is that real entropy and p-adic negentropies characterize the same entanglement with coefficients in an extension of rationals.

NTU for hyperbolic and ordinary phases is definetely the central idea. How the invariance of angles under conformal transformations does relate to this? Could one perhaps define a discretized version of conformal symmetry preserving the phases defined by the angles between vectors assignable with the tangent spaces of discretized geometric structures and thus respecting NTU? Of should one apply conformal symmetry at Lie algebra level only?

9.4.2 NTU and the correspondence between real and p-adic physics

p-Adic real correspondence is certainly the basic problem of p-adicization and adelization. One can make several general questions about p-adic real correspondence and canonical identification inspired by p-adic mass calculations.

How generally p-adic real correspondence does apply? Could canonical identification for group invariants combined with direct identification of ordinary and hyperbolic phases identified as roots of unity and e apply at WCW and imbedding space level having maximally symmetric geometries? Could this make sense even at space-time level as a correspondence induced from imbedding space level [L17]? Does canonical identification apply locally for the discretizations of space-time surface or only globally for the parameters characterizing PEs (string world sheets and partonic 2-surfaces by SH), which are general coordinate invariant and Poincare invariant quantities?

The following vision seems to be the most feasible one found hitherto.

1. Preservation of symmetries and continuity compete. Lorenz transformations do not commute with canonical identification. This suggests that canonical identification applies only to Lorentz invariants formed from quantum numbers. This is enough in the case of scattering amplitudes. Canonical identification applies only to isometry invariants at the level of WCW and the phases/exponents of ordinary/hyperbolic angles correspond to numbers in the algebraic extension common to extensions of rationals and various p-adics.

2. Canonical identification applies at the level of momentum space and maps p-adic Lorentz invariants of scattering amplitudes to their real counterparts. Phases of angles and their hyperbolic counterparts should correspond to parameters defining extension and should be mapped as such to their p-adic counterparts.

3. The constraints coming from GCI and symmetries do not allow local correspondence but allow to consider its discretized version at space-time leve induced by the correspondence at the level of imbedding space.

   This requires the restriction of isometries and other symmetries to algebraic subgroups defined by the extension of rationals. This would imply reduction of symmetry due to finite cognitive/measurement resolution and should be acceptable. If one wants to realize the ideas about imagination, discretization must be applied also for the space-time interior meaning partial breaking of SH and giving rise to dark matter degrees freedom in TGD sense. SH could apply in real sector for realizable imaginations only. Note that the number of algebraic points of space-time surface is expected to be relatively small.

The correspondence must be considered at the level of imbedding space, space-time, and WCW.

1. At the level of imbedding space p-adic–real correspondence is induced by points in extension of rationals and is totally discontinuous. This requires that space-time dimension is smaller than imbedding space dimension.

2. At space-time level the correspondence involves field equations derivable from a local variational principle make sense also p-adically although the action itself is ill-defined as 4-D integral. The notion of p-adic PE makes sense by strong form of holography applied to 2-surfaces in the intersection. p-Adically however only the vanishing of Noether currents for a sub-algebra of the super-symplectic algebra might make sense. This condition is stronger than the vanishing of Noether charges defined by 3-D integrals.

3. Correspondence at the level of WCW can make sense and reduces to that for string world sheets and partonic 2-surfaces by SH. Real and p-adic 4-surfaces would be obtained by algebraic continuation as PEs from 2-surfaces by assuming that the space-time surface contains

subset of points of imbedding space belonging to the extension of rationals [L17]. p-Adic pseudo constants make p-adic continuation easy. Real continuation need not exist always. p-Adic WCW would be considerably larger than real WCW and make possible a predictive quantum theory of imagination and cognition.

What I have called intersection of realities and p-adicities can be identified as the set of 2-surfaces plus algebraic discretization of space-time interior. Also the values of induced spinor fields at the points of discretization must be given. The parameters characterizing the extremals (say coefficients of polynomials) - WCW coordinates - would be in extension of rationals inducing a finite-D extension of p-adic number fields.

The hierarchy of algebraic extensions induces an evolutionary hierarchy of adeles. The interpretation could be as a mathematical correlate for cosmic evolution realized at the level of the core of WCW defined by the intersection? 2-surfaces could be called space-time genes.

4. Also the p-adic variant Kähler action or at least the exponent of Kähler action defining vacuum functional should be obtainable by algebraic continuation. The weakest condition states that the ratios of action exponents for the maxima of Kähler function to the sum of action exponents for maxima belong to the extension. Without this condition the hopes of satisfying NTU seem rather meager.

9.4.3 NTU at space-time level

What about NTU at space-time level? NTU requires a correspondence between real and p-adic numbers and the details of this corresponds have been a long standing problem.

1. The recent view about the correspondence between real PEs to their p-adic counterparts does not demand discrete local correspondence assumed in the earlier proposal [K112]. The most abstract approach would give up the local correspondence at space-time level altogether, and restrict the preferred coordinates of WCW (having maximal group of isometries) to numbers in the extension of rationals considered. WCW would be discretized.

   Intuitively a more realistic view is a correspondence at space-time level in the sense that real and p-adic space-time sheets intersect at points belonging to the extension of rationals and defining “cognitive representations”. Only some p-adic space-time surfaces would have real counterpart.

2. The strongest form of NTU would require that the allowed points of imbedding space belonging an extension of rationals are mapped as such to corresponding extensions of p-adic number fields (no canonical identification). At imbedding space level this correspondence would be extremely discontinuous. The “spines” of space-time surfaces would however contain only a subset of points of extension, and a natural resolution length scale could emerge and prevent the fluctuation. This could be also seen as a reason for why space-times surfaces must be 4-D. The fact that the curve $x^n + y^n = z^n$ has no rational points for $n > 2$, raises the hope that the resolution scale could emerge spontaneously.

3. The notion of monadic geometry discussed in detail in [L17] would realize this idea. Define first a number theoretic discretization of imbedding space in terms of points, whose coordinates in group theoretically preferred coordinate system belong to the extension of rationals considered. One can say that these algebraic points are in the intersection of reality and various p-adicities. Overlapping open sets assigned with this discretization define in the real sector a covering by open sets. In p-adic sector compact-open-topology allows to assign with each point 8^{th} Cartesian power of algebraic extension of p-adic numbers. These compact open sets define analogs for the monads of Leibniz and p-adic variants of field equations make sense inside them.

   The monadic manifold structure of H is induced to space-time surfaces containing discrete subset of points in the algebraic discretization with field equations defining a continuation to space-time surface in given number field, and unique only in finite measurement resolution. This approach would resolve the tension between continuity and symmetries in p-adic–real correspondence: isometry groups would be replaced by their sub-groups with parameters in extension of rationals considered and acting in the intersection of reality and p-adicities.

The Galois group of extension acts non-trivially on the "spines" of space-time surfaces. Hence the number theoretical symmetries act as physical symmetries and define the orbit of given space-time surface as a kind of covering space. The coverings assigned to the hierarchy of Planck constants would naturally correspond to Galois coverings and dark matter would represent number theoretical physics.

This would give rise to a kind of algebraic hierarchy of adelic 4-surfaces identifiable as evolutionary hierarchy: the higher the dimension of the extension, the higher the evolutionary level.

9.4.4 NTU and WCW

p-Adic–real correspondence at the level of WCW

It has not been obvious whether one should perform p-adicization and adelization at the level of WCW. Minimalist could argue that scattering amplitudes are all we want and that their p-adicization and adelization by algebraic continuation can be tolerated only if it can give powerful enough constraints on the amplitudes.

1. The anti-commutations for fermionic oscillator operators are number theoretically universal. Supersymmetry suggests that also WCW bosonic degrees of freedom satisfy NTU. This could mean that the coordinates of p-adic WCW consist of super-symplectic invariants mappable by canonical identification to their real counterparts plus phases and their hyperbolic counterparts expressible as genuinely algebraic numbers common to all number fields. This kind of coordinates are naturally assignable to symmetric spaces [L17].

2. Kähler structure should be mapped from p-adic to real sector and vice versa. Vacuum functional identified as exponent of action should be NTU. Algebraic continuation defined by SH involves p-adic pseudo constants. All p-adic continuations by SH should correspond to the same value of exponent of action obtained by algebraic continuation from its real value. The degeneracy associated with p-adic pseudo-constants would be analogous to gauge invariance - imagination in TGD inspired theory of consciousness.

3. Ist it possible have NTU for WCW functional integration? Or is it enough to realize NTU for scattering amplitudes only. What seems clear that functional integral must reduce to a discrete sum. Physical intuition suggests a sum over maxima of Kähler function forming a subset of PEs representing stationary points. One cannot even exclude the possibility that the set of PEs is discrete and that one can sum over all of them.

   Restriction to maximum/stationary phase approximation gives rise to sum over exponents multiplied with Gaussian determinants. The determinant of Kähler metric however cancels the Gaussian determinants, and one obtains only a sum over the exponents of action.

   The breaking of strong NTU could happen: consider only p-adic mass calculations. This breaking is however associated with the parts of quantum states assignable to the boundaries of CD, not with the vacuum functional.

NTU for functional integral

Number theoretical vision relies on NTU. In fermionic sector NTU is necessary: one cannot speak about real and p-adic fermions as separate entities and fermionic anti-commutation relations are indeed number theoretically universal.

What about NTU in case of functional integral? There are two opposite views.

1. One can define p-adic variants of field equations without difficulties if preferred extremals are minimal surface extremals of Kähler action so that coupling constants do not appear in the solutions. If the extremal property is determined solely by the analyticity properties as it is for various conjectures, it makes sense independent of number field. Therefore there would be no need to continue the functional integral to p-adic sectors. This in accordance with the

philosophy that thought cannot be put in scale. This would be also the option favored by pragmatist.

2. Consciousness theorist might argue that also cognition and imagination allow quantum description. The supersymmetry NTU should apply also to functional integral over WCW (more precisely, its sector defined by CD) involved with the definition of scattering amplitudes.

1. Key observations

The general vision involves some crucial observations.

1. Only the expressions for the scatterings amplitudes should should satisfy NTU. This does not require that the functional integral satisfies NTU.

2. Since the Gaussian and metric determinants cancel in WCW Kähler metric the contributions form maxima are proportional to action exponentials $exp(S_k)$ divided by the $\sum_k exp(S_k)$. Loops vanish by quantum criticality.

3. Scattering amplitudes can be defined as sums over the contributions from the maxima, which would have also stationary phase by the double extremal property made possible by the complex value of α_K. These contributions are normalized by the vacuum amplitude.

   It is enough to require NTU for $X_i = exp(S_i)/\sum_k exp(S_k)$. This requires that $S_k - S_l$ has form $q_1 + q_2 i\pi + q_3 log(n)$. The condition brings in mind homology theory without boundary operation defined by the difference $S_k - S_l$. NTU for both S_k and $exp(S_k)$ would only values of general form $S_k = q_1 + q_2 i\pi + q_3 log(n)$ for S_k and this looks quite too strong a condition.

4. If it is possible to express the 4-D exponentials as single 2-D exponential associated with union of string world sheets, vacuum functional disappears completely from consideration! There is only a sum over discretization with the same effective action and one obtains purely combinatorial expression.

2. What does one mean with functional integral?

The definition of functional integral in WCW is one of the key technical problems of quantum TGD [K109]. NTU states that the integral should be defined simultaneously in all number fields in the intersection of real and p-adic worlds defined by string world sheets and partonic 2-surfaces with WCW coordinates in algebraic extension of rationals and allowing by strong holography continuation to 4-D space-time surface. NTU is powerful constraint and could help in this respect.

1. Path integral is not in question. Rather, the functional integral is analogous to Wiener integral and perhaps allows identification as a genuine integral in the real sector. In p-adic sectors algebraic continuation should give the integral and here number theoretical universality gives excellent hopes. The integral would have exactly the same form in real and p-adic sector and expressible solely in terms of algebraic numbers characterizing algebraic extension and finite roots of e and roots of unity $U_n = exp(i2\pi/n)$ in algebraic extension of p-adic numbers.

   Since vacuum functional $exp(S)$ is exponential of complex action S, the natural idea is that only rational powers e^q and roots of unity and phases $exp(i2\pi q)$ are involved and there is no dependence on p-adic prime p! This is true in the integer part of q is smaller than p so that one does not obtain e^{kp}, which is ordinary p-adic number and would spoil the number theoretic universality. This condition is not possible to satisfy for all values of p unless the value of Kähler function is smaller than 2. One might consider the possibility that the allow primes are above some minimum value.

   The minimal solution to NTU conditions is that the ratios of action exponentials for maxima of Kähler function to the sum of these exponentials belong to the extension of rationals considered.

2. What does one mean with functional integral? TGD is expected to be an integrable in some sense. In integrable QFTs functional integral reduces to a sum over stationary points of the action: typically only single point contributes - at least in good approximation.

   For real α_K and Λ vacuum functional decomposes to a product of exponents of real contribution from Euclidian regions ($\sqrt{g_4}$ real) and imaginary contribution Minkowskian regions ($\sqrt{g_4}$ imaginary). There would be no exchange of momentum between Minkowskian and Euclidian regions. For complex values of α_K [K35] situation changes and Kähler function as real part of action receives contributions from both Euclidian and Minkowskian regions. The imaginary part of action has interpretation as analog of Morse function and action as it appears in QFTs. Now saddle points must be considered.

   PEs satisfy extremely strong conditions [K8, K10]. All classical Noether charges for a sub-algebra associated with super-symplectic algebra and isomorphic to the algebra itself vanish at both ends of CD. The conformal weights of this algebra are $n > 0$-ples of those for the entire algebra. What is fascinating that the condition that the preferred extremals are minimal surface extremals of Kähler action could solve these conditions and guarantee also NTU at the level of space-time surfaces. Supersymplectic boundary conditions at the ends of CD would however pose number theoretic conditions on the coupling parameters. In p-adic case the conditions should reduce to purely local conditions since p-adic charges are not well-defined as integrals.

3. In TGD framework one is constructing zero energy states rather calculating the matrix elements of S-matrix in terms of path integral. This gives certain liberties but a natural expectation is that functional integral as a formal tool at least is involved.

   Could one *define* the functional integral as a discrete sum of contributions of standard form for the preferred extremals, which correspond to maxima in Euclidian regions and associated stationary phase points in Minkowskian regions? Could one assume that WCW spinor field is concentrated along single maximum/stationary point.

   Quantum classical correspondence suggests that in Cartan algebra isometry charges are equal to the quantal charges for quantum states expressible in number theoretically universal manner in terms of fermionic oscillator operators or WCW gamma matrices? Even stronger condition would be that classical correlation functions are identical with quantal ones for allowed space-time surfaces in the quantum superposition. Could the reduction to a discrete sum be interpreted in terms of a finite measurement resolution?

4. In QFT Gaussian determinants produce problems because they are often poorly defined. In the recent case they could also spoil the NTU based on the exceptional properties of e. In the recent case however Gaussian determinant and metric determinant for Kähler metric cancel each other and this problem disappears. One could obtain just a sum over products of roots of e and roots of unity. Thus also Kähler structure seems to be crucial for the dream about NTU.

9.4.5 Breaking of NTU at the level of scattering amplitudes

NTU in strong sense could be broken at the level of scattering amplitudes. At space-time level the breaking does not look natural in the recent framework. Consider only p-adic mass calculations predicting that mass scale depends on p-adic prime. Also for the action strong form of NTU might fail for small p-adic primes since the value of the real part of action would be larger than than p. Should one allow this? What does one actually mean with NTU in the case of action?

Canonical identification is an important element of p-adic mass calculations and might also be needed to map p-adic variants of scattering amplitudes to their real counterparts. The breaking of NTU would take place, when the canonical real valued image of the p-adic scattering amplitude differs from the real scattering amplitude. The interpretation would be in terms of finite measurement resolution. By the finite measurement/cognitive resolution characterized by p one cannot detect the difference.

The simplest form of the canonical identification is $x = \sum_n x_n p^n \to \sum x_n p^{-n}$. Product xy and sum $x + y$ do not in general map to product and sum in canonical identification. The interpretation would be in terms of a finite measurement resolution: $(xy)_R = x_R y_R$ and $(x+y)_R = x_R + y_R$ only modulo finite measurement resolution. p-Adic scattering amplitudes are obtained by algebraic continuation from the intersection by replacing algebraic number valued parameters (such as momenta) by general p-adic numbers. The real images of these amplitudes under canonical identification are in general not identical with real scattering amplitudes the interpretation being in terms of a finite measurement resolution.

In p-adic thermodynamics NTU in the strong sense fails since thermal masses depend on p-adic mass scale. NTU can be broken by the fermionic matrix elements in the functional integral so that the real scattering amplitudes differ from the canonical images of the p-adic scattering amplitudes. For instance, the elementary particle vacuum functionals in the space of Teichmueller parameters for the partonic 2-surfaces and string world sheets should break NTU [K23].

9.4.6 NTU and the spectrum of Kähler coupling strength

During years I have made several attempts to understand coupling evolution in TGD framework. The most convincing proposal has emerged rather recently and relates the spectrum of $1/\alpha_K$ to that for the zeros of Riemann zeta [K35] and to the evolution of the electroweak U(1) couplings strength.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K43]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.

   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K113] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking.

   The problem is that the dynamics in Minkowskian and Euclidian regions decouple completely and if Euclidian regions serve as space-time correlates for physical objects, there would be no exchanges of classical charges between physical objects. Should one conclude that α_K must be complex?

2. p-Adic mass calculations for 2 decades ago [K51] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity. Needless to say these attempts were premature and a hoc.

3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [?] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that

Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

Quite recently the number theoretic interpretation of coupling constant evolution [K109] [L9] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K104]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.

   Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.

   The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s = 2$. The trivial poles for $s = 2n$, $n = 1, 2, ..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff} = n \times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s = 2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?

   It turns out that at p-adic length scale $k = 131$ ($p \simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K109]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k = 127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w = w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `http://tinyurl.com/gwjs85b`) with real coefficients (element of $GL(2, R)$) so that one as $\zeta_F((as + b)/(cs + d))$. Rather general arguments force it to be and element of $GL(2, Q)$, $GL(2, Z)$ or maybe even $SL(2, Z)$ ($ad - bc = 1$) satisfying additional constraints. Since TGD

predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2, Z)$ and by a scaling factor K.

Could one understand the general qualitative features of color and weak coupling contant evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs+d$ and color confinement with the zero of $as+b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as+b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as+b/)(cs+d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p \simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a, b, c, d). In the sequel this vision is discussed in more detail.

9.4.7 Other applications of NTU

NTU in the strongest form says that all numbers involved at "basic level" (whatever this means!) of adelic TGD are products of roots of unity and of power of a root of e. This is extremely powerful physics inspired conjecture with a wide range of possible mathematical applications.

1. For instance, vacuum functional defined as an exponent of action for preferred externals would be number of this kind. One could define functional integral as adelic operation in all number fields: essentially as sum of exponents of action for stationary preferred extremals since Gaussian and metric determinants potentially spoiling NTU would cancel each other leaving only the exponent.

2. The implications of NTU for the zeros of Riemann zeta [L9] will be discussed in more detail in the Appendix. Suffice it to say that the observations about Fourier transform for the distribution of loci of non-trivial zeros of zeta together with NTU leads to explicit proposal for the algebraic for of zeros of zeta. The testable proposal is that zeros decompose to disjoint classes $C(p)$ labelled by primes p and the condition that p^{iy} is root of unity in given class $C(p)$.

3. NTU generalises to all Lie groups. Exponents $exp(in_iJ_i/n)$ of lie-algebra generators define generalisations of number theoretically universal group elements and generate a discrete subgroup of compact Lie group. Also hyperbolic "phases" based on the roots $e^{m/n}$ are possible and make possible discretized NTU versions of all Lie-groups expected to play a key role in adelization of TGD.

   NTU generalises also to quaternions and octonions and allows to define them as number theoretically universal entities. Note that ordinary p-adic variants of quaternions and octonions do not give rise to a number field: inverse of quaternion can have vanishing p-adic variant of norm squared satisfying $\sum_n x_n^2 = 0$.

   NTU allows to define also the notion of Hilbert space as an adelic notion. The exponents of angles characterising unit vector of Hilbert space would correspond to roots of unity.

9.4.8 Going to the roots of p-adicity

The basic questions raised by the p-adic mass calculations concern the origin of preferred p-adic primes and of p-adic length scale hypothesis. One can also ask whether there might be a natural origin for p-adicity at the level of WCW.

Preferred primes as ramified primes for extensions of rationals?

Preferred primes as ramified primes for extensions of rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and to me the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of *ramification of primes* (http://tinyurl.com/hddljlf) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Ramification is completely analogous to the degeneracy of some roots of polynomial and corresponds to criticality if the polynomial corresponds to criticality (catastrophe theory of Thom is one application). Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this sloppy language of physicist): as one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called *integral closure* (http://tinyurl.com/js6fpvr) over integers of K decompose to products of prime ideals of L (prime ideal is a more rigorous manner to express primeness). Note that the general ideal is analog of integer.

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

   Prime ideals of K can be decomposed to products of prime ideals of L: $P = \prod P_i^{e_i}$, where e_i is the ramification index. If $e_i > 1$ is true for some i, *ramification* occurs. P_i:s in question are like co-inciding roots of polynomial, which for in thermodynamics and Thom's catastrophe theory corresponds to criticality. Ramification could therefore be a natural aspect of quantum criticality and ramified primes P are good candidates for preferred primes for a given extension of rationals. Note that the ramification make sense also for extensions of given extension of rationals.

2. A physical analogy for the decomposition of ideals to ideals of extension is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, the physical analog would be the number of elementary particles of type i in the state (http://tinyurl.com/h9528pl). Unramified prime P would be analogous a state with e fermions. Maximally ramified prime would be analogous to Bose-Einstein condensate of e bosons. General ramified prime would be analogous to an e-particle state containing both fermions and condensed bosons. This is of course just a formal analogy.

3. There are two further basic notions related to ramification and characterizing it. *Relative discriminant* is the ideal divided by all ramified ideals in K (integer of K having no ramified prime factors) and relative different for P is the ideal of L divided by all ramified P_i:s (product of prime factors of P in L). These ideals represent the analogs of product of preferred primes P of K and primes P_i of L dividing them. These two integers ideals would characterize the ramification.

In TGD framework the extensions of rationals (http://tinyurl.com/h9528pl) and p-adic number fields (http://tinyurl.com/zq22tvb) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would gradually proceed to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

How could ramification relate to p-adic and adelic physics and could it explain preferred primes?

1. Ramified p-adic prime $P = P_i^e$ would be replaced with its e:th root P_i in p-adicization. Same would apply to general ramified primes. Each un-ramified prime of K is replaced with $e = K : L$ primes of L and ramified primes P with $\#\{P_i\} < e$ primes of L: the increase of algebraic dimension is smaller. An interesting question relates to p-adic length scale. What happens to p-adic length scales. Is p-adic prime effectively replaced with e:th root of p-adic prime: $L_p \propto p^{1/2}L_1 \rightarrow p^{1/2e}L_1$? The only physical option is that the p-adic temperature for P would be scaled down $T_p = 1/n \rightarrow 1/ne$ for its e:th root (for fermions serving as fundamental particles in TGD one actually has $T_p = 1$). Could the lower temperature state be more stable and select the preferred primes as maximimally ramified ones? What about general ramified primes?

2. This need not be the whole story. Some algebraic extensions would be more favored than others and p-adic view about realizable imaginations could be involved. p-Adic pseudo constants are expected to allow p-adic continuations of string world sheets and partonic 2-surfaces to 4-D preferred extremals with number theoretic discretization. For real continuations the situation is more difficult. For preferred extensions - and therefore for corresponding ramified primes - the number of real continuations - realizable imaginations - would be especially large.

   The challenge would be to understand why primes near powers of 2 and possibly also of other small primes would be favored. Why for them the number of realizable imaginations would be especially large so that they would be winners in number theoretical fight for survival?

Can one make this picture more concrete? What kind of algebraic extensions could be considered?

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naive generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. It would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (`http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

   Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n:th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with

$\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

4. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \to x+1$ transforms $(x^n-1)/(x-1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a manner that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the imbedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

The origin of p-adic length scale hypothesis?

p-Adic length scale hypothesis emerged from p-adic length scale hypothesis. A possible generalization of this hypothesis is that p-adic primes near powers of prime are physically favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [I2] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present: this is discussed in TGD inspired theory of music harmony and genetic code [K80]. See also [L19, L14].

One explanation would be that for preferred primes the number of p-adic space-time sheets representable also as real space-time sheets is maximal. Imagined worlds would be maximally realizable. Preferred p-adic primes would correspond to ramified primes for extensions with the property that the number of realizable imaginations is especially large for them. Why primes satisfying p-adic length scale hypothesis or its generalization would appear as ramified primes for extensions, which are winners in number theoretical evolution?

Also the weak form of NMP (WNMP) applying also to the purely number theoretic form of NMP [K56] might come in rescue here.

1. Entanglement negentropy for a NE [K56] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretical entanglement negentropy (NEN). If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured, which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real PE as p-adic PE (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. WNMP [K56, K106] suggests a more feasible explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. WNMP allows "free will" so that all dimensions $n-k$, $k=0,1,...n-1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n-k$ is power of prime. For $n=2^k=M_k+1$, where M_k is Mersenne prime $n-1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n=2^k-r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p=2$ can be replaced by any prime.

9.5 p-Adic physics and consciousness

p-Adic physics as physics of cognition and imagination is an important thread in TGD inspired theory of consciousness. In the sequel I describe briefly the basic of TGD inspired theory of consciousness as generalization of quantum measurement theory to ZEO (ZEO), describe the definition of self, consider the question whether NMP is needed as a separate principle or whether it is implied is in statistical sense by the unavoidable statistical increase of $n=h_{eff}/h$ if identified as a factor of the dimension of Galois group extension of rationals defining the adeles, and finally summarize the vision about how p-adic physics serves as a correlate of cognition and imagination.

9.5.1 From quantum measurement theory to a theory of consciousness

The notion of self can be seen as a generalization of the poorly defined definition of the notion of observer in quantum physics. In the following I take the role of skeptic trying to be as critical as possible.

The original definition of self was as a subsystem able to remain unentangled under state function reductions associated with subsequent quantum jumps. The density matrix was assumed to define the universal observable. Note that a density matrix, which is power series of a product of matrices representing commuting observables has in the generic case eigenstates, which are simultaneous eigenstates of all observables. Second aspect of self was assumed to be the integration of subsequent quantum jumps to coherent whole giving rise to the experienced flow of time.

The precise identification of self allowing to understand both of these aspects turned out to be difficult problem. I became aware the solution of the problem in terms of ZEO (ZEO) only rather recently (2014).

1. Self corresponds to a sequence of quantum jumps integrating to single unit as in the original proposal, but these quantum jumps correspond to state function reductions to a fixed boundary of causal diamond CD leaving the corresponding parts of zero energy states invariant - "small" state function reductions. The parts of zero energy states at second boundary of CD change and even the position of the tip of the opposite boundary changes: one actually has wave function over positions of second boundary (CD sizes roughly) and this wave function changes. In positive energy ontology these repeated state function reductions would have no effect on the state (Zeno effect) but in TGD framework there occurs a change for the second boundary and gives rise to the experienced flow of time and its arrow and self: self is generalized Zeno effect.

2. The first quantum jump to the opposite boundary corresponds to the act of "free will" or birth of re-incarnated self. Hence the act of "free will" changes the arrow of psychological time at some level of hierarchy of CDs. The first reduction to the opposite boundary of CD means "death" of self and "re-incarnation" of time-reversed self at opposite boundary at which the the temporal distance between the tips of CD increases in opposite direction. The

sequence of selves and time reversed selves is analogous to a cosmic expansion for CD. The repeated birth and death of mental images could correspond to this sequence at the level of sub-selves.

3. This allows to understand the relationship between subjective and geometric time and how the arrow of and flow of clock time (psychological time) emerge. The average distance between the tips of CD increases on the average as along as state function functions occur repeatedly at the fixed boundary: situation is analogous to that in diffusion. The localization of contents of conscious experience to boundary of CD gives rise to the illusion that universe is 3-dimensional. The possibility of memories made possibly by hierarchy of CDs demonstrates that this is not the case. Self is simply the sequence of state function reductions at same boundary of CD remaining fixed and the lifetime of self is the total growth of the average temporal distance between the tips of CD.

One can identify several rather abstract state function reductions selecting a sector of WCW.

1. There are quantum measurements inducing localization in the moduli space of CDs with passive boundary and states at it fixed. In particular, a localization in the moduli characterizing the Lorentz transform of the upper tip of CD would be measured. The measured moduli characterize also the analog of symplectic form in M^4 strongly suggested by twistor lift of TGD - that is the rest system (time axis) and spin quantization axes. Of course, also other kinds of reductions are possible.

2. Also a localization to an extension of rationals defining the adeles should occur. Could the value of $n = h_{eff}/h$ be observable? The value of n for given space-time surface at the active boundary of CD could be identified as the order of the smallest Galois group containing all Galois groups assignable to 3-surfaces at the boundary. The superposition of space-time surface would not be eigenstate of n at active boundary unless localization occurs. It is not obvious whether this is consistent with a fixe value of n at passive boundary.

   The measured value of n could be larger or smaller than the value of n at the passive boundary of CD but in statistical sense n would increase by the analogy with diffusion on half line defined by non-negative integers. The distance from the origin unavoidably increases in statistical sense. This would imply evolution as increase of maximal value of negentropy and generation of quantum coherence in increasingly longer scales.

3. A further abstract choice corresponds to the the replacement of the roles of active and passive boundary of CD changing the arrow of clock time and correspond to a death of self and re-incarnation as time-reversed self.

Can one assume that these measurements reduce to measurements of a density matrix of either entangled system as assumed in the earlier formulation of NMP, or should one allow both options. This question actually applies to all quantum measurements and leads to a fundamental philosophical questions unavoidable in all consciousness theories.

1. Do all measurements involve entanglement between the moduli or extensions of two CDs reduced in the measurement of the density matrix? Non-diagonal entanglement would allow final states states, which are not eigenstates of moduli or of n: this looks strange. This could also lead to an infinite regress since it seems that one must assume endless hierarchy of entangled CDs so that the reduction sequence would proceed from top to bottom. It looks natural to regard single CD as a sub-Universe.

   For instance, if a selection of quantization axis of color hypercharge and isospin (localization in the twistor space of CP_2) is involved, one would have an outcome corresponding to a quantum superposition of measurements with different color quantization axis!

   Going philosophical, one can also argue, that the measurement of density matrix is only a reaction to environment and does not allow intentional free will.

2. Can one assume that a mere localization in the moduli space or for the extension of rationals (producing an eigenstate of n) takes place for a fixed CD - a kind of self measurement possible

for even unentangled system? If there is entanglement in these degrees of freedom between two systems (say CDs), it would be reduced in these self measurements but the outcome would not be an eigenstate of density matrix. An interpretation as a realization of intention would be approriate.

3. If one allows both options, the interpretation would be that state function reduction as a measurement of density matrix is only a reaction to environment and self-measurement represents a realization of intention.

4. Self measurements would occur at higher level say as a selection of quantization axis, localization in the moduli space of CD, or selection of extension of rationals. A possible general rule is that measurements at space-time level are reactions as measurements of density matrix whereas a selection of a sector of WCW would be an intentional action. This because formally the quantum states at the level of WCW are as modes of classical WCW spinor field single particle states.

5. If the selections of sectors of WCW at active boundary of CD commute with observables, whose eigenstates appear at passive boundary (briefly *passive observables*) meaning that time reversal commutes with them - they can occur repeatedly during the reduction sequence and self as a generalized Zeno effect makes sense.

   If the selections of WCW sectors at active boundary do not commute with passive observables then volition as a choice of sector of WCW must change the arrow of time. Libet's findings show that conscious choice induces neural activity for a fraction of second before the conscious choice. This would imply the correspondences *"big" measurement changing the arrow of time - self-measurement at the level of WCW - intentional action* and *"small" measurement - measurement at space-time level - reaction.*

   Self as a generalized Zeno effect makes sense only if there are active commuting with passive observables. If the passive observables form a maximal set, the new active observables commuting with them must emerge. The increase of the size of extension of rationals might generate them by expanding the state space so that self would survive only as long at it evolves.

   Otherwise there would be only single unitary time evolution followed by a reduction to opposite boundary. This makes sense only if the sequence of "big" reductions for sub-selves can give rise to the time flow experienced by self: the birth and death of mental images would give rise to flow of time of self.

A hierarchical process starting from given CD and proceeding downwards to shorter scales and stopping when the entanglement is stable is highly suggestive and favors self measurements. What stability could mean will be discussed in the next section. CDs would be a correlate for self hierarchy. One can say also something about the anatomy and correlates of self hierarchy.

1. Self experiences its sub-selves as mental images and even we would represent mental images of some higher level collective self. Everything is conscious but consciousness can be lost or at least it is not possible to have memory about it. The flow of consciousness for a given self could be due to the quantum jump sequences performed by its sub-selves giving rise to mental images.

2. By quantum classical correspondence self has also space-time correlates. One can visualize sub-self as a space-time sheet "glued" by topological sum to the space-time sheet of self. Subsystem is not described as a tensor factor as in the standard description of subsystems. Also sub-selves of selves can entangle negentropically and this gives rise to a sharing of mental images about which stereo vision would be basic example. Quite generally, one could speak of stereo consciousness. Also the experiences of sensed presence [J3] could be understood as a sharing of mental images between brain hemispheres, which are not themselves entangled. This is possible also between different brains. In the normal situation brain hemispheres are entangled.

3. At the level of 8-dimensional imbedding space the natural correlate of self would be CD (causal diamond). At the level of space-time the correlate would be space-time sheet or light-like 3-surface. The contents of consciousness of self would be determined by the space-time sheets in the interior of CD. Without further restrictions the experience of self would be essentially four-dimensional. Memories would be like sensory experiences except that they would be about the geometric past and for some reason are not usually colored by sensory qualia. For instance .1 second time scale defining sensory chronon corresponds to the secondary p-adic time scale characterizing the size of electron's CD (Mersenne prime M_{127}), which suggests that Cooper pairs of electrons are essential for the sensory qualia.

9.5.2 NMP and self

The view about Negentropy Maximization Principle (NMP) [K56] has co-evolved with the notion of self and I have considered many variants of NMP.

1. The original formulation of NMP was in positive energy ontology and made same predictions as standard quantum measurement theory. The new element was that the density matrix of sub-system defines the fundamental observable and the system goes to its eigenstate in state function reduction. As found, the localizations at to WCW sectors define what might be called self-measurements and identifiable as active volitions rather than reactions.

2. In p-adic physics one can assign with rational and even algebraic entanglement probabilities number theoretical entanglement negentropy (NEN) satisfying the same basic axioms as the ordinary Shannon entropy but having negative values and therefore having interpretation as information. The definition of p-adic negentropy (real valued) reads as $S_p = -\sum P_k log(|P_k|_p)$, where $|.|_p$ denotes p-adic norm. The news is that $N_p = -S_p$ can be positive and is positive for rational entanglement probabilities. Real entanglement entropy S is always non-negative.

   NMP would force the generation of negentropic entanglement (NE) and stabilize it. NNE resources of the Universe - one might call them Akashic records- would steadily increase.

3. A decisive step of progress was the realization is that NTU forces all states in adelic physics to have entanglement coefficients in some extension of rationals inducing finite-D extension of p-adic numbers. The same entanglement can be characterized by real entropy S and p-adic negentropies N_p, which can be positive. One can define also total p-adic negentropy: $N = \sum_p N_p$ for all p and total negentropy $N_{tot} = N - S$.

   For rational entanglement probabilities it is easy to demonstrate that the generalization of adelic theorem holds true: $N_{tot} = N - S = 0$. NMP based on N_{tot} rather than N would not say anything about rational entanglement. For extensions of rationals it is easy to find that $N - S > 0$ is possible if entanglement probabilities are of form X_i/n with $|X_i|_p = 1$ and n integer [L16]. Should one identify the total negentropy as difference $N_{tot} = N - S$ or as $N_{tot} = N$?

   Irrespective of answer, large p-adic negentropy seems to force large real entropy: this nicely correlates with the paradoxical finding that living systems tend to be entropic although one would expect just the oppositecite [L16]: this relates in very interesting manner to the work of biologists Jeremy England [I3]. The negentropy would be cognitive negentropy and not visible for ordinary physics.

4. The latest step in the evolution of ideas NMP was the question whether NMP follows from number theory alone just as second law follows form probability theory! This irritates theoretician's ego but is victory for theory. The dimension n of extension is positive integer and cannot but grow in statistical sense in evolution! Since one expects that the maximal value of negentropy (define as $N - S$) must increase with n. Negentropy must increase in long run.

Number theoretic entanglement can be stable

Number theoretical Shannon entropy can serve as a measure for genuine information assignable to a pair of entanglement systems [K56]. Entanglement with coefficients in the extension is always negentropic if entanglement negentropy comes from p-adic sectors only. It can be negentropic if negentropy is defined as the difference of p-adic negentropy and real entropy.

The diagonalized density matrix need not belong to the algebraic extension since the probabilities defining its diagonal elements are eigenvalues of the density matrix as roots of N:th order polynomial, which in the generic case requires n-dimensional algebraic extension of rationals. One can argue that since diagonalization is not possible, also state function reduction selecting one of the eigenstates is impossible unless a phase transition increasing the dimension of algebraic extension used occurs simultaneously. This kind of NE could give rise to cognitive entanglement.

There is also a special kind of NE, which can result if one requires that density matrix serves a universal observable in state function reduction. The outcome of reduction must be an eigen space of density matrix, which is projector to this subspace acting as identity matrix inside it. This kind NE allows all unitarily related basis as eigenstate basis (unitary transformations must belong to the algebraic extension). This kind of NE could serve as a correlate for "enlightened" states of consciousness. Schrödingers cat is in this kind of state stably in superposition of dead and alive and state basis obtained by unitary rotation from this basis is equally good. One can say that there are no discriminations in this state, and this is what is claimed about "enlightened" states too.

The vision about number theoretical evolution suggests that NMP forces the generation of NE resources as NE assignable to the "passive" boundary of CD for which no changes occur during sequence of state function reductions defining self. It would define the unchanging self as negentropy resources, which could be regarded as kind of Akashic records. During the next "re-incarnation" after the first reduction to opposite boundary of CD the NE associated with the reduced state would serve as new Akashic records for the time reversed self. If NMP reduces to the statistical increase of $h_{eff}/h = n$ the consciousness information contents of the Universe increases in statistical sense. In the best possible world of SNMP it would increase steadily.

Does NMP reduce to number theory?

The heretic question that emerged quite recently is whether NMP is actually needed at all! Is NMP a separate principle or could NMP reduced to mere number theory [K56]? Consider first the possibility that NMP is not needed at all as a separate principle.

1. The value of $h_{eff}/h = n$ should increase in the evolution by the phase transitions increasing the dimension of the extension of rationals. $h_{eff}/h = n$ has been identified as the number of sheets of some kind of covering space. The Galois group of extension acts on number theoretic discretizations of the monadic surface and the orbit defines a covering space. Suppose n is the number of sheets of this covering and thus the dimension of the Galois group for the extension of rationals or factor of it.

2. It has been already noticed that the "big" state function reductions giving rise to death and reincarnation of self could correspond to a measurement of $n = h_{eff}$ implied by the measurement of the extension of the rationals defining the adeles. The statistical increase of n follows automatically and implies statistical increase of maximal entanglement negentropy. Entanglement negentropy increases in statistical sense.

   The resulting world would not be the best possible one unlike for a strong form of NMP demanding that negentropy does increaes in "big" state function reductions. n also decrease temporarily and they seem to be needed. In TGD inspired model of bio-catalysis the phase transition reducing the value of n for the magnetic flux tubes connecting reacting bio-molecules allows them to find each other in the molecular soup. This would be crucial for understanding processes like DNA replication and transcription.

3. State function reduction corresponding to the measurement of density matrix could occur to an eigenstate/eigenspace of density matrix only if the corresponding eigenvalue and eigenstate/eigenspace is expressible using numbers in the extension of rationals defining the adele considered. In the generic case these numbers belong to N-dimensional extension of the original extension. This can make the entanglement stable with respect to state the measurements of density matrix.

A phase transition to an extension of an extension containing these coefficients would be required to make possible reduction. A step in number theoretic evolution would occur. Also an entanglement of measured state pairs with those of measuring system in containing the extension of extension would make possible the reduction. Negentropy could be reduced but higher-D extension would provide potential for more negentropic entanglement and NMP would hold true in the statistical sense.

4. If one has higher-D eigen space of density matrix, p-adic negentropy is largest for the entire subspace and the sum of real and p-adic negentropies vanishes for all of them. For negentropy identified as total p-adic negentropy SNMP would select the entire sub-space and NMP would indeed say something explicit about negentropy.

Or is NMP needed as a separate principle?

Hitherto I have postulated NMP as a separate principle [K56]. Strong form of NMP (SNMP) states that Negentropy does not decrease in "big" state function reductions corresponding to death and re-incarnations of self.

One can however argue that SNMP is not realistic. SNMP would force the Universe to be the best possible one, and this does not seem to be the case. Also ethically responsible free will would be very restricted since self would be forced always to do the best deed that is increase maximally the negentropy serving as information resources of the Universe. Giving up separate NMP altogether would allow to have also "Good" and "Evil".

This forces to consider what I christened weak form of NMP (WNMP). Instead of maximal dimension corresponding to N-dimensional projector self can choose also lower-dimensional sub-spaces and 1-D sub-space corresponds to the vanishing entanglement and negentropy assumed in standard quantum measurement theory. As a matter fact, this can also lead to larger negentropy gain since negentropy depends strongly on what is the large power of p in the dimension of the resulting eigen sub-space of density matrix. This could apply also to the purely number theoretical reduction of NMP.

WNMP suggests how to understand the notions of Good and Evil. Various choices in the state function reduction would correspond to Boolean algebra, which suggests an interpretation in terms of what might be called emotional intelligence [K106]. Also it turns out that one can understand how p-adic length scale hypothesis - actually its generalization - emerges from WNMP [K109].

1. One can start from ordinary quantum entanglement. It corresponds to a superposition of pairs of states. Second state corresponds to the internal state of the self and second state to a state of external world or biological body of self. In negentropic quantum entanglement each is replaced with a pair of sub-spaces of state spaces of self and external world. The dimension of the sub-space depends on which pair is in question. In state function reduction one of these pairs is selected and deed is done. How to make some of these deeds good and some bad? Recall that WNMP allows only the possibility to generate NNE but does not force it. WNMP would be like God allowing the possibility to do good but not forcing good deeds.

   Self can choose any sub-space of the subspace defined by $k \leq N$-dimensional projector and 1-D subspace corresponds to the standard quantum measurement. For $k = 1$ the state function reduction leads to vanishing negentropy, and separation of self and the target of the action. Negentropy does not increase in this action and self is isolated from the target: kind of price for sin.

   For the maximal dimension of this sub-space the negentropy gain is maximal. This deed would be good and by the proposed criterion NE corresponds to conscious experience with positive emotional coloring. Interestingly, there are $2^k - 1$ possible choices, which is almost the dimension of Boolean algebra consisting of k independent bits. The excluded option corresponds to 0-dimensional sub-space - empty set in set theoretic realization of Boolean algebra. This could relate directly to fermionic oscillator operators defining basis of Boolean algebra - here Fock vacuum would be the excluded state. The deed in this sense would be a choice of how loving the attention towards system of external world is.

2. A map of different choices of k-dimensional sub-spaces to k-fermion states is suggestive. The realization of logic in terms of emotions of different degrees of positivity would be mapped to many-fermion states - perhaps zero energy states with vanishing total fermion number. State function reductions to k-dimensional spaces would be mapped to k-fermion states: quantum jumps to quantum states!

   The problem brings in mind quantum classical correspondence in quantum measurement theory. The direction of the pointer of the measurement apparatus (in very metaphorical sense) corresponds to the outcome of state function reduction, which is now 1-D subspace. For ordinary measurement the pointer has k positions. Now it must have $2^k - 1$ positions. To the discrete space of k pointer positions one must assign fermionic Clifford algebra of second quantized fermionic oscillator operators. The hierarchy of Planck constants and dark matter suggests the realization. Replace the pointer with its space-time k-sheeted covering and consider zero energy energy states made of pairs of k-fermion states at the sheets of the n-sheeted covering? Dark matter would be therefore necessary for cognition. The role of fermions would be to "mark" the k space-time sheets in the covering.

The cautious conclusion is that NMP as a separate principle is not necessary and follows in statistical sense from the unavoidable increase of $n = h_{eff}/h$ identified as dimension of extension of rationals define the adeles if this extension or at least the dimension of its Galois group is observable.

9.5.3 p-Adic physics as correlate of cognition and imagination

The items in the following list give motivations for the proposal that p-adic physics could serve as a correlate for cognition and imagination.

1. By the total disconnectedness of the p-adic topology, p-adic world decomposes naturally into blobs, objects. This happens also in sensory perception. The pinary digits of p-adic number can be assigned to a p-tree. Parisi proposed in the model of spin glass [B20] that p-adic numbers could relate to the mathematical description of cognition and also Khrennikov [J1] has developed this idea. In TGD framework that idea is taken to space-time level: p-adic space-time sheets represent thought bubbles and they correlate with the real ones since they form cognitive reprentations of the real world. SH allows a concrete realization of this.

2. p-Adic non-determinism due to p-adic pseudo constants suggests interpretation in terms of imagination. Given 2-surfaces could allow completion to p-adic preferred extremal but not to a real one so that pure "non-realizable" imagination is in question.

3. Number theoretic negentropy has interpretation as negentropy characterizing information content of entanglement. The superposition of state pairs could be interpreted as a quantum representation for a rule or abstracted association containing its instances as state pairs. Number theoretical negentropy characterizes the relationship of two systems and should not be confused with thermodynamical entropy, which characterizes the uncertainty about the state of single system.

The original vision was that p-adic non-determinism could serve as a correlate for cognition, imagination, and intention. The recent view is much more cautious. Imagination need not completely reduce to p-adic non-determinism since it has also real physics correlates - maybe as partial realizations of SH as in nerve pulse pattern, which does not propagate down to muscles.

A possible interpretation for the solutions of the p-adic field equations would be as geometric correlates of cognition, imagination, and perhaps even intentionality. Plans, intentions, expectations, dreams, and possibly also cognition as imagination in general could have p-adic cognitive space-time sheets as their geometric correlates. A deep principle seems to be involved: incompleteness is the characteristic feature of p-adic physics but the flexibility made possible by this incompleteness is absolutely essential for imagination and cognitive consciousness in general.

The most feasible view is that the intersections of p-adic and real space-time surfaces define cognitive representations of real space-time surfaces (PEs, [K14, K8, K10]). One could also say that real space-time surface represents sensory aspects of conscious experience and p-adic space-time surfaces its cognitive aspects. Both real and p-adics rather than real or p-adics.

The identification of p-adic pseudo constants as correlates of imagination at space-time level is indeed a further natural idea.

1. The construction of PEs by SH from the data at 2-surfaces is like boundary value problem with number theoretic discretization of space-time surface as additional data. PE property in real context implies strong correlations between string world sheets and partonic 2-surfaces by boundary conditions a them. One cannot choose these 2-surfaces completely independently in real context.

2. In p-adic sectors the integration constants are replaced with pseudo-constants depending on finite number of pinary digits of variables depending on coordinates normal to string world sheets and partonic 2-surfaces. The fixing of the discretization of space-time surface would allow to fix the p-adic pseudo-constants. Once the number theoretic discretization of space-time surface is fixed, the p-adic pseudo-constants can be fixed. Pseudo-constant could allow a large number of p-adic configurations involving string world sheets, partonic 2-surfaces, and number theoretic discretization but not allowed in real context.

   Could these p-adic PEs correspond to imaginations, which in general are not realizable? Could the realizable intentional actions belong to the intersection of real and p-adic WCWs? Could one identify non-realistic imaginations as the modes of WCW spinor fields for which 2-surfaces are not extendable to real space-time surfaces and are localized to 2-surfaces? Could they allow only a partial continuation to real space-time surface. Could nerve pulse pattern representing imagined motor action and not proceeding to the level of muscles correspond to a partially real PE?

   Could imagination and problem solving be search for those collections of string world sheets and partonic 2-surfaces, which allow extension to (realization as) real PEs? If so, p-adic physics would be there as an independent aspect of existence and this is just the original idea. Imagination could be realized in state function reduction, which always selects only those 2-surfaces, which allow continuation to real space-time surfaces. The distinction between only imaginable and also realizable would be the extendability by using strong form of holography.

3. An interesting question is why elementary particles are characterized by preferred p-adic primes (primes near powers of 2, in particular Mersenne primes). Could the number of realizable imaginations for these primes be especially large?

I have the feeling that this view allows respectable mathematical realization of imagination in terms of adelic quantum physics. It is remarkable that SH derivable from - you can guess, SGCI (the Big E again!), plays an absolutely central role in it.

9.6 Appendix: Super-symplectic conformal weights and zeros of Riemann zeta

Since fermions are the only fundamental particles in TGD one could argue that the conformal weight of for the generating elements of supersympelectic algebra could be negatives for the poles of fermionic zeta ζ_F. This demands $n > 0$ as does also the fractal hierarchy of supersymplectic symmetry breakings. NTU of Riemann zeta in some sense is strongly suggested if adelic physics is to make sense.

For ordinary conformal algebras there are only finite number of generating elements ($-2 \leq n \leq 2$). If the radial conformal weights for the generators of g consist of poles of ζ_F, the situation changes. ζ_F is suggested by the observation that fermions are the only fundamental particles in TGD.

1. Riemann Zeta $\zeta(s) = \prod_p (1/(1-p^{-s})$ identifiable formally as a partition function $\zeta_B(s)$ of arithmetic boson gas with bosons with energy $log(p)$ and temperature $1/s = 1/(1/2+iy)$ should be replaced with that of arithmetic fermionic gas given in the product representation by $\zeta_F(s) = \prod_p (1+p^{-s})$ so that the identity $\zeta_B(s))/\zeta_F(s) = \zeta_B(2s)$ follows. This gives

$$\frac{\zeta_B(s)}{\zeta_B(2s)} .$$

$\zeta_F(s)$ has zeros at zeros s_n of $\zeta(s)$ and at the pole $s = 1/2$ of $zeta(2s)$. $\zeta_F(s)$ has poles at zeros $s_n/2$ of $\zeta(2s)$ and at pole $s = 1$ of $\zeta(s)$.

The spectrum of $1/T$ would be for the generators of algebra $\{(-1/2+iy)/2, n > 0, -1\}$. In p-adic thermodynamics the p-adic temperature is $1/T = 1/n$ and corresponds to "trivial" poles of ζ_F. Complex values of temperature does not make sense in ordinary thermodynamics. In ZEO quantum theory can be regarded as a square root of thermodynamics and complex temperature parameter makes sense.

2. If the spectrum of conformal weights of the generating elements of the algebra corresponds to poles serving as analogs of propagator poles, it consists of the "trivial" conformal $h = n > 0$- the standard spectrum with $h = 0$ assignable to massless particles excluded - and "non-trivial" $h = -1/4 + iy/2$. There is also a pole at $h = -1$.

   Both the non-trivial pole with real part $h_R = -1/4$ and the pole $h = -1$ correspond to tachyons. I have earlier proposed conformal confinement meaning that the total conformal weight for the state is real. If so, one obtains for a conformally confined two-particle states corresponding to conjugate non-trivial zeros in minimal situation $h_R = -1/2$ assignable to N-S representation.

   In p-adic mass calculations ground state conformal weight must be $-5/2$ [K51]. The negative fermion ground state weight could explain why the ground state conformal weight must be tachyonic $-5/2$. With the required 5 tensor factors one would indeed obtain this with minimal conformal confinement. In fact, arbitrarily large tachyonic conformal weight is possible but physical state should always have conformal weights $h > 0$.

3. $h = 0$ is not possible for generators, which reminds of Higgs mechanism for which the naïve ground states corresponds to tachyonic Higgs. $h = 0$ conformally confined massless states are necessarily composites obtained by applying the generators of Kac-Moody algebra or super-symplectic algebra to the ground state. This is the case according to p-adic mass calculations [K51], and would suggest that the negative ground state conformal weight can be associated with super-symplectic algebra and the remaining contribution comes from ordinary super-conformal generators. Hadronic masses, whose origin is poorly understood, could come from super-symplectic degrees of freedom. There is no need for p-adic thermodynamics in super-symplectic degrees of freedom.

9.6.1 A general formula for the zeros of zeta from NTU

Dyson's comment about Fourier transform of Riemann Zeta [A47] (`http://tinyurl.com/hjbfsuv`) is interesting from the point of NTU for Riemann zeta.

1. The numerical calculation of Fourier transform for the imaginary parts iy of zeros $s = 1/2+iy$ of zeta shows that it is concentrated at discrete set of frequencies coming as $log(p^n)$, p prime. This translates to the statement that the zeros of zeta form a 1-dimensional quasicrystal, a discrete structure Fourier spectrum by definition is also discrete (this of course holds for ordinary crystals as a special case). Also the logarithms of powers of primes would form a quasicrystal, which is very interesting from the point of view of p-adic length scale hypothesis. Primes label the "energies" of elementary fermions and bosons in arithmetic number theory, whose repeated second quantization gives rise to the hierarchy of infinite primes [K94]. The energies for general states are logarithms of integers.

2. Powers p^n label the points of quasicrystal defined by points $log(p^n)$ and Riemann zeta has interpretation as partition function for boson case with this spectrum. Could p^n label also the points of the dual lattice defined by iy.

3. The existence of Fourier transform for points $log(p_i^n)$ for any vector y_a in class $C(p)$ of zeros labelled by p requires $p_i^{iy_a}$ to be a root of unity inside $C(p)$. This could define the sense in

which zeros of zeta are universal. This condition also guarantees that the factor $n^{-1/2-iy}$ appearing in zeta at critical line are number theoretically universal ($p^{1/2}$ is problematic for Q_p: the problem might be solved by eliminating from p-adic analog of zeta the factor $1-p^{-s}$.

(a) One obtains for the pair (p_i, s_a) the condition $log(p_i)y_a = q_{ia}2\pi$, where q_{ia} is a rational number. Dividing the conditions for (i,a) and (j,a) gives

$$p_i = p_j^{q_{ia}/q_{ja}}$$

for every zero s_a so that the ratios q_{ia}/q_{ja} do not depend on s_a. From this one easily deduce $p_i^M = p_j^N$, where M and N are integers so that one ends up with a contradiction.

(b) Dividing the conditions for (i,a) and (i,b) one obtains

$$\frac{y_a}{y_b} = \frac{q_{ia}}{q_{ib}}$$

so that the ratios q_{ia}/q_{ib} do not depend on p_i. The ratios of the imaginary parts of zeta would be therefore rational number which is very strong prediction and zeros could be mapped by scaling y_a/y_1 where y_1 is the zero which smallest imaginary part to rationals.

(c) The impossible consistency conditions for (i,a) and (j,a) can be avoided if each prime and its powers correspond to its own subset of zeros and these subsets of zeros are disjoint: one would have infinite union of sub-quasicrystals labelled by primes and each p-adic number field would correspond to its own subset of zeros: this might be seen as an abstract analog for the decomposition of rational to powers of primes. This decomposition would be natural if for ordinary complex numbers the contribution in the complement of this set to the Fourier trasform vanishes. The conditions $(i,a)$and (i,b) require now that the ratios of zeros are rationals only in the subset associated with p_i.

For the general option the Fourier transform can be delta function for $x = log(p^k)$ and the set $\{y_a(p)\}$ contains N_p zeros. The following argument inspires the conjecture that for each p there is an infinite number N_p of zeros $y_a(p)$ in class $C(p)$ satisfying

$$p^{iy_a(p)} = u(p) = e^{\frac{r(p)}{m(p)}i2\pi} \quad ,$$

where $u(p)$ is a root of unity that is $y_a(p) = 2\pi(m(a)+r(p))/log(p)$ and forming a subset of a lattice with a lattice constant $y_0 = 2\pi/log(p)$, which itself need not be a zero.

In terms of stationary phase approximation the zeros $y_a(p)$ associated with p would have constant stationary phase whereas for $y_a(p_i \neq p))$ the phase $p^{iy_a(p_i)}$ would fail to be stationary. The phase e^{ixy} would be non-stationary also for $x \neq log(p^k)$ as function of y.

1. Assume that for $x = qlog(p)$, where q not a rational, the phases e^{ixy} fail to be roots of unity and are random implying the vanishing/smallness of $F(x)$.

2. Assume that for a given p all powers p^{iy} for $y \notin \{y_a(p)\}$ fail to be roots of unity and are also random so that the contribution of the set $y \notin \{y_a(p)\}$ to $F(p)$ vanishes/is small.

3. For $x = log(p^{k/m})$ the Fourier transform should vanish or be small for $m \neq 1$ (rational roots of primes) and give a non-vanishing contribution for $m = 1$. One has

$$F(x = log(p^{k/m}) = \sum_{1\leq a\leq N(p)} e^{k\frac{M(a,p)}{mN(p)}i2\pi}u(p) \quad ,$$
$$u(p) = e^{\frac{r(p)}{m(p)}i2\pi} \quad .$$

Obviously one can always choose $N(a,p) = N(p)$.

4. For the simplest option $N(p) = 1$ one would obtain delta function distribution for $x = log(p^k)$. The sum of the phases associated with $y_a(p)$ and $-y_a(p)$ from the half axes of the critical line would give

$$F(x = log(p^n)) \propto X(p^n) \equiv 2cos(n\frac{r(p)}{m(p)}2\pi) \ .$$

The sign of F would vary.

5. For $x = log(p^{k/m})$ the value of Fourier transform is expected to be small by interference effects if $M(a,p)$ is random integer, and negligible as compared with the value at $x = log(p^k)$. This option is highly attractive. For $N(p) > 1$ and $M(a,p)$ a random integer also $F(x = log(p^k)$ is small by interference effects. Hence it seems that this option is the most natural one.

6. The rational $r(p)/m(p)$ would characterize given prime (one can require that $r(p)$ and $m(p)$ have no common divisors). $F(x)$ is non-vanishing for all powers $x = log(p^n)$ for $m(p)$ odd. For $p = 2$, also $m(2) = 2$ allows to have $|X(2^n)| = 2$. An interesting ad hoc ansatz is $m(p) = p$ or $p^{s(p)}$. One has periodicity in n with period $m(p)$ that is logarithmic wave. This periodicity serves as a test and in principle allows to deduce the value of $r(p)/m(p)$ from the Fourier transform.

What could one conclude from the data (`http://tinyurl.com/hjbfsuv`)?

1. The first graph gives $|F(x = log(p^k)|$ and second graph displays a zoomed up part of $|F(x = log(p^k)|$ for small powers of primes in the range $[2, 19]$. For the first graph the eighth peak $(p = 11)$ is the largest one but in the zoomed graphs this is not the case. Hence something is wrong or the graphs correspond to different approximations suggesting that one should not take them too seriously.

   In any case, the modulus is not constant as function of p^k. For small values of p^k the envelope of the curve decreases and seems to approach constant for large values of p^k (one has $x < 15$ $(e^{15} \simeq 3.3 \times 10^6)$.

2. According to the first graph $|F(x)|$ decreases for $x = klog(p) < 8$, is largest for small primes, and remains below a fixed maximum for $8 < x < 15$. According to the second graph the amplitude decreases for powers of a given prime (say $p = 2$). Clearly, the small primes and their powers have much larger $|F(x)|$ than large primes.

There are many possible reasons for this behavior. Most plausible reason is that the sums involved converge slowly and the approximation used is not good. The inclusion of only 10^4 zeros would show the positions of peaks but would not allow reliable estimate for their intensities.

1. The distribution of zeros could be such that for small primes and their powers the number of zeros is large in the set of 10^4 zeros considered. This would be the case if the distribution of zeros $y_a(p)$ is fractal and gets "thinner" with p so that the number of contributing zeros scales down with p as a power of p, say $1/p$, as suggested by the envelope in the first figure.

2. The infinite sum, which should vanish, converges only very slowly to zero. Consider the contribution $\Delta F(p^k, p_1)$ of zeros not belonging to the class $p_1 \neq p$ to $F(x = log(p^k)) = \sum_{p_i} \Delta F(p^k, p_i)$, which includes also $p_i = p$. $\Delta F(p^k, p_i)$, $p \neq p_1$ should vanish in exact calculation.

   (a) By the proposed hypothesis this contribution reads as

   $$\begin{aligned} \Delta F(p, p_1) &= \sum_a cos\left[X(p^k, p_1)(M(a, p_1) + \tfrac{r(p_1)}{m(p_1)})2\pi)\right] \ . \\ X(p^k, p_1) &= \tfrac{log(p^k)}{log(p_1)} \ . \end{aligned}$$

   Here a labels the zeros associated with p_1. If p^k is "approximately divisible" by p^1 in other words, $p^k \simeq np_1$, the sum over finite number of terms gives a large contribution since interference effects are small, and a large number of terms are needed to give a nearly vanishing contribution suggested by the non-stationarity of the phase. This happens in several situations.

(b) The number $\pi(x)$ of primes smaller than x goes asymptotically like $\pi(x) \simeq x/log(x)$ and prime density approximately like $1/log(x) - 1/log(x)^2$ so that the problem is worst for the small primes. The problematic situation is encountered most often for powers p^k of small primes p near larger prime and primes p (also large) near a power of small prime (the envelope of $|F(x)|$ seems to become constant above $x \sim 10^3$).

(c) The worst situation is encountered for $p = 2$ and $p_1 = 2^k - 1$ - a Mersenne prime and $p_1 = 2^{2^k} + 1$, $k \leq 4$ - Fermat prime. For $(p, p_1) = (2^k, M_k)$ one encounters $X(2^k, M_k) = (log(2^k)/log(2^k - 1)$ factor very near to unity for large Mersennes primes. For $(p, p_1) = (M_k, 2)$ one encounters $X(M_k, 2) = (log(2^k - 1)/log(2) \simeq k$. Examples of Mersennes and Fermats are $(3,2), (5,2), (7,2), (17,2), (31,2), (127,2), (257,2), ...$ Powers 2^k, $k = 2, 3, 4, 5, 7, 8, ..$ are also problematic.

(d) Also twin primes are problematic since in this case one has factor $X(p = p_1 + 2, p_1) = \frac{log(p_1+2)}{log(p_1)}$. The region of small primes contains many twin prime pairs: (3,5), (5,7), (11,13), (17,19), (29,31),....

These observations suggest that the problems might be understood as resulting from including too small number of zeros.

3. The predicted periodicity of the distribution with respect to the exponent k of p^k is not consistent with the graph for small values of prime unless the periodic $m(p)$ for small primes is large enough. The above mentioned effects can quite well mask the periodicity. If the first graph is taken at face value for small primes, $r(p)/m(p)$ is near zero, and $m(p)$ is so large that the periodicity does not become manifest for small primes. For $p = 2$ this would require $m(2) > 21$ since the largest power $2^n \simeq e^{15}$ corresponds to $n \sim 21$.

To summarize, the prediction is that for zeros of zeta should divide into disjoint classes $\{y_a(p)\}$ labelled by primes such that within the class labelled by p one has $p^{iy_a(p)} = e^{(r(p)/m(p))i2\pi}$ so that has $y_a(p) = [M(a,p) + r(p)/m(p))]2\pi/log(p)$.

9.6.2 More precise view about zeros of Zeta

There is a very interesting blog post by Mumford (`http://tinyurl.com/zemw27o`), which leads to much more precise formulation of the idea and improved view about the Fourier transform hypothesis: the Fourier transform or its generalization must be defined for all zeros, not only the non-trivial ones and trivial zeros give a background term allowing to understand better the properties of the Fourier transform.

Mumford essentially begins from Riemann's "explicit formula" in von Mangoldt's form.

$$\sum_p \sum_{n \geq 1} log(p)\delta_{p^n}(x) = 1 - \sum_k x^{s_k - 1} - \frac{1}{x(x^2-1)} \quad ,$$

where p denotes prime and s_k a non-trivial zero of zeta. The left hand side represents the distribution associated with powers of primes. The right hand side contains sum over cosines

$$\sum_k x^{s_k - 1} = 2\frac{\sum_k cos(log(x)y_k)}{x^{1/2}} \quad ,$$

where y_k ithe imaginary part of non-trivial zero. Apart from the factor $x^{-1/2}$ this is just the Fourier transform over the distribution of zeros.

There is also a slowly varying term $1 - \frac{1}{x(x^2-1)}$, which has interpretation as the analog of the Fourier transform term but sum over trivial zeros of zeta at $s = -2n$, $n > 0$. The entire expression is analogous to a "Fourier transform" over the distribution of all zeros. Quasicrystal is replaced with union on 1-D quasicrystals.

Therefore the distribution for powers of primes is expressible as "Fourier transform" over the distribution of both trivial and non-trivial zeros rather than only non-trivial zeros as suggested by numerical data to which Dyson [A47] referred to (`http://tinyurl.com/hjbfsuv`). Trivial zeros give a slowly varying background term large for small values of argument x (poles at $x = 0$ and

$x = 1$ - note that also $p = 0$ and $p = 1$ appear effectively as primes) so that the peaks of the distribution are higher for small primes.

The question was how can one obtain this kind of delta function distribution concentrated on powers of primes from a sum over terms $cos(log(x)y_k)$ appearing in the Fourier transform of the distribution of zeros.

Consider $x = p^n$. One must get a constructive interference. Stationary phase approximation is in terms of which physicist thinks. The argument was that a destructive interference occurs for given $x = p^n$ for those zeros for which the cosine does not correspond to a real part of root of unity as one sums over such y_k: random phase approximation gives more or less zero. To get something nontrivial y_k must be proportional to $2\pi \times n(y_k)/log(p)$ in class $C(p)$ to which y_k belongs. If the number of these y_k:s in $C(p)$ is infinite, one obtains delta function in good approximation by destructive interference for other values of argument x.

The guess that the number of zeros in $C(p)$ is infinite is encouraged by the behaviors of the densities of primes one hand and zeros of zeta on the other hand. The number of primes smaller than real number x goes like

$$\pi(x) = N(primes < x) \sim \frac{x}{log(x)}$$

in the sense of distribution. The number of zeros along critical line goes like

$$N(zeros < t) = (t/2\pi) \times log(\frac{t}{2\pi})$$

in the same sense. If the real axis and critical line have same metric measure then one can say that the number of zeros in interval T per number of primes in interval T behaves roughly like

$$\frac{N(zeros < T)}{N(primes < T)} = log(\frac{T}{2\pi}) \times \frac{log(T)}{2\pi}$$

so that at the limit of $T \to \infty$ the number of zeros associated with given prime is infinite. This asumption of course makes the argument a poor man's argument only.

9.6.3 Possible relevance for TGD

What this speculative picture from the point of view of TGD?

1. A possible formulation for NTU for the poles of fermionic Riemann zeta $\zeta_F = \zeta(s)/\zeta(2s)$ could be as a condition that is that the exponents $p^{ks_a(p)/2} = p^{k/4}p^{iky_a(p)/2}$ exist in a number theoretically universal manner for the zeros $s_a(p)$ for given p-adic prime p and for some subset of integers k. If the proposed conditions hold true, exponent reduces $p^{k/4}e^{k(r(p/m(p)i2\pi}$ requiring that k is a multiple of 4. The number of the non-trivial generating elements of super-symplectic algebra in the monomial creating physical state would be a multiple of 4. These monomials would have real part of conformal weight -1. Conformal confinement suggests that these monomials are products of pairs of generators for which imaginary parts cancel.

2. Quasi-crystal property might have an application to TGD. The functions of light-like radial coordinate appearing in the generators of supersymplectic algebra could be of form r^s, s zero of zeta or rather, its imaginary part. The eigenstate property with respect to the radial scaling rd/dr is natural by radial conformal invariance.

   The idea that arithmetic QFT assignable to infinite primes is behind the scenes in turn suggests light-like momenta assignable to the radial coordinate have energies with the dual spectrum $log(p^n)$. This is also suggested by the interpretation of ζ as square root of thermodynamical partition function for boson gas with momentum $log(p)$ and analogous interpretation of ζ_F.

   The two spectra would be associated with radial scalings and with light-like translations of light-cone boundary respecting the direction and light-likeness of the light-like radial vector. $log(p^n)$ spectrum would be associated with light-like momenta whereas p-adic mass scales would characterize states with thermal mass. Note that generalization of p-adic length scale hypothesis raises the scales defined by p^n to a special physical position: this might relate to ideal structure of adeles.

3. Finite measurement resolution suggests that the approximations of Fourier transforms over the distribution of zeros taking into account only a finite number of zeros might have a physical meaning. This might provide additional understand about the origins of generalized p-adic length scale hypothesis stating that primes $p \simeq p_1^k$, p_1 small prime - say Mersenne primes - have a special physical role.

Chapter i

Appendix

A-1 Introduction

Originally this appendix was meant to be a purely technical summary of basic facts but in its recent form it tries to briefly summarize those basic visions about TGD which I dare to regarded stabilized. I have added illustrations making it easier to build mental images about what is involved and represented briefly the key arguments. This chapter is hoped to help the reader to get fast grasp about the concepts of TGD.

The basic properties of embedding space and related spaces are discussed and the relationship of CP_2 to the standard model is summarized. The basic vision is simple: the geometry of the embedding space $H = M^4 \times CP_2$ geometrizes standard model symmetries and quantum numbers. The assumption that space-time surfaces are basic objects, brings in dynamics as dynamics of 3-D surfaces based on the induced geometry. Second quantization of free spinor fields of H induces quantization at the level of H, which means a dramatic simplification.

The notions of induction of metric and spinor connection, and of spinor structure are discussed. Many-sheeted space-time and related notions such as topological field quantization and the relationship many-sheeted space-time to that of GRT space-time are discussed as well as the recent view about induced spinor fields and the emergence of fermionic strings. Also the relationship to string models is discussed briefly.

Various topics related to p-adic numbers are summarized with a brief definition of p-adic manifold and the idea about generalization of the number concept by gluing real and p-adic number fields to a larger book like structure analogous to adele [L22, L23]. In the recent view of quantum TGD [L45], both notions reduce to physics as number theory vision, which relies on $M^8 - H$ duality [L35, L36] and is complementary to the physics as geometry vision.

Zero energy ontology (ZEO) [L34] [K114] has become a central part of quantum TGD and leads to a TGD inspired theory of consciousness as a generalization of quantum measurement theory having quantum biology as an application. Also these aspects of TGD are briefly discussed.

A-2 Embedding space $M^4 \times CP_2$

Space-times are regarded as 4-surfaces in $H = M^4 \times CP_2$ the Cartesian product of empty Minkowski space - the space-time of special relativity - and compact 4-D space CP_2 with size scale of order 10^4 Planck lengths. One can say that embedding space is obtained by replacing each point m of empty Minkowski space with 4-D tiny CP_2. The space-time of general relativity is replaced by a 4-D surface in H which has very complex topology. The notion of many-sheeted space-time gives an idea about what is involved.

Fig. 1. Embedding space $H = M^4 \times CP_2$ as Cartesian product of Minkowski space M^4 and complex projective space CP_2. `http://tgdtheory.fi/appfigures/Hoo.jpg`

Denote by M^4_+ and M^4_- the future and past directed lightcones of M^4. Denote their intersection, which is not unique, by CD. In zero energy ontology (ZEO) [L34, L38] [K114] causal diamond

(CD) is defined as cartesian product $CD \times CP_2$. Often I use CD to refer just to $CD \times CP_2$ since CP_2 factor is relevant from the point of view of ZEO.

Fig. 2. Future and past light-cones M^4_+ and M^4_-. Causal diamonds (CD) are defined as their intersections. `http://tgdtheory.fi/appfigures/futurepast.jpg`

Fig. 3. Causal diamond (CD) is highly analogous to Penrose diagram but simpler. `http://tgdtheory.fi/appfigures/penrose.jpg`

A rather recent discovery was that CP_2 is the only compact 4-manifold with Euclidian signature of metric allowing twistor space with Kähler structure. M^4 is in turn is the only 4-D space with Minkowskian signature of metric allowing twistor space with Kähler structure [A45] so that $H = M^4 \times CP_2$ is twistorially unique.

One can loosely say that quantum states in a given sector of "world of classical worlds" (WCW) are superpositions of space-time surfaces inside CDs and that positive and negative energy parts of zero energy states are localized and past and future boundaries of CDs. CDs form a hierarchy. One can have CDs within CDs and CDs can also overlap. The size of CD is characterized by the proper time distance between its two tips. One can perform both translations and also Lorentz boosts of CD leaving either boundary invariant. Therefore one can assign to CDs a moduli space and speak about wave function in this moduli space.

In number theoretic approach it is natural to restrict the allowed Lorentz boosts to some discrete subgroup of Lorentz group and also the distances between the tips of CDs to multiples of CP_2 radius defined by the length of its geodesic. Therefore the moduli space of CDs discretizes. The quantization of cosmic recession velocities for which there are indications, could relate to this quantization.

A-2.1 Basic facts about CP_2

CP_2 as a four-manifold is very special. The following arguments demonstrate that it codes for the symmetries of standard models via its isometries and holonomies.

CP_2 as a manifold

CP_2, the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space C^3 under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \ . \tag{A-2.1}$$

Here λ is any non-zero complex number. Note that CP_2 can be also regarded as the coset space $SU(3)/U(2)$. The pair z^i/z^j for fixed j and $z^i \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three coordinate charts covering CP_2, the charts being holomorphically related to each other (e.g. CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homoeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to S^2. Therefore CP_2 is obtained by "adding the 2-sphere at infinity to R^4".

Besides the standard complex coordinates $\xi^i = z^i/z^3$, $i = 1, 2$ the coordinates of Eguchi and Freund [A36] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \ , \\ \xi^2 &= x + iy \ . \end{aligned} \tag{A-2.1}$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= rexp(i\frac{(\Psi + \Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= rexp(i\frac{(\Psi - \Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{A-2.1}$$

The ranges of the variables r, Θ, Φ, Ψ are $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively.

Considered as a real four-manifold CP_2 is compact and simply connected, with Euler number Euler number 3, Pontryagin number 3 and second $b = 1$.

Fig. 4. CP_2 as manifold. `http://tgdtheory.fi/appfigures/cp2.jpg`

Metric and Kähler structure of CP_2

In order to obtain a natural metric for CP_2, observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \rightarrow exp(i\alpha)z^i$ on the sphere S^5: $\sum z^i\bar{z}^i = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits of the isometries. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5.

The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.2}$$

where the Hermitian, in fact Kähler metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K \ , \tag{A-2.3}$$

where the function K, Kähler function, is defined as

$$\begin{aligned} K &= log(F) \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-2.3}$$

The Kähler function for S^2 has the same form. It gives the S^2 metric $dzd\bar{z}/(1+r^2)^2$ related to its standard form in spherical coordinates by the coordinate transformation $(r, \phi) = (tan(\theta/2), \phi)$.

The representation of the CP_2 metric is deducible from S^5 metric is obtained by putting the angle coordinate of a geodesic sphere constant in it and is given

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2\sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} \ , \tag{A-2.4}$$

where the quantities σ_i are defined as

$$\begin{aligned} r^2\sigma_1 &= Im(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_2 &= -Re(\xi^1 d\xi^2 - \xi^2 d\xi^1) \ , \\ r^2\sigma_3 &= -Im(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) \ . \end{aligned} \tag{A-2.3}$$

R denotes the radius of the geodesic circle of CP_2. The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A \ , \tag{A-2.4}$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} \ , \quad & e^1 &= \frac{r\sigma_1}{\sqrt{F}} \ , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} \ , \quad & e^3 &= \frac{r\sigma_3}{F} \ . \end{aligned} \tag{A-2.5}$$

The explicit representations of vierbein vectors are given by

$$\begin{array}{llllll} e^0 & = & \frac{dr}{F} \ , & e^1 & = & \frac{r(sin\Theta cos\Psi d\Phi + sin\Psi d\Theta)}{2\sqrt{F}} \ , \\ e^2 & = & \frac{r(sin\Theta sin\Psi d\Phi - cos\Psi d\Theta)}{2\sqrt{F}} \ , & e^3 & = & \frac{r(d\Psi + cos\Theta d\Phi)}{2F} \ . \end{array} \tag{A-2.5}$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = \frac{dr^2}{F^2} + \frac{r^2}{4F^2}(d\Psi + cos\Theta d\Phi)^2 + \frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2) \ . \tag{A-2.5}$$

From this expression one finds that at coordinate infinity $r = \infty$ line element reduces to $\frac{r^2}{4F}(d\Theta^2 + sin^2\Theta d\Phi^2)$ of S^2 meaning that 3-sphere degenerates metrically to 2-sphere and one can say that CP_2 is obtained by adding to R^4 a 2-sphere at infinity.

The vierbein connection satisfying the defining relation

$$de^A = -V^A_B \wedge e^B \ , \tag{A-2.6}$$

is given by

$$\begin{array}{llllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ . \end{array} \tag{A-2.7}$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{array}{llllll} R_{01} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , & R_{23} & = & e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} & = & e^0 \wedge e^2 - e^3 \wedge e^1 \ , & R_{31} & = & -e^0 \wedge e^2 + e^3 \wedge e^1 \ , \\ R_{03} & = & 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , & R_{12} & = & 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \tag{A-2.8}$$

Metric defines a real, covariantly constant, and therefore closed 2-form J

$$J = -is_{a\bar{b}} d\xi^a d\bar{\xi}^b \ , \tag{A-2.9}$$

the so called Kähler form. Kähler form J defines in CP_2 a symplectic structure because it satisfies the condition

$$J^k_{\ r} J^{rl} = -s^{kl} \ . \tag{A-2.10}$$

The condition states that J and g give representations of real unit and imaginary units related by the formula $i^2 = -1$.

Kähler form is expressible locally in terms of Kähler gauge potential

$$J = dB \ , \tag{A-2.11}$$

where B is the so called Kähler potential, which is not defined globally since J describes homological magnetic monopole.

$dJ = ddB = 0$ gives the topological half of Maxwell equations (vanishing of magnetic charges and Faraday's induction law) and self-duality $*J = J$ reduces the remaining equations to $dJ = 0$. Hence the Kähler form can be regarded as a curvature form of a $U(1)$ gauge potential B carrying a magnetic charge of unit $1/2g$ (g denotes the gauge coupling).

The magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representations of J and B are given by

$$\begin{aligned} B &= 2re^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r}{F^2} dr \wedge (d\Psi + cos\Theta d\Phi) + \frac{r^2}{2F} sin\Theta d\Theta \wedge d\Phi \ . \end{aligned} \tag{A-2.10}$$

The vierbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1, 1).

Useful coordinates for CP_2 are the so called canonical (or symplectic or Darboux) coordinates in which the Kähler potential and Kähler form have very simple expressions

$$\begin{aligned} B &= \sum_{k=1,2} P_k dQ_k \ , \\ J &= \sum_{k=1,2} dP_k \wedge dQ_k \ . \end{aligned} \tag{A-2.10}$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned} P_1 &= -\frac{1}{1+r^2} \ , \\ P_2 &= -\frac{r^2 cos\Theta}{2(1+r^2)} \ , \\ Q_1 &= \Psi \ , \\ Q_2 &= \Phi \ . \end{aligned} \tag{A-2.8}$$

Spinors In CP_2

CP_2 doesn't allow spinor structure in the conventional sense [A27]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of CP_2 play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space M. The parallel propagation around a closed curve with a base point x leads to a rotated vierbein at x: $e^A = R^A_B e^B$ and one can associate to each closed path an element of $SO(4)$.

Consider now a one-parameter family of closed curves $\gamma(v) : v \in (0, 1)$ with the same base point x and $\gamma(0)$ and $\gamma(1)$ trivial paths. Clearly these paths define a sphere S^2 in M and the element $R^A_B(v)$ defines a closed path in $SO(4)$. When the sphere S^2 is contractible to a point e.g., homologically trivial, the path in $SO(4)$ is also contractible to a point and therefore represents a trivial element of the homotopy group $\Pi_1(SO(4)) = Z_2$.

For a homologically nontrivial 2-surface S^2 the associated path in $SO(4)$ can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group Spin(4) (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallel propagate also spinors and by the above construction associate a closed path of Spin(4) to the surface S^2. Now, however this path corresponds to a lift of the corresponding $SO(4)$ path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the non-allowed -1-factor associated with the parallel transport of the spinor around the sphere S^2 by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating -1-factor. For a $U(1)$ gauge potential this factor is given by the exponential

$exp(i2\Phi)$, where Φ is the magnetic flux through the surface. This factor has the value -1 provided the $U(1)$ potential carries half odd multiple of Dirac charge $1/2g$. In case of CP_2 the required gauge potential is half odd multiple of the Kähler potential B defined previously. In the case of $M^4 \times CP_2$ one can in addition couple the spinor components with different chiralities independently to an odd multiple of $B/2$.

Geodesic sub-manifolds of CP_2

Geodesic sub-manifolds are defined as sub-manifolds having common geodesic lines with the embedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors h^k_α (understood as vectors of H) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to H and X^4.

In [A63] a general characterization of the geodesic sub-manifolds for an arbitrary symmetric space G/H is given. Geodesic sub-manifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra g of the group G. The Lie triple system t is defined as a subspace of g characterized by the closedness property with respect to double commutation

$$[X,[Y,Z]] \in t \;\text{ for }\; X,Y,Z \in t \ . \tag{A-2.9}$$

$SU(3)$ allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that $SU(3)$ allows two nonequivalent $SU(2)$ algebras corresponding to subgroups $SO(3)$ (orthogonal 3×3 matrices) and the usual isospin group $SU(2)$. By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic sub-manifold of CP_2.

Standard representatives for the geodesic spheres of CP_2 are given by the equations

$$S^2_I \ : \ \xi^1 = \bar{\xi}^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Psi = 0) \ ,$$

$$S^2_{II} \ : \ \xi^1 = \xi^2 \ \text{ or equivalently } \ (\Theta = \pi/2, \Phi = 0) \ .$$

The non-equivalence of these sub-manifolds is clear from the fact that isometries act as holomorphic transformations in CP_2. The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for S^2_I. S^2_{II} is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

A-2.2 CP_2 geometry and Standard Model symmetries

Identification of the electro-weak couplings

The delicacies of the spinor structure of CP_2 make it a unique candidate for space S. First, the coupling of the spinors to the $U(1)$ gauge potential defined by the Kähler structure provides the missing $U(1)$ factor in the gauge group. Secondly, it is possible to couple different H-chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [B24] and in particular that the right handed neutrinos decouple completely from the electro-weak interactions.

To begin with, recall that the space H allows to define three different chiralities for spinors. Spinors with fixed H-chirality $e = \pm 1$, CP_2-chirality l, r and M^4-chirality L, R are defined by the condition

$$\begin{aligned} \Gamma\Psi &= e\Psi \ , \\ e &= \pm 1 \ , \end{aligned} \tag{A-2.9}$$

where Γ denotes the matrix $\Gamma_9 = \gamma_5 \otimes \gamma_5$, $1 \otimes \gamma_5$ and $\gamma_5 \otimes 1$ respectively. Clearly, for a fixed H-chirality CP_2- and M^4-chiralities are correlated.

The spinors with H-chirality $e = \pm 1$ can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite H-chirality one can identify the vielbein group of CP_2 as the electro-weak group: $SO(4)$ having as its covering group $SU(2)_L \times SU(2)_R$.

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+ 1_+ + n_- 1_-) \ . \qquad \text{(A-2.10)}$$

Here V and B denote the projections of the vielbein and Kähler gauge potentials respectively and $1_{+(-)}$ projects to the spinor H-chirality $+(-)$. The integers $n_\pm$ are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection V and of B are given by the equations

$$\begin{array}{llllll} V_{01} & = & -\frac{e^1}{r} \ , & V_{23} & = & \frac{e^1}{r} \ , \\ V_{02} & = & -\frac{e^2}{r} \ , & V_{31} & = & \frac{e^2}{r} \ , \\ V_{03} & = & (r - \frac{1}{r})e^3 \ , & V_{12} & = & (2r + \frac{1}{r})e^3 \ , \end{array} \qquad \text{(A-2.11)}$$

and

$$B = 2re^3 \ , \qquad \text{(A-2.12)}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying Σ_3^0 and Σ_2^1 as the diagonal (neutral) Lie-algebra generators of $SO(4)$, one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 \ , \qquad \text{(A-2.13)}$$

where one have defined

$$\begin{array}{lll} I_L^1 & = & \frac{(\Sigma_{01} - \Sigma_{23})}{2} \ , \\ I_L^2 & = & \frac{(\Sigma_{02} - \Sigma_{13})}{2} \ . \end{array} \qquad \text{(A-2.13)}$$

A_{ch} is clearly left handed so that one can perform the identification of the gauge potential as

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} \ , \qquad \text{(A-2.14)}$$

where $W^\pm$ denotes the charged intermediate vector boson.

The covariantly constant curvature tensor is given by

$$\begin{array}{l} R_{01} = -R_{23} = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \\ R_{02} = -R_{31} = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \\ R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2 \ , \\ R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \ . \end{array} \qquad \text{(A-2.15)}$$

The charged part of the curvature tensor is left handed.

This is to be compared with the Weyl tensor, which defines a representation of quaternionic imaginary units.

$$W_{03} = W_{12} \equiv 2I_3 = 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ ,$$
$$W_{01} = W_{23} \equiv I_1 = -e^0 \wedge e^1 - e^2 \wedge e^3 \ ,$$
$$W_{02} = W_{31} \equiv I_2 = -e^0 \wedge e^2 - e^3 \wedge e^1 \ . \qquad \text{(A-2.16)}$$

The charged part of the Weyl tensor is right-handed and that the relative sign of the two terms in the curvature tensor and Weyl tensor are opposite.

Consider next the identification of the neutral gauge bosons γ and Z^0 as appropriate linear combinations of the two functionally independent quantities

$$X = re^3 \ ,$$
$$Y = \frac{e^3}{r} \ , \qquad \text{(A-2.16)}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define

$$\bar{\gamma} = aX + bY \ ,$$
$$\bar{Z}^0 = cX + dY \ , \qquad \text{(A-2.16)}$$

where the normalization condition

$$ad - bc = 1 \ ,$$

is satisfied. The physical fields γ and Z^0 are related to $\bar{\gamma}$ and $\bar{Z}^0$ by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$A_{nc} = [(c+d)2\Sigma_{03} + (2d-c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma}$$
$$+ [(a-b)2\Sigma_{03} + (a-2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 \ . \qquad \text{(A-2.15)}$$

Identifying Σ_{12} and $\Sigma_{03} = 1 \times \gamma_5\Sigma_{12}$ as vectorial and axial Lie-algebra generators, respectively, the requirement that γ couples vectorially leads to the condition

$$c = -d \ . \qquad \text{(A-2.16)}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \qquad \text{(A-2.17)}$$

Here the electromagnetic charge Q_{em} and the weak isospin are defined by

$$Q_{em} = \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} \ ,$$
$$I_L^3 = \frac{(\Sigma^{12} - \Sigma^{03})}{2} \ . \qquad \text{(A-2.17)}$$

The fields γ and Z^0 are defined via the relations

$$\gamma = 6d\bar{\gamma} = \frac{6}{(a+b)}(aX + bY) \ ,$$
$$Z^0 = 4(a+b)\bar{Z}^0 = 4(X - Y) \ . \qquad \text{(A-2.17)}$$

The value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{3b}{2(a+b)} \ , \tag{A-2.18}$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electro-weak interactions.

The determination of the value of the Weinberg angle is a dynamical problem. The original approach was based on the assumption that it makes sense to talk about electroweak action defined at fundamental level and introduce a symmetry breaking by adding an additional term proportional to Kähler action. The recent view is that Kähler action plus volume term defines the fundamental action.

The Weinberg angle is completely fixed if one requires that the electroweak action contains no cross term of type γZ^0. This leads to a definite value for the Weinberg angle.

One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle. As a matter fact, color gauge action identifying color gauge field as proportional to $H^A J_{\alpha\beta}$ is proportional to Kähler action. A possible interpretation would be as a sum of electroweak and color gauge interactions.

To evaluate the value of the Weinberg angle one can express the neutral part F_{nc} of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_+1_+ + n_-1_-) \ , \tag{A-2.19}$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \end{aligned} \tag{A-2.18}$$

in terms of the fields γ and Z^0 (photon and Z- boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - sin^2\theta_W Q_{em}) \ . \tag{A-2.19}$$

Evaluating the expressions above, one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} \ . \end{aligned} \tag{A-2.19}$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + sin^2\theta_W Z^0) \ . \tag{A-2.20}$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + fJ^{\alpha\beta}J_{\alpha\beta} \ , \\ L_{sym} &= \frac{1}{4g^2}Tr(F^{\alpha\beta}F_{\alpha\beta}) \ , \end{aligned} \tag{A-2.20}$$

where the trace is taken in spinor representation, in terms of γ and Z^0 one obtains for the coefficient X of the γZ^0 cross term (this coefficient must vanish) the expression

$$X = -\frac{K}{2g^2} + \frac{fp}{18} \ ,$$
$$K = Tr\left[Q_{em}(I_L^3 - sin^2\theta_W Q_{em})\right] \ , \quad \text{(A-2.20)}$$

This parameter can be calculated by substituting the values of quark and lepton charges and weak isospins.

In the general case the value of the coefficient K is given by

$$K = \sum_i \left[-\frac{(18+2n_i^2)sin^2\theta_W}{9} \right] \ , \quad \text{(A-2.21)}$$

where the sum is over the spinor chiralities, which appear as elementary fermions and n_i is the integer describing the coupling of the spinor field to the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9\sum_i 1}{(fg^2 + 2\sum_i(18+n_i^2))} \ . \quad \text{(A-2.22)}$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} \ . \quad \text{(A-2.23)}$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is not far from the typical value 9/24 of GUTs at high energies [B9]. The experimental value at the scale length scale of the electron can be deduced from the ratio of W and Z boson masses as $sin^2\theta_W = 1-(m_W/m_Z)^2 \simeq .22290$. This ratio and also the weak boson masses depend on the length scale.

If one interprets the additional term proportional to J as color action, one could perhaps interpret the value of Weinberg angle as expressing a connection between strong and weak coupling constant evolution. The limit $f \rightarrow 0$ should correspond to an infinite value of color coupling strength and at this limit one would have $sin^2\theta_W = \frac{9}{28}$ for $f/g^2 \rightarrow 0$. This does not make sense since the Weinberg angle is in the standard model much smaller in QCD scale Λ corresponding roughly to pion mass scale. The Weinberg angle is in principle predicted by the p-adic coupling constant evolution fixed by the number theoretical vision of TGD.

One could however have a sum of electroweak action, correction terms changing the value of Weinberg angle, and color action and coupling constant evolution could be understood in terms of the coupling parameters involved.

Electroweak symmetry breaking

One of the hardest challenges in the development of the TGD based view of weak symmetry breaking was the fact that classical field equations allow space-time surfaces with finite but arbitrarily large size. For a fixed space-time surface, the induced gauge fields, including classical weak fields, are long ranged. On the other hand, the large mass for weak bosons would require a short correlation length. How can one understand this together with the fact that a photon has a long correlation length?

In zero energy ontology quantum states are superpositions of space-time surfaces as analogs of almost unique Bohr orbits of particles identified as 3-D surfaces. For some reason the superposition should be such that the quantum averages of weak gauge boson fields vanish below the weak scale whereas the quantum average of electromagnetic fields is non-vanishing.

This is indeed the case.

1. The supersymplectic symmetries form isometries of the world of classical worlds (WCW) and they act in CP_2 degrees of freedom as symplectic transformations leaving the CP_2 symplectic form J invariant and therefore also its contribution to the electromagnetic field since this part is the same for all space-time surfaces in the superposition of space-time surfaces as a representation of supersymplectic isometry group (as a special case a representation of color group).

2. In TGD, color and electroweak symmetries acting as holonomies are not independent and for the $SU(2)_L$ part of induced spinor connection the symplectic transformations induces $SU(2)_L \times U(1)_R$ gauge transformation. This suggests that the quantum expectations of the induced weak fields over the space-time surfaces vanish above the quantum coherence scale. The averages of W and of the left handed part of Z^0 should therefore vanish.

3. $\langle Z^0 \rangle$ should vanish. For $U(1)_R$ part of Z^0, the action of gauge transformation is trivial in gauge theory. Now however the space-time surface changes under symplectic transformations and this could make the average of the right-handed part of Z^0 vanishing. The vanishing of the average of the axial part of the Z^0 is suggested by the partially conserved axial current hypothesis.

One can formulate this picture quantitatively.

1. The electromagnetic field [L49] contains, besides the induced Kähler form, also the induced curvature form R_{12}, which couples vectorially. Conserved vector current hypothesis suggests that the average of R_{12} is non-vanishing. One can express the neutral part of the induced gauge field in terms of induced spinor curvature and Kähler form J as

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) = J + 2e^0 \wedge e^3 \ , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) \ , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) = 3J - 2e^0 \wedge e^3 \quad , \end{aligned} \tag{A-2.22}$$

2. The induced fields γ and Z^0 (photon and Z- boson) can be expressed as

$$\begin{aligned} \gamma &= 3J - sin^2\theta_W R_{12} \ , \\ Z^0 &= 2R_{03} = 2(J + 2e^0 \wedge e^3) && \text{(A-2.22)} \\ per. & && \text{(A-2.23)} \end{aligned}$$

The condition $\langle Z^0 \rangle = 0$ gives $2\langle e^0 \wedge e^3 \rangle = -2J$ and this in turn gives $\langle R_{12} \rangle = 4J$. The average over γ would be

$$\langle \gamma \rangle = (3 - 4sin^2\theta_W)J \ .$$

For $sin^2\theta_W = 3/4\ langle \gamma \rangle$ would vanish.

The quantum averages of classical weak fields quite generally vanish. What about correlation functions?

1. One expects that the correlators of classical weak fields as color invariants, and perhaps even symplectic invariants, are non-vanishing below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as hadron.

2. The intuitive picture is that in longer length scales one has disjoint 3-surfaces with a size scale of Compton length. If the states associated with two disjoint 3-surfaces are separately color invariant there are no correlations in color degrees of freedom and correlators reduce to the products of expectations of classical weak fields and vanish. This could also hold when the 3-surfaces are connected by flux tube bonds.

   Below the Compton length weak bosons would thus behave as correlated massless fields. The Compton lengths of weak bosons are proportional to the value of effective Planck constant h_{eff} and in living systems the Compton lengths are proposed to be even of the order of cell size. This would explain the mysterious chiral selection in living systems requiring large parity violation.

3. What about the averages and correlators of color gauge fields? Classical color gauge fields are proportional to the products of Hamiltonians of color isometries induced Kähler form and the expectations of color Hamiltonians give vanishing average above Compton length and therefore vanishing average. Correlators are non-vanishing below the hadron scale. Gluons do not propagate in long scales for the same reason as weak bosons. This is implied by color confinement, which has also classical description in the sense that 3-surfaces have necessarily a finite size.

   A large value of h_{eff} allows colored states even in biological scales below the Compton length since in this kind of situation the points in the correlation function belong to the same 3-surface representing particle, such as dark hadron.

Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

1. Symmetries must be realized as purely geometric transformations.

2. Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [B11] .

The action of the reflection P on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \otimes \gamma^0 \Psi \ . \tag{A-2.24}$$

in the representation of the gamma matrices for which γ^0 is diagonal. It should be noticed that W and Z^0 bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in CP_2 is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) \ , \\ \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \gamma^1\gamma^3 \otimes 1\Psi \ . \end{aligned} \tag{A-2.23}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in CP_2:

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k \ , \\ \Psi &\rightarrow \Psi^\dagger\gamma^2\gamma^0 \otimes 1 \ . \end{aligned} \tag{A-2.23}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

A-3 Induction procedure and many-sheeted space-time

Since the classical gauge fields are closely related in TGD framework, it is not possible to have space-time sheets carrying only single kind of gauge field. For instance, em fields are accompanied by Z^0 fields for extremals of Kähler action.

Classical em fields are always accompanied by Z^0 field and some components of color gauge field. For extremals having homologically non-trivial sphere as a CP_2 projection em and Z^0 fields are the only non-vanishing electroweak gauge fields. For homologically trivial sphere only W fields are non-vanishing. Color rotations does not affect the situation.

For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has U(1) holonomy by 2-dimensionality of the CP_2 projection. Color gauge field has $U(1)$ holonomy for all space-time surfaces and quantum classical correspondence suggest a weak form of color confinement meaning that physical states correspond to color neutral members of color multiplets.

A-3.1 Induction procedure for gauge fields and spinor connection

Induction procedure for gauge potentials and spinor structure is a standard procedure of bundle theory. If one has embedding of some manifold to the base space of a bundle, the bundle structure can be induced so that it has as a base space the imbedded manifold, whose points have as fiber the fiber if embedding space at their image points. In the recent case the embedding of space-time surface to embedding space defines the induction procedure. The induced gauge potentials and gauge fields are projections of the spinor connection of the embedding space to the space-time surface (see `http://tgdtheory.fi/appfigures/induct.jpg`).

Induction procedure makes sense also for the spinor fields of embedding space and one obtains geometrization of both electroweak gauge potentials and of spinors. The new element is induction of gamma matrices which gives their projections at space-time surface.

As a matter fact, the induced gamma matrices cannot appear in the counterpart of massless Dirac equation. To achieve super-symmetry, Dirac action must be replaced with Kähler-Dirac action for which gamma matrices are contractions of the canonical momentum currents of Kähler action with embedding space gamma matrices. Induced gamma matrices in Dirac action would correspond to 4-volume as action.

Fig. 9. Induction of spinor connection and metric as projection to the space-time surface. `http://tgdtheory.fi/appfigures/induct.jpg`.

A-3.2 Induced gauge fields for space-times for which CP$_2$ projection is a geodesic sphere

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional CP$_2$ projection, only vacuum extremals and space-time surfaces for which CP$_2$ projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing W fields and homologically non-trivial sphere to non-vanishing W fields but vanishing γ and Z^0. This can be verified by explicit examples.

$r = \infty$ surface gives rise to a homologically non-trivial geodesic sphere for which e_0 and e_3 vanish imply the vanishing of W field. For space-time sheets for which CP$_2$ projection is $r = \infty$ homologically non-trivial geodesic sphere of CP_2 one has

$$\gamma = (\frac{3}{4} - \frac{sin^2(\theta_W)}{2})Z^0 \simeq \frac{5Z^0}{8} \quad .$$

The induced W fields vanish in this case and they vanish also for all geodesic sphere obtained by $SU(3)$ rotation.

$Im(\xi^1) = Im(\xi^2) = 0$ corresponds to homologically trivial geodesic sphere. A more general representative is obtained by using for the phase angles of standard complex CP_2 coordinates constant values. In this case e^1 and e^3 vanish so that the induced em, Z^0, and Kähler fields vanish but induced W fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D CP$_2$ projection color rotations and weak symmetries commute.

A-3.3 Many-sheeted space-time

TGD space-time is many-sheeted: in other words, there are in general several space-sheets which have projection to the same M^4 region. Second manner to say this is that CP_2 coordinates are many-valued functions of M^4 coordinates. The original physical interpretation of many-sheeted space-time time was not correct: it was assumed that single sheet corresponds to GRT space-time and this obviously leads to difficulties since the induced gauge fields are expressible in terms of only four embedding space coordinates.

Fig. 10. Illustration of many-sheeted space-time of TGD. http://tgdtheory.fi/appfigures/manysheeted.jpg

Superposition of effects instead of superposition of fields

The first objection against TGD is that superposition is not possible for induced gauge fields and induced metric. The resolution of the problem is that it is effects which need to superpose, not the fields.

Test particle topologically condenses simultaneously to all space-time sheets having a projection to same region of M^4 (that is touches them). The superposition of effects of fields at various space-time sheets replaces the superposition of fields.This is crucial for the understanding also how GRT space-time relates to TGD space-time, which is also in the appendix of this book).

Wormhole contacts

Wormhole contacts are key element of many-sheeted space-time. One does not expect them to be stable unless there is non-trivial Kähler magnetic flux flowing through then so that the throats look like Kähler magnetic monopoles.

Fig. 11. Wormhole contact. http://tgdtheory.fi/appfigures/wormholecontact.jpg

Since the flow lines of Kähler magnetic field must be closed this requires the presence of another wormhole contact so that one obtains closed monopole flux tube decomposing to two Minkowskian pieces at the two space-time sheets involved and two wormhole contacts with Euclidian signature of the induced metric. These objects are identified as space-time correlates of elementary particles and are clearly analogous to string like objects.

The relationship between the many-sheeted space-time of TGD and of GRT space-time

The space-time of general relativity is single-sheeted and there is no need to regard it as surface in H although the assumption about representability as vacuum extremal gives very powerful constraints in cosmology and astrophysics and might make sense in simple situations.

The space-time of GRT can be regarded as a long length scale approximation obtained by lumping together the sheets of the many-sheeted space-time to a region of M^4 and providing it with an effective metric obtained as sum of M^4 metric and deviations of the induced metrics of various space-time sheets from M^4 metric. Also induced gauge potentials sum up in the similar manner so that also the gauge fields of gauge theories would not be fundamental fields.

Fig. 12. The superposition of fields is replaced with the superposition of their effects in many-sheeted space-time. http://tgdtheory.fi/appfigures/fieldsuperpose.jpg

Space-time surfaces of TGD are considerably simpler objects that the space-times of general relativity and relate to GRT space-time like elementary particles to systems of condensed matter physics. Same can be said about fields since all fields are expressible in terms of embedding space coordinates and their gradients, and general coordinate invariance means that the number of bosonic field degrees is reduced locally to 4. TGD space-time can be said to be a microscopic description whereas GRT space-time a macroscopic description. In TGD complexity of space-time topology replaces the complexity due to large number of fields in quantum field theory.

Topological field quantization and the notion of magnetic body

Topological field quantization also TGD from Maxwell's theory. TGD predicts topological light rays ("massless extremals (MEs)") as space-time sheets carrying waves or arbitrary shape propagating with maximal signal velocity in single direction only and analogous to laser beams and carrying light-like gauge currents in the generi case. There are also magnetic flux quanta and electric flux quanta. The deformations of cosmic strings with 2-D string orbit as M^4 projection gives rise to magnetic flux tubes carrying monopole flux made possible by CP_2 topology allowing homological Kähler magnetic monopoles.

Fig. 13. Topological quantization for magnetic fields replaces magnetic fields with bundles of them defining flux tubes as topological field quanta. `http://tgdtheory.fi/appfigures/field.jpg`

The imbeddability condition for say magnetic field means that the region containing constant magnetic field splits into flux quanta, say tubes and sheets carrying constant magnetic field. Unless one assumes a separate boundary term in Kähler action, boundaries in the usual sense are forbidden except as ends of space-time surfaces at the boundaries of causal diamonds. One obtains typically pairs of sheets glued together along their boundaries giving rise to flux tubes with closed cross section possibly carrying monopole flux.

These kind of flux tubes might make possible magnetic fields in cosmic scales already during primordial period of cosmology since no currents are needed to generate these magnetic fields: cosmic string would be indeed this kind of objects and would dominated during the primordial period. Even superconductors and maybe even ferromagnets could involve this kind of monopole flux tubes.

A-3.4 Embedding space spinors and induced spinors

One can geometrize also fermionic degrees of freedom by inducing the spinor structure of $M^4 \times CP_2$.

CP_2 does not allow spinor structure in the ordinary sense but one can couple the opposite H-chiralities of H-spinors to an $n = 1$ ($n = 3$) integer multiple of Kähler gauge potential to obtain a respectable modified spinor structure. The em charges of resulting spinors are fractional (integer valued) and the interpretation as quarks (leptons) makes sense since the couplings to the induced spinor connection having interpretation in terms electro-weak gauge potential are identical to those assumed in standard model.

The notion of quark color differs from that of standard model.

1. Spinors do not couple to color gauge potential although the identification of color gauge potential as projection of $SU(3)$ Killing vector fields is possible. This coupling must emerge only at the effective gauge theory limit of TGD.

2. Spinor harmonics of embedding space correspond to triality $t = 1$ ($t = 0$) partial waves. The detailed correspondence between color and electroweak quantum numbers is however not correct as such and the interpretation of spinor harmonics of embedding space is as representations for ground states of super-conformal representations. The wormhole pairs associated with physical quarks and leptons must carry also neutrino pair to neutralize weak quantum numbers above the length scale of flux tube (weak scale or Compton length). The total color quantum numbers or these states must be those of standard model. For instance, the color quantum numbers of fundamental left-hand neutrino and lepton can compensate each other for the physical lepton. For fundamental quark-lepton pair they could sum up to those of physical quark.

The well-definedness of em charge is crucial condition.

1. Although the embedding space spinor connection carries W gauge potentials one can say that the embedding space spinor modes have well-defined em charge. One expects that this is true for induced spinor fields inside wormhole contacts with 4-D CP_2 projection and Euclidian signature of the induced metric.

2. The situation is not the same for the modes of induced spinor fields inside Minkowskian region and one must require that the CP_2 projection of the regions carrying induced spinor field is such that the induced W fields and above weak scale also the induced Z^0 fields vanish in order to avoid large parity breaking effects. This condition forces the CP_2 projection to be 2-dimensional. For a generic Minkowskian space-time region this is achieved only if the

spinor modes are localized at 2-D surfaces of space-time surface - string world sheets and possibly also partonic 2-surfaces.

3. Also the Kähler-Dirac gamma matrices appearing in the modified Dirac equation must vanish in the directions normal to the 2-D surface in order that Kähler-Dirac equation can be satisfied. This does not seem plausible for space-time regions with 4-D CP_2 projection.

4. One can thus say that strings emerge from TGD in Minkowskian space-time regions. In particular, elementary particles are accompanied by a pair of fermionic strings at the opposite space-time sheets and connecting wormhole contacts. Quite generally, fundamental fermions would propagate at the boundaries of string world sheets as massless particles and wormhole contacts would define the stringy vertices of generalized Feynman diagrams. One obtains geometrized diagrammatics, which brings looks like a combination of stringy and Feynman diagrammatics.

5. This is what happens in the the generic situation. Cosmic strings could serve as examples about surfaces with 2-D CP_2 projection and carrying only em fields and allowing delocalization of spinor modes to the entire space-time surfaces.

A-3.5 About induced gauge fields

In the following the induced gauge fields are studied for general space-time surface without assuming the preferred extremal property (Bohr orbit property). Therefore the following arguments are somewhat obsolete in their generality.

Space-times with vanishing em, Z^0, or Kähler fields

The following considerations apply to a more general situation in which the homologically trivial geodesic sphere and extremal property are not assumed. It must be emphasized that this case is possible in TGD framework only for a vanishing Kähler field.

Using spherical coordinates (r, Θ, Ψ, Φ) for CP_2, the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta) d\Phi) + \frac{r^2}{2F} sin(\Theta) d\Theta \wedge d\Phi \ , \\ F &= 1 + r^2 \ . \end{aligned} \tag{A-3.0}$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3+2p)\frac{r}{F^2} dr \wedge (d\Psi + cos(\Theta) d\Phi) + (3+p)\frac{r^2}{2F} sin(\Theta) d\Theta \wedge d\Phi \ , \\ p &= sin^2(\Theta_W) \ , \end{aligned} \tag{A-3.0}$$

where Θ_W denotes Weinberg angle.

1. The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\begin{aligned} \Psi &= k\Phi \ , \\ (3+2p)\frac{1}{r^2 F}(d(r^2)/d\Theta)(k + cos(\Theta)) &+ (3+p) sin(\Theta) = 0 \ , \end{aligned} \tag{A-3.0}$$

hold true. The conditions imply that CP_2 projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$\begin{aligned} r &= \sqrt{\frac{X}{1-X}} \ , \\ X &= D\left[|\frac{k+u}{C}|\right]^{\epsilon} \ , \\ u &\equiv cos(\Theta) \ , \ C = k + cos(\Theta_0) \ , \ D = \frac{r_0^2}{1+r_0^2} \ , \ \epsilon = \frac{3+p}{3+2p} \ , \end{aligned} \qquad \text{(A-3.-1)}$$

where C and D are integration constants. $0 \leq X \leq 1$ is required by the reality of r. $r = 0$ would correspond to $X = 0$ giving $u = -k$ achieved only for $|k| \leq 1$ and $r = \infty$ to $X = 1$ giving $|u+k| = [(1+r_0^2)/r_0^2)]^{(3+2p)/(3+p)}$ achieved only for

$$sign(u+k) \times [\frac{1+r_0^2}{r_0^2}]^{\frac{3+2p}{3+p}} \leq k+1 \ ,$$

where $sign(x)$ denotes the sign of x.

The expressions for Kähler form and Z^0 field are given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \ , \\ Z^0 &= -\frac{6}{p} J \ . \end{aligned} \qquad \text{(A-3.-1)}$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range Z^0 vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

2. The vanishing of Z^0 fields is achieved by the replacement of the parameter ϵ with $\epsilon = 1/2$ as becomes clear by considering the condition stating that Z^0 field vanishes identically. Also the relationship $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$ is useful.

3. The vanishing Kähler field corresponds to $\epsilon = 1, p = 0$ in the formula for em neutral space-times. In this case classical em and Z^0 fields are proportional to each other:

$$\begin{aligned} Z^0 &= 2e^0 \wedge e^3 = \frac{r}{F^2}(k+u)\frac{\partial r}{\partial u} du \wedge d\Phi = (k+u) du \wedge d\Phi \ , \\ r &= \sqrt{\frac{X}{1-X}} \ , \ X = D|k+u| \ , \\ \gamma &= -\frac{p}{2} Z^0 \ . \end{aligned} \qquad \text{(A-3.-2)}$$

For a vanishing value of Weinberg angle ($p = 0$) em field vanishes and only Z^0 field remains as a long range gauge field. Vacuum extremals for which long range Z^0 field vanishes but em field is non-vanishing are not possible.

The effective form of CP_2 metric for surfaces with 2-dimensional CP_2 projection

The effective form of the CP_2 metric for a space-time having vanishing em,Z^0, or Kähler field is of practical value in the case of vacuum extremals and is given by

$$\begin{aligned} ds^2_{eff} &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s^{eff}_{\Theta\Theta}d\Theta^2 + s^{eff}_{\Phi\Phi}d\Phi^2] \ , \\ s^{eff}_{\Theta\Theta} &= X \times \left[\frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X\right] \ , \\ s^{eff}_{\Phi\Phi} &= X \times \left[(1-X)(k+u)^2 + 1 - u^2\right] \ , \end{aligned} \qquad \text{(A-3.-3)}$$

and is useful in the construction of vacuum embedding of, say Schwartchild metric.

Topological quantum numbers

Space-times for which either em, Z^0, or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers (ω_1 and ω_2) are frequency type parameters, two (k_1 and k_2) are wave vector like quantum numbers, two of the quantum numbers (n_1 and n_2) are integers. The parameters ω_i and n_i will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of CP_2 coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

Under rather general conditions the coordinates Ψ and Φ can be written in the form

$$\begin{aligned} \Psi &= \omega_2 m^0 + k_2 m^3 + n_2\phi + \text{Fourier expansion} \ , \\ \Phi &= \omega_1 m^0 + k_1 m^3 + n_1\phi + \text{Fourier expansion} \ . \end{aligned} \qquad \text{(A-3.-3)}$$

m^0,m^3 and ϕ denote the coordinate variables of the cylindrical M^4 coordinates) so that one has $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$. The regions of the space-time surface with given values of the vacuum parameters ω_i,k_i and n_i and m and C are bounded by the surfaces at which space-time surface becomes ill-defined, say by $r > 0$ or $r < \infty$ surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters r_0 and Θ_0. At $r = \infty$ surfaces n_2,ω_2 and m can change since all values of Ψ correspond to the same point of CP_2: at $r = 0$ surfaces also n_1 and ω_1 can change since all values of Φ correspond to same point of CP_2, too. If $r = 0$ or $r = \infty$ is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global embedding for, say a constant magnetic field. Although global embedding exists it decomposes into regions with different values of the vacuum parameters and the coordinate u in general possesses discontinuous derivative at $r = 0$ and $r = \infty$ surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteeing the absence of edges are satisfied.

For instance, the vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \ , \qquad \text{(A-3.-2)}$$

is satisfied. In particular, the ratio ω_2/ω_1 is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter n_1 and n_2 (ω_1 and ω_2) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

A-4 The relationship of TGD to QFT and string models

The recent view of the relationship of TGD to QFT and string models has developed slowly during years and it seems that in a certain sense TGD means a return to roots: instead of QFT like description involving path integral one would have wave mechanics for 3-surfaces.

A-4.1 TGD as a generalization of wave mechanism obtained by replacing point-like particles with 3-surfaces

The first vision of TGD was as a generalization of quantum field theory (string models) obtained by replacing pointlike particles (strings) as fundamental objects with 3-surfaces.

The later work has revealed that TGD could be seen as a generalization of the wave mechanism based on the replacement of a point-like particle with 3-D surface. This is due to holography implied by general coordinate invariance. The definition of the metric of the "world of classical worlds" (WCW) must assign a unique or at least almost unique space-time surface to a given 3-surface. This 4-surface is analogous to Bohr orbit so that also Bohr orbitology becomes an exact part of quantum physics. The failure of strict determinism forces to replace 3-surfaces with 4-surfaces and this leads to zero energy ontology (ZEO) in which quantum states are superpositions of space-time surfaces [K43, K26, K84] [L39, L45].

Fig. 5. TGD replaces point-like particles with 3-surfaces. `http://tgdtheory.fi/appfigures/particletgd.jpg`

A-4.2 Extension of superconformal invariance

The fact that light-like 3-surfaces are effectively metrically 2-dimensional and thus possess generalization of 2-dimensional conformal symmetries with light-like radial coordinate defining the analog of second complex coordinate suggests that this generalization could work and extend the super-conformal symmetries to their 4-D analogs.

The boundary $\delta M^4_+ = S^2 \times R_+$- of 4-D light-cone M^4_+ is also metrically 2-dimensional and allows extended conformal invariance. Also the group of isometries of light-cone boundary and of light-like 3-surfaces is infinite-dimensional since the conformal scalings of S^2 can be compensated by S^2-local scaling of the light-like radial coordinate of R_+. These simple facts mean that 4-dimensional Minkowski space and 4-dimensional space-time surfaces are in a completely unique position as far as symmetries are considered.

In fact, this leads to a generalization of the Kac-Moody type symmetries of string models. $\delta M^4_+ \times CP_2$ allows huge supersymplectic symmetries for which the radial light-like coordinate of δM^4_+ plays the role of complex string coordinate in string models. These symmetries are assumed to act as isometries of WCW.

A-4.3 String-like objects and strings

String like objects obtained as deformations of cosmic strings $X^2 \times Y^2$, where X^2 is minimal surface in M^4 and Y^2 a holomorphic surface of CP_2 are fundamental extremals of Kähler action having string world sheet as M^4 projections. Cosmic strings dominate the primordial cosmology of the TGD Universe and the inflationary period corresponds to the transition to radiation dominated cosmology for which space-time sheets with 4-D M^4 projection dominate.

Also genuine string-like objects emerge from TGD. The conditions that the em charge of modes of induces spinor fields is well-defined requires in the generic case the localization of the modes at 2-D surfaces -string world sheets and possibly also partonic 2-surfaces. This in Minkowskian space-time regions.

Fig. 6. Well-definedness of em charge forces the localization of induced spinor modes to 2-D surfaces in generic situations in Minkowskian regions of space-time surface. `http://tgdtheory.fi/appfigures/fermistring.jpg`

A-4.4 TGD view of elementary particles

The TGD based view about elementary particles has two key aspects.

1. The space-time correlates of elementary particles are identified as pairs of wormhole contacts with Euclidean signature of metric and having 4-D CP_2 projection. Their throats behave effectively as Kähler magnetic monopoles so that wormhole throats must be connected by Kähler magnetic flux tubes with monopole flux so that closed flux tubes are obtained.

2. At the level of H Fermion number is carried by the modes of the induced spinor field. In space-time regions with Minkowski signature the modes are localized at string world sheets connecting the wormhole contacts.

Fig. 7. TGD view about elementary particles. a) Particle orbit corresponds to a 4-D generalization of a world line or b) with its light-like 3-D boundary (holography). c) Particle world lines have Euclidean signature of the induced metric. d) They can be identified as wormhole contacts. e) The throats of wormhole contacts carry effective Kähler magnetic charges so that wormhole contacts must appear as pairs in order to obtain closed flux tubes. f) Wormhole contacts are accompanied by fermionic strings connecting the throats at the same sheet: the strings do not extend inside the wormhole contacts. `http://tgdtheory.fi/appfigures/elparticletgd.jpg`

Particle interactions involve both stringy and QFT aspects.

1. The boundaries of string world sheets correspond to fundamental fermions. This gives rise to massless propagator lines in generalized Feynman diagrammatics. One can speak of "long" string connecting wormhole contacts and having a hadronic string as a physical counterpart. Long strings should be distinguished from wormhole contacts which due to their super-conformal invariance behave like "short" strings with length scale given by CP_2 size, which is 10^4 times longer than Planck scale characterizing strings in string models.

2. Wormhole contact defines basic stringy interaction vertex for fermion-fermion scattering. The propagator is essentially the inverse of the superconformal scaling generator L_0. Wormhole contacts containing fermion and antifermion at its opposite throats behave like virtual bosons so that one has BFF type vertices typically.

3. In topological sense one has 3-vertices serving as generalizations of 3-vertices of Feynman diagrams. In these vertices 4-D "lines" of generalized Feynman diagrams meet along their 3-D ends. One obtains also the analogs of stringy diagrams but stringy vertices do not have the usual interpretation in terms of particle decays but in terms of propagation of particles along two different routes.

Fig. 8. a) TGD analogs of Feynman and string diagrammatics at the level of space-time topology. b) The 4-D analogs of both string diagrams and QFT diagrams appear but the interpretation of the analogs stringy diagrams is different. `http://tgdtheory.fi/appfigures/tgdgraphs.jpg`

A-5 About the selection of the action defining the Kähler function of the "world of classical worlds" (WCW)

The proposal is that space-time surfaces correspond to preferred extremals of some action principle, being analogous to Bohr orbits, so that they are almost deterministic. The action for the preferred extremal would define the Kähler function of WCW [K43, K84].

How unique is the choice of the action defining WCW Kähler metric? The problem is that twistor lift strongly suggests the identification of the preferred extremals as 4-D surfaces having 4-D generalization of complex structure and that a large number of general coordinate invariant actions constructible in terms of the induced geometry have the same preferred extremals.

A-5.1 Could twistor lift fix the choice of the action uniquely?

The twistor lift of TGD [L25] [L39, L40, L41] generalizes the notion of induction to the level of twistor fields and leads to a proposal that the action is obtained by dimensional reduction of the action having as its preferred extremals the counterpart of twistor space of the space-time surface identified as 6-D surface in the product $T(M^4) \times T(CP_2)$ twistor spaces of $T(M^4)$ and $T(CP_2)$

of M^4 and CP_2. Only M^4 and CP_2 allow a twistor space with Kähler structure [A45] so that TGD would be unique. Dimensional reduction is forced by the condition that the 6-surface has S^2-bundle structure characterizing twistor spaces and the base space would be the space-time surface.

1. Dimensional reduction of 6-D Kähler action implies that at the space-time level the fundamental action can be identified as the sum of Kähler action and volume term (cosmological constant). Other choices of the action do not look natural in this picture although they would have the same preferred extremals.

2. Preferred extremals are proposed to correspond to minimal surfaces with singularities such that they are also extremals of 4-D Kähler action outside the singularities. The physical analogue are soap films spanned by frames and one can localize the violation of the strict determinism and of strict holography to the frames.

3. The preferred extremal property is realized as the holomorphicity characterizing string world sheets, which generalizes to the 4-D situation. This in turn implies that the preferred extremals are the same for any general coordinate invariant action defined on the induced gauge fields and induced metric apart from possible extremals with vanishing CP_2 Kähler action.

   For instance, 4-D Kähler action and Weyl action as the sum of the tensor squares of the components of the Weyl tensor of CP_2 representing quaternionic imaginary units constructed from the Weyl tensor of CP_2 as an analog of gauge field would have the same preferred extremals and only the definition of Kähler function and therefore Kähler metric of WCW would change. One can even consider the possibility that the volume term in the 4-D action could be assigned to the tensor square of the induced metric representing a quaternionic or octonionic real unit.

Action principle does not seem to be unique. On the other hand, the WCW Kähler form and metric should be unique since its existence requires maximal isometries.

Unique action is not the only way to achieve this. One cannot exclude the possibility that the Kähler gauge potential of WCW in the complex coordinates of WCW differs only by a complex gradient of a holomorphic function for different actions so that they would give the same Kähler form for WCW. This gradient is induced by a symplectic transformation of WCW inducing a U(1) gauge transformation. The Kähler metric is the same if the symplectic transformation is an isometry.

Symplectic transformations of WCW could give rise to inequivalent representations of the theory in terms of action at space-time level. Maybe the length scale dependent coupling parameters of an effective action could be interpreted in terms of a choice of WCW Kähler function, which maximally simplifies the computations at a given scale.

1. The 6-D analogues of electroweak action and color action reducing to Kähler action in 4-D case exist. The 6-D analog of Weyl action based on the tensor representation of quaternionic imaginary units does not however exist. One could however consider the possibility that only the base space of twistor space $T(M^4)$ and $T(CP_2)$ have quaternionic structure.

2. Kähler action has a huge vacuum degeneracy, which clearly distinguishes it from other actions. The presence of the volume term removes this degeneracy. However, for minimal surfaces having CP_2 projections, which are Lagrangian manifolds and therefore have a vanishing induced Kähler form, would be preferred extremals according to the proposed definition. For these 4-surfaces, the existence of the generalized complex structure is dubious.

   For the electroweak action, the terms corresponding to charged weak bosons eliminate these extremals and one could argue that electroweak action or its sum with the analogue of color action, also proportional Kähler action, defines the more plausible choice. Interestingly, also the neutral part of electroweak action is proportional to Kähler action.

Twistor lift strongly suggests that also M^4 has the analog of Kähler structure. M^8 must be complexified by adding a commuting imaginary unit i. In the E^8 subspace, the Kähler structure of E^4 is defined in the standard sense and it is proposed that this generalizes to M^4 allowing also generalization of the quaternionic structure. M^4 Kähler structure violates Lorentz invariance but could be realized at the level of moduli space of these structures.

The minimal possibility is that the M^4 Kähler form vanishes: one can have a different representation of the Kähler gauge potential for it obtained as generalization of symplectic transformations acting non-trivially in M^4. The recent picture about the second quantization of spinors of $M^4 \times CP_2$ assumes however non-trivial Kähler structure in M^4.

A-5.2 Two paradoxes

TGD view leads to two apparent paradoxes.

1. If the preferred extremals satisfy 4-D generalization of holomorphicity, a very large set of actions gives rise to the same preferred extremals unless there are some additional conditions restricting the number of preferred extremals for a given action.

2. WCW metric has an infinite number of zero modes, which appear as parameters of the metric but do not contribute to the line element. The induced Kähler form depends on these degrees of freedom. The existence of the Kähler metric requires maximal isometries, which suggests that the Kähler metric is uniquely fixed apart from a conformal scaling factor Ω depending on zero modes. This cannot be true: galaxy and elementary particle cannot correspond to the same Kähler metric.

Number theoretical vision and the hierarchy of inclusions of HFFs associated with supersymplectic algebra actings as isometries of WcW provide equivalent realizations of the measurement resolution. This solves these paradoxes and predicts that WCW decomposes into sectors for which Kähler metrics of WCW differ in a natural way.

The hierarchy subalgebras of supersymplectic algebra implies the decomposition of WCW into sectors with different actions

Supersymplectic algebra of $\delta M^4_+ \times CP_2$ is assumed to act as isometries of WCW [L45]. There are also other important algebras but these will not be discussed now.

1. The symplectic algebra A of $\delta M^4_+ \times CP_2$ has the structure of a conformal algebra in the sense that the radial conformal weights with non-negative real part, which is half integer, label the elements of the algebra have an interpretation as conformal weights.

   The super symplectic algebra A has an infinite hierarchy of sub-algebras [L45] such that the conformal weights of sub-algebras $A_{n(SS)}$ are integer multiples of the conformal weights of the entire algebra. The superconformal gauge conditions are weakened. Only the subalgebra $A_{n(SS)}$ and the commutator $[A_{n(SS)}, A]$ annihilate the physical states. Also the corresponding classical Noether charges vanish for allowed space-time surfaces.

   This weakening makes sense also for ordinary superconformal algebras and associated Kac-Moody algebras. This hierarchy can be interpreted as a hierarchy symmetry breakings, meaning that sub-algebra $A_{n(SS)}$ acts as genuine dynamical symmetries rather than mere gauge symmetries. It is natural to assume that the super-symplectic algebra A does not affect the coupling parameters of the action.

2. The generators of A correspond to the dynamical quantum degrees of freedom and leave the induced Kähler form invariant. They affect the induced space-time metric but this effect is gravitational and very small for Einsteinian space-time surfaces with 4-D M^4 projection.

   The number of dynamical degrees of freedom increases with $n(SS)$. Therefore WCW decomposes into sectors labelled by $n(SS)$ with different numbers of dynamical degrees of freedom so that their Kähler metrics cannot be equivalent and cannot be related by a symplectic isometry. They can correspond to different actions.

Number theoretic vision implies the decomposition of WCW into sectors with different actions

The number theoretical vision leads to the same conclusion as the hierarchy of HFFs. The number theoretic vision of TGD based on M^8-H duality [L45] predicts a hierarchy with levels labelled by the degrees $n(P)$ of rational polynomials P and corresponding extensions of rationals characterized by Galois groups and by ramified primes defining p-adic length scales.

These sequences allow us to imagine several discrete coupling constant evolutions realized at the level H in terms of action whose coupling parameters depend on the number theoretic parameters.

1. Coupling constant evolution with respect to $n(P)$

The first coupling constant evolution would be with respect to $n(P)$.

1. The coupling constants characterizing action could depend on the degree $n(P)$ of the polynomial defining the space-time region by M^8-H duality. The complexity of the space-time surface would increase with $n(P)$ and new degrees of freedom would emerge as the number of the rational coefficients of P.

2. This coupling constant evolution could naturally correspond to that assignable to the inclusion hierarchy of hyperfinite factors of type II_1 (HFFs). I have indeed proposed [L45] that the degree $n(P)$ equals to the number $n(braid)$ of braids assignable to HFF for which super symplectic algebra subalgebra $A_{n(SS)}$ with radial conformal weights coming as $n(SS)$-multiples of those of entire algebra A. One would have $n(P)=n(braid)=n(SS)$. The number of dynamical degrees of freedom increases with n which just as it increases with $n(P)$ and $n(SS)$.

3. The actions related to different values of $n(P)=n(braid)=n(SS)$ cannot define the same Kähler metric since the number of allowed space-time surfaces depends on $n(SS)$.

   WCW could decompose to sub-WCWs corresponding to different actions, a kind of theory space. These theories would not be equivalent. A possible interpretation would be as a hierarchy of effective field theories.

4. Hierarchies of composite polynomials define sequences of polynomials with increasing values of $n(P)$ such that the order of a polynomial at a given level is divided by those at the lower levels. The proposal is that the inclusion sequences of extensions are realized at quantum level as inclusion hierarchies of hyperfinite factors of type II_1.

   A given inclusion hierarchy corresponds to a sequence $n(SS)_i$ such that $n(SS)_i$ divides $n(SS)_{i+1}$. Therefore the degree of the composite polynomials increases very rapidly. The values of $n(SS)_i$ can be chosen to be primes and these primes correspond to the degrees of so called prime polynomials [L42] so that the decompositions correspond to prime factorizations of integers. The "densest" sequence of this kind would come in powers of 2 as $n(SS)_i=2^i$. The corresponding p-adic length scales (assignable to maximal ramified primes for given $n(SS)_i$) are expected to increase roughly exponentially, say as 2^{r2^i}. $r=1/2$ would give a subset of scales $2^{r/2}$ allowed by the p-adic length scale hypothesis. These transitions would be very rare.

   A theory corresponding to a given composite polynomial would contain as sub-theories the theories corresponding to lower polynomial composites. The evolution with respect to $n(SS)$ would correspond to a sequence of phase transitions in which the action genuinely changes. For instance, color confinement could be seen as an example of this phase transition.

5. A subset of p-adic primes allowed by the p-adic length scale hypothesis $p\simeq 2^k$ defining the proposed p-adic length scale hierarchy could relate to n_S changing phase transition. TGD suggests a hierarchy of hadron physics corresponding to a scale hierarchy defined by Mersenne primes and their Gaussian counterparts [K57, K58]). Each of them would be characterized by a confinement phase transition in which n_S and therefore also the action changes.

2. Coupling constant evolutions with respect to ramified primes for a given value of $n(P)$

For a given value of $n(P)$, one could have coupling constant sub-evolutions with respect to the set of ramified primes of P and dimensions $n = h_{eff}/h_0$ of algebraic extensions. The action would only change by U(1) gauge transformation induced by a symplectic isometry of WCW. Coupling parameters could change but the actions would be equivalent.

The choice of the action in an optimal manner in a given scale could be seen as a choice of the most appropriate effective field theory in which radiative corrections would be taken into account. One can interpret the possibility to use a single choice of coupling parameters in terms of quantum criticality.

The range of the p-adic length scales labelled by ramified primes and effective Planck constants h_{eff}/h_0 is finite for a given value of $n(SS)$.

The first coupling constant evolution of this kind corresponds to ramified primes defining p-adic length scales for given $n(SS)$.

1. Ramified primes are factors of the discriminant $D(P)$ of P, which is expressible as a product of non-vanishing root differents and reduces to a polynomial of the n coefficients of P. Ramified primes define p-adic length scales assignable to the particles in the amplitudes scattering amplitudes defined by zero energy states.

   P would represent the space-time surface defining an interaction region in $N-$-particle scattering. The N ramified primes dividing $D(P)$ would characterize the p-adic length scales assignable to these particles. If $D(P)$ reduces to a single ramified prime, one has elementary particle [L42], and the forward scattering amplitude corresponds to the propagator.

   This would give rise to a multi-scale p-adic length scale evolution of the amplitudes analogous to the ordinary continuous coupling constant evolution of n-point scattering amplitudes with respect to momentum scales of the particles. This kind of evolutions extend also to evolutions with respect to $n(SS)$.

2. According to [L42], physical constraints require that $n(P)$ and the maximum size of the ramified prime of P correlate.

   A given rational polynomial of degree $n(P)$ can be always transformed to a polynomial with integer coefficients. If the integer coefficients are smaller than $n(P)$, there is an upper bound for the ramified primes. This assumption also implies that finite fields become fundamental number fields in number theoretical vision [L42].

3. p-Adic length scale hypothesis [L46] in its basic form states that there exist preferred primes $p \simeq 2^k$ near some powers of 2. A more general hypothesis states that also primes near some powers of 3 possibly also other small primes are preferred physically. The challenge is to understand the origin of these preferred scales.

   For polynomials P with a given degree $n(P)$ for which discriminant $D(P)$ is prime, there exists a maximal ramified prime. Numerical calculations suggest that the upper bound depends exponentially on $n(P)$.

   Could these maximal ramified primes satisfy the p-adic length scale hypothesis or its generalization? The maximal prime defines a fixed point of coupling constant evolution in accordance with the earlier proposal. For instance, could one think that one has $p \simeq 2^k$, $k = n(SS)$? Each p-adic prime would correspond to a p-adic coupling constant sub-evolution representable in terms of symplectic isometries.

Also the dimension n of the algebraic extension associated with P, which is identified in terms of effective Planck constant $h_{eff}/h_0 = n$ labelling different phases of the ordinary matter behaving like dark matter, could give rise to coupling constant evolution for given $n(SS)$. The range of allowed values of n is finite. Note however that several polynomials of a given degree can correspond to the same dimension of extension.

Number theoretic discretization of WCW and maxima of WCW Kähler function

Number theoretic approach involves a unique discretization of space-time surface and also of WCW. The question is how the points of the discretized WCW correspond to the preferred extremals.

1. The exponents of Kähler function for the maxima of Kähler function, which correspond to the universal preferred extremals, appear in the scattering amplitudes. The number theoretical approach involves a unique discretization of space-time surfaces defining the WCW coordinates of the space-time surface regarded as a point of WCW.

   In [L45] it is assumed that these WCW points appearing in the number theoretical discretization correspond to the maxima of the Kähler function. The maxima would depend on the action and would differ for ghd maxima associated with different actions unless they are not related by symplectic WCW isometry.

2. The symplectic transformations of WCW acting as isometries are assumed to be induced by the symplectic transformations of $\delta M^4_+ \times CP_2$ [K43, K26]. As isometries they would naturally permute the maxima with each other.

A-6 Number theoretic vision of TGD

Physics as number theory vision is complementary to the physics as geometry vision and has developed gradually since 1993. Langlands program is the counterpart of this vision in mathematics [L44].

The notion of p-adic number fields emerged with the motivation coming from the observation that elementary particle mass scales and mass ratios could be understood in terms of the so-called p-adic length scale hypothesis [K62, K51, K23]. The fusion of the various p-adic physics leads to what I call adelic physics [L22, L23]. Later the hypothesis about hierarchy of Planck constants labelling phases of ordinary matter behaving like dark matter emerged [K29, K30, K31, K31].

Eventually this led to that the values of effective Planck constant could be identified as the dimension of an algebraic extension of rationals assignable to polynomials with rational coefficients. This led to the number theoretic vision in which so-called M^8-H duality [L35, L36] plays a key role. M^8 (actually a complexification of real M^8) is analogous to momentum space so that the duality generalizes momentum position duality for point-like particles. M^8 has an interpretation as complexified octonions.

The dynamics of 4-surfaces in M^8 is coded by polynomials with rational coefficients, whose roots define mass shells H^3 of $M^4 \subset M^8$. It has turned out that the polynomials satisfy stringent additional conditions and one can speak of number theoretic holography [L42, L44]. Also the ordinary 3$\rightarrow$ 4 holography is needed to assign 4-surfaces with these 3-D mass shells. The number theoretic dynamics is based on the condition that the normal space of the 4-surface in M^8 is associative (quaternionic) and contains a commutative complex sub-space. This makes it possible to assign to this surface space-time surface in $H = M^4 \times CP_2$.

At the level of H the space-time surfaces are by holography preferred extremals and are assumed to be determined by the twistor lift of TGD [L25] giving rise to an action which is sum of the Kähler action and volume term. The preferred extremals would be minimal surfaces analogous to soap films spanned by frames. Outside frames they would be simultaneous extremals of the Kähler action, which requires a generalization of the holomorphy characterizing string world sheets.

In the following only p-adic numbers and hierarchy of Planck constants will be discussed.

A-6.1 p-Adic numbers and TGD

p-Adic number fields

p-Adic numbers (p is prime: 2, 3, 5, ...) can be regarded as a completion of the rational numbers using a norm, which is different from the ordinary norm of real numbers [A25]. p-Adic numbers are representable as power expansion of the prime number p of form

$$x = \sum_{k\geq k_0} x(k)p^k , \; x(k) = 0, ..., p-1 \; . \quad \text{(A-6.1)}$$

The norm of a p-adic number is given by

$$|x| \;=\; p^{-k_0(x)} \; . \qquad \text{(A-6.2)}$$

Here $k_0(x)$ is the lowest power in the expansion of the p-adic number. The norm differs drastically from the norm of the ordinary real numbers since it depends on the lowest pinary digit of the p-adic number only. Arbitrarily high powers in the expansion are possible since the norm of the p-adic number is finite also for numbers, which are infinite with respect to the ordinary norm. A convenient representation for p-adic numbers is in the form

$$x \;=\; p^{k_0}\varepsilon(x) \;, \qquad \text{(A-6.3)}$$

where $\varepsilon(x) = k + ...$ with $0 < k < p$, is p-adic number with unit norm and analogous to the phase factor $exp(i\phi)$ of a complex number.

The distance function $d(x, y) = |x - y|_p$ defined by the p-adic norm possesses a very general property called ultra-metricity:

$$d(x, z) \;\leq\; max\{d(x, y), d(y, z)\} \; . \qquad \text{(A-6.4)}$$

The properties of the distance function make it possible to decompose R_p into a union of disjoint sets using the criterion that x and y belong to same class if the distance between x and y satisfies the condition

$$d(x, y) \;\leq\; D \; . \qquad \text{(A-6.5)}$$

This division of the metric space into classes has following properties:

1. Distances between the members of two different classes X and Y do not depend on the choice of points x and y inside classes. One can therefore speak about distance function between classes.
2. Distances of points x and y inside single class are smaller than distances between different classes.
3. Classes form a hierarchical tree.

Notice that the concept of the ultra-metricity emerged in physics from the models for spin glasses and is believed to have also applications in biology [B20]. The emergence of p-adic topology as the topology of the effective space-time would make ultra-metricity property basic feature of physics.

Canonical correspondence between p-adic and real numbers

The basic challenge encountered by p-adic physicist is how to map the predictions of the p-adic physics to real numbers. p-Adic probabilities provide a basic example in this respect. Identification via common rationals and canonical identification and its variants have turned out to play a key role in this respect.

1. Basic form of the canonical identification

There exists a natural continuous map $I : R_p \rightarrow R_+$ from p-adic numbers to non-negative real numbers given by the "pinary" expansion of the real number for $x \in R$ and $y \in R_p$ this correspondence reads

$$\begin{aligned} y \;&=\; \sum_{k>N} y_k p^k \rightarrow x = \sum_{k<N} y_k p^{-k} \;, \\ y_k \;&\in\; \{0, 1, .., p-1\} \; . \end{aligned} \qquad \text{(A-6.5)}$$

This map is continuous as one easily finds out. There is however a little difficulty associated with the definition of the inverse map since the pinary expansion like also decimal expansion is not unique (1 = 0.999...) for the real numbers x, which allow pinary expansion with finite number of pinary digits

$$\begin{aligned} x &= \sum_{k=N_0}^{N} x_k p^{-k} \ , \\ x &= \sum_{k=N_0}^{N-1} x_k p^{-k} + (x_N - 1)p^{-N} + (p-1)p^{-N-1} \sum_{k=0,..} p^{-k} \ . \end{aligned} \tag{A-6.4}$$

The p-adic images associated with these expansions are different

$$\begin{aligned} y_1 &= \sum_{k=N_0}^{N} x_k p^{k} \ , \\ y_2 &= \sum_{k=N_0}^{N-1} x_k p^{k} + (x_N - 1)p^{N} + (p-1)p^{N+1} \sum_{k=0,..} p^{k} \\ &= y_1 + (x_N - 1)p^N - p^{N+1} \ , \end{aligned} \tag{A-6.3}$$

so that the inverse map is either two-valued for p-adic numbers having expansion with finite pinary digits or single valued and discontinuous and non-surjective if one makes pinary expansion unique by choosing the one with finite pinary digits. The finite pinary digit expansion is a natural choice since in the numerical work one always must use a pinary cutoff on the real axis.

2. The topology induced by canonical identification

The topology induced by the canonical identification in the set of positive real numbers differs from the ordinary topology. The difference is easily understood by interpreting the p-adic norm as a norm in the set of the real numbers. The norm is constant in each interval $[p^k, p^{k+1})$ (see **Fig. A-6.1**) and is equal to the usual real norm at the points $x = p^k$: the usual linear norm is replaced with a piecewise constant norm. This means that p-adic topology is coarser than the usual real topology and the higher the value of p is, the coarser the resulting topology is above a given length scale. This hierarchical ordering of the p-adic topologies will be a central feature as far as the proposed applications of the p-adic numbers are considered.

Ordinary continuity implies p-adic continuity since the norm induced from the p-adic topology is rougher than the ordinary norm. p-Adic continuity implies ordinary continuity from right as is clear already from the properties of the p-adic norm (the graph of the norm is indeed continuous from right). This feature is one clear signature of the p-adic topology.

Fig. 14. The real norm induced by canonical identification from 2-adic norm. `http://tgdtheory.fi/appfigures/norm.png`

The linear structure of the p-adic numbers induces a corresponding structure in the set of the non-negative real numbers and p-adic linearity in general differs from the ordinary concept of linearity. For example, p-adic sum is equal to real sum only provided the summands have no common pinary digits. Furthermore, the condition $x +_p y < max\{x, y\}$ holds in general for the p-adic sum of the real numbers. p-Adic multiplication is equivalent with the ordinary multiplication only provided that either of the members of the product is power of p. Moreover one has $x \times_p y < x \times y$ in general. The p-Adic negative -1_p associated with p-adic unit 1 is given by $(-1)_p = \sum_k (p-1)p^k$ and defines p-adic negative for each real number x. An interesting possibility is that p-adic linearity might replace the ordinary linearity in some strongly nonlinear systems so these systems would look simple in the p-adic topology.

These results suggest that canonical identification is involved with some deeper mathematical structure. The following inequalities hold true:

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |x|_p|y|_R \leq (xy)_R &\leq x_R y_R \ , \end{aligned} \tag{A-6.3}$$

where $|x|_p$ denotes p-adic norm. These inequalities can be generalized to the case of $(R_p)^n$ (a linear vector space over the p-adic numbers).

$$\begin{aligned} (x+y)_R &\leq x_R + y_R \ , \\ |\lambda|_p|y|_R \leq (\lambda y)_R &\leq \lambda_R y_R \ , \end{aligned} \tag{A-6.3}$$

where the norm of the vector $x \in T_p^n$ is defined in some manner. The case of Euclidian space suggests the definition

$$(x_R)^2 = (\sum_n x_n^2)_R \ . \tag{A-6.4}$$

These inequalities resemble those satisfied by the vector norm. The only difference is the failure of linearity in the sense that the norm of a scaled vector is not obtained by scaling the norm of the original vector. Ordinary situation prevails only if the scaling corresponds to a power of p.

These observations suggests that the concept of a normed space or Banach space might have a generalization and physically the generalization might apply to the description of some non-linear systems. The nonlinearity would be concentrated in the nonlinear behavior of the norm under scaling.

3. Modified form of the canonical identification

The original form of the canonical identification is continuous but does not respect symmetries even approximately. This led to a search of variants which would do better in this respect. The modification of the canonical identification applying to rationals only and given by

$$I_Q(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{A-6.5}$$

is uniquely defined for rationals, maps rationals to rationals, has also a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$. It has turned out that it is this map which most naturally appears in the applications. The map is obviously continuous locally since p-adically small modifications of r and s mean small modifications of the real counterparts.

Canonical identification is in a key role in the successful predictions of the elementary particle masses. The predictions for the light elementary particle masses are within extreme accuracy same for I and I_Q but I_Q is theoretically preferred since the real probabilities obtained from p-adic ones by I_Q sum up to one in p-adic thermodynamics.

4. Generalization of number concept and notion of embedding space

TGD forces an extension of number concept: roughly a fusion of reals and various p-adic number fields along common rationals is in question. This induces a similar fusion of real and p-adic embedding spaces. Since finite p-adic numbers correspond always to non-negative reals n-dimensional space R^n must be covered by 2^n copies of the p-adic variant R_p^n of R^n each of which projects to a copy of R_+^n (four quadrants in the case of plane). The common points of p-adic and real embedding spaces are rational points and most p-adic points are at real infinity.

Real numbers and various algebraic extensions of p-adic number fields are thus glued together along common rationals and also numbers in algebraic extension of rationals whose number belong to the algebraic extension of p-adic numbers. This gives rise to a book like structure with rationals and various algebraic extensions of rationals taking the role of the back of the book. Note that Neper number is exceptional in the sense that it is algebraic number in p-adic number field Q_p satisfying $e^p \ mod \ p = 1$.

Fig. 15. Various number fields combine to form a book like structure. `http://tgdtheory.fi/appfigures/book.jpg`

For a given p-adic space-time sheet most points are literally infinite as real points and the projection to the real embedding space consists of a discrete set of rational points: the interpretation in terms of the unavoidable discreteness of the physical representations of cognition is natural. Purely local p-adic physics implies real p-adic fractality and thus long range correlations for the real space-time surfaces having enough common points with this projection.

p-Adic fractality means that M^4 projections for the rational points of space-time surface X^4 are related by a direct identification whereas CP_2 coordinates of X^4 at these points are related by I, I_Q or some of its variants implying long range correlates for CP_2 coordinates. Since only a discrete set of points are related in this manner, both real and p-adic field equations can be satisfied and there are no problems with symmetries. p-Adic effective topology is expected to be a good approximation only within some length scale range which means infrared and UV cutoffs. Also multi-p-fractality is possible.

The notion of p-adic manifold

The notion of p-adic manifold is needed in order to fuse real physics and various p-adic physics to a larger structure which suggests that real and p-adic number fields should be glued together along common rationals bringing in mind adeles. The notion is problematic because p-adic topology is totally disconnected implying that p-adic balls are either disjoint or nested so that ordinary definition of manifold using p-adic chart maps fails. A cure is suggested to be based on chart maps from p-adics to reals rather than to p-adics (see the appendix of the book)

The chart maps are interpreted as cognitive maps, "thought bubbles".

Fig. 16. The basic idea between p-adic manifold. `http://tgdtheory.fi/appfigures/padmanifold.jpg`

There are some problems.

1. Canonical identification does not respect symmetries since it does not commute with second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map arithmetic operations which requires pinary cutoff below which chart map takes rationals to rationals so that commutativity with arithmetics and symmetries is achieved in finite resolution: above the cutoff canonical identification is used

2. Canonical identification is continuous but does not map smooth p-adic surfaces to smooth real surfaces requiring second pinary cutoff so that only a discrete set of rational points is mapped to their real counterparts by chart map requiring completion of the image to smooth preferred extremal of Kähler action so that chart map is not unique in accordance with finite measurement resolution

3. Canonical identification violates general coordinate invariance of chart map: (cognition-induced symmetry breaking) minimized if p-adic manifold structure is induced from that for p-adic embedding space with chart maps to real embedding space and assuming preferred coordinates made possible by isometries of embedding space: one however obtains several in-equivalent p-adic manifold structures depending on the choice of coordinates: these cognitive representations are not equivalent.

A-6.2 Hierarchy of Planck constants and dark matter hierarchy

Hierarchy of Planck constants was motivated by the "impossible" quantal effects of ELF em fields on vertebrate cyclotron energies $E = hf = \hbar \times eB/m$ are above thermal energy is possible only if $\hbar$ has value much larger than its standard value. Also Nottale's finding that planetary orbits migh be understood as Bohr orbits for a gigantic gravitational Planck constant.

Hierachy of Planck constant would mean that the values of Planck constant come as integer multiples of ordinary Planck constant: $h_{eff} = n \times h$. The particles at magnetic flux tubes characterized by h_{eff} would correspond to dark matter which would be invisible in the sense that only particle with same value of h_{eff} appear in the same vertex of Feynman diagram.

Hierarchy of Planck constants would be due to the non-determism of the Kähler action predicting huge vacuum degeneracy allowing all space-time surfaces which are sub-manfolds of any $M^4 \times Y^2$, where Y^2 is Lagrangian sub-manifold of CP_2. For agiven Y^2 one obtains new manifolds Y^2 by applying symplectic transformations of CP_2.

Non-determinism would mean that the 3-surface at the ends of causal diamond (CD) can be connected by several space-time surfaces carrying same conserved Kähler charges and having same values of Kähler action. Conformal symmetries defined by Kac-Moody algebra associated with the embedding space isometries could act as gauge transformations and respect the light-likeness property of partonic orbits at which the signature of the induced metric changes from Minkowskian to Euclidian (Minkowskianb space-time region transforms to wormhole contact say). The number of conformal equivalence classes of these surfaces could be finite number n and define discrete physical degree of freedom and one would have $h_{eff} = n \times h$. This degeneracy would mean "second quantization" for the sheets of n-furcation: not only one but several sheets can be realized.

This relates also to quantum criticality postulated to be the basic characteristics of the dynamics of quantum TGD. Quantum criticalities would correspond to an infinite fractal hierarchy of broken conformal symmetries defined by sub-algebras of conformal algebra with conformal weights coming as integer multiples of n. This leads also to connections with quantum criticality and hierarchy of broken conformal symmetries, p-adicity, and negentropic entanglement which by consistency with standard quantum measurement theory would be described in terms of density matrix proportional $n \times n$ identity matrix and being due to unitary entanglement coefficients (typical for quantum computing systems).

Formally the situation could be described by regarding space-time surfaces as surfaces in singular n-fold singular coverings of embedding space. A stronger assumption would be that they are expressible as as products of n_1 -fold covering of M^4 and n_2-fold covering of CP_2 meaning analogy with multi-sheeted Riemann surfaces and that M^4 coordinates are n_1-valued functions and CP_2 coordinates n_2 -valued functions of space-time coordinates for $n = n_1 \times n_2$. These singular coverings of embedding space form a book like structure with singularities of the coverings localizable at the boundaries of causal diamonds defining the back of the book like structure.

Fig. 17. Hierarchy of Planck constants. `http://tgdtheory.fi/appfigures/planckhierarchy.jpg`

A-6.3 $M^8 - H$ duality as it is towards the end of 2021

The view of M^8-H duality (see Appendix **??**) has changed considerably towards the end 2021 [L39] after the realization that this duality is the TGD counterpart of momentum position duality of wave mechanics, which is lost in QFTs. Therefore M^8 and also space-time surface is analogous to momentum space. This forced us to give up the original simple identification of the points $M^4 \subset M^4 \times E^4 = M^8$ and of $M^4 \times CP_2$ so that it respects Uncertainty Principle (UP).

The first improved guess for the duality map was the replacement with the inversion $p^k \rightarrow m^k = \hbar_{eff} p^k/p^2$ conforming in spirit with UP but turned out to be too naive.

The improved form [L39] of the M^8-H duality map takes mass shells $p^2 = m^2$ of $M^4 \subset M^8$ to cds with size $L(m) = \hbar_{eff}/m$ with a common center. The slicing by mass shells is mapped to a Russian doll like slicing by cds. Therefore would be no CDs in M^8 contrary to what I believed first.

Quantum classical correspondence (QCC) inspires the proposal that the point $p^k \in M^8$ is mapped to a geodesic line corresponding to momentum p^k starting from the common center of cds. Its intersection with the opposite boundary of cd with size $L(m)$ defines the image point. This is not yet quite enough to satisfy UP but the additional details [L39] are not needed in the sequel.

The 6-D brane-like special solutions in M^8 are of special interest in the TGD inspired theory of consciousness. They have an M^4 projection which is $E = E_n$ 3-ball. Here E_n is a root of the real polynomial P defining $X^4 \subset M^8_c$ (M^8 is complexified to M^8_c) as a "root" of its octonionic continuation [L35, L36]. E_n has an interpretation as energy, which can be complex. The original interpretation was as moment of time. For this interpretation, $M^8 - H$ duality would be a linear identification and these hyper planes would be mapped to hyperplanes in $M^4 \subset H$.

This motivated the term "very special moment in the life of self" for the image of the $E = E_n$ section of $X^4 \subset M^8$ [L32]. This notion does not make sense at the level M^8 anymore.

The modified M^8-H duality forces us to modify the original interpretation [L39]. The point $(E_n, p = 0)$ is mapped $(t_n = \hbar_{eff}/E_n, 0)$. The momenta (E_n, p) in $E = E_n$ plane are mapped to the boundary of cd and correspond to a continuous time interval at the boundary of CD: "very special moment" becomes a "very special time interval".

The quantum state however corresponds to a set of points corresponding to quark momenta, which belong to a cognitive representation and are therefore algebraic integers in the extension determined by the polynomial. These active points in E_n are mapped to a discrete set at the boundary of cd(m). A "very special moment" is replaced with a sequence of "very special moments".

So called Galois confinement [L37] forces the total momenta for bound states of quarks and antiquarks to be rational integers invariant under Galois group of extension of rationals determined by the polynomial P [L39]. These states correspond to states at boundaries of sub-CDs so that one obtains a hierarchy. Galois confinement provides a universal number theoretic mechanism for the formation of bound states.

A-7 Zero energy ontology (ZEO)

ZEO is implied by the holography forced in the TGD framework by general coordinate invariance.

A-7.1 Basic motivations and ideas of ZEO

The following gives a brief summary of ZEO [L34] [K114].

1. In ZEO quantum states are not 3-dimensional but superpositions of 4-dimensional deterministic time evolutions connecting ordinary initial 3-dimensional states. By holography they are equivalent to pairs of ordinary 3-D states identified as initial and final states of time evolution. One can say that in the TGD framework general coordinate invariance implies holography and the slight failure of its determinism in turn forces ZEO.

   Quantum jumps replace this state with a new one: a superposition of deterministic time evolutions is replaced with a new superposition. Classical determinism of individual time evolution is not violated and this solves the basic paradox of quantum measurement theory. There are two kinds of quantum jumps: ordinary ("big") state function reductions (BSFRs) changing the arrow of time and "small" state function reductions (SSFRs) (weak measurements) preserving it and giving rise to the analog of Zeno effect [L34].

2. To avoid getting totally confused it is good to emphasize some aspects of ZEO.

   (a) ZEO does not mean that physical states in the usual 3-D sense as snapshots of time evolution would have zero energy state pairs defining zero energy states as initial and final states have same conserved quantities such as energy. Conservation implies that one can adopt the conventions that the values of conserved quantities are opposite for these states so that their sum vanishes: one can think that incoming and outgoing particles come from geometric past and future is the picture used in quantum field theories.

   (b) ZEO means two times: subjective time as sequence of quantum jumps and geometric time as space-time coordinate. These times are identifiable but are strongly correlated.

3. In BSFRs the arrow of time is changed and the time evolution in the final state occurs backwards with respect to the time of the external observer. BSFRs can occur in all scales since TGD predicts a hierarchy of effective Planck constants with arbitrarily large values. There is empirical support for BSFRs.

   (a) The findings of Minev et al [L30] in atomic scale can be explained by the same mechanism [L30]. In BSFR a final zero energy state as a superposition of classical deterministic time evolutions emerges and for an observer with a standard arrow of time looks like a superposition of deterministic smooth time evolutions leading to the final state. Interestingly, once this evolution has started, it cannot be stopped unless one changes the stimulus signal inducing the evolution in which case the process does not lead to anywhere: the interpretation would be that BSFR back to the initial state occurs!

(b) Libets' experiments about active aspects of consciousness [J2] can be understood. Subject person raises his finger and neural activity starts before the conscious decision to do so. In the physicalistic framework it is thought to lead to raising of the finger. The problem with the explanation is that the activity beginning .5 seconds earlier seems to be dissipation with a reversed arrow of time: from chaotic and disordered to ordered at around .15 seconds. ZEO explanation is that macroscopic quantum jump occurred and generated a signal proceeding backwards in time and generated neural activity and dissipated to randomness.

(c) Earthquakes involve a strange anomaly: they are preceded by ELF radiation. One would expect that they generate ELF radiation. The identification as BSFR would explain the anomaly [L31]. In biology the reversal of the arrow of time would occur routinely and be a central element of biological self-organization, in particular self-organized quantum criticality (see [L33, L53].

A-7.2 Some implications of ZEO

ZEO has profound implications for understanding self-organization and self-organized quantum criticality in terms of dissipation with non-standard arrow of time looking like generation of structures [L33, L53]. ZEO could also allow understanding of what planned actions - like realizing the experiment under consideration - could be.

1. Second law in the standard sense does not favor - perhaps even not allow - realization of planned actions. ZEO forces a generalization of thermodynamics: dissipation with a non-standard arrow of time for a subsystem would look like self-organization and planned action and its realization.

   Could most if not all planned action be like this - induced by BSFR in the geometric future and only apparently planned? There would be however the experience of planning and realizing induced by the signals from geometric future by a higher level in the hierarchy of conscious entities predicted by TGD! In long time scales we would be realizing our fates or wishes of higher level conscious entities rather than agents with completely free will.

2. The notion of magnetic body (MB) serving as a boss of ordinary matter would be central. MB carries dark matter as $h_{eff} = nh_0$ phases of ordinary matter with n serving as a measure for algebraic complexity of extension of rationals as its dimension and defining a kind of universal IQ. There is a hierarchy of these phases and MBs labelled by extension of rationals and the value of n.

   MBs would form a hierarchy of bosses - a realization for master slave hierarchy. Ordinary matter would be at the bottom and its coherent behavior would be induced from quantum coherence at higher levels. BSFR for higher level MB would give rise to what looks like planned actions and experienced as planned action at the lower levels of hierarchy. One could speak of planned actions inducing a cascade of planned actions in shorter time scales and eventually proceeding to atomic level.

A-8 Some notions relevant to TGD inspired consciousness and quantum biology

Below some notions relevant to TGD inspired theory of consciousness and quantum biology.

A-8.1 The notion of magnetic body

Topological field quantization inspires the notion of field body about which magnetic body is especially important example and plays key role in TGD inspired quantum biology and consciousness theory. This is a crucial departure fromt the Maxwellian view. Magnetic body brings in third level to the description of living system as a system interacting strongly with environment. Magnetic body would serve as an intentional agent using biological body as a motor instrument and sensory receptor. EEG would communicated the information from biological body to magnetic body and Libet's findings from time delays of consciousness support this view.

The following pictures illustrate the notion of magnetic body and its dynamics relevant for quantum biology in TGD Universe.

Fig. 18. Magnetic body associated with dipole field. `http://tgdtheory.fi/appfigures/fluxquant.jpg`

Fig. 19. Illustration of the reconnection by magnetic flux loops. `http://tgdtheory.fi/appfigures/reconnect1.jpg`

Fig. 20. Illustration of the reconnection by flux tubes connecting pairs of molecules. `http://tgdtheory.fi/appfigures/reconect2.jpg`

Fig. 21. Flux tube dynamics. a) Reconnection making possible magnetic body to "recognize" the presence of another magnetic body, b) braiding, knotting and linking of flux tubes making possible topological quantum computation, c) contraction of flux tube in phase transition reducing the value of h_{eff} allowing two molecules to find each other in dense molecular soup. `http://tgdtheory.fi/appfigures/fluxtubedynamics.jpg`

A-8.2 Number theoretic entropy and negentropic entanglement

TGD inspired theory of consciousness relies heavily p-Adic norm allows an to define the notion of Shannon entropy for rational probabilities (and even those in algebraic extension of rationals) by replacing the argument of logarithm of probability with its p-adic norm. The resulting entropy can be negative and the interpretation is that number theoretic entanglement entropy defined by this formula for the p-adic prime minimizing its value serves as a measure for conscious information. This negentropy characterizes two-particle system and has nothing to do with the formal negative negentropy assignable to thermodynamic entropy characterizing single particle. Negentropy Maximization Principle (NMP) implies that number theoretic negentropy increases during evolution by quantum jumps. The condition that NMP is consistent with the standard quantum measurement theory requires that negentropic entanglement has a density matrix proportional to unit matrix so that in 2-particle case the entanglement matrix is unitary.

Fig. 22. Schrödinger cat is neither dead or alive. For negentropic entanglement this state would be stable. `http://tgdtheory.fi/appfigures/cat.jpg`

A-8.3 Life as something residing in the intersection of reality and p-adicities

In TGD inspired theory of consciousness p-adic space-time sheets correspond to space-time correlates for thoughts and intentions. The intersections of real and p-adic preferred extremals consist of points whose coordinates are rational or belong to some extension of rational numbers in preferred embedding space coordinates. They would correspond to the intersection of reality and various p-adicities representing the "mind stuff" of Descartes. There is temptation to assign life to the intersection of realities and p-adicities. The discretization of the chart map assigning to real space-time surface its p-adic counterpart would reflect finite cognitive resolution.

At the level of "world of classical worlds" (WCW) the intersection of reality and various p-adicities would correspond to space-time surfaces (or possibly partonic 2-surfaces) representable in terms of rational functions with polynomial coefficients with are rational or belong to algebraic extension of rationals.

The quantum jump replacing real space-time sheet with p-adic one (vice versa) would correspond to a buildup of cognitive representation (realization of intentional action).

Fig. 23. The quantum jump replacing real space-time surface with corresponding p-adic manifold can be interpreted as formation of though, cognitive representation. Its reversal would correspond to a transformation of intention to action. http://tgdtheory.fi/appfigures/padictoreal.jpg

A-8.4 Sharing of mental images

The 3-surfaces serving as correlates for sub-selves can topologically condense to disjoint large space-time sheets representing selves. These 3-surfaces can also have flux tube connections and this makes possible entanglement of sub-selves, which unentangled in the resolution defined by the size of sub-selves. The interpretation for this negentropic entanglement would be in terms of sharing of mental images. This would mean that contents of consciousness are not completely private as assumed in neuroscience.

Fig. 24. Sharing of mental images by entanglement of subselves made possible by flux tube connections between topologically condensed space-time sheets associated with mental images. http://tgdtheory.fi/appfigures/sharing.jpg

A-8.5 Time mirror mechanism

Zero energy ontology (ZEO) is crucial part of both TGD and TGD inspired consciousness and leads to the understanding of the relationship between geometric time and experience time and how the arrow of psychological time emerges. One of the basic predictions is the possibiity of negative energy signals propagating backwards in geometric time and having the property that entropy basically associated with subjective time grows in reversed direction of geometric time. Negative energy signals inspire time mirror mechanism (see **Fig.** http://tgdtheory.fi/appfigures/timemirror.jpg or **Fig.** 24 in the appendix of this book) providing mechanisms of both memory recall, realization of intentational action initiating action already in geometric past, and remote metabolism. What happens that negative energy signal travels to past and is reflected as positive energy signal and returns to the sender. This process works also in the reverse time direction.

Fig. 25. Zero energy ontology allows time mirror mechanism as a mechanism of memory recall. Essentially "seeing" in time direction is in question. http://tgdtheory.fi/appfigures/timemirror.jpg

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REFERENCES

Mathematics

[A1] A directory of all known zeta functions. Available at: https://secamlocal.ex.ac.uk/~mwatkins/zeta/directoryofzetafunctions.htm.

[A2] Atyiah-Singer index-theorem. Available at: https://en.wikipedia.org/wiki/Atiyah-Singer_index_theorem.

[A3] Coalgebra. Available at: https://en.wikipedia.org/wiki/Coalgebra.

[A4] Frobenius theorem (differential topology). Available at: https://en.wikipedia.org/wiki/Frobenius_theorem_(differential_topology).

[A5] Icosahedral graph. Wolfram MathWorld. Available at: https://mathworld.wolfram.com/IcosahedralGraph.html.

[A6] Knot theory. Available at: https://en.wikipedia.org/wiki/Knot_theory.

[A7] Langlands program. Available at: https://en.wikipedia.org/wiki/Langlands_program.

[A8] Mellin transform. Available at: https://en.wikipedia.org/wiki/Mellin_transform.

[A9] Octonions. Available at: https://en.wikipedia.org/wiki/Octonions.

[A10] Pythagorean triangles. Available at: https://en.wikipedia.org/wiki/Pythagorean_triangle.

[A11] Quantum group. Available at: https://en.wikipedia.org/wiki/Quantum_group.

[A12] Quaternions. Available at: https://en.wikipedia.org/wiki/Quaternion.

[A13] Riemann-Hilbert problem. Available at: https://en.wikipedia.org/wiki/Riemann-Hilbert_problem.

[A14] Torus knot. Available at: https://en.wikipedia.org/wiki/Torus_knot.

[A15] Weil conjectures. Available at: https://en.wikipedia.org/wiki/Weil_conjectures.

[A16] Why are there 1024 Hamiltonian cycles on an icosahedron? Available at: https://tinyurl.com/pmghcwd.

[A17] Yangian symmetry. Available at: https://en.wikipedia.org/wiki/Yangian.

[A18] Zeta function. Available at: https://en.wikipedia.org/wiki/Zeta_function.

[A19] Khrennikov A. *p-Adic Valued Distributions in Mathematical Physics*. Kluwer, Dordrecht, 1994.

[A20] Prastaro A. Geometry of PDEs: I: Integral bordism groups in PDEs. *J Math Anal & Appl*, 319(2):547–566, 2006. Available at: https://www.sciencedirect.com/science/article/pii/S0022247X05005998.

[A21] Prastaro A. Geometry of PDEs: II: Integral bordism groups in PDEs. *J Math Anal & Appl.* Available at: https://www.sciencedirect.com/science/article/pii/S0022247X05008115, 321(2):930–948, 2006. Available at: https://www.sciencedirect.com/science/article/pii/S0022247X05005998.

[A22] Robinson A. *Nonstandard Analysis.* North-Holland, Amsterdam, 1974.

[A23] Redfern EJ Allenby RB. *Introduction to Number Theory with Computing.* Edward Arnold, 1989.

[A24] Khrennikov AYu. p-Adic Probability and Statistics. *Dokl Akad Nauk*, (6), 1992.

[A25] Shafarevich IR Borevich ZI. *Number Theory.* Academic Press, 1966.

[A26] Freund PG Brekke L. p-Adic Numbers in Physics. *Phys Rep*, 233(1), 1993.

[A27] Pope CN. Eigenfunctions and $Spin^c$ Structures on CP_2, 1980.

[A28] Kreimer D Connes A. *Hofp algebras, renormalization, and non-commutative geometry*, volume 1999. Kluwer, 1998. Available at: https://arxiv.org/abs/hep-th/9912092.

[A29] Delborough R Daboul J. Matrix Representations of Octonions and Generalizations, 1999. Available at: https://arxiv.org/abs/hep-th/9906065.

[A30] Freed DS. The Geometry of Loop Groups, 1985.

[A31] Heckmann GJ Duistermaat JJ. *Inv Math*, 69, 1982.

[A32] Frenkel E. Recent Advances in Langlands program. *AMS*, 41(2):151–184, 2004.

[A33] Frenkel E. Lectures on Langlands program and conformal field theory, 2005. Available at: https://arxiv.org/abs/hep-th/0512172.

[A34] Witten E. Quantum field theory and the Jones polynomial. *Comm Math Phys*, 121:351–399, 1989.

[A35] Zeeman EC. *Catastrophe Theory.* Addison-Wessley Publishing Company, 1977.

[A36] Hanson J Eguchi T, Gilkey B. *Phys Rep*, 66, 1980.

[A37] Eisenhart. *Riemannian Geometry.* Princeton University Press, 1964.

[A38] Murty Ram M Esmonde J. *Problems in Algebraic Number Theory.* Springer Verlag, New York, 1991.

[A39] Brillhart J et al. Factorizations of $b^m \pm 1$. *AMS*, 1990.

[A40] Gouvea FQ. *p-Adic Numbers: An Introduction.* Springer Verlag, 1997.

[A41] Jones FR. *Braid groups, Hecke algebras and type II_1 factors.* 1983.

[A42] Shapiro ZYa Gelfand IM, Minklos RA. *Representations of the rotation and Lorentz groups and their applications.* Pergamon Press, 1963.

[A43] Pope CN Gibbons GW. CP_2 as gravitational instanton. *Comm Math Phys*, 55, 1977.

[A44] Pope CN Hawking SW. Generalized Spin Structures in Quantum Gravity. *Phys Lett*, (1), 1978.

[A45] N. Hitchin. Kählerian twistor spaces. *Proc London Math Soc*, 8(43):133–151, 1981.. Available at: https://tinyurl.com/pb8zpqo.

[A46] Besse AL. Arthur L (Besse is a pseudonym for a group of mathematicians mostly in France). *Einstein Manifolds.* Springer Verlag, 1987.

[A47] Baez J. Quasicrystals and the Riemann Hypothesis. The n-Category Cafe, 2013. Available at: https://golem.ph.utexas.edu/category/2013/06/quasicrystals_and_the_riemann.html..

[A48] Dixmier J. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981. First published in French in 1957: Les Algebres d'Operateurs dans l'Espace Hilbertien, Paris: Gauthier-Villars.

[A49] Milnor J. *Topology form Differential Point of View.* The University Press of Virginia, Virginia, 1965.

[A50] Baez JC. The Octonions. *Bull Amer Math Soc*, 39(2002), 2001. Available at: https://math.ucr.edu/home/baez/Octonions/octonions.html.

[A51] Rotelli P Leo de S. A New Definition of Hypercomplex Analyticity, 1997. Available at: https://arxiv.org/pdf/funct-an/9701004.pdf.

[A52] Barnsley M. *Fractals Everywhere.* Academic Press, 1988.

[A53] Eichler M. *Introduction to the theory of algebraic numbers and functions.* Academic Press, New York, 1966.

[A54] Kontsevich M. Operads and Motives in Deformation Quantization, 1999. textnormalAvailable at: https://arxiv.org/abs/math/9904055.

[A55] Reid M. McKay correspondence. Available at: https://arxiv.org/abs/alg-geom/9702016.

[A56] Spivak M. *Differential Geometry I,II,III,IV.* Publish or Perish, Boston, 1970.

[A57] Schroeder MR. *Number Theory in Science and Communication.* Springer Verlag, 1990.

[A58] Cartier P. A Mad Day's Work: From Grothendienck to Connes and Kontsevich: the Evolution of Concepts of Space and Symmetry. *AMS*, 38(4):389–408, 2001.

[A59] Hajac PM. On and around the Drinfeld double. Available at: https://www.impan.pl/~burgunde/WSBC09/Ddouble_Hajac.pdf.

[A60] Elwes R. The Ultimate logic: to infinity and beyond, 2011. Available at: https://tinyurl.com/3va48jq.

[A61] Harvey R. *Spinors and Calibrations.* Academic Press, New York, 1990.

[A62] Thom R. *Comm Math Helvet*, 28, 1954.

[A63] Helgason S. *Differential Geometry and Symmetric Spaces.* Academic Press, New York, 1962.

[A64] Majid S. *Foundations of Quantum Group Theory.* 2005. Available at: https://lib.mexmat.ru/books/1236.

[A65] Sawin S. Links, Quantum Groups, and TQFT's, 1995.Available at: https://arxiv.org/abs/q-alg/9506002.

[A66] Manogue CA Schray J. Octonionic representations of Clifford algebras and triality, 1994. Available at: https://arxiv.org/abs/hep-th/9407179.

[A67] White S Solla S, Sorkin G. *Configuration space analysis for optimization problems.* Springer Verlag, Berlin, 1986.

[A68] Abhyankar SS. Algebraic Geometry for Scientists and Engineers. *Math Surveys & Monographs*, 35, 1980.

[A69] Durt T. A new expression for mutually unbiased bases in prime power dimensions, 2004. Available at: https://arxiv.org/abs/quant-ph/0409090.

[A70] Smith T. D4-D5-E6 Physics, 1997. Available at: https://galaxy.cau.edu/tsmith/d4d5e6hist.html. The homepage contains a lot of information and ideas about the possible relationship of octonions and quaternions to physic.

[A71] Lieb EH Temperley NH. Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices:some exact results for the percolation problem. *Proc R Soc London*, 322, 1971.

[A72] Jones V. In and around the origin of quantum groups, 2003. Available at: https://arxiv.org/abs/math/0309199.

[A73] Wallace. *Differential Topology.* W. A. Benjamin, New York, 1968.

[A74] Vandoren S Wit de B, Rocek M. Hypermultiplets, Hyperkähler Cones and Quaternion-Kähler Geometry, 2001. Available at: https://arxiv.org/pdf/hep-th/0101161.pdf.

Theoretical Physics

[B1] Global Scaling. Available at: https://www.dr-nawrocki.de/globalscalingengl2.html.

[B2] Korteweg-de Vries equation. Available at: https://en.wikipedia.org/wiki/Korteweg\OT1\textendashde_Vries_equation.

[B3] Lax pair. Available at: https://en.wikipedia.org/wiki/Lax_pair.

[B4] Montonen Olive Duality. Available at: https://en.wikipedia.org/wiki/Montonen-Olive_duality.

[B5] Non-linear Schrödinger equation. Available at: https://en.wikipedia.org/wiki/Non-linear_Schr\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{o\global\mathchardef\accent@spacefactor\spacefactor}\let\begingroup\endgroup\relax\let\ignorespaces\relax\accent127o\egroup\spacefactor\accent@spacefactordinger_equation.

[B6] Sine-Gordon equation. Available at: https://en.wikipedia.org/wiki/Sine-Gordon.

[B7] Israel Gelfand, 2009. Available at: https://terrytao.wordpress.com/2009/10/07/israel-gelfand/#more-2860.

[B8] Lakthakia A. *Beltrami Fields in Chiral Media*, volume 2. World Scientific, Singapore, 1994.

[B9] Zee A. *The Unity of Forces in the Universe.* World Sci Press, Singapore, 1982.

[B10] John Baez. This Weeks Finds (Week 259), 2007. Available at: https://math.ucr.edu/home/baez/week259.html.

[B11] Drell S Björken J. *Relativistic Quantum Fields.* Mc Graw-Hill, New York, 1965.

[B12] Chowdbury D. *Spin Glasses and other Frustrated Systems.* World Scientific, 1986.

[B13] Rapoport D. Stochastic processes in conformal Riemann-Cartan-Weyl gravitation, 1991. Available at: https://link.springer.com/article/10.1007/BF00675614.

[B14] Witten E Dolan L, Nappi CR. Yangian Symmetry in $D = 4$ superconformal Yang-Mills theory, 2004. Available at: https://arxiv.org/abs/hep-th/0401243.

[B15] Plefka J Drummond J, Henn J. Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, 2009. Available at: https://cdsweb.cern.ch/record/1162372/files/jhep052009046.pdf.

[B16] Huang Y-T Elvang H. Scattering amplitudes, 2013. Available at: https://arxiv.org/pdf/1308.1697v1.pdf.

[B17] Arkani-Hamed N et al. Scattering amplitides and the positive Grassmannian. Available at: https://arxiv.org/pdf/1212.5605v1.pdf.

[B18] Arkani-Hamed N et al. The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM, 2010. Available at: https://arxiv.org/abs/1008.2958.

[B19] Minev ZK et al. To catch and reverse a quantum jump mid-flight, 2019. Available at: https://arxiv.org/abs/1803.00545.

[B20] Parisi G. *Field Theory, Disorder and Simulations*. World Scientific, 1992.

[B21] Haken H. *Synergetics*. Springer Verlag, 1978.

[B22] Strominger A Horowitz G, Maldacena J, 1996. Available at: https://arxiv.org/abs/hep-th/9603019.

[B23] Klebanov IR. TASI Lectures: Introduction to the AdS/CFT Correspondence, 2000. Available at: https://arxiv.org/abs/hep-th/0009139.

[B24] Huang K. *Quarks,Leptons* & *Gauge Fields*. World Scientific, 1982.

[B25] Tosi MP Lundqvist S, March NH. *Order and Chaos in Nonlinear Physical Systems*, volume 295. Plenum, New York, 1988.

[B26] Arkani-Hamed N and Maldacena J. Cosmological Collider Physics, 2015.Available at:https://arxiv.org/pdf/1503.08043v1.

[B27] Lister R. Simulated Annealing: Quasi-fractals and Quasi-failures, 1997. Available at: https://www.csu.edu.au/ci/vol2/lister/node1.html.

[B28] Penrose R. The Central Programme of Twistor Theory. *Chaos, Solitons & Fractals*, 10, 1999.

[B29] Cutkosky RE. *J Math Phys*, (1):429–433, 1960.

Particle and Nuclear Physics

[C1] Lamb shift. Available at: https://en.wikipedia.org/wiki/Lamb_shift.

[C2] Weiner N Kaplan DB, Nelson AE. Neutrino Oscillations as a Probe of Dark Energy, 2004. Available at: https://arxiv.org/abs/hep-ph/0401099.

[C3] Turner MS Ressell MT. *Astrophys*, 14, 1990.

Condensed Matter Physics

[D1] A Bibliography of $1/f$ noise. Available at: https://linkage.rockefeller.edu/wli/1fnoise.

[D2] Wilczek F Mackenzie R. *Rev Mod Phys . A*, 3:2827, 1988.

[D3] Golovko VA. Why does superfluid helium leak out of an open container?, 2012. Available at: https://arxiv.org/pdf/1103.0517.pdf.

Cosmology and Astro-Physics

[E1] Nottale L Da Rocha D. Gravitational Structure Formation in Scale Relativity, 2003. Available at: https://arxiv.org/abs/astro-ph/0310036.

[E2] Tajmar M et al. Experimental Detection of Gravimagnetic London Moment, 2006. Available at: https://arxiv.org/abs/gr-gc0603033. See also the popular article "Towards new tests of general relativity", at https://www.esa.int/SPECIALS/GSP/SEM0L6OVGJE_0.html.

[E3] Weinberg S. *Gravitation and Cosmology.* Wiley, New York, 1967.

[E4] Matos de CJ Tajmar M. Local Photon and Graviton Mass and Its Consequences, 2006. Available at: https://arxiv.org/abs/gr-gc0603032.

Fringe Physics

[H1] Godin SM Roshchin VV. An Experimental Investigation of the Physical Effects in a Dynamic Magnetic System. *New Energy Technologies*, 1, 2001.

Biology

[I1] Fiaxat JD. A hypothesis on the rhythm of becoming. *World Futures*, 36:31–36, 1993.

[I2] Fiaxat JD. The hidden rhythm of evolution, 2014. Available at: `https://byebyedarwin.blogspot.fi/p/english-version_01.html`.

[I3] England J Perunov N, Marsland R. Statistical Physics of Adaptation, 2014. Available at: `https://arxiv.org/pdf/1412.1875v1.pdf`.

Neuroscience and Consciousness

[J1] Khrennikov AY. Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social, and anomalous phenomena. *Found Phys*, 29:1065–2098, 1999.

[J2] Libet B. Readiness potentials preceding unrestricted spontaneous and preplanned voluntary acts, 1982. Available at: `https://tinyurl.com/jqp1`. See also the article *Libet's Research on Timing of Conscious Intention to Act: A Commentary* of Stanley Klein at `https://tinyurl.com/jqp1`.

[J3] Lavallee F C Persinger MA. Theoretical and Experimental Evidence of Macroscopic Entanglement between Human Brain Activity and Photon Emissions: Implications for Quantum Consciousness and Future Applications. *J Consc Expl & Res*, 1(7):785–807, 2010.

Books related to TGD

[K1] Pitkänen M. Overall View About TGD from Particle Physics Perspective. In *Topological Geometrodynamics: Overview: Part II. https://tgdtheory.fi/tgdhtml/Btgdview2.html*.

[K2] Pitkänen M. *Topological Geometrodynamics.* 1983. Thesis in Helsinki University 1983.

[K3] Pitkänen M. DNA as Topological Quantum Computer. In *Quantum - and Classical Computation in TGD Universe. https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: `https://tgdtheory.fi/pdfpool/dnatqc.pdf`, 2015.

[K4] Pitkänen M. Topological Quantum Computation in TGD Universe. In *Quantum - and Classical Computation in TGD Universe.* *https://tgdtheory.fi/tgdhtml/Btgdcomp.html*. Available at: https://tgdtheory.fi/pdfpool/tqc.pdf, 2015.

[K5] Pitkänen M. Philosophy of Adelic Physics. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/adelephysics.pdf, 2017.

[K6] Pitkänen M. About Absolute Galois Group. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/agg.pdf, 2023.

[K7] Pitkänen M. About Nature of Time. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timenature.pdf, 2023.

[K8] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/prext.pdf, 2023.

[K9] Pitkänen M. About the New Physics Behind Qualia. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/newphys.pdf, 2023.

[K10] Pitkänen M. About twistor lift of TGD? In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/hgrtwistor.pdf, 2023.

[K11] Pitkänen M. Appendix A: Quantum Groups and Related Structures. In *Hyper-finite Factors and Dark Matter Hierarchy: Part I.* Available at: https://tgdtheory.fi/pdfpool/bialgebra.pdf, 2023.

[K12] Pitkänen M. Are dark photons behind biophotons? In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biophotonslian.pdf, 2023.

[K13] Pitkänen M. Are higher structures needed in the categorification of TGD? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/nstructures.pdf, 2023.

[K14] Pitkänen M. Basic Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/class.pdf, 2023.

[K15] Pitkänen M. Basic Properties of CP_2 and Elementary Facts about p-Adic Numbers. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://www.tgdtheory.fi/public_html/pdfpool/append.pdf, 2023.

[K16] Pitkänen M. *Bio-Systems as Conscious Holograms.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/holography.html., 2023.

[K17] Pitkänen M. *Bio-Systems as Self-Organizing Quantum Systems.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioselforg.html., 2023.

[K18] Pitkänen M. Can one apply Occam's razor as a general purpose debunking argument to TGD? In *Topological Geometrodynamics: Overview: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/simplicity.pdf, 2023.

[K19] Pitkänen M. Category Theory and Quantum TGD. In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/categorynew.pdf, 2023.

[K20] Pitkänen M. Classical TGD. In *Topological Geometrodynamics: Overview: Part I:https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdclass.pdf, 2023.

[K21] Pitkänen M. Comments on the recent experiments by the group of Michael Persinger. In *TGD and EEG: Part I. https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/persconsc.pdf, 2023.

[K22] Pitkänen M. Conscious Information and Intelligence. In *TGD Inspired Theory of Consciousness: Part II. https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/intsysc.pdf, 2023.

[K23] Pitkänen M. Construction of elementary particle vacuum functionals. In *p-Adic Physics. https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/elvafu.pdf, 2023.

[K24] Pitkänen M. Construction of Quantum Theory: M-matrix. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/towards.pdf, 2023.

[K25] Pitkänen M. Construction of Quantum Theory: Symmetries. In *Quantum TGD: Part I. https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/quthe.pdf, 2023.

[K26] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/compl1.pdf, 2023.

[K27] Pitkänen M. Cosmic Strings. In *Physics in Many-Sheeted Space-Time: Part II. https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cstrings.pdf, 2023.

[K28] Pitkänen M. Could categories, tensor networks, and Yangians provide the tools for handling the complexity of TGD? In *TGD and Hyper-finite Factors. https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/Yangianagain.pdf, 2023.

[K29] Pitkänen M. Criticality and dark matter: part I. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark1.pdf, 2023.

[K30] Pitkänen M. Criticality and dark matter: part II. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark2.pdf, 2023.

[K31] Pitkänen M. Criticality and dark matter: part III. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/qcritdark3.pdf, 2023.

[K32] Pitkänen M. Dark Forces and Living Matter. In *Bio-Systems as Self-Organizing Quantum Systems. https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/darkforces.pdf, 2023.

[K33] Pitkänen M. Dark Matter Hierarchy and Hierarchy of EEGs. In *TGD and EEG: Part I. https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/eegdark.pdf, 2023.

[K34] Pitkänen M. Dark Nuclear Physics and Condensed Matter. In *TGD and Nuclear Physics. https://tgdtheory.fi/tgdhtml/Bnucl.html*. Available at: https://tgdtheory.fi/pdfpool/exonuclear.pdf, 2023.

[K35] Pitkänen M. Does Riemann Zeta Code for Generic Coupling Constant Evolution? In *TGD as a Generalized Number Theory: Part III. https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/fermizeta.pdf, 2023.

[K36] Pitkänen M. Does TGD Predict a Spectrum of Planck Constants? In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/Planck, 2023.

[K37] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Topological Geometrodynamics: Overview: Part II. https://tgdtheory.fi/tgdhtml/Btgdoverview2*. Available at: https://tgdtheory.fi/pdfpool/vNeumannnew, 2023.

[K38] Pitkänen M. From Principles to Diagrams. In *Quantum TGD: Part III. https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/diagrams.pdf, 2023.

[K39] Pitkänen M. Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory. In *TGD as a Generalized Number Theory: Part I. https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/mblocks.pdf, 2023.

[K40] Pitkänen M. *Genes and Memes*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/genememe.html., 2023.

[K41] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. In *Dark Matter and TGD: https://tgdtheory.fi/tgdhtml/Bdark.html*. Available at: https://tgdtheory.fi/pdfpool/tensornet.pdf, 2023.

[K42] Pitkänen M. *Hyper-finite Factors and Dark Matter Hierarchy*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/neuplanck.html., 2023.

[K43] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/kahler.pdf., 2023.

[K44] Pitkänen M. Knots and TGD. In *Quantum Physics as Infinite-Dimensional Geometry. https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/knotstgd.pdf, 2023.

[K45] Pitkänen M. Langlands Program and TGD. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/Langlands.pdf, 2023.

[K46] Pitkänen M. Langlands Program and TGD: Years Later. In *TGD as a Generalized Number Theory: Part II. https://tgdtheory.fi/tgdhtml/Btgdnumber2.html*. Available at: https://tgdtheory.fi/pdfpool/langlandsnew.pdf, 2023.

[K47] Pitkänen M. Life and Death, and Consciousness. In *TGD Inspired Theory of Consciousness: Part I. https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lifedeath.pdf, 2023.

[K48] Pitkänen M. Macroscopic Quantum Phenomena and CP_2 Geometry. In *TGD and Condensed Matter. https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/super.pdf, 2023.

[K49] Pitkänen M. *Magnetospheric Consciousness*. Online book. Available at: https://www.tgdtheory.fi/tgdhtml/magnconsc.html., 2023.

[K50] Pitkänen M. Many-Sheeted DNA. In *Genes and Memes: Part I. https://tgdtheory.fi/tgdhtml/Bgenememe1.html*. Available at: https://tgdtheory.fi/pdfpool/genecodec.pdf, 2023.

[K51] Pitkänen M. Massless states and particle massivation. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mless.pdf, 2023.

[K52] Pitkänen M. *Mathematical Aspects of Consciousness Theory.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/mathconsc.html, 2023.

[K53] Pitkänen M. Matter, Mind, Quantum. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/conscic.pdf, 2023.

[K54] Pitkänen M. More about TGD Inspired Cosmology. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cosmomore.pdf, 2023.

[K55] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infmotives.pdf, 2023.

[K56] Pitkänen M. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/nmpc.pdf, 2023.

[K57] Pitkänen M. New Physics Predicted by TGD: Part I. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys1.pdf, 2023.

[K58] Pitkänen M. New Physics Predicted by TGD: Part II. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/TGDnewphys2.pdf, 2023.

[K59] Pitkänen M. Non-standard Numbers and TGD. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/infsur.pdf, 2023.

[K60] Pitkänen M. Nuclear String Hypothesis. In *TGD and Nuclear Physics.* *https://tgdtheory.fi/tgdhtml/Bnucl.html*. Available at: https://tgdtheory.fi/pdfpool/nuclstring.pdf, 2023.

[K61] Pitkänen M. Number theoretic vision, Hyper-finite Factors and S-matrix. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/UandM.pdf, 2023.

[K62] Pitkänen M. *p-Adic length Scale Hypothesis.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/padphys.html., 2023.

[K63] Pitkänen M. p-Adic Numbers and Generalization of Number Concept. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/padmat.pdf, 2023.

[K64] Pitkänen M. p-Adic Particle Massivation: Hadron Masses. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/mass3.pdf, 2023.

[K65] Pitkänen M. p-Adic Physics as Physics of Cognition and Intention. In *TGD Inspired Theory of Consciousness: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdconsc2.html*. Available at: https://tgdtheory.fi/pdfpool/cognic.pdf, 2023.

[K66] Pitkänen M. p-Adic Physics: Physical Ideas. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/phblocks.pdf, 2023.

[K67] Pitkänen M. Physics as a Generalized Number Theory. In *Topological Geometrodynamics: Overview: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdnumber.pdf, 2023.

[K68] Pitkänen M. Quantum Adeles. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/galois.pdf, 2023.

[K69] Pitkänen M. Quantum Antenna Hypothesis. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/tubuc.pdf, 2023.

[K70] Pitkänen M. Quantum Arithmetics and the Relationship between Real and p-Adic Physics. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/qarithmetics.pdf, 2023.

[K71] Pitkänen M. Quantum Astrophysics. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/qastro.pdf, 2023.

[K72] Pitkänen M. Quantum gravity, dark matter, and prebiotic evolution. In *Evolution in TGD Universe.* *https://tgdtheory.fi/tgdhtml/Btgdevolution.html*. Available at: https://tgdtheory.fi/pdfpool/hgrprebio.pdf, 2023.

[K73] Pitkänen M. Quantum Hall effect and Hierarchy of Planck Constants. In *TGD and Condensed Matter.* *https://tgdtheory.fi/tgdhtml/BTGDcondmat.html*. Available at: https://tgdtheory.fi/pdfpool/anyontgd.pdf, 2023.

[K74] Pitkänen M. *Quantum Hardware of Living Matter.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/bioware.html., 2023.

[K75] Pitkänen M. Quantum Mind and Neuroscience. In *TGD and EEG: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPN.pdf, 2023.

[K76] Pitkänen M. Quantum Mind in TGD Universe. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPC.pdf, 2023.

[K77] Pitkänen M. Quantum Mind, Magnetic Body, and Biological Body. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPB.pdf, 2023.

[K78] Pitkänen M. Quantum Model for Bio-Superconductivity: I. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondI.pdf, 2023.

[K79] Pitkänen M. Quantum Model for Bio-Superconductivity: II. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/biosupercondII.pdf, 2023.

[K80] Pitkänen M. Quantum Model for Hearing. In *TGD and EEG: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdeeg2.html*. Available at: https://tgdtheory.fi/pdfpool/hearing.pdf, 2023.

[K81] Pitkänen M. Quantum Model for Nerve Pulse. In *TGD and EEG: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdeeg1.html*. Available at: https://tgdtheory.fi/pdfpool/nervepulse.pdf, 2023.

[K82] Pitkänen M. *Quantum TGD.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdquantum.html., 2023.

[K83] Pitkänen M. Quantum Theory of Self-Organization. In *Bio-Systems as Self-Organizing Quantum Systems.* *https://tgdtheory.fi/tgdhtml/BbioSO.html*. Available at: https://tgdtheory.fi/pdfpool/selforgac.pdf, 2023.

[K84] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwnew.pdf, 2023.

[K85] Pitkänen M. Riemann Hypothesis and Physics. In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/riema.pdf, 2023.

[K86] Pitkänen M. Some questions related to the twistor lift of TGD. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistquestions.pdf, 2023.

[K87] Pitkänen M. SUSY in TGD Universe. In *p-Adic Physics.* *https://tgdtheory.fi/tgdhtml/Bpadphys.html*. Available at: https://tgdtheory.fi/pdfpool/susychap.pdf, 2023.

[K88] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds". In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/wcwsymm.pdf, 2023.

[K89] Pitkänen M. TGD and Astrophysics. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/astro.pdf, 2023.

[K90] Pitkänen M. TGD and Cosmology. In *Physics in Many-Sheeted Space-Time: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdclass2.html*. Available at: https://tgdtheory.fi/pdfpool/cosmo.pdf, 2023.

[K91] Pitkänen M. *TGD and EEG.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdeeg.html, 2023.

[K92] Pitkänen M. *TGD and Fringe Physics.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/freenergy.html., 2023.

[K93] Pitkänen M. TGD and Nuclear Physics. In *TGD and Nuclear Physics.* *https://tgdtheory.fi/tgdhtml/Bnucl.html*. Available at: https://tgdtheory.fi/pdfpool/padnucl.pdf, 2023.

[K94] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionc.pdf, 2023.

[K95] Pitkänen M. TGD as a Generalized Number Theory: p-Adicization Program. In *Quantum Physics as Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visiona.pdf, 2023.

[K96] Pitkänen M. TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/visionb.pdf, 2023.

[K97] Pitkänen M. TGD Based View about Classical Fields in Relation to Consciousness Theory and Quantum Biology. In *TGD and Quantum Biology: Part I.* *https://tgdtheory.fi/tgdhtml/Bqbio1.html*. Available at: https://tgdtheory.fi/pdfpool/maxtgdc.pdf, 2023.

[K98] Pitkänen M. *TGD Based View About Living Matter and Remote Mental Interactions.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdlian.html, 2023.

[K99] Pitkänen M. *TGD Inspired Theory of Consciousness.* Online book. Available at: https://www.tgdtheory.fi/tgdhtml/tgdconsc.html., 2023.

[K100] Pitkänen M. TGD View about Coupling Constant Evolution. In *Quantum TGD: Part II.* *https://tgdtheory.fi/tgdhtml/Btgdquantum2.html*. Available at: https://tgdtheory.fi/pdfpool/ccevolution.pdf, 2023.

[K101] Pitkänen M. The classical part of the twistor story. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/twistorstory.pdf, 2023.

[K102] Pitkänen M. The Geometry of the World of the Classical Worlds. In *Topological Geometrodynamics: Overview: Part I:https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgeom.pdf, 2023.

[K103] Pitkänen M. The Recent View about Twistorialization in TGD Framework. In *Quantum TGD: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdquantum3.html*. Available at: https://tgdtheory.fi/pdfpool/smatrix.pdf, 2023.

[K104] Pitkänen M. The Relationship Between TGD and GRT. In *Physics in Many-Sheeted Space-Time: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdclass1.html*. Available at: https://tgdtheory.fi/pdfpool/tgdgrt.pdf., 2023.

[K105] Pitkänen M. Three new physics realizations of the genetic code and the role of dark matter in bio-systems. In *Genes and Memes: Part II.* *https://tgdtheory.fi/tgdhtml/Bgenememe2.html*. Available at: https://tgdtheory.fi/pdfpool/dnatqccodes.pdf, 2023.

[K106] Pitkänen M. Time and Consciousness. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/timesc.pdf, 2023.

[K107] Pitkänen M. *Topological Geometrodynamics: an Overview.* Online book.Available at: https://www.tgdtheory.fi/tgdhtml/tgdview.html., 2023.

[K108] Pitkänen M. Topological Geometrodynamics: Basic Visions. In *TGD Inspired Theory of Consciousness: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdconsc1.html*. Available at: https://tgdtheory.fi/pdfpool/lianPTGD.pdf, 2023.

[K109] Pitkänen M. Unified Number Theoretical Vision. In *TGD as a Generalized Number Theory: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdnumber1.html*. Available at: https://tgdtheory.fi/pdfpool/numbervision.pdf, 2023.

[K110] Pitkänen M. Was von Neumann Right After All? In *TGD and Hyper-finite Factors.* *https://tgdtheory.fi/tgdhtml/BHFF.html*. Available at: https://tgdtheory.fi/pdfpool/vNeumann.pdf, 2023.

[K111] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry.* *https://tgdtheory.fi/tgdhtml/Btgdgeom.html*. Available at: https://tgdtheory.fi/pdfpool/cspin.pdf, 2023.

[K112] Pitkänen M. What p-Adic Icosahedron Could Mean? And What about p-Adic Manifold? In *TGD as a Generalized Number Theory: Part III.* *https://tgdtheory.fi/tgdhtml/Btgdnumber3.html*. Available at: https://tgdtheory.fi/pdfpool/picosahedron.pdf, 2023.

[K113] Pitkänen M. Why TGD and What TGD is? In *Topological Geometrodynamics: an Overview.* *https://tgdtheory.fi/tgdhtml/Btgdview1.html*. Available at: https://tgdtheory.fi/pdfpool/WhyTGD.pdf, 2023.

[K114] Pitkänen M. Zero Energy Ontology. In *Quantum TGD: Part I.* *https://tgdtheory.fi/tgdhtml/Btgdquantum1.html*. Available at: https://tgdtheory.fi/pdfpool/ZEO.pdf, 2023.

Articles about TGD

[L1] Pitkänen M. A Strategy for Proving Riemann Hypothesis. Available at: https://www.emis.math.ca/EMIS/journals/AMUC/, 2003.

[L2] Pitkänen M. Further Progress in Nuclear String Hypothesis. Available at: https://tgdtheory.fi/public_html/articles/nuclstring.pdf., 2007.

[L3] Pitkänen M. TGD Inspired Theory of Consciousness. Available at: https://tgdtheory.fi/tgdconsc.pdf., 2008.

[L4] Pitkänen M. Topological Geometrodynamics: What Might Be The Basic Principles. Available at: https://tgdtheory.fi/public_html/articles/tgd2008.pdf., 2008.

[L5] Pitkänen M. Topological Geometrodynamics: What Might Be the Basic Principles. Available at: https://tgdtheory.fi/public_html/articles/tgd2008.pdf., 2008.

[L6] Pitkänen M. CMAP representations about TGD, and TGD inspired theory of consciousness and quantum biology. Available at: https://www.tgdtheory.fi/tgdglossary.pdf., 2014.

[L7] Pitkänen M. Geometric theory of harmony. Available at: https://tgdtheory.fi/public_html/articles/harmonytheory.pdf., 2014.

[L8] Pitkänen M. Could one Define Dynamical Homotopy Groups in WCW? Available at: https://tgdtheory.fi/public_html/articles/dynatopo.pdf., 2015.

[L9] Pitkänen M. Could one realize number theoretical universality for functional integral? Available at: https://tgdtheory.fi/public_html/articles/ntu.pdf., 2015.

[L10] Pitkänen M. Positivity of N = 4 scattering amplitudes from number theoretical universality. Available at: https://tgdtheory.fi/public_html/articles/positivity.pdf., 2015.

[L11] Pitkänen M. The vanishing of conformal charges as a gauge conditions selecting preferred extremals of Kähler action. Available at: https://tgdtheory.fi/public_html/articles/variationalhamed.pdf., 2015.

[L12] Pitkänen M. What went wrong with symmetries? Available at: https://tgdtheory.fi/public_html/articles/whatiswrong.pdf., 2015.

[L13] Pitkänen M. About minimal surface extremals of Kähler action. Available at: https://tgdtheory.fi/public_html/articles/minimalkahler.pdf., 2016.

[L14] Pitkänen M. Combinatorial Hierarchy: two decades later. Available at: https://tgdtheory.fi/public_html/articles/CH.pdf., 2016.

[L15] Pitkänen M. Holography and Quantum Error Correcting Codes: TGD View. Available at: https://tgdtheory.fi/public_html/articles/tensornet.pdf., 2016.

[L16] Pitkänen M. Is the sum of p-adic negentropies equal to real entropy? Available at: https://tgdtheory.fi/public_html/articles/adelicinfo.pdf., 2016.

[L17] Pitkänen M. p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework. Available at: https://tgdtheory.fi/public_html/articles/padicgeom.pdf., 2016.

[L18] Pitkänen M. TGD Inspired Comments about Integrated Information Theory of Consciousness. Available at: https://tgdtheory.fi/public_html/articles/tononikoch.pdf., 2016.

[L19] Pitkänen M. Why Mersenne primes are so special? Available at: https://tgdtheory.fi/public_html/articles/whymersennes.pdf., 2016.

[L20] Pitkänen M. About parity violation in hadron physics. Available at: https://tgdtheory.fi/public_html/articles/hadroPbreak.pdf., 2017.

[L21] Pitkänen M. p-Adicization and adelic physics. Available at: https://tgdtheory.fi/public_html/articles/adelicphysics.pdf., 2017.

[L22] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L23] Pitkänen M. Philosophy of Adelic Physics. Available at: https://tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L24] Pitkänen M. Quantum computations without definite causal structure: TGD view. Available at: https://tgdtheory.fi/public_html/articles/causalstructure.pdf., 2017.

[L25] Pitkänen M. Questions about twistor lift of TGD. Available at: https://tgdtheory.fi/public_html/articles/twistquestions.pdf., 2017.

[L26] Pitkänen M. Re-examination of the basic notions of TGD inspired theory of consciousness. Available at: https://tgdtheory.fi/public_html/articles/conscrit.pdf., 2017.

[L27] Pitkänen M. Some questions related to the twistor lift of TGD. Available at: https://tgdtheory.fi/public_html/articles/graviconst.pdf., 2017.

[L28] Pitkänen M. TGD view about coupling constant evolution. Available at: https://tgdtheory.fi/public_html/articles/ccevolution.pdf., 2018.

[L29] Pitkänen M. The Recent View about Twistorialization in TGD Framework. Available at: https://tgdtheory.fi/public_html/articles/smatrix.pdf., 2018.

[L30] Pitkänen M. Copenhagen interpretation dead: long live ZEO based quantum measurement theory! Available at: https://tgdtheory.fi/public_html/articles/Bohrdead.pdf., 2019.

[L31] Pitkänen M. Earhquakes and volcanic eruptions as macroscopic quantum jumps in zero energy ontology. Available at: https://tgdtheory.fi/public_html/articles/earthquakes.pdf., 2019.

[L32] Pitkänen M. $M^8 - H$ duality and consciousness. Available at: https://tgdtheory.fi/public_html/articles/M8Hconsc.pdf., 2019.

[L33] Pitkänen M. Quantum self-organization by h_{eff} changing phase transitions. Available at: https://tgdtheory.fi/public_html/articles/heffselforg.pdf., 2019.

[L34] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https://tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L35] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: https://tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L36] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: https://tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L37] Pitkänen M. About the role of Galois groups in TGD framework. https://tgdtheory.fi/public_html/articles/GaloisTGD.pdf., 2021.

[L38] Pitkänen M. Some questions concerning zero energy ontology. https://tgdtheory.fi/public_html/articles/zeonew.pdf., 2021.

[L39] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L40] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[L41] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[L42] Pitkänen M. Finite Fields and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L43] Pitkänen M. Intersection form for 4-manifolds, knots and 2-knots, smooth exotics, and TGD. https://tgdtheory.fi/public_html/articles/finitefieldsTGD.pdf., 2022.

[L44] Pitkänen M. Some New Ideas Related to Langlands Program *viz.* TGD. https://tgdtheory.fi/public_html/articles/Langlands2022.pdf., 2022.

[L45] Pitkänen M. Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole. https://tgdtheory.fi/public_html/articles/fusionTGD.pdf., 2022.

[L46] Pitkänen M. Two objections against p-adic thermodynamics and their resolution. https://tgdtheory.fi/public_html/articles/padmass2022.pdf., 2022.

[L47] Pitkänen M. Holography and Hamilton-Jacobi Structure as 4-D generalization of 2-D complex structure. https://tgdtheory.fi/public_html/articles/HJ.pdf., 2023.

[L48] Pitkänen M. New result about causal diamonds from the TGD view point of view. https://tgdtheory.fi/public_html/articles/CDconformal.pdf., 2023.

[L49] Pitkänen M. Reduction of standard model structure to CP_2 geometry and other key ideas of TGD. https://tgdtheory.fi/public_html/articles/cp2etc.pdf., 2023.

[L50] Pitkänen M. Symmetries and Geometry of the "World of Classical Worlds" . https://tgdtheory.fi/public_html/articles/wcwsymm.pdf., 2023.

[L51] Pitkänen M. A fresh look at M^8H duality and Poincare invariance. https://tgdtheory.fi/public_html/articles/TGDcritics.pdf., 2024.

[L52] Pitkänen M. What gravitons are and could one detect them in TGD Universe? https://tgdtheory.fi/public_html/articles/whatgravitons.pdf., 2024.

[L53] Pitkänen M and Rastmanesh R. Homeostasis as self-organized quantum criticality. Available at: https://tgdtheory.fi/public_html/articles/SP.pdf., 2020.

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